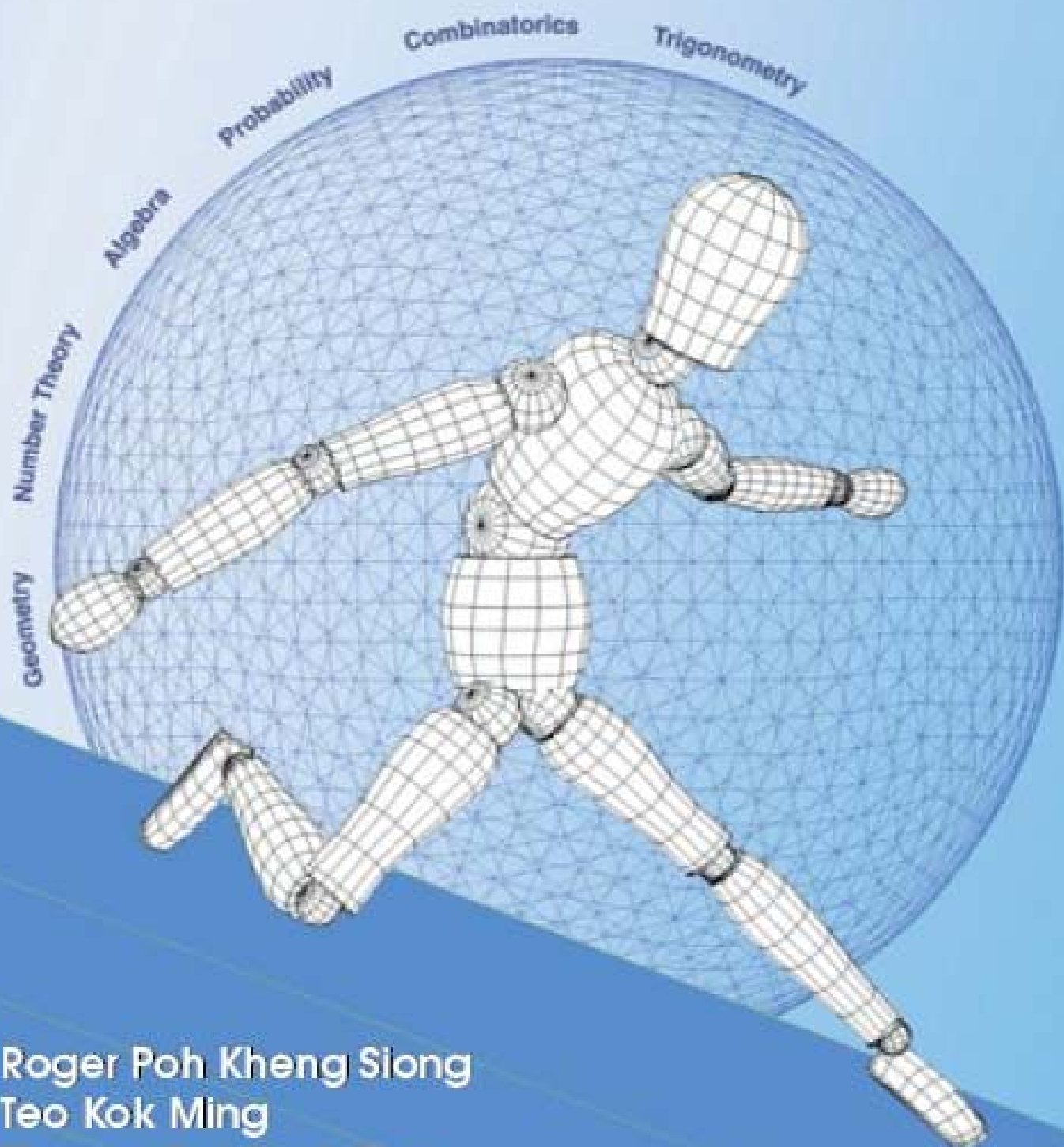


SINGAPORE MATHEMATICAL OLYMPIADS 2008



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Published by Singapore Mathematical Society

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2008

(Junior Section)

Tuesday, 27 May 2008

0930 – 1200 hrs

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answers on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.

For the other short questions, write your answer in the answer sheet and shade the appropriate bubble below your answer.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

- 1 How many zeroes does the product of 25^5 , 150^4 and 2008^3 end with?
- (A) 5
(B) 9
(C) 10
(D) 12
(E) 13
- 2 Given that $\sqrt{2x+y} + \sqrt{x^2-9} = 0$, find the value(s) of $y-x$.
- (A) -9
(B) -6
(C) -9 or 9
(D) -3 or 3
(E) None of the above
- 3 The number of integers between 208 and 2008 ending with 1 is
- (A) 101
(B) 163
(C) 179
(D) 180
(E) 200
- 4 The remainder when $7^{2008} + 9^{2008}$ is divided by 64 is
- (A) 2
(B) 4
(C) 8
(D) 16
(E) 32
- 5 John has two 20 cent coins and three 50 cent coins in his pocket. He takes two coins out of his pocket, at random, one after the other without replacement. What is the probability that the total value of the two coins taken out is 70 cents?
- (A) $\frac{6}{25}$
(B) $\frac{3}{10}$
(C) $\frac{12}{25}$
(D) $\frac{3}{5}$
(E) $\frac{13}{20}$

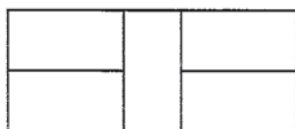
- 6 In the following sum, O represent the digit 0. A , B , X and Y each represents distinct digit. How many possible digits can A be?

$$\begin{array}{r} A O O B A O O B \\ + B O O A B O O A \\ \hline X X O X Y X O X X \end{array}$$

- (A) 6
(B) 7
(C) 8
(D) 9
(E) 10
- 7 The least integer that is greater than $(2 + \sqrt{3})^2$ is
- (A) 13
(B) 14
(C) 15
(D) 16
(E) 17
- 8 Let x , y and z be non-negative numbers. Suppose $x + y = 10$ and $y + z = 8$. Let $S = x + z$. What is the sum of the maximum and the minimum value of S ?
- (A) 16
(B) 18
(C) 20
(D) 24
(E) 26
- 9 How many integer solutions (x, y, z) are there to the equation $xyz = 2008$?
- (A) 30
(B) 60
(C) 90
(D) 120
(E) 150
- 10 The last two digits of 9^{2008} is
- (A) 01
(B) 21
(C) 41
(D) 61
(E) 81

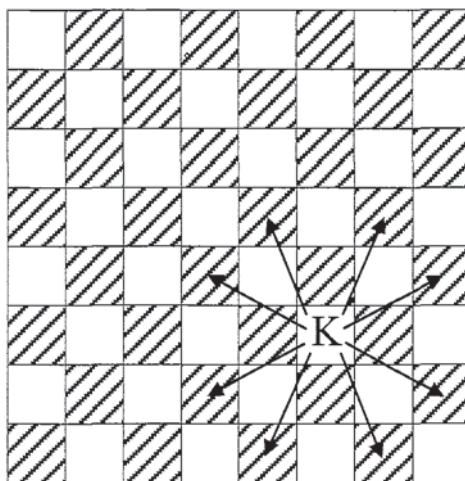
Short Questions

- 11 Find the remainder when $x^{2008} + 2008x + 2008$ is divided by $x + 1$.
- 12 Find the maximum value of $\sqrt{x - 144} + \sqrt{722 - x}$.
- 13 Five identical rectangles of area 8 cm^2 are arranged into a large rectangle as shown.



Find the perimeter of this large rectangle.

- 14 60 students were interviewed. Of these, 33 liked swimming and 36 liked soccer. Find the greatest possible number of students who neither liked swimming nor soccer.
- 15 As shown in the picture, the knight can move to any of the indicated squares of the 8×8 chessboard in 1 move. If the knight starts from the position shown, find the number of possible landing positions after 20 consecutive moves.



- 16 Given that $\alpha + \beta = 17$ and $\alpha\beta = 70$, find the value of $|\alpha - \beta|$.
- 17 Evaluate (in simplest form)

$$\sqrt{2008 + 2007 \sqrt{2008 + 2007 \sqrt{2008 + 2007 \sqrt{\dots}}}}$$

- 18 Find the sum of all the positive integers less than 999 that are divisible by 15.

- 19 A brand of orange juice is available in shop A and shop B at an original price of \$2.00 per bottle. Shop A provides the "buy 4 get 1 free" promotion and shop B provides 15% discount if one buys 4 bottles or more. Find the minimum cost (in cents) if one wants to buy 13 bottles of the orange juice.

- 20 Anna randomly picked five integers from the following list

53, 62, 66, 68, 71, 82, 89

and discover that the average value of the five integers she picked is still an integer. If two of the integers she picked were 62 and 89, find the sum of the remaining three integers.

- 21 Suppose the equation $||x - a| - b| = 2008$ has 3 distinct real roots and $a \neq 0$. Find the value of b .

- 22 Find the value of the integer n for the following pair of simultaneous equations to have no solution.

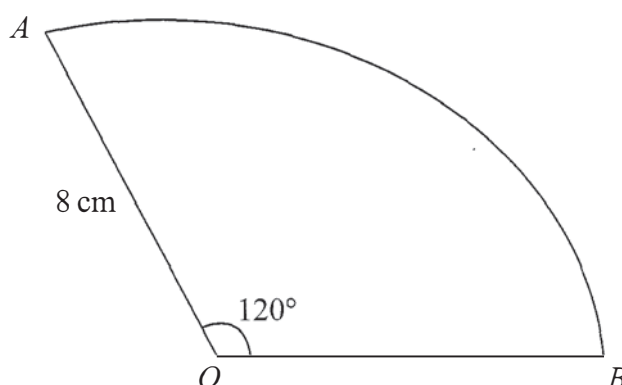
$$\begin{aligned} 2x &= 1 + ny, \\ nx &= 1 + 2y. \end{aligned}$$

- 23 There are 88 numbers $a_1, a_2, a_3, \dots, a_{88}$ and each of them is either equals to -3 or -1 . Given that $a_1^2 + a_2^2 + \dots + a_{88}^2 = 280$, find the value of $a_1^4 + a_2^4 + \dots + a_{88}^4$.

- 24 Find the value of $\frac{\frac{1}{2} - \frac{1}{3}}{\frac{1}{3} - \frac{1}{4}} \times \frac{\frac{1}{4} - \frac{1}{5}}{\frac{1}{5} - \frac{1}{6}} \times \frac{\frac{1}{6} - \frac{1}{7}}{\frac{1}{7} - \frac{1}{8}} \times \dots \times \frac{\frac{1}{2004} - \frac{1}{2005}}{\frac{1}{2005} - \frac{1}{2006}} \times \frac{\frac{1}{2006} - \frac{1}{2007}}{\frac{1}{2007} - \frac{1}{2008}}$.

- 25 An integer is chosen from the set $\{1, 2, 3, \dots, 499, 500\}$. The probability that this integer is divisible by 7 or 11 is $\frac{m}{n}$ in its lowest terms. Find the value of $m + n$.

- 26 The diagram shows a sector OAB of a circle, centre O and radius 8 cm, in which $\angle AOB = 120^\circ$. Another circle of radius r cm is to be drawn through the points O , A and B . Find the value of r .



- 27 The difference between the highest common factor and the lowest common multiple of x and 18 is 120. Find the value of x .
- 28 Let α and β be the roots of $x^2 - 4x + c = 0$, where c is a real number. If $-\alpha$ is a root of $x^2 + 4x - c = 0$, find the value of $\alpha\beta$.
- 29 Let m, n be integers such that $1 < m \leq n$. Define
- $$f(m, n) = \left(1 - \frac{1}{m}\right) \times \left(1 - \frac{1}{m+1}\right) \times \left(1 - \frac{1}{m+2}\right) \times \dots \times \left(1 - \frac{1}{n}\right).$$
- If $S = f(2, 2008) + f(3, 2008) + f(4, 2008) + \dots + f(2008, 2008)$, find the value of $2S$.
- 30 Let a and b be the roots of $x^2 + 2000x + 1 = 0$ and let c and d be the roots of $x^2 - 2008x + 1 = 0$. Find the value of $(a + c)(b + c)(a - d)(b - d)$.
- 31 4 black balls, 4 white balls and 2 red balls are arranged in a row. Find the total number of ways this can be done if all the balls of the same colour do not appear in a consecutive block.
- 32 Given that n is a ten-digit number in the form $\overline{2007x2008y}$ where x and y can be any of the digits 0, 1, 2, ..., 9. How many such numbers n are there that are divisible by 33?
- 33 In triangle ABC , $AB = (b^2 - 1)$ cm, $BC = a^2$ cm and $AC = 2a$ cm, where a and b are positive integers greater than 1. Find the value of $a - b$.
- 34 How many positive integers n , where $10 \leq n \leq 100$, are there such that $\frac{n^2 - 9}{n^2 - 7}$ is a fraction in its lowest terms?
- 35 Let n be a positive integer such that $n^2 + 19n + 48$ is a perfect square. Find the value of n .

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2008

(Junior Section Solutions)

1. Answer: (E)
 $25^5 = 5^{10}$, $150^4 = 2^4 \times 3^4 \times 5^8$ and $2008^3 = 2^9 \times 251^3$. So the product contains factors 2^{13} and 5^{18} , which produce 10^{13} .
2. Answer: (C)
For $\sqrt{2x+y} + \sqrt{x^2-9} = 0$, $\sqrt{2x+y} = 0$ and $\sqrt{x^2-9} = 0$. So we have $x = 3$ or -3 and $y = -2x = -6$ or $6 \Rightarrow y - x = -9$ or 9 .
3. Answer: (D)
The first integer is 211 and the last is 2001. So $211 + (n-1)10 = 2001 \Rightarrow n = 180$.
4. Answer: (A)
 $7^{2008} = (8-1)^{2008} = 64k_1 + 1$ for some integers k_1 . Similarly, we have $9^{2008} = (8+1)^{2008} = 64k_2 + 1$ for some integers k_2 . Hence the remainder is 2.
5. Answer: (D)
 $\frac{3}{5} \times \frac{2}{4} + \frac{2}{5} \times \frac{3}{4} = \frac{3}{5}$.
6. Answer: (A)
It is clear that $X = 1$. So $A + B = 11 \Rightarrow Y = 2$. Hence $A + B$ can be $3 + 8$ or $4 + 7$ or $5 + 6$ or $6 + 5$ or $7 + 4$ or $8 + 3$.
7. Answer: (B)
 $(2 + \sqrt{3})^2 = 4 + 2\sqrt{3} + 3 = 7 + 2\sqrt{3}$ and $(2 - \sqrt{3})^2 = 7 - 2\sqrt{3}$. So $(2 + \sqrt{3})^2 + (2 - \sqrt{3})^2 = 14$. Since $0 < 2 - \sqrt{3} < 1$, $0 < (2 - \sqrt{3})^2 < 1 \Rightarrow$ the least integer that is greater than $(2 + \sqrt{3})^2$ is 14.
8. Answer: (C)
 $x + y + z = 9 + \frac{S}{2}$. So $x = 1 + \frac{S}{2}$, $y = 9 - \frac{S}{2}$ and $z = -1 + \frac{S}{2}$. Since $x, y, z \geq 0$, we have $2 \leq S \leq 18$.
9. Answer: (D)
 $2008 = 2^3 \times 251$. Consider $|x| = 2^{p_1} \times 251^{q_1}$, $|y| = 2^{p_2} \times 251^{q_2}$ and $|z| = 2^{p_3} \times 251^{q_3}$. Then $p_1 + p_2 + p_3 = 3$ and $q_1 + q_2 + q_3 = 1$, so the number of *positive* integer solutions is ${}^5C_2 \times {}^3C_2 = 30$. Together with the 4 possible distributions of the signs:

$(+, +, +), (+, -, -), (-, +, -), (-, -, +)$, the equation has $30 \times 4 = 120$ integer solutions.

10. Answer: (B)
Here we try to find $9^{2008} \pmod{100}$. We see that the $9^2 \pmod{100} = 81$, $9^4 \pmod{100} = 61$, $9^6 \pmod{100} = 41$, $9^8 \pmod{100} = 21$ and $9^{10} \pmod{100} = 01$. Hence $9^{2008} \equiv 9^8 \pmod{100} = 21$.
11. Answer: 1
The remainder is $(-1)^{2008} + 2008(-1) + 2008 = 1$.
12. Answer: 34
From AM-GM: $\sqrt{ab} \leq \frac{a+b}{2}$, we have $\sqrt{a} + \sqrt{b} \leq \sqrt{2(a+b)}$. Hence $\sqrt{x-144} + \sqrt{722-x} \leq \sqrt{2(722-144)} = 34$.
13. Answer: 28
Let width of each rectangle = b , then length = $2b$. So $2b^2 = 8 \Rightarrow b = 2$. Hence perimeter = $14b = 28$.
14. Answer: 24
Let $A = \{\text{students who liked swimming}\}$ and $B = \{\text{students who liked soccer}\}$. The greatest possible number of students who neither liked swimming nor soccer occurs when $A \subset B$, hence $60 - 36 = 24$.
15. Answer: 32
The knight can move to *any* squares on the chessboard within 20 moves (in fact 4 moves are enough). Note that the knight starts from a white square, so after 20 consecutive moves it can be in any of the $64 \div 2 = 32$ white squares.
16. Answer: 3
Note that α and β are the roots of the equation $x^2 - 17x + 70 = 0$. Solving we have $x = 7$ or 10 . Hence $|\alpha - \beta| = 3$.
17. Answer: 2008
Let $x = \sqrt{2008 + 2007 \sqrt{2008 + 2007 \sqrt{2008 + 2007 \sqrt{\dots}}}}$, which is clearly positive. Now $x^2 = 2008 + 2007x \Rightarrow (x - 2008)(x + 1) = 0$. Hence the only solution is $x = 2008$.
18. Answer: 33165
The required sum is $15 + 30 + 45 + \dots + 990 = 15(2211) = 33165$.
19. Ans: 2160
In order to get the best price, the number of bottles bought in shop A should be a multiple of 4. We see that in the 3 cases:

- (i) 0 from shop A and 13 from shop B : $2600 \times 85\% = 2210$,
- (ii) 4 from shop A and 8 from shop B : $800 + 1600 \times 85\% = 2160$,
- (iii) 8 from shop A and 3 from shop B : $1600 + 600 = 2200$,

the lowest price is 2160.

20. Answer: 219

If we take modulo 5, the seven integers 53, 62, 66, 68, 71, 82, 89 give 3, 2, 1, 3, 1, 2, 4. Hence the remaining three integers can only be 66, 71 and 82.

21. Answer: 2008

The equation is equivalent to $|x - a| = b \pm 2008$.

Case 1: If $b < 2008$, then $|x - a| = b - 2008$ has no real root since $b - 2008 < 0$, and $|x - a| = b + 2008$ has at most 2 real roots.

Case 2: If $b > 2008$, then both $|x - a| = b - 2008$ and $|x - a| = b + 2008$ has 2 real roots, which gives 4 *distinct* real roots $a \pm (b - 2008)$ and $a \pm (b + 2008)$, since $a + b + 2008 > a + b - 2008 > a - b + 2008 > a - b - 2008$.

Case 3: If $b = 2008$, then $|x - a| = b - 2008 = 0$ has only 1 real root $x = a$, and $|x - a| = b + 2008 = 4016$ has 2 real roots $x = a \pm 4016$.

Hence $b = 2008$.

22. Answer: -2

Adding the 2 equations and simplifying gives $(n + 2)(x - y) = 2$. Hence if $n = -2$, we get $0 = 2$ which is impossible.

23. Answer: 2008

Let m of them be -3 and n of them be -1. Then $m + n = 88$ and $(-3)^2 m + (-1)^2 n = 280$. Solving, $m = 24$, $n = 64$. Hence $(-3)^4 m + (-1)^4 n = 2008$.

24. Answer: 1004

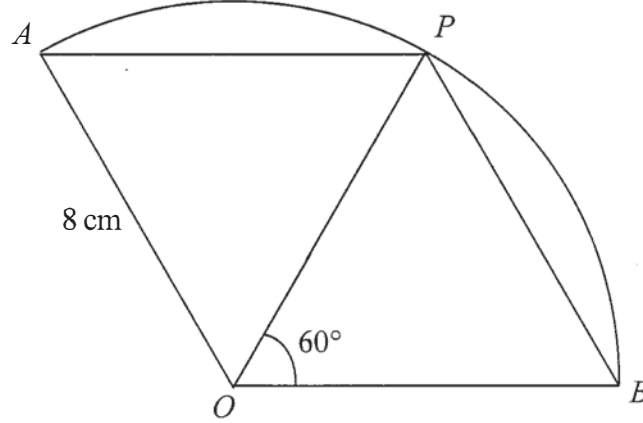
$\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$. So the numerator = $\frac{1}{2 \times 3 \times 4 \times 5 \times \dots \times 2006 \times 2007}$ and the denominator = $\frac{1}{3 \times 4 \times 5 \times 6 \times \dots \times 2007 \times 2008}$ which gives $\frac{2008}{2} = 1004$.

25. Answer: 61

There are 71 multiples of 7, 45 multiples of 11 and 6 multiples of 77 that are less than 500. So there are $71 + 45 - 6 = 110$ numbers in the set which are multiples of 7 or 11. Hence probability = $\frac{110}{500} = \frac{11}{50}$.

26. Answer: 8

Consider the line OP such that $\triangle BOP$ is equilateral. Similarly, $\triangle AOP$ is equilateral. Hence $PA = PO = PB \Rightarrow P$ is the centre of the circle that passes through the points O, A and B . Clearly, $r = OP = 8$ cm.



27. Answer: 42

Let the HCF of x & 18 be k and let $x = ka$, $18 = kb$ where $\gcd(a, b) = 1$. We have LCM of x & 18 = kab . From information given, $kab - k = 120$. Clearly $\gcd(ab - 1, b) = 1$, so $\gcd(kab - k, kb) = k \Rightarrow \gcd(120, 18) = k \Rightarrow k = 6$. Hence $b = 3$ and $x = 42$.

28. Answer: 0

α is a root of the equation $x^2 - 4x + c = 0$ so $\alpha^2 - 4\alpha + c = 0$ and $-\alpha$ is a root of the equation $x^2 + 4x - c = 0$ so $(-\alpha)^2 + 4(-\alpha) - c = 0$. We have $2c = 0 \Rightarrow \alpha\beta = c = 0$.

29. Answer: 2007

$$f(m, n) = \left(\frac{m-1}{m}\right) \times \left(\frac{m}{m+1}\right) \times \left(\frac{m+1}{m+2}\right) \times \dots \times \left(\frac{n-1}{n}\right) = \frac{m-1}{n}.$$

$$\Rightarrow f(2, n) + f(3, n) + f(4, n) + \dots + f(n, n) = \frac{1 + 2 + 3 + \dots + (n-1)}{n} = \frac{n-1}{2}.$$

$$\Rightarrow f(2, 2008) + f(3, 2008) + f(4, 2008) + \dots + f(2008, 2008) = \frac{2007}{2} = S.$$

30. Answer: 32064

Note that $ab = cd = 1$, $a + b = -2000$ and $c + d = 2008$. So we have

$$\begin{aligned} (a+c)(b+c)(a-d)(b-d) &= [ab + (a+b)c + c^2][ab - (a+b)d + d^2] \\ &= (1 - 2000c + c^2)(1 + 2000d + d^2) \\ &= (8c)(4008d) \\ &= 32064. \end{aligned}$$

31. Answer: 2376

Using the Principle of Inclusion and Exclusion, number of ways = $\frac{10!}{4!4!2!} - 2 \frac{7!}{4!2!} - \frac{9!}{4!4!} + 2 \frac{6!}{4!} + \frac{4!}{2!} - 3! = 3150 - 210 - 630 + 60 + 12 - 6 = 2376$.

32. Answer: 3

$33 = 3 \times 11$. So 3 must divide $2 + 7 + x + 2 + 8 + y \Rightarrow 19 + x + y = 3k_1$. 11 must divide $(7 + 2 + y) - (2 + x + 8) \Rightarrow y - x - 1 = 11k_2 \Rightarrow -10 \leq 11k_2 \leq 8 \Rightarrow k_2 = 0$. So $y = x + 1 \Rightarrow 20 + 2x = 3k_1$. When $k_1 = 8$, $x = 2$ and $y = 3$. When $k_1 = 10$, $x = 5$ and $y = 6$. When $k_1 = 12$, $x = 8$ and $y = 9$.

33. Answer: 0

Using Triangle Inequality, $a^2 + 2a > b^2 - 1 \Rightarrow (a + 1 - b)(a + 1 + b) > 0$. Since $a + 1 + b > 0 \Rightarrow a + 1 - b > 0 \Rightarrow a - b \geq 0$. Using Triangle Inequality again, $2a + b^2 - 1 > a^2 \Rightarrow (a - 1 - b)(a - 1 + b) < 0$. Since $a - 1 + b > 0 \Rightarrow a - 1 - b < 0 \Rightarrow a - b \leq 0$. Hence $a - b = 0$.

34. Answer: 46

$\gcd(n^2 - 9, n^2 - 7) = 1 \Rightarrow \gcd(n^2 - 9, 2) = 1$. Hence $n^2 - 9$ must be an odd number $\Rightarrow n$ is even. Since $10 \leq n \leq 100$, there are $\frac{100 - 10}{2} + 1 = 46$ possible positive integers n .

35. Answer: 33

$n^2 + 19n + 48 = (n + 3)(n + 16)$. If $n + 3 = 13k$ for some integer k , then $n + 16 = 13(k + 1)$. However, $(n + 3)(n + 16) = 169k(k + 1)$ cannot be a square. So 13 must not divide $n + 3 \Rightarrow \gcd(n + 3, n + 16) = 1 \Rightarrow$ they should be both perfect squares. Let $n + 3 = m^2$. Since $n + 16 \geq (m + 1)^2 = m^2 + 2m + 1 = n + 3 + 2m + 1$, we have $m \leq 6$. Checking $m = 2, 3, 4, 5$ and 6 , $n + 16 = m^2 + 13 = 17, 22, 29, 38$ and 49 respectively. Hence $m = 6$ and $n = 33$.

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2008

(Junior Section, Round 2)

Saturday, 28 June 2008

0930-1230

Instructions to contestants

1. Answer ALL 5 questions.
2. Show all the steps in your working.
3. Each question carries 10 mark.
4. No calculators are allowed.

1. In $\triangle ABC$, $\angle ACB = 90^\circ$, D is the foot of the altitude from C to AB and E is the point on the side BC such that $CE = BD/2$. Prove that $AD + CE = AE$.
2. Let a, b, c, d be positive real numbers such that $cd = 1$. Prove that there is an integer n such that $ab \leq n^2 \leq (a + c)(b + d)$.
3. In the quadrilateral $PQRS$, A, B, C and D are the midpoints of the sides PQ, QR, RS and SP respectively, and M is the midpoint of CD . Suppose H is the point on the line AM such that $HC = BC$. Prove that $\angle BHM = 90^\circ$.
4. Six distinct positive integers a, b, c, d, e, f are given. Jack and Jill calculated the sums of each pair of these numbers. Jack claims that he has 10 prime numbers while Jill claims that she has 9 prime numbers among the sums. Who has the correct claim?

5. Determine all primes p such that

$$5^p + 4 \cdot p^4$$

is a perfect square, i.e., the square of an integer.

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2008
(Junior Section, Round 2 Solutions)

1. Let $AD = q$, $BD = p$. Then $CE = p/2$ and $p + q = AB = c$. By similar triangles, we also have $CA = b = \sqrt{cq}$. Let F be the point on AE so that $EF = CE$. From $\triangle ACE$, we have

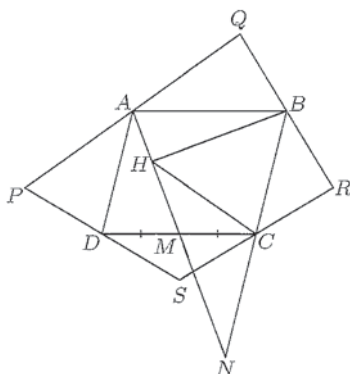
$$AE^2 = CA^2 + CE^2, \quad \text{i.e.,} \quad (AF + p/2)^2 = cq + p^2/4 = (p + q)q + p^2/4.$$

Therefore

$$AF(AF + p) = q(q + p).$$

From here it follows that $AF = q$ and we are done.

2. We need to prove that $\sqrt{ab} + 1 \leq \sqrt{(a+c)(b+d)}$. By squaring both sides and simplify, this is equivalent to $\sqrt{ab} \leq (bc + ad)/2$. Since $(bc + ad)/2 \geq \sqrt{abcd} = \sqrt{ab}$, the proof is complete.
3. First observe that $ABCD$ is a parallelogram by Varignon's theorem. Let the extensions of AM and BC meet at N . Since AD is parallel to CN , $\angle MAD = \angle MNC$. Since $\angle AMD = \angle NMC$ and $MD = MC$, $\triangle MAD$ and $\triangle MNC$ are congruent, so that $CN = DA = CB = HC$. Thus H lies on the circle centred at C with diameter BN . Hence $\angle BHM = 90^\circ$.



4. Suppose k of the 6 numbers are even. Since the sum of two even or two odd numbers are even, and the sum of two distinct positive integers is > 2 , the only even prime, we see that the number of primes among the sums is at most $k(6 - k)$. By checking for $k = 0, 1, \dots, 6$, we see that the maximum value of $k(6 - k)$ is 9 attained when $k = 3$. Thus Jack's answer is definitely wrong. Jill's answer is correct because 9 primes can be obtained from the following 6 numbers: 2, 4, 8, 3, 15, 39.

5. Let $5^p + 4 \cdot p^4 = q^2$. Then

$$5^p = (q - 2p^2)(q + 2p^2).$$

Thus

$$q - 2p^2 = 5^s, \quad q + 2p^2 = 5^t \quad \text{where } 0 \leq s < t \text{ and } s + t = p$$

Eliminating q , we get $4p^2 = 5^s(5^{t-s} - 1)$. If $s > 0$, then $5 \mid 4p^2$. Thus $p = 5$ and the given expression is indeed a square. If $s = 0$, then $t = p$ and we have $5^p = 4p^2 + 1$. We shall prove by induction that $5^k > 4k^2 + 1$ for every integer $k \geq 2$. The inequality certainly holds for $k = 2$. So we assume that it holds for some $k \geq 2$. Note that

$$\frac{4(k+1)^2 + 1}{4k^2 + 1} = \frac{4k^2 + 1}{4k^2 + 1} + \frac{8k}{4k^2 + 1} + \frac{4}{4k^2 + 1} < 1 + 1 + 1 < 5 \quad \text{for } k \geq 2.$$

Therefore

$$5^{k+1} = 5 \cdot 5^k > 5(4k^2 + 1) > 4(k+1)^2 + 1.$$

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2008
(Senior Section)

Tuesday, 27 May 2008

0930 – 1200 hrs

Important:

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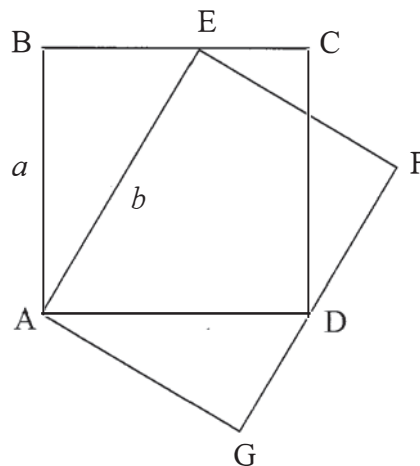
No calculators are allowed.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Multiple Choice Questions

1. Find the value of $\frac{1+3+5+7+\dots+99}{2+4+6+8+\dots+100}$.
- (A) $\frac{48}{49}$
- (B) $\frac{49}{50}$
- (C) $\frac{50}{51}$
- (D) $\frac{98}{99}$
- (E) $\frac{100}{101}$
2. Suppose that x and y are real numbers that satisfy all the following three conditions: $3x-2y=4-p$; $4x-3y=2+p$; $x>y$. What are the possible values of p ?
- (A) $p > -1$
- (B) $p < 1$
- (C) $p < -1$
- (D) $p > 1$
- (E) p can be any real number
3. If $f(x) = x^2 + \sqrt{1-x^2}$ where $-1 \leq x \leq 1$, find the range of $f(x)$.
- (A) $\frac{1}{2} \leq f(x) \leq 1$
- (B) $1 \leq f(x) \leq \frac{5}{4}$
- (C) $1 \leq f(x) \leq \frac{1+2\sqrt{3}}{4}$
- (D) $\frac{\sqrt{3}}{2} \leq f(x) \leq 1$
- (E) $\frac{1}{2} \leq f(x) \leq \frac{\sqrt{3}}{2}$

4. If a and b are integers and $\sqrt{7-4\sqrt{3}}$ is one of the roots of the equation $x^2 + ax + b = 0$, find the value of $a + b$.
- (A) -3
 (B) -2
 (C) 0
 (D) 2
 (E) 3
5. A bag contains 30 balls that are numbered $1, 2, \dots, 30$. Two balls are randomly chosen from the bag. Find the probability that the sum of the two numbers is divisible by 3.
- (A) $\frac{1}{2}$
 (B) $\frac{1}{3}$
 (C) $\frac{7}{29}$
 (D) $\frac{9}{29}$
 (E) $\frac{11}{87}$
6. ABCD is a square with $AB = a$, and AEFG is a rectangle such that E lies on side BC and D lies on side FG. If $AE = b$, what is the length of side EF?
- (A) $\frac{b}{a}$
 (B) $\frac{3a^2}{2b}$
 (C) $\frac{4a^2}{3b}$
 (D) $\frac{\sqrt{2}a^2}{b}$
 (E) $\frac{a^2}{b}$

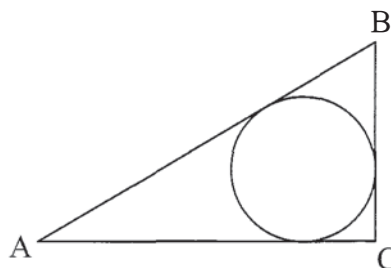


7. Find the value of $\sin^4 \frac{\pi}{8} + \sin^4 \frac{3\pi}{8} + \sin^4 \frac{5\pi}{8} + \sin^4 \frac{7\pi}{8}$.

(A) 1
 (B) $\frac{3}{2}$
 (C) $\frac{7}{4}$
 (D) 2
 (E) $\frac{5}{2}$

8. A circle with radius x cm is inscribed inside a triangle ABC, where $\angle ACB$ is a right angle. If $AB = 9$ cm and the area of the triangle ABC is 36 cm^2 , find the value of x .

(A) 2.2
 (B) 2.6
 (C) 3
 (D) 3.4
 (E) 3.8



9. How many positive integers n are there such that $7n + 1$ is a perfect square and $3n + 1 < 2008$?

(A) 6
 (B) 9
 (C) 12
 (D) 15
 (E) 18

10. Find the minimum value of $(a + b)\left(\frac{1}{a} + \frac{4}{b}\right)$, where a and b range over all positive real numbers.

(A) 3
 (B) 6
 (C) 9
 (D) 12
 (E) 15

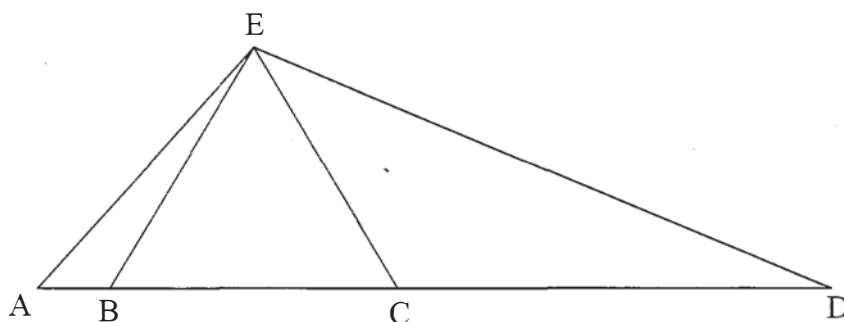
Short Questions

11. Find the smallest integer n such that $n(\sqrt{101} - 10) > 1$.
12. Given that x and y are positive real numbers such that $(x + y)^2 = 2500$ and $xy = 500$, find the exact value of $x^3 + y^3$.
13. Find the smallest positive integer N such that $2^n > n^2$ for every integer n in $\{N, N + 1, N + 2, N + 3, N + 4\}$.
14. The lengths of the sides of a quadrilateral are 2006 cm, 2007 cm, 2008 cm and x cm. If x is an integer, find the largest possible value of x .
15. Find the number of positive integers x that satisfy the inequality $\left| 3 + \log_x \frac{1}{3} \right| < \frac{8}{3}$.
16. Two bullets are placed in two consecutive chambers of a 6-chamber pistol. The cylinder is then spun. The pistol is fired but the first shot is a blank. Let p denote the probability that the second shot is also a blank if the cylinder is spun after the first shot and let q denote the probability that the second shot is also a blank if the cylinder is not spun after the first shot. Find the smallest integer N such that
$$N \geq \frac{100p}{q}.$$
17. Find the value of $\left(\log_{\sqrt{2}}(\cos 20^\circ) + \log_{\sqrt{2}}(\cos 40^\circ) + \log_{\sqrt{2}}(\cos 80^\circ) \right)^2$.
18. Find the number of ways for 5 persons to be seated at a rectangular table with 6 seats, 2 each on the longer sides and 1 each on the shorter sides. The seats are not numbered.
19. Find the remainder when $(x - 1)^{100} + (x - 2)^{200}$ is divided by $x^2 - 3x + 2$.
20. Suppose that ABC is a triangle and D is a point on side AB with $AD = BD = CD$. If $\angle ACB = x^\circ$, find the value of x .

21. If x, y and z are positive integers such that $27x + 28y + 29z = 363$, find the value of $10(x + y + z)$.

22. Find the value of $\frac{\tan 40^\circ \tan 60^\circ \tan 80^\circ}{\tan 40^\circ + \tan 60^\circ + \tan 80^\circ}$.

23. In the figure below, ADE is a triangle with $\angle AED = 120^\circ$, and B and C are points on side AD such that BCE is an equilateral triangle. If $AB = 4$ cm, $CD = 16$ cm and $BC = x$ cm, find the value of x .



24. Suppose that x and y are positive integers such that $x + 2y = 2008$ and xy has the maximum value. Find the value of $x - y$.

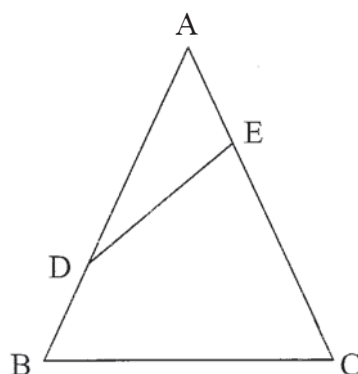
25. If $\cos(2A) = -\frac{\sqrt{5}}{3}$, find the value of $6\sin^6 A + 6\cos^6 A$.

26. Let N be the positive integer for which the sum of its two smallest factors is 4 and the sum of its two largest factors is 204. Find the value of N .

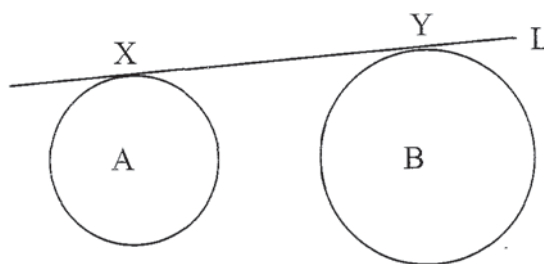
27. If $S = \sum_{k=1}^{99} \frac{(-1)^{k+1}}{\sqrt{k(k+1)}(\sqrt{k+1} - \sqrt{k})}$, find the value of $1000S$.

28. A teacher wrote down three positive integers on the whiteboard: 1125, 2925, N , and asked her class to compute the least common multiple of the three numbers. One student misread 1125 as 1725 and computed the least common multiple of 1725, 2925 and N instead. The answer he obtained was the same as the correct answer. Find the least possible value of N .

29. The figure below shows a triangle ABC where $AB = AC$. D and E are points on sides AB and AC , respectively, such that $AB = 4DB$ and $AC = 4AE$. If the area of the quadrilateral $BCED$ is 52 cm^2 and the area of the triangle ADE is $x \text{ cm}^2$, find x .



30. The figure below shows two circles with centres A and B , and a line L which is a tangent to the circles at X and Y . Suppose that $XY = 40 \text{ cm}$, $AB = 41 \text{ cm}$ and the area of the quadrilateral $ABYX$ is 300 cm^2 . If a and b denote the areas of the circles with centre A and centre B respectively, find the value of $\frac{b}{a}$.



31. Find the maximum value of $3 \sin\left(x + \frac{\pi}{9}\right) + 5 \sin\left(x + \frac{4\pi}{9}\right)$, where x ranges over all real numbers.

32. Find the number of 11-digit positive integers such that the digits from left to right are non-decreasing. (For example, 12345678999, 55555555555, 23345557889.)
33. Find the largest positive integer n such that $\sqrt{n-100} + \sqrt{n+100}$ is a rational number.
34. Let $S = \{ 1, 2, 3, \dots, 20 \}$ be the set of all positive integers from 1 to 20. Suppose that N is the smallest positive integer such that exactly eighteen numbers from S are factors of N , and the only two numbers from S that are not factors of N are consecutive integers. Find the sum of the digits of N .
35. Let a_1, a_2, a_3, \dots be the sequence of all positive integers that are relatively prime to 75, where $a_1 < a_2 < a_3 < \dots$. (The first five terms of the sequence are: $a_1 = 1, a_2 = 2, a_3 = 4, a_4 = 7, a_5 = 8$.) Find the value of a_{2008} .

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2008

(Senior Section Solutions)

1. Answer: (C)
Since the numbers of terms in the numerator and denominator are the same, we have $\frac{1+3+5+7+\cdots+99}{2+4+6+8+\cdots+100} = \frac{100}{102} = \frac{50}{51}$.
2. Answer: (D)
Solving the simultaneous equations $3x - 2y = 4 - p$ and $4x - 3y = 2 + p$, we obtain $x = 8 - 5p$, $y = 10 - 7p$. Since $x > y$, $8 - 5p > 10 - 7p$, so $p > 1$.
3. Answer: (B)
Let $x^2 + \sqrt{1-x^2} = M$. Then $(x^2 - M)^2 = 1 - x^2$, which leads to $x^4 - (2M-1)x^2 + M^2 - 1 = 0$. For real x , we must have $(2M-1)^2 - 4(M^2 - 1) > 0$, which gives $M \leq \frac{5}{4}$.
Note that $f(-1) = f(0) = f(1) = 1$. Assume that $f(x) < 1$ for some $x \neq -1, 0, 1$. Then $x^2 + \sqrt{1-x^2} < 1$, or $\sqrt{1-x^2} < 1 - x^2$, which is not possible since $-1 < x < 1$. Thus $f(x) \geq 1$, and hence $1 \leq f(x) \leq \frac{5}{4}$.
4. Answer: (A)
Note that $\sqrt{7-4\sqrt{3}} = \sqrt{(2-\sqrt{3})^2} = 2-\sqrt{3}$. Since $\sqrt{7-4\sqrt{3}}$ is a root of the equation $x^2 + ax + b = 0$, we have $(7-4\sqrt{3}) + (2-\sqrt{3})a + b = 0$. Rearranging the terms, we obtain $(7+2a+b) - (4+a)\sqrt{3} = 0$. This implies that $7+2a+b=0$ and $4+a=0$, which give $a=-4$ and $b=1$. Thus $a+b=-3$.
5. Answer: (B)
Let $S_1 = \{1, 4, 7, \dots, 28\}$, $S_2 = \{2, 5, 8, \dots, 29\}$, $S_3 = \{3, 6, 9, \dots, 30\}$. Note that each of the three sets has 10 elements. Now for any two distinct numbers a and b in $\{1, 2, 3, \dots, 28, 29, 30\}$, their sum $a+b$ is divisible by 3 if and only if one of the following conditions holds:
(i) the two numbers a and b belongs to S_1 and S_2 respectively;
(ii) both numbers a and b belong to S_3 .
Therefore the required probability = $\frac{{}^{10}C_2 + {}^{10}C_1 {}^{10}C_1}{{}^{10}C_3} = \frac{45+100}{(15)(29)} = \frac{1}{3}$.

6. Answer: (E)

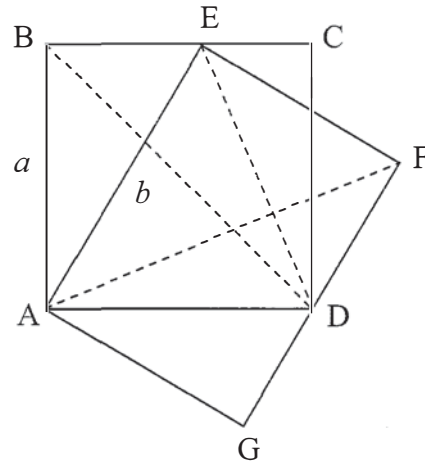
Note that

$$\begin{aligned}\text{area of } \triangle ABD &= \text{area of } \triangle AED \\ &= \text{area of } \triangle AEF.\end{aligned}$$

Therefore

$$\begin{aligned}a^2 &= \text{area of square } ABCD \\ &= 2 \times \text{area of } \triangle ABD \\ &= 2 \times \text{area of } \triangle AEF \\ &= \text{area of rectangle } AEF G.\end{aligned}$$

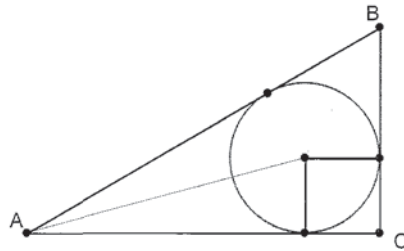
$$\text{Hence } EF = \frac{a^2}{b}.$$



7. Answer: (B)

$$\begin{aligned}& \sin^4 \frac{\pi}{8} + \sin^4 \frac{3\pi}{8} + \sin^4 \frac{5\pi}{8} + \sin^4 \frac{7\pi}{8} \\ &= 2 \left(\sin^4 \frac{\pi}{8} + \sin^4 \frac{3\pi}{8} \right) = 2 \left(\sin^4 \frac{\pi}{8} + \cos^4 \frac{\pi}{8} \right) \\ &= 2 \left[\left(\sin^2 \frac{\pi}{8} + \cos^2 \frac{\pi}{8} \right)^2 - 2 \sin^2 \frac{\pi}{8} \cos^2 \frac{\pi}{8} \right] \\ &= 2 \left(1 - \frac{1}{2} \sin^2 \frac{\pi}{4} \right) = \frac{3}{2}.\end{aligned}$$

8. Answer: (C)



Note that area of $\triangle ABC = (AB)x + x^2$. Thus we obtain $36 = 9x + x^2$, or $x^2 + 9x - 36 = 0$. Solving the equation gives $x = 3$ or $x = -12$, so the radius is 3cm.

9. Answer: (E)

First note that $3n + 1 < 2008$ implies that $n < 669$.

Since $7n + 1$ is a perfect square, we let $7n + 1 = m^2$. Then $n = \frac{m^2 - 1}{7}$, which is an integer. Thus 7 is a factor of $m^2 - 1 = (m - 1)(m + 1)$. Since 7 is a prime number,

this implies that 7 is a factor of $m - 1$ or $m + 1$; that is, $m = 7k - 1$ or $7k + 1$ for some integer k .

If $m = 7k - 1$, then $n = \frac{m^2 - 1}{7} = \frac{49k^2 - 14k}{7} = 7k^2 - 2k$. As $n < 669$, this gives $7k^2 - 2k < 669$, which implies that $k = 1, 2, \dots, 9$.

Similarly, if $m = 7k + 1$, then $n = \frac{m^2 - 1}{7} = \frac{49k^2 + 14k}{7} = 7k^2 + 2k < 669$, so we obtain $k = 1, 2, 3, \dots, 9$.

Hence there are 18 such positive integers.

10. Answer: (C)

$(a + b)\left(\frac{1}{a} + \frac{4}{b}\right) = 1 + 4 + \frac{b}{a} + \frac{4a}{b} = 5 + \frac{b^2 + 4a^2}{ab}$. Let $m = \frac{b^2 + 4a^2}{ab}$. Then we

have $4a^2 - mba + b^2 = 0$. This implies that for any positive values of m and b , $m^2b^2 - 16b^2 \geq 0$. Thus we have $m^2 - 16 \geq 0$, or $m \geq 4$.

Therefore $(a + b)\left(\frac{1}{a} + \frac{4}{b}\right) = 5 + m \geq 9$. Hence the minimum value is 9.

11. Answer: 21

Solving the inequality $n(\sqrt{101} - 10) > 1$, we obtain

$$n > \frac{1}{\sqrt{101} - 10} = \frac{\sqrt{101} + 10}{1} = \sqrt{101} + 10.$$

As n is an integer and $10 < \sqrt{101} < 11$, we have $n \geq 21$.

12. Answer: 50000

$$\begin{aligned} x^3 + y^3 &= (x + y)(x^2 - xy + y^2) = (x + y)[(x + y)^2 - 3xy] \\ &= 50[2500 - 3(500)] = 50000. \end{aligned}$$

13. Answer: 5

Using guess and check, we obtain $N = 5$.

14. Answer: 6020

By triangle inequality, $x < 2006 + 2007 + 2008 = 6021$. Since x is an integer, it follows that the largest possible value of x is 6020.

15. Answer: 25

We have $-\frac{8}{3} < 3 + \log_x \frac{1}{3} < \frac{8}{3}$

$$\begin{aligned}
-\frac{17}{3} &< -\frac{1}{\log_3 x} < -\frac{1}{3} \\
\frac{3}{17} &< \log_3 x < 3 \\
3^{3/17} &< x < 27.
\end{aligned}$$

As $1 < 3^{3/17} < 2$, we see that integer solutions of the last inequality are $x = 2, 3, \dots, 26$. Hence there are 25 of them.

16. Answer: 89

$p = \text{Prob}(2^{\text{nd}} \text{ shot blank} \mid \text{cylinder spun again}) = 4/6$, and

$q = \text{Prob}(2^{\text{nd}} \text{ shot blank} \mid \text{cylinder not spun again}) = 3/4$.

Therefore $\frac{100p}{q} = \frac{800}{9} \approx 88.8$, and hence the smallest $N = 89$.

17. Answer: 36

We have

$$\begin{aligned}
&\log_{\sqrt{2}}(\cos 20^\circ) + \log_{\sqrt{2}}(\cos 40^\circ) + \log_{\sqrt{2}}(\cos 80^\circ) \\
&= \log_{\sqrt{2}}(\cos 20^\circ \cdot \cos 40^\circ \cdot \cos 80^\circ) \\
&= \log_{\sqrt{2}}\left(\cos 20^\circ \cdot \frac{1}{2}(\cos 120^\circ + \cos 40^\circ)\right) \\
&= \log_{\sqrt{2}}\left(-\frac{1}{4}\cos 20^\circ + \frac{1}{2}\cos 40^\circ \cos 20^\circ\right) \\
&= \log_{\sqrt{2}}\left(-\frac{1}{4}\cos 20^\circ + \frac{1}{4}(\cos 60^\circ + \cos 20^\circ)\right) \\
&= \log_{\sqrt{2}} \frac{1}{8} = -\frac{\log_2 8}{\log_2(2^{1/2})} = -6.
\end{aligned}$$

Hence the answer is $(-6)^2 = 36$.

18. Answer: 360

The empty seat can be considered to be a person. Now the number of ways of sitting at a round table is $(6-1)! = 120$. Note that each arrangement at a round table can be matched to 3 different arrangements at the rectangular table. Thus the number of ways of sitting at the rectangular table is $3 \times 120 = 360$.

19. Answer: 1

By the division algorithm, we have

$$(x-1)^{100} + (x-2)^{200} = (x^2 - 3x + 2)q(x) + ax + b,$$

where $q(x)$ is the quotient, and $ax + b$ is the remainder with constants a and b .

Note that $x^2 - 3x + 2 = (x-1)(x-2)$. Therefore

$$(x-1)^{100} + (x-2)^{200} = (x-1)(x-2)q(x) + ax + b,$$

Putting $x = 1$, we obtain

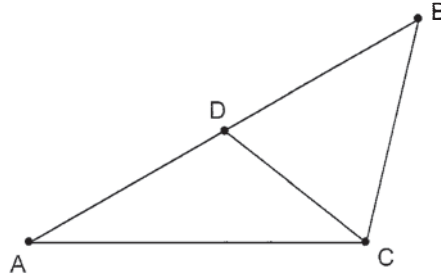
$$(1-2)^{200} = a + b, \text{ or } a + b = 1.$$

Putting $x = 2$, we obtain

$$(2-1)^{100} = 2a + b, \text{ or } 2a + b = 1.$$

Solving the two equations gives $a = 0$ and $b = 1$.

20. Answer: 90



First note that since $\triangle DAC$ and $\triangle DBC$ are isosceles triangles, we have $\angle DAC = \angle DCA$ and $\angle DBC = \angle DCB$. Now by considering the sum of angles in $\triangle ABC$, we have

$$\angle DAC + \angle DBC + \angle DCA + \angle DCB = 180^\circ,$$

$$2\angle DCA + 2\angle DCB = 180^\circ.$$

Therefore $\angle ACB = \angle DCA + \angle DCB = 90^\circ$.

21. Answer: 130

Since x, y and z are positive, we have

$$27(x + y + z) < 27x + 28y + 29z < 29(x + y + z).$$

That is, $27(x + y + z) < 363 < 29(x + y + z)$.

Therefore it follows that

$$x + y + z < 13.4 \text{ and } x + y + z > 12.5.$$

Since x, y and z are integers, we obtain $x + y + z = 13$. Hence $10(x + y + z) = 130$.

22. Answer: 1

We show that more generally, if $\triangle ABC$ is acute-angled, then

$$\frac{\tan A \cdot \tan B \cdot \tan C}{\tan A + \tan B + \tan C} = 1.$$

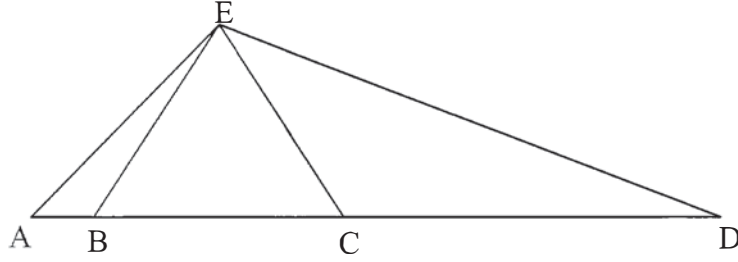
We have

$$\begin{aligned} & \tan A + \tan B + \tan C \\ &= \tan A + \tan B + \tan(180^\circ - (A + B)) \\ &= \tan A + \tan B - \tan(A + B) \\ &= \tan A + \tan B - \frac{\tan A + \tan B}{1 - \tan A \tan B} \\ &= (\tan A + \tan B) \left(1 - \frac{1}{1 - \tan A \cdot \tan B} \right) \end{aligned}$$

$$\begin{aligned}
&= (\tan A + \tan B) \left(\frac{-\tan A \cdot \tan B}{1 - \tan A \cdot \tan B} \right) \\
&= (\tan A \cdot \tan B) \left(-\frac{\tan A + \tan B}{1 - \tan A \cdot \tan B} \right) \\
&= \tan A \cdot \tan B \cdot \tan(180^\circ - (A + B)) \\
&= \tan A \cdot \tan B \cdot \tan C,
\end{aligned}$$

so the result follows.

23. Answer: 8.



We have

$$\begin{aligned}
\angle ABE &= 180^\circ - 60^\circ = 120^\circ = \angle ECD \text{ and} \\
\angle AEB &= 180^\circ - (\angle BAE + 120^\circ) = \angle CDE.
\end{aligned}$$

Therefore $\triangle ABE$ is similar to $\triangle ECD$, and it follows that $AB:BE = EC:CD$.
Hence $BC^2 = (BE)(EC) = (AB)(CD) = 64 \text{ cm}^2$, so $BC = 8 \text{ cm}$.

24. Answer: 502

Since $x + 2y = 2008$, $x \cdot 2y$ has the maximum value if and only if

$$x = 2y = \frac{2008}{2} = 1004. \text{ Note that } 2xy \text{ has the maximum value if and only if } xy$$

does. Thus $x - y = 1004 - 502 = 502$.

25. Answer: 4

Using the identity $(a^3 + b^3) = (a + b)(a^2 - ab + b^2)$ with $a = \sin^2 A$ and $b = \cos^2 A$, we have

$$\begin{aligned}
\sin^6 A + \cos^6 A &= (\sin^2 A + \cos^2 A)((\sin^2 A + \cos^2 A)^2 - 3\sin^2 A \cos^2 A) \\
&= 1 - 3\sin^2 A \cos^2 A = 1 - \frac{3}{4} \sin^2 2A \\
&= 1 - \frac{3}{4} (1 - \cos^2 2A) = 1 - \frac{3}{4} \left(1 - \frac{5}{9} \right) = \frac{2}{3}.
\end{aligned}$$

Hence $6\sin^6 A + 6\cos^6 A = 4$.

26. Answer: 153

Note that 1 is certainly the smallest factor of N , so 1 and 3 are the two smallest factors of N . Consequently, N and $\frac{N}{3}$ are the two largest factors of N . Thus we obtain the equation $N + \frac{N}{3} = 204$. Solving the equation gives $N = 153$.

27. Answer: 1100

$$\begin{aligned} S &= \frac{\sqrt{2} + \sqrt{1}}{\sqrt{1 \times 2}} - \frac{\sqrt{3} + \sqrt{2}}{\sqrt{2 \times 3}} + \frac{\sqrt{4} + \sqrt{3}}{\sqrt{3 \times 4}} - \cdots + \frac{\sqrt{100} + \sqrt{99}}{\sqrt{99 \times 100}} \\ &= \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} \right) - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \right) + \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} \right) - \cdots + \left(\frac{1}{\sqrt{99}} + \frac{1}{\sqrt{100}} \right) \\ &= 1 + \frac{1}{\sqrt{100}} = 1 + \frac{1}{10} = \frac{11}{10}. \end{aligned}$$

Hence $1000S = 1100$.

28. Answer: 2875

We look at the prime factorizations of 1125, 2925 and 1725. We have

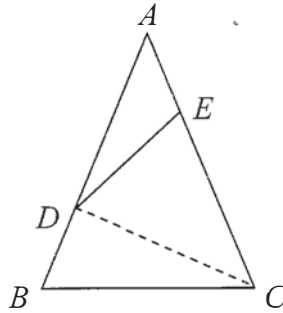
$$1125 = 3^2 \times 5^3, \quad 2925 = 3^2 \times 5^2 \times 13, \quad 1725 = 3 \times 5^2 \times 23.$$

Since

$$\text{LCM}(3^2 \times 5^3, 3^2 \times 5^2 \times 13, N) = \text{LCM}(3 \times 5^2 \times 23, 3^2 \times 5^2 \times 13, N),$$

we see that the least possible value of N is $5^3 \times 23 = 2875$.

29. Answer: 12



Consider $\triangle ADE$ and $\triangle CDE$. Since $CE = 3AE$, we see that

$$\text{area of } \triangle CDE = 3 \times \text{area of } \triangle ADE = 3x \text{ cm}^2.$$

Similarly, we have

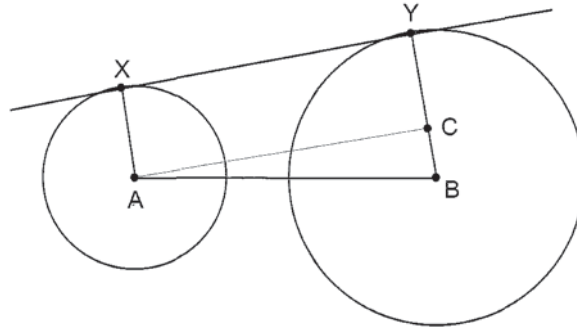
$$\text{area of } \triangle BCD = \frac{1}{3} \times \text{area of } \triangle ACD.$$

Now area of $\triangle ACD = \text{area of } \triangle ADE + \text{area of } \triangle CDE = 4x \text{ cm}^2$, so

$$\text{area of } \triangle BCD = \frac{4x}{3} \text{ cm}^2.$$

Thus we obtain the equation $3x + \frac{4x}{3} = 52$, which gives $x = 12$.

30. Answer: 16



Let the radii of the circles with centres A and B be x cm and y cm respectively, and let C be the point on the line segment BY such that AC is parallel to XY. Then $AC = 40$ cm and $BC = (y - x)$ cm. Since $\triangle ABC$ is a right-angled triangle, we have $(y - x)^2 + 40^2 = 41^2$ by Pythagorean theorem, so

$$y - x = 9.$$

Now consider the trapezium ABYX. We have $300 = \frac{40}{2}(x + y)$, which gives

$$x + y = 15.$$

Solving the two simultaneous equations, we obtain $x = 3$ and $y = 12$. Hence

$$\frac{b}{a} = \left(\frac{12}{3}\right)^2 = 16.$$

31. Answer: 7

$$\begin{aligned} & 3 \sin\left(x + \frac{\pi}{9}\right) + 5 \sin\left(x + \frac{4\pi}{9}\right) \\ &= 3 \sin\left(x + \frac{\pi}{9}\right) + 5 \sin\left(x + \frac{\pi}{9} + \frac{\pi}{3}\right) \\ &= 3 \sin\left(x + \frac{\pi}{9}\right) + 5 \sin\left(x + \frac{\pi}{9}\right) \cos \frac{\pi}{3} + 5 \cos\left(x + \frac{\pi}{9}\right) \sin \frac{\pi}{3} \\ &= \frac{11}{2} \sin\left(x + \frac{\pi}{9}\right) + \frac{5\sqrt{3}}{2} \cos\left(x + \frac{\pi}{9}\right). \end{aligned}$$

Hence the maximum value is $\sqrt{\left(\frac{11}{2}\right)^2 + \left(\frac{5\sqrt{3}}{2}\right)^2} = 7$.

32. Answer: 75582

We note that each 19-digit binary sequence containing exactly eight '0's and eleven '1's can be matched uniquely to such an 11-digit positive integer in the following way: Each '1' will be replaced by a digit from 1 to 9, and one more than the number of '0's to the left of a particular '1' indicates the value of the digit. For example, 011000001010111111 is matched to 22789999999, and

1110000000011111111 is matched to 11199999999. It follows that the required number is $\binom{19}{8} = 75582$.

33. Answer: 2501

First we note that if $\sqrt{n-100} + \sqrt{n+100}$ is a rational number, then both $\sqrt{n-100}$ and $\sqrt{n+100}$ are rational. Since n is a positive integer, this implies that $\sqrt{n-100}$ and $\sqrt{n+100}$ are integers. Let $\sqrt{n-100} = k$ and $\sqrt{n+100} = \ell$.

Squaring both sides of the equations, we obtain $n - 100 = k^2$ and $n + 100 = \ell^2$. This gives

$$200 = \ell^2 - k^2 = (\ell - k)(\ell + k). \quad \text{--- (1)}$$

Now for n to be the largest positive integer for which $\sqrt{n-100} + \sqrt{n+100}$ is rational, $\ell - k = \sqrt{n+100} - \sqrt{n-100}$ should be as small as possible. We rule out $\ell - k = 1$ since it leads to ℓ and k being non-integers from equation (1). Thus we have $\ell - k = 2$, and it follows from equation (1) that $\ell + k = 100$. Hence $\ell = 51$ and $k = 49$, and $n = k^2 + 100 = 49^2 + 100 = 2501$.

34. Answer: 36

We first find out which two consecutive numbers from S are not factors of N . Clearly 1 is a factor of N . Note that if an integer k is not a factor of N , then $2k$ is not a factor of N either. Therefore for $2 \leq k \leq 10$, since $2k$ is in S , k must be a factor of N , for otherwise, there would be at least three numbers from S (the two consecutive numbers including k , and $2k$) that are not factors of N . Hence 2, 3, ..., 10 are factors of N . Then it follows that $12 = 3 \times 4$, $14 = 2 \times 7$, $15 = 3 \times 5$, $18 = 2 \times 9$, $20 = 4 \times 5$ are also factors of N . Consequently, since the two numbers from S that are not factors of N are consecutive, we deduce that 11, 13, and 19 are factors of N as well. Thus we conclude that 16 and 17 are the only two consecutive numbers from S that are not factors of N . Hence

$$N = 2^3 \times 3^2 \times 5 \times 7 \times 11 \times 13 \times 19 = 6846840,$$

so the sum of digits of $N = 2 \times (6 + 8 + 4) = 36$.

35. Answer: 3764

Let $U = \{1, 2, 3, \dots, 74, 75\}$ be the set of integers from 1 to 75. We first find the number of integers in U that are relatively prime to $75 = 3 \times 5^2$. Let $A = \{n \in U : 3 \text{ is a factor of } n\}$ and $B = \{n \in U : 5 \text{ is a factor of } n\}$. Then $A \cup B$ is the set of integers in U that are not relatively prime to 75. Note that $|A| = 25$, $|B| = 15$ and $|A \cap B| = |\{n \in U : 15 \text{ is a factor of } n\}| = 5$. By the principle of inclusion-exclusion, $|A \cup B| = |A| + |B| - |A \cap B| = 35$. Therefore the number of integers in U that are relatively prime to 75 is $|U| - |A \cup B| = 75 - 35 = 40$. Thus

$$a_1 = 1, a_2 = 2, a_3 = 4, a_4 = 7, a_5 = 8, \dots, a_{40} = 74.$$

Now by division algorithm, any non-negative integer can be written uniquely in the form $75k + r$ for some integers k and r , where $k \geq 0$ and $0 \leq r \leq 74$. Note that $\gcd(75k + r, 75) = 1$ if and only if $\gcd(r, 75) = 1$. Therefore the sequence $\{a_n\}_{n \geq 1}$

can be written in the form $a_n = 75k + a_i$, where $k \geq 0$ and $1 \leq i \leq 40$. Indeed, k is given by $k = \left\lfloor \frac{n}{40} \right\rfloor$ and i is the remainder when n is divided by 40. Thus for $n = 2008$, $k = \left\lfloor \frac{2008}{40} \right\rfloor = 50$ and $i = 8$. Hence

$$a_{2008} = 75 \times 50 + a_8 = 3750 + 14 = 3764.$$

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2008
(Senior Section, Round 2)

Saturday, 28 June 2008

0930-1230

Instructions to contestants

1. Answer ALL 5 questions.
2. Show all the steps in your working.
3. Each question carries 10 mark.
4. No calculators are allowed.

1. Let $ABCD$ be a trapezium with $AD \parallel BC$. Suppose K and L are, respectively, points on the sides AB and CD such that $\angle BAL = \angle CDK$. Prove that $\angle BLA = \angle CKD$.

2. Determine all primes p such that

$$5^p + 4 \cdot p^4$$

is a perfect square, i.e., the square of an integer.

3. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ so that

(i) $f(2u) = f(u+v)f(v-u) + f(u-v)f(-u-v)$ for all $u, v \in \mathbb{R}$, and

(ii) $f(u) \geq 0$ for all $u \in \mathbb{R}$.

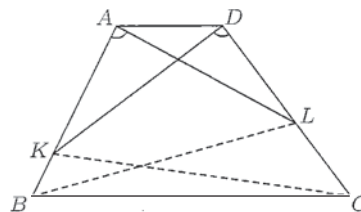
4. There are 11 committees in a club. Each committee has 5 members and every two committees have a member in common. Show that there is a member who belongs to 4 committees.

5. Let $a, b, c \geq 0$. Prove that

$$\frac{(1+a^2)(1+b^2)(1+c^2)}{(1+a)(1+b)(1+c)} \geq \frac{1}{2}(1+abc).$$

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2008
 (Senior Section, Round 2 Solutions)

1. It's clear that $ABLK$ is cyclic. Thus $\angle ADL + \angle AKL = 180^\circ$. Therefore $\angle BKL + \angle BCL = (180^\circ - \angle AKL) + (180^\circ - \angle ADL) = 180^\circ$, so that $BCLK$ is also cyclic. Hence $\angle ABL = \angle DCK$ and $\angle BLA = 180^\circ - \angle ABL - \angle BAL = 180^\circ - \angle DCK - \angle CDK = \angle CKD$.



2. Let $5^p + 4 \cdot p^4 = q^2$. Then

$$5^p = (q - 2p^2)(q + 2p^2).$$

Thus

$$q - 2p^2 = 5^s, \quad q + 2p^2 = 5^t \quad \text{where } 0 \leq s < t \text{ and } s + t = p$$

Eliminating q , we get $4p^2 = 5^s(5^{t-s} - 1)$. If $s > 0$, then $5 \mid 4p^2$. Thus $p = 5$ and the given expression is indeed a square. If $s = 0$, then $t = p$ and we have $5^p = 4p^2 + 1$. We shall prove by induction that $5^k > 4k^2 + 1$ for every integer $k \geq 2$. The inequality certainly holds for $k = 2$. So we assume that it holds for some $k \geq 2$. Note that

$$\frac{4(k+1)^2 + 1}{4k^2 + 1} = \frac{4k^2 + 1}{4k^2 + 1} + \frac{8k}{4k^2 + 1} + \frac{4}{4k^2 + 1} < 1 + 1 + 1 < 5 \quad \text{for } k \geq 2.$$

Thus

$$5^{k+1} = 5 \cdot 5^k > 5(4k^2 + 1) > 4(k+1)^2 + 1.$$

3. f is either constantly 0 or constantly $1/2$.

Clearly, either of the constant functions above satisfies the requirements. Conversely, suppose the given conditions hold. Setting $u = v$, we have

$$f(2u) = f(2u)f(0) + f(0)f(-2u).$$

Case 1. $f(0) = 0$.

Then $f(2u) = 0$ for all $u \in \mathbb{R}$ and hence f is constantly 0.

Case 2. $f(0) = c \neq 0$.

Then

$$f(-2u) = \frac{1-c}{c}f(2u) \quad \text{for all } u \in \mathbb{R}.$$

Setting $u = 0$, we have in particular $c = 1 - c$. Hence $c = 1/2$. It follows that $f(-u) = f(u)$ for all $u \in \mathbb{R}$. Therefore,

$$f(2u) = f(u+v)f(u-v) + f(u-v)f(u+v) = 2f(u+v)f(u-v).$$

Setting $u = 0$, we have $1/2 = f(0) = 2f(v)f(-v) = 2(f(v))^2$. Thus $f(v) = 1/2$ for all $v \in \mathbb{R}$ since $f(v) \geq 0$.

4. Form an incidence matrix A where the rows are indexed by the committees and the columns are indexed by the members and where the (i, j) entry, $a_{ij} = 1$ if member j is in committee i . Then there are five 1's in each row and for each i, j , there exists k such that $a_{ik} = a_{jk} = 1$. Thus there are fifty five 1's in A . We need to prove that there is column with four 1's. Without loss of generality, assume that $a_{1i} = 1$ for $i = 1, \dots, 5$. Consider the submatrix B formed by the first five columns and the last ten rows. Each row of B has at least one 1. Hence B has ten 1's. If there is a column with three 1's, then A has a column with four 1's and we are done. If not, then every column has two 1's and thus each of the corresponding columns in A has three 1's. By consider all the other rows, we see that if no column has four 1's, then every column must have exactly three 1's. But this is impossible as there are fifty five 1's in A but $3 \nmid 55$.

5. First, for any real number t , we have

$$2(1+t^2)^3 = (1+t^3)(1+t)^3 + (1-t^3)(1-t)^3 \geq (1+t^3)(1+t)^3,$$

with equality if and only if $t = 1$. In particular, for nonnegative t ,

$$2^{\frac{1}{3}} \frac{1+t^2}{1+t} \geq (1+t^3)^{\frac{1}{3}},$$

with equality if and only if $t = 1$. Thus

$$\begin{aligned} 2 \frac{(1+a^2)(1+b^2)(1+c^2)}{(1+a)(1+b)(1+c)} &\geq (1+a^3)^{\frac{1}{3}}(1+b^3)^{\frac{1}{3}}(1+c^3)^{\frac{1}{3}} \\ &= (1+a^3+b^3+c^3+a^3b^3+b^3c^3+c^3a^3+a^3b^3c^3)^{\frac{1}{3}} \\ &\geq (1+3abc+3a^2b^2c^2+a^3b^3c^3)^{\frac{1}{3}} \\ &\geq 1+abc. \end{aligned}$$

Equality holds if and only if $a = b = c = 1$.

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2008

(Open Section, Round 1)

Wednesday, 28 May 2008

0930-1200

Instructions to contestants

1. Answer ALL 25 questions.
2. Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.
3. No steps are needed to justify your answers.
4. Each question carries 1 mark.
5. No calculators are allowed.

1. Determine the number of three-element subsets of the set $\{1, 2, 3, 4, \dots, 120\}$ for which the sum of the three elements is a multiple of 3.
2. There are 10 students taking part in a mathematics competition. After the competition, they discover that each of them solves exactly 3 problems and any 2 of them solve at least 1 common problem. What is the minimum number of students who solve a common problem which is solved by most students?

3. Evaluate the sum

$$\sum_{n=1}^{6237} \lfloor \log_2(n) \rfloor,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x .

4. Determine the number of positive integer divisors of 998^{49999} that are not the divisors of 998^{49998} .
5. Let $p(x)$ be a polynomial with real coefficients such that for all real x ,

$$2(1 + p(x)) = p(x - 1) + p(x + 1)$$

and $p(0) = 8, p(2) = 32$. Determine the value of $p(40)$.

6. In the triangle ABC , $AC = 2BC$, $\angle C = 90^\circ$ and D is the foot of the altitude from C onto AB . A circle with diameter AD intersects the segment AC at E . Find $AE : EC$.
7. In the triangle ABC , $AB = 8, BC = 7$ and $CA = 6$. Let E be the point on BC such that $\angle BAE = 3\angle EAC$. Find $4AE^2$.
8. In the triangle ABC , the bisectors of $\angle A$ and $\angle B$ meet at the incentre I , the extension of AI meets the circumcircle of triangle ABC at D . Let P be the foot of the perpendicular from B onto AD , and Q a point on the extension of AD such that $ID = DQ$. Determine the value of $(BQ \times IB)/(BP \times ID)$.

9. In a convex quadrilateral $ABCD$, $\angle BAC = \angle CAD$, $\angle ABC = \angle ACD$, the extensions of AD and BC meet at E , and the extensions of AB and DC meet at F . Determine the value of

$$\frac{AB \cdot DE}{BC \cdot CE}.$$

10. For any positive integer n , let N_n be the set of integers from 1 to n , i.e., $N_n = \{1, 2, 3, \dots, n\}$. Now assume that $n \geq 10$. Determine the maximum value of n such that the following inequality

$$\max_{\substack{a, b \in A \\ a \neq b}} |a - b| \leq 10$$

holds for each $A \subseteq N_n$ with $|A| \geq 10$.

11. How many four-digit numbers greater than 5000 can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 if only the digit 4 may be repeated?
12. Three girls A, B and C, and nine boys are to be lined up in a row. Let n be the number of ways this can be done if B must lie between A and C, and A, B must be separated by exactly 4 boys. Determine $\lfloor n/7! \rfloor$.
13. Determine the number of 4-element subsets $\{a, b, c, d\}$ of $\{1, 2, 3, 4, \dots, 20\}$ such that $a + b + c + d$ is divisible by 3.
14. Find how many three digit numbers, lying between 100 and 999 inclusive, have two and only two consecutive digits identical.
15. Find the maximum natural number which are divisible by 30 and have exactly 30 different positive divisors.
16. Determine the number of 0's at the end of the value of the product $1 \times 2 \times 3 \times 4 \times \dots \times 2008$.
17. Let a_k be the coefficient of x^k in the expansion of $(1 + 2x)^{100}$, where $0 \leq k \leq 100$. Find the number of integers $r : 0 \leq r \leq 99$ such that $a_r < a_{r+1}$.
18. Let a_k be the coefficient of x^k in the expansion of

$$(x + 1) + (x + 1)^2 + (x + 1)^3 + (x + 1)^4 + \dots + (x + 1)^{99}.$$

Determine the value of $\lfloor a_4/a_3 \rfloor$.

19. Let a, b, c, d, e be five numbers satisfying the following conditions:

$$a + b + c + d + e = 0, \quad \text{and}$$

$$abc + abd + abe + acd + ace + ade + bcd + bce + bde + cde = 2008.$$

Find the value of $a^3 + b^3 + c^3 + d^3 + e^3$.

20. Let a_1, a_2, \dots be a sequence of rational numbers such that $a_1 = 2$ and for $n \geq 1$

$$a_{n+1} = \frac{1 + a_n}{1 - a_n}.$$

Determine $30 \times a_{2008}$.

21. Find the number of eight-digit integers comprising the eight digits from 1 to 8 such that $(i + 1)$ does not immediately follow i for all i that runs from 1 to 7.
22. Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ where a_0, a_1, a_2, a_3 and a_4 are constants with $a_4 \neq 0$. When divided by $x - 2003$, $x - 2004$, $x - 2005$, $x - 2006$ and $x - 2007$, respectively, $f(x)$ leaves a remainder of 24, -6 , 4, -6 and 24. Find the value of $f(2008)$.
23. Find the number of 10-letter permutations comprising 4 a 's, 3 b 's, 3 c 's such that no two adjacent letters are identical.
24. Let $f(x) = x^3 + 3x + 1$, where x is a real number. Given that the inverse function of f exists and is given by

$$f^{-1}(x) = \left(\frac{x - a + \sqrt{x^2 - bx + c}}{2} \right)^{1/3} + \left(\frac{x - a - \sqrt{x^2 - bx + c}}{2} \right)^{1/3}$$

where a , b and c are positive constants, find the value of $a + 10b + 100c$.

25. Between 1 and 8000 inclusive, find the number of integers which are divisible by neither 14 nor 21 but divisible by either 4 or 6.

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2008
(Open Section, Round 1 Solutions)

1. Answer: 93640.

For $i = 0, 1, 2$, let $A_i = \{x \mid 1 \leq x \leq 120 \text{ and } x \equiv i \pmod{3}\}$. Then $|A_i| = 40$. If $\{a, b, c\}$ is a 3-element subset of the given set, then 3 divides $a + b + c$ if and only if exactly one of the following conditions holds: (i) all a, b, c are in A_0 , or in A_1 or in A_2 , (ii) one of the a, b, c is in A_0 , another in A_1 , and the third one in A_2 . The number of 3-element subsets of A_i is $\binom{40}{3}$. For each choice of a in A_0 , b in A_1 and c in A_2 , we get a 3-element subset such that 3 divides $a + b + c$. Thus the total number of 3-element subsets $\{a, b, c\}$ such that 3 divides $a + b + c$ is equal to $3\binom{40}{3} + 40^3 = 93640$.

2. Answer: 5.

Without loss of generality, we may assume that every problem is solved by some student. Let A be one of the students. By assumption each of the 9 other students solves at least one common problem as A . By the pigeonhole principle, there are at least 3 students who solve a common problem which is also solved by A . Therefore the minimum number of students who solve a common problem which is solved by most students is at least 4. If the minimum number is exactly 4, then each problem is solved by exactly 4 students. If there are n problems in the competition, then $4n = 30$. But this is a contradiction since 4 does not divide 30. Hence the minimum number of students who solve a problem which is solved by most students is at least 5. We shall show that 5 is in fact the minimum in the following example, where we write (123) to mean that a student solves problems 1, 2 and 3:

$$\begin{array}{ccccc} (123) & (134) & (145) & (156) & (162) \\ (235) & (245) & (246) & (346) & (356) \end{array}$$

3. Answer: 66666.

Note that $\lfloor \log_2(1) \rfloor = 0$, and $\lfloor \log_2(n) \rfloor = k$ if $2^k \leq n < 2^{k+1}$. Since $6237 = 2^{12} + 2141$, we have $\lfloor \log_2(n) \rfloor = 12$ for $2^{12} \leq n \leq 6237$, and there are 2142 such n 's. Thus

$$\begin{aligned} \sum_{n=1}^{6237} \lfloor \log_2(n) \rfloor &= 0 + 1(2^2 - 2) + 2(2^3 - 2^2) + \cdots + 11(2^{12} - 2^{11}) + (12)(2142) \\ &= (11)2^{12} - (2 + 2^2 + \cdots + 2^{11}) + (12)(2142) \\ &= (11)2^{12} - 2(2^{11} - 1) + (12)(2142) \\ &= (20)2^{11} + (12)(2142) + 2 = 66666. \end{aligned}$$

4. Answer: 99999.

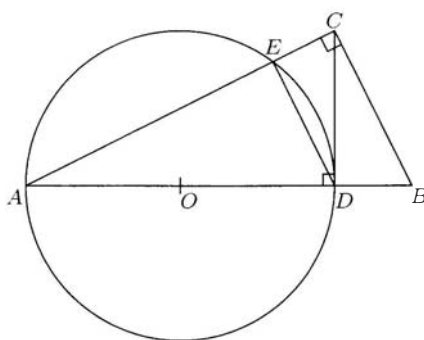
Note that $998 = 2 \times 499$, and 499 is a prime. Any divisor of 998^{49999} has the form $d = 2^a 499^b$, where a and b are positive integers between 0 and 49999. This divisor d does not divide 998^{49998} only in two cases which are either $a = 49999$ or $b = 49999$. In the first case, d can be $2^{49999}, 2^{49999}499, 2^{49999}499^2, \dots, 2^{49999}499^{49999}$. In the second case, d can be $499^{49999}, 499^{49999}2, 499^{49999}2^2, \dots, 499^{49999}2^{49999}$. In each case, there are 50000 possible values, but the number $2^{49999}499^{49999}$ is counted twice. Thus the total number of required divisors is $2 \times 50000 - 1 = 99999$.

5. Answer: 2008.

Let $p(x) = q(x) + x^2$. Substituting this into the given functional equation, we get $q(x) - q(x-1) = q(x+1) - q(x)$ for all real x . Since $q(x)$ is a polynomial, we must have $q(x) - q(x-1) \equiv b$, where b is a real constant. (The polynomial $Q(x) \equiv q(x) - q(x-1)$ can't take the same value at an infinite number of distinct points as it is eventually monotonic, unless it is a constant polynomial.) Next let $q(x) = r(x) + bx$. Substituting this into $q(x) - q(x-1) = b$, we get $r(x) = r(x-1)$ so that $r(x) \equiv c$, where c is a real constant. Therefore, $p(x) = x^2 + bx + c$. It can be easily verified that any polynomial $p(x) = x^2 + bx + c$ satisfies the given function equation. Using $p(0) = 8$ and $p(2) = 32$, we get $p(x) = x^2 + 10x + 8$. Thus $p(40) = 2008$.

6. Answer: 4.

Since AD is a diameter, we have $\angle AED = 90^\circ$ so that DE is parallel to BC .



As $\triangle ADC$ is similar to $\triangle ACB$, we have $AC/AD = AB/AC$ so that $AC^2 = AD \times AB$. Similarly, $BC^2 = AB \times BD$. Thus $AE/EC = AD/BD = (AC/BC)^2 = 4$.

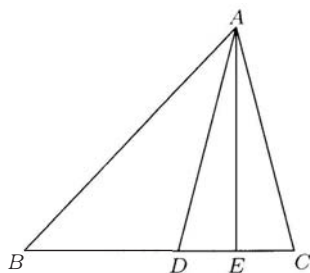
7. Answer: 135.

In general, we let $AB = c$, $AC = b$ and $BC = a$. Let AD be the bisector of $\angle A$. Using the angle-bisector theorem, $BD/CD = c/b$. Thus $BD = ac/(b+c)$ and $CD = ab/(b+c)$. By Stewart's theorem,

$$\left(\frac{ab}{b+c}\right)c^2 + \left(\frac{ac}{b+c}\right)b^2 = aAD^2 + \frac{abc}{(b+c)^2}a^2,$$

which after solving for AD^2 gives

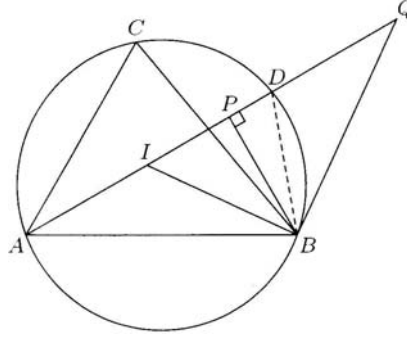
$$AD^2 = bc \left[1 - \left(\frac{a}{b+c} \right)^2 \right].$$



Substituting the values of $a = 7, b = 6, c = 8$, we get $AD = 6$. Thus $\triangle ACD$ is isosceles and the bisector AE of $\angle CAD$ is perpendicular to BC and E is the midpoint of CD . As $DC = 3$, we have $EC = 3/2$. Using Pythagoras' theorem, we get $AE^2 = AC^2 - EC^2 = 6^2 - (\frac{3}{2})^2 = 135/4$. Therefore, $4AE^2 = 135$.

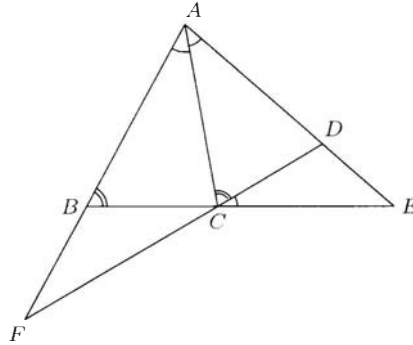
8. Answer: 2.

Let's prove that $\triangle BPQ$ is similar to $\triangle IBQ$.



First $\angle IAB = \angle CAD = \angle CBD$. As $\angle IBA = \angle IBC$, we have $\angle IAB + \angle IBA = \angle CBD + \angle IBC$ so that $\angle DIB = \angle DBI$, thus $DI = DB = DQ$. This means $\triangle IBQ$ is a right-angled triangle with $\angle IBQ = 90^\circ$. As $\angle Q$ is a common angle, we thus have $\triangle BPQ$ is similar to $\triangle IBQ$. Hence, $BQ/BP = IQ/IB = 2ID/IB$. Consequently, $(BQ \times IB)/(BP \times ID) = 2$.

9. Answer: 1.



Since $\angle ACE$ is an exterior angle for the triangle ABC , we have $\angle ACE = \angle ABC + \angle BAC$, hence $\angle ACE = \angle ACD + \angle CAD$. It follows that $\angle CAD = \angle DCE$, so the triangles CED and AEC are similar. For this, we obtain $CE/AE = DE/CE$. But $\angle BAC = \angle CAE$, so we get $BC/CE = AB/AE$. Using these equalities, we get $(AB \cdot DE)/(BC \cdot CE) = 1$.

10. Answer: 99.

First assume that $n \geq 100$. Consider the following set A :

$$A = \{1 + 11i : i = 0, 1, 2, \dots, 9\}.$$

Note that $|A| = 10$, $A \subseteq N_n$ and

$$\bigwedge_{\substack{a, b \in A \\ a \neq b}} |a - b| = 11.$$

Hence the answer is not larger than 99.

Now consider the case that $n \leq 99$. For $i = 1, 2, 3, \dots, 9$,

$$P_i = \{11(i-1) + j : 1 \leq j \leq 11\}.$$

Note that

$$N_n \subseteq \bigcup_{1 \leq i \leq 9} P_i.$$

Now assume that A is any subset of N_n with $|A| \geq 10$. By Pigeonhole Principle, $|A \cap P_i| \geq 2$ for some $i : 1 \leq i \leq 9$. Let $c_1, c_2 \in A \cap P_i$. Thus

$$\bigwedge_{\substack{a, b \in A \\ a \neq b}} |a - b| \leq |c_1 - c_2| \leq (11(i-1) + 11) - (11(i-1) + 1) = 10.$$

11. Answer: 2645.

Let \overline{abcd} represent the integer $a \times 10^3 + b \times 10^2 + c \times 10 + d$.

Note that $\overline{abcd} > 5000$ iff $a \geq 5$ and b, c, d are not all 0. a must be a number in $\{5, 6, 7, 8, 9\}$. Suppose that a is selected from $\{5, 6, 7, 8, 9\}$.

If 4 is not repeated, then the number of ways to choose b, c, d is $9 \times 8 \times 7$;

If 4 appears exactly twice, then the number of ways to choose b, c, d is $\binom{3}{2} \times 8$;

If 4 appears exactly three times, then the number of ways to choose b, c, d is 1.

Hence the answer is

$$5 \times (9 \times 8 \times 7 + \binom{3}{2} \times 8 + 1) = 2645.$$

12. Answer: 3024.

Let Φ be the set of arrangements of these girls and boys under the condition that A, B must be separated by exactly 4 boys. Form a block P with A, B at the two ends and exactly 4 boys between them. The number of ways to form such a block P is

$$2 \times \binom{9}{4} \times 4!.$$

Then we have

$$|\Phi| = 2 \times \binom{9}{4} \times 4! \times 7!,$$

as we can consider such a block P as one item, and there are still 5 boys and one girl (i.e., C).

Note that in any arrangement in Φ , C is outside the block between A and B . In exactly half of the arrangements of Φ , B is between A and C . Hence

$$n = |\Phi|/2 = \binom{9}{4} \times 4! \times 7!.$$

Hence the answer is $\binom{9}{4} \times 4! = 3024$.

13. Answer: 11901.

For $i = 0, 1, 2$, let

$$A_i = \{j : 1 \leq j \leq 20, j \equiv i \pmod{3}\}.$$

Note that $|A_0| = 6$, $|A_1| = 7$ and $|A_2| = 7$.

It can be shown that for $a, b, c, d \in \{1, 2, 3, 4, \dots, 20\}$, $a + b + c + d$ is divisible by 3 iff

- (i) $|\{a, b, c, d\} \cap A_i| = 2$ for all $i = 1, 2$; or
- (ii) $|\{a, b, c, d\} \cap A_0| = 1$ and $|\{a, b, c, d\} \cap A_i| = 3$ for some $i : 1 \leq i \leq 2$; or
- (iii) $|\{a, b, c, d\} \cap A_0| = 2$ and $|\{a, b, c, d\} \cap A_i| = 1$ for all $i = 1, 2$; or
- (iv) $|\{a, b, c, d\} \cap A_0| = 4$.

Thus the number of 4-element subsets $\{a, b, c, d\}$ of $\{1, 2, 3, 4, \dots, 20\}$ such that $a + b + c + d$ is divisible by 3 is

$$\begin{aligned} & \binom{|A_1|}{2} \binom{|A_2|}{2} + \binom{|A_0|}{1} \binom{|A_1|}{3} + \binom{|A_0|}{1} \binom{|A_2|}{3} + \binom{|A_0|}{2} \times |A_1| \times |A_2| + \binom{|A_0|}{4} \\ &= \binom{7}{2}^2 + 6 \binom{7}{3} \times 2 + \binom{6}{2}^2 \times 7^2 + \binom{6}{4} = 11901. \end{aligned}$$

14. Answer: 162.

There are two possible formats for three digit numbers to have two and only two consecutive digits identical:

- (i) \overline{aac} where $a \neq 0$ and $c \neq a$, or
- (ii) \overline{abb} where $a \neq 0$ and $a \neq b$.

Thus the number of such integers is

$$9 \times 9 + 9 \times 9 = 162.$$

15. Answer: 11250.

Let a be a natural number divisible by 30. Thus a can be expressed as

$$a = 2^{n_2} 3^{n_3} 5^{n_5} \prod_{1 \leq i \leq r} p_i^{k_i},$$

where $r \geq 0$, $k_i \geq 0$, and n_2, n_3, n_5 are positive integers. The number of positive divisors of a is

$$(n_2 + 1)(n_3 + 1)(n_5 + 1) \prod_{1 \leq i \leq r} (k_i + 1).$$

Since $30 = 2 \times 3 \times 5$, we have $r = 0$ if

$$(n_2 + 1)(n_3 + 1)(n_5 + 1) \prod_{1 \leq i \leq r} (k_i + 1) = 30.$$

Hence $30|a$ and a has exactly 30 positive divisors iff $a = 2^{n_2} 3^{n_3} 5^{n_5}$ and $(n_2 + 1)(n_3 + 1)(n_5 + 1) = 30$, where n_2, n_3, n_5 are all positive. Further, $(n_2 + 1)(n_3 + 1)(n_5 + 1) = 30$, where n_2, n_3, n_5 are all positive, iff $\{n_2, n_3, n_5\} = \{1, 2, 4\}$.

The maximum value of such a is $a = 2 \times 3^2 \times 5^4 = 11250$.

16. Answer: 500.

Let m be the maximum integer such that 2^m is a factor of $1 \times 2 \times 3 \times 4 \times \cdots \times 2008$, and n the maximum integer such that 5^n is a factor of $1 \times 2 \times 3 \times 4 \times \cdots \times 2008$.

Then the number of 0's at the end of $1 \times 2 \times 3 \times 4 \times \cdots \times 2008$ is equal to $\min\{m, n\}$. It is obvious that $m \geq n$.

Observe that $2008 < 5^5$. For $1 \leq k \leq 4$, the number, denoted by a_k , of integers in $\{1, 2, 3, 4, \dots, 2008\}$, denoted by A , divisible by 5^k is

$$a_k = \left\lfloor 2008/5^k \right\rfloor.$$

So $a_1 = 401$, $a_2 = 80$, $a_3 = 16$, $a_4 = 3$. Thus, there are exactly 3 numbers in A divisible by 5^4 ;

exactly $16 - 3 = 13$ numbers in A divisible by 5^3 but not by 5^4 ;

exactly $80 - 16 = 64$ numbers in A divisible by 5^2 but not by 5^3 ;

exactly $401 - 80 = 321$ numbers in A divisible by 5 but not by 5^2 .

Hence, the maximum integer n such that 5^n is a factor of $1 \times 2 \times 3 \times 4 \times \cdots \times 2008$ is

$$n = 4 \times 3 + 3 \times 13 + 2 \times 64 + 1 \times 321 = 500.$$

17. Answer: 67.

Let a_k be the coefficient of x^k in the expansion of $(1 + 2x)^{100}$. Then

$$a_k = 2^k \binom{100}{k}.$$

Thus

$$\frac{a_{r+1}}{a_r} = 2 \binom{100}{r+1} / \binom{100}{r} = \frac{2(100-r)}{r+1}.$$

Solving the inequality

$$\frac{2(100-r)}{r+1} > 0$$

gives the solution $r < 199/3$. It implies that

$$a_0 < a_1 < \cdots < a_{66} < a_{67} \geq a_{68} \geq \cdots \geq a_{100}.$$

Hence the answer is 67.

18. Answer: 19.

Note that

$$a_k = \binom{1}{k} + \binom{2}{k} + \binom{3}{k} + \binom{4}{k} + \cdots + \binom{99}{k} = \binom{100}{k+1}.$$

Thus

$$a_4/a_3 = \frac{\binom{100}{5}}{\binom{100}{4}} = \frac{96}{5}.$$

Thus the answer is 19.

19. Answer: 6024.

Note that $(a+b+c+d+e)^3 = a^3+b^3+c^3+d^3+e^3+3a^2(b+c+d+e)+3b^2(c+d+e+a)+3c^2(d+e+a+b)+3d^2(e+a+b+c)+3e^2(a+b+c+d)+6(abc+abd+abe+\dots)$. Since $a+b+c+d+e=0$, $b+c+d+e=-a$, so $3a^2(b+c+d+e)=-3a^3$. Similarly, we get $-3b^3, -3c^3, -3d^3$ and $-3e^3$. Thus, we have $2(a^3+b^3+c^3+d^3+e^3)=6(abc+abd+abe+\dots)$, so $a^3+b^3+c^3+d^3+e^3=3 \times 2008 = 6024$.

20. Answer: 10.

We have

$$\begin{aligned} a_{n+1} &= \frac{1+a_n}{1-a_n} \\ a_{n+2} &= \frac{1+a_{n+1}}{1-a_{n+1}} = \frac{1+\frac{1+a_n}{1-a_n}}{1-\frac{1+a_n}{1-a_n}} = -\frac{1}{a_n} \\ a_{n+4} &= -\frac{1}{a_{n+2}} = a_n. \end{aligned}$$

Thus $a_{2008} = a_{2004} = \dots = a_4 = \frac{1}{3}$.

21. Answer: 16687.

We may proceed by using the principle of inclusion and exclusion as follows. Let the universal set S be the set of 8-digit integers comprising the 8 digits from 1 to 8. Let $P_i (1 \leq i \leq 7)$ be the property that $(i+1)$ immediately follows i in an element of S . Hence $W(0) = 8!$, $W(i) = \binom{7}{i}(8-i)!$ for $1 \leq i \leq 7$. Then the answer $= E(0) = W(0) - W(1) + W(2) - \dots - W(7) = 16687$. The answer can also be obtained from either one of the two expressions, namely, $\frac{1}{8}D_9$ or $D_8 + D_7$, where D_n is the number of derangements of the integers from 1 to n and $D_n = n![1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!}]$.

22. Answer: 274.

$$\begin{aligned} f(x) &= \frac{(x-2004)(x-2005)(x-2006)(x-2007)(24)}{(-1)(-2)(-3)(-4)} \\ &+ \frac{(x-2003)(x-2005)(x-2006)(x-2007)(-6)}{(1)(-1)(-2)(-3)} \\ &+ \frac{(x-2003)(x-2004)(x-2006)(x-2007)(4)}{(2)(1)(-1)(-2)} \\ &+ \frac{(x-2003)(x-2004)(x-2005)(x-2007)(-6)}{(3)(2)(1)(-1)} \\ &+ \frac{(x-2003)(x-2004)(x-2005)(x-2006)(24)}{(4)(3)(2)(1)}. \end{aligned}$$

Therefore $f(2008) = 274$.

23. Answer: 248.

We shall use the *RP hex function* (where *RP* stands for Roger Poh) to deal with this question. Let $\#(p, q, r)$ denote the number of permutations of p copies of A , q copies of B and r copies of C such that no two adjacent letters are identical. Among these $\#(p, q, r)$ such permutations, let $\#(p; q, r)$ denote the number of those permutations which do not begin with A . Hence, we have

$$\begin{aligned} \#(p, q, r) &= \#(p-1; q, r) + \#(q-1; p, r) + \#(r-1; p, q) \quad \text{and} \\ \#(p; q, r) &= \#(q-1; p, r) + \#(r-1; p, q). \end{aligned}$$

For this question, we are required to evaluate $\#(4, 3, 3)$.

$$\begin{aligned}
\#(4, 3, 3) &= \#(3; 3, 3) + 2\#(2; 4, 3) \\
\#(3; 3, 3) &= 2\#(2; 3, 3) = 4\#(2; 2, 3) = 116 \\
\#(2; 2, 3) &= \#(1; 2, 3) + \#(2; 2, 2) = 9 + 20 = 29 \\
\#(1; 2, 3) &= \#(1; 1, 3) + \#(2; 1, 2) = 2 + 7 = 9 \\
\#(1; 1, 3) &= \#(1, 3) + \#(2; 1, 1) = \#(2; 1, 1) = 2 \\
\#(2; 1, 1) &= 2\#(2, 1) = 2 \\
\#(2; 1, 2) &= \#(2, 2) + \#(1; 2, 1) = 2 + \#(1; 2, 1) = 7 \\
\#(1; 2, 1) &= \#(1; 1, 1) + \#(1, 2) = 2\#(1, 1) + 1 = 5 \\
\#(2; 2, 2) &= 2\#(1; 2, 2) = 4\#(1; 2, 1) = 20 \\
\#(2; 4, 3) &= \#(3; 2, 3) + \#(2; 4, 2) = 45 + 21 = 66 \\
\#(3; 2, 3) &= \#(1; 3, 3) + \#(2; 3, 2) = 2\#(2; 1, 3) + \#(2; 3, 2) \\
\#(2; 1, 3) &= \#(2, 3) + \#(2; 1, 2) = 1 + 7 = 8 \\
\#(3; 2, 3) &= 16 + 29 = 45 \\
\#(2; 4, 2) &= \#(3; 2, 2) + \#(1; 2, 4) = 2\#(1; 3, 2) + \#(1; 2, 4) \\
&= 18 + \#(1; 2, 4) = 18 + 3 = 21 \\
\#(4; 3, 3) &= 116 + 2(66) = 248.
\end{aligned}$$

24. Answer: 521.

Let $y = x^3 + 3x + 1$. Then $x^3 + 3x + (1 - y) = 0$. Let A, B be constants such that

$$(x^3 + A^3 + B^3) - 3ABx \equiv x^3 + 3x + (1 - y).$$

Hence

$$A^3 + B^3 = 1 - y \quad (1)$$

$$-3AB = 3 \quad (2)$$

By (2), we have $B = -1/A$. Substitute this into (1) and simplify,

$$A^6 + (y - 1)A^3 - 1 = 0.$$

Hence

$$A^3 = \frac{1 - y \pm \sqrt{(y - 1)^2 + 4}}{2} = \frac{1 - y \pm \sqrt{y^2 - 2y + 5}}{2}.$$

Therefore $B = \left(\frac{1 - y \mp \sqrt{y^2 - 2y + 5}}{2} \right)^{1/3}$. Hence

$$\begin{aligned}
x &= -A - B = \left(\frac{y - 1 + \sqrt{y^2 - 2y + 5}}{2} \right)^{1/3} + \left(\frac{y - 1 - \sqrt{y^2 - 2y + 5}}{2} \right)^{1/3} \\
&\equiv \left(\frac{y - a + \sqrt{y^2 - by + c}}{2} \right)^{1/3} + \left(\frac{y - a - \sqrt{y^2 - by + c}}{2} \right)^{1/3}
\end{aligned}$$

Thus $a = 1, b = 2, c = 5$ and $a + 10b + 100c = 521$.

25. Answer: 2287.

Let $S = \{1, \dots, 8000\}$, $A = \{x \in S : 4 \mid x\}$, $B = \{x \in S : 6 \mid x\}$, $C = \{x \in S : 14 \mid x\}$,
 $D = \{x \in S : 21 \mid x\}$.

$$\begin{aligned}
& |(A \cup B) \cap (C \cup D)'| \\
&= |A \cup B| - |(A \cup B) \cap (C \cup D)| \\
&= |A| + |B| - |A \cap B| - |A \cap C| \cup (B \cap C) \cup (A \cap D) \cup (B \cap D)| \\
&= |A| + |B| - |A \cap B| - |A \cap C| - |B \cap C| - |A \cap D| - |B \cap D| \\
&\quad + |A \cap B \cap C| + |A \cap C \cap D| + |A \cap B \cap C \cap D| + |A \cap B \cap C \cap D| \\
&\quad + |B \cap C \cap D| + |A \cap B \cap D| - |A \cap B \cap C \cap D| - |A \cap B \cap C \cap D| \\
&\quad - |A \cap B \cap C \cap D| - |A \cap B \cap C \cap D| + |A \cap B \cap C \cap D| \\
&= \lfloor 8000/4 \rfloor + \lfloor 8000/6 \rfloor - \lfloor 8000/12 \rfloor - \lfloor 8000/28 \rfloor - \lfloor 8000/42 \rfloor - \lfloor 8000/84 \rfloor \\
&\quad - \lfloor 8000/42 \rfloor + 3 \times \lfloor 8000/84 \rfloor + \lfloor 8000/42 \rfloor - \lfloor 8000/84 \rfloor \\
&= 2287.
\end{aligned}$$

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2008
 (Open Section, Round 2)

Saturday, 5 July 2008

0900-1330

Instructions to contestants

1. Answer ALL 5 questions.
2. Show all the steps in your working.
3. Each question carries 10 mark.
4. No calculators are allowed.

1. Find all pairs of positive integers (n, k) so that $(n + 1)^k - 1 = n!$.
2. In the acute triangle ABC , M is a point in the interior of the segment AC and N is a point on the extension of the segment AC such that $MN = AC$. Let D and E be the feet of the perpendiculars from M and N onto the lines BC and AB respectively. Prove that the orthocentre of $\triangle ABC$ lies on the circumcircle of $\triangle BED$.
3. Let n, m be positive integers with $m > n \geq 5$ and with m depending on n . Consider the sequence a_1, a_2, \dots, a_m where

$$\begin{aligned} a_i &= i && \text{for } i = 1, \dots, n \\ a_{n+j} &= a_{3j} + a_{3j-1} + a_{3j-2} && \text{for } j = 1, \dots, m - n \end{aligned}$$

with $m - 3(m - n) = 1$ or 2 , i.e., $a_m = a_{m-k} + a_{m-k-1} + a_{m-k-2}$ where $k = 1$ or 2 . (Thus if $n = 5$, the sequence is $1, 2, 3, 4, 5, 6, 15$ and if $n = 8$, the sequence is $1, 2, 3, 4, 5, 6, 7, 8, 6, 15, 21$.) Find $S = a_1 + a_2 + \dots + a_m$ if (i) $n = 2007$, (ii) $n = 2008$.

4. Let $0 < a, b < \pi/2$. Show that

$$\frac{5}{\cos^2 a} + \frac{5}{\sin^2 a \sin^2 b \cos^2 b} \geq 27 \cos a + 36 \sin a.$$

5. Consider a 2008×2008 chess board. Let M be the smallest number of rectangles that can be drawn on the chess board so that the sides of every cell of the board is contained in the sides of one of the rectangles. Find the value of M . (For example, for the 2×3 chess board, the value of M is 3.)

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2008

(Open Section, Round 2 Solutions)

1. If p is a prime factor of $n + 1$, it is also a factor of $n! + 1$. However, none of the numbers $1, \dots, n$ is a factor of $n! + 1$. So we must have $p = n + 1$, i.e., $n + 1$ must be prime. For $n + 1 = 2, 3, 5$, we have solutions $(n, k) = (1, 1), (2, 1)$ and $(4, 2)$. Let us show that there are no other solutions. Indeed, suppose that $n + 1$ is a prime number ≥ 7 . Then $n = 2m$, $m > 2$. Now $2 < m < n$ and thus $n^2 = 2mn$ divides $n!$. Thus n^2 divides

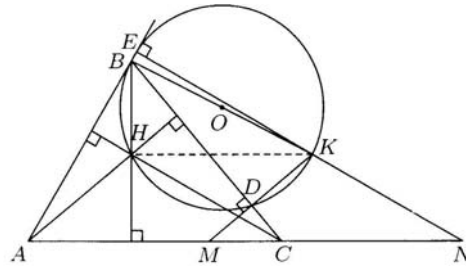
$$(n + 1)^k - 1 = n^k + kn^{k-1} + \dots + \frac{k(k-1)}{2}n^2 + nk.$$

Thus $n^2 \mid kn$ and, in particular $k \geq n$. It follows that

$$n! = (n + 1)^k - 1 > n^k \geq n^n \geq n!,$$

a contradiction.

2. Let K be the point of intersection of MD and NE . It is easy to see that the circle with diameter BK is the circumcircle of $\triangle BED$. As AH is parallel to MK and CH is parallel to NK , we have $\angle HAC = \angle KMN$ and $\angle ACH = \angle MNK$. Since $AC = MN$, we thus have $\triangle AHC$ is congruent to $\triangle MKN$.



Therefore the distance from K onto AC equals to the distance from H onto AC . But H and K are on the same side with respect to the line AC , it follows that HK is parallel to AC . Therefore HK is perpendicular to BH and H lies on the circle with diameter BK circumscribing about $\triangle BED$.

3. We shall solve the problem for general n . Initially, let a_1, a_2, \dots, a_n be the *active* sequence. An *operation* removes the first three terms and append their sum as the last term of the sequence. Thus after one operation on the initial sequence, the active sequence is a_4, a_5, \dots, a_{n+1} . Such operations will stop when the active sequence is of length at most 2. In this case, the last term of the active sequence is a_m . The following observations are obvious.

1. After each operation, the length of the active sequence decreases by 2 and the sum of its terms is preserved.
2. If the length of an active sequence is $3k$ and the last term is a_p , then after k operations, the length of the active sequence is k and the first term is a_{p+1} .

To compute S , we have two cases:

Case (i). n odd. Let k and r be integers such that $n = 3^k + 2r < 3^{k+1}$, i.e., $0 \leq r < 3^k$. Note that $3r < n$. The first $3r$ terms, a_1, \dots, a_{3r} is called the *first block* and the sum of its terms is $M = 1 + 2 + \dots + 3r = 3r(3r+1)/2$. Now apply the above observations. The sum of the terms of any active sequence is $N = 1 + 2 + \dots + n = n(n+1)/2$. After r operations, the length of the active sequence is 3^k and its first term is a_{3r+1} . This active sequence is called the *second block*. After 3^{k-1} operations, the length of the active sequence is 3^{k-1} . This active sequence is called the *third block*. Repeat this until the length of the active sequence is 1. This is now the $(k+2)^{\text{nd}}$ block. Thus we have a block whose sum is M and $k+1$ blocks whose sum is N . Hence $S = M + (k+1)N$. Since $3^6 = 729$, $n = 2007 = 3^6 + 2(639)$, we have $S = 15943599$.

Case (ii). n even. Let k and r be integers such that $n = 2(3^k) + 2r < 2(3^{k+1})$, i.e., $0 \leq r < 2(3^k)$. Note that $3r < n$. The first $3r$ terms, a_1, \dots, a_{3r} is called the *first block* and the sum of its terms is M . As in case i, after r operations, the length of the active sequence is $2(3^k)$ and its first term is a_{3r+1} . This active sequence is called the *second block*. After $2(3^{k-1})$ operations, the length of the active sequence is $2(3^{k-1})$. This active sequence is called the *third block*. Repeat this until the length of the active sequence is 2. This is now the $(k+2)^{\text{nd}}$ block. Thus we have a block whose sum is M and $k+1$ blocks whose sum is N . Hence $S = M + (k+1)N$. Since $3^6 = 729$, $2008 = 2(3^6) + 2(275)$. Thus $r = 275$ and $k = 6$ and so $S = 14459977$.

4. First note that by AM-GM,

$$\frac{\cos^2 a}{\sin^2 a \sin^2 b \cos^2 b} + \frac{\sin^2 a}{\cos^2 a} \geq \frac{2}{\sin b \cos b}.$$

Thus

$$\begin{aligned} \text{LHS} &= \left(\frac{5}{\cos^2 a} + \frac{5}{\sin^2 a \sin^2 b \cos^2 b} \right) (\cos^2 a + \sin^2 a) \\ &= 5 + 5 \left(\frac{\cos^2 a}{\sin^2 a \sin^2 b \cos^2 b} + \frac{\sin^2 a}{\cos^2 a} \right) + \frac{5}{\sin^2 b \cos^2 b} \\ &\geq 5 + \frac{10}{\sin b \cos b} + \frac{5}{\sin^2 b \cos^2 b} \geq 5 \left(1 + \frac{1}{\sin b \cos b} \right)^2 \\ &\geq 5 \left(1 + \frac{2}{\sin 2b} \right)^2 \geq 45 \geq 27 \cos a + 36 \sin a. \end{aligned}$$

The last inequality follows because if $\sin x = 3/5$, then $\cos x = 4/5$ and $1 \geq \sin(a+x) = \sin x \cos a + \cos x \sin a$.

5. The answer is $M = 2009$. All the horizontal sides can be covered by 1004 pieces of 1×2008 rectangles except the boundary of the chess board which can be covered by the boundary rectangle. The remaining vertical sides can be covered by 1004 pieces of 2008×1 rectangles. Thus $M \leq 2009$.

Now suppose that the chess board has been covered by M rectangles in the desired way. Let a of the rectangles have their top and bottom on the top and bottom of the board, b of the rectangles have their top on the top of the board, c of the rectangles have their bottom on the bottom of the board and d of the rectangles have neither their top nor bottom on the top or bottom of the board.

Since there are 2007 internal horizontal lines, we have $b + c + 2d \geq 2007$. Since there are 2009 vertical lines intersecting the top of the board, we have $2a + 2b \geq 2009$ or $a + b \geq 1005$. Similarly, $a + c \geq 1005$. Thus $2a + b + c \geq 2010$. Hence $2(a + b + c + d) \geq 4017$ i.e., $M = a + b + c + d \geq 2009$.