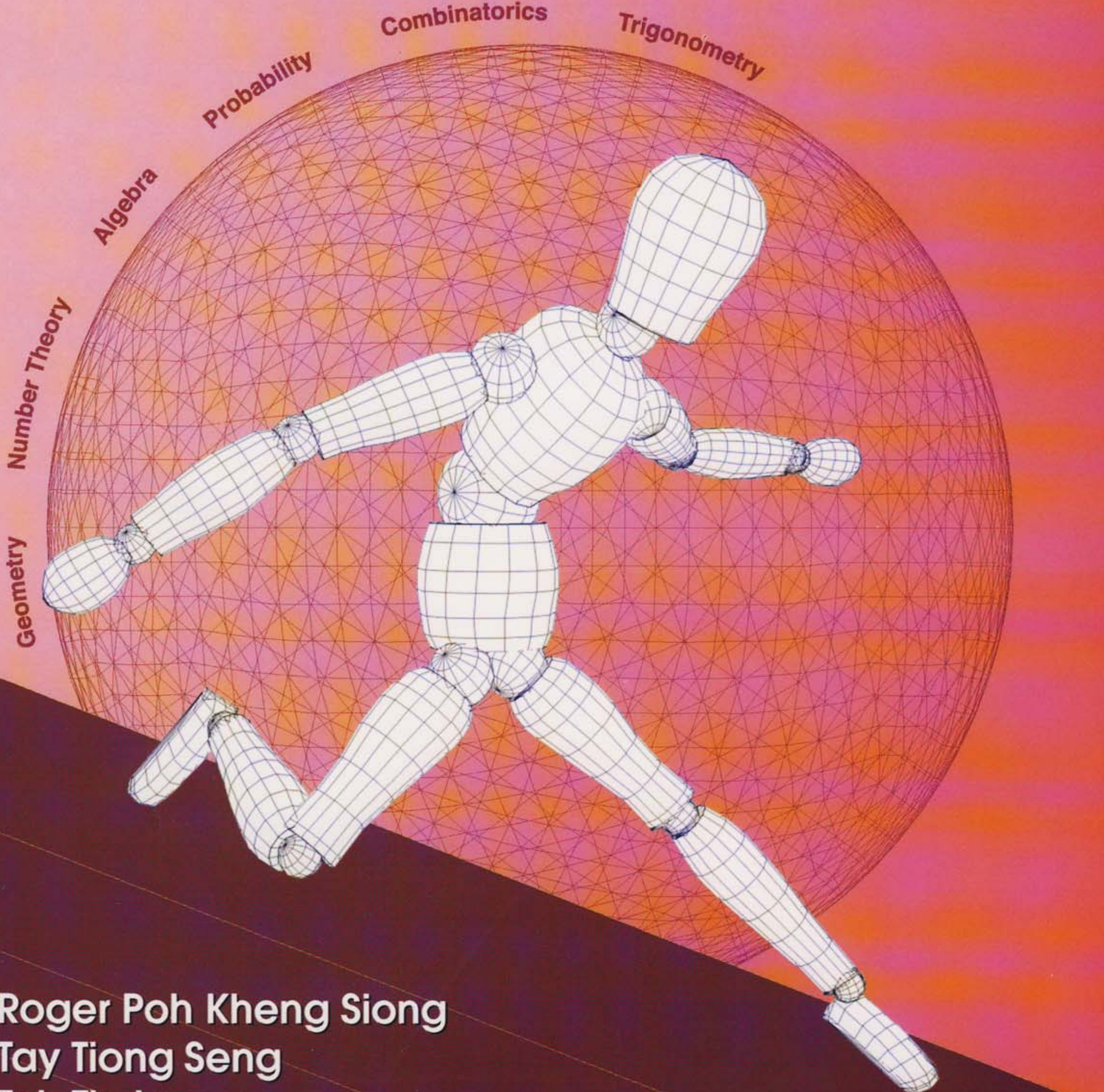
SINGAPORE MATHEMATICAL OLYMPIADS

2007



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Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Junior Section)

Tuesday, 29 May 2007 | 0930-1200

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter only the letters (A, B, C, D, or E) corresponding to the correct answers in the answer sheet.

For the other short questions, write your answers in answer sheet and shade the appropriate bubbles below your answers.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

1. Among the four statements on integers below,

"If $a < b$ then $a^2 < b^2$"; "$a^2 > 0$ is always true";

"$-a < 0$ is always true"; "If $ac^2 < bc^2$ then $a < b$",

how many of them are correct?

(A) 0; (B) 1; (C) 2; (D) 3; (E) 4.

2. Which of the following numbers is odd for any integer values of k?

(A) $2007 + k^3$; (B) $2007 + 7k$; (C) $2007 + 2k^2$; (D) $2007 + 2007k$; (E) $2007k$.

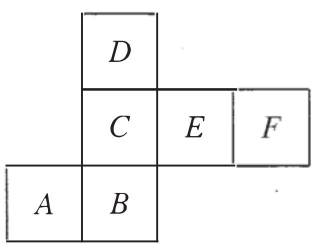
3. In a school, all 300 Secondary 3 students study either Geography, Biology or both Geography and Biology. If 80% study Geography and 50% study Biology, how many students study both Geography and biology?

(A) 30; (B) 60; (C) 80; (D) 90; (E) 150.

4. An unbiased six-sided dice is numbered 1 to 6. The dice is thrown twice and the two scores added. Which of the following events has the highest probability of occurrence?

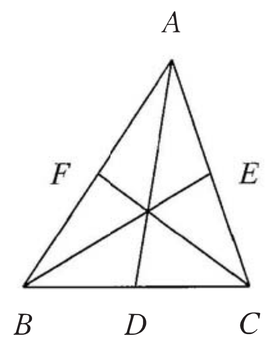
(A) The total score is a prime number; (B) The total score is a multiple of 4;
(C) The total score is a perfect square; (D) The total score is 7;
(E) The total score is a factor of 12.

5. The cardboard below can be cut out and folded to make a cube. Which face will then be opposite the face marked A?



(A) B; (B) C; (C) D; (D) E; (E) F.

6. How many triangles can you find in the following figure?

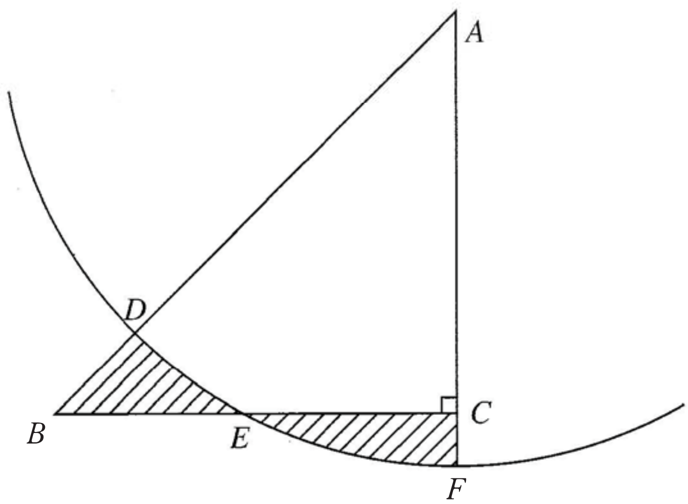


(A) 7; (B) 10; (C) 12; (D) 16; (E) 20.

7. Suppose x_1, x_2 and x_3 are roots of $(11-x)^3 + (13-x)^3 = (24-2x)^3$. What is the sum of $x_1 + x_2 + x_3$?

(A) 30; (B) 36; (C) 40; (D) 42; (E) 44.

8. In the following right-angled triangle ABC, $AC = BC = 1$ and DEF is an arc of a circle with center A. Suppose the shaded areas BDE and CEF are equal and $AD = \frac{x}{\sqrt{\pi}}$. What is the value of x?



(A) 1; (B) 2; (C) 3; (D) 4; (E) 5.

9. Suppose

$$\frac{1}{x} = \frac{2}{y+z} = \frac{3}{z+x} = \frac{x^2 - y - z}{x+y+z}.$$

What is the value of $\frac{z-y}{x}$?

(A) 1; (B) 2; (C) 3; (D) 4; (E) 5.

10. Suppose $x^2 - 13x + 1 = 0$. What is the last digit of $x^4 + x^{-4}$?

(A) 1; (B) 3; (C) 5; (D) 7; (E) 9.

11. In a triangle ABC, it is given that $AB = 1$ cm, $BC = 2007$ cm and $AC = a$ cm, where a is an integer. Determine the value of a.

12. Find the value (in the simplest form) of $\sqrt{21 + 12\sqrt{3}} - \sqrt{21 - 12\sqrt{3}}$.

13. Find the value of

$$\frac{2007^2 + 2008^2 - 1993^2 - 1992^2}{4}.$$

14. Find the greatest integer N such that

$$N \leq \sqrt{2007^2 - 20070 + 31}.$$

15. Suppose that x and y are non-zero real numbers such that

$$\frac{x}{3} = y^2 \quad \text{and} \quad \frac{x}{9} = 9y.$$

Find the value of $x + y$.

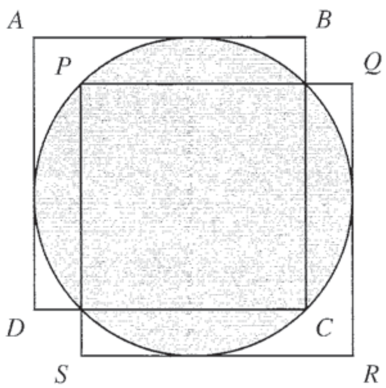
16. Evaluate the sum

$$\frac{2007}{1 \times 2} + \frac{2007}{2 \times 3} + \cdots + \frac{2007}{2006 \times 2007}.$$

17. Find the sum of the digits of the product

$$(\underbrace{111111111\ldots111}_{2007\ 1's}) \times 2007$$

18. The diagram shows two identical squares, $ABCD$ and $PQRS$, overlapping each other in such a way that their edges are parallel, and a circle of radius $(2-\sqrt{2})$ cm covered within these squares. Find the length of the square $ABCD$.

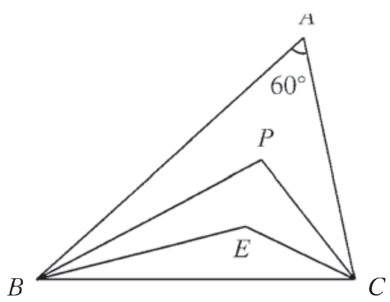


19. When 2007 bars of soap are packed into N boxes of equal size, where N is an integer strictly between 200 and 300, there are extra 5 bars remaining. Find N.

20. Suppose that $a+x^2=2006$, $b+x^2=2007$ and $c+x^2=2008$ and $abc=3$. Find the value of

$$\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}.$$

21. The diagram below shows a triangle ABC in which $\angle A=60^\circ$, BP and BE trisect $\angle ABC$; and CP and CE trisect $\angle ACB$. Let the angle $\angle BPE$ be x°. Find x.



22. Suppose that $x - y = 1$. Find the value of $x^4 - xy^3 - x^3y - 3x^2y + 3xy^2 + y^4$.

23. How many ordered pairs of integers (m, n) where $0 < m < n < 2008$ satisfy the equation $2008^2 + m^2 = 2007^2 + n^2$?

24. If $x + \sqrt{xy} + y = 9$ and $x^2 + xy + y^2 = 27$, find the value of $x - \sqrt{xy} + y$.

25. Appending three digits at the end of 2007, one obtains an integer N of seven digits. In order to get N to be the minimal number which is divisible by 3, 5 and 7 simultaneously, what are the three digits that one would append?

26. Find the largest integer n such that $n^{6021} < 2007^{2007}$.

27. Find the value of

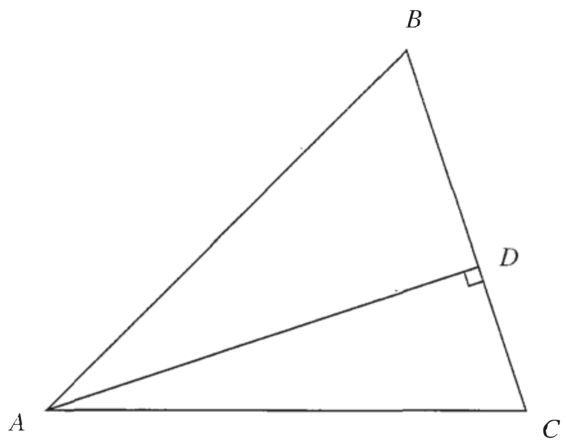
$$\frac{x^4 - 6x^3 - 2x^2 + 18x + 23}{x^2 - 8x + 15}$$

when $x = \sqrt{19 - 8\sqrt{3}}$.

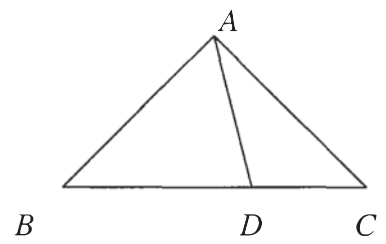
28. Find the value of a such that the two equations $x^2 + ax + 1 = 0$ and $x^2 - x - a = 0$ have one common real root.

29. Odd integers starting from 1 are grouped as follows: (1), (3, 5), (7, 9, 11), (13, 15, 17, 19), . . . , where the n-th group consists of n odd integers. How many odd integers are in the same group which 2007 belongs to?

30. In $\triangle ABC$ $\angle BAC = 45°$. D is a point on BC such that AD is perpendicular to BC. If $BD = 3$ cm and $DC = 2$ cm, and the area of the $\triangle ABC$ is x cm^2. Find the value of x.

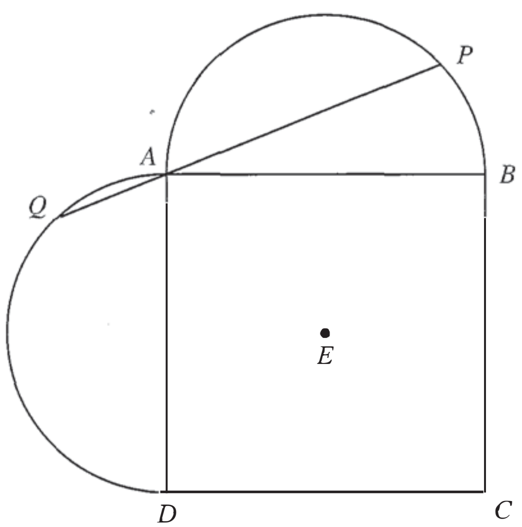


31. In $\triangle ABC$ (see below), $AB = AC = \sqrt{3}$ and D is a point on BC such that $AD = 1$. Find the value of $BD \cdot DC$.



32. Find the last digit of $2^{2^{2007}} + 1$.

33. In the following diagram, $ABCD$ is a square, and E is the center of the square $ABCD$. P is a point on a semi-circle with diameter AB. Q is a point on a semi-circle with diameter AD. Moreover, Q, A and P are collinear (that is, they are on the same line). Suppose $QA = 14$ cm, $AP = 46$ cm, and $AE = x$ cm. Find the value of x.



34. Find the smallest positive integer n such that $n(n + 1)(n + 2)$ is divisible by 247.

35. Find the largest integer N such that both $N + 496$ and $N + 224$ are perfect squares.

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(Junior Section Solutions)

1. Ans: (B)

   Only the last statement is correct: $ac^2 < bc^2$ implies $c^2 > 0$, hence $a < c$. For other statements, counterexamples can be take as $a = -2, b = -1$; $a = 0$ and $a = 0$ respectively.

2. Ans: (C)

   Because $2k^2$ is always even, thus $2007 + 2k^2$ is always odd. For other statements, either $k = 1$ or $k = 0$ gives a counterexample.

3. Ans: (D)

   Use Inclusion and Exclusion Principle.

4. Ans: (A)

   The number of occurrences for each event is: $15, 9, 7, 6$ and 12 respectively.

5. Ans: (D)

   Just imagine.

6. Ans: (D)

   Just count: Label the "center" O. There are 6 triangles like $\triangle AFO$; 3 like $\triangle AOB$; 6 like $\triangle ABD$ and 1 like $\triangle ABC$. Total: 16.

7. Ans: (B)

   Let $a = 11 - x$ and $b = 13 - x$. We have $a^3 + b^3 = (a+b)^3$. Simplify: $3ab(a+b) = 0$. Replacing a, b back in terms of x, we found the three roots are $11, 12$ and 13. Thus $x_1 + x_2 + x_3 = 36$.

8. Ans: (B)

   Since the area of sector ADF and $\triangle ABC$ are equal, we have

   $$\frac{1}{2}\left(\frac{x}{\sqrt{\pi}}\right)^2\frac{\pi}{4} = \frac{1}{2}.$$

   The result follows.

9. Ans: (B)

$$\frac{1}{x} = \frac{2}{y+z} \text{ and } \frac{1}{x} = \frac{3}{z+x}$$

tells us

$$\frac{y}{x} + \frac{z}{x} = 2 \text{ and } \frac{z}{x} + 1 = 3.$$

Thus $y = 0$ and $\frac{z}{x} = 2$. Note we didn't use the last equality, but $x = -1; y = 0$ and $z = -2$ satisfy all conditions.

10. Ans: (D)

By assumption $x + \frac{1}{x} = 13$. Thus $x^2 + \frac{1}{x^2} = 13^2 - 2 = 167$. Similarly, $x^4 + \frac{1}{x^4} = 167^2 - 2$, whose last digit is 7.

11. Ans: 2007.

Use $a < 2007 + 1$ and $2007 < a + 1$.

12. Ans: 6.

Use $21 \pm 12\sqrt{3} = (\sqrt{12} \pm 3)^2$.

13. Ans: 30000.

Using $a^2 - b^2 = (a+b)(a-b)$, $2008^2 - 1993^2 = 4001 \times 15$ and $2007^2 - 1992^2 = 3999 \times 15$. The result follows.

14. Ans: 2002.

By completion of square, $2007^2 - 20070 + 31 = (2007 - 5)^2 + 6$. The result follows.

15. Ans: 2214.

Eliminating x, we get $3y^2 = 81y$. Then $y = 27$ since $y \neq 0$. Thus $x = 2187$. The result follows.

16. Ans: 2006.

Using $\frac{1}{k \cdot (k+1)} = \frac{1}{k} - \frac{1}{k+1}$, we get

$$2007 \cdot (1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{2006} - \frac{1}{2007}) = 2006.$$

17. Ans: 18063.

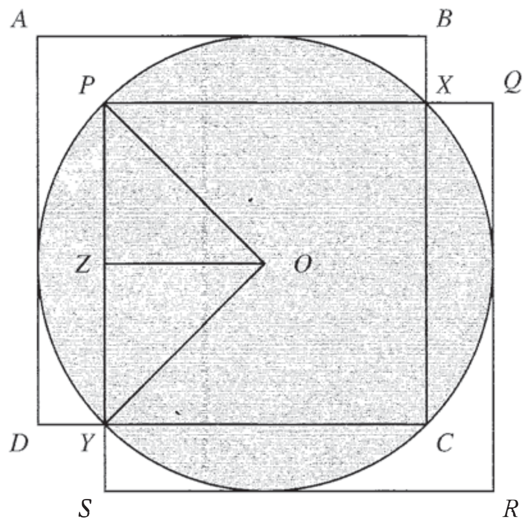
Since there is no carry, this is simply 9×2007. Or observe it is actually

$$222\underbrace{999999999\ldots999}_{2004\ 9's}777.$$

18. Ans: 1.

As the diagram below shows, $PY = \sqrt{2}OP$ and $YS = OP - \frac{1}{2}PY$. Thus,

$$PS = PY + YS = \frac{2+\sqrt{2}}{2}OP = \frac{2+\sqrt{2}}{2}(2-\sqrt{2}) = 1.$$



19. Ans: 286.

By assumption, $2007 - 5 = N \cdot k$ for some integer k. Factorize $2002 = 2 \cdot 7 \cdot 11 \cdot 13$. Since $200 < N < 300$, the only possibility is $N = 2 \cdot 11 \cdot 13 = 286$ and $k = 7$.

20. Ans: 1.

Rewrite the expression as

$$\frac{a^2+b^2+c^2-bc-ca-ab}{abc} = \frac{(a-b)^2+(b-c)^2+(c-a)^2}{2abc}.$$

Since $a - b = -1$, $b - c = -1$ and $c - a = 2$, the result follows.

21. Ans: 50.

$$\angle BPC = 180° - (\angle PBC + \angle PCB) = 180° - \frac{2}{3}(\angle ABC + \angle ACB) = 180° - \frac{2}{3}120° = 100°.$$

Observe that PE bisects $\angle BPC$, the result follows.

22. Ans: 1.

We manipulate the expression and replace $x - y$ by 1 whenever necessary:

$$\begin{aligned} & x^4 - xy^3 - x^3y - 3x^2y + 3xy^2 + y^4 \\ = \ & x^3(x-y) - y^3(x-y) - 3xy(x-y) \\ = \ & (x-y)(x^2 + xy + y^2) - 3xy \\ = \ & (x-y)^2 \\ = \ & 1. \end{aligned}$$

23. Ans: 3.

Since $2008^2 - 2007^2 = n^2 - m^2$, we have $4015 = (n+m)(n-m)$, i.e. $5 \cdot 11 \cdot 73 = (n+m)(n-m)$. There're four possibilities: $n + m = 4015, n - m = 1$; $n + m = 803, n - m = 5$; $n + m = 365, n - m = 11$ and $n + m = 73, n - m = 55$. But the first one $n = 2008$, $m = 2007$ is ruled out by assumption, the remaining pairs are $n = 404, m = 399$; $n = 188, m = 177$ and $n = 64, m = 9$.

24. Ans: 3.

Since $(x+\sqrt{xy}+y)(x-\sqrt{xy}+y) = (x+y)^2-(\sqrt{xy})^2 = x^2+xy+y^2$, we have $9\cdot(x-\sqrt{xy}+y) = 27$. The result follows.

25. Ans: 075.

It suffices to make it divisible by 105 after appending. As $2007000 = 105 \times 19114 + 30$, the least number that we need to add is 75.

26. Ans: 12.

We need the maximal n such that $(n^3)^{2007} < 2007^{2007}$. We need $n^3 < 2007$. By calculation $12^3 = 1728 < 2007 < 2197 = 13^3$. The result follows.

27. Ans: 5.

Observe $x = \sqrt{(4-\sqrt{3})^2} = 4 - \sqrt{3}$ and $x^2 - 8x + 13 = 19 - 8\sqrt{3} - 32 + 8\sqrt{3} + 13 = 0$. Use long division,

$$x^4 - 6x^3 - 2x^2 + 18x + 23 = (x^2 - 8x + 13)(x^2 + 2x + 1) + 10.$$

Thus the original expression equals

$$\frac{(x^2 - 8x + 13)(x^2 + 2x + 1) + 10}{x^2 - 8x + 13 + 2} = \frac{10}{2} = 5.$$

28. Ans: 2.

Using $ax + 1 = -x - a$, we have $x = -1$ or $a = -1$. But when $a = 1$, the original equation has no real roots. Thus $x = -1$, we have $a = 2$.

29. Ans: 45.

2007 is the 1004-th odd number. If 2007 is in the group $k + 1$, then $1 + 2 + \cdots + k < 1004 \leq 1 + 2 + \ldots (k + 1)$. Thus

$$\frac{k(k+1)}{2} < 1004 \leq \frac{(k+1)(k+2)}{2}.$$

We get $k = 44$. So 2007 is in group 45 which has 45 odd integers.

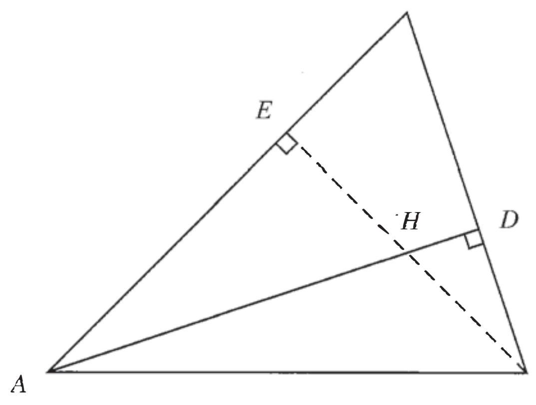
30. Ans: 15.

Construct CE which is perpendicular to AB where E is on AB. Since $\angle BAC = 45°$ (given), $AE = CE$. Thus the right $\triangle AEH$ is congruent to right $\triangle CEB$ So $AH = CB = 5$.

Next $\triangle ADB$ is similar to $\triangle CDH$, thus

$$\frac{BD}{AD} = \frac{HD}{CD} = \frac{AD - AH}{CD} \text{ which implies } \frac{3}{AD} = \frac{AD - 5}{2}.$$

we get $AD^2 - 5AD - 6 = 0$ Solving $AD = 6$ only as $AD > 0$. The result follows.



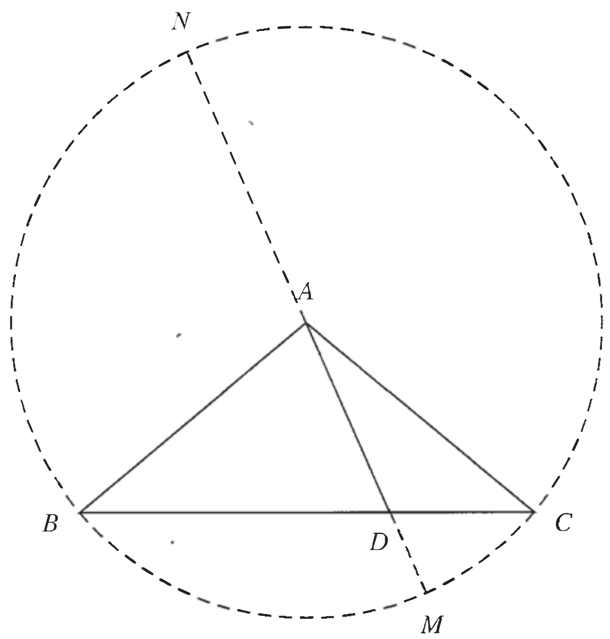
Note, if one is familiar with trigonometry, one may have an alternative solution as follows: Let α and β denote $\angle BAD$ and $\angle DAC$ respectively. Using $\tan(\alpha + \beta) = \tan 45^\circ = 1$, one get

$$\frac{\frac{3}{AD} + 2AD}{1 - \frac{3}{AD}\frac{2}{AD}} = 1,$$

consequently $AD = 6$.

31. Ans: 2.

Construct a circle with A as the center and $AB = AC = \sqrt{3}$ as the radius. Extend AD to meet the circumference at M and N as shown.



Using the Intersecting Chord Theorem

$$BD \cdot DC = MD \cdot ND = (\sqrt{3} - 1)(\sqrt{3} + 1) = 2.$$

32. Ans: 7.

Observe that for $n \geq 2$, 2^{2^n} always ends with a 6. The result follows.

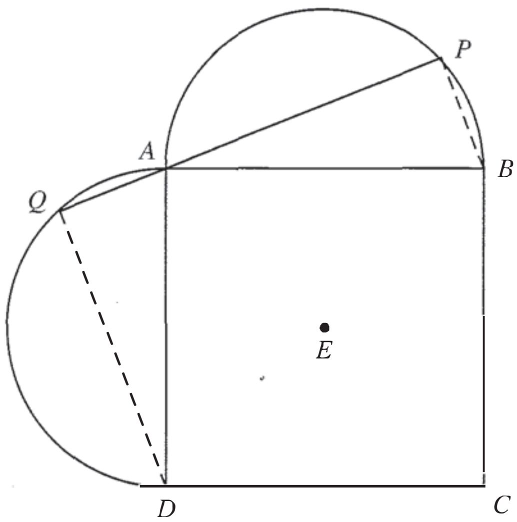
33. Ans: 34.

Join QD and PB, it is easy to show that right $\triangle DQA$ is congruent to right $\triangle APB$. Thus $PB = QA = 14$.

Apply Pythagoras' Theorem to $\triangle APB$,

$$(\sqrt{2}x)^2 = 46^2 + 14^2.$$

Solve, $x = 34$.



34. Ans: 37.

Since $247 = 13 \cdot 19$, one of $n, n+1, n+2$ is divisible by 13, call it a and one by 19 call it b. Clearly $|b - a| \leq 2$.

Let $b = 19c$. When $c = 1$, since $|b - a| \leq 2$, a is among $17, 18, 19, 20, 21$. But none is divisible by 13, hence we try $c = 2$. Now $b = 38$ and a is among $36, 37, 38, 39, 40$, hence $a = 39$. Thus the least $n = 37$.

35. Ans: 4265.

Let $N + 496 = a^2$ and $N + 224 = b^2$ for some positive integers a and b. Then $a^2 - b^2 = 496 = 224 = 272 = 2^4 \cdot 17$. Thus $17|(a+b)(a-b)$. If $17|a - b$ then $a - b \geq 17$ and $a + b \leq 16$, impossible. Thus $17|a + b$.

We have five possibilities for $(a + b, a - b)$: $(17, 16), (34, 8), (68, 4), (136, 2), (272, 1)$. Solve and discard non-integer solutions, we have $(a, b) = (21, 13), (36, 32)$ and $(69, 67)$. Thus the largest N is $69^2 - 496 = 4265$.

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Singapore Mathematical Olympiad (SMO) 2007

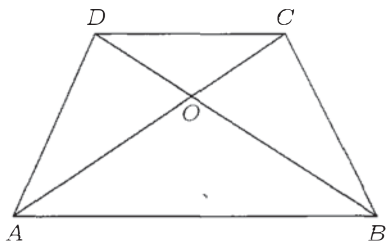
(Junior Section, Round 2)

Saturday, 30 June 2007 **0930-1230**

Instructions to contestants

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

1. In the f llowing figure, $AB \parallel DC$, $AB = b$, $CD = a$ and $a < b$. Let S be the area of the trapezium $ABCD$. Suppose the area of $\triangle BOC$ is $2S/9$. Find the value of a/b.



2. Equilateral triangles ABE and BCF are erected externally on the sides AB and BC of a parallelogram $ABCD$. Prove that $\triangle DEF$ is equilateral.

3. Let n be a positive integer and d be the greatest common divisor of $n^2 + 1$ and $(n+1)^2 + 1$. Find all the possible values of d. Justify your answer.

4. The difference between the product and the sum of two different integers is equal to the sum of their GCD (greatest common divisor) and LCM (least common multiple). Find all these pairs of numbers. Justify your answer.

5. For any positive integer n, let $f(n)$ denote the nth positive nonsquare integer, i.e., $f(1) = 2$, $f(2) = 3$, $f(3) = 5$, $f(4) = 6$, etc. Prove that

$$f(n) = n + \{\sqrt{n}\}$$

where $\{x\}$ denotes the integer closest to x. (For example, $\{\sqrt{1}\} = 1$, $\{\sqrt{2}\} = 1$, $\{\sqrt{3}\} = 2$, $\{\sqrt{4}\} = 2$.)

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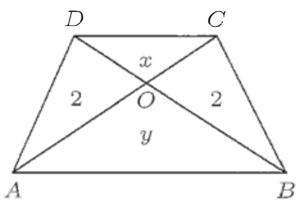
Singapore Mathematical Olympiad (SMO) 2007

(Junior Section, Round 2 Solutions)

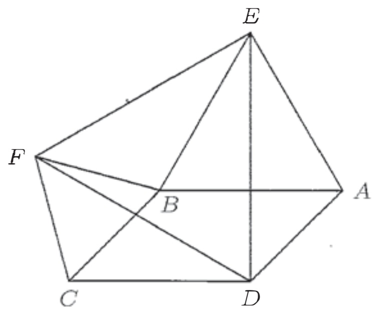
1. Without loss of generality, let $S = 9$. Then $[BOC] = 2$. Since $[ABD] = [ABC]$, we have $[AOD] = [BOC] = 2$. Let $[DOC] = x$ and $[AOB] = y$. Then $x/2 = 2/y$, i.e., $xy = 4$. Also $x + y = 5$. Thus $x(5 - x) = 4$. Solving, we get $x = 1$ and $y = 4$. Since $\triangle DOC \sim \triangle BOA$, we have $x/y = a^2/b^2$. Thus $a/b = 1/2$.



2. We have

$$\begin{aligned}\angle EBF &= 240° - \angle ABC = 240° - (180° - \angle BCD) \\ &= 60° + \angle BCD = \angle DCF\end{aligned}$$

Also $FB = FC$ and $BE = BA = CD$. Thus $\triangle FBE \cong \triangle FCD$. Therefore $FE = FD$. Similarly $\triangle EAD \cong \triangle DCF$. Therefore $ED = DF$. Thus $\triangle DEF$ is equilateral.



3. For $n = 1$ and $n = 2$, the gcd are 1 and 5, respectively. Any common divisor d of $n^2 + 1$ and $(n + 1)^2 + 1$ divides their difference, $2n + 1$. Hence d divides $4(n^2 + 1) - (2n + 1)(2n - 1) = 5$. Thus the possible values are 1 and 5.

4. Let the integers be x and y and assume that $x > y$. First we note that x and y are both nonzero and that their GCD and LCM are both positive by definition. Let M be the GCD, then $|x| = Ma$ and $|y| = Mb$, where a and b are coprime integers. Thus the LCM of x and y is Mab. If $y = 1$, then $M = 1$ and it's easily checked that there is no solution. When $y > 1$, $xy > x + y$. Thus $xy - (x + y) = M + Mab$. After substituting for x and y and simplifying, we have

$$ab(M-1) = 1 + a + b \quad \Rightarrow \quad M = 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{ab} \quad \Rightarrow \quad 1 < M \leq 4.$$

If $M = 2$, then $ab - a - b = 1$, i.e., $(a-1)(b-1) = 2$. Thus $a = 3$, $b = 2$ or $x = 6$, $y = 4$. Similarly, when $M = 3$, we get $2ab - a - b = 1$. Multiplying throughout by 2 and then factorize, we get $(2a-1)(2b-1) = 3$ which gives $x = 6$ and $y = 3$. When $M = 4$, we get $x = y = 4$ which is rejected as x and y are distinct.
Next we consider the case $x > 0 > y$. Then $x = Ma$ and $y = -Mb$. Using similar arguments, we get $x + y - xy = M + Mab$. Thus $M = 1 + \frac{1}{ab} + \frac{1}{a} - \frac{1}{b}$ which yields $1 \leq M \leq 2$. When $M = 1$, we get $a = 1 + b$. Thus the solutions are $b = t$, $a = 1 + t$ or $x = 1 + t$, $y = -t$, where $t \in \mathbb{N}$. When $M = 2$, the equation simplifies to $(a-1)(b+1) = 0$. Thus we get $a = 1$ and b arbitrary as the only solution. The solutions are $x = 2$, $y = -2t$, where $t \in \mathbb{N}$.
Finally, we consider the case $0 > x > y$. Here $x = -Ma$, $y = -Mb$ and $M^2ab + Ma + Mb = M + Mab$. Since $M2ab \geq Mab$ and $Ma + Mb > M$, there is no solution.
Thus the solutions are $(6,3)$, $(6,4)$, $(1+t, -t)$ and $(2, -2t)$ where $t \in \mathbb{N}$.

5. If $f(n) = n + k$, then there are exactly k square numbers less than $f(n)$. Thus $k^2 < f(n) < (k+1)^2$. Now we show that $k = \{\sqrt{n}\}$. We have

$$k^2 + 1 \leq f(n) = n + k \leq (k+1)^2 - 1.$$

Hence

$$\left(k - \frac{1}{2}\right)^2 + \frac{3}{4} = k^2 - k + 1 \leq n \leq k^2 + k = \left(k + \frac{1}{2}\right)^2 - \frac{1}{4}.$$

Therefore

$$k - \frac{1}{2} < \sqrt{n} < k + \frac{1}{2},$$

so that $\{\sqrt{n}\} = k$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Senior Section)

Tuesday, 29 May 2007 | 0930 – 1200 hrs

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answers in the answer sheet by shading the bubbles containing the letters (A, B, C, D *or* E) *corresponding to the correct answers.*

For the other short questions, write your answers in answer sheet and shade the appropriate bubbles below your answers.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Multiple Choice Questions

1. Find the sum of the digits of the product $\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\ldots\left(1+\frac{1}{2006}\right)\left(1+\frac{1}{2007}\right)$.

   (A) 5
   (B) 6
   (C) 9
   (D) 10
   (E) 13

2. A bag contains x green and y red sweets. A sweet is selected at random from the bag and its colour noted. It is then replaced into the bag together with 10 additional sweets of the same colour. A second sweet is next randomly drawn. Find the probability that the second sweet is red.

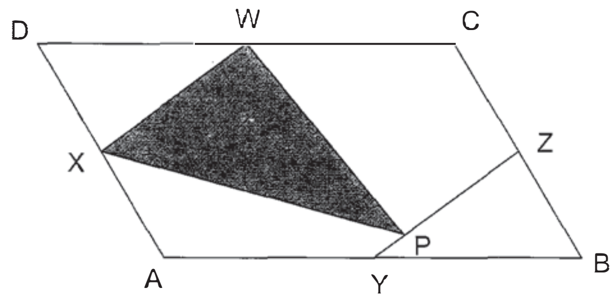
   (A) $\frac{y+10}{x+y+10}$
   (B) $\frac{y}{x+y+10}$
   (C) $\frac{y}{x+y}$
   (D) $\frac{x}{x+y}$
   (E) $\frac{x+y}{x+y+10}$

3. What is the remainder when the number

$$(\underbrace{999\,999\,999\ldots999}_{2008\ 9\text{'s}})^{2007}-(\underbrace{333\,333\,333\ldots333}_{2008\ 3\text{'s}})^{2007}$$

   is divided by 11?
   (A) 0
   (B) 2
   (C) 4
   (D) 6
   (E) None of the above

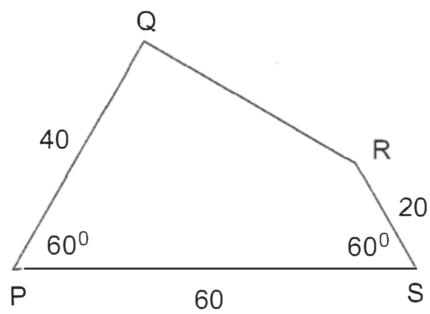
4. W, X, Y and Z are the midpoints of the four sides of parallelogram ABCD. P is a point on the line segment YZ. What percent of the area of parallelogram ABCD is triangle PXW?



(A) 50%
(B) 45%
(C) 30%
(D) 25%
(E) 20%

5. Four rods are connected together with flexible joints at their ends to make a quadrilateral as shown. Rods $PQ = 40$ cm, $RS = 20$ cm, $PS = 60$ cm and $\angle QPS = \angle RSP = 60^{o}$. Find $\angle QRS$.

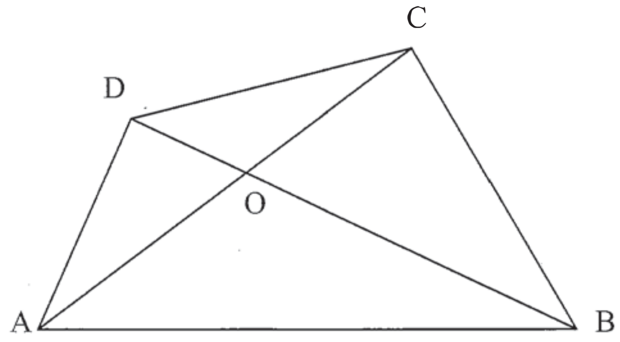
(A) 100^{o}
(B) 105^{o}
(C) 120^{o}
(D) 135^{o}
(E) 150^{o}



6. When 2007 bars of soap are packed into N boxes, where N is a positive integer, there is a remainder of 5. How many possible values of N are there?
(A) 14
(B) 16
(C) 18
(D) 20
(E) 13

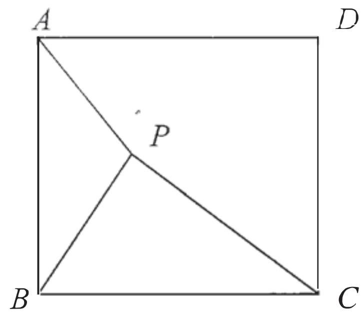
7. Suppose a_n denotes the last two digits of 7^n. For example, $a_2 = 49$, $a_3 = 43$. Find the value of $a_1 + a_2 + a_3 + \dots\dots\dots + a_{2007}$
(A) 50189
(B) 50199
(C) 50209
(D) 50219
(E) 50229

8. The diagram below shows a quadrilateral ABCD where AB = 10, BC = 6, CD = 8 and DA = 2. The diagonals AC and BD intersect at the point O and that $\angle COB = 45^0$. Find the area of the quadrilateral ABCD.



(A) 28
(B) 29
(C) 30
(D) 31
(E) 32

9. In the following diagram, $ABCD$ is a square with $PA = a$, $PB = 2a$ and $PC = 3a$. Find $\angle APB$.



(A) 120^0
(B) 130^0
(C) 135^0
(D) 140^0
(E) 145^0

10. What is the largest possible prime value of $n^2 - 12n + 27$, where n ranges over all positive integers?
(A) 91
(B) 37
(C) 23
(D) 17
(E) 7

Short Questions

11. Suppose that $\log_2[\log_3(\log_4 a)] = \log_3[\log_4(\log_2 b)] = \log_4[\log_2(\log_3 c)] = 0$. Find the value of $a + b + c$.

12. Find the unit digit of $17^{17} \times 19^{19} \times 23^{23}$.

13. Given that $x + y = 12$ and $xy = 50$, find the exact value of $x^2 + y^2$.

14. Suppose that $(21.4)^a = (0.00214)^b = 100$. Find the value of $\frac{1}{a} - \frac{1}{b}$.

15. Find the value of $100(\sin 253^\circ \sin 313^\circ + \sin 163^\circ \sin 223^\circ)$

16. The letters of the word MATHEMATICS are rearranged in such a way that the first four letters of the arrangement are all vowels. Find the total number of distinct arrangements that can be formed in this way.
(Note: The vowels of English language are A, E, I, O, U)

17. Given a set $S = \{1, 2, 3, \ldots, 199, 200\}$. The subset $A = \{a, b, c\}$ of S is said to be "*nice*" if $a + c = 2b$. How many "*nice*" subsets does S have?
(Note: The order of the elements inside the set does not matter. For example, we consider $\{a, b, c\}$ or $\{a, c, b\}$ or $\{c, b, a\}$ to be the same set.)

18. Find the remainder when $2^{55} + 1$ is divided by 33.

19. Given that the difference between two 2-digit numbers is 58 and these last two digits of the squares of these two numbers are the same, find the smaller number.

20. Evaluate $256 \sin 10^0 \sin 30^0 \sin 50^0 \sin 70^0$.

21. Find the greatest integer less than or equal to $\left(2+\sqrt{3}\right)^3$.

22. Suppose that x_1, x_2 and x_3 are the three roots of $(11 - x)^3 + (13 - x)^3 = (24 - 2x)^3$. Find the value of $x_1 + x_2 + x_3$.

23. In ΔABC, $\angle$CAB = 30° and $\angle$ABC = 80°. The point M lies inside the triangle such that $\angle$MAC = 10° and $\angle$MCA = 30°. Find $\angle$BMC in degrees.

24. How many positive integer n less than 2007 can we find such that $\left[\frac{n}{2}\right] + \left[\frac{n}{3}\right] + \left[\frac{n}{6}\right] = n$ where $[x]$ is the greatest integer less than or equal to x? (For example, [2.5] = 2; [5] = 5; [-2.5] = -3 etc.)

25. In ΔABC, let $AB = c$, $BC = a$ and $AC = b$. Suppose that $\frac{b}{c-a} - \frac{a}{b+c} = 1$, find the value of the greatest angle of ΔABC in degrees.

26. Find the number of integers N satisfying the following two conditions:
    (i) $1 \le N \le 2007$; and
    (ii) either N is divisible by 10 or 12 (or both).

27. Suppose a and b are the roots of $x^2 + x \sin \alpha + 1 = 0$ while c and d are the roots of the equation $x^2 + x \cos \alpha - 1 = 0$. Find the value of $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2}$.

28. A sequence $\{a_n\}$ is defined by $a_1 = 2$, $a_n = \frac{1 + a_{n-1}}{1 - a_{n-1}}$, $n \ge 2$. Find the value of $-2008\, a_{2007}$.

29. Let x, y and z be three real numbers such that $xy + yz + xz = 4$. Find the least possible value of $x^2 + y^2 + z^2$.

30. P is the set {1, 2, 3, …, 14, 15 }. If A = $\{a_1, a_2, a_3\}$ is a subset of P where $a_1 < a_2 < a_3$ such that $a_1 + 6 \le a_2 + 3 \le a_3$. How many such subsets are there of P ?

31. It is given that x and y are two real numbers such that

$$(x+y)^4+(x-y)^4=4112 \text{ and } x^2-y^2=16.$$

Find the value of x^2+y^2.

32. Let A be an angle such that $\tan 8A=\dfrac{\cos A-\sin A}{\cos A+\sin A}$. Suppose $A=x^{\circ}$ for some positive real number x. Find the smallest possible value of x.

33. Find the minimum value of $\sum_{k=1}^{100}|n-k|$, where n ranges over all positive integers.

34. Find the number of pairs of positive integers (x, y) are there which satisfy the equation $2x + 3y = 2007$.

35. If $S=\dfrac{1}{1+1^2+1^4}+\dfrac{2}{1+2^2+2^4}+\dfrac{3}{1+3^2+3^4}+\dots\dots\dots+\dfrac{200}{1+200^2+200^4}$, find the value of $80402\times S$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Senior Section Solution)

1. (A)

$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)..\left(1+\frac{1}{2006}\right)\left(1+\frac{1}{2007}\right) = \frac{3}{2}\times\frac{4}{3}\times\frac{5}{4}\cdots\frac{2008}{2007} = \frac{2008}{2} = 1004$,

so the sum of the digits is 5.

2. (C)

Probability $= \left(\frac{x}{x+y}\times\frac{y}{x+y+10}\right)+\left(\frac{y}{x+y}\times\frac{y+10}{x+y+10}\right)$, which upon simplification yields (C) as the answer.

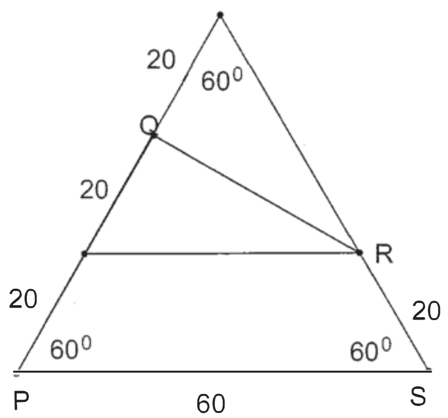
3. (A)

Observe that each of the two numbers is divisible by 11, hence the difference is also divisible by 11. Hence the remainder is zero.

4. (D)

Parallelogram WXYZ is half of ABCD. Triangle PXW is half of WXYZ. Thus, PXW is a quarter of ABCD. Hence (D)

5. (E)



We extend the given figure to the above, into an equilateral triangle. It is not possible to show that $\angle PQR = 90^{\mathrm{o}}$. Hence, $\angle QRS = 30^{\mathrm{o}} + 120^{\mathrm{o}}$.

6. (A)

2007 – 5 = 2002. N is a factor of 2002 and 2002 = 2 x 7 x 11 x 13.

There are altogether 2x2x2x2 = 16 factors of N. However, N must exceed 5. So, N cannot be 1 or 2. Hence there are 14 possible choices of N.

7. (B)

Observe that a_n repeats itself as shown in the following table.

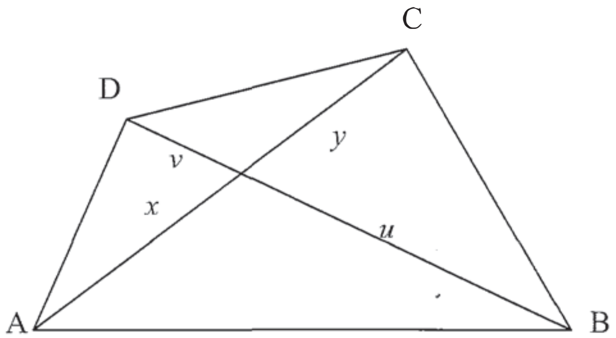
n	1	2	3	4	5	6	7	....
a_n	07	49	43	01	07	49	43	....

Further, $a_1 + a_2 + a_3 + a_4 = 100$.

Notice that 2007 = 4 x 501 + 3.

Hence the sum equals 501 x 100 + 7 + 49 + 43 = 50199.

8. (D)



Area of quadrilateral = $\frac{1}{2}(xu + yu + yv + xv)\sin 45^0$.

Using Cosine Rule, we have

$$x^2 + u^2 + 2xu\cos 45^0 = 10^2 \quad (1)$$

$$u^2 + y^2 - 2uy\cos 45^0 = 6^2 \quad (2)$$

$$y^2 + v^2 + 2yv\cos 45^0 = 8^2 \quad (3)$$

$$x^2 + v^2 - 2xv\cos 45^0 = 2^2 \quad (4)$$

(1) – (2) + (3) – (4):

$$2(xu + yu + yv + vx)\cos 45^0 = 10^2 - 6^2 + 8^2 - 2^2$$

Hence area of quadrilateral = $\frac{100 - 36 + 64 - 4}{4} = 31$.

9. (C)

Let $AB = x$ and $\angle PAB = \alpha$. By using Cosine Rule and Pythagoras' Theorem we have

$$x^2 + a^2 - 2ax\cos\alpha = (2a)^2$$

$$(x - a\cos\alpha)^2 + (x - a\sin\alpha)^2 = (3a)^2.$$

We obtain from the above two equations that

$$\cos\alpha = \frac{x^2 - 3a^2}{2ax} \quad \text{and} \quad \sin\alpha = \frac{x^2 - 5a^2}{2ax}.$$

Using $\sin^2\alpha + \cos^2\alpha = 1$, we have

$$\left(x^2 - 3a^2\right)^2 + \left(x^2 - 5a^2\right)^2 = 4a^2x^2$$

Therefore $x^4 - 10a^2x^2 + 17a^4 = 0$.

Solving, we have $x^2 = \left(5 \pm 2\sqrt{2}\right)a^2$, but since $x > a$, we have $x^2 = \left(5 + 2\sqrt{2}\right)a^2$.

Hence $\cos \angle APB = -\frac{\sqrt{2}}{2}$, so the required angle is 135^0.

10. (E)
Note that $n^2 - 12n + 27 = (n-9)(n-3)$. For this number to be a prime, either $n = 10$, in which case $n^2 - 12n + 27 = 7$; or $n = 2$, in which case $n^2 - 12n + 27 = 7$. Since a prime number cannot be factorized in other ways, we know that 7 is the only answer.

11. Answer: 89
$\log_2 [\log_3 (\log_4 a)] = 0$ implies $\log_3 (\log_4 a) = 1$. Hence $\log_4 a = 3$, hence $a = 64$. Similarly, $b = 16$ and $c = 9$. Therefore $a + b + c = 89$.

12. Answer: 1
$17^{17} \equiv 7^{17} \equiv 7 \pmod{10}$
$19^{19} \equiv 9^{19} \equiv 9 \pmod{10}$
$23^{23} \equiv 3^{23} \equiv 7 \pmod{10}$
Since $7 \times 9 \times 7 = 441 \equiv 1 \pmod{10}$, the unit digit is 1.

13. Answer: 44
Use the identity $x^2 + y^2 = (x + y)^2 - 2xy = 144 - 100 = 44$.
(Note: In this case, one may not be able to obtain the answer by guess-and-check; the values of x and y are not even real numbers)

14. Answer: 2
We have $a = \frac{2}{\log_{10} 21.4}$ and $b = \frac{2}{(\log_{10} 21.4) - 4}$. Hence direct computation yields $\frac{1}{a} - \frac{1}{b} = 2$.

15. Answer: 50

$$\begin{aligned}
&100(\sin 253^\circ \sin 313^\circ + \sin 163^\circ \sin 223^\circ) \\
&= 100(\sin 73^\circ \sin 47^\circ - \sin 17^\circ \sin 43^\circ) \\
&= 100 (\sin 47^\circ \cos 17^\circ - \cos 47^\circ \sin 17^\circ) \\
&= 100 \sin (47^\circ - 17^\circ) = 100 \sin 30^\circ \\
&= 50
\end{aligned}$$

16. Answer: 15 120

Total number of rearrangements = $\frac{4!}{2!} \times \frac{7!}{2!2!} = 12 \times 1260 = 15120.$

17. Answer: 9900

Since $a + c$ is even, a and c have the same parity (either both odd or both even). There are 100 odd and 100 even numbers in S. Once a and c are chosen, b is determined. The number of "*nice*" subsets is $2\binom{100}{2} = 100(99) = 9900$

18: Answer: 0

Note that $x^{11} + 1$ is divisible by $x + 1$ by Factor Theorem. Hence $2^{55} + 1 = 32^{11} + 1$ is divisible by $32 + 1 = 33$, i.e. $2^{55} + 1$ is divisible by 33.

19. Answer: 21

Let the numbers be n and m where $n > m$.

$n - m = 58$

$n^2 - m^2 = (n - m)(n + m) = 58(n + m)$ is a multiple of 100, that is,

$58(n + m) = 100k$ for some integer k. This simplifies to

$29(n + m) = 50k$

Since gcd(29, 50) = 1, we must have $n + m = 50p$ for some positive integer p.

Solving the above simultaneously with $n - m = 58$ and bearing in mind that both m and n are two digit numbers, only $n = 79$ and $m = 21$ satisfy the question.

Thus the smaller number is 21.

20. Answer: 16

$256 \sin 10^0 \sin 30^0 \sin 50^0 \sin 70^0$

$$= \frac{2 \times 256 \sin 10^0 \cos 10^0 \sin 30^0 \cos 40^0 \cos 20^0}{2 \cos 10^0}$$

$$= \frac{128 \sin 20^0 \cos 20^0 \cos 40^0}{2 \cos 10^0}$$

$$= \frac{64 \sin 40^0 \cos 40^0}{2 \cos 10^0}$$

$$= \frac{32 \sin 80^0}{2 \cos 10^0}$$

$$= 16.$$

21. Answer: 51

Using binomial theorem it is easy to see that

$$\left(2 + \sqrt{3}\right)^3 + \left(2 - \sqrt{3}\right)^3 = 52,$$

and that $0 < 2 - \sqrt{3} < 1$, or $0 < \left(2 - \sqrt{3}\right)^3 < 1$ so that we have

$51 < \left(2 + \sqrt{3}\right)^3 < 52.$

22. Answer: 36

Let $a = 11 - x$ and $b = 13 - x$. Hence the equation becomes

$$a^3 + b^3 = (a + b)^3 \qquad (1)$$

Using the binomial expansion

$$(a + b)^3 = a^3 + b^3 + 3ab\,(a + b),$$

Equation (1) becomes

$$ab\,(a + b) = 0 \qquad (2)$$

Thus the solution of (1) becomes

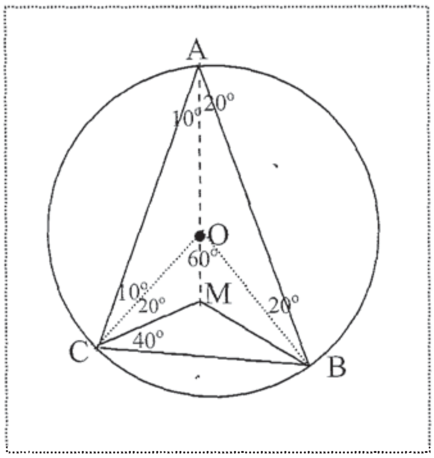
$a = 0$, $b = 0$ or $a + b = 0$

or equivalently, the solution of the original equation becomes $x = 11$, 12 or 13.

Hence $x_1 + x_2 + x_3 = 11 + 12 + 13 = 36$.

23. Answer: 110^0.

Construct a circumcircle of the triangle ABC, with O as the centre as shown.



Note that $\angle ACB = 70^o$

Since OC = OB and $\angle COB = 60^o \Rightarrow \angle OCB = 60^o$

ΔCOB is an equilateral triangle.

Thus $\angle OCA = 70^o - 60^o = 10^o = \angle OAC$

But $\angle MAC = 10^o$ (given). So AOM lies on a straight line.

$\angle AOC = 160^o \Rightarrow \angle COM = 20^o$.

Since $\angle MCA = 30^o$ (given) and $\angle OCA = 10^o$, thus $\angle MCO = 20^o$. That means ΔMCO is an isosceles triangle and $\angle MCB = 70 - 30 = 40^o$

So BM is the perpendicular bisector of the equilateral triangle OBC. Thus

$\angle OBM = 60^o \div 2 = 30^o$

$\angle BMC = 180^o - 40^o - 30^o = 110^o$

24. Answer: 334

$$\left[\frac{n}{2}\right] \le \frac{n}{2}, \quad \left[\frac{n}{3}\right] \le \frac{n}{3}, \quad \left[\frac{n}{6}\right] \le \frac{n}{6}$$

Given $\left[\frac{n}{2}\right]+\left[\frac{n}{3}\right]+\left[\frac{n}{6}\right]=n$ but we have $\frac{n}{2}+\frac{n}{3}+\frac{n}{6}=n$

So $\left[\frac{n}{2}\right]=\frac{n}{2}$, $\left[\frac{n}{3}\right]=\frac{n}{3}$, $\left[\frac{n}{6}\right]=\frac{n}{6}$

Thus n must be a multiple of 6

There are $\left[\frac{2007}{6}\right]=334$ of them.

25. Answer: 120^0.

It is given that $\frac{b}{c-a}-\frac{a}{b+c}=1$. This can be rearranged into $b^2+a^2-c^2=-ab$.

By using cosine rule for triangle, $\cos C=\frac{b^2+a^2-c^2}{2ab}=-\frac{1}{2}$. Hence $C=120^0$ and must be the greatest angle.

26. Answer: 334

Number of integers divisible by 10 = $\left[\frac{2007}{10}\right]=200$.

Number of integers divisible by 12 = $\left[\frac{2007}{12}\right]=167$

Number of integers divisible by both 12 and 10 = $\left[\frac{2007}{60}\right]=33$.

By the principle of inclusion and exclusion, the number of integers divisible by either 10 or 12 (or both) = 200 + 167 – 33 = 334.

27. Answer: 1

We have $ab=1$. Hence $a^2=\frac{1}{b^2}$ and $b^2=\frac{1}{a^2}$.

Also, $cd=-1$. Hence $d^2=\frac{1}{c^2}$ and $c^2=\frac{1}{d^2}$.

$$
\begin{aligned}
\text{Hence } \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2} &= a^2+b^2+c^2+d^2 \\
&= (a+b)^2-2ab+(c+d)^2-2cd \\
&= \sin^2\alpha-2+\cos^2\alpha+2 \\
&= 1
\end{aligned}
$$

28. Answer: 1004

Since $a_1 = 2$, $a_2 = -3$, $a_3 = -\frac{1}{2}$, $a_4 = \frac{1}{3}$, $a_5 = 2$, thus we consider

$$a_{n+4} = \frac{1+a_{n+3}}{1-a_{n+3}} = \frac{1+\frac{1+a_{n+2}}{1-a_{n+2}}}{1-\frac{1+a_{n+2}}{1-a_{n+2}}} = -\frac{1}{a_{n+2}} = -\left(\frac{1-a_{n+1}}{1+a_{n+1}}\right)$$

$$= -\frac{1-\frac{1+a_n}{1-a_n}}{1+\frac{1+a_n}{1-a_n}} = a_n$$

$a_{2007} = a_{4\text{x}501+3} = a_3 = -\frac{1}{2}$

$-2008\, a_{2007} = 1004$

29. Answer: 4

Use the identity

$x^2 + y^2 + z^2 - (xy + yz + xz) = \frac{1}{2}\left((x-y)^2 + (y-z)^2 + (x-z)^2\right) \geq 0$, the answer follows immediately.

30. Answer: 165

Supposing there are x_1 numbers smaller than a_1, x_2 numbers between a_1 and a_2 , x_3 numbers between a_2 and a_3 and x_4 numbers greater than a_3 .

Finding the number of possible subsets of A is equivalent to finding the number of integer solutions of the equation $x_1 + x_2 + x_3 + x_4 = 12$ with the conditions that $x_1 \geq 0$, $x_2 \geq 2$, $x_3 \geq 2$ and $x_4 \geq 0$.

The number of solution of this latter equation is equivalent to the number of solutions of the equation $y_1 + y_2 + y_3 + y_4 = 8$, where y_1, y_2, y_3 and y_4 are nonnegative integers.

Hence the answer is $\binom{11}{3} = 165$.

31. Answer: 34

By binomial theorem, one sees that

$$(x+y)^4 + (x-y)^4 = 2(x^4 + 6x^2y^2 + y^4)$$

Thus the first equation becomes

$$2(x^4 + 6x^2y^2 + y^4) = 4112$$

$$x^4 + 6x^2y^2 + y^4 = 2056$$

$$(x^2 - y^2)^2 + 8x^2y^2 = 2056$$

$$(16)^2 + 8x^2y^2 = 2056$$

$$x^2y^2 = 2256$$

Therefore,

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = 1156$$

Hence $x^2 + y^2 = \sqrt{1156} = 34$.

32. Answer: 5

Since $\dfrac{\sin 8A}{\cos 8A} = \tan 8A = \dfrac{\cos A - \sin A}{\cos A + \sin A}$, by rearranging we obtain

$$\sin 8A\,(\cos A + \sin A) = \cos 8A\,(\cos A - \sin A)$$

$$\sin 8A \cos A + \cos 8A \sin A = \cos 8A \cos A - \sin 8A \sin A$$

$$\sin (8A + A) = \cos (8A + A)$$

$$\sin 9A = \cos 9A$$

which reduces to

$$\tan 9A = 1$$

The smallest possible of $9A = 45^0$, which gives $x = 5$.

33. Answer: 2500

It is given that n is a positive integer.

For $n \geq 100$, $\sum_{k=1}^{100} |n - k| = 100n - 5050$, so that its minimum value is 4950 which occurs at $n = 100$.

For $n < 100$, $\sum_{k=1}^{100} |n - k| = n^2 - 101n + 5050$. If n is a positive integer, its minimum value can occur at either $n = 50$ or $n = 51$ only. By direct checking, its minimum value is 2500.

34. Answer: 334

$2x + 3y = 2007 \Rightarrow 2x = 2007 - 3y = 3(669 - y)$.

Since 2 and 3 are relatively prime, it follows that x is divisible by 3. Write $x = 3t$, where t is a positive integer.

The equation reduces to $y = 669 - 2t$.

Since $669 - 2t > 0$, we have $t < 334.5$, and since t is a positive integer, we have $1 \leq t \leq 334$.

Conversely, for any positive integer t satisfying $1 \leq t \leq 334$, it is easily seen that $(3t, 669 - 2t)$ is a pair of positive integers which satisfy the given equation.

Therefore there are 334 pairs of positive integers satisfying the given equations.

35. Answer: 40 200

Note that $\dfrac{k}{1+k^2+k^4} = \dfrac{1}{2}\left[\dfrac{1}{k(k-1)+1} - \dfrac{1}{k(k+1)+1}\right]$ for all integers k.

Hence the required sum can be found by the method of difference

$$= \frac{1}{2}\left(\frac{1}{1\times 0+1} - \frac{1}{1\times 2+1} + \frac{1}{2\times 1+1} - \frac{1}{2\times 3+1} + \frac{1}{3\times 2+1} - \frac{1}{3\times 4+1} + \ldots\ldots + \ldots\ldots + \frac{1}{200\times 199+1} - \frac{1}{200\times 201+1}\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{40201}\right)$$

$$= \frac{20100}{40201}$$

Hence 80402 x S = 40 200.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Senior Section, Round 2)

Saturday, 30 June 2007 **0930-1230**

Instructions to contestants

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

1. It is given that x, y, z are 3 real numbers such that

$$\frac{x-y}{2+xy}+\frac{y-z}{2+yz}+\frac{z-x}{2+zx}=0.$$

Is it true that at least two of the three numbers must be equal? Justify your answer.

2. For any positive integer n, let $f(n)$ denote the nth positive nonsquare integer, i.e., $f(1)=2$, $f(2)=3$, $f(3)=5$, $f(4)=6$, etc. Prove that

$$f(n)=n+\{\sqrt{n}\}$$

where $\{x\}$ denotes the integer closest to x. (For example, $\{\sqrt{1}\}=1$, $\{\sqrt{2}\}=1$, $\{\sqrt{3}\}=2$, $\{\sqrt{4}\}=2$.)

3. In the equilateral triangle ABC, M, N are the midpoints of the sides AB, AC, respectively. The line MN intersects the circumcircle of $\triangle ABC$ at K and L and the lines CK and CL meet the line AB at P and Q, respectively. Prove that $PA^2\cdot QB=QA^2\cdot PB$.

4. Thirty two pairs of identical twins are lined up in an 8×8 formation. Prove that it is possible to choose 32 persons, one from each pair of twins, so that there is at least one chosen person in each row and in each column.

5. Find the maximum and minimum of $x+y$ such that

$$x+y=\sqrt{2x-1}+\sqrt{4y+3}.$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Senior Section, Round 2 Solutions)

1. Yes. Multiplying both sides by $(2+xy)(2+yz)(2+zx)$, we get

$$F := (x-y)(2+yz)(2+zx)+(y-z)(2+xy)(2+zx)+(z-x)(2+zx)(2+xy)=0.$$

Now regard F as a polynomial in x. Since $F=0$ when $x=y$, $x-y$ is a factor of F. Similarly, $y-z$ and $z-x$ are also factors of F. Since F is of degree 3,

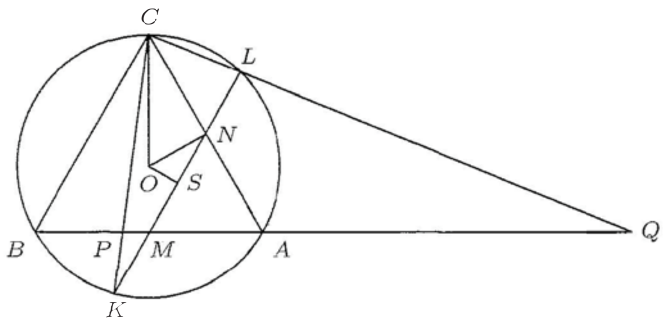
$$F \equiv k(x-y)(y-z)(z-x)$$

for some constant k. By letting $x=1$, $y=-1$, $z=0$, we have $k=2$. Thus

$$F \equiv 2(x-y)(y-z)(z-x)$$

2. See Junior Section Round 2, Question 5.

3. Let the radius of the circumcircle be $R=4t$ and the centre be O. Let S be the foot of the perpendicular from O to MN. Then $OS=ON/2=t$, $KS=\sqrt{16t^2-t^2}=\sqrt{15}t$, $MS=\sqrt{4t^2-t^2}=\sqrt{3}t$. Thus $KM=KS-MS=(\sqrt{15}-\sqrt{3})t$. Since $\triangle PMK \sim \triangle PBC$ and $BC=\sqrt{24}t$, we have $PM/PB=MK/BC=(\sqrt{5}-1)/4$, i.e., $PM=(\sqrt{5}-1)PB/4$. Thus $PA=PM+MA=PM+BM=2PM+PB=(\sqrt{5}+1)PB/2$ and $BA=PA+PB=(\sqrt{5}+3)PB/2$. Therefore $PA^2=PB\cdot BA$. Similarly, $QA^2=QB\cdot BA$. Thus $PA^2\cdot QB=QA^2\cdot PB$.



Second solution: Note that $\angle BCL = \angle CBK$ and $\angle BKC = 60°$. Thus $\angle BCL + \angle BCP = \angle CBK + \angle BCK = 180° - \angle BKC = 120°$. This implies that $\angle BQC = 180° - \angle CBQ - \angle BCQ = 120° - \angle BCL = \angle BCP$, and therefore $\triangle BCQ \sim \triangle BPC$. Thus, $CQ/PC = BQ/BC = BC/BP$, i.e., $\left(\frac{CQ}{PC}\right)^2 = \frac{BQ}{BC} \cdot \frac{BC}{BP} = \frac{BQ}{BP}$. Now since $KL \parallel BC$, A is also midpoint along the arc of KL. Thus, $\angle KCA = \angle ACL$. By the angle bisector theorem on $\triangle PCQ$, we have $CQ/CP = AQ/AP$. Combining, we get $\left(\frac{AQ}{AP}\right)^2 = \left(\frac{CQ}{PC}\right)^2 = \frac{BQ}{BP}$, i.e., $PA^2 \cdot QB = QA^2 \cdot QB$.

4. Suppose on the contrary that this is not possible. We say that a row or a column is *covered* if it contains one of the chosen persons. Choose 32 persons so that the total number of covered rows and columns is maximum. Without loss of generality, assume that column 1 is not covered. Then no pair of twins can be in column 1. The counterparts of these 8 persons must be chosen. By the maximality condition, these 8 persons must be the only chosen persons either in their row or column. If a row contains only 1 chosen person, then there are 6 other persons (excluding those in column 1) in that row who are not chosen. Since 32 persons are not chosen, we have at most 4 such rows. Similarly, there are at most 3 such columns and we have a contradiction because we have 8 such rows or columns.

   2nd solution: We'll show that at most 10 persons are needed. Note that if a set of people cover all the rows and columns, then they will still have the same property if two rows or two columns are interchanged. Thus if the configuration obtained by interchanging some pairs of rows and some pairs of columns can be covered by a set of people, then the same set of people will cover the original configuration. We also let M_i denote the configuration obtained by deleting the first i rows and columns of the original configuration. Let us denote a person by (a, b) if he is in the a-th column b-th row. Choose $(1, 1)$. In M_1, there is a person who is not the counterpart of $(1, 1)$. Without loss of generality, let this be $(2, 2)$. Select $(2, 2)$. In general, if no two of $(1, 1)$, $(2, 2)$, ..., (i, i), $i < 6$ are twins, then in M_i there is a person who is not among their counterparts. Let this person be $(i + 1, i + 1)$ and choose this person. In this way, we can choose (i, i), $i = 1, 2, \ldots, 6$. Denote the counterparts of these 6 persons by X and the first 6 persons of the 7th and 8th rows and columns by A_1, A_2, A_3, A_4, respectively. Let $|X \cap A_i| = a_i$. Then $a_1 + a_2 + a_3 + a_4 \leq 6$, Assume that $a_1 \geq a_2 \geq a_3 \geq a_4$. Suppose $a_1 = 6$. Choose $(7, 7)$ and choose a person in A_2 and a person in A_4 who are not twins and are not the counterpart of $(7, 7)$. These 9 people will cover all the row and columns. If $a_1 < 6$, then $a_2 \leq 3$, $a_3 \leq 2$ and $a_4 \leq 1$. Then there exist $p_i \in A_i$ such that $p_i \notin X$, $i = 1, 2, 3, 4$ and not two of p_1, p_2, p_3, p_4 are twins. Choose these 4 persons as well and the 10 chosen people will cover all the rows and columns.

5. We have

$$(x+y)/3 = (\sqrt{2x-1} + \sqrt{y+3/4} + \sqrt{y+3/4})/3 \le \sqrt{(2x+2y+1/2)/3}.$$

and

$$(x+y) = \sqrt{2x-1} + \sqrt{y+3/4} + \sqrt{y+3/4} \ge \sqrt{2x+2y+1/2}.$$

Let $t = x+y$. Then we have

$$\begin{aligned} & 2t^2 - 12t - 3 \le 0 \quad \Rightarrow \quad 3 - \sqrt{21/2} \le t \le 3 + \sqrt{21/2} \\ \text{and} \quad & 2t^2 - 4t - 1 \ge 0 \quad \Rightarrow \quad t \ge 1 + \sqrt{3/2}, \quad t \le 1 - \sqrt{3/2}. \end{aligned}$$

Since $t \ge 0$, the maximum value is $3 + \sqrt{21/2}$ and the minimum value is $1 + \sqrt{3/2}$. Note that the maximum value is attained by the solution of

$$x + y = 3 + \sqrt{21/2} \quad \text{and} \quad 2x - 1 = \sqrt{y + 3/4}$$

while the minimum is attained by the solution of

$$x + y = 1 + \sqrt{3/2} \quad \text{and} \quad y + 3/4 = 0.$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Open Section, Round 1)

Wednesday, 30 May 2007 **0930-1200**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

1. Let A be any k-element subset of the set $\{1, 2, 3, 4, \cdots, 100\}$. Determine the minimum value of k such that we can always guarantee the existence of two numbers a and b in A such that $|a - b| \leq 4$.

2. Determine the number of those 0-1 binary sequences of ten 0's and ten 1's which do not contain three 0's together.

3. Let A be the set of any 20 points on the circumference of a circle. Joining any two points in A produces one chord of this circle. Suppose every three such chords are not concurrent. Find the number of regions within the circle which are divided by all these chords.

4. In each of the following 7-digit natural numbers:

$$1001011, 5550000, 3838383, 7777777,$$

every digit in the number appears at least 3 times. Find the number of such 7-digit natural numbers.

5. Let $A = \{1, 2, 3, 4, \ldots, 1000\}$. Let m be the number of 2-element subsets $\{a, b\}$ of A such that $a \times b$ is divisible by 6. Find the value of $\lfloor m/10 \rfloor$. (Here and in subsequent questions $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

6. Find the number of non-negative integer solutions of the following inequality:

$$x + y + z + u \leq 20.$$

7. In how many different ways can 7 different prizes be awarded to 5 students in such a way that each student has at least one prize?

8. Let ABC be any triangle. Let D and E be the points respectively in the segments of AB and BC such that $AD = 7DB$ and $BE = 10EC$. Assume that AE and CD meet at point F. Determine $\lfloor k \rfloor$, where k is the real number such that $AF = k \times FE$.

9. Let $S = \{1, 2, 3, 4, \cdots, 50\}$. A 3-element subset $\{a, b, c\}$ of S is said to be *good* if $a + b + c$ is divisible by 3. Determine the number of 3-elements of S which are good.

10. Let $x_1, x_2, \ldots, x_{1970}$ be positive integers satisfying $x_1 + x_2 + \cdots + x_{1970} = 2007$. Determine the largest possible value of $x_1^3 + x_2^3 + \cdots + x_{1970}^3$.

11. Determine the largest value of a such that a satisfies the equations $a^2 - bc - 8a + 7 = 0$ and $b^2 + c^2 + bc - 6a + 6 = 0$ for some real numbers b and c.

12. Determine the number of distinct integers among the numbers

$$\left\lfloor \frac{1^2}{2007} \right\rfloor, \quad \left\lfloor \frac{2^2}{2007} \right\rfloor, \quad \cdots \quad , \quad \left\lfloor \frac{2007^2}{2007} \right\rfloor.$$

13. Determine the number of pairs (a, b) of integers with $1 \leq b < a \leq 200$ such that the sum $(a + b) + (a - b) + ab + a/b$ is a square of a number.

14. This question has been deleted.

15. In an acute-angled triangle ABC, points D, E, and F are the feet of the perpendiculars from A, B, and C onto BC, AC and AB, respectively. Suppose $\sin A = \frac{3}{5}$ and $BC = 39$, find the length of AH, where H is the intersection AD with BE.

16. Let O be the centre of the circumcircle of $\triangle ABC$, P and Q the midpoints of AO and BC, respectively. Suppose $\angle CBA = 4\angle OPQ$ and $\angle ACB = 6\angle OPQ$. Find the size of $\angle OPQ$ in degrees.

17. In $\triangle ABC$, $AC > AB$, the internal angle bisector of $\angle A$ meets BC at D, and E is the foot of the perpendicular from B onto AD. Suppose $AB = 5$, $BE = 4$ and $AE = 3$. Find the value of the expression $\left(\frac{AC+AB}{AC-AB}\right)ED$.

18. Find the value of

$$\prod_{k=1}^{45} \tan(2k-1)^\circ.$$

19. Find the radius of the circle inscribed in a triangle of side lengths $50, 120, 130$.

20. Suppose that $0 < a < b < c < d = 2a$ and

$$(d-a)\left(\frac{a^2}{b-a} + \frac{b^2}{c-b} + \frac{c^2}{d-c}\right) = (a+b+c)^2.$$

Find bcd/a^3.

21. Let f be a function so that

$$f(x) - \frac{1}{2}f\left(\frac{1}{x}\right) = \log x$$

for all $x > 0$, where log denotes logarithm base 10. Find $f(1000)$.

22. Let O be an interior point of $\triangle ABC$. Extend AO to meet the side BC at D. Similarly, extend BO and CO to meet CA and AB respectively at E and F. If $AO = 30$, $FO = 20$, $BO = 60$, $DO = 10$ and $CO = 20$, find EO.

23. For each positive integer n, let a_n denote the number of n-digit integers formed by some or all of the digits 0, 1, 2, and 3 which contain neither a block of 12 nor a block of 21. Evaluate a_9.

24. Let S be any nonempty set of k integers. Find the smallest value of k for which there always exist two distinct integers x and y in S such that $x + y$ or $x - y$ is divisible by 2007.

25. Let P be a 40-sided convex polygon. Find the number of triangles S formed by the vertices of P such that any two vertices of S are separated by at least two other vertices of P.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Open Section, Round 1 Solutions)

1. Ans: 21

   If A is the set of multiples of 5 in $\{1, 2, 3, \cdots, 100\}$, then $|A| = 20$ and $|a-b| \geq 5$ for every two numbers in A. Thus, if $|A| = 20$, the existence such two numbers cannot be guaranteed. However, if $|A| = 21$, by the pigeonhole principle, there must be two numbers a, b in one of the following twenty sets:

   $$\{1, 2, 3, 4, 5\}, \{6, 7, 8, 9, 10\}, \cdots, \{96, 97, 98, 99, 100\},$$

   and so $|a - b| \leq 4$. Thus the answer is 21.

2. Ans: 24068

   In such a binary sequence, 0's either appear singly or in blocks of 2. If the sequence has exactly m blocks of double 0's, then there are $10 - 2m$ single 0's. The number of such binary sequences is

   $$\binom{11}{m} \times \binom{11-m}{10-2m}.$$

   Thus the answer is

   $$\sum_{m=0}^{5} \binom{11}{m} \times \binom{11-m}{10-2m} = 24068.$$

3. Ans: 5036

   Consider such a figure as a plane graph G. Then the answer is equal to the number of interior faces of G. The number of vertices in G is $20 + \binom{20}{4}$. The sum of all degrees is

   $$20 \times 21 + 4 \times \binom{20}{4}$$

   and so the number of edges in G is

   $$10 \times 21 + 2 \times \binom{20}{4}.$$

   Hence the number of interior faces, by Euler's formula, is

   $$\binom{20}{4} + 10 \times 21 - 20 + 1 = 5036.$$

Second Solution: Let P_n be the number of such regions with n points on the circumference. Then $P_1 = 1$, $P_2 = 2$ and in general, for $n \geq 2$, $P_{n+1} = P_n + n + \sum_{i=1}^{n-2} i(n-i)$. This can be obtained as follows. Suppose there are $n+1$ points $a_0, \ldots, a_n$ in that order on the circumference. The chords formed by $a_1, \ldots, a_n$ create P_n regions. The chord a_0a_1 adds one region. The chord a_0a_1 adds $1 + 1 \times (n-2)$ regions as this chord intersects the existing chords in $1 \times (n-2)$ points. Similarly, the chord a_0a_2 adds $1 + 2 \times (n-3)$ regions, etc. From this, it is easy to show that

$$\begin{aligned}
P_n &= 1 + \binom{n}{2} + \binom{n-2}{2} + 2\binom{n-3}{2} + \cdots + (n-3)\binom{2}{2} \\
&= 1 + \binom{n}{2} + \binom{n-1}{3} + 1\binom{n-3}{2} + \cdots + (n-4)\binom{2}{2} \\
&= \cdots = 1 + \binom{n}{2} + \binom{n-1}{3} + \binom{n-2}{3} + \cdots + \binom{3}{3} = 1 + \binom{n}{2} + \binom{n}{4}
\end{aligned}$$

Thus $P_{20} = 1 + \binom{20}{2} + \binom{20}{4} = 5036$.

4. Ans: 2844

If only one digit appears, then there are 9 such numbers. If the two digits that appear are both nonzero, then the number of such numbers is

$$2 \times \binom{7}{3}\binom{9}{2} = 2520.$$

If one of two digits that appear is 0, then the number of such numbers is

$$\left(\binom{6}{4} + \binom{6}{3}\right) \times \binom{9}{1} = 315.$$

Hence the answer is $9 + 2520 + 315 = 2844$.

5. Ans: 20791

Let

$$A_6 = \{k \in A : 6 \mid k\}; \quad A_2 = \{k \in A : 2 \mid k, 6 \nmid k\}; \quad A_3 = \{k \in A : 3 \mid k, 6 \nmid k\}.$$

Note that

$$|A_6| = \left\lfloor \frac{1000}{6} \right\rfloor = 166; \qquad |A_2| = \left\lfloor \frac{1000}{2} \right\rfloor - \left\lfloor \frac{1000}{6} \right\rfloor = 334;$$
$$|A_3| = \left\lfloor \frac{1000}{3} \right\rfloor - \left\lfloor \frac{1000}{6} \right\rfloor = 167.$$

For the product $a \times b$ to be divisible by 6, either (i) one or both of them are in A_6 or (ii) one is in A_2 and the other is in A_3. Hence

$$m = \binom{166}{2} + 166 \times (1000 - 166) + 334 \times 167 = 207917.$$

6. Ans: 10626
Let $v = 20 - (x + y + z + u)$. Then $v \geq 0$ if and only if $x + y + z + u \leq 20$. Hence the answer is equal to the number of non-negative integer solutions of the following equation:

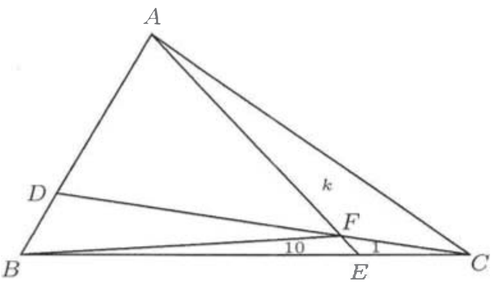
$$x + y + z + u + v = 20$$

and the answer is $\binom{24}{4} = 10626$.

7. Ans: 16800
Either one student receives three prizes; or two students are each awarded two prizes. Thus the answer is

$$\binom{7}{3} \times 5! + \binom{7}{2} \times \binom{5}{2} \times \frac{1}{2} \times 5! = 16800.$$

8. Ans: 77
Assume that $[CEF] = 1$. Then, $[AFC] = k$ and $[BFC] = 11$. Since $AD = 7DB$, $k = [AFC] = 7[BFC] = 77$.



9. Ans: 6544
For $i = 0, 1, 2$, let

$$A_i := \{1 \leq k \leq 50 : 3 \mid (k - i)\}.$$

Then $|A_0| = 16$, $|A_1| = 17$ and $|A_2| = 17$. It can be shown that $3 \mid (a+b+c)$ if and only if either $\{a, b, c\} \subseteq A_i$ for some i or $\{a, b, c\} \cap A_i \neq \emptyset$ for all $i = 1, 2, 3$. Thus the answer is

$$\binom{16}{3} + \binom{17}{3} + \binom{17}{3} \quad 16 \times 17 \times 17 = 6544.$$

10. Ans: 56841
We may assume $x_1 \le x_2 \le \cdots \le x_{1970}$. For $1 \le x_i \le x_j$ with $i \le j$, we have $x_i^3 + x_j^3 \le x_i^3 + x_j^3 + 3(x_j - x_i)(x_j + x_i - 1) = (x_i - 1)^3 + (x_j + 1)^3$. Thus when $x_1 = x_2 = \cdots = x_{1969} = 1$ and $x_{1970} = 38$, the expression $x_1^3 + x_2^3 + \cdots + x_{1970}^3$ attains its maximum value of $1969 + 38^3 = 56841$.

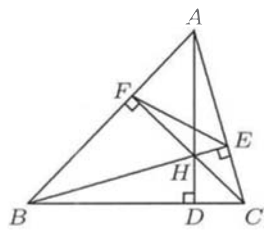
11. Ans: 9
Substituting the first equation $bc = a^2 - 8a + 7$ into the second equation, we have $(b+c)^2 = (a-1)^2$ so that $b + c = \pm(a-1)$. That means b and c are roots of the quadratic equation $x^2 \mp (a-1)x + (a^2 - 8a + 7) = 0$. Thus its discriminant $\triangle = [\mp(a-1)]^2 - 4(a^2 - 8a + 7) \ge 0$, or equivalently, $1 \le a \le 9$. For $b = c = 4$, $a = 9$ satisfies the two equations. Thus the largest value of a is 9.

12. Ans: 1506
Let $a_i = \lfloor i^2/2007 \rfloor$. Note that $\frac{(n+1)^2}{2007} - \frac{n^2}{2007} = \frac{2n+1}{2007} \le 1$ if and only if $n \le 2006/2 = 1003$. Since, $a_1 = 0$, and $a_{1003} = 501$, we see that $a_1, \ldots, a_{1003}$ assume all the values from 0 to 501. We also conclude that $a_{1004}, \ldots, a_{2007}$ are mutually distinct integers. Therefore, the answer is $502 + 1004 = 1506$.

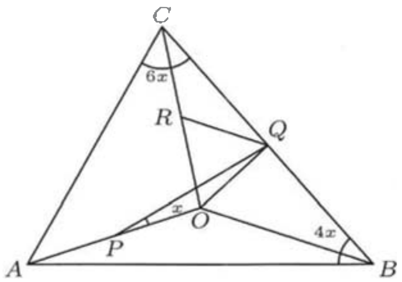
13. Ans: 112
Since the number $(a+b) + (a-b) + ab + a/b = (a/b)(b+1)^2$ is a perfect square with b and $b+1$ relatively prime, the number a/b must be a perfect square. Let $a/b = n^2$. As $a > b$, the number $n \ge 2$ so that $a = bn^2 \le 200$. From this, a can be determined once b and n are chosen. Hence, it suffices to count the number of pairs of (b, n) satisfying $bn^2 \le 200$ with $b \ge 1$ and $n \ge 2$. Hence the answer is $\lfloor 200/2^2 \rfloor + \cdots + \lfloor 200/14^2 \rfloor = 112$.

14. This question has been deleted.

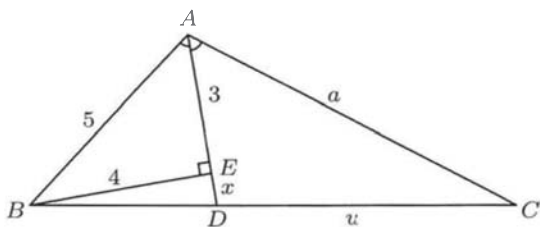
15. Ans: 52
First we know that $AE = AB\cos A$ and $AF = AC\cos A$. By cosine rule, $EF^2 = AE^2 + AF^2 - 2AE \times AF\cos A = \cos^2 A(AB^2 + AC^2 - 2AB \times AC\cos A) = BC^2\cos^2 A$. Therefore $EF = BC\cos A$. It is easy to see that A, E, H, F lie on a circle with diameter AH. Thus $AH = \frac{EF}{\sin A} = BC\cot A = 39 \times \frac{4}{3} = 52$.



16. Ans: 12
Let R be the midpoint of OC. Then $PO = RO = RQ$. Let $\angle OPQ = x$. Then $\angle POR = 2\angle B = 8x$ and $\angle BOA = 2\angle C = 12x$. Also, $\angle ROQ = \angle A = 180^\circ - 10x$. It follows that $\angle PQO = 180^\circ - x - 8x - (180^\circ - 10x) = x$ and therefore $PO = OQ$. Thus $\triangle OQR$ is equilateral, so $\angle ROQ = 180^\circ - 10x = 60^\circ$ and $x = 12^\circ$.



17. Ans: 3
Let $DE = x$, $AC = a > 5$ and $CD = u$. Then $BD = \sqrt{4^2 + x^2}$ and $\cos\angle CAD = \cos\angle BAD = 3/5$. By the cosine rule applied to $\triangle ADC$, we have $u^2 = (3+x)^2 + a^2 - 2a(3+x)(3/5)$. Using the angle bisector theorem, we have $\frac{u^2}{4^2+x^2} = \frac{a^2}{5^2}$. Thus $25[(3+x)^2 + a^2 - 2a(3+x)(3/5)] = a^2(16 + x^2)$.



This can be simplified to $(a^2 - 25)x^2 + 30(a-5)x - 9(a-5)^2 = 0$. Since $a > 5$, we can cancel a common factor $(a-5)$ to get $(a+5)x^2 + 30x - 9(a-5) = 0$, or equivalently $(x+3)((a+5)x - 3(a-5)) = 0$. Thus $x = 3(a-5)/(a+5)$. From this, we obtain $\left(\frac{AC+AB}{AC-AB}\right) ED = \left(\frac{a+5}{a-5}\right) x = 3$.

18. Ans: 1
Note that $\tan(90^\circ - \theta) = 1/\tan\theta$ for $0^\circ < \theta < 90^\circ$and that the product is positive. Setting $j = 46 - k$,

$$\prod_{k=1}^{45} \tan(2k-1)^\circ = \prod_{j=1}^{45} \tan(90 - (2j-1))^\circ = \frac{1}{\prod_{j=1}^{45} \tan(2j-1)^\circ} = 1.$$

19. Ans: 20
The triangle is a right triangle with area $A = 5 \times 120/2 = 300$. The semiperimeter is $s = \frac{1}{2} \times (50 + 120 + 130) = 150$. Hence the inradius is $A/s = 20$.

20. Ans: 4
Set

$$\mathbf{u} = (\sqrt{b-a}, \sqrt{c-b}, \sqrt{d-c})$$

$$\text{and} \quad \mathbf{v} = \Big(\frac{a}{\sqrt{b-a}}, \frac{b}{\sqrt{c-b}}, \frac{c}{\sqrt{d-c}}\Big).$$

Then $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|$ and hence $\mathbf{u} = \alpha\mathbf{v}$ for some $\alpha \geq 0$. Thus

$$\frac{b-a}{a} = \frac{c-b}{b} = \frac{d-c}{c}.$$

Hence $b/a = c/b = d/c$, i.e., a, b, c, d is a geometric progression $a, ra, r^2a, r^3a = 2a$. Now $bcd/a^3 = r^6 = (r^3)^2 = 4$.

21. Ans: 2
Putting $1/x$ in place of x in the given equation yields

$$f\Big(\frac{1}{x}\Big) - \frac{1}{2}f(x) = -\log x.$$

Solving for $f(x)$, we obtain $f(x) = \frac{2}{3}\log x$. Hence $f(1000) = 2$.

22. Ans: 20
We have

$$\frac{OD}{AD} = \frac{[BOC]}{[ABC]}, \quad \frac{OE}{BE} = \frac{[AOC]}{[ABC]}, \quad \frac{OF}{CF} = \frac{[AOB]}{[ABC]}.$$

Thus

$$\frac{OD}{AD} + \frac{OE}{BE} + \frac{OF}{CF} = 1 \quad \Rightarrow \quad OE = 20.$$

23. Ans: 53808

For $n \geq 3$, among the a_n such integers, let b_n denote the number of those that end with 1. By symmetry, the number of those that end with 2 is also equal to b_n. Also the number of those that end with 0 or 3 are both a_{n-1}. Thus

$$a_n = 2a_{n-1} + 2b_n.$$

Among the b_n integers that end with 1, the number of those that end with 11 is b_{n-1} while the number of those that end with 01 or 31 are both a_{n-2}. Thus

$$b_n = b_{n-1} + 2a_{n-2}.$$

Solving, we get $a_n = 3a_{n-1} + 2a_{n-2}$. Since $a_1 = 3$ and $a_2 = 10$, we get $a_9 = 73368$.

24. Ans: 1005

Partition all the possible remainders when divided by 2006 as follows:

$$(0),\ (1, 2006),\ (2, 3005),\ \ldots\ ,\ (1003, 1004).$$

Suppose S has 1005 elements. If S has two elements with their difference divisible by 2007, we are done. Otherwise the elements of S have distinct remainders when divided by 2007. By the pigeonhole principle, S has two integers whose sum is divisible by 2007. The set $\{1, 2, \ldots, 1004\}$ does not have 2 elements, x, y such that $x + y$ or $x - y$ is divisible by 2007.

25. Ans: 7040

For better understanding, we consider the general case when P has n vertices, where $n \geq 9$. We first count the number of such triangles S having a particular vertex A. The number is $\binom{n-7}{2}$. (This can be obtained as follows. Let the triangle be ABC ordered clockwise. Then B has a "left" neighbour and C has a "right" neighbour. The location of of B and C are uniquely determined by their neighbours. Besides A, B, C and the two vertices to the left and two vertices to the right of A, the two neighbours can be chosen from the remaining $n - 7$ vertices. Thus the required answer is $\frac{n}{3}\binom{n-7}{2} = 7040$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Open Section, Round 2)

Saturday, 30 June 2007 **0930-1230**

Instructions to contestants

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

1. Let $a_1, a_2, \ldots, a_n$ be n real numbers whose squares sum to 1. Prove that for any integer $k \geq 2$, there exists n integers $x_1, x_2, \ldots, x_n$, each with absolute value $\leq k-1$ and not all 0, such that
$$\Big|\sum_{i=1}^{n} a_i x_i\Big| \leq \frac{(k-1)\sqrt{n}}{k^n - 1}.$$

2. If $a_1,\ a_2,\ \ldots a_n$ are distinct integers, prove that $(x-a_1)(x-a_2)\ldots(x-a_n)-1$ cannot be expressed as a product of two polynomials, each with integer coefficients and of degree at least 1.

3. Let A_1, B_1 be two points on the base AB of an isosceles triangle ABC, with $\angle C > 60^\circ$, such that $\angle A_1CB_1 = \angle ABC$. A circle externally tangent to the circumcircle of $\triangle A_1B_1C$ is tangent to the rays CA and CB at points A_2 and B_2, respectively. Prove that $A_2B_2 = 2AB$.

4. Let $\mathbb{N}$ be the set of positive integers, i.e., $\mathbb{N} = \{1, 2, \ldots\}$. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that
$$f\big(f(m) + f(n)\big) = m + n \quad \text{for all } m, n \in \mathbb{N}.$$

5. Find the largest positive integer x such that x is divisible by all the positive integers $\leq \sqrt[3]{x}$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2007

(Open Section, Round 2 Solutions)

1. Without loss of generality, we may assume that all the a_i are positive, else we just change the sign of x_i. Since

$$\left(\frac{\sum_{i=1}^n a_i}{n}\right)^2 \leq \frac{\sum_{i=1}^n a_i^2}{n},$$

we have $\sum_{i=1}^n a_i \leq \sqrt{n}$. There are k^n integer sequences $(t_1, t_2, \ldots, t_n)$ satisfying $0 \leq t_i \leq k-1$ and for each such sequence we have $0 \leq \sum_{i=1}^n a_i t_i \leq (k-1)\sqrt{n}$. Now divide the interval $[0, (k-1)\sqrt{n}]$ into $k^n - 1$ equal parts. By the pigeonhole principle, there must exist 2 nonnegative sequences $(y_1, y_2, \ldots, y_n)$ and $(z_1, z_2, \ldots, z_n)$ such that $\left|\sum_{i=1}^n a_i y_i - \sum_{i=1}^n a_i z_i\right| \leq \frac{(k-1)\sqrt{n}}{k^n - 1}$. Set $x_i = y_i - z_i$ to satisfy the condition.

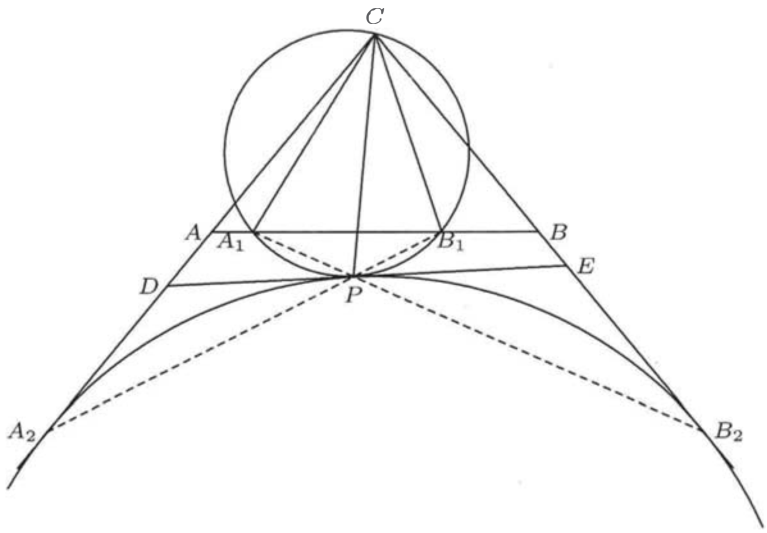
2. Suppose to the contrary that $(x-a_1)(x-a_2)\cdots(x-a_n) - 1 = f(x)g(x)$ for some polynomials $f(x)$ and $g(x)$ with integer coefficients and $\deg(f(x)), \deg(g(x)) \geq 1$. Then $f(a_i)g(a_i) = -1$ for $i = 1, 2, \ldots, n$ implies that $f(a_i) = 1$ and $g(a_i) = -1$ or $f(a_i) = -1$ and $g(a_i) = 1$. Therefore, if we set $h(x) = f(x) + g(x)$, then $h(a_i) = 0$ for all $i = 1, 2, \ldots, n$. As $\deg(h(x)) \leq \max(\deg(f(x)), \deg(g(x))) < n$, the polynomial equation $h(x) = 0$ cannot have n distinct roots. It follows that $h(x)$ must be the zero polynomial. Thus $f(x) = -g(x)$, and therefore

$$(x-a_1)(x-a_2)\ldots(x-a_n) - 1 = -(g(x))^2 \leq 0$$

for all real values of x. But this leads to a contradiction since we can choose a value for x large enough so that $(x-a_1)(x-a_2)\ldots(x-a_n) - 1$ is positive.

2nd solution: We start off as in the first solution. Then instead of defining $h(x)$, we proceed as follows. Let $f(a_i) = 1$, $1 \leq i \leq k$ and $f(a_i) = -1$, $k+1 \leq i \leq n$. Then, $g(a_i) = -1$, $1 \leq i \leq k$ and $g(a_i) = 1$, $k+1 \leq i \leq n$. Therefore $\deg(f(x) - 1) = \deg f(x) \geq \max(k, n-k) \geq \frac{k+(n-k)}{2} = \frac{n}{2}$. Similarly $\deg g(x) \geq \frac{n}{2}$. However $\deg f(x) + \deg g(x) = n$, and thus n is even with $\deg f(x) = \deg g(x) = k = \frac{n}{2}$. Thus $f(x) - 1 = b_1(x-a_1)(x-a_2)\cdots(x-a_k)$, and $g(x) + 1 = b_2(x-a_1)(x-a_2)\cdots(x-a_k)$ for some $b_1, b_2 \in \mathbb{Z}$. Together we get $f(x)g(x) + f(x) - g(x) - 1 = b_1 b_2[(x-a_1)(x-a_2)\cdots(x-a_k)]^2$. By comparing coefficient of the x^n term, $b_1 b_2 = 1$. This give us $f(x) - 1 = g(x) + 1$. Similarly, we have $f(x) + 1 = g(x) - 1$, a contradiction.

3. Let the point of contact of the two circles be P. First we show that A_1, P and B_2 are collinear. Let the common tangent at P meet CA at D and CB at E. Let $\angle ABC = b = \angle CAB = \angle A_1CB_1$, $\angle ACB = c$, $\angle A_1CP = x$ and $\angle B_1CP = y$. Then $x + y = b$ and $2b + c = 180°$. We have $\angle PB_1A = x$, $\angle B_1PE = y$. Therefore, by considering PB_1BE, $\angle BEP = 2x$. Hence $\angle EPB_2 = x$ and consequently, $\angle B_1PB_2 = x + y = b$. This implies that A_1, P and B_2 are collinear. Similarly A_2, P and B_1 are collinear. Then $\triangle A_1BC \sim \triangle A_1CB_1$, and $\triangle CAB_1 \sim \triangle A_1CB_1$, whence $\triangle A_1BC \sim \triangle CAB_1$. Thus $AC/AB_1 = A_1B/BC$ and $BC^2 = A_1B \cdot AB_1$, since $AC = BC$. Also $\triangle AA_2B_1 \sim \triangle BA_1B_2$. Thus $AB_1/AA_2 = BB_2/A_1B$ and whence $AA_2^2 = A_1B \cdot AB_1$. Thus B is the midpoint of CB_2. Since $AB \parallel A_2B_2$, we have $A_2B_2 = 2AB$ as required.



2nd solution: Let Γ_1 be the circumcircle of $\triangle A_1B_1C$ and its centre be O_1, let the other circle Γ_2 has center O_2, and the point of contact of the two circles be P. Now since CA_2 and CB_2 are tangent to Γ_2, we have $CA_2 = CB_2$. Together with $CA = CB$, we have $AA_2 = BB_2$. This implies that $\triangle CAB \sim \triangle CA_2B_2$ and $AB \parallel A_2B_2$. Now $\angle A_2O_2B_2 = 180° - \angle ACB = 2\angle CAB = 2\angle A_1CB_1 = \angle A_1O_1B_1$. Thus, isosceles triangles $A_1O_1B_1$ and $A_2O_2B_2$ are similar. Since $AB \parallel A_2B_2$, $A_1O_1 \parallel O_2B_2$, also note that O_1PO_2 is a straight line. Therefore, we have $2\angle A_1B_1P = \angle A_1O_1P = \angle PO_2B_2 = 2\angle PA_2B_2$. This implies that B_1PA_2 is a straight line. Similarly, A_1PB_2 is a straight line. Now we let Γ_1 intersects CA and CB at D and E respectively, and let DA_1 intersects EB_1 at G. By Pascal's Theorem on Γ_1 and the hexagons CEB_1PA_1D, we have A_2, G and B_2 collinear. Using the fact that $\triangle ACB_1 \sim \triangle CA_1B_1 \sim B_1A_1C$, we have $\angle GA_1B_1 = \angle DA_1A = \angle DCB_1 = \angle CA_1B_1$. Similarly, $\angle GB_1A_1 = \angle CB_1A_1$. This implies that G is the image of C under reflection of line AB. Since G is on A_2B_2, A_2B_2 is twice as far as AB from C. Thus, $A_2B_2 = 2AB$.

3rd solution: Let Γ_1 be the circumcircle of $\triangle A_1B_1C$ and its centre be O_1, let the other circle Γ_2 has center O_2. Now since CA_2 and CB_2 are tangent to Γ_2, we have $CA_2 = CB_2$. This implies that $\triangle CAB \sim \triangle CA_2B_2$. Let us perform inversion with center C and radius CA. Let the image of A_1, A_2, B_1, B_2 under this inversion be A_1', A_2', B_1', B_2' respectively. A, B and C remain invariant. The inversion keeps every line that passes through C invariant. Now the image of the line AA_1B_1B is the circumcircle of $\triangle CAB$, let it be Γ_3, and the image of Γ_1 is the line A_1B_1. Thus the image of Γ_2 is tangent to A_1B_1, AC and BC and is thus the incircle of $\triangle ABC$ and touches the sides AC and BC at A_2' and B_2', respectively. Thus $\frac{CA_2'}{CA} = \frac{CB_2'}{CB} = \frac{1}{2}$, which implies that $\frac{CA_2}{CA} = \frac{CB_2}{CB} = 2$. Hence A and B are the midpoints of A_2C and B_2C, respectively. Thus $A_2B_2 = 2AB$.

4. We show that f is the identity function. First we observe that f is an injective function:

$$\begin{aligned} f(m) = f(n) \quad &\Rightarrow \quad f(m) + f(n) = f(n) + f(n) \\ &\Rightarrow \quad f\big(f(m) + f(n)\big) = f\big(f(n) + f(n)\big) \\ &\Rightarrow \quad m + n = n + n \\ &\Rightarrow \quad m = n \end{aligned}$$

Let $k > 1$ be arbitrary. From the original equation, we have the equations

$$f(f(k+1)+f(k-1)) = (k+1)+(k-1) = 2k, \quad \text{and} \quad f(f(k)+f(k)) = k+k = 2k.$$

Since f is injective, we have

$$f(k+1) + f(k-1) = f(k) + f(k) \quad \text{or} \quad f(k+1) - f(k) = f(k) - f(k-1).$$

This characterizes f as an arithmetic progression, so we may write $f(n) = b + (n-1)t$ where $b = f(1)$ and t is the common difference. The original equation becomes $b + [(b + (m-1)t) + (b + (n-1)t) - 1]t = m + n$, which simplifies to $(3b - 2t - 1) + (m + n)t = m + n$. Comparing coefficients, we conclude that $t = 1$ and $b = 1$. Thus $f(n) = n$, as claimed. Clearly, this function satisfies the original functional equation.

5. The answer is $x = 420$.

Let $p_1, p_2, p_3, \ldots$ be all the primes arranged in increasing order. By Bertrand's Postulate, we have $p_i < p_{i+1} < 2p_i$ for all $i \in \mathbb{N}$, thus we have $p_{k+1} < 2p_k < 4p_{k-1} < 8p_{k-2}$ which implies that $64p_kp_{k-1}p_{k-2} > {p_{k+1}}^3$.

Let $p_k \leq \sqrt[3]{x} < p_{k+1}$ for some $k \in \mathbb{N}$. Note that $p_i \mid x$ for $i = 1, 2, \ldots, k$. Suppose $k \geq 5$, then $\sqrt[3]{x} \geq p_5 = 11$. Since $11 > 2^3$ and $11 > 3^2$, we have $2^33^2 \mid x$. Since $k \geq 5$, $\gcd(p_kp_{k-1}p_{k-2}, 2^33^2) = 1$ and thus $2^33^2p_kp_{k-1}p_{k-2} \mid x$. This means we have $x \geq 72p_kp_{k-1}p_{k-2} > 64p_kp_{k-1}p_{k-2} > {p_{k+1}}^3$, implying $p_{k+1} < \sqrt[3]{x}$, which is a contradiction.Thus $k < 5$ and consequently, $\sqrt[3]{x} < 11$ or $x < 1331$.

Next, we notice that the integer 420 is divisible by all positive integers $\leq \sqrt[3]{420}$, thus $x \geq 420 \Rightarrow \sqrt[3]{x} > 7$. It then follows that x is divisible by $2^2 \cdot 3 \cdot 5 \cdot 7 = 420$.

Finally, suppose $\sqrt[3]{x} \geq 9$. We then have $2^3 \cdot 3^2 \cdot 5 \cdot 7 \mid x$, i.e., $x \geq 2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520$, which is a contradiction since $x < 1331$. Thus $\sqrt[3]{x} < 9.=$ or $x < 729$. Since $420 \mid x$ and $x < 729$, we have $x = 420$.

Alternatively, since $x < 1331$ and $420 \mid x$, we only need to check the cases $x = 420, 840, 1260$.