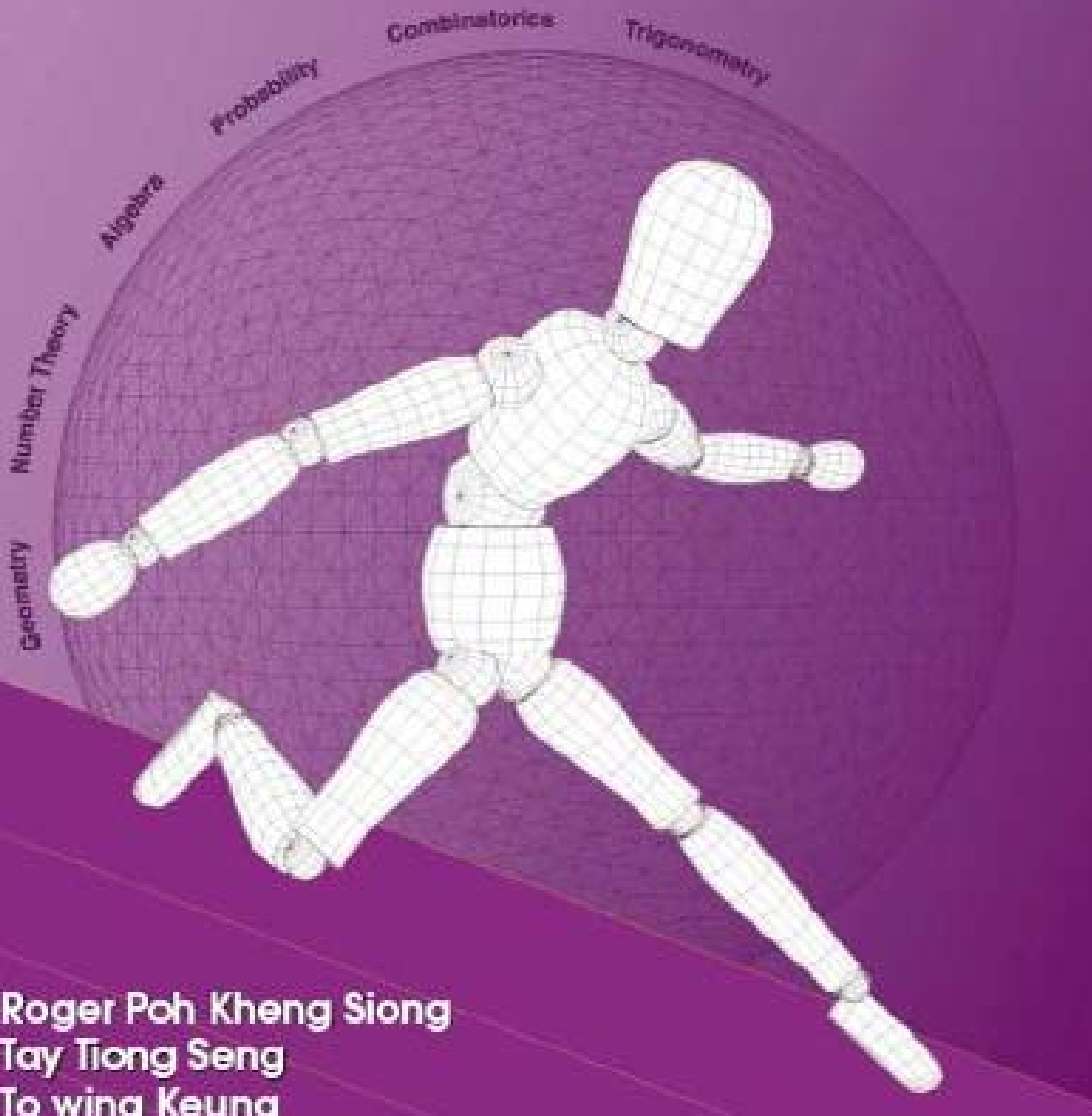
SINGAPORE MATHEMATICAL OLYMPIADS 2006



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Singapore Mathematical Olympiad (SMO) 2006

(Junior Section)

Tuesday, 30 May 2006 0930-1200

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter only the letters (A, B, C, D, or E) corresponding to the correct answers in the answer sheet.

For the other short questions, write your answers in answer sheet and shade the appropriate bubbles below your answers.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

1. What are the last two digits of $1 \times 2 \times 3 \times 4 \times \cdots \times 2004 \times 2005 \times 2006$?

   (A) 00; (B) 20; (C) 30; (D) 50; (E) 60.

2. Let x be a real number. What is the minimum value of $x^2 - 4x + 3$?

   (A) -3; (B) -1; (C) 0; (D) 1; (E) 3.

3. James calculates the sum of the first n positive integers and finds that the sum is 5053. If he has counted one integer twice, which one is it?

   (A) 1; (B) 2; (C) 3; (D) 4; (E) 5.

4. Which of the following is a possible number of diagonals of a convex polygon?

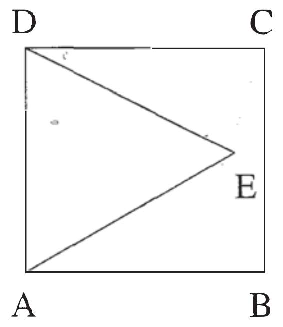
   (A) 21; (B) 32; (C) 45; (D) 54; (E) 63.

5. What is the largest positive integer n satisfying $n^{200} < 5^{300}$?

   (A) 9; (B) 10; (C) 11; (D) 12; (E) 13.

6. The diagram shows an equilateral triangle ADE inside a square $ABCD$. What is the value of

$$\frac{\text{area of } \Delta ADE}{\text{area of } \Delta DEC}?$$



(A) $\frac{\sqrt{3}}{4}$; (B) $\frac{1}{4}$; (C) $\frac{\sqrt{3}}{2}$; (D) $\sqrt{3}$; (E) 2.

7. What is the value of

$$(x+1)(x+2006)\left[\frac{1}{(x+1)(x+2)} + \frac{1}{(x+2)(x+3)} + \ldots + \frac{1}{(x+2005)(x+2006)}\right]?$$

(A) $x + 2004$; (B) 2005; (C) $x + 2006$; (D) 2006; (E) 2007.

8. Suppose that only one of the following pairs (x, y) yields the positive integer $\sqrt{x^2 + y^2}$. Then

(A) $x = 25530, y = 29464$; (B) $x = 37615, y = 26855$; (C) $x = 15123, y = 32477$;
(D) $x = 28326, y = 28614$; (E) $x = 22536, y = 27462$.

9. The value of

$$\frac{1}{3+1} + \frac{2}{3^2+1} + \frac{4}{3^4+1} + \frac{8}{3^8+1} + \cdots + \frac{2^{2006}}{3^{2^{2006}}+1}$$

is:

(A) $\frac{1}{2}$; (B) $\frac{1}{2} - \frac{2^{2005}}{3^{2^{2005}}-1}$; (C) $\frac{1}{2} - \frac{2^{2006}}{3^{2^{2006}}-1}$; (D) $\frac{1}{2} - \frac{2^{2007}}{3^{2^{2007}}-1}$; (E) None of the above.

10. Suppose that p and q are prime numbers and they are roots of the equation $x^2 - 99x + m = 0$ for some m. What is the value of $\frac{p}{q} + \frac{q}{p}$?

(A) 9413; (B) $\frac{9413}{194}$; (C) $\frac{9413}{99}$; (D) $\frac{9413}{97}$; (E) None of the above.

11. What is the remainder when $2006 \times 2005 \times 2004 \times 2003$ is divided by 7?

12. If $\dfrac{139}{22} = a + \dfrac{1}{b + \frac{1}{c}}$, where a, b and c are positive integers, find the value of $a + b + c$.

13. Let x be a positive real number. Find the minimum value of $x + \dfrac{1}{x}$.

14. Find the value (in the simplest form) of $\sqrt{45 + 20\sqrt{5}} + \sqrt{45 - 20\sqrt{5}}$.

15. Let n be the number

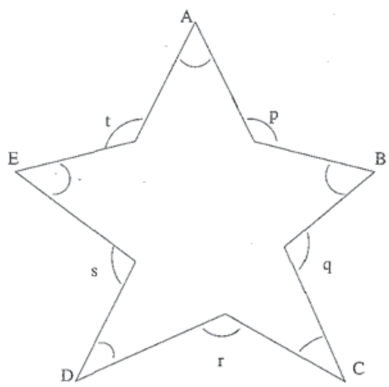
$$(\underbrace{999\,999\,999\,\ldots\,999}_{2006\text{ 9's}})^2 - (\underbrace{666\,666\,666\,\ldots\,666}_{2006\text{ 6's}})^2.$$

Find the reminder when n is divided by 11.

16. Given that $w > 0$ and that $w - \dfrac{1}{w} = 5$, find the value of $(w + \dfrac{1}{w})^2$.

17. N pieces of candy are made and packed into boxes, with each box containing 45 pieces. If N is a non-zero perfect cube and 45 is one of its factors, what is the least possible number of boxes that can be packed?

18. Consider the following "star" figure.



Given that $\angle p + \angle q + \angle r + \angle s + \angle t = 500^\circ$ and $\angle A + \angle B + \angle C + \angle D + \angle E = x^\circ$, find the value of x.

19. Given that n is a positive integer and $S = 1 + 2 + 3 + \cdots + n$. The units digit of S cannot be some numbers. Find the sum of these numbers.

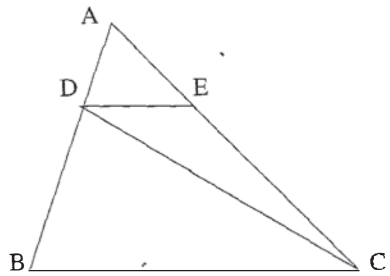
20. Let $m = 76^{2006} - 76$. Find the remainder when m is divided by 100.

21. Let $ABCDEF$ be a hexagon such that the diagonals AD, BE and CF intersect at the point O, and the area of the triangle formed by any three adjacent points is 2 (for example, area of ΔBCD is 2). Find the area of the hexagon.

22. Let C be a circle with radius 2006. Suppose n points are placed inside the circle and the distance between any two points exceed 2006. What is the largest possible n?

23. Let x and y be positive real numbers such that $x^3 + y^3 + \frac{1}{27} = xy$. Find the value of $\frac{1}{x}$.

24. In this question, $S_{\Delta XYZ}$ denotes the area of ΔXYZ. In the following figure, if $DE//BC$, $S_{\Delta ADE} = 1$ and $S_{\Delta ADC} = 4$, find $S_{\Delta DBC}$.



25. What is the product of the real roots of the equation

$$\frac{x^2 + 90x + 2027}{3} = \sqrt{x^2 + 90x + 2055}?$$

26. There are four piles of stones: One with 6 stones, two with 8, and one with 9. Five players numbered 1, 2, 3, 4 and 5 take turns, in the order of their numbers, choosing one of the piles and dividing it into two smaller piles. The loser is the player who cannot do this. State the number of the player who loses.

27. Let $m \neq n$ be two real numbers such that $m^2 = n + 2$ and $n^2 = m + 2$. Find the value of $4mn - m^3 - n^3$.

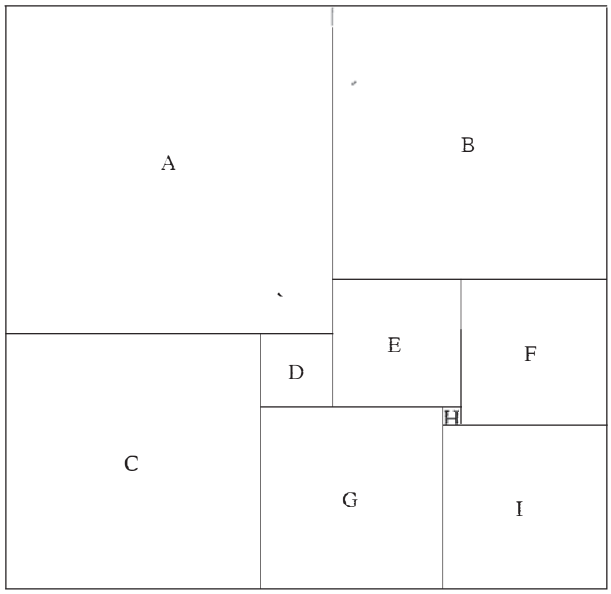
28. There are a few integer values of a such that $\frac{a^2 - 3a - 3}{a - 2}$ is an integer. Find the sum of all these integer values of a.

29. How may pairs of integers (x, y) satisfy the equation

$$\sqrt{x} + \sqrt{y} = \sqrt{200600}?$$

30. The '4' button on my calculator is spoilt, so I cannot enter numbers which contain the digit 4. Moreover, my calculator does not display the digit 4 if 4 is part of an answer either. Thus I cannot enter the calculation 2×14 and do not attempt to do so. Also, the result of multiplying 3 by 18 is displayed as 5 instead of 54 and the result of multiplying 7 by 7 is displayed as 9 instead of 49. If I multiply a positive one-digit number by a positive two-digit number on my calculator and it displays 26, how many possibilities could I have multiplied?

31. The following rectangle is formed by nine pieces of squares of different sizes. Suppose that each side of the square E is of length 7cm. Let the area of the rectangle be x cm^2. Find the value of x.



32. Suppose that n is a positive integer, and a, b are positive real numbers with $a + b = 2$. Find the smallest possible value of

$$\frac{1}{1+a^n} + \frac{1}{1+b^n}.$$

33. What is the largest positive integer n for which $n^3 + 2006$ is divisible by $n + 26$?

34. Suppose that the two roots of the equation

$$\frac{1}{x^2-10x-29}+\frac{1}{x^2-10x-45}-\frac{2}{x^2-10x-69}=0$$

are α and β. Find the value of $\alpha+\beta$.

35. Suppose that a, b, x and y are real numbers such that

$$ax+by=3, \quad ax^2+by^2=7, \quad ax^3+by^3=16 \quad \text{and} \quad ax^4+by^4=42.$$

Find the value of ax^5+by^5.

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Singapore Mathematical Olympiad (SMO) 2006

(Junior Section Solutions)

1. Ans: (A)

   Because 100 is one of the factors.

2. Ans: (B)

   Use $x^2 - 4x + 3 = (x-2)^2 - 1$.

3. Ans: (C)

   Use $1 + 2 + \cdots + 100 = 5050$.

4. Ans: (D)

   The number of diagonals of an n-side polygon is $\frac{n(n-3)}{2}$. Hence (D).

5. Ans: (C)

   Because $n^2 < 5^3 = 125$.

6. Ans: (D)

   Let a be the length of AB. Then

$$\frac{\text{area of } \Delta ADE}{\text{area of } \Delta DEC} = \frac{\frac{1}{2}a \cdot a \cdot \sin 60^\circ}{\frac{1}{2}a \cdot a \cdot \sin 30^\circ} = \sqrt{3}.$$

7. Ans: (B)

   Use

$$\frac{1}{(x+1)(x+2)} + \frac{1}{(x+2)(x+3)} + \dots + \frac{1}{(x+2005)(x+2006)} =$$

$$\frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{x+2} - \frac{1}{x+3} + \dots + \frac{1}{x+2005} - \frac{1}{x+2006} = \frac{1}{x+1} - \frac{1}{x+2006} =$$

$$\frac{2005}{(x+1)(x+2006)}$$

8. Ans: (A)

Since $x^2 + y^2$ is a perfect square, its last digit must be 0, 1, 4, 5, 6,or 9. Hence (C) and (D) are not suitable. For (B) and (e), the last two digits are 50 and 60 resp. However, if a number ends with 0, its square must end with two 0's. Thus the ans is (A) where $\sqrt{x^2 + y^2}$ turns out to be 38986.

9. Ans: (D)

Since

$$\frac{2^{k+1}}{3^{2^{k+1}} - 1} = \frac{2^k}{3^{2^k} - 1} - \frac{2^k}{3^{2^k} + 1},$$

$$\frac{2^k}{3^{2^k} + 1} = \frac{2^k}{3^{2^k} - 1} - \frac{2^{k+1}}{3^{2^{k+1}} - 1}$$

becomes a telescope sum. Thus the result is $\frac{1}{2} - \frac{2^{2007}}{3^{2^{2007}} - 1}$.

10. Ans: (B)

$p + q = 99$. The only possible prime solutions are 2, 97.

11. Ans: 3.

$2006 \times 2005 \times 2004 \times 2003 \equiv 4 \times 3 \times 2 \equiv 3 \pmod 7$.

12. Ans: 16.

Easy calculation shows that $a = 6$, $b = 3$ and $c = 7$.

13. Ans: 2.

$$x + \frac{1}{x} \geq 2\sqrt{x \cdot \frac{1}{x}} = 2,$$

and = holds when $x = 1$.

14. Ans: 10.

$\sqrt{45 + 20\sqrt{5}} + \sqrt{45 - 20\sqrt{5}} = \sqrt{(5 + 2\sqrt{5})^2} + \sqrt{(5 - 2\sqrt{5})^2} = 10.$

15. Ans: 0.

Observe that 11 divides both $\underbrace{999\,999\,999\,\ldots\,999}_{2006\text{ 9's}}$ and $\underbrace{666\,666\,666\,\ldots\,666}_{2006\text{ 6's}}$.

16. Ans: 29.

Use that $(w + \frac{1}{w})^2 = (w - \frac{1}{w})^2 + 4$.

17. Ans: 75.

The least non-zero perfect cube of the form $45m = 3^2 \cdot 5m$ is $3^3 \times 5^3$. Thus the least possible number of boxes that can be packed is $3 \times 5^2 = 75$.

18. Ans: 140.

Use angle sum of polygons,

$$x^\circ + 5 \times 360^\circ - (\angle p + \angle q + \angle r + \angle s + \angle t) = 8 \times 180^\circ,$$

$x = 140$.

19. Ans: 22.

$S = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$. The units digit of S can only be $0, 1, 3, 5, 6, 8$. Thus S cannot end with $2, 4, 7$ or 9. The sum is 22.

20. Ans: 0.

$m = 76 \cdot (76^{2005} - 1) = 76 \times 75 \times k$ for some k. Thus 100 divides m.

21. Ans: 12.

Since area of ΔABC = area of ΔFAB, $FC//AB$. Similarly $FC//ED$.

By the same reason, $FE//AD//BC$ and $AF//BE//CD$. Thus $OFDE$, $ODBC$ and $OBAF$ are all parallelogram and each of them has area 4. Thus the total area of the hexagon is 12.

22. Ans: 5.

Clearly, none of the points is in the centre O. Label the points $A_1, A_2, \ldots A_n$ in clockwise direction. Note that no two points lie on the same radius. Now, for any two points A_i, A_{i+1} ($A_{n+1} = A_0$), the angle $\angle A_iOA_{i+1} > 60$. Therefore, $n < 6$. For $n = 5$, just take the points to be the vertices of a suitable regular pentagon, then it will satisfy our condition. Therefore, $n = 5$.

23. Ans: 3.

Using the fact that AM≥ GM, we get $x^3 + y^3 + 1/27 \geq 3\sqrt[3]{x^3y^3(1/27)} = xy$ and = holds only when $x^3 = 1/27$. Thus $x = 1/3$.

24. Ans: 12.

$S_{\triangle ADE} : S_{\triangle CDE} = 1 : 3$. Hence $AE : EC = AD : DB = 1 : 3$. Thus $AE : AC = AD : AB = 1 : 4$. Hence $S_{\triangle ABC} = 16$ and $S_{\triangle DBC} = 12$.

25. Ans: 2006.

Let $u = \sqrt{x^2 + 90x + 2055}$. Then $u^2 - 28 = 3u$. Solve to find the positive root $u = 7$. Hence $x^2 + 90x + 2006 = 0$, so the product of the real roots is 2006.

26. Ans: 3.

Observe that we begin with 4 piles of stones and end up with $6+8+8+9 = 31$ piles of (one) stones. At the end of each turn, there is exactly one more pile of stones than the beginning of the turn. Thus, there can be exactly $31 - 4 = 27$ legal turns. Hence, Player 3 is the first player who cannot make a move.

27. Ans: 0.

Since $m \neq n$ and $m^2 - n^2 = (n - m)$, we get $m + n = -1$. Thus $m^2 + n^2 = (m + n) + 4 = 3$ and $mn = \frac{1}{2}[(m + n)^2 - m^2 - n^2] = -1$. Thus $4mn - m^3 - n^3 = 4mn - (m + n)(m^2 + n^2 - mn) = -4 - (-1)(3 - (-1)) = 0$.

28. Ans: 8.

Since $\dfrac{a^2 - 3a - 3}{a - 2} = a - 1 - \dfrac{5}{a - 2}$, thus $a = -3, 1, 3, 7$. Their sum is 8.

29. Ans: 11.

Since $x = 200600 + y - 20\sqrt{2006y}$ is an integer, $y = 2006u^2$ for some positive integer u. Similarly $x = 2006v^2$ for some positive integer v. Thus $u + v = 10$. There are 11 pairs in total, namely $(0, 10), (1, 9), \ldots, (10, 0)$.

30. Ans: 6.

Since the product is at most a three-digit number, the possible answers I should get is 26, 426, 246 or 264. Since

$$\begin{aligned}
26 &= \mathbf{1} \times \mathbf{26} = \mathbf{2} \times \mathbf{13} \\
426 &= 1 \times 426 = 2 \times 213 = 3 \times 142 = \mathbf{6} \times \mathbf{71} \\
246 &= 1 \times 246 = 2 \times 123 = \mathbf{3} \times \mathbf{82} = 6 \times 41 \\
264 &= 1 \times 264 = 2 \times 132 = \mathbf{3} \times \mathbf{88} = 4 \times 66 \\
&= 6 \times 44 = \mathbf{8} \times \mathbf{33} = 11 \times 24 = 12 \times 22,
\end{aligned}$$

there are 6 possibilities.

31. Ans: 1056.

Let us use $|X|$ to denote the length of one side of square X. Thus $|E| = 7$. Let $|H|$ be a and $|D|$ be b. Then $|F|$ is $a + 7$, $|B|$ is $a + 14$, $|I|$ is $2a + 7$, $|G|$ is $3a + 7$, $|C|$ is $3a + b + 7$, and $|A|$ is $3a + 2b + 7$. Hence we have

$$(3a + 2b + 7) + (a + 14) = (3a + b + 7) + (3a + 7) + (2a + 7)$$

$$(3a + 2b + 7) + (3a + b + 7) = (a + 14) + (a + 7) + (2a + 7).$$

Therefore $a = 1$ and $b = 4$. Hence the area of the rectangle is $32 \times 33 = 1056$.

32. Ans: 1.

Note that $ab \leq 1$ and $(ab)^n \leq 1$.

$$\frac{1}{1 + a^n} + \frac{1}{1 + b^n} = \frac{1 + a^n + b^n + 1}{1 + a^n + b^n + (ab)^n} \geq 1.$$

When $a = b = 1$, we get 1. Thus, the smallest value is 1.

33. Ans: 15544.

$n^3 + 2006 = (n + 26)(n^2 - 26n + 676) - 15570$. So if $n + 26$ divides $n^3 + 2006$, $n + 26$ must divide 15570. Thus the largest n is 15544.

34. Ans: 10.

Let $y = x^2 - 10x - 29$. Then $x^2 - 10x - 45 = y - 16$, $x^2 - 10x - 69 = y - 40$. Thus $\frac{1}{y} + \frac{1}{y - 16} - \frac{2}{y - 40} = 0$. Therefore $y = 10$. So $y = x^2 - 10x - 29 = 10$. Hence $x^2 - 10x - 39 = 0$. Therefore $\alpha + \beta = 10$.

35. Ans: 20.

$ax^3 + by^3 = (ax^2 + by^2)(x + y) - (ax + by)xy$. Thus, we get $16 = 7(x + y) - 3xy$. Similarly, $ax^4 + by^4 = (ax^3 + by^3)(x + y) - (ax^2 + by^2)xy$. So, we get $42 = 16(x + y) - 7xy$. Solving, we get $x + y = -14$ and $xy = -38$. Therefore,

$$ax^5 + by^5 = (ax^4 + by^4)(x + y) - (ax^3 + by^3)xy = 42(-14) - 16(-38) = 20.$$

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Singapore Mathematical Olympiad (SMO) 2006

(Junior Section, Special Round)

Saturday, 24 June 2006 0930– 1230

Important:

Attempt as many questions as you can.

No calculators are allowed.

Show all the steps in your working.

Each question carries 10 marks.

1. Find all integers x, y that satisfy the equation

$$x + y = x^2 - xy + y^2.$$

2. The fraction $\frac{2}{3}$ can be expressed as a sum of two distinct unit fractions: $\frac{1}{2} + \frac{1}{6}$. Show that the fraction $\frac{p-1}{p}$, where $p \geq 5$ is a prime, cannot be expressed as a sum of two distinct unit fractions.

3. Suppose that each of n people knows exactly one piece of information, and all n pieces are different. Every time person A phones person B, A tells B everything he knows, while B tells A nothing. What is the minimum of phone calls between pairs of people needed for everyone to know everything?

4. In $\triangle ABC$, the bisector of $\angle B$ meets AC at D and the bisector of $\angle C$ meets AB at E. These bisectors intersect at O and $OD = OE$. If $AD \neq AE$, prove that $\angle A = 60°$.

5. You have a large number of congruent equilateral triangular tiles on a table and you want to fit n of them together to make a convex equiangular hexagon (i.e., one whose interior angles are $120°$). Obviously, n cannot be any positive integer. The first three feasible n are 6, 10 and 13. Show that 12 is not feasible but 14 is.

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Singapore Mathematical Olympiad (SMO) 2006
(Junior Section, Special Round Solutions)

1. Solving for y, we get:

$$y = \frac{x+1 \pm \sqrt{-3(x-1)^2+4}}{2}.$$

Thus $3(x-1)^2 \leq 4$, i.e.,

$$1 - \frac{2}{\sqrt{3}} \leq x \leq 1 + \frac{2}{\sqrt{3}}.$$

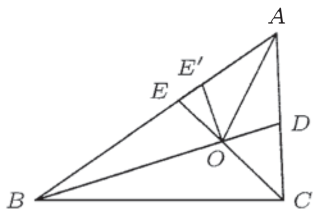
Thus $x = 0, 1, 2$ and $(x, y) = (0,0), (0,1), (1,0), (1,2), (2,1), (2,2)$ are all the solutions.

2. Note that $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$. Since $\frac{1}{2}$ and $\frac{1}{3}$ are the two largest unit fractions and $\frac{p-1}{p} > \frac{5}{6}$ for all $p \geq 7$, the result is true for $p \geq 7$. Suppose $\frac{4}{5} = \frac{1}{a} + \frac{1}{b}$, with $a > b$. Then $\frac{4}{5} = \frac{1}{a} + \frac{1}{b} < \frac{2}{b}$. Therefore $2b < 5$, i.e., $b = 2$ and there is no solution for a.

3. We claim that the minimum of calls needed is $2n-2$. Let A be a particular person, the $2n-2$ calls made by A to each of the persons and vice versa will leave everybody informed. Thus at most $2n-2$ calls are needed.

Next we prove that we need at least $2n-2$ calls. Suppose that there is a sequence of calls that leaves everybody informed. Let B be the first person to be fully informed and that he receives his last piece of information at the pth call. Then each of the remaining $n-1$ people must have placed at least one call prior to p so that B can be fully informed. Also these people must received at least one call after p since they were still not fully informed at the pth call. Thus we need at least $2(n-1)$ calls.

4.

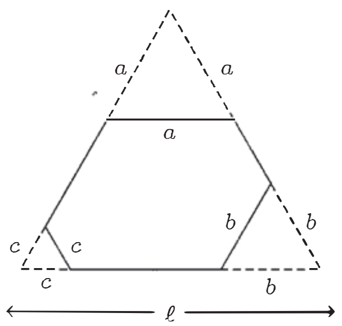


Assume that $AE > AD$. Let $\angle A = 2a$, $\angle B = 2b$, $\angle C = 2c$ and $\angle ADO = x$. Now AO bisects $\angle A$. Let E' be the point on AB such that $OE' = OE$. Since $AE > AD$, E' lies between A and E. We have $\triangle AE'O \equiv \triangle ADO$ (SAS). Thus $OE' = OD = OE$ and $x = \angle ADO = \angle AE'O = \angle BEO$. From $\triangle ABD$ and $\triangle BEC$, we have $2a + x + b = 180^\circ$ and $x + 2b + c = 180^\circ$. Thus $2a = b + c$ and so $\angle BAC = 2a = 60^\circ$.

5. Assume that the tiles are of side length 1. Note the number of tiles required to form an equilateral triangle of length x is $1 + 3 + \cdots + (2x - 1) = x^2$. The triangle formed by extending the alternate sides of the hexagon must be an equilateral triangle of side length say ℓ. The hexagon is formed by removing the three corner equilateral triangles of side lengths a, b, c and $\ell > a + b, a + c, b + c$. An equilateral triangle of side length x contains x^2 tiles. Thus n is feasible if and only if

$$n = \ell^2 - a^2 - b^2 - c^2 \quad \text{and} \quad \ell > a + b,\ a + c,\ b + c$$

Take $\ell = 5$, $a = 3$, $b = c = 1$. Then $n = 14$ and so is feasible.

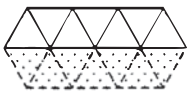


Next we show that $n = 12$ is not feasible. Let $a \geq b \geq c$. For fixed ℓ, we want to find a lower bound for n. For this purpose, we may assume that $a + b = \ell - 1$. Thus $b \leq (\ell - 1)/2$. If $a =\ \ - 1 - k$, then $b, c = k \leq (\ell - 1)/2$. Thus

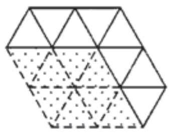
$$n \geq \ell^2 - 2k^2 - (\ell - 1 - k)^2 = (2\ell - 1) - (3k^2 - 2k(\ell - 1)).$$

Since $1 \leq k \leq (\ell - 1)/2$, we see that the maximum value of $3k^2 - 2k($
either $k = 1$ or $k = ($ $n \geq 4\ell - 6 = A$ (when $k = 1$) and $n \geq (\ell^2 + 6\ell - 3)/4 = B$ when $k = (\ell - 1)/2$. Thus $n \geq 6$ for $\ell = 3$, $n \geq 10$ for $\ell = 4$, $n \geq 13$ for $\ell = 5$, $n \geq 18$ for $\ell = 6$. For $\ell \geq 7$, $n \geq 22$. Thus we only have to check the case $\ell =$
$\ell = 3$, we have $a = b = c = 1$. For $\ell = 4$, we have $(a, b, c) = (2, 1, 1), (1, 1, 1)$. These give $n = 6, 10, 13$. Thus 12 is not feasible.

(Note: You can show that 14 is feasible by drawing a hexagon with 14 tiles. It is possible to show that 12 is not feasible by brute force. One of the sides must be at least of length 2. If one side has length 3, we need at least 14 tiles. In Fig. 1, the top side is of length 3 and the 7 tiles in the unshaded region must be present. No matter what you do, the 7 tiles in the shaded region must also be present. In fact this is the smallest hexagon with one side of length 3. If two adjacent sides are of length 2, then we need at least 16 tiles (Fig 2). If three consecutive sides are of lengths 2, 1, 2, then we need at least 13 tiles (Fig 3). The only other case is 2, 1, 1, 2, 1, 1 which gives 10 tiles (Fig 4). Thus 12 is not feasible.)



(1)



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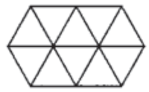


(4)

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2006

(Senior Section)

Tuesday, 30 May 2006 — 0930-1200

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answers in the answer sheet by shading the bubbles containing the letters (A, B, C, D *or* E) *corresponding to the correct answers.*

For the other short questions, write your answers in answer sheet and shade the appropriate bubbles below your answers.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

1. Let $p = 2^{3009}$, $q = 3^{2006}$ and $r = 5^{1003}$. Which of the following statements is true?

   (A) $p < q < r$ (B) $p < r < q$ (C) $q < p < r$ (D) $r < p < q$

   (E) $q < r < p$

2. Which of the following numbers is the largest?

   (A) 30^{20} (B) 10^{30} (C) $30^{10} + 20^{20}$ (D) $(30 + 10)^{20}$ (E) $(30 \times 20)^{10}$

3. What is the last digit of the number

$$2^2 + 20^{20} + 200^{200} + 2006^{2006}?$$

   (A) 2 (B) 4 (C) 6 (D) 8 (E) 0

4. Let x be a number such that $x + \frac{1}{x} = 4$. Find the value of $x^3 + \frac{1}{x^3}$.

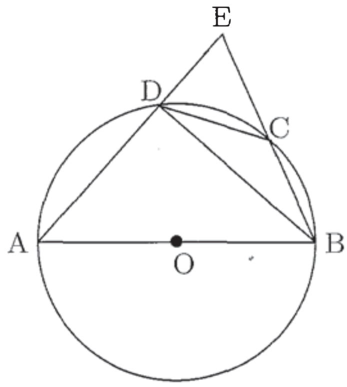
   (A) 48 (B) 50 (C) 52 (D) 54 (E) None of the above

5. Consider the two curves $y = 2x^3 + 6x + 1$ and $y = -\dfrac{3}{x^2}$ in the Cartesian plane. Find the number of distinct points at which these two curves intersect.

(A) 1 (B) 2 (C) 3 (D) 0 (E) 5

6. In the following figure, AB is the diameter of a circle with centre at O. It is given that $AB = 4$ cm, $BC = 3$ cm, $\angle ABD = \angle DBE$. Suppose the area of the quadrilateral $ABCD$ is x cm^2 and the area of $\triangle DCE$ is y cm^2. Find the value of the ratio $\dfrac{x}{y}$.

(A) 7 (B) 8 (C) 4 (D) 5 (E) 6



7. Five students A, B, C, D and E form a team to take part in a 5-leg relay competition. If A cannot run the first leg and D cannot run the last leg, how many ways can we arrange them to run the relay?

(A) 74 (B) 76 (C) 78 (D) 80 (E) 82

8. There are n balls in a box, and the balls are numbered $1, 2, 3, \ldots, n$ respectively. One of the balls is removed from the box, and it turns out that the sum of the numbers on the remaining balls in the box is 5048. If the number on the ball removed from the box is m, find the value of m.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

9. Suppose a, b, c are real numbers such that $a+b+c=0$ and $abc=-100$. Let $x=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. Which of the following statements is true?

(A) $x>0$ (B) $x=0$ (C) $-1<x<0$ (D) $-100<x<-1$
(E) $x<-100$

10. Let a and b be positive real numbers such that

$$\frac{1}{a}-\frac{1}{b}-\frac{1}{a+b}=0.$$

Find the value of $\left(\frac{b}{a}+\frac{a}{b}\right)^2$.

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

11. Find the value of

$$\frac{2006^2-1994^2}{1600}.$$

12. Find the smallest natural number n which satisfies the inequality

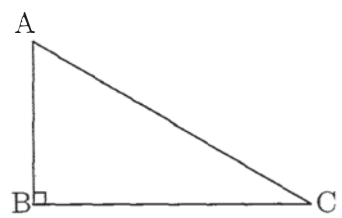
$$2006^{1003}<n^{2006}.$$

13. Find the smallest integer greater than $(1+\sqrt{2})^3$.

14. Find the number of pairs of positive integers (x, y) which satisfy the equation

$$20x+6y=2006.$$

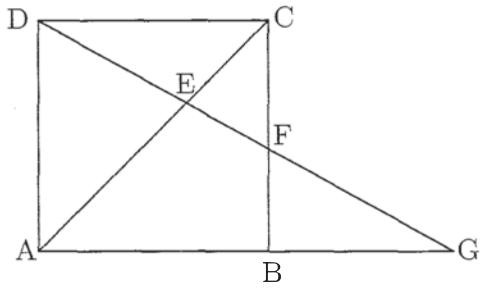
15. $\triangle ABC$ is a right-angled triangle with $\angle ABC=90°$. A circle C_1 is drawn with AB as diameter, and another circle C_2 is drawn with BC as diameter. The circles C_1 and C_2 meet at the points B and P. If $AB=5$ cm, $BC=12$ cm and $BP=x$ cm, find the value of $\frac{2400}{x}$.



16. Evaluate

$$\frac{1}{\log_2 12\sqrt{5}}+\frac{1}{\log_3 12\sqrt{5}}+\frac{1}{\log_4 12\sqrt{5}}+\frac{1}{\log_5 12\sqrt{5}}+\frac{1}{\log_6 12\sqrt{5}}.$$

17. In the diagram below, $ABCD$ is a square. The points A, B and G are collinear. The line segments AC and DG intersect at E, and the line segments DG and BC intersect at F. Suppose that $DE = 15$ cm, $EF = 9$ cm, and $FG = x$ cm. Find the value of x.



18. Find the sum of the coefficients of the polynomial

$$(4x^2 - 4x + 3)^4(4 + 3x - 3x^2)^2.$$

19. Different positive 3-digit integers are formed from the five digits 1, 2, 3, 5, 7, and repetitions of the digits are allowed. As an example, such positive 3-digit integers include 352, 577, 111, etc. Find the sum of all the distinct positive 3-digit integers formed in this way.

20. Find the value of $\dfrac{1}{\sin 10^\circ} - 4\sin 70^\circ$.

21. Let $w = 1 + \sqrt[5]{2} + \sqrt[5]{4} + \sqrt[5]{8} + \sqrt[5]{16}$. Find the value of $(1 + \frac{1}{w})^{30}$.

22. Suppose A and B are two angles such that

$$\sin A + \sin B = 1 \quad \text{and} \quad \cos A + \cos B = 0.$$

Find the value of $12\cos 2A + 4\cos 2B$.

23. Consider the 800-digit integer

$$234523452345\cdots2345.$$

The first m digits and the last n digits of the above integer are crossed out so that the sum of the remaining digits is 2345. Find the value of $m+n$.

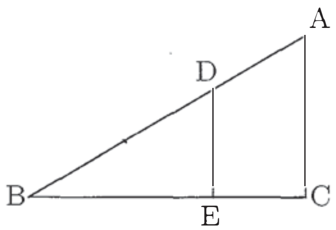
24. Let a and b be two integers. Suppose that $\sqrt{7-4\sqrt{3}}$ is a root of the equation $x^2+ax+b=0$. Find the value of $b-a$.

25. Suppose x and y are integers such that

$$(x-2004)(x-2006)=2^y.$$

Find the largest possible value of $x+y$.

26. In the following diagram, $\angle ACB=90^\circ$, $DE \perp BC$, $BE=AC$, $BD=\frac{1}{2}$ cm, and $DE+BC=1$ cm. Suppose $\angle ABC=x^\circ$. Find the value of x.



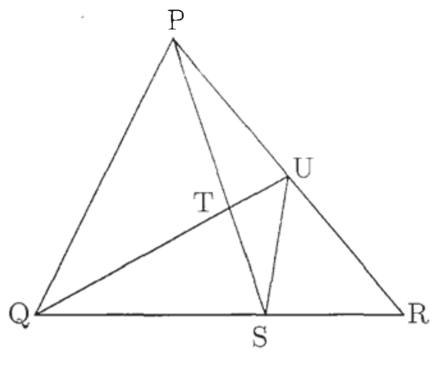
27. If

$$f(x)=\frac{1}{\sqrt[3]{x^2+2x+1}+\sqrt[3]{x^2-1}+\sqrt[3]{x^2-2x+1}}$$

for all positive integers x, find the value of

$$f(1)+f(3)+f(5)+\cdots+f(997)+f(999).$$

28. In the figure below, S is a point on QR and U is a point on PR. The line segments PS and QU intersect at the point T. It is given that $PT=TS$ and $QS=2RS$. If the area of $\triangle PQR$ is 150 cm^2 and the area of $\triangle PSU$ is x cm^2. Find the value of x.



29. Let a and b be two integers. Suppose $x^2 - x - 1$ is a factor of the polynomial $ax^5 + bx^4 + 1$. Find the value of a.

30. If $\sin\theta - \cos\theta = \dfrac{\sqrt{6}-\sqrt{2}}{2}$, find the value of $24(\sin^3\theta - \cos^3\theta)^2$.

31. How many ordered pairs of positive integers (x, y) satisfy the equation

$$x\sqrt{y} + y\sqrt{x} + \sqrt{2006xy} - \sqrt{2006x} - \sqrt{2006y} - 2006 = 0?$$

32. Find the remainder when the integer

$$1 \times 3 \times 5 \times 7 \times \cdots \times 2003 \times 2005$$

is divided by 1000.

33. Let $f : \mathbb{N} \to \mathbb{Q}$ be a function, where $\mathbb{N}$ denotes the set of natural numbers, and $\mathbb{Q}$ denotes the set of rational numbers. Suppose that $f(1) = \dfrac{3}{2}$, and

$$f(x+y) = \left(1 + \frac{y}{x+1}\right) f(x) + \left(1 + \frac{x}{y+1}\right) f(y) + x^2 y + xy + xy^2$$

for all natural numbers x, y. Find the value of $f(20)$.

34. Suppose $x_0, x_1, x_2, \cdots$ is a sequence of numbers such that $x_0 = 1000$, and

$$x_n = -\frac{1000}{n}(x_0 + x_1 + x_2 + \cdots + x_{n-1})$$

for all $n \geq 1$. Find the value of

$$\frac{1}{2^2}x_0 + \frac{1}{2}x_1 + x_2 + 2x_3 + 2^2 x_4 + \cdots + 2^{997}x_{999} + 2^{998}x_{1000}.$$

35. Let p be an integer such that both roots of the equation

$$5x^2 - 5px + (66p - 1) = 0$$

are positive integers. Find the value of p.

Singapore Mathematical Society

Singapore Mathematical Olympiad 2006

(Senior Section Solutions)

1. Ans: D

$$\begin{aligned} p &= 2^{3009} = 2^{3\times 1003} = (2^3)^{1003} = 8^{1003}. \\ q &= 3^{2006} = 3^{2\times 1003} = (3^2)^{1003} = 9^{1003}. \\ r &= 5^{1003}. \end{aligned}$$

Thus, we have $r < p < q$.

2. Ans: D

$$\begin{aligned} 30^{20} &= (30^2)^{10} = 900^{10}. \\ 10^{30} &= (10^3)^{10} = 1000^{10}. \\ 30^{10} + 20^{20} &= 30^{10} + (20^2)^{10} = 30^{10} + 400^{10} < 2 \cdot 400^{10}. \\ &\qquad < (2 \times 400)^{10} = 800^{10}. \\ (30+10)^{20} &= 40^{20} = (40^2)^{10} = 1600^{10}. \\ (30 \times 20)^{10} &= 600^{10}. \end{aligned}$$

Therefore, the largest number is $(30+10)^{20} = 1600^{10}$.

3. Ans: E

First we observe that $2^2 = 4$. Since 20^{20} is divisible by 10, it follows that its last digit is 0. Similarly, the last digit of 200^{200} is 0. The last digit of 2006^{2006} is the same as that of 6^{2006}. Since $6 \times 6 = 36$, it follows that the last digit of any positive integral power of 6 is 6. Thus the last digit of 2006^{2006} is 6. Now, $4+0+0+6 = 10$. Thus the last digit of $2^2 + 20^{20} + 200^{200} + 2006^{2006}$ is 0.

4. Ans: C

$$\begin{aligned} x + \frac{1}{x} = 4 &\Longrightarrow (x + \frac{1}{x})^3 = 4^3 = 64 \\ &\Longrightarrow x^3 + 3x + 3 \cdot \frac{1}{x} + \frac{1}{x^3} = 64 \\ &\Longrightarrow x^3 + \frac{1}{x^3} + 3(x + \frac{1}{x}) = 64 \\ &\Longrightarrow x^3 + \frac{1}{x^3} = 64 - 3(4) = 52. \end{aligned}$$

5. Ans: A

To find the intersection points of the two curves, we solve their equations simultaneously.

$$\begin{cases} y = 2x^3 + 6x + 1, \\ y = -\dfrac{3}{x^2}. \end{cases}$$

Thus, we have

$$\begin{aligned} 2x^3 + 6x + 1 = -\frac{3}{x^2} &\Longrightarrow 2x^5 + 6x^3 + x^2 + 3 = 0 \\ &\Longrightarrow (2x^3+1)(x^2+3) = 0 \\ &\Longrightarrow 2x^3 + 1 = 0, \quad \text{since } x^2 + 3 > 0 \\ &\Longrightarrow x^3 = -\frac{1}{2} \\ &\Longrightarrow x = -\frac{1}{\sqrt[3]{2}} \\ &\Longrightarrow y = -\frac{3}{\sqrt[3]{4}}. \end{aligned}$$

Thus, the only point of intersection is $\left(-\frac{1}{\sqrt[3]{2}}, -\frac{3}{\sqrt[3]{4}}\right)$.

6. Ans: A

Observe that $\triangle ABD \cong \triangle EBD$. Thus, $BE = AB = 4$, $AD = DE$. Hence

$$\text{Area of } \triangle ACE = 2 \times \text{ Area of } \triangle DCE.$$

Since $BC = 3$, $EC = EB - BC = 4 - 3 = 1$. Thus,

$$\begin{aligned} \text{Area of } \triangle ABC &= 3 \times \text{ Area of } \triangle ACE. \\ &= 6 \times \text{ Area of } \triangle DCE. \end{aligned}$$

Thus,

$$\begin{aligned} \text{Area of } ABCD &= \text{Area of } \triangle ABC + \text{ Area of } \triangle ACD \\ &= 6 \times \text{ Area of } \triangle DCE + \text{ Area of } DCE \\ &= 7 \times \text{ Area of } \triangle DCE. \end{aligned}$$

7. Ans: C

Total number of ways is

$$5! - 4! - 4! + 3! = 120 - 24 - 24 + 6 = 78.$$

Alternatively, we consider the following cases:

Case (i): A and D do not run the first or last leg. In this case, the number of arrangements is $3 \times 2 \times 3! = 36$.

Case (ii): D runs the first leg. In this case, number of arrangements is $4! = 24$.

Case (iii): D does not run the first leg and A runs the last leg. In this case, number of ways is $3 \times 3! = 18$.

Therefore, total number of ways $= 36 + 24 + 18 = 78$.

8. Ans: B

Let m be the number of the removed ball. Then we have $1 \leq m \leq n$, and

$$\begin{aligned}
&\Longrightarrow \quad 1 + 2 + 3 + \cdots + n - m = 5048 \\
&\Longrightarrow \frac{n(n+1)}{2} - m = 5048 \\
&\Longrightarrow 1 \leq \frac{n(n+1)}{2} - 5048 \leq n \\
&\Longrightarrow \frac{(n-1)n}{2} \leq 5048 \text{ and } \frac{n(n+1)}{2} \geq 5049.
\end{aligned}$$

Observe that $\frac{99 \times 100}{2} = 4950$, $\frac{100 \times 101}{2} = 5050$ and $\frac{101 \times 102}{2} = 5151$. It follows that $n = 100$. Hence $m = \frac{100 \times 101}{2} - 5048 = 2$.

9. Ans: A

Observe that

$$0 = (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca).$$

Since $abc = -100$, it follows that $a \neq 0$, $b \neq 0$ and $c \neq 0$. Thus, $a^2 + b^2 + c^2 > 0$, and it follows that $ab + bc + ca < 0$. Therefore,

$$x = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab + bc + ca}{abc} = \frac{ab + bc + ca}{-100} > 0.$$

10. Ans: B

Let $x = \dfrac{b}{a}$. Then $b = ax$. Hence we have

$$\begin{aligned}\frac{1}{a} - \frac{1}{b} - \frac{1}{a+b} = 0 &\Longrightarrow \frac{1}{a} - \frac{1}{ax} - \frac{1}{a+ax} = 0\\ &\Longrightarrow \frac{1}{a}\left(1 - \frac{1}{x} - \frac{1}{1+x}\right) = 0\\ &\Longrightarrow x(x+1) - (x+1) - x = 0\\ &\Longrightarrow x^2 - x - 1 = 0\\ &\Longrightarrow x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}.\end{aligned}$$

Since a and b are positive, it follows that $x > 0$, and thus $x = \dfrac{1+\sqrt{5}}{2}$. Then

$$\begin{aligned}\left(\frac{b}{a} + \frac{a}{b}\right)^2 = \left(x + \frac{1}{x}\right)^2 &= \left(\frac{1+\sqrt{5}}{2} + \frac{2}{1+\sqrt{5}}\right)^2\\ &= \left(\frac{1+\sqrt{5}}{2} + \frac{2}{1+\sqrt{5}} \cdot \frac{1-\sqrt{5}}{1-\sqrt{5}}\right)^2\\ &= \left(\frac{1+\sqrt{5}}{2} + \frac{2-2\sqrt{5}}{-4}\right)^2\\ &= \left(\frac{1+\sqrt{5}}{2} - \frac{1}{2} + \frac{\sqrt{5}}{2}\right)^2 = (\sqrt{5})^2 = 5.\end{aligned}$$

11. Ans: 30

$$\begin{aligned}\frac{2006^2 - 1994^2}{1600} &= \frac{(2006+1994) \times (2006-1994)}{1600}\\ &= \frac{4000 \times 12}{1600}\\ &= 30.\end{aligned}$$

12. Ans: 45

$$2006^{1003} < n^{2006} \Longleftrightarrow 2006 < n^2.$$

Since $44^2 = 1936 < 2006$ and $45^2 = 2025 > 2006$, it follows that the smallest natural number n satisfying $2006^{1003} < n^{2006}$ is 45.

13. Ans: 15

$$(1+\sqrt{2})^3 = 1 + 3\sqrt{2} + 3 \times 2 + 2\sqrt{2} = 7 + 5\sqrt{2}.$$

Observe that $1.4 < \sqrt{2} < 1.5$. Thus

$$\begin{aligned} & 7 + 5 \times 1.4 < 7 + 5\sqrt{2} < 7 + 5 \times 1.5 \\ \Longrightarrow \quad & 14 < 7 + 5\sqrt{2} < 14.5. \end{aligned}$$

Therefore, the smallest integer greater than $(1+\sqrt{2})^3$ is 15.

14. Ans: 34

$$20x + 6y = 2006 \Longleftrightarrow 10x + 3y = 1003.$$

The solutions are

$$(1, 331), (4, 321), (7, 311), \cdots, (94, 21), (97, 11), (100, 1).$$

Thus there are 34 solutions.

15. Ans: 520

Drop the perpendicular from B to AC meeting AC at Q. Then $\angle AQB = 90°$, and thus Q lies on C_1. Similarly, Q lies on C_2. Thus, $Q = P$. Now, $\triangle ABP \sim \triangle ACB$. Thus

$$\frac{BP}{AB} = \frac{BC}{AC} \Longrightarrow \frac{x}{5} = \frac{12}{\sqrt{12^2 + 5^2}} \Longrightarrow x = \frac{60}{13}.$$

Therefore,

$$\frac{2400}{x} = 2400 \times \frac{13}{60} = 520.$$

16. Ans: 2

$$\begin{aligned} & \frac{1}{\log_2 12\sqrt{5}} + \frac{1}{\log_3 12\sqrt{5}} + \frac{1}{\log_4 12\sqrt{5}} + \frac{1}{\log_5 12\sqrt{5}} + \frac{1}{\log_6 12\sqrt{5}} \\ &= \log_{12\sqrt{5}} 2 + \log_{12\sqrt{5}} 3 + \log_{12\sqrt{5}} 4 + \log_{12\sqrt{5}} 5 + \log_{12\sqrt{5}} 6 \\ &= \log_{12\sqrt{5}} 2 \times 3 \times 4 \times 5 \times 6 \\ &= \log_{12\sqrt{5}} 720 \\ &= 2, \text{ since } (12\sqrt{5})^2 = 720. \end{aligned}$$

17. Ans: 16

Let $AB = y$ cm. Since $\triangle DCF \sim \triangle GBF$, we have

$$\frac{DC}{DF} = \frac{GB}{GF} \Longrightarrow \frac{y}{24} = \frac{GB}{x} \Longrightarrow GB = \frac{xy}{24}.$$

Since $\triangle DCE \sim \triangle GAE$, we have

$$\frac{DC}{DE} = \frac{GA}{GE} \Longrightarrow \frac{y}{15} = \frac{y+GB}{9+x} \Longrightarrow GB = \frac{y(9+x)}{15} - y.$$

Hence we have

$$\frac{xy}{24} = \frac{y(9+x)}{15} - y \Longrightarrow \frac{x}{24} = \frac{9+x}{15} - 1 \Longrightarrow x = 16.$$

18. Ans: 1296

Let

$$(4x^2 - 4x + 3)^4(4 + 3x - 3x^2)^2 = A_{12}x^{12} + \cdots + A_1 x + A_0.$$

Substitute $x = 1$ to obtain

$$\begin{aligned} A_{12} + A_{11} + \cdots + A_1 + A_0 &= (0 - 0 + 3)^4(4 + 0 - 0)^2 \\ &= 1296. \end{aligned}$$

19. Ans: 49950

Number of three-digit integers formed $= 5^3 = 125$. Observe that each of the five digits $1, 2, 3, 5, 7$ appears 25 times in the first, second and third digits of the integers formed. Thus,

$$\begin{aligned} \text{sum} &= 25 \times (1+2+3+5+7) \times 100 \\ &\quad + 25 \times (1+2+3+5+7) \times 10 \\ &\quad + 25 \times (1+2+3+5+7) \\ &= 49950. \end{aligned}$$

20. Ans: 2

$$\begin{aligned} \frac{1}{\sin 10^\circ} - 4\sin 70^\circ &= \frac{1 - 4\sin 70^\circ \sin 10^\circ}{\sin 10^\circ} \\ &= \frac{1 - 2(\cos 60^\circ - \cos 80^\circ)}{\sin 10^\circ} \\ &= \frac{1 - 2(\frac{1}{2} - \cos 80^\circ)}{\sin 10^\circ} \\ &= \frac{2\cos 80^\circ}{\sin 10^\circ} \\ &= \frac{2\sin 10^\circ}{\sin 10^\circ} \\ &= 2. \end{aligned}$$

Here we have used the formula:

$$\sin A \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}.$$

21. Ans: 64

Let $y = \sqrt[5]{2}$. Then $y^5 = 2$. Therefore,

$$\begin{aligned}
\left(1+\frac{1}{w}\right)^{30} &= \left(1+\frac{1}{1+y+y^2+y^3+y^4}\right)^{30} \\
&= \left(1+\frac{y-1}{y^5-1}\right)^{30} \\
&= \left(1+\frac{y-1}{2-1}\right)^{30} \\
&= y^{30} \\
&= (y^5)^6 = 2^6 = 64.
\end{aligned}$$

22. Ans: 8

$$\sin A + \sin B = 1 \Longrightarrow \sin A = 1 - \sin B.$$
$$\cos A + \cos B = 0 \Longrightarrow \cos A = -\cos B.$$

Thus,

$$\begin{aligned}
\cos^2 A + \sin^2 A = 1 &\Longrightarrow (-\cos B)^2 + (1-\sin B)^2 = 1 \\
&\Longrightarrow \cos^2 B + 1 - 2\sin B + \sin^2 B = 1 \\
&\Longrightarrow \sin B = \frac{1}{2}.
\end{aligned}$$

Thus, $\sin A = 1 - \frac{1}{2} = \frac{1}{2}$. Therefore,

$$\begin{aligned}
12\cos 2A + 4\cos 2B &= 12(1-2\sin^2 A) + 4(1-2\sin^2 B) \\
&= 12\left(1-2\left(\frac{1}{2}\right)^2\right) + 4\left(1-2\left(\frac{1}{2}\right)^2\right) \\
&= 8.
\end{aligned}$$

23. Ans: 130

Note that $2+3+4+5=14$. Thus the sum of the 800 digits is $200 \times 14 = 2800$. Thus we need to cross out digits with a sum equal to $2800 - 2345 = 455$.

Observe that $455 = 32 \times 14 + 7$. Thus we have to cross out 32 blocks of four digits '2345' either from the front or the back, a '2' from the front that remains and a '5' from the back that remains. Thus, $m + n = 32 \times 4 + 2 = 130$.

24. Ans: 5

First we simplify $\sqrt{7-4\sqrt{3}}$, and write

$$\sqrt{7-4\sqrt{3}} = x + y\sqrt{3},$$

where x, y are rational numbers. Then

$$7 - 4\sqrt{3} = (x+y\sqrt{3})^2 = x^2 + 3y^2 + 2xy\sqrt{3}.$$

Thus, we have

$$\begin{cases} x^2+3y^2 &= 7, \\ 2xy &= \text{-4}. \end{cases}$$

Substituting the second equation into the first one, we get

$$\begin{aligned} x^2 + 3\left(-\frac{2}{x}\right)^2 = 7 &\Longrightarrow x^4 - 7x^2 + 12 = 0 \\ &\Longrightarrow (x^2-4)(x^2-3) = 0 \\ &\Longrightarrow x = 2, -2, \sqrt{3}, -\sqrt{3}. \end{aligned}$$

Since x is a rational number, $x \neq \sqrt{3}$ and $x \neq -\sqrt{3}$. When $x = 2$, $y = \dfrac{-4}{2 \times 2} = -1$. When $x = -2$, $y = 1$. Thus, $\sqrt{7-4\sqrt{3}} = 2 - \sqrt{3}$ or $-2 + \sqrt{3}$. Since $\sqrt{7-4\sqrt{3}} > 0$, it follows that $\sqrt{7-4\sqrt{3}} = 2 - \sqrt{3}$. Thus the 2nd root of the equation is $2+\sqrt{3}$, since the coefficients of the equation are integers. Therefore, $a = -(2-\sqrt{3}) - (2+\sqrt{3}) = -4$, $b = (2-\sqrt{3})(2+\sqrt{3}) = 1$. Thus, $b - a = 5$.

25. Ans: 2011

$$\begin{aligned} (x-2004)(x-2006) = 2^y &\Longrightarrow (x-2005+1)(x-2005-1) = 2^y \\ &\Longrightarrow (x-2005)^2 - 1 = 2^y \\ &\Longrightarrow (x-2005)^2 = 1 + 2^y. \end{aligned}$$

Write $n = |x - 2005|$. Then we have

$$n^2 = 1 + 2^y \Longrightarrow 2^y = (n-1)(n+1).$$

Observe that the above equality implies easily that $n \neq 0, 1$. Thus, $n-1$ and $n+1$ are positive integers, and both $n-1$ and $n+1$ are powers of 2. Write $n-1 = 2^a$. Then

$$n+1 = 2^a + 2 = 2(2^{a-1}+1).$$

Since $n \geq 2$, it follows that $n+1 \geq 3$. Moreover, since $n+1$ is a power of 2, it follows that $n+1 \geq 4$ and thus $\frac{n+1}{2} \geq 2$. Hence $2^{a-1}+1$ is a positive integer and it is also a power of 2. Clearly, $2^{a-1}+1 > 1$. Thus $2^{a-1}+1 \geq 2 \Longrightarrow a \geq 1$. If $a > 1$, then $2^{a-1}+1 > 2$ and it is odd, which is not possible. Thus we have $a = 1$. Thus,

$$n = 1 + 2 = 3.$$

Then

$$2^y = (3-1)(3+1) = 8 \Longrightarrow y = 3.$$

It follows that

$$\begin{aligned}(x-2005)^2 = 1 + 2^3 = 9 = 3^2 &\Longrightarrow x = 2005 \pm 3\\ &\Longrightarrow x = 2008 \text{ or } 2002.\end{aligned}$$

Thus the largest possible value of $x+y$ is $2008+3 = 2011$ when $x = 2008$.

26. Ans: 30

Produce BC to F such that $CF = DE$. Then $\triangle BDE$ and $\triangle ACF$ are right-angled triangles. Note that $BE = AC$ and $DE = CF$. Thus, we have

$$\triangle BDE \cong \triangle ACF.$$

Therefore, $AF = BD = \frac{1}{2}$, and $\angle FAC = \angle ABC$. Since $\angle BAC + \angle ABC = 90°$, we have

$$\angle BAC + \angle FAC = 90°.$$

Hence, $\angle BAF = 90°$. Now in $\triangle BAF$, $AF = \frac{1}{2}$, and

$$BF = BC + CF = BC + DE = 1.$$

Hence,

$$\sin x° = \frac{AF}{BF} = \frac{1}{2} \Longrightarrow x = 30.$$

Alternative solution: Let $BE = a$ and $DE = b$. Since $\angle BED = 90°$, it follows from Pythagoras Theorem that

$$a^2 + b^2 = \frac{1}{4}.$$

Since $DE + BC = 1$, we have $BC = 1 - b$. As $\triangle BDE \sim \triangle ACF$, we have

$$\frac{b}{a} = \frac{a}{1-b} \Longrightarrow a^2 = b(1-b).$$

Together with the previous equality, we have

$$b(1-b) + b^2 = \frac{1}{4} \Longrightarrow b = \frac{1}{4}$$

Hence,

$$\sin x^\circ = \frac{DE}{BD} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} \Longrightarrow x = 30.$$

27. Ans: 5

By the identity $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$, we have

$$\frac{1}{a^2 + ab + b^2} = \frac{a-b}{a^3 - b^3} \text{ if } a \neq b.$$

Let $a = \sqrt[3]{x+1}$ and $b = \sqrt[3]{x-1}$. Then

$$\begin{aligned} f(x) &= \frac{1}{(\sqrt[3]{x+1})^2 + (\sqrt[3]{x+1})(\sqrt[3]{x-1}) + (\sqrt[3]{x-1})^2} \\ &= \frac{\sqrt[3]{x+1} - \sqrt[3]{x-1}}{(x+1)-(x-1)} \\ &= \frac{1}{2}(\sqrt[3]{x+1} - \sqrt[3]{x-1}). \end{aligned}$$

Therefore,

$$\begin{aligned} &f(1) + f(3) + \cdots + f(997) + f(999) \\ =&\frac{1}{2}(\sqrt[3]{2} - 0 + \sqrt[3]{4} - \sqrt[3]{2} + \cdots + \sqrt[3]{998} - \sqrt[3]{996} + \sqrt[3]{1000} - \sqrt[3]{998}) \\ =&\frac{1}{2} \times \sqrt[3]{1000} = \frac{10}{2} = 5. \end{aligned}$$

28. Ans: 20

Suppose that the area of $\triangle PQT$ is t cm^2. Since $PT = TS$, we see that

$$\text{area of } \triangle QTS = \text{area of } \triangle PQT = t \text{ cm}^2. \quad (1)$$

Suppose the area of $\triangle PTU$ is y cm^2. Then a similar argument shows that

$$\text{area of } \triangle STU = \text{area of } \triangle PTU = y \text{ cm}^2. \quad (2)$$

Given also that $QS = 2RS$, we have

$$\text{area of } \triangle SRU = \frac{1}{2} \times \text{ area of } \triangle QSU = \frac{1}{2}(t+y). \tag{3}$$

Likewise, we have

$$\text{area of } \triangle PQS = 2 \times \text{ area of } \triangle PRS,$$

that is,

$$2t = 2\left(2y + \frac{t+y}{2}\right)$$

or

$$t = 5y. \tag{4}$$

Also,

$$\begin{aligned}\text{the total area of } \triangle PQR &= \left(2t + 2y + \frac{t+y}{2}\right) \text{ cm}^2 \\ &= 150 \text{ cm}^2.\end{aligned}$$

Thus we have

$$t + y = 60. \tag{5}$$

From (4) and (5), we obtain $y = 10$. So the area of $\triangle PSU$ is 20 cm^2.

29. Ans: 3

Let p and q be the roots of $x^2 - x - 1 = 0$. Then $p + q = 1$ and $pq = -1$. On the other hand, p and q are also roots of $ax^5 + bx^4 + 1 = 0$. Thus $ap^5 + bp^4 + 1 = 0$ and $aq^5 + bq^4 + 1 = 0$. From these two equations and $pq = -1$, we have $ap + b = -q^4$ and $aq + b = -p^4$. Therefore

$$\begin{aligned} a = \frac{p^4 - q^4}{p - q} &= (p^2 + q^2)(p + q) \\ &= ((p+q)^2 - 2pq)(p+q) \\ &= (1^2 - 2(-1)) \times 1 \\ &= 3. \end{aligned}$$

Alternative solution: Since $x^2 - x - 1$ is a factor of $ax^5 + bx + 1$, we may write

$$ax^5 + bx^4 + 1 \equiv (x^2 - x - 1)(ax^3 + cx^2 + dx - 1)$$

for some real numbers c and d. By comparing the coefficients, we have

$$\begin{aligned} x: &\quad 0 = 1 - d \Longrightarrow d = 1. \\ x^2: &\quad 0 = -1 - d - c \Longrightarrow c = -2 \\ x^3: &\quad 0 = -a - c + d \Longrightarrow a = -c + d = 2 + 1 = 3. \end{aligned}$$

30. Ans: 12

$$\begin{aligned}\sin^3\theta-\cos^3\theta &= (\sin\theta-\cos\theta)(\sin^2\theta+\sin\theta\cos\theta+\cos^2\theta)\\ &= \frac{\sqrt{6}-\sqrt{2}}{2}(1+\sin\theta\cos\theta).\end{aligned}$$

Now we have

$$\begin{aligned}(\sin\theta-\cos\theta)^2=\left(\frac{\sqrt{6}-\sqrt{2}}{2}\right)^2 &\Longrightarrow 1-2\sin\theta\cos\theta=\frac{6+2-2\sqrt{12}}{4}=2-\sqrt{3}\\ &\Longrightarrow \sin\theta\cos\theta=\frac{\sqrt{3}-1}{2}.\end{aligned}$$

Hence we have

$$\begin{aligned}\sin^3\theta-\cos^3\theta &= \frac{\sqrt{6}-\sqrt{2}}{2}\left(1+\frac{\sqrt{3}-1}{2}\right)\\ &= \frac{\sqrt{2}(\sqrt{3}-1)}{2}\left(\frac{\sqrt{3}+1}{2}\right)\\ &= \frac{\sqrt{2}(3-1)}{4}=\frac{\sqrt{2}}{2}.\end{aligned}$$

Therefore,

$$24(\sin^3\theta-\cos^3\theta)^2=24\left(\frac{\sqrt{2}}{2}\right)^2=24\times\frac{1}{2}=12.$$

31. Ans: 8

$$\begin{aligned}&x\sqrt{y}+y\sqrt{x}+\sqrt{2006xy}-\sqrt{2006x}-\sqrt{2006y}-2006=0\\ \Longleftrightarrow&(\sqrt{x}+\sqrt{y}+\sqrt{2006})(\sqrt{xy}-\sqrt{2006})=0\\ \Longleftrightarrow&\sqrt{xy}-\sqrt{2006}=0\quad(\text{since }\ \sqrt{x}+\sqrt{y}+\sqrt{2006}>0)\\ \Longleftrightarrow&xy=2006=17\times59\times2.\end{aligned}$$

Thus the solutions are

$$(1,2006),(2006,1),(2,1003),(1003,2),(34,59),(59,34),(17,118),(118,17).$$

32. Ans: 375

Let $N=1\times3\times\cdots\times2005$. We need to find the remainder when N is divided by 1000. Observe that $1000=8\times125$. Let $M=\frac{N}{125}$. We are going to find the remainder when M is divided by 8. Observe that

$$\begin{aligned}(2n-3)(2n-1)(2n+1)(2n+3) &= (4n^2-1)(4n^2-9)\\ &= 16n^4-40n^2+9\\ &\equiv 1 \pmod 8.\end{aligned}$$

Thus the product P of any 4 consecutive odd integers satisfies $P \equiv 1 \pmod 8$. Write

$$M = (1 \times 3 \times \cdots \times 123) \times (127 \times 129 \times \cdots \times 2005).$$

There are 62 factors in the expression $1 \times 3 \times \cdots \times 123$. Thus,

$$\begin{aligned} 1 \times 3 \times 5 \times \cdots \times 123 &= 1 \times 3 \times (5 \times \cdots \times 123) \\ &\equiv 3 \times 1^{15} \pmod 8 \\ &\equiv 3 \pmod 8. \end{aligned}$$

Similarly, there are 940 factors in the expression $127 \times \cdots \times 2005$. Thus,

$$\begin{aligned} 127 \times 129 \times \cdots \times 2005 &\equiv 1^{235} \pmod 8 \\ &\equiv 1 \pmod 8. \end{aligned}$$

Hence,

$$M \equiv 1 \times 3 \equiv 3 \pmod 8.$$

In other words,

$$M = 8n + 3$$

for some positive integer n. Now, we have

$$\begin{aligned} N &= 125 \times M \\ &= 125 \times (8n + 3) \\ &= 1000n + 375. \end{aligned}$$

Therefore, the remainder is 375.

33. Ans: 4305

Letting $y = 1$, one gets

$$f(x+1) = \left(1 + \frac{1}{x+1}\right) f(x) + \left(1 + \frac{x}{2}\right) \frac{3}{2} + x^2 + 2x.$$

Upon rearranging, one gets

$$\frac{f(x+1)}{x+2} - \frac{f(x)}{x+1} = x + \frac{3}{4}.$$

Then we have

$$\begin{aligned} \frac{f(n)}{n+1} - \frac{f(n-1)}{n} &= n - 1 + \frac{3}{4}, \\ \frac{f(n-1)}{n} - \frac{f(n-2)}{n-1} &= n - 2 + \frac{3}{4}, \\ &\vdots \\ \frac{f(2)}{3} - \frac{f(1)}{2} &= 1 + \frac{3}{4}. \end{aligned}$$

Adding these equalities together, we get

$$\begin{aligned}\frac{f(n)}{n+1}-\frac{f(1)}{2}&=1+2+\cdots+(n-1)+\frac{3}{4}(n-1)\\&=\frac{(n-1)n}{2}+\frac{3}{4}(n-1).\end{aligned}$$

Thus,

$$f(n)=(n+1)\left[\frac{(n-1)n}{2}+\frac{3}{4}(n-1)+\frac{1}{2}\cdot\frac{3}{2}\right]=\frac{n(n+1)(2n+1)}{4}.$$

Hence,

$$f(20)=\frac{(20)(21)(41)}{4}=4305.$$

34. Ans: 250

Observe that when $n \geq 2$,

$$\begin{cases} nx_n &= -1000(x_0+x_1+\cdots+x_{n-1}),\\ (n-1)x_{n-1} &= -1000(x_0+x_1+\cdots+x_{n-2}).\end{cases}$$

Thus, we have

$$nx_n-(n-1)x_{n-1}=-1000x_{n-1}\Longrightarrow x_n=-\left(\frac{1000-(n-1)}{n}\right)x_{n-1}.$$

It is easy to check that the above formula holds even when $n=1$. Therefore, for $1\leq n\leq 100$, we have

$$\begin{aligned}x_n&=-\frac{1000-(n-1)}{n}x_{n-1}\\&=(-1)^2\frac{1000-(n-1)}{n}\cdot\frac{1000-(n-2)}{n-1}x_{n-2}\\&=\cdots\\&=(-1)^n\frac{1000-(n-1)}{n}\cdot\frac{1000-(n-2)}{n-1}\cdots\frac{1000}{1}x_0\\&=(-1)^n\binom{1000}{n}x_0.\end{aligned}$$

Hence we have

$$
\begin{aligned}
& \frac{1}{2^2}x_0+\frac{1}{2}x_1+x_2+\cdots+2^{998}x_{1000} \\
= & \frac{1}{4}(x_0+2x_1+2^2x_2+\cdots+2^{1000}x_{1000}) \\
= & \frac{1}{4}\left(x_0-\binom{1000}{1}2x_0+\binom{1000}{2}2^2x_0-\binom{1000}{3}2^3x_0+\cdots+\binom{1000}{1000}2^{1000}x_0\right) \\
= & \frac{x_0}{4}\left(1-\binom{1000}{1}2+\binom{1000}{2}2^2-\binom{1000}{2}2^3+\cdots+\binom{1000}{1000}2^{1000}\right) \\
= & \frac{1000}{4}(1-2)^{1000} \\
= & 250.
\end{aligned}
$$

35. Ans: 76

Let u, v be the two positive integral solutions of the given equation. Then

$$
\begin{cases} u+v=p, \\ uv=\dfrac{66p-1}{5}. \end{cases}
$$

Upon eliminating p, we have

$$
\begin{aligned}
5uv=66(u+v)-1 &\Longrightarrow v(5u-66)=66u-1>0 \\
&\Longrightarrow 5u-66>0, \quad \text{since } v>0.
\end{aligned}
$$

Similarly, we have $5v-66>0$. Moreover, we have

$$
\begin{aligned}
(5u-66)(5v-66) &= 25uv-330(u+v)+66^2 \\
&= 25\times\left(\frac{66p-1}{5}\right)-330p+66^2 \\
&= -5+66^2=4351=19\times 229.
\end{aligned}
$$

Without loss of generality, we may assume that $u \geq v$. Since both 19 and 229 are prime, we must have

$$
\begin{cases} 5u-66 &= 229 \\ 5v-66 &= 19 \end{cases} \quad \text{or} \quad \begin{cases} 5u-66 &= 4351 \\ 5v-16 &= 1. \end{cases}
$$

The first set of equations imply $u=59$, $v=17$. The second set of equations does not have integral solutions. Hence, we must have $u=59$ and $v=17$. Thus, $p=u+v=76$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2006

(Senior Section, Special Round)

Saturday, 24 June 2006 — 0930– 1230

Important:

Attempt as many questions as you can.

No calculators are allowed.

Show all the steps in your working.

Each question carries 10 marks.

1. Let a, d be integers such that $a, a + d, a + 2d$ are all prime numbers larger than 3. Prove that d is a multiple of 6.

2. Let $ABCD$ be a cyclic quadrilateral, let the angle bisectors at A and B meet at E, and let the line through E parallel to side CD intersect AD at L and BC at M. Prove that $LA + MB = LM$.

3. Two circles are tangent to each other internally at a point T. Let the chord AB of the larger circle be tangent to the smaller circle at a point P. Prove that the line TP bisects $\angle ATB$.

4. You have a large number of congruent equilateral triangular tiles on a table and you want to fit n of them together to make a convex equiangular hexagon (i.e., one whose interior angles are $120°$). Obviously, n cannot be any positive integer. The first three feasible n are 6, 10 and 13. Determine if 19 and 20 are feasible.

5. It is claimed that the number

$$N = 526315789473684210$$

is a *persistent* number, that is, if multiplied by any positive integer the resulting number always contains the ten digits $0, 1, \ldots, 9$ in some order with possible repetitions.

(a) Prove or disprove the claim.

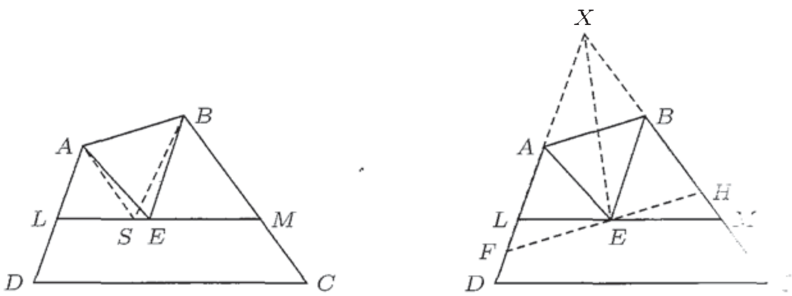
(b) Are there any persistent numbers smaller than the above number?

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2006
(Senior Section, Special Round Solutions)

1. First note that d is even. Primes larger than 3 are of the form $6n+1$ or $6n+5$. Thus two of the three primes are of the same form. Their difference is either d or $2d$ and is divisible by 6. Thus d is divisible by 3 is hence divisible by 6.

2. *First solution*: Assume that $x > y$ (See Fig.(i)). Choose a point S on the segment LM so that $LS = LA$. Clearly $\angle ASL = \angle LAS = y$. Therefore, $ASEB$ is cyclic. As $\angle LAS = y < x = \angle LAE$, it follows that S is between L and E.

On the other hand, $\angle SBM = \angle SBE + \angle EBM = \angle SAE + \angle EBM = \angle LAE - \angle LAS + y = x - y + y = x = \angle BAM$. Hence, MBS is isosceles and $MS = MB$. Therefore, $LM = LS + SM = LA + MB$.



(i) (ii)

Second solution: Produce DA and CB to meet at X (See Fig. (ii)). Draw FH parallel to AB. Draw XE with F on AD and H on BC. Then E is an excentre of $\triangle XAB$ and so XE bisects $\angle AXB$. $\triangle MLX \equiv \triangle HFX$ since they have equal angles and a common angle bisector. Hence $ME = FE$, $HE = LE$ and $HM = LF$. Since FH is parallel to AB, and AE bisects $\angle DAB$, $\angle FAE = \angle AEF$. Thus $\triangle FAE$ is isosceles, and $AF = FE$. Similarly, $BH = EH$. The rest follows easily.

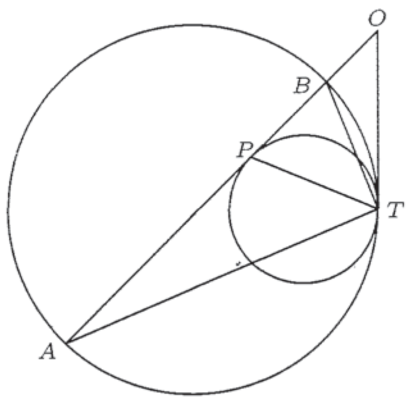
Third solution: Note that

$$\sin 2x + \sin 2y = 2\sin(2y - x)\cos x + 2\sin(2x - y)\cos y.$$

In the quadrilateral $ABML$, let $2x = \angle A$, $2y = \angle B$. Therefore $\angle L = 180^\circ - 2y$ and $\angle M = 180^\circ - 2x$. Thus $\angle LEA = 2y - x$, $\angle BEM = 2x - y$. If d is the common distance from E to AL, AB and BM, then the result follows from

$$\begin{aligned}\frac{LE+EM-LA-MB}{d} &= \frac{1}{\sin 2y}+\frac{1}{\sin 2x}-\frac{\sin(2y-x)}{\sin x\sin 2y}-\frac{\sin(2x-y)}{\sin y\sin 2x}\\ &= \frac{\sin 2x+\sin 2y}{\sin 2x\sin 2y}-2\frac{\sin(2y-x)\cos x+\sin(2x-y)\cos y}{\sin 2x\sin 2y}\\ &= 0\end{aligned}$$

3. Let the tangent at T meet the extension of the chord AB at O. Then $\angle BTO = \angle TAB$. Thus $\triangle OAT$ is similar to $\triangle OTB$ so that $\frac{TA}{TB} = \frac{OT}{OB}$. Since $OT = OP$, we have $\frac{TA}{TB} = \frac{OP}{OB}$. On the other hand, $OP^2 = OA \cdot OB$.



Therefore $\frac{TA}{TB} = \frac{OP}{OB} = \frac{OA}{OP}$. Thus $\frac{TA}{TB} = \frac{OA-OP}{OP-OB} = \frac{AP}{BP}$. Using the angle bisector theorem, we see that TP bisects $\angle ATB$.

4. See Junior Section Special Round Problem 5. You just need to check further the cases for $\ell = 5, 6$.

5. The fact is there are no persistent numbers. For any positive integer N, consider the remainder when following N numbers are divisible by N:

$$1, 11, 111, \ldots, \underbrace{11\ldots1}_{N}$$

If one of the remainders is 0, N is not persistent. If not, then two of the remainders are the same. Thus there exist two, say A, B such that $A - B = 11\ldots100\ldots0$ is divisible by N, again N is not persistent.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2006

(Open Section, Round 1)

Wednesday, 31 May 2006 0930-1200

Important:

Answer ALL 25 questions.

Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

1. How many integers are there between 0 and 10^5 having the digit sum equal to 8?

2. Given that p and q are integers that satisfy the equation $36x^2 - 4(p^2 + 11)x + 135(p + q) + 576 = 0$, find the value of $p + q$.

3. A function f is such that $f : \mathbb{R} \to \mathbb{R}$ where $f(xy+1) = f(x)f(y) - f(y) - x + 2$ for all $x, y \in \mathbb{R}$. Find $10f(2006) + f(0)$.

4. Three people A, B and C play a game of passing a basketball from one to another. Find the number of ways of passing the ball starting with A and reaching A again on the 11th pass. For example, one possible sequence of passing is

$$A \to B \to A \to B \to C \to A \to B \to C \to B \to C \to B \to A.$$

5. There are 10! permutations $s_0 s_1 \dots s_9$ of $0, 1, \dots, 9$. How many of them satisfy $s_k \geq k - 2$ for $k = 0, 1, \dots, 9$?

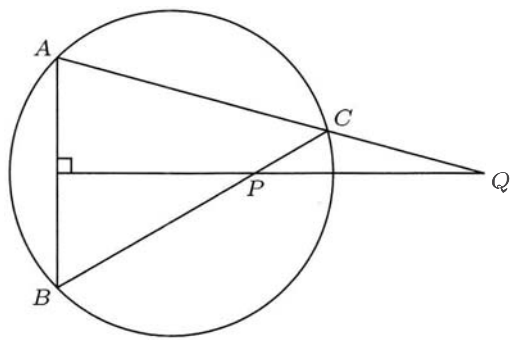
6. A triangle $\triangle ABC$ has its vertices lying on a circle $\mathbb{C}$ of radius 1, with $\angle BAC = 60°$. A circle with center I is inscribed in $\triangle ABC$. The line AI meets circle $\mathbb{C}$ again at D. Find the length of the segment ID.

7. Find the number of consecutive 0's at the end of the base 10 representation of 2006!.

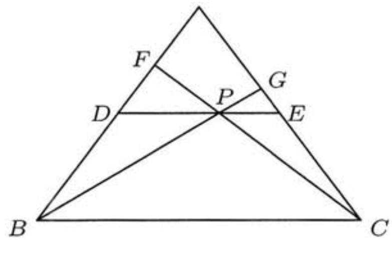
8. For any non-empty finite set A of real numbers, let $s(A)$ be the sum of the elements in A. There are exactly 61 3-element subsets A of $\{1, \ldots, 23\}$ with $s(A) = 36$. Find the number of 3-element subsets of $\{1, \ldots, 23\}$ with $s(A) < 36$.

9. Suppose f is a function satisfying $f(x + x^{-1}) = x^6 + x^{-6}$, for all $x \neq 0$. Determine $f(3)$.

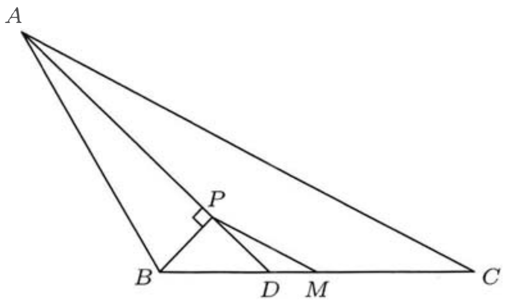
10. Points A, B, C lie on a circle centered at O with radius 7. The perpendicular bisector of AB meets the segment BC at P and the extension of AC at Q. Determine the value of $OP \cdot OQ$.



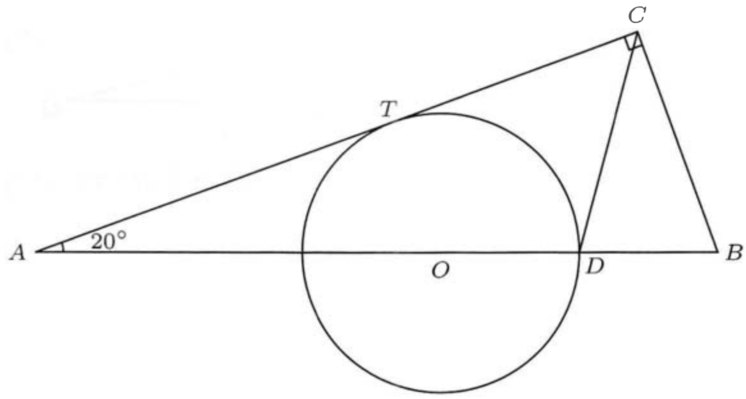
11. In the triangle ABC, $AB = AC = 1$, D and E are the midpoints of AB and AC respectively. Let P be a point on DE and let the extensions of BP and CP meet the sides AC and AB at G and F respectively. Find the value of $\frac{1}{BF} + \frac{1}{CG}$.



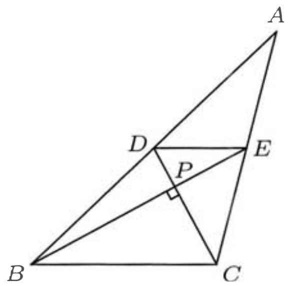
12. In the triangle ABC, $AB = 14, BC = 16, AC = 26$, M is the midpoint of BC and D is the point on BC such that AD bisects $\angle BAC$. Let P be the foot of the perpendicular from B onto AD. Determine the length of PM.



13. In the triangle ABC, $\angle A = 20°$, $\angle C = 90°$, O is a point on AB and D is the midpoint of OB. Suppose the circle centered at O with radius OD touches the side AC at T. Determine the size of $\angle BCD$ in degrees.



14. In $\triangle ABC$, D and E are the midpoints of the sides AB and AC respectively, CD and BE intersect at P with $\angle BPC = 90°$. Suppose $BD = 1829$ and $CE = 1298$. Find BC.



15. Let $X = \{1, 2, 3, \ldots, 17\}$. Find the number of subsets Y of X with odd cardinalities.

16. Find the value of $400(\cos^5 15° + \sin^5 15°) \div (\cos 15° + \sin 15°)$.

17. Find the number of real solutions of the equation

$$x^2 + \frac{1}{x^2} = 2006 + \frac{1}{2006}.$$

18. Find the largest integer n such that n is a divisor of $a^5 - a$ for all integers a.

19. Given two sets $A = \{1, 2, 3, \ldots, 15\}$ and $B = \{0, 1\}$, find the number of mappings $f : A \to B$ with 1 being the image of at least two elements of A.

20. Let $a_1, a_2, \ldots$ be a sequence satisfying the condition that $a_1 = 1$ and $a_n = 10a_{n-1} - 1$ for all $n \geq 2$. Find the minimum n such that $a_n > 10^{100}$.

21. Let P be a 30-sided polygon inscribed in a circle. Find the number of triangles whose vertices are the vertices of P such that any two vertices of each triangle are separated by at least three other vertices of P.

22. A year is called a leap year if it is either divisible by 4 but not divisible by 100, or divisible by 400. Hence, the years 2000, 2004 and 2400 are leap years while the years 2006, 2100 and 2200 are not. Find the number of leap years between 2000 and 4000 inclusive.

23. The birth date of Albert Einstein is 14 March 1879. If we denote Monday by 1, Tuesday by 2, Wednesday by 3, Thursday by 4, Friday by 5, Saturday by 6 and Sunday by 7, which day of the week was Albert Einstein born? Give your answer as an integer from 1 to 7.

24. Find the number of 7-digit integers formed by some or all of the five digits, namely, 0, 1, 2, 3, and 4, such that these integers contain none of the three blocks 22, 33 and 44.

25. Let

$$S = \sum_{r=0}^{n} \binom{3n + r}{r}.$$

Evaluate $S \div (23 \times 38 \times 41 \times 43 \times 47)$ when $n = 12$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2006

(Open Section, Round 1 Solutions)

1. Ans: 495

Each integer can be written as $\overline{x_1x_2x_3x_4x_5}$ where each $x_t = 0, 1, 2, \ldots, 9$ with $x_1 + x_2 + x_3 + x_4 + x_5 = 8$. The number of non-negative integer solutions to the above equation is 495. So there are 495 such integers.

2. Ans: 20

$$p + q = \frac{p^2 + 11}{9} \text{ and } pq = \frac{135(p + q) + 576}{36} = \frac{15(p + q)}{4} + 16.$$

So $p + q > 0$ and is a multiple of 4. Also $p + q = 1 + \frac{p^2+2}{9}$. So $p^2 + 2$ is a multiple of 9. So $p = 5, 13, 14, \ldots$. If $p = 5$, $p + q = 4$ and $pq = 31$. If $p = 13$, $p + q = 20$, $pq = 91$, $q = 7$. Thus $p + q = 20$.

3. Ans: 20071

$f(xy + 1) = f(x)f(y) - f(y) - x + 2$, so we have $f(yx + 1) = f(y)f(x) - f(x) - y + 2$. Subtracting, we have $0 = f(x) - f(y) + y - x$ or $f(x) + y = f(y) + x$. Let $y = 0$. Then $f(x) = f(0) + x$. Substitute into the given identity and putting $x = y = 0$, we get

$$f(0) + 1 = f(0)f(0) - f(0) + 2, \quad \text{or} \quad (f(0) - 1)^2 = 0.$$

Thus $f(0) = 1$ and $10f(2006) + f(0) = 20071$.

4. Ans: 682

Let a_k denote the number of ways that the k^{th} pass reach A. We have $a_1 = 0$. At each pass, the person holding the ball has 2 ways to pass the ball to. So total number of ways the ball can be passed after the k^{th} pass is 2^k The number of ways that at the $(k + 1)^{\text{th}}$ pass, A receives the ball is $a_k + 1$. So $a_{k+1} = 2^k - a_k$ Thus $a_1 = 0$, $a_2 = 2$, $a_3 = 2, \ldots, a_{11} = 682$.

5. Ans: 13122

Construct the permutation from the end. There are 3 choices each for s_9, s_8, …, s_2 and 2 choices for s_1 and 1 choice for s_0. So the answer is $2.3^8 = 13122$.

6. Ans: 1

AD bisects the angle $\angle A$ and IC bisects the angle $\angle C$. Now $\angle BCD = \angle BAD = \angle A/2$. $\angle ADC = \angle B$. Hence $\angle ICD = (\angle A + \angle C)/2$ and $\angle DIC = 180° - \angle B - (\angle A + \angle C)/2 = (\angle A + \angle C)/2$. Thus $ID = CD$. The chord CD subtends an angle of $30°$ at point A of circle $\mathbb{C}$. Hence it subtends an angle of $60°$ at the center of circle $\mathbb{C}$. Thus $ID = CD = 2\sin(60°/2) = 1$.

7. Ans: 500

If p is a prime, then the highest power of p that divides 2006! is

$$f(p) = \lfloor 2006/p \rfloor + \lfloor 2006/p^2 \rfloor + \lfloor 2006/p^3 \rfloor + \cdots.$$

(Note that the terms in the sum are eventually 0.) The number of consecutive 0's at the end of the base 10 representation of 2006! is the highest power of 10 that divides 2006!, which is $\min\{f(2), f(5)\} = f(5) = 401 + 80 + 16 + 3 = 500$.

8. Ans: 855

The map $\{a, b, c\} \to \{24-a, 24-b, 24-c\}$ is a bijection from the set of 3-element subsets of $\{1, \ldots, 23\}$ with $s(A) < 36$ onto the set of 3-element subsets of $\{1, \ldots, 23\}$ with $s(A) > 36$. The number of 3-element subsets of $\{1, \ldots, 23\}$ is $\binom{23}{3} = 1771$. Therefore, the number of 3-element subsets of $\{1, \ldots, 23\}$ with $s(A) < 36$ is

$$\frac{1}{2}\left[1771 - \text{number of 3-element subsets of } \{1, \ldots, 23\} \text{ with } s(A) = 36\right] = 855.$$

9. Ans: 322

Note that $x^6 + x^{-6} = (x^2 + x^{-2})(x^4 - 1 + x^{-4}) = ((x + x^{-1})^2 - 2)((x^2 + x^{-2})^2 - 3) = ((x + x^{-1})^2 - 2)(((x + x^{-1})^2 - 2)^2 - 3) = f(x + x^{-1})$. Thus letting $z = x + x^{-1}$, we have $f(z) = (z^2 - 2)((z^2 - 2)^2 - 3)$. Therefore, $f(3) = (3^2 - 2)((3^2 - 2)^2 - 3) = 322$.

10. Ans: 49

Let R be the foot of the perpendicular from O to BC. Since $\angle BAC = \angle COR$, we have $\angle AQO = \angle OCP$. Thus $\triangle COP$ is similar to $\triangle QOC$. Therefore $OC/OQ = OP/OC$ so that $OP \cdot OQ = OC^2 = 7^2 = 49$.

11. Ans: 3

Let $DP = x$, $PE = y$, and $BC = a$. As $\triangle FDP$ is similar to $\triangle FBC$, we have $DF/BF = x/a$. Thus $(BF - 1/2)/BF = x/a$. Solving for BF, we have $1/BF = 2(a - x)/a$.

Similarly, from the fact that $\triangle GPE$ is similar to $\triangle GBC$, we obtain $1/CG = 2(a - y)/a$. Consequently,

$$\frac{1}{BF} + \frac{1}{CG} = (4a - 2(x + y))/a = (4a - a)/a = 3.$$

12. Ans: 6

Extend BP meeting AC at E. Then ABE is an isosceles triangle with $AB = AE$ and $BP = PE$. As P and M are the midpoints of BE and BC respectively, we have PM is parallel to EC and $PM = EC/2 = (26 - 14)/2 = 6$.

13. Ans: 35

Join DT and DC. Let M be the foot of the perpendicular from D onto AC. Then OT, DM and BC are all parallel. Since D is the midpoint of OB, M is the midpoint of TC. Thus DTC is an isosceles triangle with $DT = DC$ and $\angle TDM = \angle MDC$.

As $\angle TOA = 70°$ and $\triangle OTD$ is isosceles, we have $\angle OTD = 35°$. Thus $\angle BCD = \angle MDC = \angle TDM = \angle OTD = 35°$.

14. Ans: 2006

Since D and E are the midpoints of the sides AB and AC respectively, we have DE is parallel to BC and $\triangle PDE$ is similar to $\triangle PCB$ with $PD : PC = PE : PB = DE : CB = 1 : 2$.

Let $PE = x$ and $PD = y$. Then $PB = 2x$ and $PC = 2y$. By Pythagoras' theorem applied to $\triangle PBD$ and $\triangle PCE$, we get $(2x)^2 + y^2 = 1829^2$ and $(2y)^2 + x^2 = 1298^2$. Adding these two equations, we have $x^2 + y^2 = (1829^2 + 1298^2)/5$. Thus $BC^2 = (2x)^2 + (2y)^2 = 4(1829^2 + 1298^2)/5 = 4024036$. Therefore $BC = \sqrt{4024036} = 2006$.

15. Ans: 65536

The answer is

$$\binom{17}{1} + \binom{17}{3} + \binom{17}{5} + \cdots + \binom{17}{17} = 2^{16} = 65536.$$

16. Ans: 275

$$\begin{aligned}
&400(\cos^5 15^\circ + \sin^5 15^\circ) \div (\cos 15^\circ + \sin 15^\circ)\\
&= 400(\cos^4 15^\circ - \cos^3 15^\circ \sin 15^\circ + \cos^2 15^\circ \sin^2 15^\circ - \cos 15^\circ \sin^3 15^\circ + \sin^4 15^\circ)\\
&= 400(\cos^4 15^\circ + \sin^4 15^\circ - \cos 15^\circ \sin 15^\circ + \cos^2 15^\circ \sin^2 15^\circ)\\
&= 400((\cos^2 15^\circ + \sin^2 15^\circ)^2 - \cos^2 15^\circ \sin^2 15^\circ - \cos 15^\circ \sin 15^\circ)\\
&= 400(1 - (\frac{1}{2}\sin 30^\circ)^2 - (\frac{1}{2}\sin 30^\circ))\\
&= 400(1 - 1/16 - 1/4)\\
&= 275
\end{aligned}$$

17. Ans: 4

Consider the equation $y + 1/y = a$, where $a > 0$. It can be changed into

$$y^2 - ay + 1 = 0.$$

Observe that it has two positive real solutions:

$$y = \frac{a \pm \sqrt{a^2 - 4}}{2} > 0.$$

Thus the equation

$$x^2 + \frac{1}{x^2} = 2006 + \frac{1}{2006}.$$

has four real solutions (i.e., $\pm\sqrt{2006}, \pm 1/\sqrt{2006}$).

18. Ans: 30

Note that

$$a^5 - a = a(a-1)(a+1)(a^2+1).$$

It is clear that $2 \mid a^5 - a$ and $3 \mid a^5 - a$. We can show that $5 \mid a^5 - a$ by considering the five cases: $a \equiv i \pmod 5$, $i = 0, 1, 2, 3, 4$. Thus $30 \mid a^5 - a$.

When $a = 2$, we have $a^5 - a = 30$. Thus the maximum n is 30.

19. Ans: 32752

There are 2^{15} mappings from A to B. There is only one mapping $f : A \to B$ with $f(i) = 0$ for all $i \in A$; and there are 15 mappings $f : A \to B$ with $f(i) = 0$ for all $i \in A \backslash \{k\}$ and $f(k) = 1$, for $k = 1, 2, \ldots, 15$. Thus the answer is 32752.

20. Ans: 102

Note that from $a_n = 10a_{n-1} - 1$, we have

$$a_n - \frac{1}{9} = 10\left(a_{n-1} - \frac{1}{9}\right)$$

for all $n \geq 2$. Thus,

$$a_n - \frac{1}{9} = 10^{n-1}\left(a_1 - \frac{1}{9}\right) = 10^{n-1}\frac{8}{9}$$

for all $n \geq 1$. Therefore

$$a_n = \frac{(1 + 8 \times 10^{n-1})}{9}.$$

Observe that for $n \geq 2$,

$$8 \times 10^{n-2} < a_n < 10^{n-1}.$$

Thus

$$a_{101} < 10^{100} < a_{102}.$$

That is why the answer is 102.

21. Ans: 1900

Let A be a vertex of P. First we shall count the number of such triangles having A as a vertex. After taking away A and 3 consecutive vertices of P on each side of A, we are left with 23 vertices from which we can choose two vertices in such a way that, together with A, a desired triangle can be formed. There are $\binom{2+(23-3-2)}{2} = \binom{20}{2}$ ways to do so. Hence there are $30\binom{20}{2} \div 3 = 1900$ such triangles.

22. Ans: 486

Let $S = \{x \in \mathbb{Z} \mid 2000 \leq x \leq 4000\}$, $A = \{x \in S \mid x \text{ is divisible by } 4\}$, $B = \{x \in S \mid x \text{ is divisible by } 100\}$,$C = \{x \in S \mid x \text{ is divisible by } 400\}$. The required answer is

$$|A|-|B|+|C| = \left(\frac{4000}{4} - \frac{2000}{4} + 1\right) - \left(\frac{4000}{100} - \frac{2000}{100} + 1\right) + \left(\frac{4000}{400} - \frac{2000}{400} + 1\right) = 486.$$

23. Ans: 5

Our reference day is today, 31-5-2006, Wednesday. We shall first count the number of days D from 15-5-1879 to 31-5-2006. The number of leap years between 1879 and 2005 is

$$\left(\left\lfloor\frac{2005}{4}\right\rfloor - \left\lfloor\frac{1879}{4}\right\rfloor\right) - \left(\left\lfloor\frac{2005}{100}\right\rfloor - \left\lfloor\frac{1879}{100}\right\rfloor\right) + \left(\left\lfloor\frac{2005}{400}\right\rfloor - \left\lfloor\frac{1879}{400}\right\rfloor\right) = 31.$$

From 1-1-1880 to 31-12-2005 there are $2005 - 1879 = 126$ years, of which 31 are leap years. Thus $D = 95 \times 365 + 31 \times 366 + (365 - 31 - 28 - 14) + (31 + 28 + 31 + 30 + 31) = 46464$. Since 46464 leaves a remainder of 5 when divided by 7, Albert Einstein was born on Friday. The answer is 5.

24. Ans: 29776

Let a_n denote the number of such n-digit integers. Among these a_n integers, let b_n denote the number of those which end with 2. By symmetry, the number of those which end with 3 (or 4) is also equal to b_n. Hence

$$a_n = \underbrace{2a_{n-1}}_{\text{end with 0 or 1}} + \underbrace{3b_n}_{\text{end with 2,3 or 4}} \tag{1}$$

$$b_n = \underbrace{2a_{n-2}}_{\text{end with 02 or 12}} + \underbrace{2b_{n-1}}_{\text{end with 32 or 42}} \tag{2}$$

Thus

$$\begin{aligned} a_n - 2a_{n-1} = 3b_n &= 6a_{n-2} + 6b_{n-1} \\ &= 6a_{n-2} + 2(a_{n-1} - 2a_{n-2}) \\ a_n &= 4a_{n-1} + 2a_{n-2} \end{aligned}$$

We have $a_1 = 4$, $a_2 = 4 \times 5 - 3 = 17$. By iterating we get $a_7 = 29776$.

25. Ans: 1274

By using the fact that $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$, we have

$$\begin{aligned} S &= \binom{3n}{0} + \binom{3n+1}{1} + \cdots + \binom{3n+n}{n} = \binom{3n+1}{0} + \binom{3n+1}{1} + \cdots + \binom{3n+n}{n} \\ &= \cdots = \binom{3n+n}{n-1} + \binom{3n+n}{n} = \binom{4n+1}{n}. \end{aligned}$$

Thus when $n = 12$,

$$\frac{S}{23 \times 38 \times 41 \times 43 \times 47} = 1274.$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2006

(Open Section, Special Round)

Saturday, 1 July 2006 — 0900– 1330

Important:

Attempt as many questions as you can.

No calculators are allowed.

Show all the steps in your working.

Each question carries 10 marks.

1. In the triangle ABC, $\angle A = 60^\circ$, D, M are points on the line AC and E, N are points on the line AB such that DN and EM are the perpendicular bisectors of AC and AB respectively. Let L be the midpoint of MN. Prove that $\angle EDL = \angle ELD$.

2. Show that any representation of 1 as the sum of distinct reciprocals of numbers drawn from the arithmetic progression $\{2, 5, 8, 11, \ldots\}$ such as given in the following example must have at least eight terms:

$$1 = \frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \frac{1}{20} + \frac{1}{41} + \frac{1}{110} + \frac{1}{1640}.$$

3. Consider the sequence $p_1, p_2, \ldots$ of primes such that for each $i \geq 2$, either $p_i = 2p_{i-1} - 1$ or $p_i = 2p_{i-1} + 1$. An example is the sequence $2, 5, 11, 23, 47$. Show that any such sequence has a finite number of terms.

4. Let n be a positive integer. Let $S_1, S_2, \cdots, S_k$ be a collection of $2n$-element subsets of $\{1, 2, 3, 4, \cdots, 4n-1, 4n\}$ so that $S_i \cap S_j$ contains at most n elements for all $1 \leq i < j \leq k$. Show that

$$k \leq 6^{(n+1)/2}.$$

5. Let a, b and n be positive integers. Prove that $n!$ divides

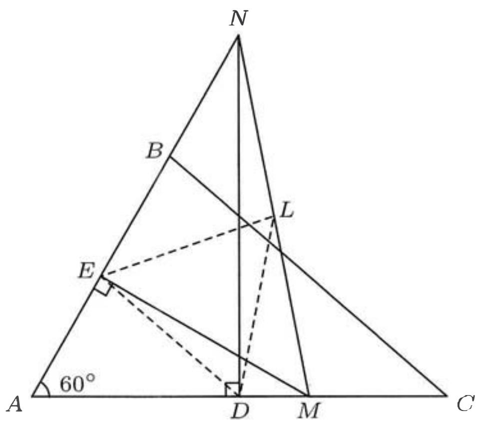
$$b^{n-1}a(a+b)(a+2b)\cdots(a+(n-1)b).$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2006

(Open Section, Special Round Solutions)

1. Set up a coordinate system so that A is the origin and AC is the x-axis. Let the coordinate of C be $(2c, 0)$ and the coordinate of B be $(2b, 2b\sqrt{3})$. Then the coordinates of D, E, N and M are $(c, 0)$, $(b, b\sqrt{3})$, $(c, c\sqrt{3})$ and $(4b, 0)$, respectively.



Thus

$$MN^2 = (c - 4b)^2 + (c\sqrt{3} - a)^2 = 4c^2 - 8bc + 16b^2.$$

Also

$$BC^2 = (2b - 2c)^2 + (2b\sqrt{3})^2 = 4c^2 - 8bc + 16b^2.$$

Therefore $MN = BC$. In the right-angled triangle EMN, $EL = \frac{1}{2}MN$. Thus $EL = \frac{1}{2}BC = ED$. That is $\angle EDL = \angle ELD$.

2. Suppose that the representation uses the reciprocals of k distinct positive integers, $x_1, \ldots, x_k$, where $x_i \equiv 2 \pmod 3$. Since $1 = \sum \frac{1}{x_i}$, we get

$$x_1 x_2 \ldots x_k = \sum X_i$$

where $X_i = \frac{x_1 x_2 \ldots x_n}{x_i}$. Thus

$$2^k \equiv k2^{k-1} \pmod 3,$$

from which we get $k \equiv 2 \pmod 3$. Hence $k = 2, 5, 8, \ldots$. Since

$$\frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \frac{1}{14} < 1$$

we see that we need at least 8 terms.

3. Except for 2 or 3, each prime is of the form $6u \pm 1$. If a prime $p = 6u + 1$ is in the chain, its successor, if any, must be of the form $2p - 1 = 2(6u) + 1$ since $2p + 1$ is divisible by 3. Hence the successors are:

$$2(6u) + 1, 2^2(6u) + 1, \ldots, 2^i(6u) + 1, \ldots$$

This sequence cannot go on forever giving primes. To prove this claim, we first note that, by Fermat's Little Theorem, there exists k such that $2^k \equiv 1 \pmod{6u+1}$. Thus

$$2^k(6u) + 1 \equiv 0 \pmod{6u+1}.$$

Hence $2^k(6u) + 1$ is not prime. A similarly argument can be given for the case $6u + 1$.

4. Let $A = \{1, 2, \cdots, 4n\}$. Let $\mathcal{F}$ be the family of subsets in A with $n + 1$-elements. Then

$$|\mathcal{F}| = \binom{4n}{n+1}.$$

Note that every $n+1$-element in S_i is also a member in $\mathcal{F}$. Since $S_i \cap S_j$ contains at most n elements in A, and any $n+1$-element in S_i is different from any $n+1$-element in S_j for all $1 \leq i < j \leq k$. Thus

$$|\mathcal{F}| \geq \sum_{i=1}^{k} \binom{2n}{n+1} = k\binom{2n}{n+1}.$$

Hence

$$\begin{aligned} k &\leq \binom{4n}{n+1} \div \binom{2n}{n+1} \\ &= \frac{4n \times (4n-1) \times \cdots \times 3n}{2n \times (2n-1) \times \cdots \times n}. \end{aligned}$$

It can be shown that

$$\frac{(4n-i)(3n+i)}{(2n-i)(n+i)} \leq \frac{4n \times 3n}{2n \times n} = 6$$

for all $0 \leq i \leq (n-1)/2$.

If n is odd, then

$$\frac{4n \times (4n-1) \times \cdots \times 3n}{2n \times (2n-1) \times \cdots \times n} = \prod_{i=0}^{(n-1)/2} \frac{(4n-i)(3n+i)}{(2n-i)(n+i)} \leq 6^{(n+1)/2}.$$

If n is even, then

$$\frac{4n \times (4n-1) \times \cdots \times 3n}{2n \times (2n-1) \times \cdots \times n} = \frac{4n - n/2}{2n - n/2} \prod_{i=0}^{(n-2)/2} \frac{(4n-i)(3n+i)}{(2n-i)(n+i)} \leq 6^{1/2} 6^{n/2} = 6^{(n+1)/2}.$$

5. We shall prove that for any prime p with $1 < p \leq n$, if $p^\alpha \mid n!$, then $p^\alpha \mid b^{n-1}a(a+b)(a+2b)\cdots(a+(n-1)b)$.

If $p \mid b$, then as

$$\alpha = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor < \sum_{k=1}^{\infty} \frac{n}{p^k} = \frac{n}{p-1} \leq n,$$

we have $\alpha \leq n-1$ so that $p^\alpha | b^{n-1}$. This shows that $p^\alpha \mid b^{n-1}a(a+b)(a+2b)\cdots(a+(n-1)b)$.

If $p \nmid b$, then there exists a positive integer b_1 such that $bb_1 \equiv 1 \pmod{p^\alpha}$. Note that $p \nmid b_1$. Thus

$$b_1^n a(a+b)(a+2b)\cdots(a+(n-1)b) \equiv ab_1(ab_1+1)\cdots(ab_1+n-1) \pmod{p^\alpha}.$$

As the right hand side of the above congruence is a product of n consecutive integers, it is divisible by $n!$. It is therefore divisible by p^α too. That is

$$p^\alpha \mid b_1^n a(a+b)(a+2b)\cdots(a+(n-1)b).$$

Since $p \nmid b_1$, we have $(p^\alpha, b_1^n) = 1$, so that $p^\alpha \mid a(a+b)(a+2b)\cdots(a+(n-1)b)$, and thus

$$p^\alpha \mid b^n a(a+b)(a+2b)\cdots(a+(n-1)b).$$