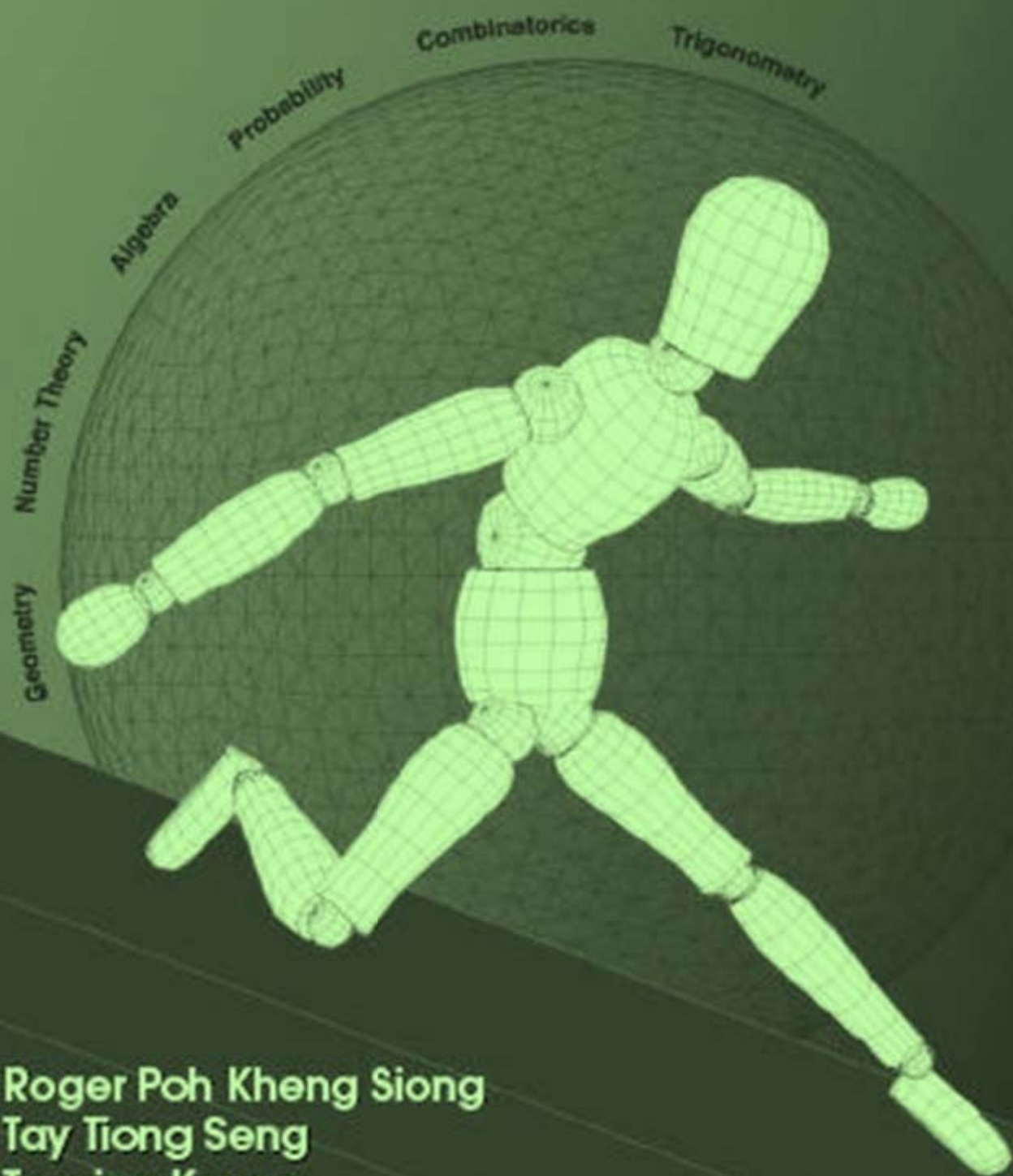


SINGAPORE MATHEMATICAL OLYMPIADS

2005



Roger Poh Kheng Siong
Tay Tiong Seng
To wing Keung
Yang Yue

Published by Singapore Mathematical Society

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2005

(Junior Section)

Tuesday, 31 May 2005

0930-1200

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter only the letters (a, b, c, d, or e) corresponding to the correct answers in the answer sheet.

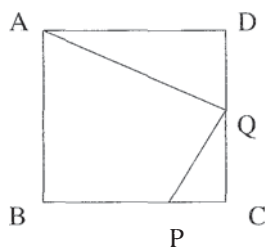
For the other short questions, write your answers in answer sheet and shade the appropriate bubbles below your answers.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

1. What is the units digit of 7^5 ?
(a) 1; (b) 3; (c) 5; (d) 7; (e) 9.
2. What is the value of $(0.5)^3 + 3 \times (0.5)^2 \times (-1.5) + 3 \times (0.5) \times (-1.5)^2 + (-1.5)^3$?
(a) 1; (b) -1; (c) 0; (d) 2; (e) -2.
3. What is the last two digits of $1 + 2 + 3 + 4 + \cdots + 2003 + 2004 + 2005$?
(a) 00; (b) 15; (c) 25; (d) 50; (e) 75.
4. How many zeroes does the number $50 \times 49 \times 48 \times \cdots \times 3 \times 2 \times 1$ end with?
(a) 8; (b) 9; (c) 10; (d) 11; (e) 12.
5. In a square $ABCD$, let P be a point on the side BC such that $BP = 3PC$ and Q be the mid-point of CD . If the area of the triangle PCQ is 5, what is the area of triangle QDA ?



(a) 5; (b) 10; (c) 15; (d) 20; (e) 25.

6. The values of p which satisfy the equations

$$\begin{aligned}p^2 + 9q^2 + 3p - pq &= 30 \\ p - 5q - 8 &= 0\end{aligned}$$

may be found by solving

(a) $x^2 - x - 6 = 0$; (b) $13x^2 - 121x - 426 = 0$; (c) $29x^2 - 101x - 174 = 0$;

(d) $29x^2 + 115x - 1326 = 0$; (e) $39x^2 - 104x - 174 = 0$.

7. Using the digits 1, 2, 3, 4 only once to form a 4-digit number, how many of them are divisible by 11?

(a) 4; (b) 5; (c) 6; (d) 7; (e) 8.

8. If a polygon has its sum of interior angles smaller than 2005° , what is the maximum number of sides of the polygon?

(a) 11; (b) 12; (c) 13; (d) 14; (e) 15.

9. The last 2 digits of

$$2005 + 2005^2 + 2005^3 + \cdots + 2005^{2005}$$

is:

(a) 00; (b) 05; (c) 25; (d) 50; (e) 75.

10. Suppose 3 distinct numbers are chosen from $1, 2, \dots, 3n$ with their sum equal to $3n$. What is the largest possible product of those 3 numbers?

(a) $n^3 - n$; (b) n^3 ; (c) $n^3 + n$; (d) $n^3 - 7n + 6$; (e) $n^3 - 7n - 6$.

11. Suppose $a \neq 0, b \neq 0$ and $\frac{b}{a} = \frac{c}{b} = 2005$. Find the value of $\frac{b+c}{a+b}$.

12. Find the exact value of $\sqrt{\frac{x-2}{6}}$ when $x = 2006^3 - 2004^3$.

13. Suppose $x - y = 1$. Find the value of

$$x^4 - xy^3 - x^3y - 3x^2y + 3xy^2 + y^4.$$

14. Simplify

$$\frac{(1 - \frac{1}{2^2})(1 - \frac{1}{3^2})(1 - \frac{1}{4^2}) \dots (1 - \frac{1}{2005^2})}{(1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4}) \dots (1 - \frac{1}{2005})}.$$

15. The figure $ABCDEF$ is a regular hexagon. Evaluate the quotient

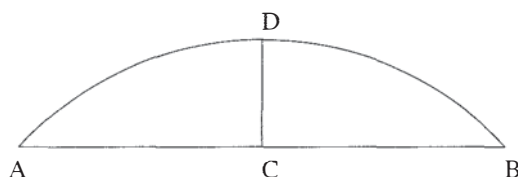
$$\frac{\text{Area of hexagon } ABCDEF}{\text{Area of triangle } ACD}.$$

16. Suppose x and y are two real numbers such that $x + y = 10$ and $x^2 + y^2 = 167$. Find the value of $x^3 + y^3$.

17. Find the value (in the simplest form) of $\sqrt{9 + 4\sqrt{5}} - \sqrt{9 - 4\sqrt{5}}$.

18. If $x > 0$ and $(x + \frac{1}{x})^2 = 25$, find the value of $x^3 + \frac{1}{x^3}$.

19. The diagram shows a segment of a circle such that CD is perpendicular bisector of the chord AB . Given that $AB = 16$ and $CD = 4$, find the diameter of the circle.



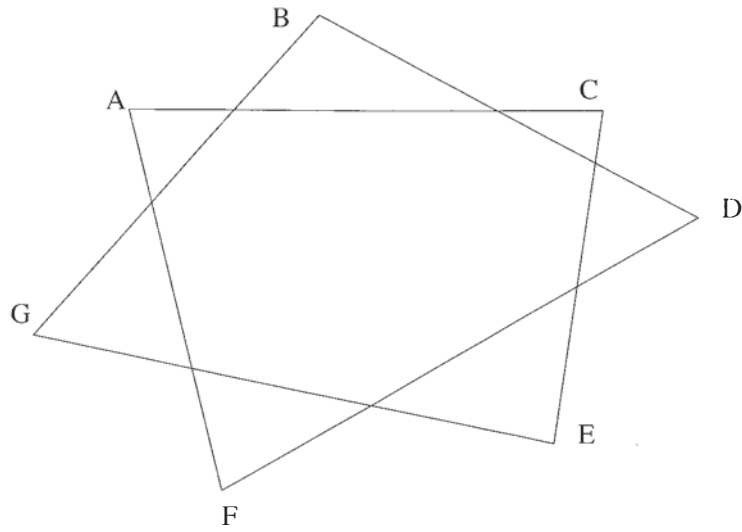
20. A palindrome number is the same either read from left to right or right to left, for example, 121 is a palindrome number. How many 5-digit palindrome numbers are there together?

21. Let p be a prime number such that the next larger number is a perfect square. Find the sum of all such prime numbers. (For example, if you think that 11 and 13 are two such prime numbers, then the sum is 24.)

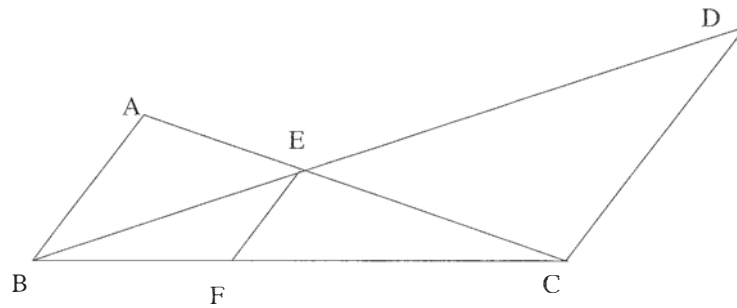
22. In the figure below, if

$$\angle A + \angle B + \angle C + \angle D + \angle E + \angle F + \angle G = x \text{ degrees,}$$

then what is x ?

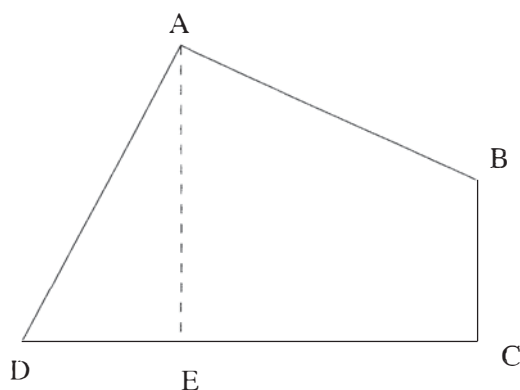


23. Given that $x = 2a^5 = 3b^2$ where a and b are positive integers, find the least possible value of x .
24. If $x^2 + x - 1 = 0$, find the value of $x^4 - 3x^2 + 3$.
25. In the diagram, $AB \parallel EF \parallel DC$. Given that $AC + BD = 250$, $BC = 100$ and $EC + ED = 150$, find CF .



26. Find the sum of all possible values of a such that the following equation has real root in x :
- $$(x - a)^2 + (x^2 - 3x + 2)^2 = 0.$$
27. If two positive integers m and n , both bigger than 1, satisfy the equation
- $$2005^2 + m^2 = 2004^2 + n^2,$$
- find the value of $m + n$.
28. Suppose $a \neq 0, b \neq 0, c \neq 0$ and $\frac{a}{b} = \frac{b}{c} = \frac{c}{a}$. Find the value of $\frac{a+b-c}{a-b+c}$.

31. In the following figure, $AD = AB$, $\angle DAB = \angle DCB = \angle AEC = 90^\circ$ and $AE = 5$. Find the area of the quadrangle $ABCD$.



32. Suppose that a, b, c are distinct numbers such that

$$(b - a)^2 - 4(b - c)(c - a) = 0,$$

find the value of $\frac{b-c}{c-a}$.

33. Find the number of even digits in the product of the two 10-digit numbers

$$2222222222 \times 9999999999.$$

34. Find an integer x that satisfies the equation

$$x^5 - 101x^3 - 999x^2 + 100900 = 0.$$

35. Determine the second smallest prime factor of

$$\frac{1^3 + 1}{1 + 1} + \frac{2^3 + 1}{2 + 1} + \frac{3^3 + 1}{3 + 1} + \cdots + \frac{2005^3 + 1}{2005 + 1}.$$

Singapore Mathematical Society

Singapore Mathematical Olympiad 2005

(Junior Section Solutions)

1 Ans: (d)

$7^5 = (7^4) \times 7 \equiv 7 \pmod{10}$ since $7^4 \equiv 1 \pmod{10}$.

2 Ans: (b).

Let $f(a, b) = (a + b)^3 = a^3 + 3a^2b + 3b^2 + b^3$. Then $f(0.5, -1.5) = (0.5 - 1)^3 = -1$.

3 Ans: (b).

$1 + 2 + \dots + 2005 = \frac{1}{2} \times 2005 \times 2006$. Hence the last two digits is 15.

4 Ans: (e).

By counting the number of factor 5 in the product, we easily see that the number $50!$ ends with $\frac{50}{5} + \frac{50}{25} = 12$ zeros.

5 Ans: (d).

The two triangles are similar and $\frac{PC}{QD} = 2$. Hence Area $QDA = 4 \times$ Area $PCQ = 20$.

6 Ans: (a).

Substituting $q = \frac{1}{5}(p - 8)$ into the first equation, we get (a).

7 Ans: (e).

If the first digit is 1, then the number is divisible by 11 implies that the 3rd digit is 4 and there are two ways to put 2 and 3. The cases when the first digit are 2, 3, 4 respectively can be dealt with similarly. Hence the total number is $4 \times 2 = 8$.

8 Ans: (c).

The sum of interior angles is $(n - 2) \times 180^\circ < 2005^\circ$. Hence the maximum n is 13.

9 Ans: (b).

The last 2 digits of 2005^n is 25 except $n = 1$ which gives 5. Hence the last 2 digits is $5 + 2004 \times 25 \pmod{100} = 05$.

10 Ans: (a).

From the identity $4ab = (a+b)^2 - (a-b)^2$, one sees that when $a+b$ is fixed, the product is the largest when $|a-b|$ is the smallest.

Now let $a < b < c$ be 3 distinct positive integers such that $a+b+c = 3n$ and the product abc is the biggest possible. If a and b differ more than 2, then by increasing a by 1 and decreasing b by 1, the product abc will become bigger. Hence a and b differ by at most 2. Similarly b and c differ by at most 2.

On the other hand, if $c = b+2 = a+4$, then one can increase the value of abc by decreasing c by 1 and increasing a by 1. So this case is ruled out.

Since $a+b+c$ is divisible by 3, it is not possible that $c = b+1 = a+3$ or $c = b+2 = a+3$.

So we must have $c = b+1 = a+2$. Hence the three numbers are $n-1$, n and $n+1$. Thus, their product is $(n-1)n(n+1) = n^3 - n$.

11 Ans: 2005.

$b = 2005a$ and $c = 2005b \implies b+c = 2005(a+b)$. Hence $\frac{b+c}{a+b} = 2005$.

12 Ans: 2005.

Let $n = 2005$. Then $x = (n+1)^3 - (n-1)^3 = 6n^2 + 2$. Hence $\sqrt{\frac{x-2}{6}} = 2005$.

13 Ans: 1.

The expression $= (x-y)[(x-y)(x^2+xy+y^2) - 3xy]$. Substituting $x-y = 1$, one sees that the expression $= (x-y)^2 = 1$.

14 Ans: 1003.

The expression $= \prod_{i=2}^{2005} (1 + \frac{1}{i}) = \prod_{i=2}^{2005} \frac{i+1}{i} = 1003$.

15 Ans: 3.

Let O be the center of the regular hexagon. Then $\text{Area } OCD = \text{Area } OAC = \frac{1}{6} \times \text{Area } ABCDEF$. The result follows.

16 Ans: 2005.

$167 = (x+y)^2 - 2xy \implies xy = -\frac{67}{2}$. Hence $x^3 + y^3 = (x+y)(x^2 - xy + y^2) = 2005$.

17 Ans: 4.

Completing the squares, one sees that the expression $= \sqrt{(2+\sqrt{5})^2} - \sqrt{(2-\sqrt{5})^2} = 4$.

18 Ans: 110.

As $x > 0$, we have $x + \frac{1}{x} = 5$. Thus, $x^3 + \frac{1}{x^3} = (x + \frac{1}{x})((x + \frac{1}{x})^2 - 3) = 110$.

19 Ans: 20.

Let r be the radius. Then $r^2 = (r - 4)^2 + 8^2 \implies r = 10$. Thus the diameter is 20.

20 Ans: 900.

Such a palindrome number is completely determined by its first three digits. There are 9, 10, 10 choices of the 1st, 2nd and 3rd digits respectively, hence total = 900.

21 Ans: 3.

If $p + 1 = n^2$, $p = (n + 1)(n - 1)$. Since p is prime, $n = 2$. Hence $p = 3$ is unique.

22 Ans: 540.

By dividing up into triangles with respect to a center O , one sees that there are seven triangles formed (e.g. OAC , OBD , etc.). The middle angle at O is added up twice. Hence $x = 7 \times 180 - 720 = 540$.

23 Ans: 15552.

Observe that $3 \mid a$ and $2 \mid b$. Set $b = 2b_1$. Then we see that $2 \mid a$. Thus the smallest possible value of a is 6. Hence $x = 2(6^5) = 15552$.

24 Ans: 2.

Using long division, we get $x^4 - 3x^2 + 3 = (x^2 + x - 1)(x^2 - x - 1) + 2$.

25 Ans: 60.

From the equalities $\frac{CE}{CF} = \frac{AE}{BF} = \frac{AC}{BC}$ and $\frac{DE}{CF} = \frac{BE}{BF} = \frac{BD}{BC}$, we get

$$\frac{DE + CE}{CF} = \frac{BE + AE}{BF} = \frac{AC + BD}{BC}.$$

Thus $\frac{150}{CF} = \frac{250}{100}$, and we have $CF = 60$.

26 Ans: 3.

$x^2 - 3x + 2 = x - a = 0 \implies a = 1$ or $a = 2$.

27 Ans: 211.

Since $(m + n)(m - n) = 4009 = 211 \times 19$ which is a product of primes. Hence $m + n = 211$.

28 Ans: 1.
Letting $\frac{a}{b} = k$, we get $a = bk, b = ck$ and $c = ak$. Thus $a = ak^3$. Thus $k = 1$ and $a = b = c$. The result follows.

29 Ans: 57.
One can count directly that the maximum number of regions is 57.

30 Ans: 3.
Observe that the stop numbers $1, 9, 25, 49, \dots$ are at the lower right corners. The point $(0, -n)$ is at the stop number $(2n+1)^2 - (n+1) = 4n^2 + 3n$. When $n = 22$, we have $4(22^2) + 3(22) = 2002$. So the point $(0, -22)$ is the 2002-th stop. Thus the point $(3, -22)$ is the 2005-th stop.

31 Ans: 25.
Upon rotating AED anticlockwise 90° through A , we get a square. Thus the area is $5^2 = 25$.

32 Ans: 1.
Observe that $((b-c) + (c-a))^2 - 4(b-c)(c-a) = 0$. Hence $((b-c) - (c-a))^2 = 0$. Thus $b-c = c-a$.

33 Ans: 10.

$$\begin{aligned} 2222222222 \times (10^{10} - 1) &= 22222222220000000000 - 2222222222 \\ &= 22222222217777777778. \end{aligned}$$

34 Ans: 10.
Upon rewriting the equation as $(x^2 - 101)(x^3 - 999) + 1 = 0$, one sees that $x = 10$ is the only integral solution.

35 Ans: 11.
The sum $= \sum_{r=1}^{2005} (r^2 - r + 1) = \frac{1}{6} \times 2005 \times 2006 \times 4011 - \frac{1}{2} \times 2005 \times 2006 + 2005$. Upon simplifying, we see that the sum $= 5 \times 401 \times 134009 = 5 \times 11 \times 401 \times 121819$. Since 2, 3, 7 does not divide 401 and 121819, the second smallest prime factor is 11.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2005

(Senior Section)

Tuesday, 31 May 2005

0930-1200

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answers in the answer sheet by shading the bubbles containing the letters (A, B, C, D or E) corresponding to the correct answers.

For the other short questions, write your answers in answer sheet and shade the appropriate bubbles below your answers.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

1. What is the smallest positive prime factor of the integer $2005^{2007} + 2007^{2005}$?

(A) 5 (B) 7 (C) 2 (D) 11 (E) 3

2. What is the value of

$$\frac{2005^2 + 2 \times 2005 \times 1995 + 1995^2}{800}?$$

(A) 20000 (B) 2000 (C) 200000 (D) 2000000 (E) None of the above

3. Let p be a real number such that the equation $2y^2 - 8y = p$ has only one solution. Then

(A) $p < 8$ (B) $p = 8$ (C) $p > -8$ (D) $p = -8$ (E) $p < -8$

4. What is the sum of the last two digits of the integer $1! + 2! + 3! + \cdots + 2005!$?

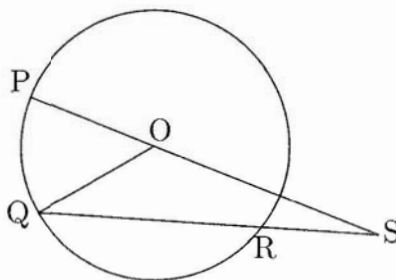
(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

5. $5002^{2005 \log_{5002} 2005}$ is equal to

(A) 1 (B) 2005^{2005} (C) 5002^{2005} (D) 2005^{5002} (E) 5002^{5002}

6. In the diagram, P , Q and R are three points on the circle whose centre is O . The lines PO and QR are produced to meet at S . Suppose that $RS = OP$, and $\angle PSQ = 12^\circ$ and $\angle POQ = x^\circ$. Find the value of x .

(A) 36 (B) 42 (C) 48 (D) 54 (E) 60



7. Let x and y be positive real numbers. What is the smallest possible value of $\frac{16}{x} + \frac{108}{y} + xy$?

(A) 16 (B) 18 (C) 24 (D) 30 (E) 36

8. In the Cartesian plane, the graph of the function $y = \frac{2x-1}{x-1}$ is reflected about the line $y = -x$. What is the equation for the graph of the image of the reflection?

(A) $y = \frac{x-1}{2x+1}$ (B) $y = \frac{x-1}{x+2}$ (C) $y = \frac{2x-1}{x+1}$ (D) $y = \frac{x+1}{x-2}$
 (E) $y = \frac{-1-x}{2+x}$

9. Simplify $\sqrt{2 \left(1 + \sqrt{1 + \left(\frac{x^4 - 1}{2x^2} \right)^2} \right)}$, where x is any positive real number.

(A) $\frac{x^2+1}{\sqrt{2}x}$ (B) $\frac{x^2+1}{x}$ (C) $\sqrt{\frac{x^2+1}{2x^2}}$ (D) x^2+1 (E) $\frac{x^2-1}{\sqrt{2}x}$

10. Let x and y be real numbers such that

$$x^2 + y^2 = 2x - 2y + 2.$$

What is the largest possible value of $x^2 + y^2$?

(A) $10 + 8\sqrt{2}$ (B) $8 + 6\sqrt{2}$ (C) $6 + 4\sqrt{2}$ (D) $4 + 2\sqrt{2}$
 (E) None of the above

11. Find the greatest integer less than $(2 + \sqrt{3})^4$.

12. $\triangle ABC$ is a triangle such that $\angle C = 90^\circ$. Suppose $AC = 156$ cm, $AB = 169$ cm and the perpendicular distance from C to AB is x cm. Find the value of x .

13. Find the sum of all the real numbers x that satisfy the equation

$$(3^x - 27)^2 + (5^x - 625)^2 = (3^x + 5^x - 652)^2.$$

14. Three positive integers are such that they differ from each other by at most 6. It is also known that the product of these three integers is 2808. What is the smallest integer among them?

15. Simplify $\frac{2005^2(2004^2 - 2003)}{(2004^2 - 1)(2004^3 + 1)} \times \frac{2003^2(2004^2 + 2005)}{2004^3 - 1}$.

16. Consider the function $f(x) = \frac{1}{3^x + \sqrt{3}}$. Find the value of

$$\sqrt{3}[f(-5)+f(-4)+f(-3)+f(-2)+f(-1)+f(0)+f(1)+f(2)+f(3)+f(4)+f(5)+f(6)].$$

17. Let A and B be two positive four-digit integers such that $A \times B = 16^5 + 2^{10}$. Find the value of $A + B$.

18. A triangle $\triangle ABC$ is inscribed in a circle of radius 4 cm. Suppose that $\angle A = 60^\circ$, $AC - AB = 4$ cm, and the area of $\triangle ABC$ is x cm². Find the value of x^2 .

19. Let a , b and c be real numbers such that

$$a = 8 - b \quad \text{and} \quad c^2 = ab - 16.$$

Find the value of $a + c$.

20. Let $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8$ be positive integers such that

$$a_1^2 + (2a_2)^2 + (3a_3)^2 + (4a_4)^2 + (5a_5)^2 + (6a_6)^2 + (7a_7)^2 + (8a_8)^2 = 204.$$

Find the value of $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8$.

21. Find the value of the positive integer n if

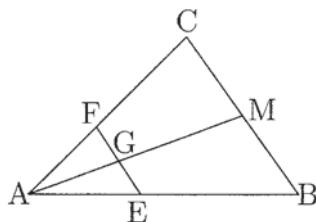
$$\frac{1}{\sqrt{4} + \sqrt{5}} + \frac{1}{\sqrt{5} + \sqrt{6}} + \frac{1}{\sqrt{6} + \sqrt{7}} + \cdots + \frac{1}{\sqrt{n} + \sqrt{n+1}} = 10.$$

22. Let A and B be two positive prime integers such that

$$\frac{1}{A} - \frac{1}{B} = \frac{192}{2005^2 - 2004^2}.$$

Find the value of B .

23. In $\triangle ABC$, $AB : AC = 4 : 3$ and M is the midpoint of BC . E is a point on AB and F is a point on AC such that $AE : AF = 2 : 1$. It is also given that EF and AM intersect at G with $GF = 72$ cm and $GE = x$ cm. Find the value of x .



24. It is given that $x = \frac{1}{2 - \sqrt{3}}$. Find the value of

$$x^6 - 2\sqrt{3}x^5 - x^4 + x^3 - 4x^2 + 2x - \sqrt{3}.$$

25. Let a , b and c be the lengths of the three sides of a triangle. Suppose a and b are the roots of the equation

$$x^2 + 4(c + 2) = (c + 4)x,$$

and the largest angle of the triangle is x° . Find the value of x .

26. Find the largest real number x such that

$$\frac{x^2 + x - 1 + |x^2 - (x - 1)|}{2} = 35x - 250.$$

27. How many ways can the word MATHEMATICS be partitioned so that each part contains at least one vowel? For example, MA-THE-MATICS, MATHE-MATICS, MATHEM-A-TICS and MATHEMATICS are such partitions.

28. Consider a sequence of real numbers $\{a_n\}$ defined by

$$a_1 = 1 \text{ and } a_{n+1} = \frac{a_n}{1 + na_n} \text{ for } n \geq 1.$$

Find the value of $\frac{1}{a_{2005}} - 2000000$.

29. For a positive integer k , we write

$$(1+x)(1+2x)(1+3x)\cdots(1+kx) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k,$$

where a_0, a_1, \dots, a_k are the coefficients of the polynomial. Find the smallest possible value of k if $a_0 + a_1 + a_2 + \cdots + a_{k-1}$ is divisible by 2005.

30. Find the largest positive number x such that

$$(2x^3 - x^2 - x + 1)^{1 + \frac{1}{2x+1}} = 1.$$

31. How many ordered pairs of integers (x, y) satisfy the equation

$$x^2 + y^2 = 2(x + y) + xy?$$

32. Find the number of ordered 7-tuples of positive integers $(a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ that have both of the following properties:

- (i) $a_n + a_{n+1} = a_{n+2}$ for $1 \leq n \leq 5$, and
- (ii) $a_6 = 2005$.

33. Let n be a positive integer such that one of the roots of the quadratic equation

$$4x^2 - (4\sqrt{3} + 4)x + \sqrt{3}n - 24 = 0$$

is an integer. Find the value of n .

34. Consider the simultaneous equations

$$\begin{cases} xy + xz = 255 \\ xz - yz = 224. \end{cases}$$

Find the number of ordered triples of positive integers (x, y, z) that satisfy the above system of equations.

35. Find the total number of positive four-digit integers N satisfying both of the following properties:

- (i) N is divisible by 7, and
- (ii) when the first and last digits of N are interchanged, the resulting positive integer is also divisible by 7. (Note that the resulting integer need not be a four-digit number.)

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2005
(Senior Section, Special Round)

Saturday, 9 July 2005

0930– 1230

Attempt as many questions as you can.

No calculators are allowed.

Show the steps in your calculations.

Each question carries 9 marks.

1. The digits of a 3-digit number are interchanged so that none of the digits retain their original positions. The difference of the two numbers is a 2-digit number and is a perfect square.
2. Consider the nonconvex quadrilateral $ABCD$ with $\angle C > 180^\circ$. Let the side DC extended meet AB at F and the side BC extended meet AD at E . A line intersects the interiors of the sides AB, AD, BC, CD at points K, L, J, I , respectively. Prove that if $DI = CF$ and $BJ = CE$, then $KJ = IL$.
3. Let S be a subset of $\{1, 2, 3, \dots, 24\}$ with $|S| = 10$. Show that S has two 2-element subsets $\{x, y\}$ and $\{u, v\}$ such that $x + y = u + v$.
4. Determine whether there exists a positive integer n such that $n!$ begins with 2005. Justify your answer.

Singapore Mathematical Society

Singapore Mathematical Olympiad 2005

(Senior Section Solutions)

1. Ans: C

Since 2005 and 2007 are odd numbers, so are 2005^{2007} and 2007^{2005} . Thus, $2005^{2007} + 2007^{2005}$ is even, and it is divisible by the smallest prime number 2.

2. Ans: A

$$\frac{2005^2 + 2 \times 2005 + 1995 + 1995^2}{800} = \frac{(2005 + 1995)^2}{800} = \frac{400^2}{800} = 20000.$$

3. Ans: D

Since the equation $2y^2 - 8y - p = 0$ has only one solution, its discriminant

$$(-8)^2 - 4 \times 2 \times (-p) = 0.$$

Thus $p = -8$.

4. Ans: B

One easily checks that for $n \geq 10$, $n!$ will end with two zeros. Also,

$$\begin{aligned} & 1! + 2! + 3! + 4! + 5! + 6! + 7! + 8! + 9! \\ &= 1 + 2 + 6 + 24 + 120 + 720 + 5040 + 40320 + 362880 \\ &= 409113. \end{aligned}$$

Thus the sum of the last two digits is $1+3=4$.

5. Ans: B

$$5002^{2005 \log_{5002} 2005} = 5002^{\log_{5002} 2005^{2005}} = 2005^{2005}.$$

6. Ans: A

Since $RS = OP$, we have $RS = OR$. Hence, $\angle ROS = \angle RSO = 12^\circ$. Then

$$\angle OQR = \angle ORQ = 2 \times 12^\circ = 24^\circ.$$

Thus,

$$\begin{aligned} \angle POQ &= 180^\circ - \angle QOR - \angle ROS \\ &= 180^\circ - (180^\circ - 24^\circ - 24^\circ) - 12^\circ \\ &= 36^\circ. \end{aligned}$$

7. Ans: E

Using Arithmetic Mean \geq Geometric Mean, we have

$$\begin{aligned}\frac{1}{3}\left(\frac{16}{x} + \frac{108}{y} + xy\right) &\geq \sqrt[3]{\frac{16}{x} \times \frac{108}{y} \times xy} = 12 \\ \implies \frac{16}{x} + \frac{108}{y} + xy &\geq 36.\end{aligned}$$

Equality is achieved when $\frac{16}{x} = \frac{108}{y} = xy = \frac{36}{3} = 12$, which is satisfied when $x = \frac{4}{3}$ and $y = 9$. Thus the smallest possible value is 36.

8. Ans: E

We simply replace x by $-y$ and y by $-x$ to get

$$\begin{aligned}-x &= \frac{2(-y) - 1}{-y - 1} \implies xy + x = -2y - 1 \\ &\implies y = \frac{-1 - x}{2 + x}.\end{aligned}$$

9. Ans: B

$$\begin{aligned}\sqrt{2\left(1 + \sqrt{1 + \left(\frac{x^4 - 1}{2x^2}\right)^2}\right)} &= \sqrt{2\left(1 + \sqrt{\frac{4x^4 + x^8 - 2x^4 + 1}{4x^4}}\right)} \\ &= \sqrt{2\left(1 + \sqrt{\left(\frac{4x^4 + 1}{2x^2}\right)^2}\right)} \\ &= \sqrt{2\left(\frac{2x^2 + x^4 + 1}{2x^2}\right)} \\ &= \sqrt{\frac{(x^2 + 1)^2}{x^2}} \\ &= \frac{x^2 + 1}{x}, \quad \text{since } x > 0.\end{aligned}$$

10. Ans: C

$$\begin{aligned}x^2 + y^2 - 2x + 2y = 2 &\iff (x^2 - 2x + 1) + (y^2 + 2y + 1) = 4 \\ &\iff (x - 1)^2 + (y + 1)^2 = 2^2.\end{aligned}$$

Therefore, the points (x, y) in the Cartesian plane satisfying the equation consist of points on the circle centred at $(1, -1)$ and of radius 2. For a point (x, y) on the circle, it is easy to see that its distance $\sqrt{x^2 + y^2}$ from the origin $(0, 0)$ is greatest

when it is the point in the fourth quadrant lying on the line passing through $(0,0)$ and $(1, -1)$. In other words,

$$(x, y) = (1 + \sqrt{2}, -1 - \sqrt{2}), \text{ and } x^2 + y^2 = (1 + \sqrt{2})^2 + (-1 - \sqrt{2})^2 = 6 + 4\sqrt{2}.$$

11. Ans: 193

$$\begin{aligned} (2 + \sqrt{3})^4 &= [(2 + \sqrt{3})^2]^2 = (4 + 4\sqrt{3} + 3)^2 \\ &= (7 + 4\sqrt{3})^2 = 49 + 56\sqrt{3} + 48 \approx 193.9. \end{aligned}$$

Thus, the greatest integer less than $(2 + \sqrt{3})^4$ is 193.

12. Ans: 60

By Pythagoras' Theorem, $BC = \sqrt{169^2 - 156^2} = 65$. Let the perpendicular from C to AB meet AB at D . Then $\triangle ABC \sim \triangle ACD$. Thus,

$$\frac{CD}{AC} = \frac{BC}{AB} \implies \frac{x}{156} = \frac{65}{169} \implies x = 60.$$

13. Ans: 7

Let $a = 3^x - 27$ and $b = 5^x - 625$, so that the equation becomes

$$a^2 + b^2 = (a + b)^2 \implies 2ab = 0 \implies a = 0 \text{ or } b = 0.$$

When $a = 0$, we have $3^x - 27 = 0 \implies x = 3$.

When $b = 0$, we have $5^x - 625 = 0 \implies x = 4$.

Hence, the sum is $3 + 4 = 7$.

14. Ans: 12

Let x be the smallest integer. Then we have

$$x^3 \leq 2808 \leq x(x + 6).$$

The first inequality implies that $x \leq 14$, while the second inequality implies that $x \geq 11$. If $x = 13$, then the product of the other integers is $2^3 \times 3^3 = 216$. This is impossible, since the smallest factor of 216 greater than 13 is 18 but $18^2 = 324 > 216$. Therefore, we must have $x = 12$. The other two integers are easily seen to be 13 and 18.

15. Ans: 1

Let $t = 2004$. Then the expression is equal to

$$\begin{aligned} &\frac{(t+1)^2(t^2-t+1)}{(t^2-1)(t^3+1)} \times \frac{(t-1)^2(t^2+t+1)}{t^3-1} \\ &= \frac{(t+1)^2(t^2-t+1)}{(t+1)(t-1)(t+1)(t^2-t+1)} \times \frac{(t-1)^2(t^2+t+1)}{(t-1)(t^2+t+1)} = 1. \end{aligned}$$

16. Ans: 6

$$f(x)+f(1-x) = \frac{1}{3^x + \sqrt{3}} + \frac{1}{3^{1-x} + \sqrt{3}} = \frac{\sqrt{3}}{3^x \sqrt{3} + 3} + \frac{3^x}{3 + 3^x \sqrt{3}} = \frac{3^x + \sqrt{3}}{3 + 3^x \sqrt{3}} = \frac{1}{\sqrt{3}}.$$

Therefore,

$$\begin{aligned} & \sqrt{3}[f(-5) + f(6)] + \sqrt{3}[f(-4) + f(5)] + \sqrt{3}[f(-3) + f(4)] \\ & + \sqrt{3}[f(-2) + f(3)] + \sqrt{3}[f(-1) + f(2)] + \sqrt{3}[f(0) + f(1)] \\ & = \sqrt{3}\left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) = 6. \end{aligned}$$

17. Ans: 2049

$$16^5 + 2^{10} = (2^4)^5 + 2^{10} = (2^{10} + 1)2^{10} = 1025 \times 1024 = 41 \times 5^2 \times 2^{10}.$$

Without loss of generality, we may assume that $A \leq B$. It follows that $1000 \leq A \leq 1024$, and a direct check shows that 1024 is the only number between 1000 and 1024 that divides $41 \times 5^2 \times 2^{10}$. Thus, $A = 1024$ and $B = 1025$. Therefore,

$$A + B = 1024 + 1025 = 2049.$$

18. Ans: 192

Let $AB = c$ cm, $BC = a$ cm and $CA = b$ cm. By the cosine rule,

$$a^2 = b^2 + c^2 - 2bc \cos 60^\circ = b^2 + c^2 - bc = (b - c)^2 + bc = 4^2 + bc = 16 + bc.$$

Let O be the centre of the circle. Then $\angle BOC = 2\angle A = 120^\circ$. Moreover, $OB = OC = 4$ cm. Therefore, by the cosine rule,

$$a^2 = 4^2 + 4^2 - 2 \times 4 \times 4 \cos 120^\circ = 48 \implies a = 4\sqrt{3}.$$

Then $bc = a^2 - 4 = 48 - 16 = 32$. Thus,

$$\text{area } (\triangle ABC) = \frac{1}{2}bc \sin 60^\circ = \frac{1}{2} \times 32 \times \frac{\sqrt{3}}{2} = 8\sqrt{3}.$$

Therefore, $x = 8\sqrt{3}$ and $x^2 = (8\sqrt{3})^2 = 192$.

19. Ans: 4

$$\begin{aligned} c^2 &= ab - 16 = (8 - b)b - 16 = -b^2 + 8b - 16 \\ \implies c^2 + b^2 - 8b + 16 &= 0 \implies c^2 + (b - 4)^2 = 0 \implies c = 0 \text{ and } b = 4. \end{aligned}$$

Since $b = 4$, it follows that $a = 8 - 4 = 4$. Thus, $a + c = 4 + 0 = 4$. In this solution, we have used the fact that the sum of two non-negative numbers is zero only when both numbers are zero.

20. Ans: 8

Observe that

$$a_1^2 + (2a_2)^2 + \cdots + (8a_8)^2 \geq 1^2 + 2^2 + \cdots + 8^2 = 204.$$

So the only possibility is that $a_i = 1$ for each $i = 1, 2, \dots, 8$. Hence $a_1 + \cdots + a_8 = 8$.

21. Ans: 143

Observe that for any positive number x ,

$$\frac{1}{\sqrt{x} + \sqrt{x+1}} = \frac{\sqrt{x+1} - \sqrt{x}}{(\sqrt{x+1})^2 - (\sqrt{x})^2} = \sqrt{x+1} - \sqrt{x}.$$

Thus we have

$$\begin{aligned} & \frac{1}{\sqrt{4} + \sqrt{5}} + \frac{1}{\sqrt{5} + \sqrt{6}} + \cdots + \frac{1}{\sqrt{n} + \sqrt{n+1}} \\ &= (\sqrt{5} - \sqrt{4}) + (\sqrt{6} - \sqrt{5}) + \cdots + (\sqrt{n+1} - \sqrt{n}) \\ &= \sqrt{n+1} - \sqrt{4}. \end{aligned}$$

Hence, $\sqrt{n+1} - \sqrt{4} = 10 \implies \sqrt{n+1} = 12 \implies n+1 = 144 \implies n = 143$.

22. Ans: 211

$$\frac{1}{A} - \frac{1}{B} = \frac{192}{2005^2 - 2004^2} \iff \frac{B-A}{A \times B} = \frac{192}{4009} \iff 4009 \times (B-A) = 192 \times A \times B.$$

The prime factorization of 4009 is given by $4009 = 19 \times 211$. Since $192 = 2^6 \times 3$ and $B > A$, it follows that $A = 19$ and $B = 211$.

23. Ans: 108

Let the area of $\triangle AGF$ be S_1 and the area of $\triangle AGE$ be S_2 . Let the area of $\triangle AMC$ be S , so that the area of $\triangle ABM$ is also S , since M is the midpoint of BC . Now,

$$\frac{S_1}{S} = \frac{AG \cdot AF}{AM \cdot AC} \quad \text{and} \quad \frac{S_2}{S} = \frac{AG \cdot AE}{AM \cdot AB}.$$

Therefore,

$$\frac{S_2}{S_1} = \frac{AG \cdot AE \cdot AM \cdot AC}{AM \cdot AB \cdot AG \cdot AF} = \frac{3}{4} \times 2 = \frac{3}{2}.$$

Note that $\frac{S_2}{S_1} = \frac{GE}{GF} = \frac{x}{72}$. Therefore, $\frac{3}{2} = \frac{x}{72} \implies x = 108$.

24. Ans: 2

$$x = \frac{1}{2 - \sqrt{3}} = \frac{1}{2 - \sqrt{3}} \cdot \frac{2 + \sqrt{3}}{2 + \sqrt{3}} = \frac{2 + \sqrt{3}}{4 - 3} = 2 + \sqrt{3}.$$

Then $x - 2 = \sqrt{3} \implies x - \sqrt{3} = 2 \implies x^2 - 4x + 1 = 0$ and $x^2 - 2\sqrt{3}x - 1 = 0$.
Thus,

$$\begin{aligned} & x^6 - 2\sqrt{3}x^5 - x^4 + x^3 - 4x^2 + 2x - \sqrt{3} \\ &= x^4(x^2 - 2\sqrt{3}x - 1) + x(x^2 - 4x + 1) + x - \sqrt{3} \\ &= 0 + 0 + 2 + \sqrt{3} - \sqrt{3} = 2. \end{aligned}$$

25. Ans: 90

Since a, b are the roots of the equation $x^2 - (c + 4)x + 4(c + 2) = 0$, it follows that

$$a + b = c + 4 \quad \text{and} \quad ab = 4(c + 2).$$

Then $a^2 + b^2 = (a + b)^2 - 2ab = (c + 4)^2 - 8(c + 2) \implies a^2 + b^2 = c^2$. Hence the triangle is right-angled, and $x = 90$.

26. Ans: 25

Observe that $\frac{1}{2}(a + b + |a - b|) = \max(a, b)$. Also, $x^2 > x - 1$ since $x^2 - x + 1 = (x - \frac{1}{2})^2 + \frac{3}{4} > 0$. Thus, we have

$$\frac{1}{2}[x^2 + x - 1 + |x^2 - (x - 1)|] = \max(x^2, x - 1) = x^2.$$

Thus the equation becomes $x^2 = 35x - 250 \implies (x - 25)(x - 10) = 0 \implies x = 25, 10$.
Therefore, the largest value of x is 25.

27. Ans: 36

To determine the number of partitions, we just have to decide the number of ways of inserting a partition between two consecutive vowels if we insert a partition at all. Thus the total number of partitions is $4 \times 3 \times 3 = 36$.

28. Ans: 9011

For $n \geq 1$, $a_{n+1} = \frac{a_n}{1 + na_n} \implies \frac{1}{a_{n+1}} = \frac{1 + na_n}{a_n} = \frac{1}{a_n} + n$. Therefore,

$$\begin{aligned} \frac{1}{a_{2005}} &= \frac{1}{a_{2004}} + 2004 = \frac{1}{a_{2003}} + 2003 + 2004 \\ &= \dots \\ &= \frac{1}{a_1} + (1 + 2 + 3 + \dots + 2004) = 1 + \frac{2004 \times 2005}{2} = 2009011. \end{aligned}$$

Hence, $\frac{1}{a_{2005}} - 2000000 = 9011$.

29. Ans: 401

By comparing the coefficients of x^k , it is easy to see that $a_k = k!$. Letting $x = 1$, we get

$$a_0 + a_1 + \cdots + a_k = (1+1)(1+2)(1+3)\cdots(1+k) = (k+1)!$$

Therefore, $a_0 + a_1 + \cdots + a_{k-1} = (k+1)! - k! = k \cdot k!$. Note that $2005 = 5 \times 401$, where 401 is a prime number. It is easy to see that a prime number p divides $k!$ only if $k \geq p$. Therefore, the smallest possible value of k is 401.

30. Ans: 1

Since the exponent on the left side of the given equation is non-zero if $x > 0$, it follows that the equation is possible only when

$$2x^3 - x^2 - x + 1 = 1 \implies 2x^3 - x^2 - x = 0 \implies x(x-1)(2x+1) = 0 \implies x = 0, 1, -\frac{1}{2}.$$

Since $x > 0$, it follows that $x = 1$.

31. Ans: 6

We can rearrange the equation to get

$$x^2 - x(2+y) + y^2 - 2y = 0.$$

As a quadratic equation in the variable x , the discriminant of the above equation is given by

$$\text{Discriminant} = (2+y)^2 - 4(1)(y^2 - 2y) = 4 + 12y - 4y^2 = 16 - 4(y-2)^2.$$

Since x is an integer, the discriminant must be a perfect square and thus it is non-negative. Hence,

$$16 - 4(y-2)^2 \geq 0 \implies (y-2)^2 \leq \frac{16}{4} \implies |y-2| \leq 2 \implies y = 0, 2, 4.$$

Since y is an integer, it follows that its only possible values are given by $y = 0, 2, 4$. When $y = 1, 3$, the discriminant is 12, which is not a perfect square. Thus, $y = 0, 2, 4$.

When $y = 0$, we have $x^2 - 2x = 0 \implies x = 0, 2$.

When $y = 2$, we have $x^2 - 4x = 0 \implies x = 0, 4$.

When $y = 4$, we have $x^2 - 6x + 8 = 0 \implies x = 2, 4$.

Thus, there are 6 solutions, namely $(0, 0), (2, 0), (0, 2), (4, 2), (2, 4), (4, 4)$.

32. Ans: 133

From (i), it is easy to see that the whole ordered 7-tuple is determined by a_1 and a_2 . It is easy to check that $a_6 = 3a_1 + 5a_2$. Thus by (ii), we have $3a_1 + 5a_2 = 2005$. It follows that a_1 is divisible by 5. Write $a_1 = 5k$. Then we have $15k + 5a_2 = 2005 \implies 3k + a_2 = 401$. The possible values of a_2 are $\{2, 5, 8, \dots, 398\}$, if both k

and a_2 are positive. Note that each of the above values of a_2 determines a unique positive integral value of a_1 satisfying the equation $3a_1 + 5a_2 = 2005$. Therefore, there are 133 such sequences.

33. Ans: 12

Let α be an integer satisfying the given equation, so that

$$\begin{aligned} 4\alpha^2 - (4\sqrt{3} + 4)\alpha + \sqrt{3}n - 24 &= 0 \\ \implies 4\alpha^2 - 4\alpha - 24 &= \sqrt{3}(4\alpha - n). \end{aligned}$$

Since $\sqrt{3}$ is irrational, it follows that

$$4\alpha^2 - 4\alpha - 24 = 0 \quad \text{and} \quad 4\alpha - n = 0.$$

Substituting $\alpha = \frac{n}{4}$ into the first equation, we have

$$\begin{aligned} 4\left(\frac{n}{4}\right)^2 - 4\left(\frac{n}{4}\right) - 24 &= 0 \\ \implies n^2 - 4n - 96 &= 0 \\ \implies (n - 12)(n + 8) &= 0 \\ \implies n &= 12, -8. \end{aligned}$$

Since n is a positive integer, we have $n = 12$.

34. Ans: 2

Subtracting the second equation from the first one, we have

$$\begin{aligned} xy + yz &= 255 - 224 = 31 \\ \implies y(x + z) &= 31. \end{aligned}$$

Since 31 is a prime number, we have $y = 1$ and $x + z = 31$. Together with the first given equation, we have

$$\begin{aligned} x \cdot 1 + x(31 - x) &= 255 \\ \implies (x - 15)(x - 17) &= 0 \\ \implies x &= 15 \quad \text{or} \quad 17. \end{aligned}$$

When $x = 15$, $y = 1$, we have $z = 16$.

When $x = 17$, $y = 1$, we have $z = 14$.

Therefore, there are two such solutions, namely $(15, 1, 16)$ and $(17, 1, 14)$.

35. Ans: 210

Let $abcd = 1000a + 100b + 10c + d$ be a four-digit number divisible by 7. In particular, $1 \leq a \leq 9$. We are given that $dbca = 1000d + 100b + 10c + a$ is also divisible by 7. Observe that

$$abcd - dbca = 999a - 999d = 999(a - d).$$

Since 7 does not divide 999, it follows that 7 must divide $a - d$. Thus $a \equiv d \pmod{7}$. Since 7 divides 1001, we have

$$\begin{aligned} 1000a + 100b + 10c + d &\equiv 0 \pmod{7} \\ \implies -a + 100b + 10c + d &\equiv 0 \pmod{7} \\ \implies 100b + 10c &\equiv 0 \pmod{7} \\ \implies 10(10b + c) &\equiv 0 \pmod{7} \\ \implies 10b + c &\equiv 0 \pmod{7}. \end{aligned}$$

Therefore we must have $a \equiv d \pmod{7}$ and $10b + c \equiv 0 \pmod{7}$.

Conversely, by reversing the above arguments, one easily sees that if $a \equiv d \pmod{7}$ and $10b + c \equiv 0 \pmod{7}$, then both $abcd$ and $dbca$ are divisible by 7.

Now there are 14 pairs of (a, b) satisfying $a \equiv d \pmod{7}$, which are

$$(1, 1), (2, 2), (3, 3), \dots, (9, 9), (1, 8), (2, 9), (7, 0), (8, 1), (9, 2).$$

Since $0 \leq 10b + c \leq 99$, there are 15 pairs of (b, c) satisfying $10b + c \equiv 0 \pmod{7}$ (including $(0, 0)$). In fact, this is equal to the number of non-negative integers less than 100 that are divisible by 7. Therefore, there are $14 \times 15 = 210$ numbers with the required properties.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2005

(Senior Section, Special Round Solutions)

1. Since the sums of the digits of the two numbers are the same, they leave the same remainder when they are divided by 9. Thus their difference is divisible by 9. So the possible answers are 36 or 81. It is easy to see that both cases are possible: For example: (645, 564) and (218, 182).
2. Let M and N be points on AB and AD so that $MI \parallel BC$ and $NJ \parallel DC$. Let $a = DI = CF$, and $b = BJ = CE$. Then

$$\begin{aligned} \triangle KJB &\simeq \triangle KIM, & \triangle LJN &\simeq \triangle LID \\ \triangle NJE &\simeq \triangle DCE, & \triangle BCF &\simeq \triangle MIF \end{aligned}$$

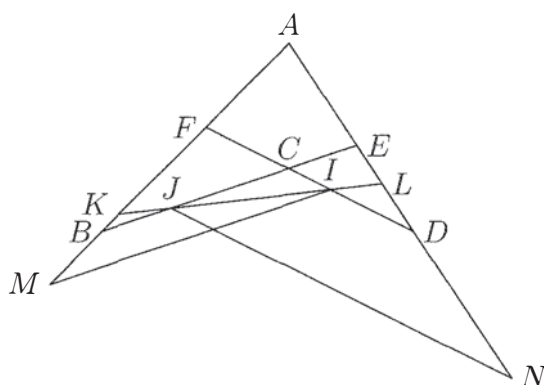
Thus

$$\begin{aligned} \frac{KJ}{b} &= \frac{KJ + JI}{MI}, & \frac{JI + IL}{NJ} &= \frac{IL}{a} \\ \frac{NJ}{b + CJ} &= \frac{a + IC}{b}, & \frac{b + CJ}{a} &= \frac{MI}{a + IC} \end{aligned}$$

Multiplying the four equations we get

$$\frac{(KJ)(JI) + (KJ)(IL)}{ab} = \frac{(IL)(JI) + (KJ)(IL)}{ab}$$

which yields $KJ = IL$.



3. Suppose that this result is not true. Observe that S has total of $\binom{10}{2} = 45$ 2-element subsets. Let $S_i = \{x_{i1}, x_{i2}\}$, $i = 1, 2, \dots, 45$, be the 45 2-element subsets of S and $s_i = x_{i1} + x_{i2}$. Then by the assumption, the 45 values s_i are mutually distinct. Since $3 \leq s_i \leq 47$, and there are exactly 45 numbers from 3 through 47, we have

$\{s_i : i = 1, \dots, 45\} = \{3, 4, \dots, 47\}$. Since there is pair summing to 3 and a pair summing to 47, the numbers $1, 2, 23, 24 \in S$. But then $1 + 24 = 2 + 23$ gives rise to a contradiction.

4. Let $m = 10001000000000$. Let $M_n = (m + n)!$, where n is an integer such that $0 \leq n \leq 10000$. Observe that if $M_n = \overline{abcde\dots}$, where a, b, c, d, \dots are the digits of M_n , then the first 4 digits of M_{n+1} are (i) \overline{abcd} , in which case the fifth digit is $a + e > e$, (ii) the first four digits of $\overline{abcd} + 1$ or (iii) the first four digits of $\overline{abcd} + 2$. The last case can happen only when $a = 9$. Therefore we see that if $a < 9$, among the numbers M_0, M_1, \dots, M_{10} , there is at least one for which the first four digits are the first four digits of $\overline{xyzw} + 1$ where \overline{xyzw} are the first four digits of M_0 . It follows that the first four digits of the numbers $M_0, M_1, \dots, M_{10000}$ includes all of $1000, 1001, \dots, 8999$. Hence the answer is yes.

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2005
(Open Section, Round 1)

Wednesday, 1 June 2005

0930-1200

Answer ALL 25 questions.

Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

1. Find the last three digits of $9^{100} - 1$.
2. Circles \mathcal{C}_1 and \mathcal{C}_2 have radii 3 and 7 respectively. The circles intersect at distinct points A and B . A point P outside \mathcal{C}_2 lies on the line determined by A and B at a distance of 5 from the center of \mathcal{C}_1 . Point Q is chosen on \mathcal{C}_2 so that PQ is tangent to \mathcal{C}_2 at Q . Find the length of the segment PQ .
3. Find the largest positive integer n such that $n!$ ends with exactly 100 zeros.
4. Find the largest value of $3k$ for which the following equation has a real root:

$$\sqrt{x^2 - k} + 2\sqrt{x^2 - 1} = x.$$

5. Find the last two digits (in order) of $7^{3^{2005}}$.
6. From the first 2005 natural numbers, k of them are arbitrarily chosen. What is the least value of k to ensure that there is at least one pair of numbers such that one of them is divisible by the other?
7. Let $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$, where a_i are nonnegative integers for $i = 0, 1, 2, \dots, n$. If $f(1) = 21$ and $f(25) = 78357$, find the value of $f(10)$.
8. If x is a real number that satisfies

$$\left\lfloor x + \frac{11}{100} \right\rfloor + \left\lfloor x + \frac{12}{100} \right\rfloor + \dots + \left\lfloor x + \frac{99}{100} \right\rfloor = 765,$$

find the value of $\lfloor 100x \rfloor$. Here $\lfloor a \rfloor$ denotes the largest integer $\leq a$.

9. The function $f(n)$ is defined for all positive integer n and take on non-negative integer values such that $f(2) = 0$, $f(3) > 0$ and $f(9999) = 3333$. Also, for all m, n ,

$$f(m+n) - f(m) - f(n) = 0 \quad \text{or} \quad 1.$$

Determine $f(2005)$.

10. It is known that the 3 sides of a triangle are consecutive positive integers and the largest angle is twice the smallest angle. Find the perimeter of this triangle.
11. A triangle $\triangle ABC$ is inscribed in a circle of radius 1, with $\angle BAC = 60^\circ$. Altitudes AD and BE of $\triangle ABC$ intersect at H . Find the smallest possible value of the length of the segment AH .
12. Let \mathbb{N} be the set of all positive integers. A function $f : \mathbb{N} \rightarrow \mathbb{N}$ satisfies $f(m+n) = f(f(m)+n)$ for all $m, n \in \mathbb{N}$, and $f(6) = 2$. Also, no two of the values $f(6), f(9), f(12)$ and $f(15)$ coincide. How many three-digit positive integers n satisfy $f(n) = f(2005)$?

13. Let f be a real-valued function so that

$$f(x, y) = f(x, z) - 2f(y, z) - 2z$$

for all real numbers x, y and z . Find $f(2005, 1000)$.

14. Let $a_1 = 2006$, and for $n \geq 2$,

$$a_1 + a_2 + \cdots + a_n = n^2 a_n.$$

What is the value of $2005 a_{2005}$?

15. Find the smallest three-digit number n such that if the three digits are a, b and c , then

$$n = a + b + c + ab + bc + ac + abc.$$

16. Let $a_1 = 21$ and $a_2 = 90$, and for $n \geq 3$, let a_n be the last two digits of $a_{n-1} + a_{n-2}$. What is the remainder of $a_1^2 + a_2^2 + \cdots + a_{2005}^2$ when it is divided by 8.
17. Find the smallest two-digit number N such that the sum of digits of $10^N - N$ is divisible by 170.

18. Find the least n such that whenever the elements of the set $\{1, 2, \dots, n\}$ are coloured red or blue, there always exist x, y, z, w (not necessarily distinct) of the same colour such that $x + y + z = w$.
19. Let x and y be positive integers such that $\frac{100}{151} < \frac{y}{x} < \frac{200}{251}$. What is the minimum value of x ?
20. Find the maximum positive integer n such that

$$n^2 \leq 160 \times 170 \times 180 \times 190.$$

21. Find the number of positive integers n such that

$$n + 2n^2 + 3n^3 + \dots + 2005n^{2005}$$

is divisible by $n - 1$.

22. Given ten 0's and ten 1's, how many 0-1 binary sequences can be formed such that no three or more than three 0's are together? For example, 01001001010011101011 is such a sequence, but the sequence 01001000101001110111 does not satisfy this condition.
23. In triangle ABC , $AB = 28$, $BC = 21$ and $CA = 14$. Points D and E are on AB with $AD = 7$ and $\angle ACD = \angle BCE$. Find the length of BE .
24. Four points in the order A, B, C, D lie on a circle with the extension of AB meeting the extension of DC at E and the extension of AD meeting the extension of BC at F . Let EP and FQ be tangents to this circle with points of tangency P and Q respectively. Suppose $EP = 60$ and $FQ = 63$. Determine the length of EF .
25. A pentagon $ABCDE$ is inscribed in a circle of radius 10 such that BC is parallel to AD and AD intersects CE at M . The tangents to this circle at B and E meet the extension of DA at a common point P . Suppose $PB = PE = 24$ and $\angle BPD = 30^\circ$. Find BM .

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2005
(Open Section, Special Round)

Saturday, 2 July 2005

0900– 1330

Attempt as many questions as you can.

No calculators are allowed.

Show the steps in your calculations.

Each question carries 10 marks.

1. An integer is square-free if it is not divisible by a^2 for any integer $a > 1$. Let S be the set of positive square-free integers. Determine, with justification, the value of

$$\sum_{k \in S} \left\lfloor \sqrt{\frac{10^{10}}{k}} \right\rfloor,$$

where $\lfloor x \rfloor$ denote the greatest integer less than or equal to x . For example $\lfloor 2.5 \rfloor = 2$.

2. Let G be the centroid of $\triangle ABC$. Through G draw a line parallel to BC and intersecting the sides AB and AC at P and Q , respectively. Let BQ intersect GC at E and CP intersect GB at F . Prove that triangles ABC and DEF are similar.
3. Let a, b, c be real numbers satisfying $a < b < c$, $a + b + c = 6$ and $ab + bc + ac = 9$. Prove that $0 < a < 1 < b < 3 < c < 4$.
4. Place 2005 points on the circumference of a circle. Two points P, Q are said to form a pair of *neighbours* if the chord PQ subtends an angle of at most 10° at the centre. Find the smallest number of pairs of neighbours.

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2005
(Open Section, Round 1 Solutions)

1. Ans: 000

$$9^{100} - 1 = (1 - 10)^{100} - 1 = 1 - \binom{100}{1}10^1 + \binom{100}{2}10^2 - \cdots + \binom{100}{100}10^{100} - 1 = 1000k$$

for some integer k . Thus, the last three digits are 000.

2. Ans: 4

P lies on the radical axis of \mathcal{C}_1 and \mathcal{C}_2 and hence has equal power with respect to both circles. Now

$$\begin{aligned} PQ^2 &= \text{Power of } P \text{ with respect to } \mathcal{C}_2 \\ &= \text{Power of } P \text{ with respect to } \mathcal{C}_1 \\ &= 5^2 - 3^2 = 16. \end{aligned}$$

3. Ans: 409

If $n!$ ends with exactly 100 zeros, then in the prime factorization of $n!$, the prime factor 5 occurs exactly 100 times (we need not worry about the prime factor 2 since it will occur more than 100 times). The number of times that 5 occurs in $n!$ is given by

$$\left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{5^2} \right\rfloor + \left\lfloor \frac{n}{5^3} \right\rfloor + \cdots.$$

Also the factor 5 occurs 24 times in $100!$. Thus the answer is about 400. Now

$$\left\lfloor \frac{400}{5} \right\rfloor + \left\lfloor \frac{400}{5^2} \right\rfloor + \left\lfloor \frac{400}{5^3} \right\rfloor + \cdots = 80 + 16 + 3 = 99.$$

It follows that $400!$ ends with exactly 99 zeros. Thus the answer is 409.

4. Ans: 4

For the equation to have a solution, k must be nonnegative, for if $k < 0$, then

$$\sqrt{x^2 - k} + 2\sqrt{x^2 - 1} \geq \sqrt{x^2 - k} > x.$$

Rewrite the given equation as $2\sqrt{x^2 - 1} = x - \sqrt{x^2 - k}$. By squaring we obtain $2x^2 + k - 4 = -2x\sqrt{x^2 - k}$. Squaring again and solving for x , we obtain

$$x^2 = \frac{(k - 4)^2}{8(2 - k)}.$$

Thus $0 \leq k < 2$ and the only possible solution is $x = \frac{4-k}{\sqrt{8(2-k)}}$. Substitute this value of x into the given equation and multiplying throughout by $\sqrt{8(2-k)}$, we obtain

$$\sqrt{(k-4)^2 - 8k(2-k)} + 2\sqrt{(k-4)^2 - 8(2-k)} = 4 - k.$$

Hence, $|3k-4| + 2k = 4-k$ and so $|3k-4| = -(3k-4)$ which holds if and only if $3k-4 \leq 0$; i.e. $3k \leq 4$. Therefore, the maximum value of $3k$ is 4.

5. Ans: 43

Note that $7^4 = 1 \pmod{100}$. Now

$$3^{2005} = (-1)^{2005} = -1 = 3 \pmod{4}.$$

Hence $3^{2005} = 4k + 3$ for some positive integer k . Thus

$$7^{3^{2005}} = 7^{4k+3} = 7^3 = 43 \pmod{100}.$$

6. Ans: 1004

Take any set 1004 numbers. Write each number in the form $2^{a_i}b_i$, where $a_i \geq 0$ and b_i is odd. Thus obtaining 1004 odd numbers b_1, \dots, b_{1004} . Since there are 1003 odd numbers in the first 2005 natural numbers, at least two of the odd numbers are the same, say $b_i = b_j$. Then $2^{a_i}b_i \mid 2^{a_j}b_j$ if $2^{a_i}b_i < 2^{a_j}b_j$. The numbers 1003, 1004, \dots , 2005 are 1003 numbers where there is no pair such that one of them is divisible by the other. So the answer is 1004.

7. Ans: 5097

We use the fact that every positive integer can be expressed uniquely in any base ≥ 2 . As $f(1) = 21$, we see that the coefficients of $f(x)$ are nonnegative integers less than 25. Therefore $f(25) = 78357$, when written in base 25, allows us to determine the coefficients. Since

$$78357 = 7 + 9 \times 25 + 5 \times 25^3,$$

we obtain $f(x) = 7 + 9x + 5x^3$. Hence $f(10) = 5097$.

8. Ans: 853

First observe that $\lfloor x + \frac{k}{100} \rfloor = \lfloor x \rfloor$ or $\lfloor x \rfloor + 1$ for $11 \leq k \leq 99$. Since there are 89 terms on the left-hand side of the equation and $89 \times 8 < 765 < 89 \times 9$, we deduce that $\lfloor x \rfloor = 8$. Now suppose $\lfloor x + \frac{k}{100} \rfloor = 8$ for $11 \leq k \leq m$ and $\lfloor x + \frac{k}{100} \rfloor = 9$ for $m+1 \leq k \leq 99$. Then

$$8(m-10) + 9(99-m) = 765,$$

which gives $m = 46$. Therefore $\lfloor x + \frac{46}{100} \rfloor = 8$ and $\lfloor x + \frac{47}{100} \rfloor = 9$, which imply $8 \leq x + \frac{46}{100} < 9$ and $9 \leq x + \frac{47}{100} < 10$ respectively. The inequalities lead to

$7.54 \leq x < 8.54$ and $8.53 \leq x < 9.53$. Consequently, we see that $8.53 \leq x < 8.54$. Hence we conclude that $\lfloor 100x \rfloor = 853$.

9. Ans: 668

The given relation implies that

$$f(m+n) \geq f(m) + f(n).$$

Putting $m = n = 1$ we obtain $f(2) \geq 2f(1)$. Since $f(2) = 0$ and $f(1)$ is non-negative, $f(1) = 0$. Next, since

$$f(3) = f(2+1) = f(2) + f(1) + \{0 \text{ or } 1\} = \{0 \text{ or } 1\},$$

where $f(3) > 0$, it follows that $f(3) = 1$. Now, it can be proved inductively that $f(3n) \geq n$ for all n . Also, if $f(3n) > n$ for some n , then $f(3m) > m$ for all $m \geq n$. So since $f(3 \times 3333) = f(9999) = 3333$, it follows that $f(3n) = n$ for $n \leq 3333$. In particular $f(3 \times 2005) = 2005$. Consequently,

$$\begin{aligned} 2005 &= f(3 \times 2005) \\ &\geq f(2 \times 2005) + f(2005) \\ &\geq 3f(2005) \end{aligned}$$

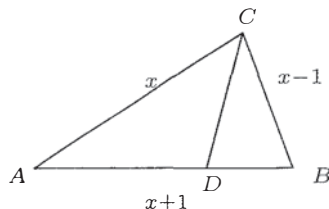
and so $f(2005) \leq 2005/3 < 669$. On the other hand,

$$f(2005) \geq f(2004) + f(1) = f(5 \times 668) = 668.$$

Therefore $f(2005) = 668$.

10. Ans: 15

Let $\angle C = 2\angle A$ and CD the bisector of $\angle C$. Let $BC = x - 1$, $CA = x$ and $AB = x + 1$. Then $\triangle ABC$ is similar to $\triangle CBD$. Thus $BD/BC = BC/AB$ so that $BD = (x - 1)^2/(x + 1)$. Also $CD/AC = CB/AB$ so that $AD = CD = x(x - 1)/(x + 1)$.



As $AB = AD + BD$, we have $x(x - 1)/(x + 1) + (x - 1)^2/(x + 1) = x + 1$. Solving this, the only positive solution is $x = 5$. Thus the perimeter of the triangle is $4 + 5 + 6 = 15$.

11. Ans: 1

$AE = AB \cos 60^\circ$. $\angle AHE = \angle ACB$. Hence

$$AH = AE / \sin \angle AHE = AB \cos 60^\circ / \sin \angle ACB = 2R \cos 60^\circ,$$

where $R = 1$ is the radius of the circle. Therefore, $AH = 1$.

12. Ans: 225

Since $f(6+n) = f(f(6)+n) = f(2+n)$ for all n , the function f is periodic with period 4 starting from 3 onwards. Now $f(6), f(5) = f(9), f(4) = f(12)$ and $f(3) = f(15)$ are four distinct values. Thus in every group of 4 consecutive positive integers ≥ 3 , there is exactly one that is mapped by f to $f(2005)$. Since the collection of three-digit positive integers can be divided into exactly 225 groups of 4 consecutive integers each, there are 225 three-digit positive integers n that satisfies $f(n) = f(2005)$.

13. Ans: 5

Setting $x = y = z$, we see that

$$f(x, x) = f(x, x) - 2f(x, x) - 2x$$

for all x . Hence $f(x, x) = -x$ for all x . Setting $y = x$ gives

$$f(x, x) = f(x, z) - 2f(x, z) - 2z$$

for all x and z . Hence $f(x, z) = -f(x, x) - 2z = x - 2z$ for all x and z . Therefore, $f(2005, 1000) = 5$.

14. Ans: 2

$$a_n = \sum_{i=1}^n a_i - \sum_{i=1}^{n-1} a_i = n^2 a_n - (n-1)^2 a_{n-1}.$$

This gives $a_n = \frac{n-1}{n+1} a_{n-1}$. Thus,

$$a_{2005} = \frac{2004}{2006} a_{2004} = \cdots = \frac{(2004)(2003) \cdots (1)}{(2006)(2005) \cdots (3)} a_1 = \frac{2}{2005}.$$

15. Ans: 199

From $100a + 10b + c = a + b + c + ab + bc + ac + abc$, we get $(99 - b - c - bc)a = b(c - 9)$. Note that LHS ≥ 0 (since $b + c + bc = (1 + b)(1 + c) - 1 \leq 99$) and RHS ≤ 0 . Thus we must have $b = c = 9$. Since $a \geq 1$, $n = 199$ is the answer.

16. Ans: 1

Note that the value of $x^2 \pmod{8}$ is completely determined by the value of $x \pmod{4}$. Furthermore, $a_n \equiv a_{n-1} + a_{n-2} \pmod{100} \equiv a_{n-1} + a_{n-2} \pmod{4}$. Thus the sequence $(a_n \pmod{4})_{n \in \mathbb{Z}^+}$ equals $(1, 2, 3, 1, 0, 1, 1, 2, 3, 1, 0, 1, 1, 2, \dots)$, which has a cycle of length 6. Also, the sum of the squares in one cycle is 0 $\pmod{8}$. Now $2005 = 6 \times 334 + 1$. Thus the required answer is 1.

17. Ans: 20

Let $N = 10a + b$. When $b \neq 0$, $10^N - N$ is a N -digit number whose first $N - 2$ digits are 9 and the last two digits are $(9 - a)$ and $(10 - b)$. Thus the sum of digits of $10^N - N$ equals $9(N - 2) + 19 - (a + b) = 89a + 8b + 1$. Thus, we need

$$\begin{aligned} a &\equiv 1 \pmod{2}; \\ -a + 3b &\equiv -1 \pmod{5}; \\ 4a + 8b &\equiv -1 \pmod{17}. \end{aligned}$$

$a = 1$ has no solution, but $a = 3, b = 9$ is a solution.

When $b = 0$, the sum of digits is $9(N - 2) + (10 - a) = 89a - 8$. Thus a is even. $a = 2$ is a solution. Thus the answer is 20 since it is < 39 .

18. Ans: 11

It can easily be checked that if 1, 2, 9, 10 are red and 3, 4, 5, 6, 7, 8 are blue, then there do not exist x, y, z, w of the same colour such that $x + y + z = w$. So $n \geq 11$.

Suppose there exists a colouring (for $n = 11$) such that there do not exist x, y, z, w of the same colour such that $x + y + z = w$. Then 1, 9 must be of a different colour from 3. Thus say 1 and 9 are red, and 3 is blue. Since $7 + 1 + 1 = 9$, 7 must then be blue. Now, $2 + 3 + 3 = 7$, so that 2 is red, and hence 4 ($= 1 + 1 + 2$) is blue. But $11 = 3 + 4 + 4 = 9 + 1 + 1$, a contradiction.

19. Ans: 3

The inequality can be transformed to

$$\frac{302y}{200} > x > \frac{251y}{200}.$$

We first need to the minimum positive integer y such that the interval $(\frac{302y}{200}, \frac{251y}{200})$ contains an integer. This minimum value of y is 2. When $y = 2$, this interval contains only one integer, that is 3. Thus the answer is 3.

20. Ans: 30499

Let $f = 160 \times 170 \times 180 \times 190$. Observe that $f = (175^2 - 15^2)(175^2 - 5^2)$. Let $x = 175^2 - 125$. Then $f = (x - 100)(x + 100) = x^2 - 10000 < x^2$. Since $x = 175^2 - 125$

and $f - (x - 1)^2 = x^2 - 1000 - (x - 1)^2 = 2x - 10001$, we have $f > (x - 1)^2$. Thus the minimum positive integer n such that $n^2 < f$ is $x - 1 = 175^2 - 126 = 30499$.

21. Ans: 16

Let $n + \dots + 2005n^{2005} = f(n)(n - 1) + a$ where $f(n)$ is a polynomial of n . If $n = 1$, we have $a = 1 + 2 + \dots + 2005 = 2005 \times 1003$. Observe that $n + \dots + 2005n^{2005}$ is divisible by $n - 1$ if and only if $a = 2005 \times 1003$ is divisible by $n - 1$. The prime factorization of a is $a = 5 \times 401 \times 17 \times 59$. So a has only $2^4 = 16$ factors. Thus there are 16 possible positive integers n such that $n + \dots + 2005n^{2005}$ is divisible by $n - 1$.

22. Ans: 24068

Consider the sequence of ten 1's. There are eleven spaces between two 1's, or before the leftmost 1 or after the rightmost 1. For each of these spaces, we can put either two 0's (double 0's) or only one 0 (single 0). If there are exactly k doubles 0's, then there are only $10 - 2k$ single 0's. The number of ways for this to happen is $\binom{11}{k} \binom{11-k}{10-2k}$. Thus the answer is

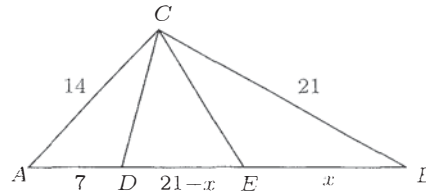
$$\sum_{k=0}^5 \binom{11}{k} \binom{11-k}{10-2k} = 24068.$$

23. Ans: 12

Let $BE = x$. By cosine rule,

$$\cos B = \frac{21^2 + 28^2 - 14^2}{2(21)(28)} = \frac{7}{8} \quad \text{and} \quad \cos A = \frac{14^2 + 28^2 - 21^2}{2(14)(28)} = \frac{11}{16}.$$

Thus $CD^2 = 14^2 + 7^2 - 2(14)(7)(\frac{11}{16}) = \frac{441}{4}$ so that $CD = \frac{21}{2}$.



Also

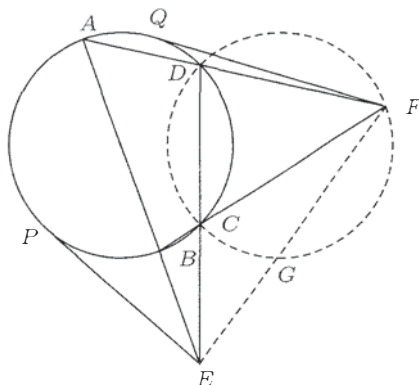
$$\cos \angle ACD = \frac{14^2 + (\frac{21}{2})^2 - 7^2}{2(14)(\frac{21}{2})} = \frac{7}{8}.$$

Thus $\angle B = \angle ACD = \angle BCE$ so that triangle BEC is isosceles. Therefore,

$$x = \frac{BC}{2 \cos B} = \frac{21}{2} \cdot \frac{8}{7} = 12.$$

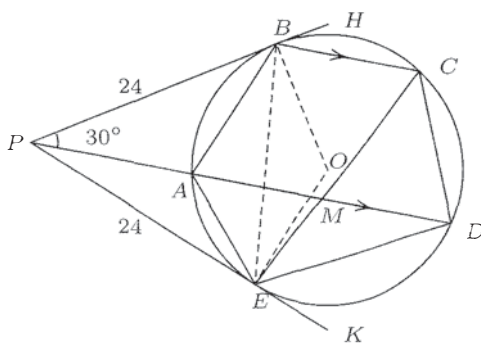
24. Ans: 87

Let the circumcircle of $\triangle CDF$ meet the line EF at G . Then G, C, D, F are concyclic. Now $\angle EBC = \angle ADC = \angle CGF$ so that E, B, C, G are also concyclic. Thus, $EP^2 = EB \cdot EA = EC \cdot ED = EG \cdot EF$ and $FQ^2 = FD \cdot FA = FC \cdot FB = FG \cdot FE$. Therefore, $EP^2 + FQ^2 = EG \cdot EF + FG \cdot FE = (EG + FG) \cdot EF = EF^2$. Consequently, $EF = \sqrt{60^2 + 63^2} = 87$.



25. Ans: 13

Let O be the centre of the circle. First we show that P, B, O, M, E lie on a common circle. Clearly P, B, O, E are concyclic. As $\angle BPM = \angle HBC = \angle BEM = 30^\circ$, points P, B, M, E are also concyclic. Thus P, B, O, M, E all lie on the circumcircle of $\triangle PBE$.



Since $\angle PBO = 90^\circ$, PO is a diameter of this circle and $PO = \sqrt{24^2 + 10^2} = 26$. This circle is also the circumcircle of $\triangle PBM$. Therefore, $BM = 26 \sin 30^\circ = 13$.

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2005
 (Open Section, Special Round Solutions)

1. We shall show that in general for any $n \in \mathbb{N}$,

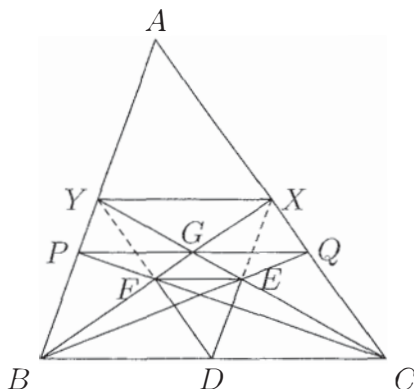
$$\sum_{k \in S} \left\lfloor \sqrt{\frac{n}{k}} \right\rfloor = n.$$

For any $k \in S$, note that $\lfloor \sqrt{n/k} \rfloor = j$ if and only if j is the largest integer such that $j^2 k \leq n$. If $S_k = \{1^2 k, 2^2 k, \dots, j^2 k\}$, where j is the largest integer such that $j^2 k \leq n$, then $\lfloor \sqrt{n/k} \rfloor = |S_k|$. Also if k and m are distinct square free integers, then $S_k \cap S_m = \emptyset$. Hence

$$\sum_{k \in S} \left\lfloor \sqrt{\frac{n}{k}} \right\rfloor = \sum_{k \in S} |S_k| = \left| \bigcup_{k \in S} S_k \right|.$$

Since every integer can be uniquely written as $j^2 k$ where k is square free, $\bigcup_{k \in S} S_k = \{1, 2, \dots, n\}$. This completes the proof.

2. First we note the centroid divides each median in the ratio 1:2. Let X be the midpoint of AC and Y be the midpoint of AB . Then $XQ/QC = XG/GB = 1/2$. So, by Ceva's theorem, X, E, D are collinear. Likewise, D, F, Y are collinear. Therefore $DE \parallel BA$ and E is the midpoint of CY . Similarly, $DF \parallel CA$ and F is the midpoint of DY . Thus $EF \parallel XY \parallel CB$. Since the corresponding sides are parallel, the two triangles are similar.



3. Firstly, $a, b, c \neq 0$ since if, for example, $c = 0$, then $a + b = 6$ and $ab = 9$ imply that $a = b = 3$, a contradiction. Also, a, b, c satisfies $x^3 - 6x^2 + 9x - abc = 0$, ie. $(x - 3)^2 = abc/x$. Since $(x - 3)^2 \geq 0$ for all x , we see that a, b, c are all of the same sign, and thus $a, b, c > 0$ as $a + b + c = 6$.

Now the graph $y = x^3 - 6x^2 + 9x - abc$ has stationary points at $x = 1$ and $x = 3$, and cuts the x -axis at $x = a, b, c$, so that $a \in (-\infty, 1)$, $b \in (1, 3)$ and $c \in (3, \infty)$. Finally, if $c > 4$, then $ab = (c - 3)^2 > 1$, so that $ab + bc + ac > 1 + 4(b + 1/b)$. But $x + 1/x$ is increasing for $x \in (1, \infty)$. Thus $1 + 4(b + 1/b) > 9$, a contradiction.

Second solution: As in the first solution, a, b and c satisfy the equation $x^3 - 6x^2 + 9x - abc = 0$, or $x(x - 3)^2 = abc$. Substitute $x = c$ to get $c(c - 3)^2 = abc$ or $(c - 3)^2 = ab$. Now, assume $c \geq 4$. This implies that $ab \geq 1$, which in turn implies $a + b > 2$ (via AM-GM). Thus $c < 4$, a contradiction. Hence $c < 4$. We get $abc = c(c - 3)^2 < 4$. ($c > 2$, since $a + b + c = 6$)

For the equation $f(x) = abc$, where $abc < 4$, $f(0) = 0$, $f(1) = 4$ implies one root between 0 and 1 $f(1) = 4$, $f(3) = 0$ implies one root between 1 and 3. $f(3) = 0$, $f(4) = 4$ implies one root between 3 and 4.

4. Let P, Q be a pair of neighbours Suppose P has k neighbours who are not neighbours of Q and Q has ℓ neighbours who are not neighbours of P , with $k \geq \ell$. Then by moving P to a point very close to Q the number of neighbouring pairs does not increase and the set of neighbours of P and Q coincide. Thus by repeating this procedure, we see that the minimum is attained when the points are divided into s clusters, $s \leq 35$, such that two points are neighbours iff they are in the same cluster. Thus, if n_i is the number of points in cluster i , then

$$\text{minimum} \geq \sum \binom{n_i}{2} \geq 25 \binom{57}{2} + 10 \binom{58}{2} = 56430.$$

The last inequality follows from the following: If the largest of the n_i 's, say a , is > 58 , then the smallest, say b , is ≤ 57 and $\binom{a}{2} + \binom{b}{2} \geq \binom{a-1}{2} + \binom{b+1}{2}$ which can be proved by direct computation. This minimum is attained when there are 35 clusters, 25 with 57 points and 10 with 58 points.