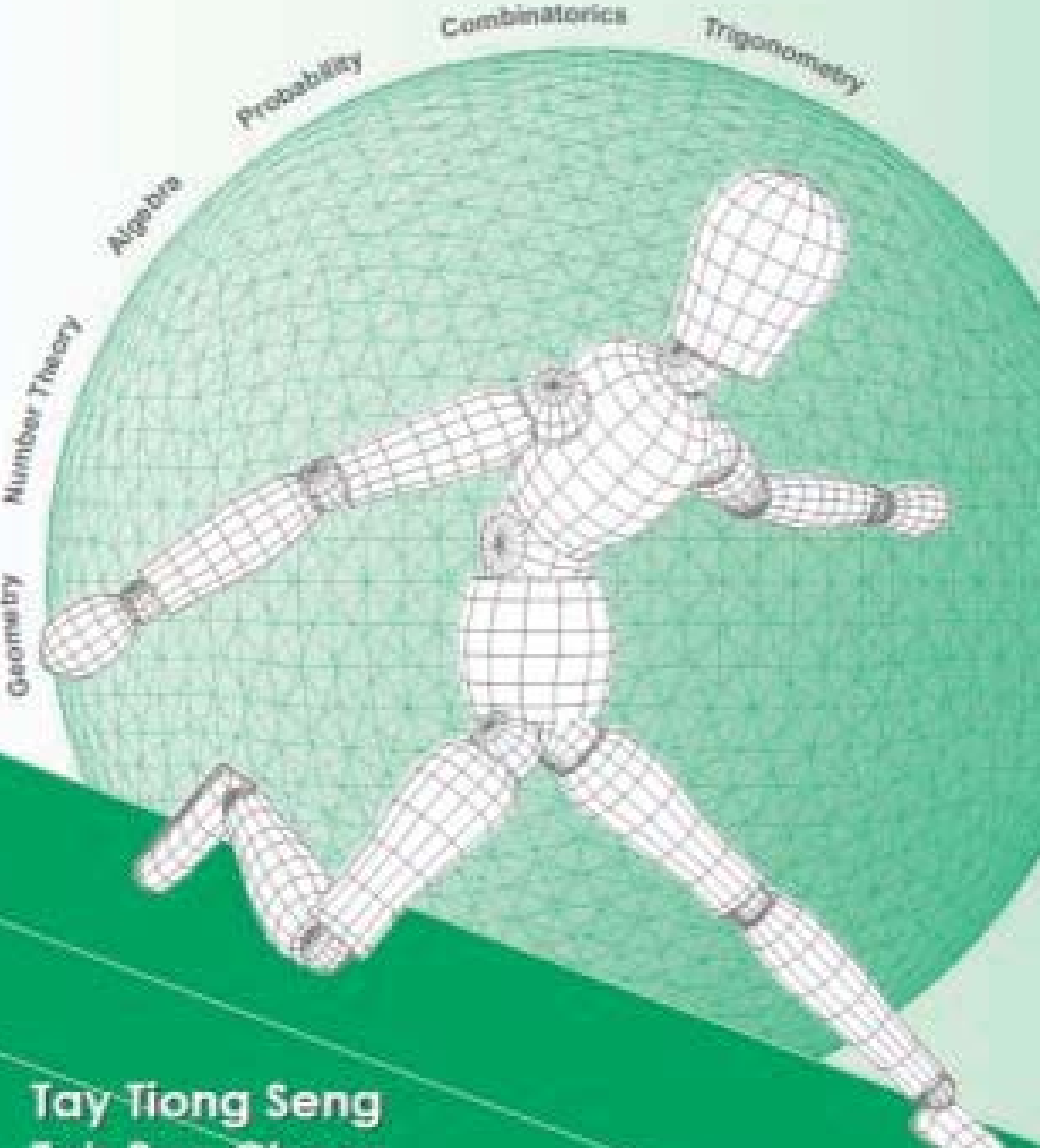
SINGAPORE MATHEMATICAL OLYMPIADS 2013



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Singapore Mathematical Olympiad (SMO) 2013

Junior Section (First Round)

Tuesday, 4 June 2013 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 35 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*
8. *Throughout this paper, let* $\lfloor x \rfloor$ *denote the greatest integer less than or equal to* x. *For example,* $\lfloor 2.1 \rfloor = 2$, $\lfloor 3.9 \rfloor = 3$.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Multiple Choice Questions

1. If $a = 8^{53}$, $b = 16^{41}$ and $c = 64^{27}$, then which of the following inequalities is true?

   (A) $a > b > c$ (B) $c > b > a$ (C) $b > a > c$ (D) $b > c > a$ (E) $c > a > b$

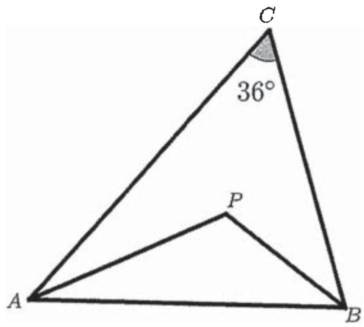
2. If a, b, c are real numbers such that $|a-b| = 1$, $|b-c| = 1$, $|c-a| = 2$ and $abc = 60$, find the value of $\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c}$.

   (A) $\frac{1}{30}$ (B) $\frac{1}{20}$ (C) $\frac{1}{10}$ (D) $\frac{1}{4}$ (E) None of the above

3. If x is a complex number satisfying $x^2 + x + 1 = 0$, what is the value of $x^{49} + x^{50} + x^{51} + x^{52} + x^{53}$?

   (A) -1 (B) $-\frac{1}{2}$ (C) 0 (D) $\frac{1}{2}$ (E) 1

4. In $\triangle ABC$, $\angle ACB = 36°$ and the interior angle bisectors of $\angle CAB$ and $\angle ABC$ intersect at P. Find $\angle APB$.



   (A) 72° (B) 108° (C) 126° (D) 136° (E) None of the above

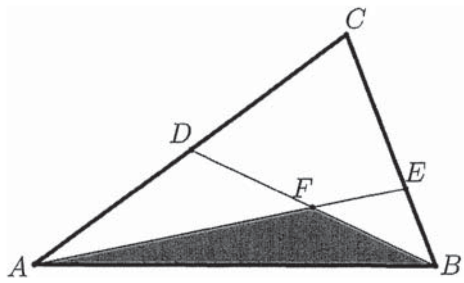
5. Find the number of integer pairs x, y such that $xy - 3x + 5y = 0$.

   (A) 1 (B) 2 (C) 4 (D) 8 (E) 16

6. Five young ladies were seated around a circular table. Miss Ong was sitting between Miss Lim and Miss Mak. Ellie was sitting between Cindy and Miss Nai. Miss Lim was between Ellie and Amy. Lastly, Beatrice was seated with Miss Poh on her left and Miss Mak on her right. What is Daisy's surname?

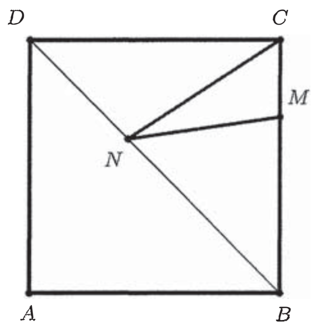
   (A) Lim (B) Mak (C) Nai (D) Ong (E) Poh

7. Given that ABC is a triangle with D being the midpoint of AC and E a point on CB such that $CE = 2EB$. If AE and BD intersect at point F and the area of $\triangle AFB = 1$ unit, find the area of $\triangle ABC$.



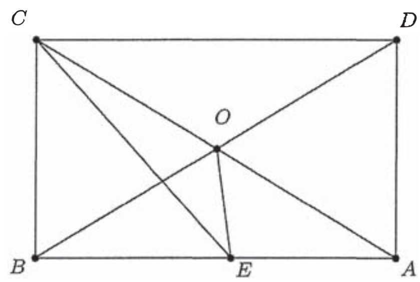
(A) 3 (B) $\frac{10}{3}$ (C) $\frac{11}{3}$ (D) 4 (E) 5

8. $ABCD$ is a square with sides 8 cm. M is a point on CB such that $CM = 2$ cm. If N is a variable point on the diagonal DB, find the least value of $CN + MN$.



(A) 8 (B) $6\sqrt{2}$ (C) 10 (D) $8\sqrt{2}$ (E) 12

9. $ABCD$ is a rectangle whose diagonals intersect at point O. E is a point on AB such that CE bisects $\angle BCD$. If $\angle ACE = 15°$, find $\angle BOE$.



(A) 60° (B) 65° (C) 70° (D) 75° (E) 80°

10. Let S be the smallest positive multiple of 15, that comprises exactly $3k$ digits with k '0's, k '3's and k '8's. Find the remainder when S is divided by 11.

(A) 0 (B) 3 (C) 5 (D) 6 (E) 8

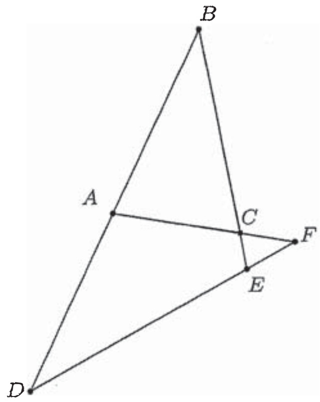
Short Questions

11. Find the value of $\sqrt{9999^2 + 19999}$.

12. If the graphs of $y = x^2 + 2ax + 6b$ and $y = x^2 + 2bx + 6a$ intersect at only one point in the xy-plane, what is the x-coordinate of the point of intersection?

13. Find the number of multiples of 11 in the sequence $99, 100, 101, 102, \cdots, 20130$.

14. In the figure below, BAD, BCE, ACF and DEF are straight lines. It is given that $BA = BC$, $AD = AF$, $EB = ED$. If $\angle BED = x°$, find the value of x.

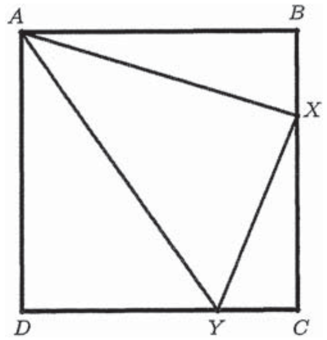


15. If $a = 1.69$, $b = 1.73$ and $c = 0.48$, find the value of

$$\frac{1}{a^2 - ac - ab + bc} + \frac{2}{b^2 - ab - bc + ac} + \frac{1}{c^2 - ac - bc + ab}.$$

16. Suppose that x_1 and x_2 are the two roots of the equation $(x-2)^2 = 3(x+5)$. What is the value of the expression $x_1x_2 + x_1^2 + x_2^2$?

17. Let $ABCD$ be a square and X and Y be points such that the lengths of XY, AX and AY are 6, 8 and 10 respectively. The area of $ABCD$ can be expressed as $\frac{m}{n}$ units where m and n are positive integers without common factors. Find the value of $m + n$.



18. Let x and y be real numbers satisfying the inequality

$$5x^2 + y^2 - 4xy + 24 \leq 10x - 1.$$

Find the value of $x^2 + y^2$.

19. A painting job can be completed by Team A alone in 2.5 hours or by Team B alone in 75 minutes. On one occasion, after Team A had completed a fraction $\frac{m}{n}$ of the job, Team B took over immediately. The whole painting job was completed in 1.5 hours. If m and n are positive integers with no common factors, find the value of $m + n$.

20. Let a, b and c be real numbers such that $\dfrac{ab}{a+b} = \dfrac{1}{3}$, $\dfrac{bc}{b+c} = \dfrac{1}{4}$ and $\dfrac{ca}{c+a} = \dfrac{1}{5}$. Find the value of $\dfrac{24abc}{ab+bc+ca}$.

21. Let x_1 and x_2 be two real numbers that satisfy $x_1 x_2 = 2013$. What is the minimum value of $(x_1 + x_2)^2$?

22. Find the value of $\sqrt{45 - \sqrt{2000}} + \sqrt{45 + \sqrt{2000}}$.

23. Find the smallest positive integer k such that $(k - 10)^{4026} \geq 2013^{2013}$.

24. Let a and b be two real numbers. If the equation $ax + (b - 3) = (5a - 1)x + 3b$ has more than one solution, what is the value of $100a + 4b$?

25. Let $S = \{1, 2, 3, \ldots, 48, 49\}$. What is the maximum value of n such that it is possible to select n numbers from S and arrange them in a circle in such a way that the product of any two adjacent numbers in the circle is less than 100?

26. Given any 4-digit positive integer x not ending in '0', we can reverse the digits to obtain another 4-digit integer y. For example if x is 1234 then y is 4321. How many possible 4-digit integers x are there if $y - x = 3177$?

27. Find the least positive integer n such that $2^8 + 2^{11} + 2^n$ is a perfect square.

28. How many 4-digit positive multiples of 4 can be formed from the digits $0, 1, 2, 3, 4, 5, 6$ such that each digit appears without repetition?

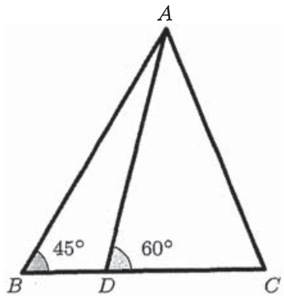
29. Let m and n be two positive integers that satisfy

$$\frac{m}{n} = \frac{1}{10 \times 12} + \frac{1}{12 \times 14} + \frac{1}{14 \times 16} + \cdots + \frac{1}{2012 \times 2014}.$$

Find the smallest possible value of $m + n$.

30. Find the units digit of $2013^1 + 2013^2 + 2013^3 + \cdots + 2013^{2013}$.

31. In $\triangle ABC$, $DC = 2BD$, $\angle ABC = 45°$ and $\angle ADC = 60°$. Find $\angle ACB$ in degrees.



32. If a and b are positive integers such that $a^2 + 2ab - 3b^2 - 41 = 0$, find the value of $a^2 + b^2$.

33. Evaluate the following sum

$$\left\lfloor \frac{1}{1} \right\rfloor + \left\lfloor \frac{1}{2} \right\rfloor + \left\lfloor \frac{2}{2} \right\rfloor + \left\lfloor \frac{1}{3} \right\rfloor + \left\lfloor \frac{2}{3} \right\rfloor + \left\lfloor \frac{3}{3} \right\rfloor + \left\lfloor \frac{1}{4} \right\rfloor + \left\lfloor \frac{2}{4} \right\rfloor + \left\lfloor \frac{3}{4} \right\rfloor + \left\lfloor \frac{4}{4} \right\rfloor + \left\lfloor \frac{1}{5} \right\rfloor + \cdots,$$

up to the 2013^{th} term.

34. What is the smallest possible integer value of n such that the following statement is always true?

In any group of $2n - 10$ *persons, there are always at least* 10 *persons who have the same birthdays.*

(For this question, you may assume that there are exactly 365 different possible birthdays.)

35. What is the smallest positive integer n, where $n \neq 11$, such that the highest common factor of $n - 11$ and $3n + 20$ is greater than 1?

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2013

Junior Section (First Round Solutions)

Multiple Choice Questions

1. Answer: (D)

   First note that $c = (8^2)^{27} = 8^{54}$, so we see that $c > a$. Next, $b = (4^2)^{41} = 4^{82}$ and $c = (4^3)^{27} = 4^{81}$. Therefore we have $b > c$. Consequently $b > c > a$.

2. Answer: (B)

$$\begin{aligned}\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}-\frac{1}{a}-\frac{1}{b}-\frac{1}{c} &= \frac{a^2+b^2+c^2-bc-ca-ab}{abc}\\ &= \frac{(a^2-2ab+b^2)+(b^2-2bc+c^2)+(c^2-2ca+a^2)}{2abc}\\ &= \frac{(a-b)^2+(b-c)^2+(c-a)^2}{2abc}\\ &= \frac{1^2+1^2+2^2}{120}=\frac{1}{20}.\end{aligned}$$

3. Answer: (A)

   Note that $x^2+x+1=\dfrac{x^3-1}{x-1}$, so $x^2+x+1=0$ implies that $x^3=1$ and $x \neq 1$. Now

$$\begin{aligned}x^{49}+x^{50}+x^{51}+x^{52}+x^{53} &= x^{49}(1+x+x^2)+x^{51}(x+x^2)\\ &= x^{49}\times 0+(x^3)^{17}(-1)\\ &= 1^{17}\times(-1)=-1.\end{aligned}$$

4. Answer: (B)

   Let $\angle CAB = x$ and $\angle ABC = y$. Then $x+y = 180° - 36° = 144°$.

   Now $\angle APB = 180° - \frac{x+y}{2} = 108°$.

5. Answer: (D)

   $xy - 3x + 5y = 0$ is equivalent to $(x+5)(y-3) = -15$.

   If $x+5=a$ and $y-3=b$, then there are eight distinct pairs of integers a, b (counting signs) such that $ab = -15$.

6. Answer: (B)

   Beatrice, being between Miss Poh and Miss Mak cannot be Miss Ong who was between Miss Lim and Miss Mak. This means that we have in order from the left, Miss Poh, Beatrice, Miss Mak, Miss Ong and Miss Lim. So Beatrice must be Miss Nai. Since Ellie was beside Miss Nai and also besides Miss Lim, she must be Miss Poh. This implies Cindy is Miss Lim and Amy was Miss Ong leaving Daisy as Miss Mak.

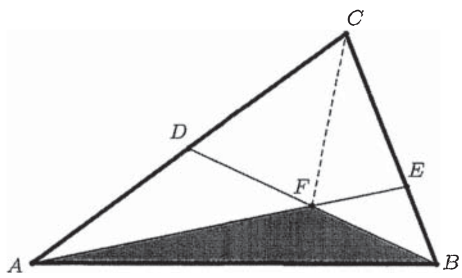
7. Answer: (D)

   Construct a line joining C and F. Then using $[XYZ]$ to denote the area of $\triangle XYZ$, we know that $[ADF] = [DCF] = x$ and if $[BFE] = z$, then $[FCE] = 2z$.

   Furthermore, we have $[ADB] = [DCB]$ i.e. $x + 1 = x + 3z$, so $z = \frac{1}{3}$.

   Also, $2 \times [AEB] = [ACE]$ i.e. $2 + 2z = 2x + 2z$, so $x = 1$.

   In conclusion, $[ABC] = 1 + 2x + 3z = 4$ units.



8. Answer: (C)

   Join A to N. By symmetry, $AN + NM = MN + CN$, and the least value occurs when ANM is a straight line. Thus the least value is

$$\sqrt{AB^2 + BM^2} = \sqrt{8^2 + 6^2} = 10.$$

9. Answer: (D)

   Since CE bisects $\angle BCD$, $\angle BCE = 45°$. Thus $\angle CEB = 45°$ also and $\triangle CBE$ is isosceles. Therefore $BC = BE$.

   Now $\angle BCO = 45° + 15° = 60°$. As $CO = BO$, we conclude that $\triangle COB$ is equilateral. Thus $BC = BO = BE$ giving us an isosceles triangle OBE. Since $\angle OBE = 30°$, thus $\angle BOE = 75°$.

10. Answer: (D)

    S being a multiple of 5 and 3 must end with '0' and has the sum of digits divisible by 3. Since $3 + 8 = 11$, the smallest positive k such that $k \times 11$ is divisible by 3 is 3. Thus $S = 300338880$ and the remainder is

$$0 - 8 + 8 - 8 + 3 - 3 + 0 - 0 + 3 = -5 \equiv 6 \pmod{11}.$$

Short Questions

11. Answer: 10000

$\sqrt{9999^2+19999} = \sqrt{9999^2+2\times 9999+1} = \sqrt{(9999+1)^2} = 10000.$

12. Answer: 3

Let (α, β) be the point of intersection of the two graphs. Then

$$\beta = \alpha^2 + 2a\alpha + 6b = \alpha^2 + 2b\alpha + 6a.$$

It follows that $2(a-b)\alpha = 6(a-b)$. Since the two graphs intersect at only one point, we see that $a - b \neq 0$ (otherwise the two graphs coincide and would have infinitely many points of intersection). Consequently $2\alpha = 6$, and hence $\alpha = 3$.

13. Answer: 1822

The the number of multiples of 11 in the sequence $1, 2, ..., n$ is equal to $\lfloor \frac{n}{11} \rfloor$. Thus the answer to this question is $\left\lfloor \frac{20130}{11} \right\rfloor - \left\lfloor \frac{98}{11} \right\rfloor = 1830 - 8 = 1822.$

14. Answer: 108

Let $\angle ABC = \alpha$ and $\angle BAC = \beta$. Since $BA = BC$, we have $\angle BCA = \angle BAC = \beta$. As $EB = ED$, it follows that $\angle EDB = \angle EBD = \angle ABC = \alpha$. Then $\angle AFD = \angle ADF = \angle EDB = \alpha$ since $AD = AF$. Note that $\angle DAF = 180° - \beta$. In $\triangle ABC$, we have $\alpha + 2\beta = 180°$; and in $\triangle ADF$, we have $2\alpha + 180° - \beta = 180°$. From the two equations, we obtain $\alpha = 36°$. By considering $\triangle BDE$, we obtain $x = 180° - 2\alpha = 108°$.

15. Answer: 20

$$\begin{aligned}
&\frac{1}{a^2-ac-ab+bc} + \frac{2}{b^2-ab-bc+ac} + \frac{1}{c^2-ac-bc+ab} \\
&= \frac{1}{(a-b)(a-c)} + \frac{2}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)} \\
&= \frac{c-b-2(c-a)-(a-b)}{(a-b)(b-c)(c-a)} \\
&= \frac{-1}{(a-b)(b-c)} = 20.
\end{aligned}$$

16. Answer: 60

The equation $(x-2)^2 = 3(x+5)$ is equivalent to $x^2 - 7x - 11 = 0$. Thus $x_1 + x_2 = 7$ and $x_1x_2 = -11$. So

$$x_1x_2 + x_1^2 + x_2^2 = (x_1+x_2)^2 - x_1x_2 = 7^2 - (-11) = 60.$$

17. Answer: 1041

Let the length of the side be s. Observe that since $6^2+8^2=10^2$ so $\angle AXY=90°$. This allows us to see that $\triangle ABX$ is similar to $\triangle XCY$. Thus $\frac{AX}{XY}=\frac{AB}{XC}$, i.e. $\frac{8}{6}=\frac{s}{s-BX}$. Solving this equation gives $s=4BX$ and we can then compute that

$$8^2=AB^2+BX^2=16BX^2+BX^2.$$

So $BX=\frac{8}{\sqrt{17}}$ and $s^2=16\times\frac{64}{17}=\frac{1024}{17}$. Thus $m+n=1041$.

18. Answer: 125

The inequality is equivalent to

$$(x-5)^2+(2x-y)^2\leq 0.$$

Thus we must have $(x-5)=0$ and $(2x-y)=0$, hence $x^2+y^2=5^2+10^2=125$.

19. Answer: 6

Suppose Team B spent t minutes on the job. Then

$$\frac{t}{75}+\frac{90-t}{150}=1.$$

Thus $t=60$ minutes and so Team A completed $\frac{30}{150}=\frac{1}{5}$ of the job. So $m+n=6$.

20. Answer: 4

Taking reciprocals, we find that $\frac{1}{a}+\frac{1}{b}=3$, $\frac{1}{b}+\frac{1}{c}=4$ and $\frac{1}{a}+\frac{1}{c}=5$. Summing the three equations, we get

$$12=2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=2\times\frac{ab+bc+ca}{abc}.$$

Hence $\frac{24abc}{ab+bc+ca}=4$.

21. Answer: 8052

$$(x_1+x_2)^2=(x_1-x_2)^2+4x_1x_2\geq 0+4\times 2013=8052.$$

If $x_1=x_2=\sqrt{2013}$, then $(x_1+x_2)^2=8052$.

22. Answer: 10

Let $x_1=\sqrt{45-\sqrt{2000}}$ and $x_2=\sqrt{45+\sqrt{2000}}$. Then $x_1^2+x_2^2=90$ and

$$x_1x_2=\sqrt{(45-\sqrt{2000})(45+\sqrt{2000})}=\sqrt{45^2-2000}=\sqrt{25}=5.$$

Thus

$$(x_1+x_2)^2=x_1^2+x_2^2+2x_1x_2=100.$$

As both x_1 and x_2 are positive, we have $x_1+x_2=10$.

23. Answer: 55

$(k-10)^{4026} = ((k-10)^2)^{2013} \geq 2013^{2013}$ is equivalent to to $(k-10)^2 \geq 2013$. As $k-10$ is an integer and $44^2 < 2013 < 45^2$, the minimum value of $k-10$ is 45, and thus the minimum value of k is 55.

24. Answer: 19

Rearranging the terms of the equation, we obtain

$$(1-4a)x = 2b+3.$$

Since the equation has more than one solution (i.e., infinitely many solutions), we must have $1-4a=0$ and $2b+3=0$. Therefore $a=\frac{1}{4}$ and $b=-\frac{3}{2}$. Consequently, $100a+4b = 19$.

25. Answer: 18

First note that the product of any two different 2-digit numbers is greater than 100. Thus if a 2-digit number is chosen, then the two numbers adjacent to it in the circle must be single-digit numbers. Note that at most nine single-digit numbers can be chosen from S, and no matter how these nine numbers $1, 2, \ldots, 9$ are arranged in the circle, there is at most one 2-digit number in between them. Hence it follows that $n \leq 18$. Now the following arrangement

$$1, 49, 2, 33, 3, 24, 4, 19, 5, 16, 6, 14, 7, 12, 8, 11, 9, 10, 1$$

shows that $n \geq 18$. Consequently we conclude that the maximum value of n is 18.

26. Answer: 48

Let $x = \overline{abcd}$ and $y = \overline{dcba}$ where $a, d \neq 0$. Then

$$\begin{aligned} y - x &= 1000 \times d - d + 100 \times c - 10 \times c + 10 \times b - 100 \times b + a - 1000 \times a \\ &= 999(d-a) + 90(c-b) = 9\left(111(d-a) + 10(c-b)\right). \end{aligned}$$

So we have $111(d-a) + 10(c-b) = 353$. Consider the remainder modulo 10, we obtain $d-a=3$, which implies that $c-b=2$. Thus the values of a and b determines the values of d and c respectively.

a can take on any value from 1 to 6, and b can take any value from 0 to 7, giving $6 \times 8 = 48$ choices.

27. Answer: 12

Let $2^8 + 2^{11} + 2^n = m^2$ and so

$$2^n = m^2 - 2^8(1+8) = (m-48)(m+48).$$

If we let $2^k = m+48$, then $2^{n-k} = m-48$ and we have

$$2^k - 2^{n-k} = 2^{n-k}(2^{2k-n} - 1) = 96 = 2^5 \times 3.$$

This means that $n-k=5$ and $2k-n=2$, giving us $n=12$.

28. Answer: 208

Note that a positive integer k is a multiple of 4 if and only if the number formed by the last two digits of k (in the same order) is a multiple of 4. There are 12 possible multiples of 4 that can be formed from the digits $0, 1, 2, 3, 4, 5, 6$ without repetition, namely

$$20, 40, 60, 12, 32, 52, 04, 24, 64, 16, 36, 56.$$

If 0 appears in the last two digits, there are 5 choices for the first digit and 4 choices for the second digit. But if 0 does not appear, there are 4 choices for the first digit and also 4 choices for the second digit. Total number is

$$4 \times 5 \times 4 + 8 \times 4 \times 4 = 208.$$

29. Answer: 10571

$$\begin{aligned} \frac{m}{n} = \frac{1}{4}\sum_{k=5}^{1006} \frac{1}{k(k+1)} &= \frac{1}{4}\sum_{k=5}^{1006} \frac{1}{k} - \frac{1}{k+1} \\ &= \frac{1}{4}\left(\frac{1}{5} - \frac{1}{1007}\right) \\ &= \frac{501}{10070}. \end{aligned}$$

Since $\gcd(501, 10070) = 1$, we have $m + n = 10571$.

30. Answer: 3

Note that the units digit of $2013^1 + 2013^2 + 2013^3 + \cdots + 2013^{2013}$ is equal to the units digit of the following number

$$3^1 + 3^2 + 3^3 + \cdots + 3^{2013}.$$

Since $3^2 = 9, 3^3 = 27, 3^4 = 81$, the units digits of the sequence of $3^1, 3^2, 3^3, 3^4, \cdots, 3^{2013}$ are

$$\underbrace{3, 9, 7, 1, 3, 9, 7, 1, \cdots, 3, 9, 7, 1}_{\text{2012 numbers}}, 3.$$

Furthermore the sum $3 + 9 + 7 + 1$ does not contribute to the units digit, so the answer is 3.

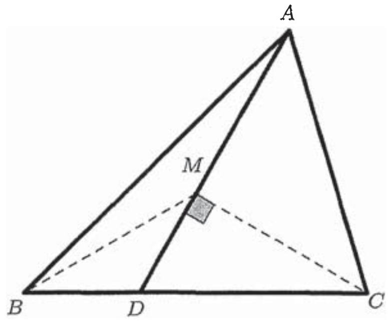
31. Answer: 75

Construct a point M on AD so that CM is perpendicular to AD. Join B and M.

Since $\angle ADC = 60°$, $\angle MCD = 30°$. As $\sin 30° = \frac{1}{2}$, so $2MD = DC$. This means that $BD = MD$ and $\triangle MDB$ is isosceles. It follows that $\angle MBD = 30°$ and $\angle ABM = 15°$.

We further observe that $\triangle MBC$ is also isosceles and thus $MB = MC$.

Now $\angle BAM = \angle BMD - \angle ABM = 15°$, giving us yet another isosceles triangle $\triangle BAM$. We now have $MC = MB = MA$, so $\triangle AMC$ is also isosceles. This allows us to calculate $\angle ACM = 45°$ and finally $\angle ACB = 30° + 45° = 75°$.



32. Answer: 221

We have $a^2+2ab-3b^2 = (a-b)(a+3b) = 41$. Since 41 is a prime number, and $a-b < a+3b$, we have $a - b = 1$ and $a + 3b = 41$. Solving the simultaneous equations gives $a = 11$ and $b = 10$. Hence $a^2 + b^2 = 221$.

33. Answer: 62

We first note that for $1 \le r < k$, $\lfloor \frac{r}{k} \rfloor = 0$ and $\lfloor \frac{k}{k} \rfloor = 1$. The total number of terms up to $\lfloor \frac{N}{N} \rfloor$ is given by $\frac{1}{2}N(N+1)$, and we have the inequality

$$\frac{62(63)}{2} = 1953 < 2013 < 2016 = \frac{63(64)}{2}.$$

So the 2013$^{\text{th}}$ term is $\lfloor \frac{60}{63} \rfloor$, and the sum up to this term is just 62.

34. Answer: 1648

By the pigeonhole principle in any group of $365 \times 9 + 1 = 3286$ persons, there must be at least 10 persons who share the same birthday.

Hence solving $2n - 10 \ge 3286$ gives $n \ge 1648$. Thus the smallest possible n is 1648 since $2 \times 1647 - 10 = 3284 < 365 \times 9$, and it is possible for each of the 365 different birthdays to be shared by at most 9 persons.

35. Answer: 64

Let $d > 1$ be the highest common factor of $n - 11$ and $3n + 20$. Then $d \mid (n - 11)$ and $d \mid (3n + 20)$. Thus $d \mid [3n + 20 - 3(n - 11)]$, i.e., $d \mid 53$. Since 53 is a prime and $d > 1$, it follows that $d = 53$. Therefore $n - 11 = 53k$, where k is a positive integer, so $n = 53k + 11$. Note that for any k, $3n+20$ is a multiple of 53 since $3n+20 = 3(53k+11)+20 = 53(3k+1)$. Hence $n = 64$ (when $k = 1$) is the smallest positive integer such that $\text{HCF}(n-11, 3n+20) > 1$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2013

(Junior Section, Round 2)

Saturday, 29 June 2013 **0930-1230**

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1. Let $a < b < c < d < e$ be real numbers. Among the 10 sums of the pairs of these numbers, the least three are 32, 36 and 37 while the largest two are 48 and 51. Find all possible values of e.

2. In the triangle ABC, points D, E, F are on the sides BC, CA and AB respectively such that FE is parallel to BC and DF is parallel to CA. Let P be the intersection of BE and DF, and Q the intersection of FE and AD. Prove that PQ is parallel to AB.

3. Find all primes that can be written both as a sum of two primes and as a difference of two primes.

4. Let a and b be positive integers with $a > b > 2$. Prove that $\frac{2^a+1}{2^b-1}$ is not an integer.

5. Six musicians gathered at a chamber music festival. At each scheduled concert some of the musicians played while the others listened as members of the audience. What is the least number of such concerts which would need to be scheduled so that for every two musicians each must play for the other in some concert?

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2013

(Junior Section, Round 2 solutions)

1. We have 37 is either $a+d$ or $b+c$ and

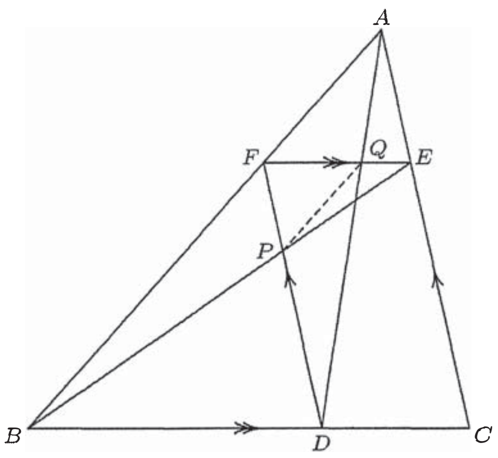
$$a+b=32, \quad a+c=36, \quad c+e=48, \quad d+e=51$$

Thus $c-b=4$, $d-c=3$ and $d-b=7$. Therefore $(a+b)+(d-b)=a+d=39$. Hence $b+c=37$. We thus have $a=15.5$, $b=16.5$, $c=20.5$, $d=23.5$ and $e=27.5$.

2. Since FE is parallel to BC and DF is parallel to CA, we have the triangles PFE, PDB and ECB are similar. Also the triangles AFQ and ABD are similar, FBD and ABC are similar. It follows that

$$\frac{DP}{PF}=\frac{BP}{PE}=\frac{BD}{DC}=\frac{BF}{FA}=\frac{DQ}{QA}$$

so that PQ is parallel to AB.



3. Let p be such a prime, then $p>2$ and is therefore odd. Thus $p=q-2=r+2$ where q,r are primes. If $r\equiv 1 \pmod 3$, then $p\equiv 0 \pmod 3$ and therefore $p=3$ and $r=1$ which is impossible. If $r\equiv 2 \pmod 3$, then $q\equiv 0 \pmod 3$ and thus $q=3$ and so $p=1$, again impossible. Thus $r\equiv 0 \pmod 3$, which means $r=3$ and hence $p=5$ and $q=7$. Thus $p=5$ is the only such prime.

4. We have $a = bm + r$ where $m = \lfloor a/b \rfloor$ and $0 \leq r < b$. Thus

$$\frac{2^a + 1}{2^b - 1} = \frac{2^a - 2^r}{2^b - 1} + \frac{2^r + 1}{2^b - 1}.$$

Note that $2^a - 2^r = 2^r(2^{a-r} - 1) = 2^r(2^{bm} - 1)$, and

$$2^{bm} - 1 = (2^b)^m - 1 = (2^b - 1)[(2^b)^{m-1} + (2^b)^{m-2} + \cdots + 1].$$

Therefore $\frac{2^a - 2^r}{2^b - 1}$ is an integer.

Observe that if $b > 2$, then $2^{b-1}(2 - 1) > 2$, i.e.,

$$2^r + 1 \leq 2^{b-1} + 1 < 2^b - 1.$$

Therefore $\frac{2^r + 1}{2^b - 1}$ is not an integer. Thus $\frac{2^a + 1}{2^b - 1}$ is not an integer.

5. Let the musicians be A, B, C, D, E, F. We first show that four concerts are sufficient. The four concerts with the performing musicians: $\{A, B, C\}$, $\{A, D, E\}$, $\{B, D, F\}$ and $\{C, E, F\}$ satisfy the requirement. We shall now prove that 3 concerts are not sufficient. Suppose there are only three concerts. Since everyone must perform at least once, there is a concert where two of the musicians, say A, B, played. But they must also play for each other. Thus we have A played and B listened in the second concert and vice versa in the third. Now C, D, E, F must all perform in the second and third concerts since these are the only times when A and B are in the audience. It is not possible for them to perform for each other in the first concert. Thus the minimum is 4.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2013

(Senior Section)

Tuesday, 4 June 2013 9:30 — 12:00

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.

For the other short questions, write your answer on the answer sheet and shade the appropriate bubbles below your answer.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1. A shop sells two kinds of products A and B. One day, a salesman sold both A and B at the same price \$2100 to a customer. Suppose product A makes a profit of 20% but product B makes a loss of 20%. Then this deal

   (A) make a profit of \$70; (B) make a loss of \$70; (C) make a profit of \$175;
   (D) make a loss of \$175; (E) makes no profit or loss.

2. How many integer solutions does the equation $(x^3 - x - 1)^{x+2013} = 1$ have?

   (A) 0; (B) 1; (C) 2; (D) 3; (E) More than 3.

3. In the xy-plane, which of the following is the reflection of the graph of

$$y = \frac{1+x}{1+x^2}$$

   about the line $y = 2x$?

   (A) $x = \frac{1+y}{1+y^2}$; (B) $x = \frac{-1+y}{1+y^2}$; (C) $x = -\frac{1+y}{1+y^2}$; (D) $x = \frac{1-y}{1+y^2}$;
   (E) None of the above.

4. Let n be a positive integer. Find the number of possible remainders when

$$2013^n - 1803^n - 1781^n + 1774^n$$

   is divided by 203.

   (A) 1; (B) 2; (C) 3; (D) 4; (E) More than 4.

5. Find the number of integers n such that the equation

$$xy^2 + y^2 - x - y = n$$

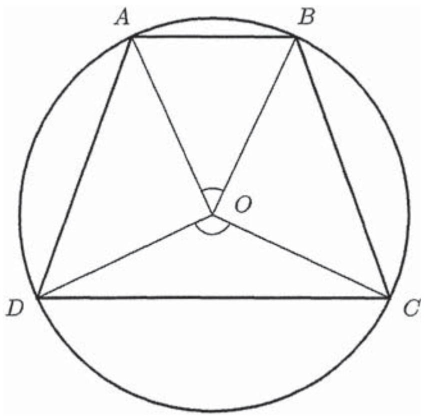
   has an infinite number of integer solutions (x, y).

   (A) 0; (B) 1; (C) 2; (D) 3; (E) More than 3.

6. If $0 < \theta < \frac{\pi}{4}$ is such that $\operatorname{cosec}\theta - \sec\theta = \frac{\sqrt{13}}{6}$, then $\cot\theta - \tan\theta$ equals

   (A) $\frac{\sqrt{13}}{6}$; (B) $\frac{\sqrt{12}}{6}$; (C) $\frac{\sqrt{5}}{6}$; (D) $\frac{13}{6}$; (E) $\frac{5}{6}$.

7. $ABCD$ is a trapezium inscribed in a circle centred at O. It is given that $AB \parallel CD$, $\angle COD = 3\angle AOB$, and $\dfrac{AB}{CD} = \dfrac{2}{5}$.

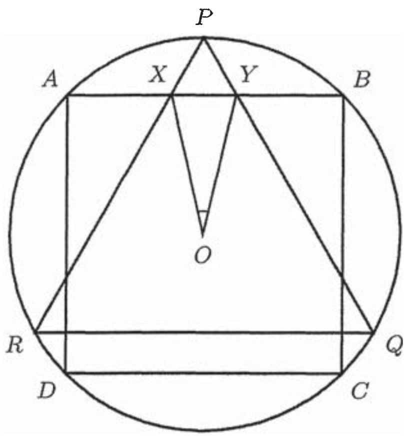


Find the ratio

$$\frac{\text{area of } \triangle BOC}{\text{area of } \triangle AOB}.$$

(A) $\frac{3}{2}$; (B) $\frac{7}{4}$; (C) $\frac{\sqrt{3}}{\sqrt{2}}$; (D) $\frac{\sqrt{5}}{2}$; (E) $\frac{\sqrt{7}}{\sqrt{2}}$.

8. A square $ABCD$ and an equilateral triangle PQR are inscribed in a circle centred at O in such a way that $AB \parallel QR$. The sides PQ and PR of the triangle meet the side AB of the square at X and Y respectively.



The value of $\tan \angle XOY$ is

(A) $\frac{1}{\sqrt{3}}$; (B) 1; (C) $\frac{\sqrt{6}-\sqrt{3}}{\sqrt{2}}$; (D) $\frac{2\sqrt{2}-2}{\sqrt{3}}$; (E) $\sqrt{3}$.

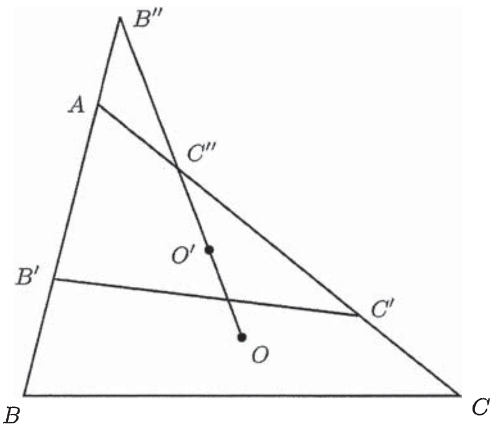
9. Two people go to the same swimming pool between 2:00p.m. and 5:00p.m. at random time and each swims for one hour. What is the chance that they meet?

(A) $\frac{1}{9}$; (B) $\frac{2}{9}$; (C) $\frac{1}{3}$; (D) $\frac{4}{9}$; (E) $\frac{5}{9}$.

10. Given a triangle $\triangle ABC$, let B' and C' be points on the sides AB and AC such that $BB' = CC'$. Let O and O' be the circumcentres (i.e., the centre of the circumscribed circle) of $\triangle ABC$ and $\triangle AB'C'$, respectively. Suppose OO' intersect lines AB' and AC' at B'' and C'', respectively. If $AB = \frac{1}{2}AC$, then

(A) $AB'' < \frac{1}{2}AC''$; (B) $AB'' = \frac{1}{2}AC''$; (C) $\frac{1}{2}AC'' < AB'' < AC''$;

(D) $AB'' = AC''$; (E) $AB'' > AC''$.



Short Questions

11. Suppose a right-angled triangle is inscribed in a circle of radius 100. Let α and β be its acute angles. If $\tan\alpha = 4\tan\beta$, find the area of the triangle.

12. Let $f(x) = \dfrac{1+10x}{10-100x}$. Set $f^n = \overbrace{f \circ f \circ \cdots \circ f}^{n \text{ terms}}$. Find the value of

$$f\left(\frac{1}{2}\right) + f^2\left(\frac{1}{2}\right) + f^3\left(\frac{1}{2}\right) + \cdots + f^{6000}\left(\frac{1}{2}\right).$$

13. Let AB and CD be perpendicular segments intersecting at point P. Suppose that $AP = 2$, $BP = 3$ and $CP = 1$. If all the points A, B, C, D lie on a circle, find the length of DP.

14. On the xy-plane, let S denote the region consisting of all points (x, y) for which

$$\left|x+\frac{1}{2}y\right| \leq 10, \quad |x| \leq 10 \quad \text{and} \quad |y| \leq 10.$$

The largest circle centred at $(0,0)$ that can be fitted in the region S has area $k\pi$. Find the value of k.

15. Given that $\sqrt[3]{17-\frac{27}{4}\sqrt{6}}$ and $\sqrt[3]{17+\frac{27}{4}\sqrt{6}}$ are the roots of the equation

$$x^2 - ax + b = 0,$$

find the value of ab.

16. Find the number of integers between 1 and 2013 with the property that the sum of its digits equals 9.

17. Let $p(x)$ be a polynomial with integer coefficients such that $p(m) - p(n)$ divides $m^2 - n^2$ for all integers m and n. If $p(0) = 1$ and $p(1) = 2$, find the largest possible value of $p(100)$.

18. Find the number of positive integer pairs (a, b) satisfying $a^2 + b^2 < 2013$ and $a^2b \mid (b^3 - a^3)$.

19. Let f and g be functions such that for all real numbers x and y,

$$g(f(x+y)) = f(x) + (x+y)g(y).$$

Find the value of $g(0) + g(1) + \cdots + g(2013)$.

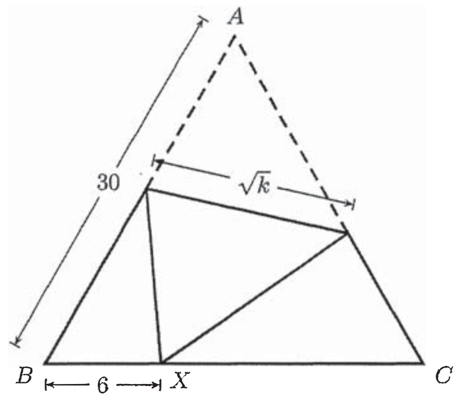
20. Each chocolate costs 1 dollar, each licorice stick costs 50 cents and each lolly costs 40 cents. How many different combinations of these three items cost a total of 10 dollars?

21. Let $A = \{1, 2, 3, 4, 5, 6\}$. Find the number of distinct functions $f : A \to A$ such that $f(f(f(n))) = n$ for all $n \in A$.

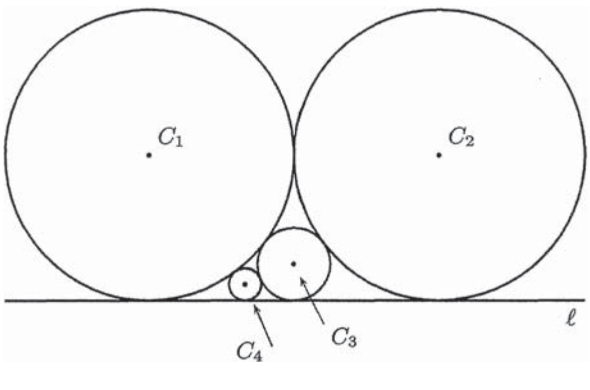
22. Find the number of triangles whose sides are formed by the sides and the diagonals of a regular heptagon (7-sided polygon). (Note: The vertices of triangles need not be the vertices of the heptagon.)

23. Six seats are arranged in a circular table. Each seat is to be painted in red, blue or green such that any two adjacent seats have different colours. How many ways are there to paint the seats?

24. $\triangle ABC$ is an equilateral triangle of side length 30. Fold the triangle so that A touches a point X on BC. If $BX = 6$, find the value of k, where $\sqrt{k}$ is the length of the crease obtained from folding.



25. As shown in the figure below, circles C_1 and C_2 of radius 360 are tangent to each other, and both tangent to straight line ℓ. If circle C_3 is tangent to C_1, C_2 and ℓ, and circle C_4 is tangent to C_1, C_3 and ℓ, find the radius of C_4.



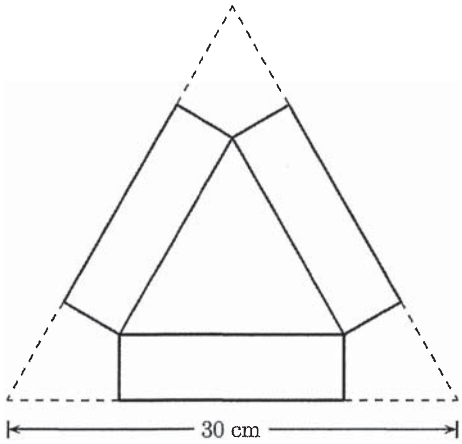
26. Set $\{x\} = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Find the number of real solutions to the equation

$$\{x\} + \{x^2\} = 1, \quad |x| \leq 10.$$

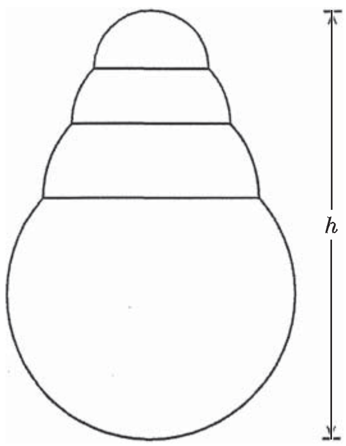
27. Find the value of $\left\lfloor \left(\frac{3+\sqrt{17}}{2} \right)^6 \right\rfloor$.

28. A regular dodecagon (12-sided polygon) is inscribed in a circle of radius 10. Find its area.

29. A triangular box is to be cut from an equilateral triangle of length 30 cm. Find the largest possible volume of the box (in cm^3).



30. A hemisphere is placed on a sphere of radius 100 cm. The second hemisphere is placed on the first one, and the third hemisphere is placed on the second one (as shown below). Find the maximum height of the tower (in cm).

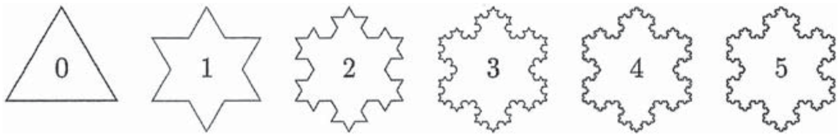


31. Let x, y, z be real numbers such that

$$x + y + z = 1 \quad \text{and} \quad x^2 + y^2 + z^2 = 1.$$

Let m denote the minimum value of $x^3 + y^3 + z^3$. Find $9m$.

32. Given an equilateral triangle of side 10, divide each side into three equal parts, construct an equilateral triangle on the middle part, and then delete the middle part. Repeat this step for each side of the resulting polygon. Find S^2, where S is the area of region obtained by repeating this procedure infinitely many times.



33. Suppose

$$\frac{1}{2013^{1000}} = a_1 + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots + \frac{a_n}{n!},$$

where n is a positive integer, $a_1, \ldots, a_n$ are nonnegative integers such that $a_k < k$ for $k = 2, \ldots, n$ and $a_n > 0$. Find the value of n.

34. Let M be a positive integer. It is known that whenever $|ax^2 + bx + c| \le 1$ for all $|x| \le 1$, then $|2ax + b| \le M$ for all $|x| \le 1$. Find the smallest possible value of M.

35. Consider integers $\{1, 2, \ldots, 10\}$. A particle is initially at 1. It moves to an adjacent integer in the next step. What is the expected number of steps it will take to reach 10 for the first time?

Solutions

1. Answer: (D).

   The original values are $2100 \div 120\% = 1750$ and $2100 \div 80\% = 2625$ respectively. Then the profit is

   $$1750 + 2625 - 2 \times 2100 = -175.$$

2. Answer: (D).

   If $x + 2013 = 0$, then $x = -2013$. Suppose $x + 2013 \neq 0$. Then $x^3 - x - 1 = \pm 1$.

   If $x^3 - x - 1 = 1$, there is no integer solution; if $x^3 - x - 1 = -1$, then $x = 0, 1, -1$. Since $x + 2013$ is even, $x = 1$ or $x = -1$.

3. Answer: (E).

   (A) and (B) are the reflections with respect to $y = x$ and $y = -x$ respectively; (C) and (D) are the rotations about the origin by $90°$ and $-90°$ respectively.

4. Answer: (A).

   For any positive integer n,

   $$\begin{aligned} 2013^n - 1803^n - 1781^n + 1774^n &= (2013^n - 1803^n) - (1781^n - 1774^n) \\ &= (2013 - 1803)u - (1781 - 1774)v = 210u - 7v, \\ 2013^n - 1803^n - 1781^n + 1774^n &= (2013^n - 1781^n) - (1803^n - 1774^n) \\ &= (2013 - 1781)x - (1803 - 1774)y = 29x - 29y. \end{aligned}$$

   So $2013^n - 1803^n - 1781^n + 1774^n$ is divisible by $7 \times 29 = 203$ for every positive integer n.

5. Answer: (C).

   Rewrite the equation as

   $$xy^2 + y^2 - x - y = (y - 1)(x(y + 1) + y) = n.$$

   If $n = 0$, then there are infinitely many integer solutions. Suppose $n \neq 0$, and the equation has infinitely many integer solutions. Then there exists a divisor k of n such that $y - 1 = k$ and $x(y + 1) + y = n/k$ for infinitely many x. It forces $y + 1 = 0$, i.e., $y = -1$. Then $n = 2$.

6. Answer: (E).

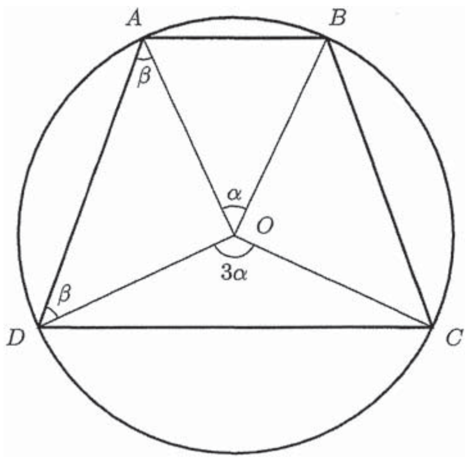
Let $k = \sin\theta\cos\theta$. Then

$$\frac{13}{36} = \left(\frac{1}{\sin\theta} - \frac{1}{\cos\theta}\right)^2 = \frac{1 - 2\sin\theta\cos\theta}{(\sin\theta\cos\theta)^2} = \frac{1-2k}{k^2}.$$

Solve the equation: $k = 6/13$ ($k = -6$ is rejected). Then $\sin 2\theta = 2k = 12/13$ and

$$\cot\theta - \tan\theta = \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta} = \frac{\cos 2\theta}{\frac{1}{2}\sin 2\theta} = \frac{5/13}{6/13} = \frac{5}{6}.$$

7. Answer: (A).



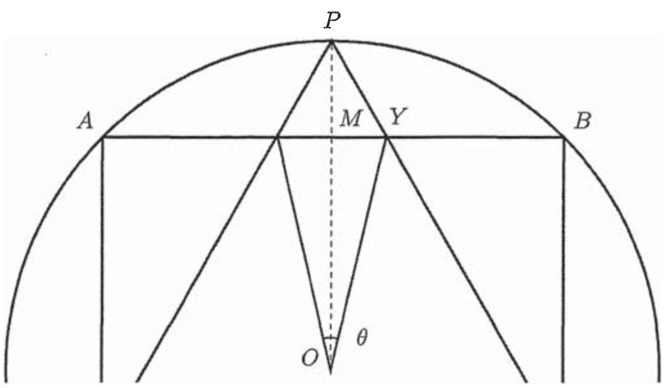
Let $\angle AOB = \alpha$ and $\angle ADO = \beta$. Then $\frac{1}{2}(\alpha + 3\alpha) = 2\beta$; that is, $\alpha = \beta$. Given that

$$\frac{5}{2} = \frac{\sin(3\alpha/2)}{\sin(\alpha/2)} = \frac{3\sin(\alpha/2) - 4\sin^3(\alpha/2)}{\sin(\alpha/2)} = 3 - 4\sin^2(\alpha/2).$$

Then $\sin^2(\alpha/2) = \frac{1}{8}$. Hence,

$$\frac{S_{\triangle BOC}}{S_{\triangle AOB}} = \frac{\sin(\pi - 2\alpha)}{\sin\alpha} = 2\cos\alpha = 2\left(1 - 2\sin^2(\alpha/2)\right) = \frac{3}{2}.$$

8. Answer: (C).



Let the side of the square be 2. Then the radius of the circle is $\sqrt{2}$. Let $\theta = \angle XOY$. So

$$\tan(\theta/2) = \frac{MY}{MO} = MY = PM \tan 30^\circ = \frac{\sqrt{2}-1}{\sqrt{3}}.$$

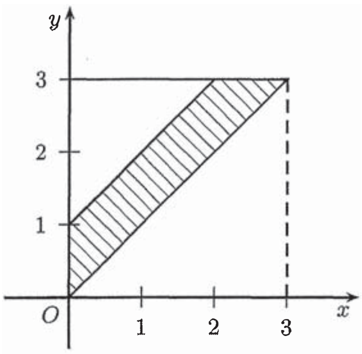
Then

$$\tan\theta = \frac{2\tan(\theta/2)}{1-\tan^2(\theta/2)} = \frac{\sqrt{6}-\sqrt{3}}{\sqrt{2}}.$$

9. Answer: (E).

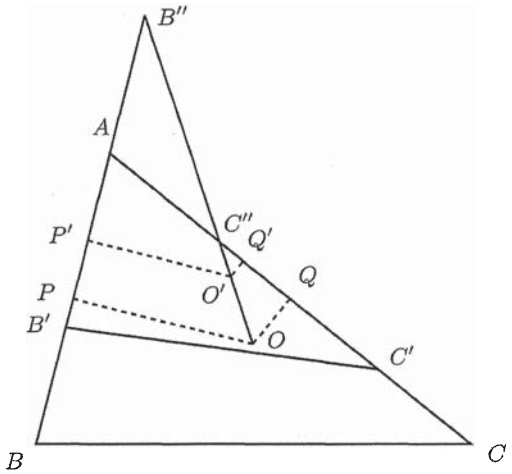
Let x and y denote the numbers of hours after 2:00p.m. that the first and the second person visits the swimming pool, respectively. Then

$$0 \le x \le y \le 3 \quad \text{and} \quad y \le x+1.$$



So the chance that they meet is $\dfrac{5/2}{9/2} = \dfrac{5}{9}$.

10. Answer: (D).



Let P and P' be the projections of O and O' on AB respectively. Then

$$PP' = AP - AP' = \frac{1}{2}AB - \frac{1}{2}AB' = \frac{1}{2}(AB - AB') = \frac{1}{2}BB'.$$

Similarly, let Q and Q' be the projections of O and O' on AC respectively, then $QQ' = \frac{1}{2}CC'$.

$$\sin\angle O'OP = \frac{PP'}{OO'} = \frac{QQ'}{OO'} = \sin\angle O'OQ \Rightarrow \angle O'OP = \angle O'OQ.$$

So $\angle AB''C'' = \angle AC''B''$. It follows that $AB'' = AC''$.

11. Answer: 8000.

$\tan\alpha = 4\tan\beta = \frac{4}{\tan\alpha} \Rightarrow \tan\alpha = 2$. The two legs are $\frac{200}{\sqrt{5}}$ and $\frac{400}{\sqrt{5}}$ respectively.

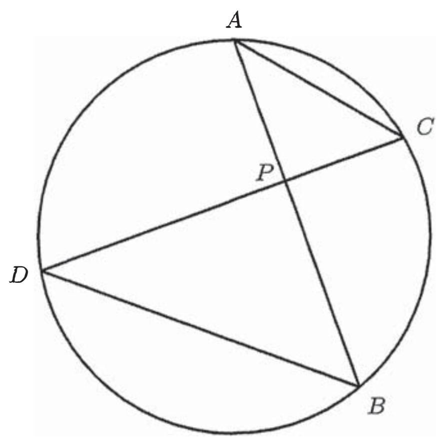
$$\text{Area} = \frac{1}{2} \times \frac{200}{\sqrt{5}} \times \frac{400}{\sqrt{5}} = 8000.$$

12. Answer: 595.

Let $f(x) = \frac{1+10x}{10-100x}$. Then $f^2(x) = -\frac{1}{100x}$, $f^3(x) = \frac{1-10x}{10+100x}$ and $f^4(x) = x$. Then

$$\begin{aligned} f\left(\frac{1}{2}\right) + f^2\left(\frac{1}{2}\right) + \cdots + f^{6000}\left(\frac{1}{2}\right) &= 1500\left[f\left(\frac{1}{2}\right) + f^2\left(\frac{1}{2}\right) + f^2\left(\frac{1}{2}\right) + f^4\left(\frac{1}{2}\right)\right] \\ &= 1500\left(-\frac{3}{10} - \frac{1}{50} + \frac{1}{15} + \frac{1}{2}\right) = 595. \end{aligned}$$

13. Answer: 6.



Since $\angle A = \angle D$ and $\angle C = \angle B$, the triangles $\triangle ACP$ and $\triangle DBP$ are similar. Then

$$\frac{DP}{BP} = \frac{AP}{CP} \Rightarrow DP = \frac{AP}{CP} \times BP = \frac{2}{1} \times 3 = 6.$$

14. Answer: 80.

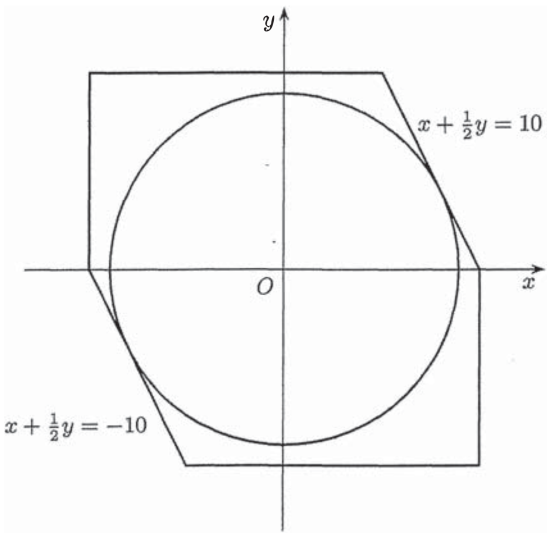
The region S is the hexagon enclosed by the lines

$$x = \pm 10, \quad y = \pm 10, \quad x + \frac{1}{2}y = \pm 10.$$

The largest circle contained in S is tangent to $x+\frac{1}{2}y=\pm 10$. Hence, its radius is the distance from the origin $(0,0)$ to $x+\frac{1}{2}y=10$:

$$r=\frac{10}{\sqrt{1+(1/2)^2}}=4\sqrt{5}.$$

The area of the largest circle is thus $\pi r^2=\pi(4\sqrt{5})^2=80\pi$.



15. Answer: 10.

Let $x_1=\sqrt[3]{17-\frac{27}{4}\sqrt{6}}$ and $x_2=\sqrt[3]{17+\frac{27}{4}\sqrt{6}}$. Then

$$b=x_1x_2=\sqrt[3]{\left(17-\frac{27}{4}\sqrt{6}\right)\left(17+\frac{27}{4}\sqrt{6}\right)}=\sqrt[3]{\frac{125}{8}}=\frac{5}{2},$$

$$\begin{aligned}a^3&=(x_1+x_2)^3=x_1^3+x_2^3+3x_1x_2(x_1+x_2)\\&=\left(17-\frac{27}{4}\sqrt{6}\right)+\left(17+\frac{27}{4}\sqrt{6}\right)+3ab=34+\frac{15}{2}a.\end{aligned}$$

Then $a=4$ and thus $ab=10$.

16. Answer: 101.

Case 1: $n<1000$. Write $n=\overline{abc}$. Then

$$a+b+c=9,\qquad a,b,c\in\{0,1,\ldots,9\}.$$

Case 2: $1000\le n<2000$. Write $n=\overline{1abc}$. Then

$$a+b+c=8,\qquad a,b,c\in\{0,1,\ldots,8\}.$$

Case 3: $2000\le n\le 2013$. Then $n=2007$.

Therefore, there are $\binom{9+3-1}{9}+\binom{8+3-1}{8}+1=55+45+1=101$ such numbers.

17. Answer: 10001.

Let $n=0$. Since $p(m)-1 \mid m^2$ for all m, $\deg p(x) \le 2$. Let $p(x)=ax^2+bx+c$. Then

$$\frac{p(m)-p(n)}{m-n}=\frac{a(m^2-n^2)+b(m-n)}{m-n}=a(m+n)+b$$

divides $m+n$. If $a\neq 0$, then $a=\pm 1$ and $b=0$; if $a=0$, then $b=\pm 1$. Thus

$$p(x)=\pm x^2+c,\ \pm x+c,\ c.$$

Since $p(0)=1$ and $p(1)=2$, we have $p(x)=x^2+1$ or $p(x)=x+1$. The largest possible value of $p(100)$ is $100^2+1=10001$.

18. Answer: 31.

Let $k=\dfrac{b^3-a^3}{a^2b}=\left(\dfrac{b}{a}\right)^2-\dfrac{a}{b}$. Then

$$\left(\frac{a}{b}\right)^3+k\left(\frac{a}{b}\right)^2-1=0.$$

The only possible positive rational solution of $x^3+kx^2-1=0$ is $x=1$; namely, $a=b$. Conversely, if $a=b$, then it is obvious that $a^2b \mid (b^3-a^3)$.

Then $2013>a^2+b^2=2a^2$ implies $a\le 31$.

19. Answer: 0.

Let $y=-x$. Then $g(f(0))=f(x)$ for all x. This shows that f is a constant function; namely $f(x)=c$ for some c. So that $g(c)=g(f(0))=f(x)=c$. For all x,y, we have

$$(x+y)g(y)=g(f(x+y))-f(x)=g(c)-c=0.$$

Since $x+y$ is arbitrary, we must have $g(y)=0$ for all y. Hence,

$$g(0)+g(1)+\cdots+g(2013)=0.$$

20. Answer: 36.

Let x, y and z denote the numbers of chocolate, licorice stick and lolly, respectively. Then

$$x+0.5y+0.4z=10.$$

For each $k=0,\ldots,10$, consider $0.5y+0.4z=k$, i.e., $5y+4z=10k$. Then

$$2 \mid y \quad\text{and}\quad 5 \mid z.$$

Set $y = 2s$ and $z = 5t$. Then $10s + 20t = 10k$, i.e., $s + 2t = k$. Then $t = 0, \ldots, \lfloor k/2 \rfloor$. So there are $\lfloor k/2 \rfloor + 1$ ways to use k dollars. The total number of ways is

$$\sum_{k=0}^{10} (\lfloor k/2 \rfloor + 1) = 1 + 1 + 2 + 2 + \cdots + 5 + 5 + 6 = 36.$$

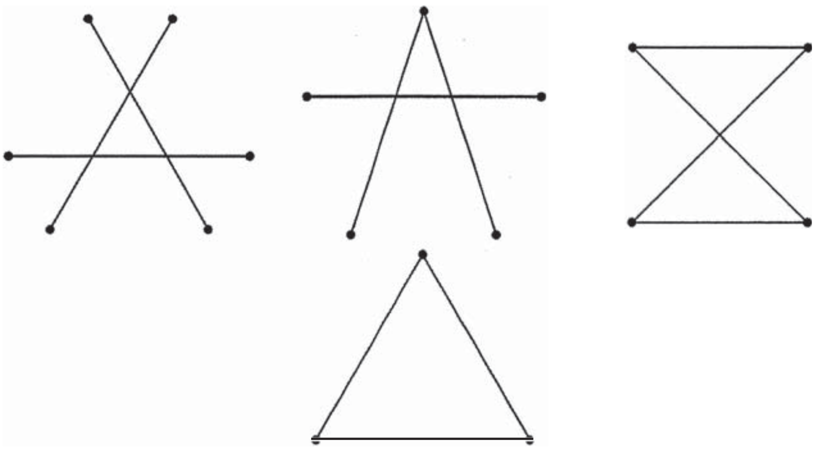
21. Answer: 81.

Suppose $f(f(f(n))) = n$ for all n. For some $k \in \{0, 1, 2\}$, there exist distinct a_i, b_i, c_i, $i \leq k$, such that $f(a_i) = b_i$, $f(b_i) = c_i$ and $f(c_i) = a_i$ and $f(n) = n$ if $n \neq a_i, b_i, c_i$. So the total number of required functions is

$$\binom{6}{6} + \binom{6}{3} \times 2 + \frac{1}{2} \times \binom{6}{3}\binom{3}{3} \times 2^2 = 81.$$

22. Answer: 287.

A triangle can be formed using 3, 4, 5 or 6 vertices.



So the total number is

$$\binom{7}{6} + 5 \times \binom{7}{5} + 2 \times 2 \times \binom{7}{4} + \binom{7}{3} = 287.$$

23. Answer: 66.

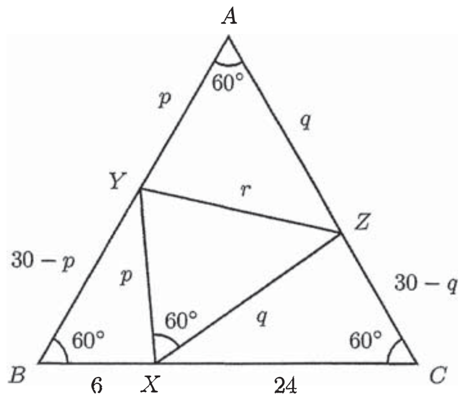
Let $n \geq 2$ be an integer, and let S_n denote the number of ways to paint n seats $a_1, \ldots, a_n$ as described, but with a_1 painted red. Consider S_{n+2} where $n \geq 2$.

Case 1: a_3 is painted red. Then there are 2 choices for a_2. Thus, the total number of ways for this case is $2S_n$.

Case 2: a_3 is not painted red. Since the colour of a_2 is uniquely determined by the colour of a_3, this is equivalent to the case when there are $(n+1)$ seats. The total number of ways for this case is S_{n+1}.

We conclude that $S_{n+2} = S_{n+1} + 2S_n$. It is clear that $S_2 = S_3 = 2$. Then $S_4 = 6$, $S_5 = 10$ and $S_6 = 22$. So the required number of ways is $3 \times 22 = 66$.

24. Answer: 343.



Apply the law of cosine on $\triangle XBY$ and $\triangle XCZ$ respectively:

$$p^2 = 6^2 + (30-p)^2 - 6(30-p),$$
$$q^2 = 24^2 + (30-q)^2 - 24(30-q).$$

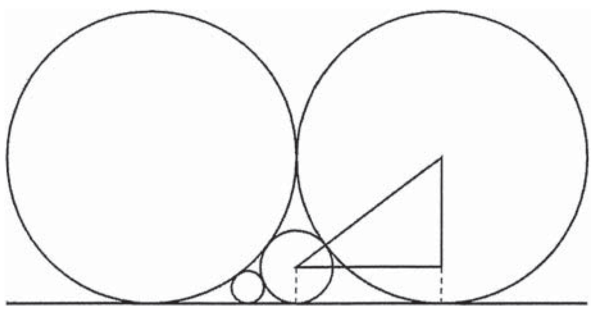
Then $p = 14$ and $q = 21$. Applying the law of cosine in $\triangle YXZ$ again to obtain

$$k = r^2 = p^2 + q^2 - pq = 14^2 + 21^2 - 14 \cdot 21 = 343.$$

25. Answer: 40.

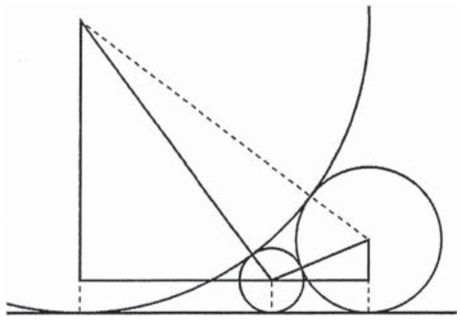
Let R be the radius of C_3. Then

$$(360-R)^2 + 360^2 = (360+R)^2 \Rightarrow R = 90.$$



Let r be the radius of C_4. Then

$$\sqrt{(360+r)^2 - (360-r)^2} + \sqrt{(90+r)^2 - (90-r)^2} = 360 \Rightarrow r = 40.$$



26. Answer: 181.

Since $\{x\}+\{x^2\}=1$, $x+x^2=n$ for some integer n. Then

$$x=\frac{-1\pm\sqrt{1+4n}}{2}.$$

$\dfrac{-1+\sqrt{1+4n}}{2}\le 10$ gives $0\le n\le 110$; $\dfrac{-1-\sqrt{1+4n}}{2}\ge -10$ implies $0\le n\le 90$.

If $\{x\}+\{x^2\}\ne 1$, then $\{x\}+\{x^2\}=0$, which happens only if x is an integer between -10 to 10. So the total number of solutions to $\{x\}+\{x^2\}=1$ is $111+91-21=181$.

27. Answer: 2040.

Let $\alpha=\dfrac{3+\sqrt{17}}{2}$ and $\beta=\dfrac{3-\sqrt{17}}{2}$. Then $\alpha\beta=-2$ and $\alpha+\beta=3$.

Set $S_n=\alpha^n+\beta^n$. Then

$$\begin{aligned}3S_{n+1}+2S_n&=(\alpha+\beta)(\alpha^{n+1}+\beta^{n+2})-\alpha\beta(\alpha^n+\beta^n)\\&=\alpha^{n+2}-\beta^{n+2}=S_{n+2}.\end{aligned}$$

Note that $|\beta|<1$. Then for even positive integer n, $\lfloor\alpha^n\rfloor=S_n+\lfloor-\beta^n\rfloor=S_n-1$.

Since $S_0=2$ and $S_1=3$, we can proceed to evaluate that $S_6=2041$.

28. Answer: 300.

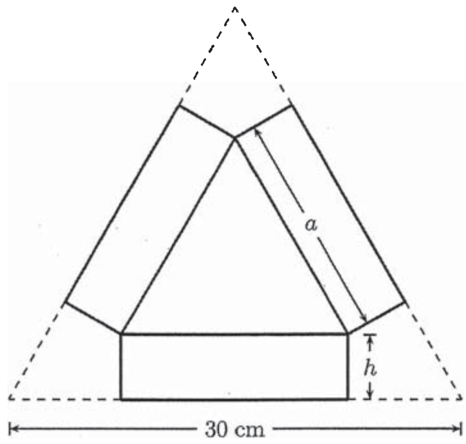
The area is $12\times\dfrac{1}{2}\times 10^2\times\sin 30^\circ=300$.

29. Answer: 500.

Let the length and the height of the box be a and h, respectively. Note that $a+2\sqrt{3}\,h=30$. Then the volume of the box is

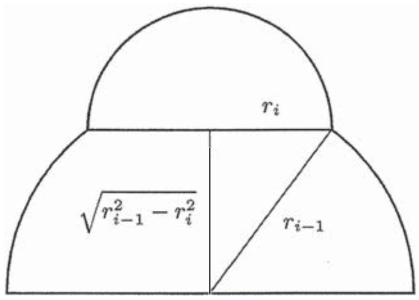
$$\frac{\sqrt{3}\,a^2}{4}h=\frac{1}{2}\left(\frac{a}{2}\cdot\frac{a}{2}\cdot 2\sqrt{3}\,h\right)\le\frac{1}{2}\left(\frac{a+2\sqrt{3}h}{3}\right)^3=\frac{1}{2}\times 10^3=500.$$

The equality holds if $a/2=2\sqrt{3}h$, i.e., $a=20$ and $h=5/\sqrt{3}$.



30. Answer: 300.

Let the radius of the ith hemisphere be r_i metre ($r_0 = 1$). Set $h_i = \sqrt{r_{i-1}^2 - r_i^2}$.



By Cauchy inequality,

$$\left(\sqrt{r_0^2 - r_1^2} + \sqrt{r_1^2 - r_2^2} + \sqrt{r_2^2 - r_3^2} + r_3\right)^2 \le 4(r_0^2 - r_1^2 + r_1^2 - r_2^2 + r_2^2 - r_3^2 + r_3^2)$$
$$= 4r_0^2 = 4.$$

The total height $h \le r_0 + \sqrt{r_0^2 - r_1^2} + \sqrt{r_1^2 - r_2^2} + \sqrt{r_2^2 - r_3^2} + r_3 \le 1 + \sqrt{4} = 3\,\text{m} = 300\,\text{cm}$.

The equality holds if $r_i = \dfrac{\sqrt{4-i}}{2}$, $i = 0, 1, 2, 3$.

31. Answer: 5.

It is clear that $|x|, |y|, |z| \le 1$. Note that

$$0 = x + y + z - x^2 - y^2 - z^2 = x(1-x) + y(1-y) + z(1-z).$$

Without loss of generality, assume that $z \le 0$. Then $x = (1-y) + (-z) \ge 0$ and similarly $y \ge 0$. Since

$$1 - z^2 = x^2 + y^2 \ge \frac{(x+y)^2}{2} = \frac{(1-z)^2}{2},$$

we have $-1/3 \le z \le 0$. On the other hand,

$$\begin{aligned} x^3+y^3+z^3 &= (x+y)\left[\frac{3(x^2+y^2)-(x+y)^2}{2}\right]+z^3 \\ &= (1-z)\left[\frac{3(1-z^2)-(1-z)^2}{2}\right]+z^3 = 1-3z^2+3z^3 \end{aligned}$$

increases as z increases. The minimum $m = 5/9$ is obtained at $z = -1/3$ and $x = y = 2/3$.

32. Answer: 4800.

Let S_n denote the area of the region obtained in the nth step. Then $S_0 = 25\sqrt{3}$ and $S_n - S_{n-1} = \dfrac{3}{4}\left(\dfrac{4}{9}\right)^n S_0$ for all $n \ge 1$. Then

$$\begin{aligned} S_n &= S_0 + \frac{3}{4}\left[\frac{4}{9}+\left(\frac{4}{9}\right)^2+\cdots+\left(\frac{4}{9}\right)^n\right] S_0 \\ &= S_0\left[1+\frac{3}{4}\cdot\frac{4}{9}\cdot\frac{1-(4/9)^n}{1-4/9}\right] = 25\sqrt{3}\left[\frac{8}{5}-\frac{3}{5}\left(\frac{4}{9}\right)^n\right]. \end{aligned}$$

As n increases, S_n tends to $S = 25\sqrt{3}\cdot\dfrac{8}{5} = 40\sqrt{3}$. So $S^2 = 4800$.

33. Answer: 60024.

Let $x = \dfrac{1}{2013^{1000}}$. Then

$$\begin{aligned} xn! &= a_1 n! + a_2\frac{n!}{2!}+\cdots+a_{n-1}\frac{n!}{(n-1)!}+a_n\frac{n!}{n!}, \\ x(n-1)! &= a_1(n-1)! + a_2\frac{(n-1)!}{2!}+\cdots+a_{n-1}\frac{(n-1)!}{(n-1)!}+\frac{a_n}{n}. \end{aligned}$$

So n is the smallest integer such that $n!x$ is an integer, i.e., $2013^{1000} \mid n!$, or equivalently $61^{1000} \mid n!$ because 61 is the largest prime divisor of 2013.

Since $\left\lfloor \dfrac{1000}{61} \right\rfloor = 16$, $n = (1000-16)\times 61 = 60024$.

34. Answer: 4.

Let a, b, c be fixed. Set $f(x) = ax^2+bx+c$. Then

$$f(-1) = a-b+c, \quad f(0) = c, \quad f(1) = a+b+c.$$

Solve the system to get

$$a = \frac{1}{2}f(-1) - f(0) + \frac{1}{2}f(1), \qquad b = -\frac{1}{2}f(-1)+\frac{1}{2}f(1).$$

Suppose $|f(x)| \leq 1$ for all $|x| \leq 1$. Then

$$\begin{aligned}|2ax+b| &= \left|\left(x-\frac{1}{2}\right)f(-1) - 2f(0)x + \left(x+\frac{1}{2}\right)f(1)\right| \\ &\leq \left|x-\frac{1}{2}\right| + 2|x| + \left|x+\frac{1}{2}\right| \\ &\leq \left|x-\frac{1}{2}\right| + \left|x+\frac{1}{2}\right| + 2 \leq 4.\end{aligned}$$

Moreover, $|2x^2-1| \leq 1$ whenever $|x| \leq 1$, and $|2x| = 4$ is achieved at $x = \pm 1$.

35. Answer: 81.

Let E_k denote the expected number of steps it takes to go from $k-1$ to k, $k = 2, \ldots, 100$. Then $E_{k+1} = \frac{1}{2}(1 + E_k + E_{k+1}) + \frac{1}{2}$, which implies $E_{k+1} = E_k + 2$.

It is clear that $E_2 = 1$. Then $E_3 = 3, E_4 = 5, \ldots, E_{10} = 17$. So

$$E = E_2 + E_3 + \cdots + E_{10} = 1 + 3 + \cdots + 17 = 81.$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2013

(Senior Section, Round 2)

Saturday, 29 June 2013 **0900-1300**

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1. In the triangle ABC, $AB > AC$, the extension of the altitude AD with D lying inside BC intersects the circumcircle ω of the triangle ABC at P. The circle through P and tangent to BC at D intersects ω at Q distinct from P with $PQ = DQ$. Prove that $AD = BD - DC$.

2. Find all pairs of integers (m, n) such that

$$m^3 - n^3 = 2mn + 8.$$

3. Let $b_1, b_2, \ldots$ be a sequence of positive real numbers such that for each $n \geq 1$,

$$b_{n+1}^2 \geq \frac{b_1^2}{1^3} + \frac{b_2^2}{2^3} + \cdots + \frac{b_n^2}{n^3}.$$

Show that there is a positive integer M such that

$$\sum_{n=1}^{M} \frac{b_{n+1}}{b_1 + b_2 + \cdots + b_n} > \frac{2013}{1013}.$$

4. In the following 6×6 array, one can choose any $k \times k$ subarray, with $1 < k \leq 6$ and add 1 to all its entries. Is it possible to perform the operation a finite number of times so that all the entries in the array are multiples of 3?

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

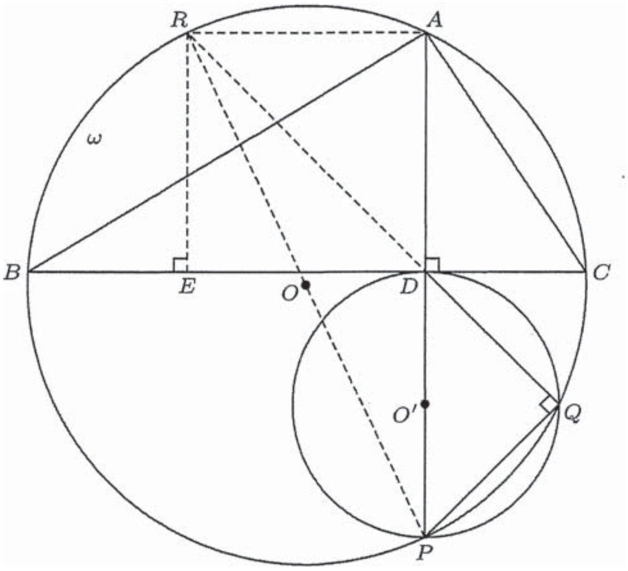
5. Let x, y be distinct real numbers such that $\frac{x^n - y^n}{x - y}$ is an integer for four consecutive positive integers n. Prove that $\frac{x^n - y^n}{x - y}$ is an integer for all positive integers n.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2013

(Senior Section, Round 2 solutions)

1.



Let the extension of QD meet ω at R. Since $\angle PQR = 90°$, PR is a diameter of ω. Thus $\angle PAR = 90°$ so that RA is parallel to BC. This means $BCAR$ is an isosceles trapezoid. Let E be the foot of the perpendicular from R onto BC. Then $BE = CD$ and $ARED$ is a rectangle. Since $\angle ADR = 45°$, $ARED$ is in fact a square so that $AD = DE$. Therefore, $BD - DC = BD - BE = DE = AD$.

2. When $m = 0$, $n = -2$ and when $n = 0$, $m = 2$. These are the two obvious solutions: $(m, n) = (0, -2), (2, 0)$. We'll show that there are no solutions when $mn \neq 0$.

Suppose $mn < 0$. If $m > 0, n < 0$, then

$$-2m|n| + 8 = m^3 + |n|^3 \geq m^2 + |n|^2 \quad \Rightarrow \quad 8 \geq (m + |n|)^2.$$

Thus $m + |n| = 2$ or $m = 1$, $n = -1$ and this is not a solution. If $m < 0$ and $n > 0$, then

$$-2|m|n + 8 = -|m|^3 - n^3 \leq -|m|^2 - |n|^2 \quad \Rightarrow \quad 8 \leq -(|m| - n)^2$$

which is impossible.

3. From Cauchy -Schwarz inequality, we have

$$\begin{aligned}
&\left(\sum_{k=1}^{n}\frac{b_k^2}{k^3}\right)\left(\sum_{k=1}^{n}k^3\right) \geq \left(\sum_{k=1}^{n}b_k^2\right)^2\\
\Rightarrow\quad &\left(\sum_{k=1}^{n}\frac{b_k^2}{k^3}\right)\left(\frac{n(n+1)}{2}\right)^2 \geq \left(\sum_{k=1}^{n}b_k^2\right)^2\\
\Rightarrow\quad &\frac{b_{n+1}}{b_1+b_2+\cdots+b_n} \geq \frac{2}{n(n+1)}\\
\Rightarrow\quad &\sum_{n=1}^{M}\frac{b_{n+1}}{b_1+b_2+\cdots+b_n} \geq \sum_{n=1}^{M}\frac{2}{n(n+1)}\\
\Rightarrow\quad &\sum_{n=1}^{M}\frac{b_{n+1}}{b_1+b_2+\cdots+b_n} \geq 2\sum_{n=1}^{M}\frac{1}{n}-\frac{1}{n+1}\\
&\qquad\qquad\qquad\qquad\qquad = 2-\frac{2}{M+1} \geq \frac{2013}{1013}
\end{aligned}$$

if $M \geq 155$.

4. The answer is no. Let the original array be A. Consider the following array

$$M = \begin{bmatrix} 0 & 1 & 1 & -1 & -1 & 0\\ -1 & 0 & 1 & -1 & 0 & 1\\ -1 & -1 & 0 & 0 & 1 & 1\\ 1 & 1 & 0 & 0 & -1 & -1\\ 1 & 0 & -1 & 1 & 0 & -1\\ 0 & -1 & -1 & 1 & 1 & 0 \end{bmatrix}.$$

Multiply the corresponding elements of the two arrays and compute the sum modulo 3. It's easy to verify that this sum is invariant under the given operation. Since the original sum is 2, one can never obtain an array where all the entries are multiples of 3.

5. Note that

$$(x+y)(x^n-y^n) = (x^{n+1}-y^{n+1}) + xy(x^{n-1}-y^{n-1}).$$

Let $t_n = \frac{x^n-y^n}{x-y}$. Then $t_0 = 0$, $t_1 = 1$ and for $n \geq 0$, $t_{n+2}+bt_{n+1}+ct_n = 0$, with $b = -(x+y)$ and $c = xy$. Then it suffices to show that $b, c \in \mathbb{Z}$.

If $c = 0$, then either $x = 0$ or $y = 0$. Say $y = 0$. Then $t_n = x^{n-1}$, with $x \neq 0$. Then $x = \frac{t_{m+1}}{t_m} \in \mathbb{Q}$. From $t_{m+1} = x^m \in \mathbb{Z}$, it follows that $x \in \mathbb{Z}$. Thus $t_n \in \mathbb{Z}$ for all n. The case $x = 0$ is similar.

We now assume that $c \neq 0$. Let $t_n \in \mathbb{Z}$ for $n = m, m+1, m+2, m+3$. Note that $c^n = (xy)^n = t_{n+1}^2 - t_n t_{n+2}$. Thus $c^m, c^{m+1} \in \mathbb{Z}$. Therefore $c = \frac{c^{m+1}}{c^m} \in \mathbb{Q}$. As before, we have $c \in \mathbb{Z}$. If both t_{m+1}, t_{m+2} are 0, then using the recurrence, we can show easily that $t_n = 0$ for all n, a contradiction. Thus one of them is nonzero. Note that, with $k = m+1$ or $m+2$, whichever is nonzero, we have

$$b = \frac{-ct_{k-1} - t_{k+1}}{t_k} \in \mathbb{Q}.$$

From the recurrence, it follows by induction that t_n can be represented as $t_n = f_{n-1}(b)$ where $f_{n-1}(X)$ is a polynomial with integer coefficients, $\deg f_{n-1} = n-1$ and with the coefficient of $X^{n-1} = \pm 1$. Since $b \in \mathbb{Q}$ is a root of the equation $f_m(X) = t_{m+1}$, $b \in \mathbb{Z}$.

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2013
(Open Section, First round)

Wednesday, 5 June 2013 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

1. The sum

$$\frac{1}{1\times2\times3}+\frac{1}{2\times3\times4}+\frac{1}{3\times4\times5}+\cdots+\frac{1}{100\times101\times102}$$

can be expressed as $\frac{a}{b}$, a fraction in its simplest form. Find $a+b$.

2. Determine the maximum value of $\frac{1+\cos x}{\sin x+\cos x+2}$, where x ranges over all real numbers.

3. Let $\tan\alpha$ and $\tan\beta$ be two solutions of the equation $x^2-3x-3=0$. Find the value of

$$\left|\sin^2(\alpha+\beta)-3\sin(\alpha+\beta)\cos(\alpha+\beta)-3\cos^2(\alpha+\beta)\right|.$$

(Note: $|x|$ denotes the absolute value of x.)

4. Suppose that $a_1, a_2, a_3, a_4, \cdots$ is an arithmetic progression with $a_1>0$ and $3a_8=5a_{13}$. Let $S_n=a_1+a_2+\cdots+a_n$ for all integers $n\geq1$. Find the integer n such that S_n has the maximum value.

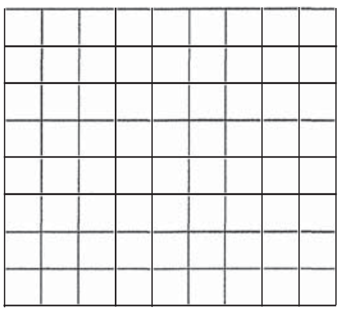
5. If $g(x)=\tan\frac{x}{2}$ for $0<x<\pi$ and $f(g(x))=\sin 2x$, find the value of k such that $kf(\frac{\sqrt{2}}{2})=36\sqrt{2}$.

6. Let $g(x)$ be a strictly increasing function defined for all $x\geq0$. It is known that the range of t satisfying

$$g(2t^2+t+5)<g(t^2-3t+2)$$

is $b<t<a$. Find $a-b$.

7. The figure below shows an 8×9 rectangular board.



How many squares are there in the above rectangular board?

8. Let a, b, c be positive real numbers such that $a+b+c=2013$. Find the maximum value of $\sqrt{3a+12}+\sqrt{3b+12}+\sqrt{3c+12}$.

9. Let $A=\cos^2 10^\circ+\cos^2 50^\circ-\sin 40^\circ\sin 80^\circ$. Determine the value of $100A$.

10. Assume that $a_i \in \{1, -1\}$ for all $i = 1, 2, \cdots, 2013$. Find the least positive number of the following expression

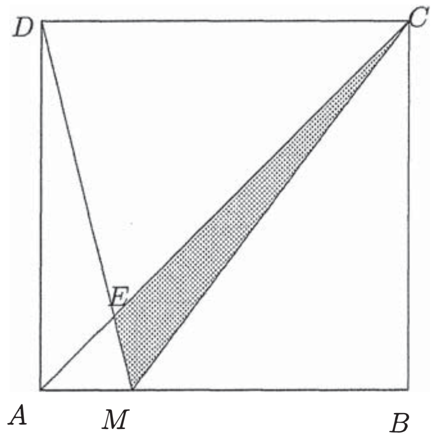
$$\sum_{1 \leq i < j \leq 2013} a_i a_j.$$

11. Let f be a function defined on non-zero real numbers such that

$$\frac{27\mathrm{f}(-x)}{x} - x^2 \mathrm{f}\left(\frac{1}{x}\right) = -2x^2,$$

for all $x \neq 0$. Find f(3).

12. In the figure below, $ABCD$ is a square with $AB = 20$ cm (not drawn to scale). Assume that M is a point such that the area of the shaded region is 40 cm^2. Find AM in centimetres.



13. In the triangle ABC, a circle passes through the point A, the midpoint E of AC, the midpoint F of AB and is tangent to the side BC at D. Suppose

$$\frac{AB}{AC} + \frac{AC}{AB} = 4.$$

Determine the size of $\angle EDF$ in degrees.

14. Let $a_1, a_2, a_3, \cdots$ be a sequence of real numbers in a geometric progression. Let $S_n = a_1 + a_2 + \cdots + a_n$ for all integers $n \geq 1$. Assume that $S_{10} = 10$ and $S_{30} = 70$. Find the value of S_{40}.

15. Find the number of three-digit numbers which are multiples of 3 and are formed by the digits 0,1,2,3,4,5,6,7 without repetition.

16. All the positive integers which are co-prime to 2012 are grouped in an increasing order in such a way that the n^{th} group has $2n-1$ numbers. So, the first three groups in this grouping are $(1), (3,5,7), (9,11,13,15,17)$. It is known that 2013 belongs to the k^{th} group. Find the value of k.

(Note: Two integers are said to be co-prime if their greatest common divisor is 1.)

17. The numbers $1, 2, 3, \cdots, 7$ are randomly divided into two non-empty subsets. The probability that the sum of the numbers in the two subsets being equal is $\frac{p}{q}$ expressed in the lowest term. Find $p+q$.

18. Find the number of real roots of the equation $\log_{10}^2 x - \lfloor \log_{10} x \rfloor - 2 = 0$.

(Note: $\lfloor x \rfloor$ denotes the greatest integer not exceeding x.)

19. In the triangle ABC, $AB = AC$, $\angle A = 90°$, D is the midpoint of BC, E is the midpoint of AC and F is a point on AB such that BE intersects CF at P and B, D, P, F lie on a circle. Let AD intersect CP at H. Given $AP = \sqrt{5} + 2$, find the length of PH.

20. Find the total number of positive integers n not more than 2013 such that $n^4 + 5n^2 + 9$ is divisible by 5.

21. In a circle ω centred at O, AA' and BB' are diameters perpendicular to each other such that the points A, B, A', B' are arranged in an anticlockwise sense in this order. Let P be a point on the minor arc $A'B'$ such that AP intersects BB' at D and BP intersects AA' at C. Suppose the area of the quadrilateral $ABCD$ is 100. Find the radius of ω.

22. A sequence $a_1, a_2, a_3, a_4, \cdots$, with $a_1 = \frac{1}{2}$, is defined by

$$a_n = 2a_n a_{n+1} + 3a_{n+1}$$

for all $n = 1, 2, 3, \cdots$. If $b_n = 1 + \frac{1}{a_n}$ for all $n = 1, 2, 3, \cdots$, find the largest integer m such that

$$\sum_{k=1}^{n} \frac{1}{\log_3 b_k} > \frac{m}{24}$$

for all positive integer $n \geq 2$.

23. Find the largest real number p such that all three roots of the equation below are positive integers:

$$5x^3 - 5(p+1)x^2 + (71p-1)x + 1 = 66p.$$

24. Let a, b, c, d be 4 distinct nonzero integers such that $a+b+c+d = 0$ and the number $M = (bc-ad)(ac-bd)(ab-cd)$ lies strictly between 96100 and 98000. Determine the value of M.

25. In the triangle ABC, $AB = 585$, $BC = 520$, $CA = 455$. Let P, Q be points on the side BC, and $R \neq A$ the intersection of the line AQ with the circumcircle ω of the triangle ABC. Suppose PR is parallel to AC and the circumcircle of the triangle PQR is tangent to ω at R. Find PQ.

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(Open Section, First round Solution)

1. Answer: 12877

   Solution. Let S be the required sum. By using method of difference,

$$\begin{aligned} S &= \frac{1}{2}\left(\frac{1}{1\times 2}-\frac{1}{2\times 3}+\frac{1}{2\times 3}-\frac{1}{3\times 4}+\cdots+\frac{1}{100\times 101}-\frac{1}{101\times 102}\right) \\ &= \frac{1}{2}\left(\frac{1}{2}-\frac{1}{10302}\right) \\ &= \frac{2575}{10302}, \end{aligned}$$

   Hence $a+b=12877$. □

2. Answer: 1

   Solution. Let $y=\dfrac{1+\cos x}{\sin x+\cos x+2}$. When $\cos x+1=0$, $y=0$. Otherwise,

$$y=\frac{1}{1+\frac{1+\sin x}{1+\cos x}}.$$

   Let $u=\dfrac{1+\sin x}{1+\cos x}$. It is clear that $u\geq 0$, and so $y\leq 1$ where the equality holds when $u=0$.

   Thus the maximum value of y is 1 when $\sin x=-1$. □

3. Answer: 3

   Solution. We have $\tan\alpha+\tan\beta=3$ and $\tan\alpha\tan\beta=-3$. Hence

$$\tan(\alpha+\beta)=\frac{3}{1-(-3)}=\frac{3}{4}.$$

   Hence

$$\begin{aligned} &\left|\sin^2(\alpha+\beta)-3\sin(\alpha+\beta)\cos(\alpha+\beta)-3\cos^2(\alpha+\beta)\right| \\ =\ & \cos^2(\alpha+\beta)\left[\left(\frac{3}{4}\right)^2-3\left(\frac{3}{4}\right)-3\right] \\ =\ & \cos^2(\alpha+\beta)\times\left(-\frac{75}{16}\right), \end{aligned}$$

   and since

$$\tan^2(\alpha+\beta)=\frac{1-\cos^2(\alpha+\beta)}{\cos^2(\alpha+\beta)}=\frac{1}{\cos^2(\alpha+\beta)}-1=\frac{9}{16},$$

   we have $\cos^2(\alpha+\beta)=\dfrac{16}{25}$. Thus, we have

$$\left|\sin^2(\alpha+\beta)-3\sin(\alpha+\beta)\cos(\alpha+\beta)-3\cos^2(\alpha+\beta)\right|=|-3|=3.$$

   □

4. Answer: 20

Solution. Let $a_n = a_1 + (n-1)d$. As

$$3a_8 = 5a_{13},$$

we have $3(a_1 + 7d) = 5(a_1 + 12d)$, and so $2a_1 + 39d = 0$. So $d < 0$ and

$$a_{20} + a_{21} = a_1 + 19d + a_1 + 20d = 0.$$

So $a_{20} > 0$ but $a_{21} < 0$, as $a_{21} = a_{20} + d$ and $a_{20} + a_{21} = 0$. Thus $a_1, a_2, a_3, a_4, \cdots$ is an decreasing sequence and

$$a_1 > a_2 > \cdots > a_{20} > 0 > a_{21} > \cdots.$$

Hence S_n has the maximum value when $n = 20$. □

5. Answer: 81

Solution. Note that $\mathrm{f}(\mathrm{g}(x)) = \sin 2x = 2\sin x \cos x = \dfrac{4\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} \cdot \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$. Hence

$$\mathrm{f}(\frac{\sqrt{2}}{2}) = \frac{2\sqrt{2}}{1+\frac{1}{2}} \cdot \frac{1-\frac{1}{2}}{1+\frac{1}{2}} = \frac{4\sqrt{2}}{9}.$$

If $k\mathrm{f}(\dfrac{\sqrt{2}}{2}) = 36\sqrt{2}$, then $k = 81$. □

6. Answer: 2

Solution. Note that $2t^2+t+5 = 2\left(t+\dfrac{1}{4}\right)^2 + \dfrac{39}{8} > 0$. Hence $\mathrm{g}(2t^2+t+5) < \mathrm{g}(t^2-3x+2)$ is true if and only if

$$2t^2 + t + 5 < t^2 - 3t + 2,$$

which is equivalent to $(t+3)(t+1) < 0$. Hence the range of t satisfying the given inequality is $-3 < t < -1$, which yields $a - b = (-1) - (-3) = 2$. □

7. Answer: 240

Solution. By counting the number of squares of different types, we obtain

$$1\times 2 + 2\times 3 + 3\times 4 + 4\times 5 + 5\times 6 + 6\times 7 + 7\times 8 + 8\times 9 = \frac{8\times 9\times 10}{3} = 240.$$

□

8. Answer: 135

Solution. Note that

$$(\sqrt{3a+12} + \sqrt{3b+12} + \sqrt{3c+12})^2 \le 3(3a + 12 + 3b + 12 + 3c + 12)$$
$$= 9(a + b + c + 12) = 9(2013 + 12) = 9 \times 2025,$$

where the equality holds if $3a + 12 = 3b + 12 = 3c + 12$, i.e., $a = b = c = 671$. Thus the answer is $3 \times 45 = 135$.

□

9. Answer: 75

Solution. Let

$$B = \sin^2 10^\circ + \sin^2 50^\circ - \cos 40^\circ \cos 80^\circ.$$

Then

$$A + B = 2 - \cos 40^\circ$$

and

$$A - B = \cos 20^\circ + \cos 100^\circ + \cos 120^\circ = 2\cos 60^\circ \cos 40^\circ + \cos 120^\circ$$
$$= \cos 40^\circ - \frac{1}{2}.$$

Thus $2A = \frac{3}{2}$ and $100A = 75$. □

10. Answer: 6

Solution. Note that

$$2\sum_{1\le i<j\le 2013} a_i a_j = (a_1 + a_2 + \cdots + a_{2013})^2 - (a_1^2 + a_2^2 + \cdots + a_{2013}^2)$$

$$= (a_1 + a_2 + \cdots + a_{2013})^2 - 2013.$$

By the given condition, $a_1 + a_2 + \cdots + a_{2013}$ is an odd number between -2013 and 2013 inclusive.

Also note that the minimum positive integer of $x^2 - 2013$ for an integer x is $45^2 - 2013 = 12$ when $x = 45$ or -45. As an illustration, $x = 45$ can be achieved by taking $a_1 = a_2 = a_3 = \cdots = a_{45} = 1$ and the others $a_{46}, a_{47}, \cdots, a_{2013}$ to consist of equal number of 1's and -1's. Thus the least value is $\frac{12}{2} = 6$. □

11. Answer: 2

Solution. Letting $x = -y$, we get

$$-\frac{27\mathrm{f}(y)}{y} - y^2\mathrm{f}\left(-\frac{1}{y}\right) = -2y^2. \qquad (1)$$

Letting $x = \frac{1}{y}$, we get

$$27y\mathrm{f}\left(-\frac{1}{y}\right) - \frac{1}{y^2}\mathrm{f}(y) = -\frac{2}{y^2}. \qquad (2)$$

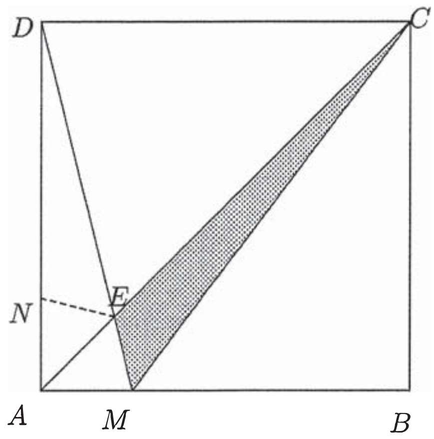
Then $27 \times (1) + y \times (2)$ gives

$$-\frac{729\mathrm{f}(y)}{y} - \frac{\mathrm{f}(y)}{y} = -54y^2 - \frac{2}{y}.$$

Solving for $\mathrm{f}(y)$, we have $\mathrm{f}(y) = \frac{1}{365}(27y^3 + 1)$. Thus $\mathrm{f}(3) = \frac{3^6+1}{365} = 2$. □

12. Answer: 5

Solution. Choose a point N on DA such that $NA = MA = x$.



It is clear that ΔNAE and ΔMAE are congruent by SAS test. Let S be the area of $\triangle NAE$. Then area of $\Delta DNE = \dfrac{20-x}{x}S$. It is also clear that areas of ΔDAE and ΔCEM are equal to 40cm^2. It follows that

$$\text{Area of } \Delta DAE = \frac{20-x}{x}S + S = \frac{20}{x}S,$$

so that $\dfrac{20}{x}S = 40\text{cm}^2$, that is, $S = 2x$.

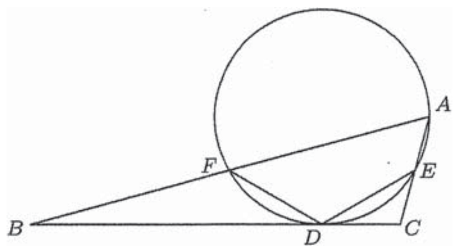
Area of $\Delta DAM = \dfrac{1}{2} \times x \times 20 = 10x$. On the other hand,

$$\begin{aligned}
\text{Area of } \Delta DAM &= \text{Area of} \Delta DAE + \text{Area of} \Delta AEM \\
&= \frac{20}{x}S + S \\
&= \frac{20+x}{x}S = \frac{20+x}{x} \times 2x = 2(20+x).
\end{aligned}$$

So $2(20+x) = 10x$, which means that $AM = x = 5\text{cm}$. □

13. Answer: 120

Solution.



Let $BC = a, CA = b$ and $AB = c$. Let $BD = a_1$ and $DC = a_2$. Using the power of B with respect to the circle, we have $a_1^2 = c^2/2$. Similarly, $a_2^2 = b^2/2$. Thus $b + c = \sqrt{2}(a_1 + a_2) =$

$\sqrt{2}a$, or $2a^2 = (b+c)^2$. Therefore,

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{2b^2+2c^2-(b+c)^2}{4bc} = \frac{1}{4}\left(\frac{b}{c}+\frac{c}{b}-2\right) = \frac{1}{4}(4-2) = \frac{1}{2}.$$

Therefore, $\angle A = 60°$. Since A, F, D, E are concyclic, $\angle EDF = 120°$.

□

14. Answer: 150

Solution. Let r be the common ratio of this geometric sequence. Thus

$$S_n = a_1(1 + r + r^2 + \cdots + r^{n-1}).$$

Thus

$$10 = a_1(1 + r + \cdots + r^9)$$

and

$$70 = a_1(1 + r + \cdots + r^{29}).$$

As

$$1 + r + \cdots + r^{29} = (1 + r + \cdots + r^9)(1 + r^{10} + r^{20}),$$

we have

$$1 + r^{10} + r^{20} = 7.$$

So r^{10} is either 2 or -3. As $r^{10} > 0$, $r^{10} = 2$.

Hence

$$S_{40} = a_1(1 + r + \cdots + r^{39}) = a_1(1 + r + \cdots + r^9)(1 + r^{10} + r^{20} + r^{30})$$
$$= 10 \times (1 + 2 + 2^2 + 2^3) = 150.$$

□

15. Answer: 106

Solution. Note that an integer is a multiple of 3 if and only if the sum of its digits is a multiple of 3. Also note that the sum of three integers a, b, c is a multiple of 3 if and only if either (i) a, b, c all have the same remainder when divided by 3, or (ii) a, b, c have the distinct remainders when divided by 3. Observe that the remainders of $0, 1, 2, 3, 4, 5, 6, 7$ when divided by 3 are $0, 1, 2, 0, 1, 2, 0, 1$ respectively.

For case (i), the only possible selections such that all the three numbers have the same remainder when divided by 3 are $\{0, 3, 6\}$ and $\{1, 4, 7\}$. With $\{0, 3, 6\}$, we have 4 possible numbers (note that a number does not begin with 0), and with $\{1, 4, 7\}$, there are 6 possible choices.

For case (ii), if the choice of the numbers does not include 0, then there are $2\times3\times2\times3! = 72$; if 0 is included, then there are $3 \times 2 \times 4$ choices.

Hence the total number of possible three-digit numbers is $72 + 24 + 10 = 106$.

□

16. Answer: 32

Solution. Note that $2012 = 2^2 \times 503$, and that 503 is a prime number. There are 1006 multiples of 2 less than or equal to 2012; there are 4 multiples of 503 less than or equal to 2012; there are 2 multiples of 1006 less than or equal to 2012. By the Principle of Inclusion and Exclusion, there are $1006 + 4 - 2 = 1008$ positive integers not more than 2012 which are not co-prime to 2012. Hence there are $2012 - 1008 = 1004$ positive integers less than 2012 which are co-prime with 2012. Thus, 2013 is the 1005^{th} number co-prime with 2012. Note also that the sum of the first n odd numbers equals n^2, and that $31^2 < 1005 < 32^2$, the number 2013 must be in the 326th group. Hence $k = 32$. □

17. Answer: 67

The total number of ways of dividing the seven numbers into two non-empty subsets is $\frac{2^7 - 2}{2} = 63$. Note that since $1 + 2 + 3 + \cdots + 7 = 28$, the sum of the numbers in each of the two groups is 14. Note also that the numbers $5, 6, 7$ cannot be in the same group since $5 + 6 + 7 = 18 > 14$. We consider three separate cases:

Case (i): Only 6 and 7 in the same group and 5 in the other group:

$\{2, 3, 4, 5\}, \{1, 6, 7\}$

Case (ii): Only 5 and 6 in the same group and 7 in the other group:

$\{1, 2, 5, 6\}, \{3, 4, 7\}$

$\{3, 5, 6\}, \{1, 2, 4, 7\}$

Case (iii): Only 5 and 7 in the same group and 6 in the other group:

$\{2, 5, 7\}, \{1, 3, 4, 6\}$

Hence there are 4 such possibilities. Thus the required probability is $\frac{4}{63}$, yielding that $p + q = 67$. □

18. Answer: 3

Solution. Let $u = \lfloor \log_{10} x \rfloor$ and $r = \log_{10} x - u$. So $0 \leq r < 1$. Thus

$$(u + r)^2 = u + 2.$$

Case 1: $r = 0$.

Then $u^2 = u + 2$ and so $u = 2$ or $u = -1$, corresponding to $x = 10^2 = 100$ and $x = 10^{-1} = 0.1$.

Case 2: $0 < r < 1$.

In this case, $u + 2$ is an integer which is not a complete square and

$$r = \sqrt{u + 2} - u.$$

As $r > 0$, we have $u \leq 2$. But $u + 2$ is not a complete square. So $u \leq 1$. As $u + 2 \geq 0$ and not a complete square, we have $u \geq 0$. Hence $u \in \{0, 1\}$.

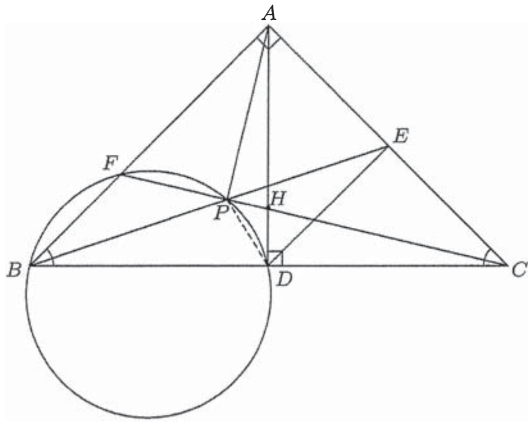
If $u = 0$, then $r = \sqrt{2} - 0 > 1$, not suitable.

If $u = 1$, then $r = \sqrt{3} - 1$. So $\log_{10} x = \sqrt{3}$ and $x = 10^{\sqrt{3}}$.

Hence the answer is 3. □

19. Answer: 1

Solution.



Join PD. Then $\angle DPC = \angle FBD = 45° = \angle DAC$ so that D, P, A, C are concyclic. Thus $\angle APC = \angle ADC = 90°$. It follows that $EA = EP = ED = EC$. Let $\angle PAH = \theta$. Then $\angle PCD = \theta$. Thus $\angle EPC = \angle ECP = 45° - \theta$ so that $\angle AEB = 90° - 2\theta$. That is $\angle ABE = 2\theta$. Thus $\tan 2\theta = AE/AB = 1/2$. From this, we get $\tan\theta = \sqrt{5} - 2$. Therefore, $PH = AP \tan\theta = (\sqrt{5}+2)(\sqrt{5}-2) = 1$. □

20. Answer: 1611

Solution. Note that $n^4 + 5n^2 + 9 = n^4 - 1 + 5n^2 + 10 = (n-1)(n+1)(n^2+1) + 5(n^2+2)$.

If $n \equiv 1$ or $4 \pmod 5$, then 5 divides $n-1$ or $n+1$.

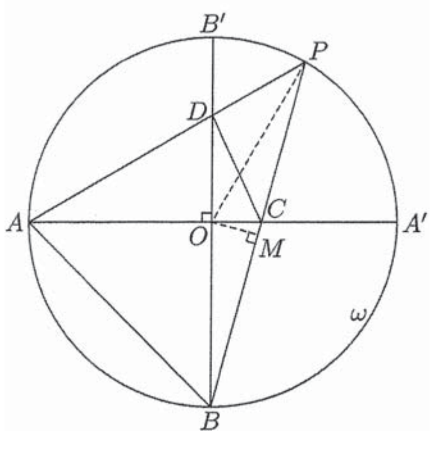
If $n \equiv 2$ or $3 \pmod 5$, then 5 divides n^2+1.

If $n \equiv 0 \pmod 5$, then 5 does not divide $(n-1)n(n^2+1)$ but divides $5(n^2+2)$, hence does not divide $n^4 + 5n^2 + 9$.

Thus, there are $2010 \div 5 = 402$ multiples of 5 from 1 to 2013. The number of integers thus required is $2013 - 402 = 1611$. □

21. Answer: 10

Solution. Since AC intersects BD at right angle, the area of the convex quadrilateral $ABCD$ is $\frac{1}{2}AC \cdot BD$. Let M be the midpoint of PB. As $\angle CAB = \angle ABD = 45°$, and $\angle BCA = \angle BOM = \angle DAB$, we have $\triangle ABC$ is similar to $\triangle BDA$. Thus $AB/BD = AC/BA$. From this, we have $(ABCD) = \frac{1}{2}AC \cdot BD = \frac{1}{2}AB^2 = OA^2$ so that $OA = 10$.



□

22. Answer: 13

Solution. Given that $a_n = 2a_n a_{n+1} + 3a_{n+1}$ we obtain $a_{n+1} = \dfrac{a_n}{2a_n+3}$. Thus we have $\dfrac{1}{a_{n+1}} = 2 + \dfrac{3}{a_n}$. We thus have $\dfrac{1}{a_{n+1}} + 1 = 3\left(1 + \dfrac{1}{a_n}\right)$ for all $n = 1, 2, 3, \cdots$. Letting $b_n = 1 + \dfrac{1}{a_n}$, it is clear that the sequence $\{b_n\}$ follows a geometric progression with first term $b_1 = 1 + \dfrac{1}{a_1} = 3$, and common ratio 3. Thus, for $n = 1, 2, 3, \cdots$, $b_n = 1 + \dfrac{1}{a_n} = 3^n$ for $n = 1, 2, 3, \cdots$.

Let $f(n) = \displaystyle\sum_{k=1}^{n} \frac{1}{n + \log_3 b_k} = \sum_{k=1}^{n} \frac{1}{n+k} > \frac{m}{24}$, $n = 2, 3, 4, \cdots$. It is clear that $f(n)$ is an increasing function since

$$f(n+1) - f(n) = \frac{1}{n+1} > 0.$$

Thus $f(n)$ is a strictly increasing sequence in n. Thus the minimum value of $f(n)$ occurs when $n = 2$.

$$f(2) = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} > \frac{m}{24},$$

forcing $m < 14$. Thus the largest value of integer m is 13. □

23. Answer: 76

Solution. Observe that $x = 1$ is always a root of the equation

$$5x^3 - 5(p+1)x^2 + (71p-1)x + 1 = 66p.$$

Thus this equation has all roots positive integers if and only if the two roots of the equation below are positive integers:

$$5x^2 - 5px + 66p - 1 = 0.$$

Let u, v be the two roots with $u \le v$. Then

$$u + v = p, \quad uv = (66p-1)/5,$$

implying that

$$5uv = 66(u+v) - 1.$$

By this expression, we know that u, v are not divisible by any one of $2, 3, 11$. We also have $5uv > 66(u+v)$, implying that

$$\frac{2}{u} \ge \frac{1}{u} + \frac{1}{v} > \frac{5}{66},$$

and so $u \le 26$. As

$$v = \frac{66u-1}{5u-66} > 0,$$

we have $5u - 66 > 0$ and so $u \ge 14$. Since u is not a multiple of any one of 2, 3, 11, we have

$$u \in \{17, 19, 23, 25\}.$$

As $v = \frac{66u-1}{5u-66}$, only when $u = 17$, $v = 59$ is an integer.

Thus, only when $p = u + v = 17 + 69 = 76$, the equation

$$5x^3 - 5(p+1)x^2 + (71p-1)x + 1 = 66p$$

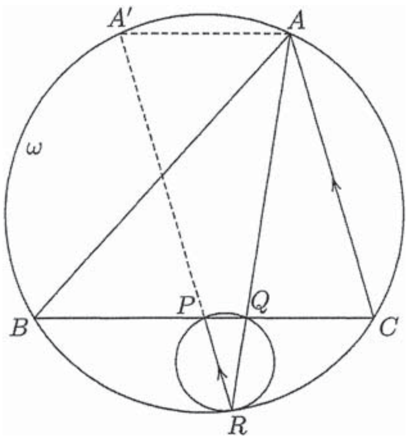
has all three roots being positive integers. □

24. Answer: 97344

Solution. First we show that M must be a square. Let $d = -a - b - c$. Then $bc - ad = bc + a(a+b+c) = (a+c)(a+b)$, $ac - bd = ac + b(a+b+c) = (b+c)(b+a)$, and $ab - cd = ab + c(a+b+c) = (c+a)(c+b)$. Therefore $M = (a+b)^2(b+c)^2(c+a)^2$. Note that $(a+b)(b+c)(c+a)$ cannot be an odd integer since two of the 3 numbers a, b, c must be of the same parity. The only squares in $(96100, 98000)$ are $311^2, 312^2, 313^2$. Since 311 and 313 are odd, the only value of M is $312^2 = 97344$. When $a = 18, b = -5, c = 6, d = -19$, it gives $M = 97344$. □

25. Answer: 64

Solution.



First by cosine rule, $\cos C = 2/7$. Reflect A about the perpendicular bisector of BC to get the point A' on ω. Then $AA'BC$ is an isosceles trapezoid with $A'A$ parallel to BC. Thus $A'A = BC - 2AC\cos C = 520 - 2 \times 455 \times 2/7 = 260$. Consider the homothety h centred at R mapping the circumcircle of PQR to ω. We have $h(Q) = A$, and $h(P) = A'$ because PQ is parallel to $A'A$. Thus A', P, R are collinear and $AA'PC$ is a parallelogram. Hence $PC = AA' = 260$, and P is the midpoint of BC. Also $PA' = CA = 455$. As $PA' \times PR = BP \times PC$, we have $455 \times PR = 260^2$ giving $PR = 1040/7$. Since the triangles PQR and $A'AR$ are similar, we have $PQ/A'A = RP/RA'$. Therefore, $PQ = 260 \times (1040/7)/(455 + 1040/7) = 64$. □

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2013

(Open Section, Round 2)

Saturday, 6 July 2013 **0900-1300**

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1. Let $a_1, a_2, \ldots$ be a sequence of integers defined recursively by $a_1 = 2013$ and for $n \geq 1$, a_{n+1} is the sum of the 2013^{th} power of the digits of a_n. Do there exist distinct positive integers i, j such that $a_i = a_j$?

2. Let ABC be an acute-angled triangle and let D, E and F be the midpoints of BC, CA and AB respectively. Construct a circle, centred at the orthocentre of triangle ABC, such that triangle ABC lies in the interior of the circle. Extend EF to intersect the circle at P, FD to intersect the circle at Q and DE to intersect the circle at R. Show that $AP = BQ = CR$.

3. Let N be a positive integer. Prove that there exists a positive integer n such that $n^{2013} - n^{20} + n^{13} - 2013$ has at least N distinct prime factors.

4. Let F be a finite nonempty set of integers and let n be a positive integer. Suppose that

- Any $x \in F$ may be written as $x = y + z$ for some $y, z \in F$;
- If $1 \leq k \leq n$ and $x_1, \ldots, x_k \in F$, then $x_1 + \cdots + x_k \neq 0$.

Show that F has at least $2n + 2$ distinct elements.

5. Let ABC be a triangle with integral side lengths such that $\angle A = 3\angle B$. Find the minimum value of its perimeter.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2013

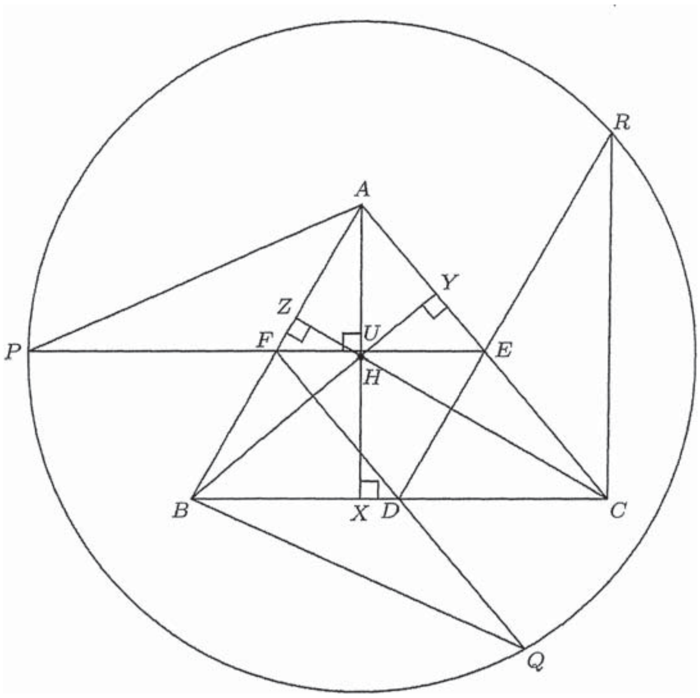
(Open Section, Round 2 solutions)

1. The answer is yes. For any positive integer n, let $f(n)$ be the sum of 2013^{th} power of the digits of n. Let $S = \{1, 2, \ldots, 10^{2017} - 1\}$, and $n = \overline{a_1 a_2 \ldots a_{2017}} \in S$. Then

$$f(n) = \sum a_i^{2013} \leq 2017 \cdot 9^{2013} < 10^4 \cdot 10^{2013} = 10^{2017} \in S.$$

Since $a_i = f^{(i)}(2013) \in S$, there exist distinct positive integers i, j such that $a_i = a_j$.

2. Let the radius of the circle be r. Let X, Y and Z be the feet of the altitudes from A, B and C respectively. Let PE intersect the altitude from A at U. We have $AP^2 = AU^2 + PU^2 = AU^2 + r^2 - UH^2 = r^2 + (AU + UH) \cdot (AU - UH) = r^2 + AH \cdot (AU - UH) = r^2 + AH \cdot (UX - UH) = r^2 + AH \cdot HX$. Similarly, $BQ = r^2 + BH \cdot HY$, and $CR = r^2 + CH \cdot HZ$. Since $AH \cdot HX = BH \cdot HY = CH \cdot HZ$, we have $AP = BQ = CR$.



3. The result is true for any nonconstant polynomial $f(n) = a_m n^m + a_{m-1} n^{m-1} + \cdots + a_0$ with integer coefficients. We may assume that $a_m > 0$. Thus there exists a positive integer n_0 such that $f(n)$ is positive and increasing on (n_0, ∞).

It suffices to show that if for some $n_1 > n_0$, $f(n_1) = p_1^{r_1} \cdots p_k^{r_k}$ has exactly k distinct prime factors, then for some $n_2 > n_1$, $f(n_2)$ has more than k prime factors. Given such an n_1, let $n_2 = n_1 + p_1^{r_1+1} \cdots p_k^{r_k+1}$. Then

$$f(n_2) \equiv p_1^{r_1} \cdots p_k^{r_k} \pmod{p_1^{r_1+1} \cdots p_k^{r_k+1}}.$$

Hence, for each j, $1 \leq j \leq k$, we have that $p_j^{r_j}$ divides $f(n_2)$ but $p_j^{r_j+1}$ does not divide $f(n_2)$. As $f(n_2) > f(n_1) = p_1^{r_1} \cdots p_k^{r_k}$, it follows that $f(n_2)$ must have at least $k+1$ prime factors.

4. Because of the second condition above, $0 \notin F$. If F contains only positive elements, let x be the smallest element in F. But then $x = y+z$, and $y, z > 0$ imply that $y, z < x$, a contradiction. Hence F contains negative elements. A similar argument shows that F contains positive elements.

Pick any positive element of F and label it as x_1. Assume that positive elements of F, $x_1, \ldots, x_k$, have been chosen. We can write $x_k = y + z$, where $y, z \in F$. We may assume that $y > 0$. Label y as x_{k+1}. Carry on in this manner to choose positive elements $x_1, x_2, \ldots$ of F, not necessarily distinct. Since F is a finite set, there exist positive integers $i < j$ such that $x_i, \ldots, x_{j-1}$ are distinct and $x_j = x_i$. There are $z_i, \ldots z_{j-1} \in F$ such that

$$\begin{aligned} x_i &= x_{i+1} + z_i \\ x_{i+1} &= x_{i+2} + z_{i+1} \\ \vdots \quad & \quad \vdots \\ x_{j-1} &= x_j + z_{j-1}. \end{aligned}$$

Since $x_i = x_j$, we see that $z_i + z_{i+1} + \cdots + z_{j-1} = 0$. By the assumption, $j - i > n$. Since the elements $x_i, \ldots, x_{j-1}$ are distinct, F contains at least $j - i \geq n+1$ positive elements. Similarly, F contains at least $n+1$ negative elements. The result follows.

5. Let the sides be a, b, c. From the sine rule, we have

$$\begin{aligned} \frac{a}{b} &= \frac{\sin 3B}{\sin B} = 4\cos^2 B - 1 \\ \frac{c}{b} &= \frac{\sin C}{\sin B} = \frac{\sin 4B}{\sin B} = 8\cos^3 B - 4\cos B \end{aligned}$$

Thus

$$2\cos B = \frac{a^2 + c^2 - b^2}{ac} \in \mathbb{Q}.$$

Hence there exist coprime positive integers p, q such that $2\cos B = \frac{p}{q}$. Hence

$$\frac{a}{b} = \frac{p^2}{q^2} - 1 \quad \Leftrightarrow \quad \frac{a}{p^2 - q^2} = \frac{b}{q^2};$$
$$\frac{c}{b} = \frac{p^3}{q^3} - \frac{2p}{q} \quad \Leftrightarrow \quad \frac{c}{p^3 - 2pq^2} = \frac{b}{q^3}.$$

Thus

$$\frac{a}{(p^2 - q^2)q} = \frac{b}{q^3} = \frac{c}{p^3 - 2pq^2} = \frac{e}{f}, \qquad \gcd(e, f) = 1.$$

Since perimeter is minimum, $\gcd(a, b, c) = 1$. From $\gcd(e, f) = 1$, we have $f \mid q^3$ and $f \mid p^3 - 2pq^2$. We'll prove that $f = 1$.

If $f > 1$, then it has a prime divisor $f' > 1$ such that $f' \mid q^3$ and $f' \mid p^3 - 2pq^2$. Thus $f' \mid q$ and $f' \mid p$, contradicting $\gcd(p, q) = 1$. Thus $f = 1$. From $\gcd(a, b, c) = 1$, we conclude that $e = 1$. Thus

$$a = (p^2 - q^2)q, \quad b = q^3, \quad c = p^3 - 2pq^2.$$

From $0° < \angle A + \angle B = 4\angle B < 180°$, we get $0° < \angle B < 45°$ and hence $\sqrt{2} < 2\cos B < 2$ implying that $\sqrt{q} < p < 2q$. The smallest positive integers satisfying this inequality is $p = 3, q = 2$. Since $a + b + c = p^2q + p(p^2 - 2q^2)$ and $p^2 - 2q^2 = 1$, we see that the minimum perimeter is achieved when $p = 3, q = 2$ and the value is 21.