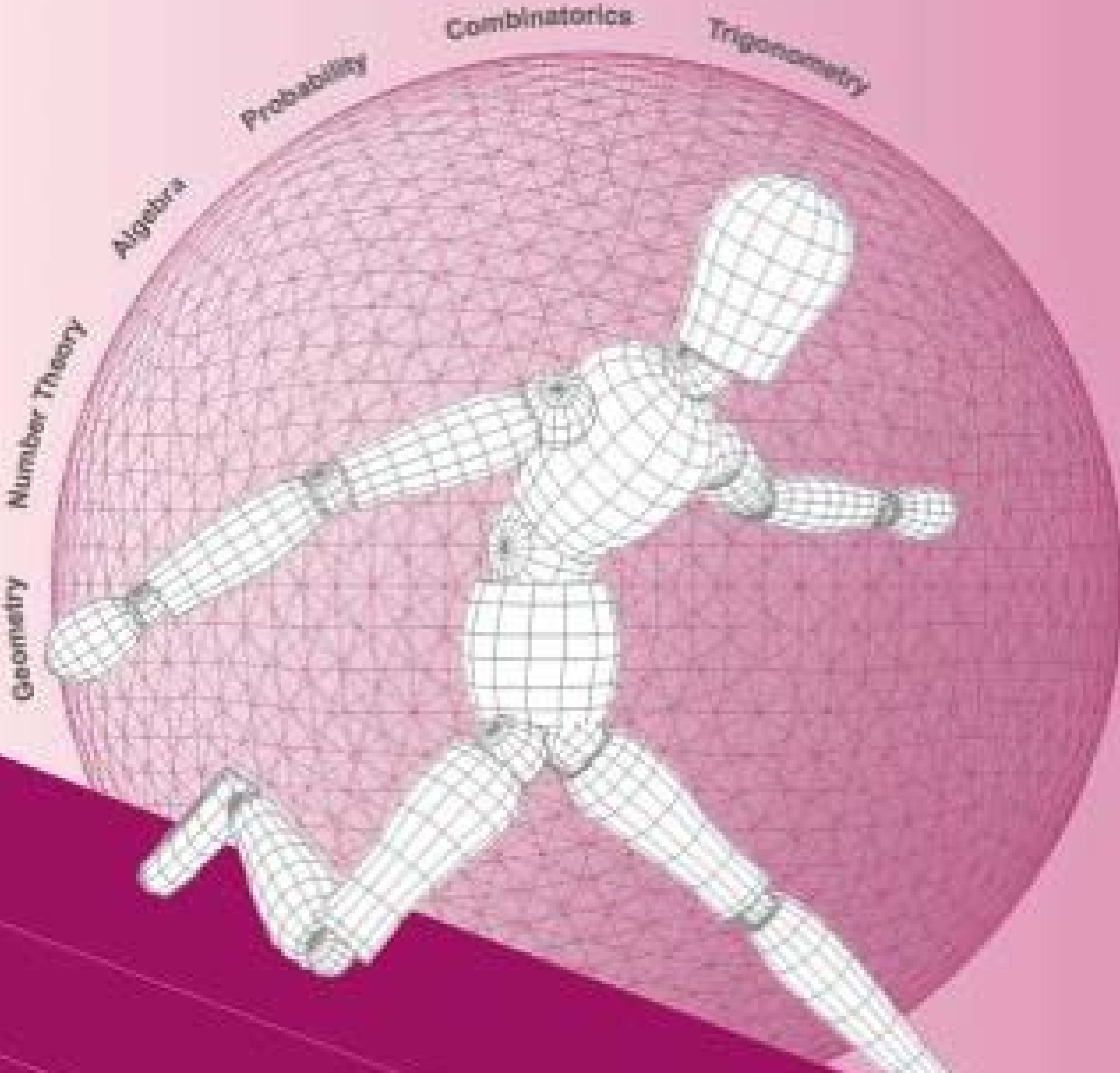
SINGAPORE MATHEMATICAL OLYMPIADS 2012



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Singapore Mathematical Olympiad (SMO) 2012

Junior Section (First Round)

Tuesday, 29 May 2012 **0930–1200 hrs**

Instructions to contestants

1. *Answer ALL 35 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*
8. *Throughout this paper, let* $\lfloor x \rfloor$ *denote the greatest integer less than or equal to* x*. For example,* $\lfloor 2.1 \rfloor = 2$, $\lfloor 3.9 \rfloor = 3$.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1. Let α and β be the roots of the quadratic equation $x^2 + 2bx + b = 1$. The smallest possible value of $(\alpha - \beta)^2$ is

   (A) 0; (B) 1; (C) 2; (D) 3; (E) 4.

2. It is known that $n^{2012} + n^{2010}$ is divisible by 10 for some positive integer n. Which of the following numbers is not a possible value for n?

   (A) 2; (B) 13; (C) 35; (D) 47; (E) 59.

3. Using the vertices of a cube as vertices, how many triangular pyramid can you form?

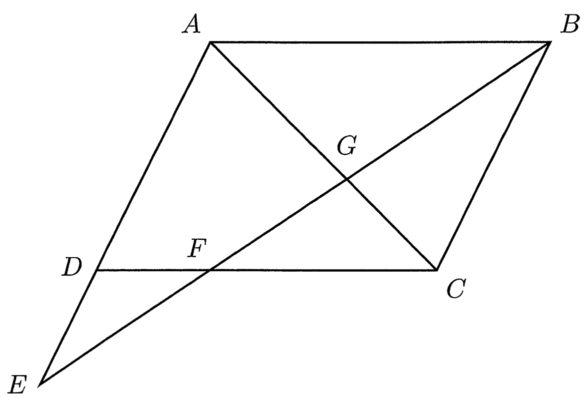
   (A) 54; (B) 58; (C) 60; (D) 64; (E) 70.

4. AB is a chord of a circle with centre O. CD is the diameter perpendicular to the chord AB, with AB closer to C than to D. Given that $\angle AOB = 90^\circ$, then the quotient

$$\frac{\text{area of } \triangle ABC}{\text{area of } \triangle AOD} = \_\_\_\_$$

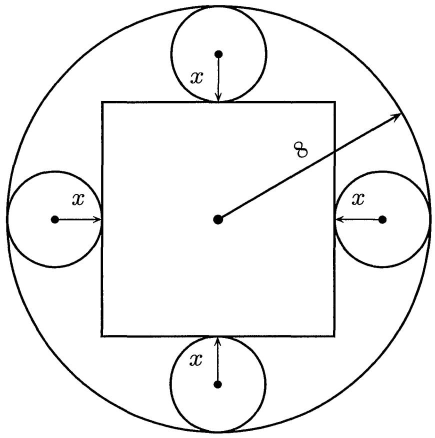
   (A) $\sqrt{2} - 1$; (B) $2 - \sqrt{2}$; (C) $\frac{\sqrt{2}}{2}$; (D) $\frac{1+\sqrt{2}}{2}$; (E) $\frac{1}{2}$.

5. The diagram below shows that $ABCD$ is a parallelogram and both AE and BE are straight lines. Let F and G be the intersections of BE with CD and AC respectively. Given that $BG = EF$, find the quotient DE AE.



   (A) $\frac{3-\sqrt{5}}{2}$; (B) $\frac{3-\sqrt{6}}{2}$; (C) $\frac{3-\sqrt{6}}{4}$; (D) $\frac{3-\sqrt{7}}{2}$; (E) $\frac{3-\sqrt{7}}{4}$.

6. Four circles each of radius x and a square are arranged within a circle of radius 8 as shown in the following figure.



What is the range of x?

(A) $0 < x < 4$; (B) $0 < x < 8(\sqrt{2}+1)$; (C) $4 - 2\sqrt{2} < x < 4$;

(D) $4 - 2\sqrt{2} < x < 8(\sqrt{2}-1)$; (E) $4 - \sqrt{2} < x < 4(\sqrt{2}+1)$.

7. Adam has a triangular field ABC with $AB = 5$, $BC = 8$ and $CA = 11$. He intends to separate the field into two parts by building a straight fence from A to a point D on side BC such that AD bisects $\angle BAC$. Find the area of the part of the field ABD.

(A) $\frac{4\sqrt{21}}{11}$; (B) $\frac{4\sqrt{21}}{5}$; (C) $\frac{5\sqrt{21}}{11}$; (D) $\frac{5\sqrt{21}}{4}$; (E) None of the above.

8. For any real number x, let $\lfloor x \rfloor$ be the largest integer less than or equal to x and $x \;\; = x - \lfloor x \rfloor$. Let a and b be real numbers with $b \neq 0$ such that

$$a = b\left\lfloor \frac{a}{b} \right\rfloor - b\left\{\frac{a}{b}\right\}.$$

Which of the following statements is incorrect?

(A) If b is an integer then a is an integer;

(B) If a is a non-zero integer then b is an integer;

(C) If b is a rational number then a is a rational number;

(D) If a is a non-zero rational number then b is a rational number;

(E) If b is an even number then a is an even number.

9. Given that

$$y = \frac{\not{x} - \not{x}}{x}$$

is an integer. Which of the following is incorrect?

(A) x can admit the value of any non-zero integer;

(B) x can be any positive number;

(C) x can be any negative number;

(D) y can take the value 2;

(E) y can take the value -2.

10. Suppose that A, B, C are three teachers working in three different schools X, Y, Z and specializing in three different subjects: Mathematics, Latin and Music. It is known that

(i) A does not teach Mathematics and B does not work in school Z;

(ii) The teacher in school Z teaches Music;

(iii) The teacher in school X does not teach Latin;

(iv) B does not teach Mathematics.

Which of the following statement is correct?

(A) B works in school X and C works in school Y;

(B) A teaches Latin and works in school Z;

(C) B teaches Latin and works in school Y;

(D) A teaches Music and C teaches Latin;

(E) None of the above.

Short Questions

11. Let a and b be real numbers such that $a > b$, $2^a + 2^b = 75$ and $2^{-a} + 2^{-b} = 12^{-1}$. Find the value of 2^{a-b}.

12. Find the sum of all positive integers x such that $\dfrac{x^3 - x + 120}{(x-1)(x+1)}$ is an integer.

13. Consider the equation

$$\sqrt{3x^2 - 8x + 1} + \sqrt{9x^2 - 24x - 8} = 3.$$

It is known that the largest root of the equation is $-k$ times the smallest root. Find k.

14. Find the four-digit number $\overline{abcd}$ satisfying

$$2(\overline{abcd}) + 1000 = \overline{dcba}.$$

(For example, if $a = 1$, $b = 2$, $c = 3$ and $d = 4$, then $\overline{abcd} = 1234$.)

15. Suppose x and y are real numbers satisfying $x^2 + y^2 - 22x - 20y + 221 = 0$. Find xy.

16. Let m and n be positive integers satisfying

$$mn^2 + 876 = 4mn + 217n.$$

Find the sum of all possible values of m.

17. For any real number x, let $\lfloor x \rfloor$ denote the largest integer less than or equal to x. Find the value of $\lfloor x \rfloor$ of the smallest x satisfying $\lfloor x^2 \rfloor - \lfloor x \rfloor^2 = 100$.

18. Suppose $x_1, x_2, \ldots, x_{49}$ are real numbers such that

$$x_1^2 + 2x_2^2 + \cdots + 49x_{49}^2 = 1.$$

Find the maximum value of $x_1 + 2x_2 + \cdots + 49x_{49}$.

19. Find the minimum value of

$$\sqrt{x^2 + (20 - y)^2} + \sqrt{y^2 + (21 - z)^2} + \sqrt{z^2 + (20 - w)^2} + \sqrt{w^2 + (21 - x)^2}.$$

20. Let A be a 4-digit integer. When both the first digit (left-most) and the third digit are increased by n, and the second digit and the fourth digit are decreased by n, the new number is n times A. Find the value of A.

21. Find the remainder when 1021^{1022} is divided by 1023.

22. Consider a list of six numbers. When the largest number is removed from the list, the average is decreased by 1. When the smallest number is removed, the average is increased by 1. When both the largest and the smallest numbers are removed, the average of the remaining four numbers is 20. Find the product of the largest and the smallest numbers.

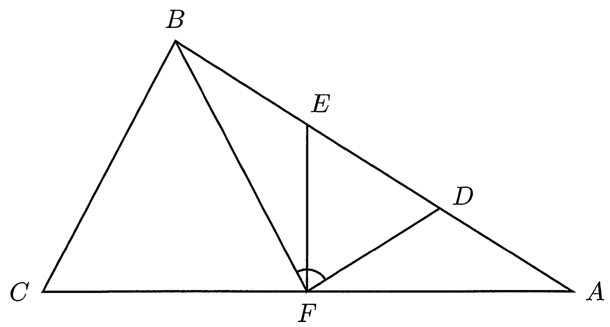
23. For each positive integer $n \geq 1$, we define the recursive relation given by

$$a_{n+1} = \frac{1}{1 + a_n}.$$

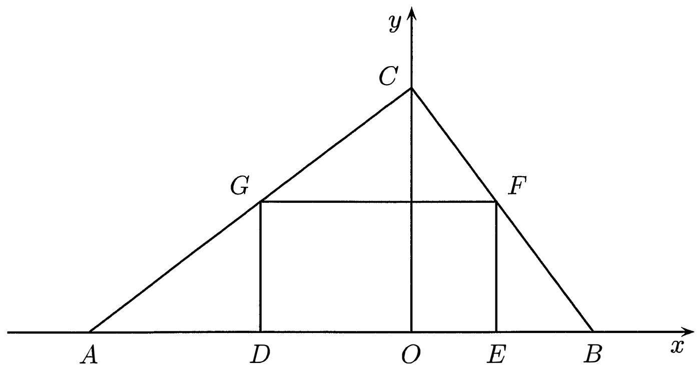
Suppose that $a_1 = a_{2012}$. Find the sum of the squares of all possible values of a_1.

24. A positive integer is called *friendly* if it is divisible by the sum of its digits. For example, 111 is friendly but 123 is not. Find the number of all two-digit friendly numbers.

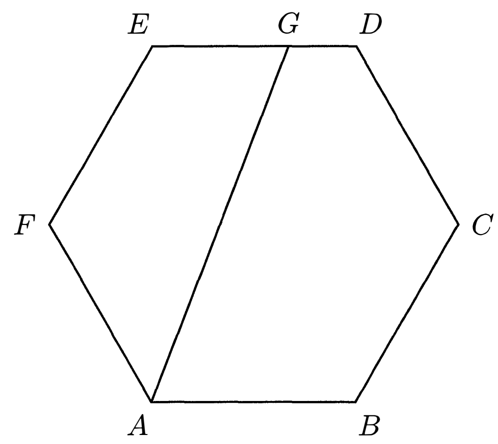
25. In the diagram below, D and E lie on the side AB, and F lies on the side AC such that $DA = DF = DE$, $BE = EF$ and $BF = BC$. It is given that $\angle ABC = 2\angle ACB$. Find x, where $\angle BFD = x°$.



26. In the diagram below, A and $B(20, 0)$ lie on the x-axis and $C(0, 30)$ lies on the y-axis such that $\angle ACB = 90°$. A rectangle $DEFG$ is inscribed in triangle ABC. Given that the area of triangle CGF is 351, calculate the area of the rectangle $DEFG$.



27. Let $ABCDEF$ be a regular hexagon. Let G be a point on ED such that $EG = 3GD$. If the area of $AGEF$ is 100, find the area of the hexagon $ABCDEF$.



28. Given a package containing 200 red marbles, 300 blue marbles and 400 green marbles. At each occasion, you are allowed to withdraw at most one red marble, at most two blue marbles and a total of at most five marbles out of the package. Find the minimal number of withdrawals required to withdraw all the marbles from the package.

29. 3 red marbles, 4 blue marbles and 5 green marbles are distributed to 12 students. Each student gets one and only one marble. In how many ways can the marbles be distributed so that Jamy and Jaren get the same colour and Jason gets a green marble?

30. A round cake is cut into n pieces with 3 cuts. Find the product of all possible values of n.

31. How many triples of non-negative integers (x, y, z) satisfying the equation

$$xyz + xy + yz + zx + x + y + z = 2012?$$

32. There are 2012 students in a secondary school. Every student writes a new year card. The cards are mixed up and randomly distributed to students. Suppose each student gets one and only one card. Find the expected number of students who get back their own cards.

33. Two players A and B play rock-paper-scissors continuously until player A wins 2 consecutive games. Suppose each player is equally likely to use each hand-sign in every game. What is the expected number of games they will play?

34. There are 2012 students standing in a circle; they are numbered $1, 2, \ldots, 2012$ clockwise. The counting starts from the first student (number 1) and proceeds around the circle clockwise. Alternate students will be eliminated from the circle in the following way: The first student stays in the circle while the second student leaves the circle. The third student stays while the fourth student leaves and so on. When the counting reaches number 2012, it goes back to number 1 and the elimination continues until the last student remains. What is the number of the last student?

35. There are k people and n chairs in a row, where $2 \leq k < n$. There is a couple among the k people. The number of ways in which all k people can be seated such that the couple is seated together is equal to the number of ways in which the $(k-2)$ people, without the couple present, can be seated. Find the smallest value of n.

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Multiple Choice Questions

1. Answer: (D).

   Since $x^2 + 2bx + (b - 1) = 0$, we have $\alpha\beta = b - 1$ and $\alpha + \beta = -2b$. Then

$$\begin{aligned}(\alpha - \beta)^2 &= (\alpha + \beta)^2 - 4\alpha\beta = (-2b)^2 - 4(b - 1)\\ &= 4(b^2 - b + 1) = 4[(b - 1\ \ 2)^2 + 3\ \ 4] \geq 3\end{aligned}$$

   The equality holds if and only if $b = 1\ \ 2$.

2. Answer: (E).

   Note that $n^{2012} + n^{2010} = n^{2010}(n^2 + 1)$.

   If $n = 2$, then $5\ \ (n^2 + 1)$ and $2\ \ n^{2010}$. If $n = 13$ or 47, then $10\ \ (n^2 + 1)$.

   If $n = 35$, then $2\ \ (n^2 + 1)$ and $5\ \ n^{2010}$. If $n = 59$, then $5 \nmid (59^2 + 1)$ and $5 \nmid n^{2010}$.

3. Answer: (B).

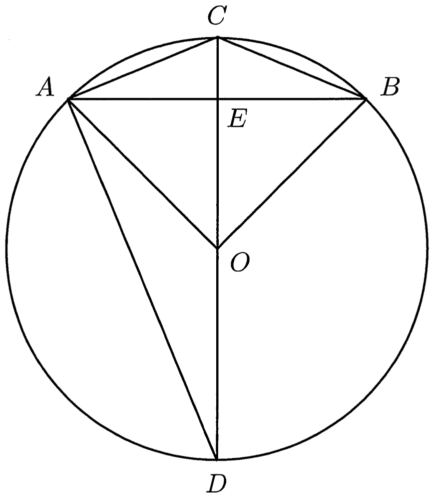
   There are 8 vertices in a cube. Any 4 vertices form a triangular pyramid unless they lie on the same plane.

$$\binom{8}{4} - 6 - 6 = 58$$

4. Answer: (B).

   Let the radius of the circle be 1. Then

$$\frac{S_{\triangle ABC}}{S_{\triangle AOD}} = \frac{AE \times CE}{S_{\triangle AOD}} = \frac{AE \times CE}{\frac{1}{2} \times AE \times OD} = \frac{2 \times CE}{OD} = \frac{2(1 - 1\ \ \sqrt{2})}{1} = 2 - \sqrt{2}$$



5. Answer: (A).

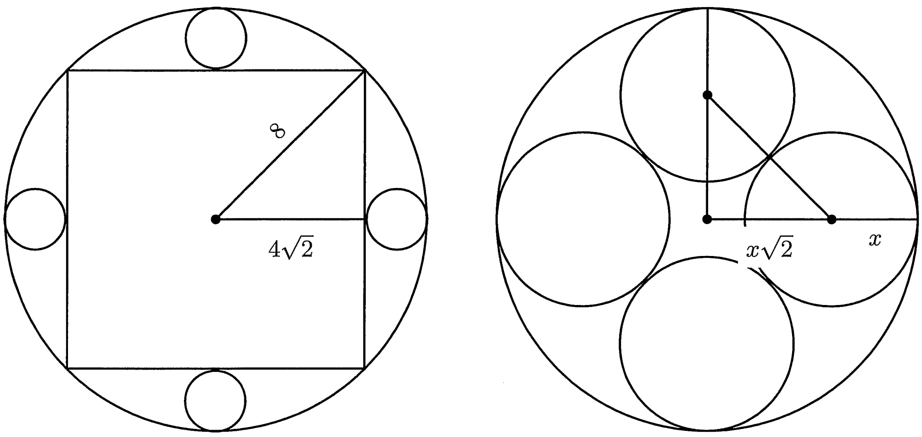
   Let $x = \dfrac{DE}{AE}$. Then $x = \dfrac{EF}{EB}$. Note that

$$1 = \frac{DC}{AB} = \frac{DF}{AB} + \frac{FC}{AB} = \frac{EF}{EB} + \frac{FG}{GB} = \frac{EF}{EB} + \frac{EB - 2EF}{EF} = x + \left(\frac{1}{x} - 2\right)$$

   Then $x + \dfrac{1}{x} - 3 = 0$, and thus $x = \dfrac{3 - \sqrt{5}}{2}$ $(0 < x < 1)$.

6. Answer: (D).

   Consider the extreme cases:



   Then $x_{\min} = \dfrac{8 - 4\sqrt{2}}{2} = 4 - 2\sqrt{2}$ and $x_{\max} = \dfrac{8}{\sqrt{2}+1} = 8(\sqrt{2} - 1)$.

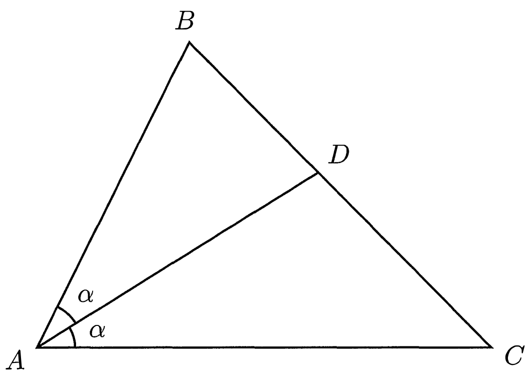
7. Answer: (D)

   By Heron's formula, $S_{\triangle ABC} = \sqrt{12(12-5)(12-8)(12-11)} = 4\sqrt{21}$.

$$\frac{S_{\triangle ABD}}{S_{\triangle ACD}} = \frac{\frac{1}{2} \times AB \times AD \times \sin\alpha}{\frac{1}{2} \times AC \times AD \times \sin\alpha} = \frac{AB}{AC} = \frac{5}{11}$$

   Then

$$S_{\triangle ABD} = \frac{5}{5+11} \times 4\sqrt{21} = \frac{5\sqrt{21}}{4}$$



8. Answer: (B).

   Note that $a = b \cdot \frac{a}{b} = b\left(\left\lfloor \frac{a}{b} \right\rfloor + \left\{ \frac{a}{b} \right\}\right) = b\left\lfloor \frac{a}{b} \right\rfloor + b\left\{ \frac{a}{b} \right\}$.

   It follows from $a = b\left\lfloor \frac{a}{b} \right\rfloor + b\left\{ \frac{a}{b} \right\}$ that $b\left\lfloor \frac{a}{b} \right\rfloor = a$. Hence, $\frac{a}{b} = \left\lfloor \frac{a}{b} \right\rfloor$ is an integer.

   Obviously, b is not necessary an integer even if a is an integer.

9. Answer: (C).

   x can take any nonzero real number. If $x > 0$, then $y = \frac{x - x}{x} = \frac{x - x}{x} = \frac{0}{x} = 0$.

   If $x < 0$, then $y = \frac{x - x}{x} = \frac{x - (-x)}{x} = \frac{2x}{x} = \frac{-2x}{x} = -2$.

10. Answer: (C).

    The assignment is as follows:

    A: in Z, teaches Music; B: in Y, teaches Latin; C: in X, teaches Mathematics.

Short Questions

11. Answer: 4.

    $\frac{75}{12} = (2^a + 2^b)(2^{-a} + 2^{-b}) = 2 + 2^{a-b} + 2^{b-a}$. Then $2^{a-b} + 2^{b-a} = \frac{17}{4} = 4 + \frac{1}{4}$.

    Since $a > b$, we obtain $2^{a-b} = 4$.

12. Answer: 25.

    Note that $\frac{x^3 - x + 120}{(x-1)(x+1)} = x + \frac{120}{x^2 - 1}$. It is an integer if and only if $(x^2 - 1)$ –120.

    Then $x^2 - 1 = \pm 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 8 \ 10 \ 12 \ 15 \ 20 \ 24 \ 30 \ 40 \ 60 \ 120$.

    Hence, $x = 0 \ 2 \ 3 \ 4 \ 5 \ 11$; and $2 + 3 + 4 + 5 + 11 = 25$.

13. Answer: 9.

Let $y = \sqrt{3x^2 - 8x + 1}$. Then the equation becomes

$$y + \sqrt{3y^2 - 11} = 3.$$

Then $\sqrt{3y^2 - 11} = 3 - y$. Squaring both sides, we have $3y^2 - 11 = 9 - 6y + y^2$; that is, $y^2 + 3y - 10 = 0$. Then $y = 2$ or $y = -5$ (rejected because $y \geq 0$).

Solve $3x^2 - 8x + 1 = 2^2$. Then $x = 3$ and $x = -3^{-1}$. Hence, $k = 3/3^{-1} = 9$.

14. Answer: 2996.

Rewrite the equation as the following:

$$\begin{array}{ccccc} & a & b & c & d \\ & a & b & c & d \\ + & 1 & 0 & 0 & 0 \\ \hline & d & c & b & a \end{array}$$

Since a is the last digit of $2d$, a is even; since $2a + 1 \leq d \leq 9$, $a \leq 4$. So $a = 2$ or $a = 4$.

If $a = 4$, then $d \geq 2a + 1 = 9$ and thus $d = 9$; but then the last digit of $2d$ would be $8 \neq a$, a contradiction.

If $a = 2$, then $d \geq 2a + 1 = 5$ and the last digit of $2d$ is 2; so $d = 6$. The equation reduces to

$$\begin{array}{cccc} & & b & c \\ & & b & c \\ + & & & 1 \\ \hline & 1 & c & b \end{array}$$

There are 2 cases: either $2c + 1 = b$ and $2b = 10 + c$, which has no integer solution; or $2c + 1 = 10 + b$ and $2b + 1 = 10 + c$, which gives $b = c = 9$.

15. Answer: 110.

Complete the square: $(x - 11)^2 + (y - 10)^2 = 0$. Then $x = 11$ and $y = 10$; thus $xy = 110$.

16. Answer: 93.

Rearranging, we have

$$mn^2 - 217n = 4mn - 876 \Rightarrow n = \frac{4mn - 876}{mn - 217} = 4 - \frac{8}{mn - 217}.$$

Then $(mn - 217) \;\; 8$. It follows that $mn - 217 = \pm 1, \pm 2, \pm 4, \pm 8$.

So $mn = 218, 216, 219, 215, 221, 213, 225, 209,$ and $n = -4, 12, 0, 8, 2, 6, 3, 5$ respectively.

Note that $m = \dfrac{mn}{n}$ is a positive integer. Then $m = \dfrac{216}{12} = 18$ or $m = \dfrac{225}{3} = 75$.

17. Answer: 50.

Write $x = \lfloor x \rfloor + \{x\}$. Then $100 \le (\lfloor x \rfloor + \{x\})^2 - \lfloor x \rfloor^2 = 2\lfloor x \rfloor\{x\} + \{x\}^2 < 2\lfloor x \rfloor + 1$.

So $\lfloor x \rfloor \ge 50$ and $x^2 \ge \lfloor x^2 \rfloor = 100 + 50^2 = 2600$. On the other hand, $x = \sqrt{2600}$ is a solution.

18. Answer: 35.

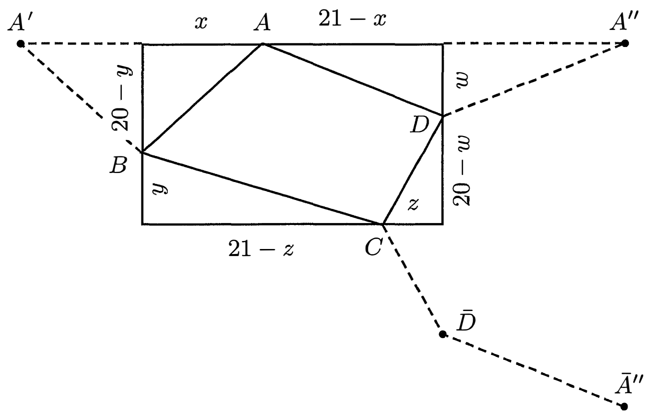
$$\begin{aligned}(x_1 + 2x_2 + \cdots + 49x_{49})^2 &= \left(1 \cdot x_1 + \sqrt{2} \cdot \sqrt{2}\,x_2 + \cdots + \sqrt{49} \cdot \sqrt{49}\,x_{49}\right)^2 \\ &\le (1 + 2 + \cdots + 49)\left(x_1^2 + 2x_2^2 + \cdots + 49x_{49}^2\right) \\ &= \frac{49 \times 50}{2} \times 1 = 35^2.\end{aligned}$$

The equality holds if and only if $x_1 = \cdots = x_{49} = \pm\frac{1}{35}$.

19. Answer: 58.

As shown below,

$$\begin{aligned}&\sqrt{x^2 + (20-y)^2} + \sqrt{y^2 + (21-z)^2} + \sqrt{z^2 + (20-w)^2} + \sqrt{w^2 + (21-x)^2} \\ &= AB + BC + CD + DA \\ &= A'B + BC + C\bar{D} + \bar{D}\bar{A}'' \\ &\le A'\bar{A}'' = \sqrt{42^2 + 40^2} = 58.\end{aligned}$$



20. Answer: 1818.

Let the 4-digit number be $A = \overline{abcd}$. Then

$$1000(a+n) + 100(b-n) + 10(c+n) + (d-n) = nA.$$

It gives $A + 909n = nA$; or equivalently, $(n-1)A = 909n$.

Note that $(n-1)$ and n are relatively prime and 101 is a prime number. We must have $(n-1) \mid 9$. So $n = 2$ or $n = 4$.

If $n = 4$, then $A = 1212$, which is impossible since $b < n$. So $n = 2$ and $A = 909 \times 2 = 1818$.

21. Answer: 4.

Note that $1024 = 2^{10} \equiv 1 \pmod{1023}$. Then

$$1021^{1022} \equiv (-2)^{1022} \equiv 2^{1022} \equiv 2^{10\times 102+2} \equiv 1024^{102} \times 2^2 \equiv 1^{102} \times 2^2 \equiv 4 \pmod{1023}$$

22. Answer: 375.

Let m and M be the smallest and the largest numbers. Then

$$\frac{m+80}{5} + 1 = \frac{M+80}{5} - 1 = \frac{m+M+80}{6}$$

Solving the system, $m = 15$ and $M = 25$. Then $mM = 375$.

23. Answer: 3.

Let $a_1 = a$. Then $a_2 = \frac{1}{1+a}$, $a_3 = \frac{1+a}{2+a}$, $a_4 = \frac{2+a}{3+2a}$, $a_5 = \frac{3+2a}{5+3a}$, ... In general,

$$a_n = \frac{F_n + F_{n-1}a}{F_{n+1} + F_n a}$$

where $F_1 = 0$, $F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 1$.
If $a_{2012} = \frac{F_{2012} + F_{2011}a}{F_{2013} + F_{2012}a} = a$, then $(a^2 + a - 1)F_{2012} = 0$.
Since $F_{2012} > 0$, we have $a^2 + a - 1 = 0$. Let α and β be the roots of $a^2 + a - 1 = 0$. Then

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (-1)^2 - 2(-1) = 3$$

24. Answer: 23.

Since $\frac{10a+b}{a+b} = \frac{9a}{a+b} + 1$, we have $(a+b)\quad 9a$.

Case 1: If $3 \nmid (a+b)$, then $(a+b)\quad a$, and thus $b = 0$. We have 10 20 40 50 70 80.

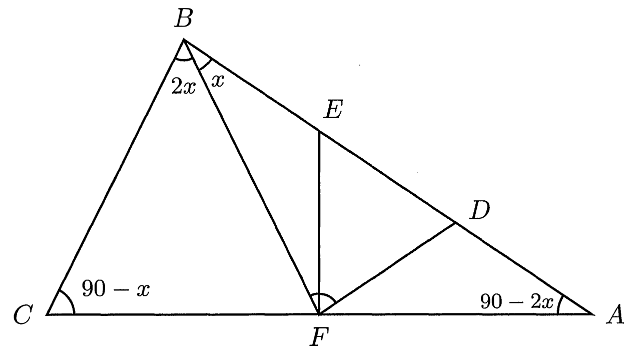
Case 2: If $3\quad (a+b)$ but $9 \nmid (a+b)$, then $(a+b)\quad 3a$ and $1 \leq \frac{3a}{a+b} \leq 3$. If $\frac{3a}{a+b} = 1$, then $2a = b$, we have 12 24 48. If $\frac{3a}{a+b} = 2$, then $a = 2b$, we have 21 42 84. If $\frac{3a}{a+b} = 3$, then $b = 0$, we have 30 60.

Case 3: If $9\quad (a+b)$, then $a+b = 9$ or 18. If $a+b = 18$, then $a = b = 9$, which is impossible. If $a + b = 9$, then we have 9 friendly numbers 18 27 36 45 54 63 72 81 90.

Therefore, in total there are 23 2-digit friendly numbers.

25. Answer: 108.

Since $DA = DE = DF$, $\angle EFA = 90°$. Let $\angle EBF = \angle EFB = x°$. Then $\angle BCF = \angle BFC = 90° - x°$ and $\angle CBF = 2x°$, $\angle BAC = 90° - 2x°$.



It is given that $3x = 2(90 - x)$. Then $x = 36$. So $x = 180 - (90 - x) - (90 - 2x) = 3x = 108$.

26. Answer: 468.

Note that $AO = \dfrac{30^2}{20} = 45$. Then the area of $\triangle ABC$ is $\dfrac{(20 + 45) \times 30}{2} = 975$.

Let the height of $\triangle CGF$ be h. Then

$$\left(\frac{h}{30}\right)^2 = \frac{351}{975} = \left(\frac{3}{5}\right)^2 \Rightarrow \frac{h}{30 - h} = \frac{3}{2}$$

Note that the rectangle $DEFG$ has the same base as $\triangle CGF$. Then its area is

$$351 \times \frac{2}{3} \times 2 = 468$$

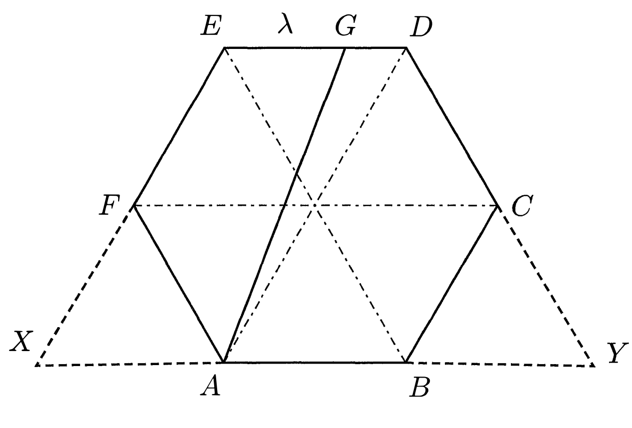
27. Answer: 240.

We may assume that $DE = 1$ and denote $\lambda = EG = 3\ 4$.

$\dfrac{EGAX}{DEXY} = \dfrac{1+\lambda}{4}$ implies $\dfrac{AGEF}{DEXY} = \dfrac{1+\lambda}{4} - \dfrac{1}{8} = \dfrac{1+2\lambda}{8}$.

Note that $\dfrac{ABCDEF}{DEXY} = \dfrac{6}{8}$. Then $\dfrac{AGEF}{ABCDEF} = \dfrac{1+2\lambda}{6} = \dfrac{5}{12}$.

If $AGEF = 100$, then $ABCDEF = 100 \times \dfrac{12}{5} = 240$.



28. Answer: 200.

Since at most one red marble can be withdrawn each time, it requires at least 200 withdrawals.

On the other hand, 200 is possible. For example,

$$150 \cdot (1,2,2) + 20 \cdot (1,0,4) + 10 \cdot (1,0,2) + 20 \cdot (1,0,0) = (200,300,400).$$

29. Answer: 3150.

Case 1: Jamy and Jaren both take red marbles. So 1 red, 4 blue and 4 green marbles are distributed to 9 students:

$$\binom{9}{1} \times \binom{8}{4} = 630.$$

Case 2: Jamy and Jaren both take blue marbles. So 3 red, 2 blue and 4 green marbles are distributed to 9 students:

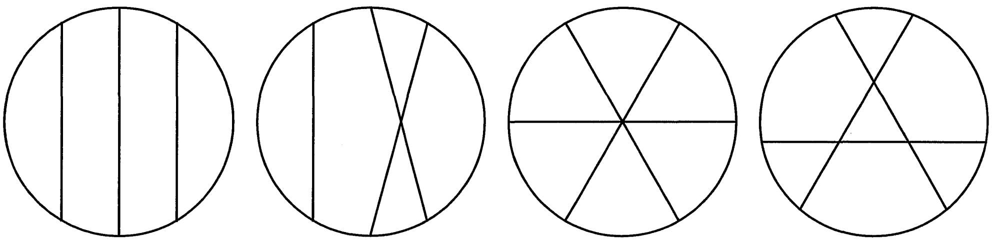
$$\binom{9}{3} \times \binom{6}{2} = 1260.$$

Case 3: Jamy and Jaren both take green marbles. So 3 red, 4 blue and 2 green marbles are distributed to 9 students. The number is the same as case 2.

$$630 + 1260 + 1260 = 3150.$$

30. Answer: 840.

With three cuts, a round cake can be cut into at least $1 + 3 = 4$ pieces, and at most $1 + 1 + 2 + 3 = 7$ pieces. Moreover, $n = 4, 5, 6, 7$ are all possible. $4 \times 5 \times 6 \times 7 = 840$.



31. Answer: 27.

$(x+1)(y+1)(z+1) = 2013 = 3 \times 11 \times 61$.

If all x, y, z are positive, there are $3! = 6$ solutions.

If exactly one of x, y, z is 0, there are $3 \times 6 = 18$ solutions.

If exactly two of x, y, z are 0, there are 3 solutions.

$6 + 18 + 3 = 27$.

32. Answer: 1.

For each student, the probability that he gets back his card is $\frac{1}{2012}$. Then the expectation of the whole class is $2012 \times \frac{1}{2012} = 1$.

33. Answer: 12.

Let E be the expectation. If A does not win, the probability is $2/3$ and the game restarts. If A wins and then does not win, the probability is $(1/3)(2/3)$ and the game restarts. The probability that A wins two consecutive games is $(1/3)(1/3)$. Then

$$E = \frac{2}{3} \times (E+1) + \frac{2}{9} \times (E+2) + \frac{1}{9} \times 2.$$

Solving the equation, we get $E = 12$.

34. Answer: 1976.

If there are $1024 = 2^{10}$ students, then the 1024^{th} student is the last one leaving the circle. Suppose $2012 - 1024 = 988$ students have left. Among the remaining 1024 students, the last student is $(2 \times 988 - 1) + 1 = 1976$.

35. Answer: 12.

$(n-1) \times 2 \times \binom{n-2}{k-2} \times (k-2)! = \binom{n}{k-2} \times (k-2)!$. Then $2 = \dfrac{n}{(n-k+1)(n-k+2)}$.

That is, $2n^2 - (4k-6)n + (2k^2 - 6k + 4 - n) = 0$. We can solve

$$n = k + \frac{\sqrt{8k-7} - 5}{4} \qquad (n > k).$$

Note that the square of any odd number has the form $8k-7$. Choose k so that $\sqrt{8k-7} - 5 = 4$, i.e., $k = 11$. Then $n = 12$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2012

(Junior Section, Round 2)

Saturday, 23 June 2012 **0930-1230**

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1. Let O be the centre of a parallelogram $ABCD$ and P be any point in the plane. Let M, N be the midpoints of AP, BP, respectively and Q be the intersection of MC and ND. Prove that O, P and Q are collinear.

2. Does there exist an integer A such that each of the ten digits $0, 1, \ldots, 9$ appears exactly once as a digit in exactly one of the numbers A, A^2, A^3.

3. In $\triangle ABC$, the external bisectors of $\angle A$ and $\angle B$ meet at a point D. Prove that the circumcentre of $\triangle ABD$ and the points C, D lie on the same straight line.

4. Determine the values of the positive integer n for which the following system of equations has a solution in positive integers $x_1, x_2, \ldots, x_n$. Find all solutions for each such n.

$$x_1 + x_2 + \cdots + x_n = 16 \tag{1}$$

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1 \tag{2}$$

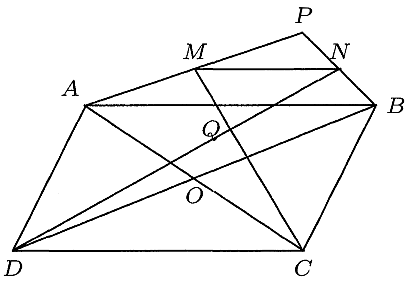
5. Suppose $S = \ a_1, a_2, \ldots, a_{15}$ is a set of 15 distinct positive integers chosen from $2, 3, \ldots, 2012$ such that every two of them are coprime. Prove that S contains a prime number. (Note" Two positive integers m, n are coprime if their only common factor is 1.)

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2012

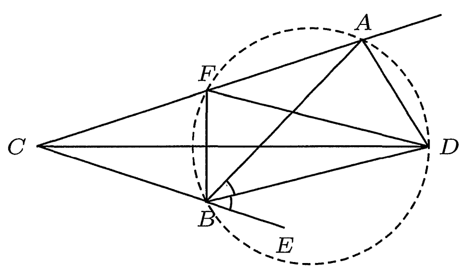
(Junior Section, Round 2 solutions)

1. Since $MN \parallel AB \parallel CD$, we have $\triangle MQN \sim \triangle CDQ$. Hence $MN = AB/2 = CD/2$. Thus $QM = CQ/2$. In $\triangle ACP$, CM is a median and Q divides CM in the ratio 1:2. Thus Q is the centroid. Hence the median PO passes through Q.



2. Since the total number of digits in A, A^2 and A^3 is 10, the total number digits in $32, 32^2, 32^3$ is 11 and the total number of digits in $20, 20^2, 20^3$ is 9, any solution A must satisfy $21 \le A \le 31$. Since the unit digits of A, A^2, A^3 are distinct, the unit digit of A can only be $2, 3, 7, 8$. Thus the only possible values of A are $22, 23, 27, 28$. None of them has the desired property. Thus no such number exists.

3. Note that CD bisects $\angle C$. If $CA = CB$, then CD is the perpendicular bisector of AB. Thus the circumcentre of $\triangle ABD$ is on CD.



If $CA \neq CB$, we may assume that $CA > CB$. Let E be a point on CB extended and F be the point on CA so that $CF = CB$. Then, since CD is the perpendicular bisector of

BF, we have $\angle AFD = \angle DBE = \angle DBA$. Thus $AFBD$ is a cyclic quadrilateral, i.e., F is on the circumcircle of $\triangle ABD$. The circumcentre lies on the perpendicular bisector of BF which is CD.

4. Without loss of generality, we may assume that $x_1 \leq x_2 \leq \cdots \leq x_n$. If $x_1 = 1$, then from (2), $n = 1$ and (1) cannot be satisfied. Thus $x_1 \geq 2$. If $x_2 = 2$, then $n = 2$ and again (1) cannot be satisfied. Thus $x_2 \geq 3$. Similarly, $x_3 \geq 4$. Thus $x_4 + \cdots + x_n \leq 7$ with $x_4 \geq 4$. Thus $n \leq 4$.

(i) $n = 1$: No solution.

(ii) $n = 2$: The only solution of $\frac{1}{x_1} + \frac{1}{x_2} = 1$ is $x_1 = x_2 = 2$ which doesn't satisfy (1). Thus there is no solution.

(iii) $n = 3$: The only solutions of $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = 1$ are $(x_1, x_2, x_3) = (2, 3, 6), (2, 4, 4)$ and $(3, 3, 3)$. They all do not satisfy (1).

(iv) $n = 4$: According to the discussion in the first paragraph, the solutions of $x_1 + \cdots + x_4 = 16$ are

$$\begin{aligned}(x_1, x_2, x_3, x_4) = {} & (2, 3, 4, 7), (2, 3, 5, 6), (2, 4, 4, 6), (2, 4, 5, 5), \\ & (3, 3, 4, 6), (3, 3, 5, 5), (3, 4, 4, 5), (4, 4, 4, 4).\end{aligned}$$

Only the last one satisfy (2).

Thus the system of equations has a solution only when $n = 4$ and for this n, the only solution is $x_1 = x_2 = x_3 = x_4 = 4$.

5. Suppose, on the contrary, that S contains no primes. For each i, let p_i be the smallest prime divisor of a_i. Then $p_1, p_2, \ldots, p_{15}$ are distinct since the numbers in S are pairwise coprime. The first 15 primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47. If p_j is the largest among $p_1, p_2, \ldots, p_{15}$, then $p_j \geq 47$ and $a_j \geq 47^2 = 47^2 = 2309 > 2012$, a contradiction. Thus S must contain a prime number.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2012

Senior Section (First Round)

Tuesday, 29 May 2012 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 35 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*
8. *Throughout this paper, let* $\lfloor x \rfloor$ *denote the greatest integer less than or equal to* x*. For example,* $\lfloor 2.1 \rfloor = 2$, $\lfloor 3.9 \rfloor = 3$.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Multiple Choice Questions

1. Suppose α and β are real numbers that satisfy the equation

$$x^2 + \left(2\sqrt{\sqrt{2}+1}\right)x + \sqrt{\sqrt{2}+1} = 0$$

Find the value of $\dfrac{1}{\alpha^3} + \dfrac{1}{\beta^3}$.

(A) $3\sqrt{\sqrt{2}+1}\,(\sqrt{2}-1) - 8$ (B) $8 - 6\sqrt{\sqrt{2}+1}\,(\sqrt{2}-1)$

(C) $3\sqrt{\sqrt{2}+1}\,(\sqrt{2}-1) + 8$ (D) $6\sqrt{\sqrt{2}+1}\,(\sqrt{2}-1) - 8$

(E) None of the above

2. Find the value of

$$\frac{2011^2 \times 2012 - 2013}{2012!} + \frac{2013^2 \times 2014 - 2015}{2014!}$$

(A) $\frac{1}{2009!} + \frac{1}{2010!} + \frac{1}{2011!} + \frac{1}{2012!}$ (B) $\frac{1}{2009!} + \frac{1}{2010!} - \frac{1}{2011!} - \frac{1}{2012!}$

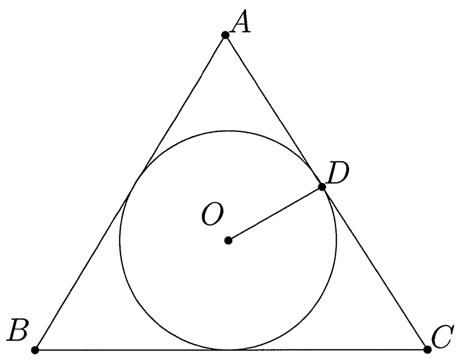
(C) $\frac{1}{2009!} + \frac{1}{2010!} - \frac{1}{2012!} - \frac{1}{2013!}$ (D) $\frac{1}{2009!} + \frac{1}{2010!} + \frac{1}{2013!} + \frac{1}{2014!}$

(E) $\frac{1}{2009!} + \frac{1}{2010!} - \frac{1}{2013!} - \frac{1}{2014!}$

3. The increasing sequence $T =$ 2 3 5 6 7 8 10 11 consists of all positive integers which are not perfect squares. What is the 2012th term of T?

(A) 2055 (B) 2056 (C) 2057 (D) 2058 (E) 2059

4. Let O be the center of the inscribed circle of triangle $\triangle ABC$ and D be the point on AC with $OD \perp AC$. If $AB = 10$ $AC = 9$ $BC = 11$, find CD.



(A) 4 (B) 4.5 (C) 5 (D) 5.5 (E) 6

5. Find the value of

$$\frac{\cos^4 75° + \sin^4 75° + 3\sin^2 75° \cos^2 75°}{\cos^6 75° + \sin^6 75° + 4\sin^2 75° \cos^2 75°}.$$

(A) $\frac{\sqrt{2}}{4}$ (B) $\frac{1}{2}$ (C) $\frac{3}{4}$ (D) 1 (E) $\cos 75° + \sin 75°$

6. If the roots of the equation $x^2 + 3x - 1 = 0$ are also the roots of the equation $x^4 + ax^2 + bx + c = 0$, find the value of $a + b + 4c$.

(A) -13 (B) -7 (C) 5 (D) 7 (E) 11

7. Find the sum of the **digits** of all numbers in the sequence $1, 2, 3, 4, \ldots, 1000$.

(A) 4501 (B) 12195 (C) 13501 (D) 499500 (E) None of the above

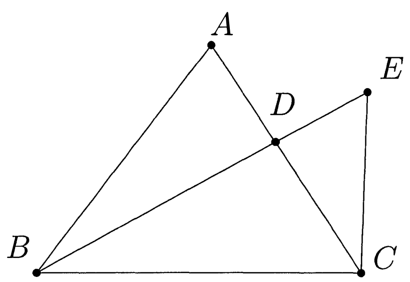
8. Find the number of real solutions to the equation

$$\frac{x}{100} = \sin x,$$

where x is measured in radians.

(A) 30 (B) 32 (C) 62 (D) 63 (E) 64

9. In the triangle $\triangle ABC$, $AB = AC$, $\angle ABC = 40°$ and the point D is on AC such that BD is the angle bisector of $\angle ABC$. If BD is extended to the point E such that $DE = AD$, find $\angle ECA$.



(A) 20° (B) 30° (C) 40° (D) 45° (E) 50°

10. Let m and n be positive integers such that $m > n$. If the last three digits of 2012^m and 2012^n are identical, find the smallest possible value of $m + n$.

(A) 98 (B) 100 (C) 102 (D) 104 (E) None of the above

Short Questions

11. Let a b c and d be four distinct positive real numbers that satisfy the equations

$$(a^{2012} - c^{2012})(a^{2012} - d^{2012}) = 2011$$

and

$$(b^{2012} - c^{2012})(b^{2012} - d^{2012}) = 2011$$

Find the value of $(cd)^{2012} - (ab)^{2012}$.

12. Determine the total number of pairs of integers x and y that satisfy the equation

$$\frac{1}{y} - \frac{1}{y+2} = \frac{1}{3 \cdot 2^x}$$

13. Given a set $S = \quad 1 \ 2 \qquad 10 \ $, a collection $\mathcal{F}$ of subsets of S is said to be *intersecting* if for any two subsets A and B in $\mathcal{F}$, we have $A \cap B \neq \emptyset$. What is the maximum size of $\mathcal{F}$?

14. The set M contains all the integral values of m such that the polynomial

$$2(m-1)x^2 - (m^2 - m + 12)x + 6m$$

has either one repeated or two distinct integral roots. Find the number of elements of M.

15. Find the minimum value of

$$\left| \sin x + \cos x + \frac{\cos x - \sin x}{\cos 2x} \right|$$

16. Find the number of ways to arrange the letters A, A, B, B, C, C, D and E in a line, such that there are no consecutive identical letters.

17. Suppose $x = 3^{\sqrt{2+\log_3 x}}$ is an integer. Determine the value of x.

18. Let $f(x)$ be the polynomial $(x-a_1)(x-a_2)(x-a_3)(x-a_4)(x-a_5)$ where a_1, a_2, a_3, a_4 and a_5 are distinct integers. Given that $f(104) = 2012$, evaluate $a_1+a_2+a_3+a_4+a_5$.

19. Suppose x, y, z and λ are positive real numbers such that

$$\begin{aligned} yz &= 6\lambda x \\ xz &= 6\lambda y \\ xy &= 6\lambda z \\ x^2 + y^2 + z^2 &= 1 \end{aligned}$$

Find the value of $(xyz\lambda)^{-1}$.

20. Find the least value of the expression $(x+y)(y+z)$, given that x, y, z are positive real numbers satisfying the equation

$$xyz(x+y+z) = 1.$$

21. For each real number x, let $f(x)$ be the minimum of the numbers $4x+1, x+2$ and $-2x+4$. Determine the maximum value of $6f(x)+2012$.

22. Find the number of pairs (A, B) of distinct subsets of $1, 2, 3, 4, 5, 6$ such that A is a proper subset of B. Note that A can be an empty set.

23. Find the sum of all the integral values of x that satisfy

$$\sqrt{x+3-4\sqrt{x-1}} + \sqrt{x+8-6\sqrt{x-1}} = 1.$$

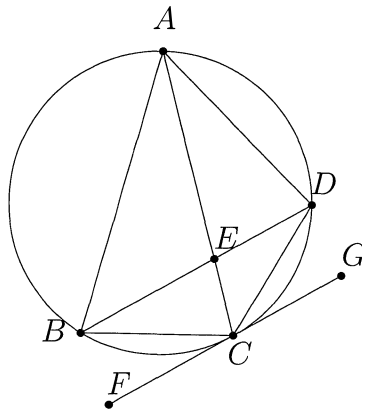
24. Given that

$$S = \left|\sqrt{x^2+4x+5} - \sqrt{x^2+2x+5}\right|,$$

for real values of x, find the maximum value of S^4.

25. Three integers are selected from the set $S = 1, 2, 3, \ldots, 19, 20$. Find the number of selections where the sum of the three integers is divisible by 3.

26. In the diagram below, $ABCD$ is a cyclic quadrilateral with $AB = AC$. The line FG is tangent to the circle at the point C, and is parallel to BD. If $AB = 6$ and $BC = 4$, find the value of $3AE$.



27. Two Wei Qi teams, A and B, each comprising of 7 members, take on each other in a competition. The players on each team are fielded in a fixed sequence. The first game is played by the first player of each team. The losing player is eliminated while the winning player stays on to play with the next player of the opposing team. This continues until one team is completely eliminated and the surviving team emerges as the final winner – thus, yielding a possible gaming outcome. Find the total number of possible gaming outcomes.

28. Given that $\mathbf{m} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$ and $\mathbf{n} = (\sqrt{2} - \sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are the usual unit vectors along the x-axis and the y-axis respectively, and $\theta \in (\pi \;\, 2\pi)$. If the length or magnitude of the vector $\mathbf{m} + \mathbf{n}$ is given by $\;\mathbf{m} + \mathbf{n} \;\; = \frac{8\sqrt{2}}{5}$, find the value of $5\cos\left(\frac{\theta}{2} + \frac{\pi}{8}\right) + 5$.

29. Given that the real numbers x, y and z satisfies the condition $x + y + z = 3$, find the maximum value of $f(x \;\, y \;\, z) = \sqrt{2x + 13} + \sqrt[3]{3y + 5} + \sqrt[4]{8z + 12}$.

30. Let $P(x)$ be a polynomial of degree 34 such that $P(k) = k \;\, (k + 1)$ for all integers $k = 0 \;\, 1 \;\, 2 \qquad 34$. Evaluate $42840 \times P(35)$.

31. Given that α is an acute angle satisfying

$$\sqrt{369 - 360\cos\alpha} + \sqrt{544 - 480\sin\alpha} - 25 = 0$$

find the value of $40\tan\alpha$.

32. Given that $a \;\, b \;\, c \;\, d \;\, e$ are real numbers such that

$$a + b + c + d + e = 8$$

and

$$a^2 + b^2 + c^2 + d^2 + e^2 = 16$$

Determine the maximum value of $\lfloor e \rfloor$.

33. Let L denote the minimum value of the quotient of a 3-digit number formed by three distinct digits divided by the sum of its digits. Determine $\lfloor 10L \rfloor$.

34. Find the last 2 digits of

$$x = 19^{17^{15^{\cdot^{\cdot^{\cdot^{3^1}}}}}}$$

35. Let $f(n)$ be the integer nearest to $\sqrt{n}$. Find the value of

$$\sum_{n=1}^{\infty} \frac{\left(\frac{3}{2}\right)^{f(n)} + \left(\frac{3}{2}\right)^{-f(n)}}{\left(\frac{3}{2}\right)^n}$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2012

Senior Section (First Round Solutions)

Multiple Choice Questions

1. Answer: (D)

   Note that $\alpha+\beta=-\left(2\sqrt{\sqrt{2}+1}\right)$ and $\alpha\beta=\sqrt{\sqrt{2}+1}$.

$$\begin{aligned}\frac{1}{\alpha^3}+\frac{1}{\beta^3}&=\frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{(\alpha\beta)^3}\\&=\frac{6-8\sqrt{\sqrt{2}+1}}{\sqrt{\sqrt{2}+1}}\\&=\frac{6(\sqrt{\sqrt{2}+1})-8(\sqrt{2}+1)}{\sqrt{2}+1}\\&=6\sqrt{\sqrt{2}+1}\left(\sqrt{2}-1\right)-8\end{aligned}$$

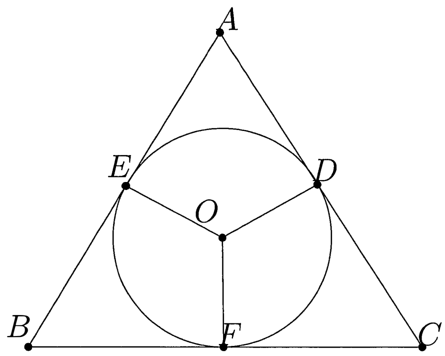
2. Answer: (E)

$$\begin{aligned}&\frac{2011^2\times 2012-2013}{2012!}+\frac{2013^2\times 2014-2015}{2014!}\\&=\frac{2011}{2010!}-\frac{2013}{2012!}+\frac{2013}{2012!}-\frac{2015}{2014!}\\&=\left(\frac{2010}{2010!}+\frac{1}{2010!}-\frac{2012}{2012!}-\frac{1}{2012!}\right)+\left(\frac{2012}{2012!}+\frac{1}{2012!}-\frac{2014}{2014!}-\frac{1}{2014!}\right)\\&=\left(\frac{1}{2009!}+\frac{1}{2010!}-\frac{1}{2011!}-\frac{1}{2012!}\right)+\left(\frac{1}{2011!}+\frac{1}{2012!}-\frac{1}{2013!}-\frac{1}{2014!}\right)\\&=\frac{1}{2009!}+\frac{1}{2010!}-\frac{1}{2013!}-\frac{1}{2014!}\end{aligned}$$

3. Answer: (C)

   Note that $44^2 = 1936$, $45^2 = 2025$ and $46^2 = 2116$. So $2, 3, \ldots, 2012$ has at most $2012 - 44$ terms. For the 2012th term, we need to add the 44 numbers from 2013 to 2056. But in doing so, we are counting $45^2 = 2025$, so the 2012th term should be $2012 + 44 + 1 = 2057$.

4. Answer: (C)



   AC is tangent to the circle at D, by constructing E and F as shown, we have $CD = CF$, $AD = AE$ and $BE = BF$. Solving for the unknowns give $CD = 5$.

5. Answer: (D)

$$\begin{aligned}
&\cos^6 x + \sin^6 x + 4\sin^2 x\cos^2 x \\
&= (\cos^2 x + \sin^2 x)(\sin^4 x + \cos^4 x - \sin^2 x\cos^2 x) + 4\sin^2 x\cos^2 x \\
&= (\sin^4 x + \cos^4 x + 3\sin^2 x\cos^2 x).
\end{aligned}$$

6. Answer: (B)

   We have the factorization

$$(x^2 + 3x - 1)(x^2 + mx - c) = x^4 + ax^2 + bx + c.$$

   Comparing coefficients give $3 + m = 0$, $-1 - c + 3m = a$ and $-3c - m = b$. We can solve these equations to obtain $a + b + 4c = -7$.

7. Answer: (C)

   Among all numbers with 3 or less digits, each i, $i = 0, 1, 2, \cdots, 9$, appears exactly 300 times. Thus the sum of the digits of all the numbers in the sequence $1, 2, 3, 4, \cdots, 999$ is

$$300(1 + 2 + \cdots + 9) = 13500,$$

   and so the answer is 13501.

8. Answer: (D)

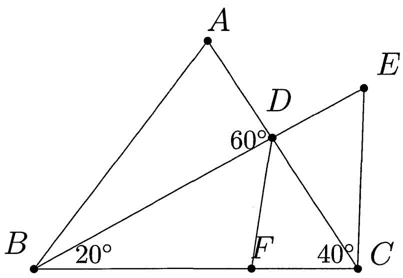
   Since $-1 \leq \sin x \leq 1$, we have $-100 \leq x \leq 100$. We also observe that $31\pi < 100 < 32\pi$.

   For each integer k with $1 \leq k \leq 16$, $\frac{x}{100} = \sin x$ has exactly two solutions in $[2(k-1)\pi \quad (2k-1)\pi]$, but it has no solutions in $((2k-1)\pi \quad 2k\pi)$. Thus this equation has exactly 32 non-negative real solutions, i.e. $x = 0$ and exactly 31 positive real solutions. Then it also has exactly 31 negative real solutions, giving a total of 63.

9. Answer: (C)

   Construct a point F on BC such that $BF = BA$. Since $\angle ABD = \angle FBD$, $\triangle ABD$ is congruent to $\triangle FBD$.

   Thus $DF = DA = DE$ and $\angle FDB = \angle ADB = 60°$. We also have $\angle EDC = \angle ADB = 60°$, which implies that $\angle FDC = 60°$ and $\triangle CFD$ is congruent to $\triangle CED$. In conclusion, $\angle ECD = \angle FCD = 40°$.



10. Answer: (D)

   We want to solve $2012^m \equiv 2012^n \pmod{1000}$ which is equivalent to

$$1000 \quad 12^n(12^{m-n} - 1)$$

   Since $(12^{m-n} - 1)$ is odd, we must have $8 \quad 12^n$ so $n \geq 2$. It remains to check that $125 \quad 12^{m-n} - 1$ i.e. $12^{m-n} \equiv 1 \pmod{125}$. Let φ be Euler's phi function. As $\varphi(125) = 125 - 25 = 100$, by Euler's theorem, we know that the smallest $m - n$ must be a factor of 100. By checking all possible factors, we can conclude $m - n = 100$ and so the smallest possible value for $m + n$ is 104 since $n \geq 2$.

Short Questions

11. Answer: 2011

    Let $A = a^{2012}$, $B = b^{2012}$, $C = c^{2012}$ and $D = d^{2012}$. Then A and B are distinct roots of the equation $(x-C)(x-D) = 2011$. Thus the product of roots, $AB = CD - 2011$ and $CD - AB = 2011$.

12. Answer: 6

    Note that x and y must satisfy

    $$2^{x+1} \cdot 3 = y(y+2).$$

    We first assume $x \geq 0$, which means both y and $y+2$ are even integers. Either $3 \quad y$ or $3 \quad y+2$. In the first case, assuming $y > 0$, we have $y = 3 \cdot 2^k$ and $y + 2 = 2(3 \cdot 2^{k-1} + 1) = 2^{x+1-k}$. The only way for this equation to hold is $k = 1$ and $x = 3$. So $(x, y, y+2) = (3, 6, 8)$.

    In the case $3 \quad y+2$, assuming $y > 0$, we have $y+2 = 3 \cdot 2^k$ and $y = 2(3 \cdot 2^{k-1} - 1) = 2^{x+1-k}$. Now the only possibility is $k = 1$ and $x = 2$, so $(x, y, y+2) = (2, 4, 6)$.

    In the two previous cases, we could also have both y and $y+2$ to be negative, giving $(x, y, y+2) = (3, -8, -6)$ or $(2, -6, -4)$.

    Finally, we consider $x < 0$ so $3 = 2^{-x-1}y(y+2)$. In this case we can only have $x = -1$ and $(x, y, y+2) = (-1, 1, 3)$ or $(-1, -3, -1)$.

    Hence possible (x, y) pairs are $(3, 6)$, $(3, -8)$, $(2, 4)$, $(2, -6)$, $(-1, 1)$ and $(-1, -3)$.

13. Answer: 512

    If $A \in \mathcal{F}$ then the complement $S \cap A \notin \mathcal{F}$. So at most half of all the subsets of S can belong to $\mathcal{F}$, that is

    $$\mathcal{F} \leq \frac{2^{10}}{2} = 512.$$

    Equality holds because we can take $\mathcal{F}$ to be all subsets of S containing 1.

14. Answer: 4

    If $m = 1$, the polynomial reduces to $-12x + 6 = 0$ which has no integral roots.

    For $m \neq 1$, the polynomial factorizes as $((2x - m)((m-1)x - 6)$, with roots $x = \frac{m}{2}$ and $x = \frac{6}{m-1}$. For integral roots, m must be even and $m - 1$ must divide 6. The only possible values are $m = -2, 0, 2$ and 4. So M has 4 elements.

15. Answer: 2

    Note that $\cos 2x = \cos^2 x - \sin^2 x$. So

    $$\left|\sin x + \cos x + \frac{\cos x - \sin x}{\cos 2x}\right| = \left|\sin x + \cos x + \frac{1}{\sin x + \cos x}\right|.$$

    Set $w = \sin x + \cos x$ and minimize $\left|w + \frac{1}{w}\right|$.

    By AM-GM inequality, if w is positive then the minimum of $w + \frac{1}{w}$ is 2; if w is negative, then the maximum of $w + \frac{1}{w}$ is -2. Therefore, the minimum of $\left|w + \frac{1}{w}\right|$ is 2.

16. Answer: 2220

We shall use the Principle of Inclusion and Exclusion. There are $\frac{8!}{2!2!2!}$ ways to arrange the letters without restriction. There are $\frac{7!}{2!2!}$ ways to arrange the letters such that both the As occur consecutively. (Note that this is the same number if we use B or C instead of A.)

There are $\frac{6!}{2!}$ ways to arrange the letters such that both As and Bs are consecutive. (Again this number is the same for other possible pairs.) Finally there are 5! ways to arrange the letters such that As, Bs and Cs occur consecutively.

For there to be no consecutive identical letters, total number of ways is

$$\frac{8!}{2!2!2!} - 3 \times \frac{7!}{2!2!} + 3 \times \frac{6!}{2!} - 5! = 2220$$

17. Answer: 9

Taking logarithm, we get $\log_3 x = \sqrt{2 + \log_3 x}$ Let $y = \log_3 x$. The only possible solution for $y = \sqrt{2+y}$ is 2. Therefore $x = 3^2 = 9$.

18. Answer: 17

The prime factorization of 2012 is $2^2 \cdot 503$. Let $b = 104$. If a_i are distinct, so are $b - a_i$, i.e. $(b - a_1)$, $(b - a_2)$, $(b - a_3)$, $(b - a_4)$ and $(b - a_5)$ must be exactly the integers $1\ -1\ 2\ -2\ 503$. Summing up, we have

$$5(104) - (a_1 + a_2 + a_3 + a_4 + a_5) = 1 - 1 + 2 - 2 + 503$$

ie. $a_1 + a_2 + a_3 + a_4 + a_5 = 17$.

19. Answer: 54

Multiplying the first three equations by x, y and z respectively, we have

$$xyz = 6\lambda x^2 = 6\lambda y^2 = 6\lambda z^2$$

Since $\lambda \neq 0$ and $x^2+y^2+z^2 = 1$, we deduce that $x^2 = y^2 = z^2 = \frac{1}{3}$, so $x = y = z = \frac{1}{\sqrt{3}}$ and $\lambda = \frac{xyz}{6x^2} = \frac{x^3}{2x^2} = \frac{1}{6\sqrt{3}}$.

Hence

$$(xyz\lambda)^{-1} = (\sqrt{3})^3 6\sqrt{3} = 54$$

20. Answer: 2

Observe that

$$(x+y)(y+z) = xy + xz + y^2 + yz = y(x+y+z) + xz = \frac{1}{xz} + xz \geq 2$$

where the equality holds if and only if $xz = 1$. Let $x = z = 1$ and $y = \sqrt{2} - 1$, then we have the minimum value 2 for $(x+y)(y+z)$.

21. Answer: 2028

Let L_1, L_2, L_3 represent the three lines $y = 4x + 1, y = x + 2$ and $y = -2x + 4$ respectively.

Observe that L_1 and L_2 intersects at $(\frac{1}{3}, \frac{7}{3})$, L_1 and L_3 intersects at $(\frac{1}{2}, 3)$, and L_2 and L_3 intersects at $(\frac{2}{3}, \frac{8}{3})$. Thus

$$f(x) = \begin{cases} 4x + 1, & x < \frac{1}{3}; \\ \frac{7}{3}, & x = \frac{1}{3}; \\ x + 2, & \frac{1}{3} < x < \frac{2}{3}; \\ \frac{8}{3}, & x = \frac{2}{3}; \\ -2x + 4, & x > \frac{2}{3}. \end{cases}$$

Thus the maximum value of $f(x)$ is $\frac{8}{3}$ and the maximum value of $6f(x) + 2012$ is 2028.

22. Answer: 665

Since B cannot be empty, the number of elements in B is between 1 to 6. After picking B with k elements, there are $2^k - 1$ possible subsets of B which qualifies for A, as A and B must be distinct. Thus the total number of possibilities is

$$\begin{aligned} \sum_{k=1}^{6} \binom{6}{k}(2^k - 1) &= \sum_{k=0}^{6} \binom{6}{k}(2^k - 1) \\ &= \sum_{k=0}^{6} \binom{6}{k} 2^k - \sum_{k=0}^{6} \binom{6}{k} \\ &= (2 + 1)^6 - (1 + 1)^6 \\ &= 3^6 - 2^6 \\ &= 665. \end{aligned}$$

23. Answer: 45

The equation can be rewritten as $\sqrt{(\sqrt{x-1} - 2)^2} + \sqrt{(\sqrt{x-1} - 3)^2} = 1$.

If $\sqrt{x-1} \geq 3$, it reduces to $\sqrt{x-1} - 2 + \sqrt{x-1} - 3 = 1$ i.e. $\sqrt{x-1} = 3$ giving $x = 10$.

If $\sqrt{x-1} \leq 2$, it reduces to $2 - \sqrt{x-1} + 3 - \sqrt{x-1} = 1$ i.e. $\sqrt{x-1} = 2$ giving $x = 5$.

If $2 < \sqrt{x-1} < 3$, i.e. $5 < x < 10$, it reduces to $\sqrt{x-1} - 2 + 3 - \sqrt{x-1} = 1$ which is true for all values of x between 5 and 10.

Hence the sum of all integral solutions is $5 + 6 + 7 + 8 + 9 + 10 = 45$.

24. Answer: 4

$$S = \left| \sqrt{(x+2)^2 + (0-1)^2} - \sqrt{(x+1)^2 + (0-2)^2} \right|.$$

Let $P = (x, 0)$, $A = (-2, 1)$ and $B = (-1, 2)$, then S represents the difference between the lengths PA and PB. S is maximum when the points P, A and B are collinear and that occurs when $P = (-3, 0)$. So

$$S = \left|\sqrt{(-1)^2 + (1)^2} - \sqrt{(-2)^2 + (0-2)^2}\right| = \left|\sqrt{2} - 2\sqrt{2}\right|.$$

Thus the maximum value of $S^4 = 4$.

25. Answer: 384

    Partition S into three subsets according to their residues modulo 3: $S_0 = \ 3, 6, \ldots, 18\ $, $S_1 = \ 1, 4, \ldots, 19\ $ and $S_2 = \ 2, 5, \ldots, 20\ $. In order for the sum of three integers to be divisible by 3, either all three must belong to exactly one S_i or all three must belong to different S_i.

    Hence total number of such choices is $\binom{6}{3} + 2\binom{7}{3} + 6 \times 7 \times 7 = 384$.

26. Answer: 10

    Since $BD \parallel FG$ and FG is tangent to the circle at C, we have

    $$\angle BCF = \angle CBE = \angle DCG = \angle BDC = \angle BAC.$$

    Furthermore

    $$\angle BEC = \angle BAC + \angle ABE = \angle CBE + \angle ABE = \angle ABC = \angle ACB.$$

    We can then conclude that $BE = BC = DC = 4$. Also, $\triangle ABE$ is similar to $\triangle DCE$. If we let $AE = x$,

    $$\frac{DE}{DC} = \frac{AE}{AB} \Longrightarrow DE = \frac{2}{3}x.$$

    By the Intersecting Chord Theorem, $AE \cdot EC = BE \cdot ED$, i.e. $x(6 - x) = 4(\frac{2}{3}x)$, which gives $x = \frac{10}{3}$, so $3AE = 3x = 10$.

27. Answer: 3432

    We use $a_1, a_2, \cdots, a_7$ and $b_1, b_2, \cdots, b_7$ to denote the players of Team A and Team B, respectively. A possible gaming outcome can be represented by a linear sequence of the above 14 terms. For instance, we may have $a_1a_2b_1b_2a_3b_3b_4b_5a_4b_6b_7a_5a_6a_7$ which indicates player 1 followed by player 2 from Team A were eliminated first, and the third player eliminated was player 1 from team B. However Team A emerged the final winner as all seven players of Team B gets eliminated with a_5, a_6 and a_7 remaining uneliminated. (Note a_6 and a_7 never actually played.) Thus, a gaming outcome can be formed by choosing 7 out of 14 possible positions for Team A, with the remaining filled by Team B players. Therefore, the total number of gaming outcomes is given by $\binom{14}{7} = \frac{14!}{(7!)^2} = 3432$.

28. Answer: 1

$$\begin{aligned}\frac{8\sqrt{2}}{5} = \mathbf{m}+\mathbf{n} &= \sqrt{4+2\sqrt{2}\cos\theta - 2\sqrt{2}\sin\theta} \\ &= 2\sqrt{1+\cos\left(\theta+\frac{\pi}{4}\right)} \\ &= 2\sqrt{2}\left|\cos\left(\frac{\theta}{2}+\frac{\pi}{8}\right)\right|\end{aligned}$$

Since $\pi < \theta < 2\pi$, we have $\frac{5}{8}\pi < \frac{\theta}{2}+\frac{\pi}{8} < \frac{9}{8}\pi$. Thus, $\cos\left(\frac{\theta}{2}+\frac{\pi}{8}\right) < 0$. This implies that $\cos\left(\frac{\theta}{2}+\frac{\pi}{8}\right) = -\frac{4}{5}$ and hence

$$5\cos\left(\frac{\theta}{2}+\frac{\pi}{8}\right)+5 = 1$$

29. Answer: 8

Using the AM-GM inequality, we have

$$\begin{aligned} f(x\ y\ z) &= \sqrt{2x+13}+\sqrt[3]{3y+5}+\sqrt[4]{8z+12} \\ &= \sqrt{\frac{2x+13}{4}}\cdot\sqrt{4}+\sqrt[3]{\frac{3y+5}{4}}\cdot\sqrt[3]{2}\cdot\sqrt[3]{2}+\sqrt[4]{\frac{8z+12}{8}}\cdot\sqrt[4]{2}\cdot\sqrt[4]{2}\cdot\sqrt[4]{2} \\ &\le \frac{\frac{2x+13}{4}+4}{2}+\frac{\frac{3y+5}{4}+2+2}{3}+\frac{\frac{8z+12}{8}+2+2+2}{4} \\ &= \frac{1}{4}(x+y+z)+\frac{29}{4} \\ &= 8 \end{aligned}$$

The equality is achieved at $x=\frac{3}{2}$, $y=1$ and $z=\frac{1}{2}$.

30. Answer: 40460

Let $n=34$ and $Q(x)=(x+1)P(x)-x$

Then $Q(x)$ is a polynomial of degree $n+1$ and $Q(k)=0$ for all $k=0\ 1\ 2\ \cdots\ n$. Thus there is a constant C such that

$$Q(x)=(x+1)P(x)-x=Cx(x-1)(x-2)\cdots(x-n)$$

Letting $x=-1$ gives

$$1=C(-1)(-2)\cdots(-n-1)=C(-1)^{n+1}(n+1)!$$

Thus $C=(-1)^{n+1}\ (n+1)!$ and

$$P(x)=\frac{1}{x+1}(x+Q(x))=\frac{1}{x+1}\left(x+\frac{(-1)^{n+1}x(x-1)(x-2)\cdots(x-n)}{(n+1)!}\right)$$

So

$$P(n+1)=\frac{1}{n+2}(n+1+\frac{(-1)^{n+1}(n+1)!}{(n+1)!})=\frac{1}{n+2}(n+1+(-1)^{n+1})=\frac{n}{n+2}$$

since $n = 34$ is even. Hence

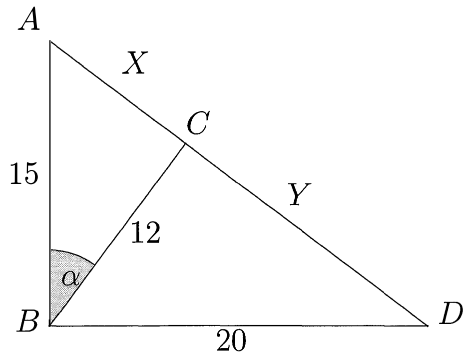
$$42840 \times P(35) = 34 \times 35 \times 36 \times \frac{34}{36} = 40460$$

31. Answer: 30

Let $X = \sqrt{369 - 360\cos\alpha}$ and $Y = \sqrt{544 - 480\sin\alpha}$. Observe that

$$X^2 = 12^2 + 15^2 - 2(12)(15)\cos\alpha$$
$$Y^2 = 12^2 + 20^2 - 2(12)(20)\cos(90^\circ - \alpha)$$

and $15^2 + 20^2 = 25^2 = (X+Y)^2$, so we can construct a right-angled triangle ABD as shown.



In particular $\angle ABC = \alpha$ and $\angle CBD = 90^\circ - \alpha$. We can check that $\triangle ACB$ is in fact similar to $\triangle ABD$. So $\angle ADB = \alpha$ and

$$40\tan\alpha = 40 \times \frac{15}{20} = 30$$

32. Answer: 3

We shall apply the following inequality:

$$4(a^2 + b^2 + c^2 + d^2) \geq (a + b + c + d)^2$$

Since $a + b + c + d = 8 - e$ and $a^2 + b^2 + c^2 + d^2 = 16 - e^2$, we have

$$4(16 - e^2) \geq (8 - e)^2$$

i.e. $e(5e - 16) \leq 0$. Thus $0 \leq e \leq 16\ 5$.

Note that if $a = b = c = d = 6\ 5$, we have $e = 16\ 5$. Hence $\lfloor e \rfloor = 3$.

33. Answer: 105

A three-digit number can be expressed as $100a + 10b + c$, and so we are minimizing

$$F(a\ b\ c) = \frac{100a + 10b + c}{a + b + c}$$

Observe that with distinct digits $a\ b\ c$, $F(a\ b\ c)$ has the minimum value when $a < b < c$. Thus we assume that $0 < a < b < c \leq 9$.

Note that

$$F(a\ b\ c) = \frac{100a + 10b + c}{a + b + c} = 1 + \frac{99a + 9b}{a + b + c}$$

We observe now that $F(a\ b\ c)$ is minimum when $c = 9$.

$$F(a\ b\ 9) = 1 + \frac{99a + 9b}{a + b + 9} = 1 + \frac{9(a + b + 9) + 90a - 81}{a + b + 9} = 10 + \frac{9(10a - 9)}{a + b + 9}$$

Now $F(a\ b\ 9)$ is minimum when $b = 8$.

$$F(a\ 8\ 9) = 10 + \frac{9(10a - 9)}{a + 17} = 10 + \frac{90(a + 17) - 1611}{a + 17} = 100 - \frac{1611}{a + 17}$$

which has the minimum value when $a = 1$, and so $L = F(1\ 8\ 9) = 10\ 5$ and $10L = 105$.

34. Answer: 59

Let $\alpha = 17^{15^{\cdot^{\cdot^{\cdot^{3^1}}}}}$ and $\beta = 15^{13^{\cdot^{\cdot^{\cdot^{3^1}}}}}$. Since $13^{\cdot^{\cdot^{\cdot^{3^1}}}}$ is odd. $\beta \equiv -1 \pmod{16}$. Now let φ be Euler's phi function, $\varphi(100) = 40$ and $\varphi(40) = 16$. By Euler's theorem,

$$\alpha = 17^{\beta} \equiv 17^{-1} \equiv 33 \pmod{40}$$

where the last congruence can be calculated by the extended Euclidean algorithm. Thus by repeated squaring, we have

$$19^{\alpha} \equiv 19^{33} \equiv 59 \pmod{100}$$

35. Answer: 5

Note that $(n + \frac{1}{2})^2 = n^2 + n + \frac{1}{4}$, so $f(n^2 + n) = n$ but $f(n^2 + n + 1) = n + 1$. So each of the sequences $(n - f(n))_{n=1}^{\infty} = (0\ 1\ 1\ 2\quad)$ and $(n + f(n))_{n=1}^{\infty} = (2\ 3\ 5\ 6\quad)$ increases by 1 for every increment of n by 1, except when $n = m^2 + m$. If $n = m^2 + m$, we have $n - f(n) = m^2$ and $(n + 1 - f(n + 1)) = m^2$, so the former sequence has every perfect square repeated once. On the other hand, if $n = m^2 + m$, we have $n + f(n) = m^2 + 2m$ but $(n + 1 + f(n + 1)) = m^2 + 2m + 2$, so the latter sequence omits every perfect square. Thus

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\left(\frac{3}{2}\right)^{f(n)} + \left(\frac{3}{2}\right)^{-f(n)}}{\left(\frac{3}{2}\right)^{n}} &= \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-f(n)} + \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n+f(n)} \\
&= \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n} + \sum_{m=1}^{\infty} \left(\frac{2}{3}\right)^{m^2} + \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n} - \sum_{m=1}^{\infty} \left(\frac{2}{3}\right)^{m^2} \\
&= 5
\end{aligned}$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2012

(Senior Section, Round 2)

Saturday, 23 June 2012 **0900-1300**

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1. A circle ω through the incentre I of a triangle ABC and tangent to AB at A, intersects the segment BC at D and the extension of BC at E. Prove that the line IC intersects ω at a point M such that $MD = ME$.

2. Determine all positive integers n such that n equals the square of the sum of the digits of n.

3. If 46 squares are colored red in a 9×9 board, show that there is a 2×2 block on the board in which at least 3 of the squares are colored red.

4. Let $a_1\ a_2 \qquad a_n\ a_{n+1}$ be a finite sequence of real numbers satisfying

$$a_0 = a_{n+1} = 0$$
$$\text{and} \quad a_{k-1} - 2a_k + a_{k+1} \le 1 \quad \text{for } k = 1\ 2 \qquad n$$

Prove that for $k = 0\ 1 \qquad n+1$,

$$a_k \le \frac{k(n+1-k)}{2}$$

5. Prove that for any real numbers $a\ b\ c\ d \ge 0$ with $a + b = c + d = 2$,

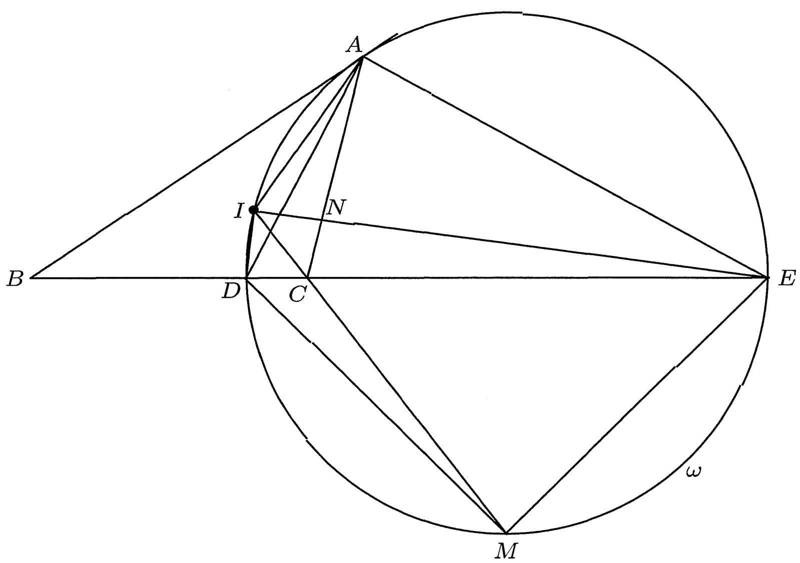
$$(a^2 + c^2)(a^2 + d^2)(b^2 + c^2)(b^2 + d^2) \le 25$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2012

(Senior Section, Round 2 solutions)

1. Join AD, ID, IA and AE. Let IE intersect AC at N. We have $\angle IAN = \angle IAB = \angle IEA$ so that the triangles NIA and AIE are similar. Thus $\angle ANI = \angle EAI = \angle IDB$. Also $\angle DCI = \angle NCI$. Therefore, the triangles DCI and NCI are congruent. Hence $\angle DIC = \angle NIC$ implying $MD = ME$.



2. Let $s(n)$ denote the sum of all the digits of n. Suppose n is a positive integer such that $s(n)^2 = n$. Let $s(n) = k$ so that $n = k^2$. Then $s(k^2) = s(n) = k$. Let $10^{r-1} \leq k < 10^r$, where r is a positive integer. That is k has exactly r digits. From $10^{r-1} \leq k$, we have $r \leq \log k + 1$. From $k < 10^r$, we have $k^2 < 10^{2r}$ so that k^2 has at most $2r$ digits. Therefore, $s(k^2) \leq 9 \times 2r = 18r \leq 18 \log k + 18$ which is less than k if $k \geq 50$. Thus the equation $s(k^2) = k$ has no solution in k if $k \geq 50$.

Let $k < 50$ and $s(k^2) = k$. Taking mod 9, we get $k^2 \equiv k \pmod 9$. Thus $k \equiv 0, 1 \pmod 9$. That is $k = 1, 9, 10, 18, 19, 27, 28, 36, 37, 45, 46$. Only when $k = 1$ and $k = 9$, we have $s(k^2) = k$. The corresponding solutions for n are $n = 1$ or 81.

3. Suppose that at most 2 squares are colored red in any 2×2 square. Then in any 9×2 block, there are at most 10 red squares. Moreover, if there are 10 red squares, then there must be 5 in each row.

Now let the number of red squares in row i of the 9×9 board be r_i. Then $r_i+r_{i+1}\le 10$, $1\le i\le 8$. Suppose that some $r_i\le 5$ with i odd. Then

$$(r_1+r_2)+\cdots+(r_{i-2}+r_{i-1})+r_i+\cdots+(r_8+r_9)\le 4\times 10+5=45.$$

On the other hand, suppose that $r_1,r_3,r_5,r_7,r_9\ge 6$. Then the sum of any 2 consecutive r_i's is ≤ 9. So

$$(r_1+r_2)+\cdots+(r_7+r_8)+r_9\le 4\times 9+9=45.$$

4. Let $b_k=\frac{k(n+1-k)}{2}$. Then $b_0=b_{n+1}=0$ and $b_{k-1}-2b_k+b_{k+1}=-1$ for $k=1,2\dots,n$. Suppose there exists an index i such that $a_i>b_i$, then the sequence $a_0-b_0,\dots,a_{n+1}-b_{n+1}$ has a positive term. Let j be the index such that $a_{j-1}-b_{j-1}<a_j-b_j$ and a_j-b_j has the largest value. Then

$$(a_{j-1}-b_{j-1})+(a_{j+1}-b_{j+1})<2(a_j-b_j).$$

Using

$$a_{k-1}-2a_k+a_{k+1}\ge -1 \quad\text{and}\quad b_{k-1}-2b_k+b_{k+1}=-1 \quad\text{for all } k$$

we obtain

$$(a_{j-1}-b_{j-1})+(a_{j+1}-b_{j+1})\ge 2(a_j-b_j)$$

a contradiction. Thus $a_k\le b_k$ for all k. Similarly, we can show that $a_k\ge -b_k$ for all k and therefore $a_k\ \le b_k$ as required.

5. First note that $(ac+bd)(ad+bc)\ge(ab-cd)^2$. To see this, we may assume $a\ge c\ge d\ge b$ since $a+b=c+d$. Then $cd-ab\ge 0$. Also we have the two obvious inequalities $ac+bd\ge cd-ab$ and $ad+bc\ge cd-ab$. Multiplying them together we get $(ac+bd)(ad+bc)\ge(ab-cd)^2$. Next

$$\begin{aligned}
&(a^2+c^2)(a^2+d^2)(b^2+c^2)(b^2+d^2)\\
&\quad=\left((ac+bd)(ad+bc)-(ab-cd)^2\right)^2+(ab-cd)^2\\
&\qquad+\left((a+b)^2(c+d)^2-1\right)(ab-cd)^2\\
&\quad=\left((ac+bd)(ad+bc)-(ab-cd)^2\right)^2+16(ab-cd)^2\\
&\quad\le\left(\frac{(ac+bd+ad+bc)^2}{4}-(ab-cd)^2\right)^2+16(ab-cd)^2 \quad\text{by AM-GM}\\
&\quad=\left(\frac{(a+b)^2(c+d)^2}{4}-(ab-cd)^2\right)^2+16(ab-cd)^2\\
&\quad=\left(4-(ab-cd)^2\right)^2+16(ab-cd)^2.
\end{aligned}$$

This final expression is an increasing function of $(ab-cd)^2$. The largest value of $(ab-cd)^2$ is 1 when $(a,b,c,d)=(1,1,0,2),(1,1,2,0),(0,2,1,1),(2,0,1,1)$. Consequently, $\left(4-(ab-cd)^2\right)^2+16(ab-cd)^2\le 25$, proving the inequality.

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2012
(Open Section, First Round)

Wednesday, 30 May 2012 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Throughout this paper, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. For example, $\lfloor 2.1 \rfloor = 2$, $\lfloor 3.9 \rfloor = 3$ (This notation is used in Questions 2, 10, 16, 17, 18 and 22).

1. The sum of the squares of 50 consecutive odd integers is 300850. Find the largest odd integer whose square is the last term of this sum.

2. Find the value of $\displaystyle\sum_{k=3}^{1000} \lfloor \log_2 k \rfloor$.

3. Given that $\mathrm{f}(x)$ is a polynomial of degree 2012, and that $\mathrm{f}(k) = \dfrac{2}{k}$ for $k = 1, 2, 3, \cdots, 2013$, find the value of $2014 \times \mathrm{f}(2014)$.

4. Find the total number of sets of positive integers (x, y, z), where x, y and z are positive integers, with $x < y < z$ such that
$$x + y + z = 203.$$

5. There are a few integers n such that $n^2 + n + 1$ divides $n^{2013} + 61$. Find the sum of the squares of these integers.

6. It is given that the sequence $(a_n)_{n=1}^{\infty}$, with $a_1 = a_2 = 2$, is given by the recurrence relation
$$\frac{2a_{n-1}a_n}{a_{n-1}a_{n+1} - a_n^2} = n^3 - n$$
for all $n = 2, 3, 4, \cdots$. Find the integer that is closest to the value of $\displaystyle\sum_{k=2}^{2011} \frac{a_{k+1}}{a_k}$.

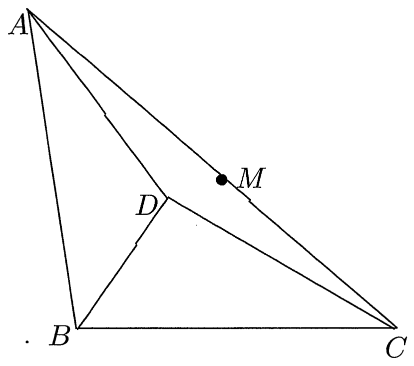
7. Determine the largest even positive integer which cannot be expressed as the sum of two composite odd positive integers.

8. The lengths of the sides of a triangle are successive terms of a geometric progression. Let A and C be the smallest and largest interior angles of the triangle respectively. If the shortest side has length 16 cm and
$$\frac{\sin A - 2\sin B + 3\sin C}{\sin C - 2\sin B + 3\sin A} = \frac{19}{9},$$
find the perimeter of the triangle in centimetres.

9. Find the least positive integral value of n for which the equation
$$x_1^3 + x_2^3 + \cdots + x_n^3 = 2002^{2002}$$
has integer solutions $(x_1, x_2, x_3, \cdots, x_n)$.

10. Let α_n be a real root of the cubic equation $nx^3 + 2x - n = 0$, where n is a positive integer. If $\beta_n = \lfloor (n+1)\alpha_n \rfloor$ for $n = 2\ 3\ 4\ \cdots$, find the value of $\frac{1}{1006}\sum_{k=2}^{2013} \beta_k$.

11. In the diagram below, the point D lies inside the triangle ABC such that $\angle BAD = \angle BCD$ and $\angle BDC = 90^o$. Given that $AB = 5$ and $BC = 6$, and the point M is the midpoint of AC, find the value of $8 \times DM^2$.



12. Suppose the real numbers x and y satisfy the equations

$$x^3 - 3x^2 + 5x = 1 \quad \text{and } y^3 - 3y^2 + 5y = 5$$

Find $x + y$.

13. The product of two of the four roots of the quartic equation $x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$ is -32. Determine the value of k.

14. Determine the smallest integer n with $n \geq 2$ such that

$$\sqrt{\frac{(n+1)(2n+1)}{6}}$$

is an integer.

15. Given that f is a real-valued function on the set of all real numbers such that for any real numbers a and b,

$$\mathrm{f}(a\mathrm{f}(b)) = ab$$

Find the value of $\mathrm{f}(2011)$.

16. The solutions to the equation $x^3 - 4\lfloor x \rfloor = 5$, where x is a real number, are denoted by $x_1\ x_2\ x_3\ \cdots x_k$ for some positive integer k. Find $\sum_{i=1}^{k} x_i^3$.

17. Determine the maximum integer solution of the equation

$$\left\lfloor \frac{x}{1!} \right\rfloor + \left\lfloor \frac{x}{2!} \right\rfloor + \left\lfloor \frac{x}{3!} \right\rfloor + \cdots + \left\lfloor \frac{x}{10!} \right\rfloor = 1001$$

18. Let A, B, C be the three angles of a triangle. Let L be the maximum value of

$$\sin 3A + \sin 3B + \sin 3C \ .$$

Determine $\lfloor 10L \rfloor$.

19. Determine the number of sets of solutions (x, y, z), where x, y and z are integers, of the the equation $x^2 + y^2 + z^2 = x^2 y^2$.

20. We can find sets of 13 distinct positive integers that add up to 2142. Find the largest possible greatest common divisor of these 13 distinct positive integers.

21. Determine the maximum number of different sets consisting of three terms which form an arithmetic progressions that can be chosen from a sequence of real numbers $a_1, a_2, \cdots, a_{101}$, where

$$a_1 < a_2 < a_3 < \cdots < a_{101}.$$

22. Find the value of the series

$$\sum_{k=0}^{\infty} \left\lfloor \frac{20121 + 2^k}{2^{k+1}} \right\rfloor .$$

23. The sequence $(x_n)_{n=1}^{\infty}$ is defined recursively by

$$x_{n+1} = \frac{x_n + (2 - \sqrt{3})}{1 - x_n(2 - \sqrt{3})},$$

with $x_1 = 1$. Determine the value of $x_{1001} - x_{401}$.

24. Determine the maximum value of the following expression

$$-\cdots - x_1 - x_2 - x_3 - x_4 - \cdots - x_{2014} -$$

where $x_1, x_2, \cdots, x_{2014}$ are distinct numbers in the set $-1, 2, 3, 4, \cdots, 2014-$.

25. Evaluate $\frac{-1}{2^{2011}} \sum_{k=0}^{1006} (-1)^k 3^k \binom{2012}{2k}$.

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2012
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1. Answer: 121

   Solution. Let the integers be $X+2\ X+4\ \cdots\ X+100$. Then

   $$(X+2)^2+(X+4)^2+\cdots+(X+100)^2=300850$$

   Let $y=X+51$ and regrouping the terms, we obtain

   $$\left[(y-49)^2+(y+49)^2\right]+\left[(y-47)^2+(y+47)^2\right]+\cdots+\left[(y-1)^2+(y+1)^2\right]=300850$$

   which simplifies to

   $$50y^2+2(1^2+3^2+5^2+7^2+\cdots+49^2)=300850$$

   Using the fact that $1^2+3^2+5^2+\cdots+(2n-1)^2=\frac{4}{3}n^3-\frac{1}{3}n$, we obtain $y=72$. Hence $X=21$, so that the required number is 121.

   □

2. Answer: 7986

   Solution. Note that $2^{k+1}-2^k=2^k$, and that $2^k\le t<2^{k+1}$ if and only if $\lfloor\log_2 t\rfloor=k$. Then the requires sum (denoted by S) can be obtained by

   $$\begin{aligned} S &= \sum_{k=2}^{9}\sum_{2^k\le t<2^{k+1}}\lfloor\log_2 t\rfloor+\lfloor\log_2 3\rfloor-\sum_{t=1001}^{1023}\lfloor\log_2 t\rfloor \\ &= \left(\sum_{k=2}^{9}k2^k\right)+1-\sum_{t=1001}^{1023}9 \\ &= 8192+1-23(9)=7986 \end{aligned}$$

   □

3. Answer: 4

   Solution. Let $\mathrm{g}(x)=x\mathrm{f}(x)-2$, hence $\mathrm{g}(x)$ is a polynomial of degree 2013. Since $\mathrm{g}(1)=\mathrm{g}(2)=\mathrm{g}(3)=\cdots=\mathrm{g}(2013)=0$, we must have

   $$\mathrm{g}(x)=\lambda(x-1)(x-2)(x-3)\cdots(x-2012)(x-2013)$$

   for some λ. Also, $\mathrm{g}(0)=-2=-\lambda\cdot 2013!$, we thus have $\lambda=\frac{2}{2013!}$ Hence,

   $$\mathrm{g}(2014)=\frac{2}{2013!}(2013!)=2014\cdot\mathrm{f}(2014)-2$$

   concluding that $2014\cdot\mathrm{f}(2014)=4$ □

4. Answer: 3333

Solution. First note that there are $\binom{202}{2} = \dfrac{202(201)}{2} = 20301$ positive integer sets (x, y, z) which satisfy the given equation. These solution sets include those where two of the three values are equal. If $x = y$, then $2x+z = 203$. By enumerating, $z = 1, 3, 5, \cdots, 201$. There are thus 101 solutions of the form (x, x, z). Similarly, there are 101 solutions of the form (x, y, x) and (x, y, y). Since $x < y < z$, the required answer is

$$\frac{1}{3!}\left(\binom{202}{2} - 3(101)\right) = \frac{20301 - 303}{6} = 3333.$$

□

5. Answer: 62

Solution. Since $n^3 - 1 = (n-1)(n^2+n+1)$, we know that n^2+n+1 divides n^3-1. Also, since $n^{2013} - 1 = (n^3)^{671} - 1$, we also know that n^2+n+1 divides $n^{2013}-1$. As

$$n^{2013} + 61 = n^{2013} - 1 + 62,$$

we must have that n^2+n+1 divides $n^{2013}+61$ if and only if n^2+n+1 divides 62.

Case (i): If $n^2+n+1 = 1$, then $n = 0, -1$.

Case (ii): If $n^2+n+1 = 2$, there is no integer solution for n.

Case (iii): If $n^2+n+1 = 31$, then $n = 6, -5$.

Case (iv): If $n^2+n+1 = 62$, there is no integer solution for n.

Thus, all the integer values of n are $0, -1, 6, -5$. Hence the sum of squares is $1+36+25 = 62$.

□

6. Answer: 3015

Solution. The recurrence relation can be written as

$$\frac{a_{n+1}}{a_n} - \frac{a_n}{a_{n-1}} = \left(\frac{1}{n-1} - \frac{1}{n}\right) - \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

Summing for $n = 2$ to N, we obtain

$$\frac{a_{N+1}}{a_N} - \frac{a_2}{a_1} = \left(1 - \frac{1}{N}\right) - \left(\frac{1}{2} - \frac{1}{N+1}\right),$$

showing that

$$\frac{a_{N+1}}{a_N} = \frac{3}{2} - \left(\frac{1}{N} - \frac{1}{N+1}\right).$$

Summing this up for $N = 2$ to $N = 2011$, we obtain

$$S = \sum_{k=2}^{2011} \frac{a_{k+1}}{a_k} = \frac{3}{2}(2010) - \left(\frac{1}{2} - \frac{1}{2012}\right) = 3014.5 + \frac{1}{2012}$$

showing that the integer closest to S is 3015. □

7. Answer: 38

Solution. Let n be an even positive integer. Then each of the following expresses n as the sum of two odd integers: $n = (n-15)+15$ $(n-25)+25$ or $(n-35)+35$. Note that at least one of $n-15$ $n-25$ $n-35$ is divisible by 3 so that it is composite, and hence n can be expressed as the sum of two composite odd numbers if $n > 38$. Indeed, it can be verified that 38 cannot be expressed as the sum of two composite odd positive integers. □

8. Answer: 76

Solution. Let the lengths of the sides of the triangle in centimetres be 16, $16r$ and $16r^2$ (where $r > 1$). Then $\dfrac{1-2r+3r^2}{r^2-2r+3} = \dfrac{19}{9}$ so that $r = \dfrac{3}{2}$. Hence, the perimeter of the triangle $= 16\left(1+\dfrac{3}{2}+\dfrac{9}{4}\right) = 76\text{cm}$ □

9. Answer: 4

Solution. Since $2002 \equiv 4(\text{mod}9)$ $4^3 \equiv 1(\text{mod}9)$ and $2002 = 667 \times 3 + 1$, it follows that $2002^{2002} \equiv 4^{667\times 3+1} \equiv 4(\text{mod}9)$. Observe that for positive integers x, the possible residues modulo 9 for x^3 are 0 ± 1 . Therefore, none of the following

$$x_1^3 \quad x_1^3 + x_2^3 \quad x_1^3 + x_2^3 + x_3^3$$

can have a residue of 4 modulo 9. However, since $2002 = 10^3 + 10^3 + 1^3 + 1^3$, it follows that

$$\begin{aligned} 2002^{2002} &= 2002 \cdot (2002^{667})^3 \\ &= (10^3 + 10^3 + 1^3 + 1^3)(2002^{667})^3 \\ &= (10 \cdot 2002^{667})^3 + (10 \cdot 2002^{667})^3 + (2002^{667})^3 + (2002^{667})^3 \end{aligned}$$

This shows that $x_1^3 + x_2^3 + x_3^3 + x_4^4 = 2002^{2002}$ is indeed solvable. Hence the least integral value of n is 4. □

10. Answer: 2015

Let $\mathrm{f}(x) = nx^3 + 2x - n$. It is easy to see that f is a strictly increasing function for $n = 2$ 3 4 $\cdots$. Further,

$$f\left(\frac{n}{n+1}\right) = n\left(\frac{n}{n+1}\right)^3 + 2\left(\frac{n}{n+1}\right) - n = \frac{n}{(n+1)^3}\left(-n^2+n+1\right) < 0$$

for all $n \geq 2$. Also, $\mathrm{f}(1) = 2 > 0$. Thus, the only real root of the equation $nx^3 + 2x - n = 0$ for $n \geq 2$ is located in the interval $\left(\frac{n}{n+1}\ 1\right)$. We must have

$$\frac{n}{n+1} < \alpha_n < 1 \Longrightarrow n < (n+1)\alpha_n < n+1$$

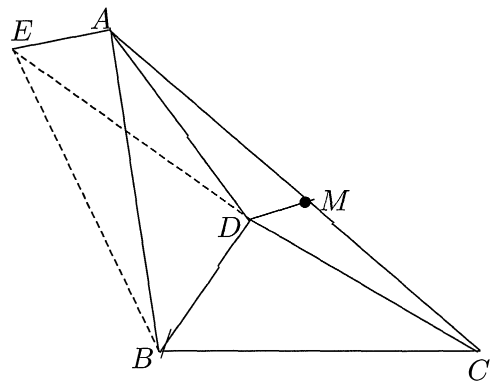
so that $\beta_n = \lfloor (n+1)\alpha_n \rfloor = n$ for all $n \geq 2$. Consequently,

$$\frac{1}{1006}\sum_{k=2}^{2013} \beta_k = \frac{1}{1006}\sum_{k=2}^{2013} k = \frac{1}{1006} \cdot \frac{2012}{2}(2+2013) = 2015$$

□

11. Answer: 22

Solution. Extend CD to E such that $CD = DE$.



It is clear that ΔCDB and ΔEDB are congruent. Hence $EB = CB = 6$ and $\angle BED = \angle BCD$. Thus, $\angle BED = \angle BCD = \angle BAD$ implies that the points B, D, A are E are concyclic. Given that $\angle BDC = 90^{\circ}$, hence $\angle EDB = 90^{\circ}$. $BDAE$ is a cyclic quadrilateral with EB as a diameter. Thus, $\angle EAB = 90^{\circ}$. In the right-angled triangle EAB, we have

$$AE = \sqrt{EB^2 - AB^2} = \sqrt{6^2 - 5^2} = \sqrt{11}.$$

Since D and M are the midpoints of EC and AC respectively, $DM = \dfrac{1}{2}AE = \dfrac{\sqrt{11}}{2}$. Thus, $8 \times DM^2 = 22$. □

12. Answer: 2

Solution. From $x^3 - 3x^2 + 5x = 1$, we have

$$(x-1)^3 + 2(x-1) = -2,$$

and from $y^3 - 3y^2 + 5y = 5$, we have

$$(y-1)^3 + 2(y-1) = 2.$$

Thus

$$\begin{aligned} 0 &= (x-1)^3 + 2(x-1) + (y-1)^3 + 2(y-1) \\ &= (x+y-2)((x-1)^2 - (x-1)(y-1) + (y-1)^2) + 2(x+y-2) \\ &= (x+y-2)(2 + (x-1)^2 - (x-1)(y-1) + (y-1)^2). \end{aligned}$$

For any real numbers x and y, we always have

$$(x-1)^2 - (x-1)(y-1) + (y-1)^2 \geq 0$$

and thus $x + y - 2 = 0$, implying that $x + y = 2$. □

13. Answer: 86

Solution. Let a, b, c, d be the four roots of $x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$ such that $ab = -32$. Then

$$\begin{cases} a + b + c + d = 18, \\ ab + ac + ad + bc + bd + cd = k, \\ abc + abd + acd + bcd = -200, \\ abcd = -1984. \end{cases}$$

Since $ab = -32$ and $abcd = -1984$, we have $cd = 62$. Then, from $abc + abd + acd + bcd = -200$ we have

$$-200 = -32c - 32d + 62a + 62b = -32(c + d) + 62(a + b).$$

Solving this equation together with the equation $a + b + c + d = 18$ gives that

$$a + b = 4, \qquad c + d = 14.$$

From $ab + ac + ad + bc + bd + cd = k$, we have

$$k = ab + ac + ad + bc + bd + cd = -32 + ac + ad + bc + bd + 62$$

$$= 30 + (a + b)(c + d) = 86.$$

□

14. Answer: 337

Solution. Assume that

$$\sqrt{\frac{(n+1)(2n+1)}{6}} = m$$

and so

$$(n + 1)(2n + 1) = 6m^2.$$

Thus 6 $(n + 1)(2n + 1)$, implying that $n \equiv 1$ or $5 (\text{mod } 6)$.

Case 1: $n = 6k + 5$.

Then $m^2 = (k+1)(12k+11)$. Since $(k+1)$ and $(12k+11)$ are relatively prime, both must be squares. So there are positive integers a and b such that $k + 1 = a^2$ and $12k + 11 = b^2$. Thus $12a^2 = b^2 + 1$. But, as $12a^2 \equiv 0 (\text{mod } 4)$ and $b^2 + 1 \equiv 1$ or $2 (\text{mod } 4)$, there are no integers a and b such that $12a^2 = b^2 + 1$. Hence Case 1 cannot happen.

Case 2: $n = 6k + 1$.

Then $m^2 = (3k + 1)(4k + 1)$. Since $3k + 1$ and $4k + 1$ are relatively prime, both must be squares. So there are positive integers a and b such that $3k + 1 = a^2$ and $4k + 1 = b^2$. Then $3b^2 = (2a - 1)(2a + 1)$. Observe that in the left-hand side, every prime factor except 3 has an even power. So neither $2a - 1$ nor $2a + 1$ can be a prime other than 3.

Now we consider positive integers a such that neither $2a - 1$ nor $2a + 1$ can be a prime other than 3. If $a = 1$, then $b = 1$ and $n = 1$. So we consider $a \geq 2$. The next smallest suitable value for a is 13. When $a = 13$, we have

$$3b^2 = 25 \times 27$$

and so $b = 15$, implying that $k = 56$ and so $n = 6k + 1 = 337$. □

15. Answer: 2011

From the recurrence relation, $\mathrm{f}(\mathrm{f}(1)\mathrm{f}(b)) = \mathrm{f}(1)b$ and $\mathrm{f}(\mathrm{f}(b)\mathrm{f}(1)) = \mathrm{f}(b)\cdot 1$. Hence, $\mathrm{f}(b) = \mathrm{f}(1)b$. By letting $b = f(1)$, we obtain $\mathrm{f}(\mathrm{f}(1)) = (\mathrm{f}(1))^2$. Also, from the given functional equation, we have $\mathrm{f}(\mathrm{f}(1)) = 1$, hence $(f(1))^2 = 1$, following that $f(1)$ is either 1 or -1. Hence $\mathrm{f}(2011) = 2011$. □

16. Answer: 19

Solution. Note that $x - 1 < \lfloor x \rfloor \leq x$. Note that if $x \geq 3$, there will be no solution as

$$x^3 - 4\lfloor x \rfloor \geq x^3 - 4x = x(x^2 - 4) \geq 3(5) = 15.$$

Also, if $x \leq -2$, there will be no solution as $x^3 - 4\lfloor x \rfloor < x^3 - 4(x-1) = x(x^2-4) + 4 \leq 4$. Hence the solution must be in the interval $(-2, 3)$.

If $\lfloor x \rfloor = -2$, then $x^3 = -3$, giving $x = \sqrt[3]{-3}$, which is a solution.

If $\lfloor x \rfloor = -1$, then $x^3 = 1$, giving $x = 1$ which contradicts with $\lfloor x \rfloor = -1$.

If $\lfloor x \rfloor = 0$, then $x^3 = 5$, hence there is no solution.

If $\lfloor x \rfloor = 1$, then $x^3 = 9$. Since $2 < \sqrt[3]{9} < 3$, there is no solution.

If $\lfloor x \rfloor = 2$, then $x^3 = 13$. Since $2 < \sqrt[3]{13} < 3$, $x = \sqrt[3]{13}$ is a solution.

Thus, the required answer is $-3 + 13 = 10$. □

17. Answer: 584

Solution. It is clear that

$$\left\lfloor \frac{x}{1!} \right\rfloor + \left\lfloor \frac{x}{2!} \right\rfloor + \left\lfloor \frac{x}{3!} \right\rfloor + \cdots + \left\lfloor \frac{x}{10!} \right\rfloor$$

is a monotone increasing function of x, and when $x = 6!$, the above expression has a value larger than 1001. Thus each solution of the equation

$$\left\lfloor \frac{x}{1!} \right\rfloor + \left\lfloor \frac{x}{2!} \right\rfloor + \left\lfloor \frac{x}{3!} \right\rfloor + \cdots + \left\lfloor \frac{x}{10!} \right\rfloor = 1001$$

is less than 6! and so if x is a solution, then

$$\left\lfloor \frac{x}{1!} \right\rfloor + \left\lfloor \frac{x}{2!} \right\rfloor + \left\lfloor \frac{x}{3!} \right\rfloor + \cdots + \left\lfloor \frac{x}{10!} \right\rfloor = \left\lfloor \frac{x}{1!} \right\rfloor + \left\lfloor \frac{x}{2!} \right\rfloor + \left\lfloor \frac{x}{3!} \right\rfloor + \left\lfloor \frac{x}{4!} \right\rfloor + \left\lfloor \frac{x}{5!} \right\rfloor.$$

As $x < 6!$, it has a unique expression of the form

$$x = a \times 5! + b \times 4! + c \times 3! + d \times 2! + e,$$

where a, b, c, d, e are non-negative integer with $a \leq 5, b \leq 4, c \leq 3, d \leq 2, e \leq 1$. Note that if

$$x = a \times 5! + b \times 4! + c \times 3! + d \times 2! + e,$$

then

$$\left\lfloor \frac{x}{1!} \right\rfloor + \left\lfloor \frac{x}{2!} \right\rfloor + \left\lfloor \frac{x}{3!} \right\rfloor + \left\lfloor \frac{x}{4!} \right\rfloor + \left\lfloor \frac{x}{5!} \right\rfloor = 206a + 41b + 10c + 3d + e.$$

Since $41b + 10c + 3d + e \leq 201$, we have $800 \leq 206a \leq 1001$ and so $a = 4$. Thus

$$41b + 10c + 3d + e = 177,$$

which implies that $b = 4$ and so on, giving that $c = d = 1$ and $e = 0$. Thus

$$x = 4 \times 5! + 4 \times 4! + 1 \times 3! + 1 \times 2! + 0 = 584.$$

As 584 is the only integer solution, the answer is 584.

□

18. Answer: 25

Solution. We shall show that $-2 \leq \sin 3A + \sin 3B + \sin 3C \leq 3\sqrt{3}/2$.

Assume that $A \leq B \leq C$. Then $A \leq 60^o$. Thus $\sin 3A \geq 0$. It is clear that $\sin 3B \geq -1$ and $\sin 3C \geq -1$. Thus $\sin 3A + \sin 3B + \sin 3C \geq -2$. Let $B = C$. Then $B = C = 90^o - A/2$. If A is very small, B and C are close to 90^o, and thus $\sin 3A + \sin 3B + \sin 3C$ is close to -2.

Now we show that $\sin 3A + \sin 3B + \sin 3C \leq 3\sqrt{3}/2$. First the upper bound can be reached when $A = B = 20^o$ and $C = 140^o$.

Let $X = 3A$, $Y = 3B$ and $Z = 3(C - 120^o)$. Then $X + Y + Z = 180^o$ and

$$\sin 3A + \sin 3B + \sin 3C = \sin X + \sin Y + \sin Z.$$

Suppose that X, Y, Z satisfy the condition that $X + Y + Z = 180^o$ such that $\sin X + \sin Y + \sin Z$ has the maximum value. We can then show that $X = Y = Z$.

Assume that $X \leq Y \leq Z$. If $X < Z$, then

$$\sin X + \sin Z = 2 \sin \frac{X+Z}{2} \cos \frac{X-Z}{2} < 2 \sin \frac{X+Z}{2},$$

implying that

$$\sin X + \sin Y + \sin Z < \sin \frac{X+Z}{2} + \sin Y + \sin \frac{X+Z}{2}$$

which contradicts the assumption that $\sin X + \sin Y + \sin Z$ has the maximum value. Hence $X = Y = Z = 60^o$, implying that $A = 20^o$, $B = 20^o$ and $C = 140^o$ and

$$\sin 3A + \sin 3B + \sin 3C = 3\sqrt{3}/2.$$

Since $\sqrt{3} \approx 1.732$, the answer is then obtained. □

19. Answer: 1

Solution. Note that $x = 0, y = 0$ and $z = 0$ is a solution of this equation. We shall show that this is its only integer solution by proving that if x, y, z is a solution of this equation and whenever x, y, z are divisible by 2^k, they are also divisible by 2^{k+1} for any $k \geq 0$.

Let $x = 2^k x'$, $y = 2^k y'$ and $z = 2^k z'$. Then $x^2 + y^2 + z^2 = x^2 y^2$ is changed to

$$x'^2 + y'^2 + z'^2 = 2^k x'^2 y'^2$$

It is easy to verify that only when x', y', z' are all even, $x'^2 + y'^2 + z'^2$ and $2^{2k} x'^2 y'^2$ have the same remainder when divided by 4. Thus x, y, z are divisible by 2^{k+1}. □

20. Answer: 21

Solution. Let d be the greatest common divisor (gcd) of these 13 distinct positive integers. Then these 13 integers can be represented as $da_1, da_2, \cdots, da_{13}$, where $\gcd(a_1, a_2, \cdots, a_{13}) = 1$. Let S denote $a_1 + a_2 + \cdots + a_{13}$. Then $Sd = 2142$. In order for d to be the largest possible, S must be the smallest. Since $S \geq 1+2+3+\cdots+13 = 91$ and that S divides 2142, and that $2142 = 2 \times 3 \times 7 \times 51$, the smallest possible value of S can be $2 \times 51 = 102$, and the largest value of d is thus 21. By choosing $(a_1, a_2, a_3, \cdots, a_{12}, a_{13}) = (1, 2, 3, \cdots, 12, 24)$, we conclude that $d = 21$ is possible. □

21. Answer: 2500

Solution. First, for the following particular sequence, there are really 2500 different three-term arithmetic progressions which can be chosen from this sequence:

$$1, 2, 3, \cdots, 101.$$

They are $s, i, 2i - s$ for all integers s, i with $1 \leq s < i \leq 51$ and $2i - t, i, t$ for all integers i and t with $52 \leq i < t \leq 101$.

Now we show that for any given sequence of real numbers $a_1 < a_2 < \cdots < a_{101}$, there are at most 2500 different three-term arithmetic progressions which can be chosen from this sequence.

Let a_s, a_i, a_t represent a three-term arithmetic progression. It is clear that $2 \leq i \leq 100$. If $2 \leq i \leq 51$, then the first term a_s has at most $i - 1$ choices, as s must be an index in $1, 2, \cdots, i-1$. If $52 \leq i \leq 100$, then the third term a_t has at most $101 - i$ choices, as t must be an index in $i+1, i+2, \cdots, 101$.

So the number of different three-term arithmetic progressions which can be chosen from this sequence is at most

$$\sum_{i=2}^{51}(i-1) + \sum_{i=52}^{100}(101-i) = 1 + 2 + \cdots + 50 + 1 + 2 + \cdots + 49 = 2500.$$

□

22. Answer: 20121

Solution. Write $x := x - \lfloor x \rfloor$. Then

$$\left\lfloor \frac{1}{2} + \left\{\frac{x}{2}\right\} \right\rfloor = \begin{cases} 0, & \text{if } \frac{x}{2} < \frac{1}{2} \\ 1, & \text{otherwise} \end{cases} = \left\lfloor 2\left\{\frac{x}{2}\right\} \right\rfloor.$$

Thus, we have

$$\left\lfloor \frac{1}{2} + \frac{x}{2} \right\rfloor = \left\lfloor 2\left\{\frac{x}{2}\right\} \right\rfloor + \left\lfloor \frac{x}{2} \right\rfloor = \lfloor x \rfloor - \left\lfloor \frac{x}{2} \right\rfloor.$$

Applying the above result for $x = \frac{n}{2^k}$,

$$\begin{aligned} \sum_{k=0}^{\infty} \left\lfloor \frac{n+2^k}{2^{k+1}} \right\rfloor &= \sum_{k=0}^{\infty} \left(\left\lfloor \frac{n}{2^k} \right\rfloor - \left\lfloor \frac{n}{2^{k+1}} \right\rfloor \right) \\ &= \left\lfloor \frac{n}{2^0} \right\rfloor \\ &= n. \end{aligned}$$

In particular, when $n = 20121$, the infinite series converges to 20121. □

23. Answer: 0

Solution. Let $x_n = \tan \alpha_n$. Since $2 - \sqrt{3} = \tan\left(\frac{\pi}{12}\right)$, it follows that

$$x_{n+1} = \tan \alpha_{n+1} = \frac{\tan \alpha_n + \tan\left(\frac{\pi}{12}\right)}{1 - \tan \alpha_n \tan\left(\frac{\pi}{12}\right)} = \tan\left(\alpha_n + \frac{\pi}{12}\right)$$

So, $x_{n+12} = \tan(\alpha_n + \pi) = \tan \alpha_n = x_n$, implying that this sequence has a period of 12. Observe that $1001 \equiv 5 (\bmod 12)$ and $401 \equiv 5 (\bmod 12)$. Consequently,

$$x_{1001} - x_{401} = x_5 - x_5 = 0$$

□

24. Answer: 2013

Solution. First it is clear that the answer is an integer between 0 and 2014. But it cannot be 2014, as $\cdots \quad x_1 - x_2 \;- x_3 \;- x_4 \;\cdots - x_{2014}$ and

$$x_1 + x_2 + \cdots + x_{2014} = 1 + 2 + \cdots + 2014 = 1007 \times 2015$$

have the same parity.

Now we just need to show that 2013 can be achieved. For any integer k,

$$(4k+2) - (4k+4) \;- (4k+5) \;- (4k+3) \;= 0$$

Thus

$$\cdots \quad \cdots \;\; 2 - 4 \;- 5 \;- 3 \;- \cdots - (4k+2) \;- (4k+4) \;- (4k+5) \;- (4k+3) \cdots$$
$$-2010 \;- 2012 \;- 2013 \;- 2011 \;- 2014 \;- 1$$
$$= \;\; 0 - 2014 \;- 1 \;= 2013$$

□

25. Answer: 1

Solution. Consider the complex number $\omega = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$. On one hand, using the binomial theorem one has

$$\begin{aligned}
\mathrm{Re}(\omega^{2012}) &= \mathrm{Re}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{2012} \\
&= \mathrm{Re}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{2012} \\
&= \left(\frac{1}{2}\right)^{2012} - \binom{2012}{2}\left(\frac{1}{2}\right)^{2010}\left(\frac{3}{2^2}\right) + \frac{2012}{4}\left(\frac{1}{2}\right)^{2008}\left(\frac{3^2}{2^4}\right) \\
&\quad + \cdots + \left(\frac{3^{1006}}{2^{2012}}\right) \\
&= \frac{1}{2^{2012}}\left[1 - 3\binom{2012}{2} + 3^2\binom{2012}{4} + \cdots + 3^{1006}\binom{2012}{2012}\right]
\end{aligned}$$

On the other hand, using the De Moivre's theorem one has

$$\mathrm{Re}(\omega^{2012}) = \mathrm{Re}\left(\cos\frac{2012\pi}{3} + i\sin\frac{2012\pi}{3}\right) = \cos\frac{2012\pi}{3} = \cos\frac{2\pi}{3} = -\frac{1}{2}$$

Thus,

$$\frac{1}{2^{2012}}\left[1 - 3\binom{2012}{2} + 3^2\binom{2012}{4} + \cdots - 3^{1004}\binom{2012}{2010} + 3^{1006}\binom{2012}{2012}\right] = -\frac{1}{2}$$

so that

$$\frac{-1}{2^{2011}}\left[1 - 3\binom{2012}{2} + 3^2\binom{2012}{4} + \cdots - 3^{1004}\binom{2012}{2010} + 3^{1006}\binom{2012}{2012}\right] = 1$$

□

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2012

(Open Section, Round 2)

Saturday, 30 June 2012 **0900-1300**

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1. The incircle with centre I of the triangle ABC touches the sides BC, CA and AB at D, E and F respectively. The line ID intersects the segment EF at K. Prove that A, K and M are collinear where M is the midpoint of BC.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ so that

$$(x+y)(f(x)-f(y)) = (x-y)f(x+y)$$

for all $x, y \in \mathbb{R}$.

3. For each $i = 1, 2, \ldots, N$, let a_i, b_i, c_i be integers such that at least one of them is odd. Show that one can find integers x, y, z such that $xa_i + yb_i + zc_i$ is odd for at least $4N/7$ different values of i.

4. Let p be an odd prime. Prove that

$$1^{p-2} + 2^{p-2} + 3^{p-2} + \cdots + \left(\frac{p-1}{2}\right)^{p-2} \equiv \frac{2-2^p}{p} \pmod{p}.$$

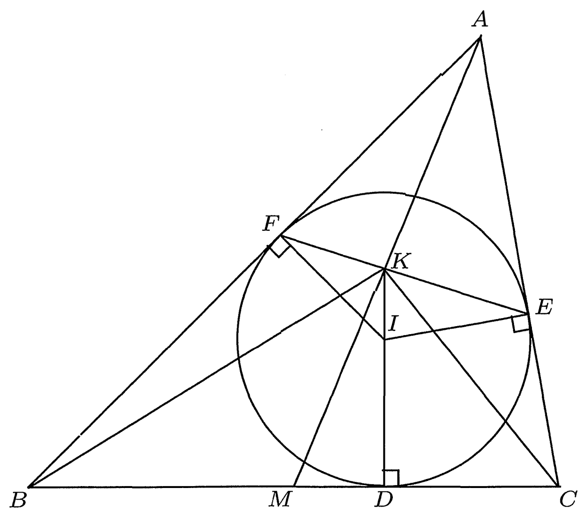
5. There are 2012 distinct points in the plane each of which is to be coloured using one of n colours so that the number of points of each colour are distinct. A set of n points is said to be multi-coloured if their colours are distinct. Determine n that maximizes the number of multi-coloured sets.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2012

(Open Section, Round 2 solutions)

1.



Let the line AK intersect BC at M. We shall prove that M is the midpoint of BC. Since $\angle FIK = \angle B$ and $\angle EIK = \angle C$, we have

$$\frac{FK}{EK} = \frac{\sin \angle FIK}{\sin \angle EIK} = \frac{\sin B}{\sin C}.$$

Also

$$\frac{FK}{\sin \angle FAK} = \frac{AF}{\sin \angle AKF} = \frac{AE}{\sin \angle AKE} = \frac{EK}{\sin \angle KAE}.$$

Therefore,

$$\frac{\sin \angle FAK}{\sin \angle KAE} = \frac{FK}{EK} = \frac{\sin B}{\sin C}.$$

Consequently,

$$\frac{BM}{CM} = \frac{BM}{AM} \cdot \frac{AM}{CM} = \frac{\sin \angle FAK}{\sin B} \cdot \frac{\sin C}{\sin \angle KAE} = 1,$$

so that $BM = CM$.

2. Suppose that f is a solution. Let

$$a = \frac{1}{2}(f(1) - f(-1)), \quad b = \frac{1}{2}(f(1) + f(-1))$$

and $g(x) = f(x) - ax - bx^2$. Then

$$(x + y)(g(x) - g(y)) = (x - y)g(x + y)$$

and $g(1) = g(-1) = 0$. Letting $y = 1$ and $y = -1$ above give

$$\begin{aligned}(x + 1)g(x) &= (x - 1)g(x + 1)\\ xg(x + 1) &= (x + 2)g(x).\end{aligned}$$

Thus

$$x(x + 1)g(x) = x(x - 1)g(x + 1) = (x - 1)(x + 2)g(x)$$

for all x. So $g(x) = 0$ for all x. Hence $f(x) = ax + bx^2$. We can check directly that any function of this form (for some $a, b \in \mathbb{R}$) satisfies the given equation.

3. Consider all the 7 triples (x, y, z), where x, y, z are either 0 or 1 but not all 0. For each i, at least one of the numbers a_i, b_i, c_i is odd. Thus among the 7 sums $xa_i+yb_i+zc_i$, 3 are even and 4 are odd. Hence there are altogether $4N$ odd sums. Thus there is choice of (x, y, z) for which at least $4N/7$ of the corresponding sums are odd. (You can think of a table where the rows are numbered $1, 2, \ldots, N$ and the columns correspond to the 7 choices of the triples (x, y, z). The 7 entries in row i are the 7 sums $xa_i + yb_i + zc_i$. Thus there are 4 odd numbers in each row, making a total of $4N$ odd sums in the table. Since there are 7 columns, one of the columns must contain at least $4N/7$ odd sums.)

4. First, for each $i = 1, 2, \cdots, \frac{p-1}{2}$,

$$\frac{2i}{p}\binom{p}{2i} = \frac{(p-1)(p-2)\cdots(p-(2i-1))}{(2i-1)!} \equiv \frac{(-1)(-2)\cdots(-(2i-1))}{(2i-1)!} \equiv -1 \pmod{p}.$$

Hence

$$\begin{aligned}\sum_{i=1}^{(p-1)/2} i^{p-2} &\equiv -\sum_{i=1}^{(p-1)/2} i^{p-2}\frac{2i}{p}\binom{p}{2i} \equiv -\frac{2}{p}\sum_{i=1}^{(p-1)/2} i^{p-1}\binom{p}{2i}\\ &\equiv -\frac{2}{p}\sum_{i=1}^{(p-1)/2}\binom{p}{2i} \pmod{p} \quad \text{(by Fermat's Little Theorem.)}\end{aligned}$$

The last summation counts the even-sized nonempty subsets of a p-element set, of which there are $2^{p-1} - 1$.

5. Let $m_1 < m_2 < \cdots < m_n$ be the number of points of each colour. We call $m_1, m_2, \ldots, m_n$ the colour distribution. Then $m_1 + \cdots + m_n = 2012$ and the number of multi-coloured sets is $M = m_1 m_2 \ldots m_n$. We have the following observations.

(i) $m_1 > 1$. For if $m_1 = 1$, then $m_1 m_2 \ldots m_n < m_2 m_3 \ldots m_{n-1}(1 + m_n)$. This means if we use $n - 1$ colours with colour distribution $m_2, m_3, \ldots, m_{n-1}, (1 + m_n)$, we obtain a larger M.

(ii) $m_{i+1} - m_i \leq 2$ for all i. For if there exists k with $m_{k+1} - m_k \geq 3$, then the colour distribution with m_k, m_{k+1} replaced by $m_k + 1, m_{k+1} - 1$ yields a larger M.

(iii) $m_{i+1} - m_i = 2$ for at most one i. For if there exist $i < j$ with $m_{i+1} - m_i = m_{j+1} - m_j = 2$, the colour distribution with m_i, m_{j+1} replaced by $m_i + 1, m_{j+1} - 1$ yields a larger M.

(iv) $m_{i+1} - m_i = 2$ for exactly one i. For if $m_{i+1} - m_i = 1$ for all i, then $m_1 + \cdots + m_n = nm_1 + \frac{n(n-1)}{2} = 2012 = 4 \cdot 503$. Thus $n(2m_1 - 1 + n) = 8 \cdot 503$. Since 503 is prime, the parity of n and $2m_1 - 1 + n$ are opposite and $2m_1 - 1 + n > n$, we have $n = 8$ and $m_1 = 248$. The colour distribution with m_1 replaced by two numbers $2, 246$ (using $n + 1$ colours) yields a larger M.

(v) $m_1 = 2$. If $m_n - m_{n-1} = 2$, then from (iv), we have $m_1 + \cdots + m_n = nm_1 + \frac{n(n-1)}{2} + 1 = 2012$. Thus $n(2m_1 - 1 + n) = 2 \cdot 2011$. Since 2011 is prime, we get $n = 2$ and $m_1 = 1005$ which will lead to a contradiction as in (iv). Thus $m_n - m_{n-1} = 1$. $m_{i+1} - m_i = 2$ for some $1 \leq i \leq n - 2$. Suppose $m_1 \geq 3$. Let $m' = m_{i+2} - 2$. Then $m_i < m' < m_{i+1}$ with replacing m_{i+2} by $2, m'$ yields a larger M. Thus $m_1 = 2$.

From the above analysis, with n colours, we see that the colour distribution 2, 3, …, $i - 1$, $i + 1$, $i + 2$, …, $n + 1$, $n + 2$, with $3 \leq i \leq n$, yields the maximum M. Now we have $\sum m_i = \frac{1}{2}(n + 1)(n + 4) - i = 2012$. Thus $n^2 + 5n - 4020 = 2i$, $3 \leq i \leq n$, i.e., $n^2 + 5n \geq 4026$ and $n^2 + 3n \leq 4020$. Thus $n = 61$ and $i = 3$. Thus the maximum is achieved when $n = 61$ with the colour distribution $2, 4, 5, 6, \ldots, 63$.