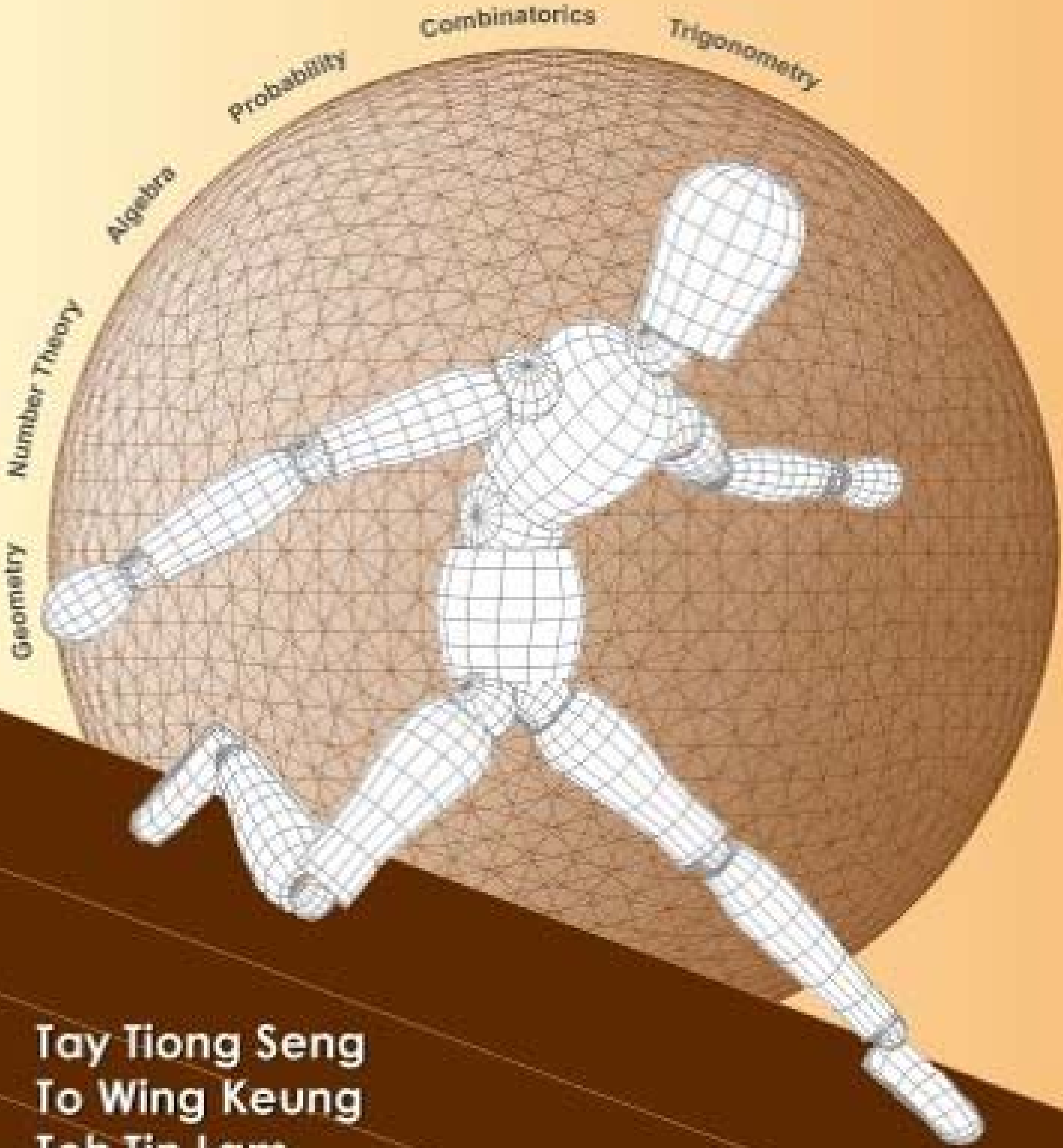
SINGAPORE MATHEMATICAL OLYMPIADS 2011



Tay Tiong Seng
To Wing Keung
Toh Tin Lam
Wang Fei

Published by Singapore Mathematical Society

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2011

(Junior Section)

Tuesday, 31 May 2011 | 0930–1200

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.

For the other short questions, write your answer on the answer sheet and shade the appropriate bubbles below your answer.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1. Calculate the following sum:

$$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\cdots+\frac{10}{2^{10}}.$$

(A) $\frac{503}{256}$; (B) $\frac{505}{256}$; (C) $\frac{507}{256}$; (D) $\frac{509}{256}$; (E) None of the above.

2. It is known that the roots of the equation

$$x^5+3x^4-4044118x^3-12132362x^2-12132363x-2011^2=0$$

are all integers. How many distinct roots does the equation have?

(A) 1; (B) 2; (C) 3; (D) 4; (E) 5.

3. A fair dice is thrown three times. The results of the first, second and third throw are recorded as x, y and z, respectively. Suppose $x+y=z$. What is the probability that at least one of x, y and z is 2?

(A) $\frac{1}{12}$; (B) $\frac{3}{8}$; (C) $\frac{8}{15}$; (D) $\frac{1}{3}$; (E) $\frac{7}{13}$.

4. Let

$$x=1\underbrace{000\cdots000}_{2011\text{ times}}1\underbrace{000\cdots000}_{2012\text{ times}}50.$$

Which of the following is a perfect square?

(A) $x-75$; (B) $x-25$; (C) x; (D) $x+25$; (E) $x+75$.

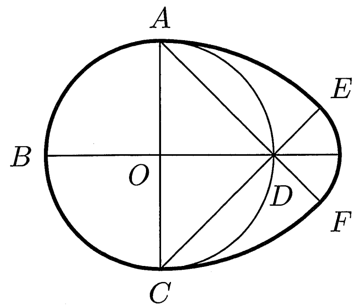
5. Suppose $N_1, N_2, \ldots, N_{2011}$ are positive integers. Let

$$X=(N_1+N_2+\cdots+N_{2010})(N_2+N_3+\cdots+N_{2011}),$$
$$Y=(N_1+N_2+\cdots+N_{2011})(N_2+N_3+\cdots+N_{2010}).$$

Which one of the following relationships always holds?

(A) $X=Y$; (B) $X>Y$; (C) $X<Y$; (D) $X-N_1<Y-N_{2011}$; (E) None of the above.

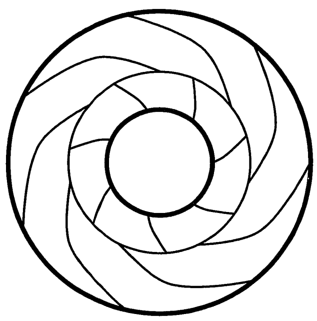
6. Consider the following egg shaped curve. $ABCD$ is a circle of radius 1 centred at O. The arc $\overset{\frown}{AE}$ is centred at C, $\overset{\frown}{CF}$ is centred at A and $\overset{\frown}{EF}$ is centred at D.



What is the area of the region enclosed by the egg shaped curve?

(A) $(3-\sqrt{2})\pi-1$; (B) $(3-\sqrt{2})\pi$; (C) $(3+\sqrt{2})\pi+1$; (D) $(3-2\sqrt{2})\pi$; (E) $(3-2\sqrt{2})\pi+1$.

7. The following annulus is cut into 14 regions. Each region is painted with one colour. What is the minimum number of colours needed to paint the annulus so that any no two adjacent regions share the same colours?



(A) 3; (B) 4; (C) 5; (D) 6; (E) 7.

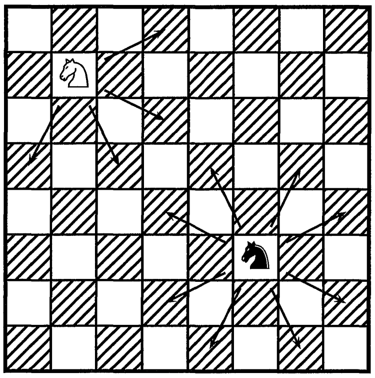
8. Let $n = (2^4-1)(3^6-1)(5^{10}-1)(7^{12}-1)$. Which of the following statements is true?

(A) n is divisible by $5, 7$ and 11 but not 13; (B) n is divisible by $5, 7$ and 13 but not 11;

(C) n is divisible by $5, 11$ and 13 but not 7; (D) n is divisible by $7, 11$ and 13 but not 5;

(E) None of the above.

9. How many ways can you place a White Knight and a Black Knight on an 8×8 chessboard such that they do not attack each other?



(A) 1680; (B) 1712; (C) 3696; (D) 3760; (E) None of the above.

10. In the set $\{1, 6, 7, 9\}$, which of the numbers appear as the last digit of n^n for infinitely many positive integers n?

(A) $1, 6, 7$ only; (B) $1, 6, 9$ only; (C) $1, 7, 9$ only; (D) $6, 7, 9$ only; (E) $1, 6, 7, 9$.

Short Questions

11. Suppose $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \sqrt{2}$ and $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0$. Find

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}.$$

12. Suppose $x = \frac{13}{\sqrt{19 + 8\sqrt{3}}}$. Find the exact value of

$$\frac{x^4 - 6x^3 - 2x^2 + 18x + 23}{x^2 - 8x + 15}.$$

13. Let $a_1 = 3$, and define $a_{n+1} = \frac{\sqrt{3}\, a_n - 1}{a_n + \sqrt{3}}$ for all positive integers n. Find a_{2011}.

14. Let a, b, c be positive real numbers such that

$$\begin{cases} a^2 + ab + b^2 = 25, \\ b^2 + bc + c^2 = 49, \\ c^2 + ca + a^2 = 64. \end{cases}$$

Find $(a + b + c)^2$.

15. Let $P(x)$ be a polynomial of degree 2010. Suppose $P(n) = \dfrac{n}{1+n}$ for all $n = 0, 1, 2, \ldots, 2010$. Find $P(2012)$.

16. Let $\lfloor x \rfloor$ be the greatest integer smaller than or equal to x. How many solutions are there to the equation $x^3 - \lfloor x^3 \rfloor = (x - \lfloor x \rfloor)^3$ on the interval $[1, 20]$?

17. Let n be the smallest positive integer such that the sum of its digits is 2011. How many digits does n have?

18. Find the largest positive integer n such that $n + 10$ is a divisor of $n^3 + 2011$.

19. Let a, b, c, d be real numbers such that

$$\begin{cases} a^2 + b^2 + 2a - 4b + 4 = 0, \\ c^2 + d^2 - 4c + 4d + 4 = 0. \end{cases}$$

Let m and M be the minimum and the maximum values of $(a-c)^2 + (b-d)^2$, respectively. What is $m \times M$?

20. Suppose $x_1, x_2, \ldots, x_{2011}$ are positive integers satisfying

$$x_1 + x_2 + \cdots + x_{2011} = x_1 x_2 \cdots x_{2011}.$$

Find the maximum value of $x_1 + x_2 + \cdots + x_{2011}$.

21. Suppose that a function $M(n)$, where n is a positive integer, is defined by

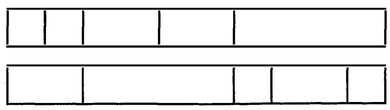
$$M(n) = \begin{cases} n - 10 & \text{if } n > 100, \\ M(M(n+11)) & \text{if } n \leq 100. \end{cases}$$

How many solutions does the equation $M(n) = 91$ have?

22. For each positive integer n, define $A_n = \dfrac{20^n + 11^n}{n!}$, where $n! = 1 \times 2 \times \cdots \times n$. Find the value of n that maximizes A_n.

23. Find the number of ways to pave a 1×10 block with tiles of sizes 1×1, 1×2 and 1×4, assuming tiles of the same size are indistinguishable. (For example, the following are two

distinct ways of using two tiles of size 1×1, two tiles of size 1×2 and one tile of size 1×4. It is not necessary to use all the three kinds of tiles.)



24. A 4×4 Sudoku grid is filled with digits so that each column, each row, and each of the four 2×2 sub-grids that composes the grid contains all of the digits from 1 to 4. For example,

4	3	1	2
2	1	3	4
1	2	4	3
3	4	2	1

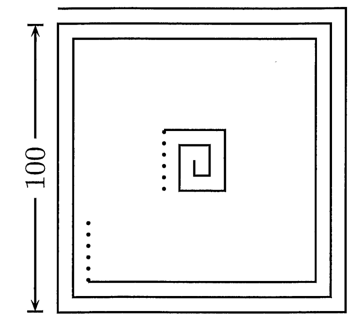
Find the total number of possible 4×4 Sudoku grids.

25. If the 13th of any particular month falls on a Friday, we call it *Friday the 13th.* It is known that Friday the 13th occurs at least once every calendar year. If the longest interval between two consecutive occurrences of Friday the 13th is x months, find x.

26. How many ways are there to put 7 identical apples into 4 identical packages so that each package has at least one apple?

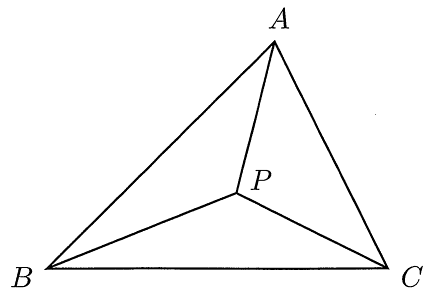
27. At a fun fair, coupons can be used to purchase food. Each coupon is worth \$5, \$8 or \$12. For example, for a \$15 purchase you can use three coupons of \$5, or use one coupon of \$5 and one coupon of \$8 and pay \$2 by cash. Suppose the prices in the fun fair are all whole dollars. What is the largest amount that you cannot purchase using only coupons?

28. Find the length of the spirangle in the following diagram, where the gap between adjacent parallel lines is 1 unit.



29. There are two fair dice and their sides are positive integers $a_1, \ldots, a_6$ and $b_1, \ldots, b_6$, respectively. After throwing them, the probability of getting a sum of $2, 3, 4, \ldots, 12$ respectively is the same as that of throwing two normal fair dice. Suppose that $a_1 + \cdots + a_6 < b_1 + \cdots + b_6$. What is $a_1 + \cdots + a_6$?

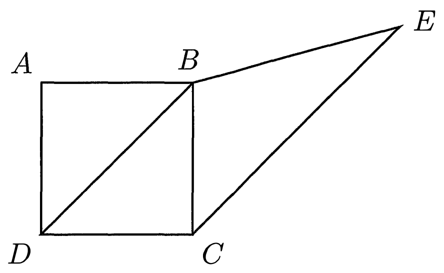
30. Consider a triangle ABC, where $AB = 20$, $BC = 25$ and $CA = 17$. P is a point on the plane. What is the minimum value of $2 \times PA + 3 \times PB + 5 \times PC$?



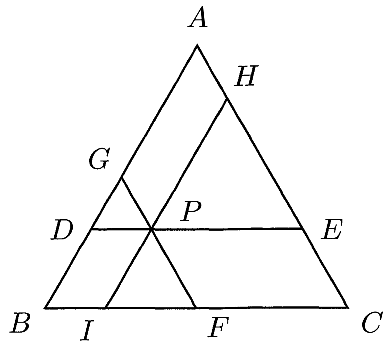
31. Given an equilateral triangle, what is the ratio of the area of its circumscribed circle to the area of its inscribed circle?

32. Let A and B be points that lie on the parabola $y = x^2$ such that both are at a distance of $8\sqrt{2}$ units from the line $y = -x - 4$. Find the square of the distance between A and B.

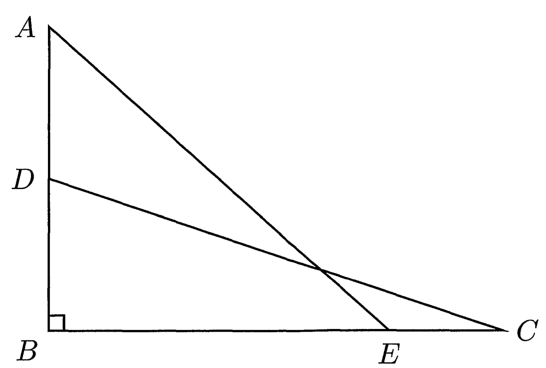
33. In the following diagram, $ABCD$ is a square, $BD \parallel CE$ and $BE = BD$. Let $\angle E = x^\circ$. Find x.



34. Consider an equilateral triangle ABC, where $AB = BC = CA = 2011$. Let P be a point inside $\triangle ABC$. Draw line segments passing through P such that $DE \parallel BC$, $FG \parallel CA$ and $HI \parallel AB$. Suppose $DE : FG : HI = 8 : 7 : 10$. Find $DE + FG + HI$.



35. In the following diagram, $AB \perp BC$. D and E are points on segments AB and BC respectively, such that $BA + AE = BD + DC$. It is known that $AD = 2$, $BE = 3$ and $EC = 4$. Find $BA + AE$.



Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2011

(Junior Section)

Multiple Choice Questions

1. Answer: (D)

   Note that $\frac{n}{2^n} = \frac{n+1}{2^{n-1}} - \frac{n+2}{2^n}$. Then

$$\begin{aligned}\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\cdots+\frac{10}{2^{10}} &= \left(\frac{2}{2^0}-\frac{3}{2^1}\right)+\left(\frac{3}{2^1}-\frac{4}{2^2}\right)+\left(\frac{4}{2^2}-\frac{5}{2^3}\right)+\cdots+\left(\frac{11}{2^9}-\frac{12}{2^{10}}\right)\\ &= \frac{2}{2^0}-\frac{12}{2^{10}}=\frac{509}{256}.\end{aligned}$$

2. Answer: (C).

   Let the roots be $n_1 \le n_2 \le n_3 \le n_4 \le n_5$. Then the polynomial can be factorized as

   $(x-n_1)(x-n_2)(x-n_3)(x-n_4)(x-n_5) = x^5-(n_1+n_2+n_3+n_4+n_5)x^4+\cdots-n_1n_2n_3n_4n_5.$

   Compare the coefficients: $n_1+n_2+n_3+n_4+n_5=-3$ and $n_1n_2n_3n_4n_5 = 2011^2$. Then

$$n_1=-2011, \quad n_2=n_3=n_4=-1, \quad n_5=2011.$$

3. Answer: (C).

   If $z=2$, then $(x,y)=(1,1)$.

   If $z=3$, then $(x,y)=(1,2),(2,1)$.

   If $z=4$, then $(x,y)=(1,3),(2,2),(3,1)$.

   If $z=5$, then $(x,y)=(1,4),(2,3),(3,2),(4,1)$.

   If $z=6$, then $(x,y)=(1,5),(2,4),(3,3),(4,2),(5,1)$.

   Out if these 15 cases, 8 of them contain at least one 2. Hence, the required probability is $\frac{8}{15}$.

4. Answer: (B).

Note that

$$\begin{aligned} x &= (10^{1012}+1)\times 10^{2014}+50 = 10^{4026}+10^{2014}+50 \\ &= (10^{2013})^2+2\times 10^{2013}\times 5+50 = (10^{2013}+5)^2+25. \end{aligned}$$

So $x-25$ is a perfect square.

5. Answer: (B).

Let $K = N_2+\cdots+N_{2010}$. Then $X=(N_1+K)(K+N_{2011})$ and $Y=(N_1+K+N_{2011})K$.

$$X-Y=(N_1K+K^2+N_1N_{2011}+KN_{2011})-(N_1K+K^2+N_{2011}K)=N_1N_{2011}>0.$$

6. Answer: (A).

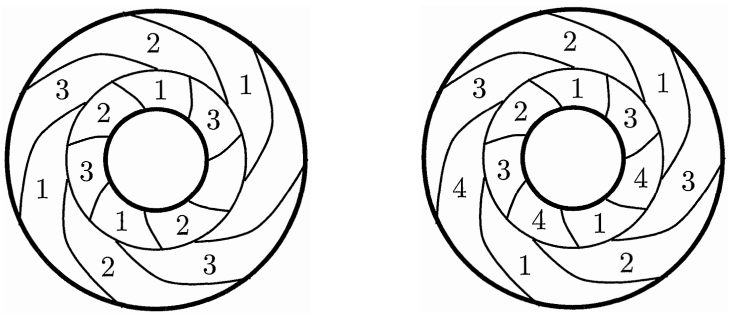
The area enclosed by AD, DE and $\overset{\frown}{AE}$ is $\dfrac{\pi(2^2)}{8}-1=\dfrac{\pi}{2}-1$.

The area of the wedge EDF is $\dfrac{\pi(2-\sqrt{2})^2}{4}=\left(\dfrac{3}{2}-\sqrt{2}\right)\pi$.

So the area of the egg is: $\dfrac{\pi}{2}+1+2\times\left(\dfrac{\pi}{2}-1\right)+\left(\dfrac{3}{2}-\sqrt{2}\right)\pi=(3-\sqrt{2})\pi-1$.

7. Answer: (B).

The left shows that 3 colours are not enough. The right is a painting using 4 colours.



8. Answer: (E).

Since $5 \mid (2^4-1)$, $7\mid(3^6-1)$, $11\mid(5^{10}-1)$, $13\mid(7^{12}-1)$, n is divisible by $5,7,11$ and 13.

9. Answer: (C).

We consider the position of the Black Knight. The number of positions being attacked by the White Knight can be counted.

2	3	4	4	4	4	3	2
3	4	6	6	6	6	4	3
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
3	4	6	6	6	6	4	3
2	3	4	4	4	4	3	2

There are $16 \times 8 + 16 \times 6 + 20 \times 4 + 8 \times 3 + 4 \times 2 = 336$ cases. Hence, the total number of cases that the Kights do not attack each other is $64 \times 63 - 336 = 3696$.

10. Answer: (E).

Note that the power of any positive integer n with last digit 1 or 6 is 1 or 6 respectively.

If the last digit of n is 9, then $n^2 \equiv 1 \pmod{10}$, and $n^n \equiv n^{10k+9} \equiv -1 \equiv 9 \pmod{10}$.

If the last digit of n is 7, then $n^4 \equiv 1 \pmod{10}$. Suppose the second last digit of n is odd. Then $n^n \equiv n^{20k+17} \equiv 7 \pmod{10}$.

Short Questions

11. Answer: 2.

$$2 = \left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right)^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + 2\left(\frac{xy}{ab} + \frac{yz}{bc} + \frac{zx}{ca}\right) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} + 2\,\frac{xyz}{abc}\left(\frac{c}{z} + \frac{a}{x} + \frac{b}{y}\right).$$

So $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 2$.

12. Answer: 5.

Note that $x = \dfrac{1}{13\sqrt{(4+\sqrt{3})^2}} = \dfrac{13(4-\sqrt{3})}{(4-\sqrt{3})(4+\sqrt{3})} = 4 - \sqrt{3}$. So $(x-4)^2 = 3$. That is,

$$x^2 - 8x + 15 = 2.$$

It follows that

$$\begin{aligned}
\frac{x^4 - 6x^3 - 2x^2 + 18x + 23}{x^2 - 8x + 15} &= x^2 + 2x - 1 + \frac{38 - 20x}{x^2 - 8x + 15} \\
&= x^2 + 2x - 1 + \frac{38 - 20x}{2} \\
&= x^2 - 8x + 18 \\
&= 2 + 3 = 5.
\end{aligned}$$

13. Answer: 3.

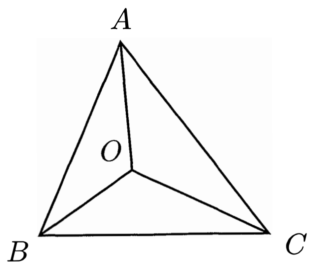
Let $f(x) = \dfrac{\sqrt{3}x - 1}{x + \sqrt{3}}$. Then $f(f(x)) = \dfrac{\sqrt{3}\frac{\sqrt{3}-1}{x+\sqrt{3}} - 1}{\frac{\sqrt{3}x-1}{x+\sqrt{3}} + \sqrt{3}} = \dfrac{3x - \sqrt{3} - x - \sqrt{3}}{\sqrt{3}x - 1 + \sqrt{3}x + 3} = \dfrac{x - \sqrt{3}}{\sqrt{3}x + 1}$.

$f(f(f(x))) = \dfrac{\frac{\sqrt{3}x-1}{x+\sqrt{3}} - \sqrt{3}}{\sqrt{3}\frac{\sqrt{3}x-1}{x+\sqrt{3}} + 1} = \dfrac{\sqrt{3}x - 1 - \sqrt{3}x - 3}{3x - \sqrt{3} + x + \sqrt{3}} = -\dfrac{1}{x}$. So $f(f(f(f(f(f(x)))))) = x$.

Since $2010 = 6 \times 335$, $a_{2011} = \underbrace{f(f(f \cdots f(f}_{\text{2010 copies}}(3)) \cdots)) = 3$.

14. Answer: 129.

Consider the following picture, where $\angle AOB = \angle BOC = \angle COA = 120^\circ$, $OA = a$, $OB = b$ and $OC = c$.



Then $|BC| = 5$, $|CA| = 7$ and $|AB| = 8$. The area of the triangle ABC is

$$\sqrt{10(10-5)(10-7)(10-8)} = 10\sqrt{3}.$$

Then $\frac{1}{2}\frac{\sqrt{3}}{2}(ab + bc + ca) = 10\sqrt{3}$. So $ab + bc + ca = 40$.

$$2(a+b+c)^2 = (a^2 + ab + b^2) + (b^2 + bc + c^2) + (c^2 + ca + a^2) + 3(ab + bc + ca) = 258.$$

Thus, $(a + b + c)^2 = 129$.

15. Answer: 0.

Define $Q(x) = (1 + x)P(x) - x$. Then $Q(x)$ is a polynomial of degree 2011. Since $Q(0) = Q(1) = Q(2) = \cdots = Q(2010) = 0$, we can write, for some constant A,

$$Q(x) = Ax(x-1)(x-2)\cdots(x-2010).$$

$1 = Q(-1) = A(-1)(-2)(-3)\cdots(-2011) = -A \cdot 2011!$. Then $Q(2012) = A \cdot 2012! = -2012$, and $P(2012) = \dfrac{Q(2012) + 2012}{2013} = 0$.

16. Answer: 9241.

Let $n = \lfloor x \rfloor$, $\{x\} = x - n$. The equation becomes $(n + \{x\})^3 - \{x\}^3 = \lfloor (n + \{x\})^3 \rfloor$. Then

$$3n\{x\}(n + \{x\}) = \lfloor 3n\{x\}(n + \{x\}) + \{x\}^3 \rfloor .$$

The right-hand side is an integer. The above holds if and only if $3n\{x\}(n+\{x\})$ is an integer.

Note that $0 \leq 3n\{x\}(n+\{x\}) < 3n(n+1)$. There are exactly $3n(n+1)$ solutions in $[n, n+1)$, $n = 1, 2, \ldots$. So on $[1, 20]$, the total number of solutions is

$$\begin{aligned} &3(1 \times 2 + 2 \times 3 + \cdots 20 \times 21) + 1 \\ =\ &(2^3 - 1^3) + (3^3 - 2^3) + \cdots + (21^3 - 20^3) - 20 + 1 \\ =\ &21^3 - 20 = 9241. \end{aligned}$$

17. Answer: 224.

For smallest possible n, we need to have 9 as the digits of n as many as possible. So n is the integer whose first digit is $2011 - 223 \times 9 = 4$ and followed by 223 9's.

18. Answer: 1001.

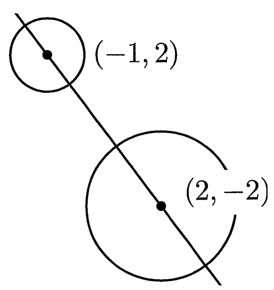
$\dfrac{n^3 + 2011}{n + 10} = n^2 - 10n + 100 + \dfrac{1011}{n + 10}$. This is an integer if and only if $(n + 10) \mid 1011$. The maximum value of n is $1011 - 10 = 1001$.

19. Answer: 16.

Complete the squares: $(a + 1)^2 + (b - 2)^2 = 1$ and $(c - 2)^2 + (d + 2)^2 = 2^2$. Each of them represents a circle.

The distance between the two centres is $\sqrt{(2 - (-1))^2 + (-2 - 2)^2} = 5$.

So $m = 5 - 1 - 2 = 2$ and $M = 5 + 1 + 2 = 8$. Thus, $m \times M = 16$.



20. Answer: 4022.

Suppose $x_1 = x_2 = \cdots = x_k = 1 < 2 \le x_{k+1} \le \cdots \le x_{2011}$. Let $M = x_1 \cdots x_{2011}$. Then

$$\begin{aligned}
M &= x_{k+1}x_{k+2}\cdots x_{2010}x_{2011} \\
&= (x_{k+1}-1)x_{k+2}\cdots x_{2010}x_{2011} + x_{k+2}\cdots x_{2010}x_{2011} \\
&\ge (x_{k+1}-1)2 + x_{k+2}\cdots x_{2010}x_{2011} \\
&\ge \cdots\cdots\cdots\cdots\cdots\cdots \\
&\ge (x_{k+1}-1)2 + \cdots + (x_{2009}-1)2 + x_{2010}x_{2011} \\
&\ge (x_{k+1}-1)2 + \cdots + (x_{2009}-1)2 + (x_{2010}-1)2 + (x_{2011}-1)2 \\
&= 2(x_{k+1} + \cdots + x_{2011} - (2011-k)) \\
&= 2(M - 2011).
\end{aligned}$$

Therefore, $M \le 4022$. On the other hand, $(1, 1, \ldots, 1, 2, 2011)$ is a solution to the equation. So the maximum value is 4022.

21. Answer: 101.

If $n \ge 102$, then $M(n) = n - 10 \ge 92$.

$M(91) = M(M(102)) = M(92) = M(M(103)) = M(93) = \cdots = M(101) = 91$.

For each $k = 1, \ldots, 10$, $M(80+k) = M(M(91+k)) = M(91) = 91$, and thus

$$\begin{gathered}
M(70+k) = M(M(81+k)) = M(91) = 91, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
M(k) = M(M(11+k)) = M(91) = 91.
\end{gathered}$$

Hence, all integers from 1 to 101 are solutions to $M(n) = 91$.

22. Answer: 19.

$\dfrac{A_{n+1}}{A_n} = \dfrac{1}{n+1}\dfrac{20^{n+1}+11^{n+1}}{20^n+11^n} = \dfrac{20 + 11\cdot(\frac{11}{20})^n}{(n+1)(1+(\frac{11}{20})^n)}$.

Then $A_{n+1} < A_n$ if $n > 10 + \dfrac{9}{1+(\frac{11}{20})^n}$; and $A_{n+1} > A_n$ if $n < 10 + \dfrac{9}{1+(\frac{11}{20})^n}$.

Note that $10 + \dfrac{9}{1+(\frac{11}{20})^n} < 10 + 9 = 19$. So $n \ge 19$ implies $A_n > A_{n+1}$.

If $10 \le n \le 18$, then $n \le 10+8 < 10+\dfrac{9}{1+(\frac{11}{20})^n}$; if $n < 10$, then $n < 10+\dfrac{9}{1+(\frac{11}{20})^n}$. Hence, $n \le 18$ implies $A_n < A_{n+1}$.

23. Answer: 169.

Let a_n be the number of ways to pave a block of $1 \times n$. Then $a_n = a_{n-1} + a_{n-2} + a_{n-4}$ with initial conditions $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ and $a_4 = 6$. Then

$$
\begin{aligned}
a_5 &= a_4 + a_3 + a_1 = 10, & a_6 &= a_5 + a_4 + a_2 = 18, \\
a_7 &= a_6 + a_5 + a_3 = 31, & a_8 &= a_7 + a_6 + a_4 = 55, \\
a_9 &= a_8 + a_7 + a_5 = 96, & a_{10} &= a_9 + a_8 + a_6 = 169.
\end{aligned}
$$

24. Answer: 288.

Consider the grid below. Suppose the left-top 2×2 sub-grid is filled in with $1, 2, 3, 4$.

If x, y, z, w are all distinct, then there are no other numbers to place in a; if $\{x, y\} = \{z, w\}$, then x', y', z, w are all distinct, and there are no other numbers for a'.

Note that $\{x, x'\} = \{1, 2\}$, $\{y, y'\} = \{3, 4\}$, $\{z, z'\} = \{2, 4\}$ and $\{w, w'\} = \{1, 3\}$. Among these $2^4 = 16$ choices, 4 of them are impossible — $\{x, y\} = \{z, w\} = \{1, 4\}$ or $\{2, 3\}$, $\{x, y\} = \{1, 4\}$ and $\{z, w\} = \{2, 3\}$, $\{x, y\} = \{2, 3\}$ and $\{z, w\} = \{1, 4\}$.

For each of the remaining 12 cases, x', y', z', w' are uniquely determined, so is the right-bottom sub-grid:

$$
\begin{aligned}
\{a\} &= \{1, 2, 3, 4\} - \{x, y\} \cup \{z, w\}, \\
\{b\} &= \{1, 2, 3, 4\} - \{x, y\} \cup \{z', w'\}, \\
\{a'\} &= \{1, 2, 3, 4\} - \{x', y'\} \cup \{z, w\}, \\
\{b'\} &= \{1, 2, 3, 4\} - \{x', y'\} \cup \{z', w'\}.
\end{aligned}
$$

Recall that that there are $4! = 24$ permutations in the left-top grid. Hence, there are $24 \times 12 = 288$ solutions.

1	2	x	x'
3	4	y	y'
z	w	a	a'
z'	w'	b	b'

25. Answer: 14.

If 13th of January falls on a particular day, represented by 0, then the 13th of February falls 3 days later, represented by $0 + 31 \equiv 3 \pmod 7$.

Case 1: The consecutive two years are non-leap years.

Jan	Feb	Mar	Apr	Mar	Jun	Jul	Aug	Sep	Oct	Nov	Dec
0	3	3	6	1	4	$\boxed{6}$	2	5	0	3	5
1	4	4	0	2	5	0	3	$\boxed{6}$	1	4	6

Case 2: The first year is a leap year.

Jan	Feb	Mar	Apr	Mar	Jun	Jul	Aug	Sep	Oct	Nov	Dec
0	3	4	0	2	5	$\boxed{0}$	3	6	1	4	6
2	5	5	1	3	6	1	4	$\boxed{0}$	2	5	0

Case 3: The second year is a leap year.

Jan	Feb	Mar	Apr	Mar	Jun	Jul	Aug	Sep	Oct	Nov	Dec
0	3	3	6	1	4	$\boxed{6}$	2	5	0	3	5
1	4	5	1	3	$\boxed{6}$	1	4	0	2	5	0

From these tables we see that the answer is 14. The longest time period occurs when the Friday of 13th falls in July of the first year and in September of the second year, while the second year is not a leap year.

26. Answer: 350.

By considering the numbers of apples in the packages, there are 3 cases:

1) $(4,1,1,1)$. $\binom{7}{4} = \frac{7\times 6\times 5\times 4}{4\times 3\times 2\times 1} = 35.$

2) $(3,2,1,1)$. $\binom{7}{3}\binom{4}{2} = \frac{7\times 6\times 5}{3\times 2\times 1} = \times\frac{4\times 3}{2\times 1} = 35\times 6 = 210.$

3) $(2,2,2,1)$. $\frac{1}{3!}\binom{7}{2}\binom{5}{2}\binom{3}{2} = \frac{1}{6}\times\frac{7\times 6}{2\times 1}\times\frac{5\times 4}{2\times 1}\times\frac{3\times 2}{2\times 1} = 105.$

So the total number of ways is $35+210+105=350$.

27. Answer: 19.

Note that $8+12=20$, $5+8+8=21$, $5+5+12=22$, $5+5+5+8=23$, $8+8+8=24$. If $n\geq 25$, write $n=5k+m$ where $20\leq m\leq 24$ and k is a positive integer. So any amount ≥ 25 can be paid exactly using coupons.

However, 19 cannot be paid exactly using these three types of coupons.

28. Answer: 10301.

The broken line is constructed using "L", with lengths $2,4,6,\ldots,200$. The last "L" is $100+101=201$. Then the total length is $2(1+2+3+\cdots+100)+201=10301$.

29. Answer: 15.

Let $P(x) = x^{a_1} + \cdots + x^{a_6}$ and $Q(x) = x^{b_1} + \cdots + x^{b_6}$. Then

$$P(x)Q(x) = (x + x^2 + \cdots + x^6)^2 = x^2(1+x)^2(1+x+x^2)^2(1-x+x^2)^2.$$

Note that $P(0) = Q(0) = 0$ and $P(1) = Q(1) = 6$, $x(1+x)(1+x+x^2)$ is a common divisor of $P(x)$ and $Q(x)$.

Since they are not normal dice,

$$P(x) = x(1+x)(1+x+x^2) = x + 2x^2 + 2x^3 + x^4,$$
$$Q(x) = x(1+x)(1+x+x^2)(1-x+x^2) = x + x^3 + x^4 + x^5 + x^6 + x^8.$$

So the numbers of the first dice are $1, 2, 2, 3, 3, 4$ and that of the second dice are $1, 3, 4, 5, 6, 8$.

Then $a_1 + \cdots + a_6 = 1 + 2 + 2 + 3 + 3 + 4 = 15$.

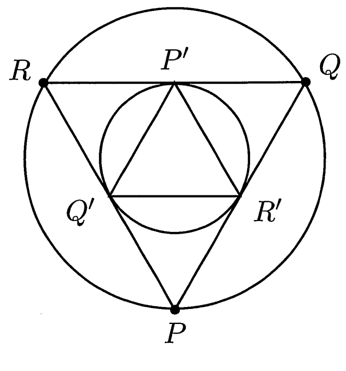
30. Answer: 109.

$2 \cdot PA + 3 \cdot PB + 5 \cdot PC = 2(PA + PC) + 3(PB + PC) \geq 2 \cdot AC + 3 \cdot BC = 2 \cdot 17 + 3 \cdot 25 = 109.$

The equality holds if and only if $P = C$.

31. Answer: 4.

Given an equilateral triangle PQR. Let C_1 be its circumscribed circle and C_2 its inscribed circle. Suppose QR, RP, PQ are tangent to C_2 at P', Q', R', respectively. The area of triangle PQR is 4 times the area of triangle $P'Q'R'$. So the area of C_1 is also 4 times the area of C_2.



32. Answer: 98.

Since $y = x^2$ and $y = -x - 4$ do not intersect, A and B must lie on a line parallel to $y = -x - 4$, namely, $y = -x + c$. The distance from $(0, -4)$ to $y = -x + c$ is

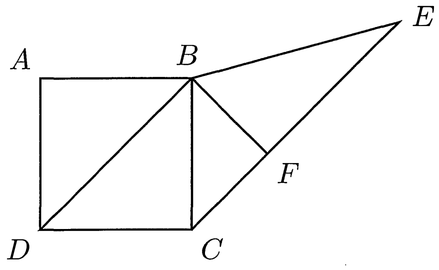
$$8\sqrt{2} = \frac{|(0) + (-4) - c|}{\sqrt{1+1}}.$$

So $c = 12$ or $c = -20$ (ignored). Substitute $y = -x + 12$ into the parabola: $x^2 = -x + 12 \Rightarrow x = 3, -4$. So A is $(3, 9)$ and B is $(-4, 16)$. Then

$$|AB|^2 = (3 - (-4))^2 + (9 - 16)^2 = 98.$$

33. Answer: 30.

Draw $BF \perp CE$, where F is on CE. If $AB = 1$, then $BF = \frac{\sqrt{2}}{2}$ and $BE = \sqrt{2}$. Thus $\angle E = 30°$.



34. Answer: 4022.

Set $DP = GP = a$, $IP = FP = b$, $EP = HP = c$. Then

$$DE + FG + HI = (a + c) + (a + b) + (b + c) = 2(a + b + c) = 2 \times 2011 = 4022.$$

35. Answer: 10.

By given, $BD + 2 + AE = BD + DC$. So $2 + AE = DC$.

Note that $AB^2 + BE^2 = AE^2$ and $BD^2 + BC^2 = DC^2$. Then

$$(2 + BD)^2 + 3^2 = AE^2, \quad BD^2 + 7^2 = (AE + 2)^2.$$

$4(AE + BD) = 32$. Then $AE + BA = \frac{32}{4} + 2 = 10$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2011

(Junior Section, Round 2)

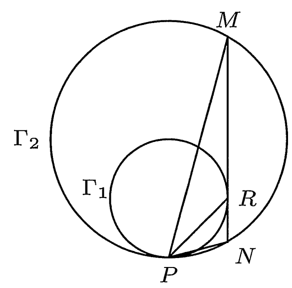
Saturday, 25 June 2011 **0930-1230**

---

1. Suppose $a, b, c, d > 0$ and $x = \sqrt{a^2 + b^2}$, $y = \sqrt{c^2 + d^2}$. Prove that

$$xy \geq ac + bd.$$

2. Two circles Γ_1, Γ_2 with radii r_1, r_2, respectively, touch internally at the point P. A tangent parallel to the diameter through P touches Γ_1 at R and intersects Γ_2 at M and N. Prove that PR bisects $\angle MPN$.



3. Let $S_1, S_2, \ldots, S_{2011}$ be nonempty sets of consecutive integers such that any 2 of them have a common element. Prove that there is an integer that belongs to every S_i, $i = 1, \ldots, 2011$. (For example, $\{2, 3, 4, 5\}$ is a set of consecutive integers while $\{2, 3, 5\}$ is not.)

4. Any positive integer n can be written in the form $n = 2^a q$, where $a \geq 0$ and q is odd. We call q the *odd part* of n. Define the sequence $a_0, a_1, \ldots,$ as follows: $a_0 = 2^{2011} - 1$ and for $m \geq 0$, a_{m+1} is the odd part of $3a_m + 1$. Find a_{2011}.

5. Initially, the number 10 is written on the board. In each subsequent moves, you can either (i) erase the number 1 and replace it with a 10, or (ii) erase the number 10 and replace it with a 1 and a 25 or (iii) erase a 25 and replace it with two 10. After sometime, you notice that there are exactly one hundred copies of 1 on the board. What is the least possible sum of all the numbers on the board at that moment?

Singapore Mathematical Society

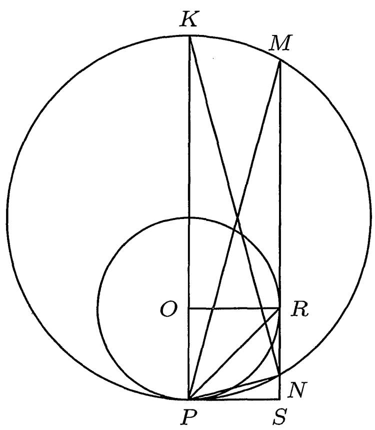
Singapore Mathematical Olympiad (SMO) 2011

(Junior Section, Round 2 solutions)

1. It's equivalent to prove $x^2y^2 \geq (ac+bd)^2$ as all the numbers are nonnegative. This is true since

$$\begin{aligned} x^2y^2 &= (a^2+b^2)(c^2+d^2) \\ &= (ac)^2+(bd)^2+a^2d^2+b^2c^2 \\ &\geq (ac)^2+(bd)^2+2(ac)(bd) \qquad \text{AM-GM} \\ &= (ac+bd)^2. \end{aligned}$$

2. Let the tangent at P meet the tangent at R at the point S. Let O be the centre of Γ_1. Then $ORST$ is a square. Hence $\angle KPR = \angle RPS = 45°$. Also $\angle NPS = \angle NKP = \angle PMS = \angle MPK$. Thus $\angle MPR = \angle RPN$.



3. Let $a_i = \max S_i$, $b_i = \min S_i$ and suppose that $t_1 = \min\{t_i\}$. For each j, if $S_1 \cap S_j \neq \emptyset$, then $a_1 \geq b_j$. Therefore $a_1 \in S_j$.

Note: Problem 4 in the Senior Section is the general version.

4. Replace 2011 by any positive odd integer n. We first show by induction that $a_m = 3^m 2^{n-m} - 1$ for $m = 0, 1, \ldots, n-1$. This is certainly true for $m = 0$. Suppose it's true for some $m < n-1$. Then $3a_m + 1 = 3^{m+1}2^{n-m} - 2$. Since $n - m > 1$, the odd part is $3^{m+1}2^{n-m-1} - 1$ which is a_{m+1}. Now $a_{n-1} = 3^{n-1}2^1 - 1$. Thus

$3a_{n-1}+1 = 3^n 2 - 2 = 2(3^n - 1)$. When n is odd, $3^n \equiv -1 \pmod 4$. Thus $4 \nmid 3^n - 1$. Hence the odd part of $2(3^n-1)$ is $\frac{3^n-1}{2}$ and this is the value of a_n.

5. Suppose the number of times that operations (i), (ii) and (iii) have been performed are x, y and z, respectively. Then the number of 1, 10 and 25 are $y-x$, $1+x-y+2z$ and $y-z$, respectively, with $-x+y=100$. Thus the sum is

$$S = y - x + 10(1+x-y+2z) + 25(y-z) = -890 + 5(5y-z).$$

Since we want the minimum values of S, y has to be as small as possible and z as large as possible. Since

$$y - x = 100,\ 1 + x - y + 2z \geq 0, \quad y - z \geq 0$$

we get, from the first equation, $y \geq 100$, from the second inequality, $2z \geq 99$ or $z \geq 50$ and $y \geq z$ from the third. Thus the minimum is achieved when $y = 100$, $x = 0$ and $z = 100$. Thus minimum $S = 1100$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2011

(Senior Section)

Tuesday, 31 May 2011 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 35 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Multiple Choice Questions

1. Suppose a, b and c are nonzero real numbers such that

$$\frac{a^2}{b^2+c^2} < \frac{b^2}{c^2+a^2} < \frac{c^2}{a^2+b^2}.$$

Which of the following statements is always true?

(A) $a < b < c$ (B) $|a| < |b| < |c|$ (C) $c < b < a$
(D) $|b| < |c| < |a|$ (E) $|c| < |b| < |a|$

2. Suppose θ is an angle between 0 and $\frac{\pi}{2}$, and $\sin 2\theta = a$. Which of the following expressions is equal to $\sin\theta + \cos\theta$?

(A) $\sqrt{a+1}$ (B) $(\sqrt{2}-1)a+1$ (C) $\sqrt{a+1}-\sqrt{a^2-a}$
(D) $\sqrt{a+1}+\sqrt{a^2-a}$ (E) None of the above

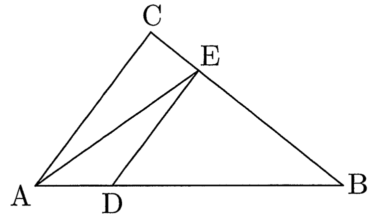
3. Let x be a real number. If $a = 2011x + 9997$, $b = 2011x + 9998$ and $c = 2011x + 9999$, find the value of $a^2 + b^2 + c^2 - ab - bc - ca$.

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

4. Suppose x, y are real numbers such that $\frac{1}{x} - \frac{1}{2y} = \frac{1}{2x+y}$. Find the value of $\frac{y^2}{x^2} + \frac{x^2}{y^2}$.

(A) $\frac{2}{3}$ (B) $\frac{9}{2}$ (C) $\frac{9}{4}$ (D) $\frac{4}{9}$ (E) $\frac{2}{9}$

5. In the figure below, ABC is a triangle, and D and E are points on AB and BC respectively. It is given that DE is parallel to AC, and $CE : EB = 1 : 3$. If the area of $\triangle ABC$ is 1440 cm^2 and the area of $\triangle ADE$ is x cm^2, what is the value of x?



(A) 288 (B) 240 (C) 320 (D) 384 (E) 270

6. Determine the value of

$$\frac{2}{\dfrac{1}{\sqrt{2}+\sqrt[4]{8}+2}+\dfrac{1}{\sqrt{2}-\sqrt[4]{8}+2}}.$$

(A) $4-\sqrt{2}$ (B) $2-2\sqrt{2}$ (C) $4+\sqrt{2}$ (D) $2\sqrt{2}+4$
(E) $4\sqrt{2}-2$

7. Let $x=\dfrac{1}{\log_{\frac{1}{3}}\frac{1}{2}}+\dfrac{1}{\log_{\frac{1}{5}}\frac{1}{4}}+\dfrac{1}{\log_{\frac{1}{7}}\frac{1}{8}}$. Which of the following statements is true?

(A) $1.5 < x < 2$ (B) $2 < x < 2.5$ (C) $2.5 < x < 3$
(D) $3 < x < 3.5$ (E) $3.5 < x < 4$

8. Determine the last two digits of 7^{5^6}.

(A) 01 (B) 07 (C) 09 (D) 43 (E) 49

9. It is given that x and y are two real numbers such that $x > 1$ and $y > 1$. Find the smallest possible value of

$$\frac{\log_x 2011+\log_y 2011}{\log_{xy} 2011}.$$

(A) 4 (B) 6 (C) 8 (D) 10 (E) 12

10. It is given that a, b and c are three real numbers such that $a+b=c-1$ and $ab=c^2-7c+14$. Find the largest possible value of a^2+b^2.

(A) 5 (B) 6 (C) 8 (D) 9 (E) 10

Short Questions

11. Find the value of

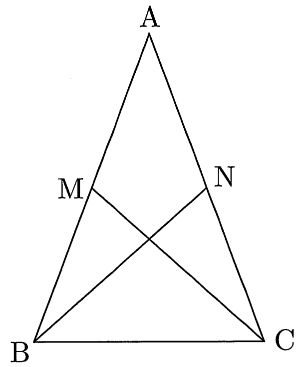
$$\frac{2011^2+2111^2-2\times 2011\times 2111}{25}.$$

12. Find the largest natural number n which satisfies the inequality

$$n^{6033} < 2011^{2011}.$$

13. Find the integer which is closest to $\dfrac{(1+\sqrt{3})^4}{4}$.

14. In the diagram below, $\triangle ABC$ is an isosceles triangle with $AB = AC$, and M and N are the midpoints of AB and AC respectively. It is given that CM is perpendicular to BN, $BC = 20$ cm, and the area of $\triangle ABC$ is x cm^2. Find the value of x.



15. Find the smallest positive integer n such that

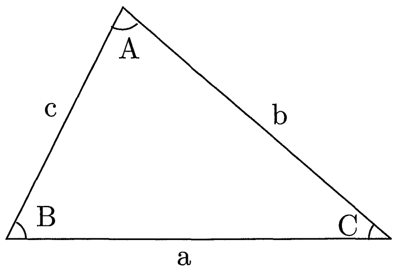
$$\sqrt{5n} - \sqrt{5n-4} < 0.01.$$

16. Find the value of

$$\frac{1}{1+11^{-2011}} + \frac{1}{1+11^{-2009}} + \frac{1}{1+11^{-2007}} + \cdots + \frac{1}{1+11^{2009}} + \frac{1}{1+11^{2011}}.$$

17. Let $x = \sin^4(\frac{\pi}{8}) + \cos^4(\frac{\pi}{8}) + \sin^4(\frac{7\pi}{8}) + \cos^4(\frac{7\pi}{8})$. Find the value of $36x$.

18. In the diagram below, the lengths of the three sides of the triangle are a cm, b cm and c cm. It is given that $\dfrac{a^2+b^2}{c^2} = 2011$. Find the value of $\dfrac{\cot C}{\cot A + \cot B}$.



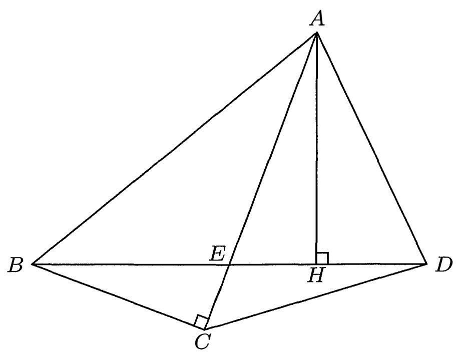
19. Suppose there are a total of 2011 participants in a mathematics competition, and at least 1000 of them are female. Moreover, given any 1011 participants, at least 11 of them are male. How many male participants are there in this competition?

20. Let $f : \mathbb{Q} \setminus \{0,1\} \to \mathbb{Q}$ be a function such that

$$x^2 f(x) + f\left(\frac{x-1}{x}\right) = 2x^2$$

for all rational numbers $x \neq 0, 1$. Here $\mathbb{Q}$ denotes the set of rational numbers. Find the value of $f(\frac{1}{2})$.

21. In the diagram below, $ABCD$ is a convex quadrilateral such that AC intersects BD at the midpoint E of BD. The point H is the foot of the perpendicular from A onto DE, and H lies in the interior of the segment DE. Suppose $\angle BCA = 90°$, $CE = 12$ cm, $EH = 15$ cm, $AH = 40$ cm and $CD = x$ cm. Find the value of x.



22. How many pairs of integers (x, y) are there such that

$$x \geq y \quad \text{and} \quad \frac{1}{x} + \frac{1}{y} = \frac{1}{211}?$$

23. It is given that a, b, c are three real numbers such that the roots of the equation $x^2 + 3x - 1 = 0$ also satisfy the equation $x^4 + ax^2 + bx + c = 0$. Find the value of $a + b + 4c + 100$.

24. It is given that m and n are two positive integers such that

$$n - \frac{m}{n} = \frac{2011}{3}.$$

Determine the smallest possible value of m.

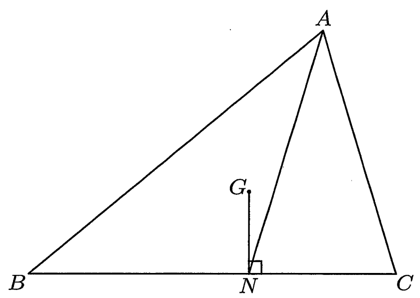
25. It is given that a, b, c are positive integers such that the roots of the three quadratic equations

$$x^2 - 2ax + b = 0, \quad x^2 - 2bx + c = 0, \quad x^2 - 2cx + a = 0$$

are all positive integers. Determine the maximum value of the product abc.

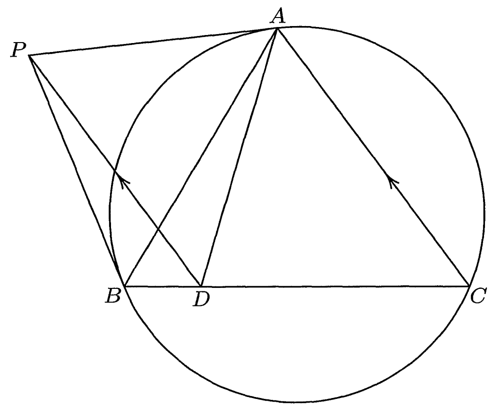
26. Suppose A, B, C are three angles such that $A \geq B \geq C \geq \frac{\pi}{8}$ and $A + B + C = \frac{\pi}{2}$. Find the largest possible value of the product $720 \times (\sin A) \times (\cos B) \times (\sin C)$.

27. In the diagram below, ABC is a triangle such that AB is longer than AC. The point N lies on BC such that AN bisects $\angle BAC$. The point G is the centroid of $\triangle ABC$, and it is given that GN is perpendicular to BC. Suppose $AC = 6$ cm, $BC = 5\sqrt{3}$ cm and $AB = x$ cm. Find the value of x.



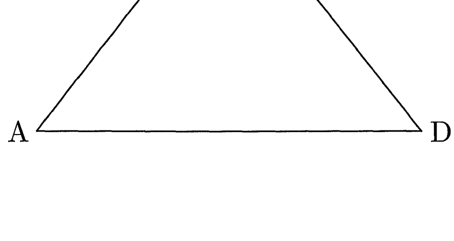
28. It is given that a, b, c and d are four positive prime numbers such that the product of these four prime numbers is equal to the sum of 55 consecutive positive integers. Find the smallest possible value of $a+b+c+d$. (Remark: The four numbers a, b, c, d are not necessarily distinct.)

29. In the diagram below, ABC is a triangle with $AB = 39$ cm , $BC = 45$ cm and $CA = 42$ cm. The tangents at A and B to the circumcircle of $\triangle ABC$ meet at the point P. The point D lies on BC such that PD is parallel to AC. It is given that the area of $\triangle ABD$ is x cm^2. Find the value of x.



30. It is given that a and b are positive integers such that a has exactly 9 positive divisors and b has exactly 10 positive divisors. If the least common multiple (LCM) of a and b is 4400, find the value of $a + b$.

31. In the diagram below, $ABCD$ is a quadrilateral such that $\angle ABC = 135°$ and $\angle BCD = 120°$. Moreover, it is given that $AB = 2\sqrt{3}$ cm, $BC = 4 - 2\sqrt{2}$ cm, $CD = 4\sqrt{2}$ cm and $AD = x$ cm. Find the value of $x^2 - 4x$.



32. It is given that p is a prime number such that

$$x^3 + y^3 - 3xy = p - 1$$

for some positive integers x and y. Determine the largest possible value of p.

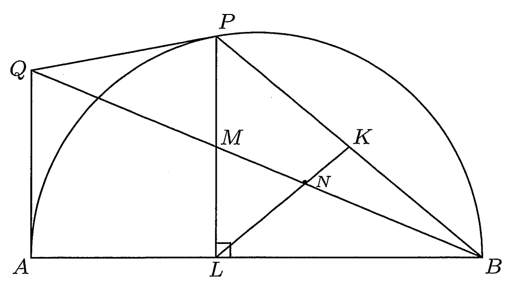
33. It is given that a, b and c are three positive integers such that

$$a^2 + b^2 + c^2 = 2011.$$

Let the highest common factor (HCF) and the least common multiple (LCM) of the three numbers a, b, c be denoted by x and y respectively. Suppose that $x + y = 388$. Find the value of $a + b + c$. (Remark: The highest common factor is also known as the greatest common divisor.)

34. Consider the set $S = \{1, 2, 3, \cdots, 2010, 2011\}$. A subset T of S is said to be a k-element RP-subset if T has exactly k elements and every pair of elements of T are relatively prime. Find the smallest positive integer k such that every k-element RP-subset of S contains at least one prime number. (As an example, $\{1, 8, 9\}$ is a 3-element RP-subset of S which does not contain any prime number.)

35. In the diagram below, P is a point on the semi-circle with diameter AB. The point L is the foot of the perpendicular from P onto AB, and K is the midpoint of PB. The tangents to the semicircle at A and at P meet at the point Q. It is given that PL intersects QB at the point M, and KL intersects QB at the point N. Suppose $\dfrac{AQ}{AB} = \dfrac{5}{12}$, $QM = 25$ cm and $MN = x$ cm. Find the value of x.



Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2011

(Senior Section Solutions)

1. Answer: (B)

   Since a, b and c are nonzero real numbers and $\dfrac{a^2}{b^2+c^2} < \dfrac{b^2}{c^2+a^2} < \dfrac{c^2}{a^2+b^2}$, we see that
   $$\frac{b^2+c^2}{a^2} > \frac{c^2+a^2}{b^2} > \frac{a^2+b^2}{c^2}.$$
   Adding 1 throughout, we obtain
   $$\frac{a^2+b^2+c^2}{a^2} > \frac{a^2+b^2+c^2}{b^2} > \frac{a^2+b^2+c^2}{c^2}.$$
   Thus $\dfrac{1}{a^2} > \dfrac{1}{b^2} > \dfrac{1}{c^2}$, which implies that $a^2 < b^2 < c^2$. So we have $|a| < |b| < |c|$.

2. Answer: (A)

   Note that
   $$(\sin\theta + \cos\theta)^2 = \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = 1 + \sin 2\theta = 1 + a.$$
   Since $0 \le \theta \le \frac{\pi}{2}$, we have $\sin\theta + \cos\theta > 0$. So $\sin\theta + \cos\theta = \sqrt{1+a}$.

3. Answer: (D)

   We have
   $$\begin{aligned} a^2+b^2+c^2-ab-bc-ca &= \frac{1}{2}\cdot\left[(a-b)^2+(b-e)^2+(c-a)^2\right] \\ &= \frac{1}{2}\cdot\left[(-1)^2+(-1)^2+2^2\right] = 3. \end{aligned}$$

4. Answer: (C)
   $$\begin{aligned} \frac{1}{x} - \frac{1}{2y} = \frac{1}{2x+y} \quad &\Rightarrow \quad \frac{2x+y}{x} - \frac{2x+y}{2y} = 1 \\ &\Rightarrow \quad 2 + \frac{y}{x} - \frac{x}{y} - \frac{1}{2} = 1 \\ &\Rightarrow \quad \frac{y}{x} - \frac{x}{y} = -\frac{1}{2}. \end{aligned}$$
   Now we have
   $$\frac{y^2}{x^2} + \frac{x^2}{y^2} = \left(\frac{y}{x} - \frac{x}{y}\right)^2 + 2 = \left(-\frac{1}{2}\right)^2 + 2 = \frac{9}{4}.$$

5. Answer: (E)

The area of $\triangle ABC$ is given to be $S = 1440$ cm^2. Let S_1 and S_2 denote the areas of $\triangle ADE$ and $\triangle DBE$ respectively. Since DE is parallel to AC, $\triangle DBE$ and $\triangle ABC$ are similar. Therefore
$$\frac{S_2}{S} = \left(\frac{BE}{BC}\right)^2 = \left(\frac{3}{4}\right)^2 = \frac{9}{16}.$$
Thus $S_2 = \frac{9}{16}S$. As DE is parallel to AC, we have $AD : DB = CE : EB = 1 : 3$. Consequently,
$$\frac{S_1}{S_2} = \frac{1}{3}.$$
Hence $S_1 = \dfrac{1}{3}S_2 = \dfrac{1}{3}\cdot\dfrac{9}{16}S = \dfrac{3}{16}\times 1440 = 270$ cm^2.

6. Answer: (A)

Let $x = 2^{\frac{1}{4}}$. Then

$$\begin{aligned}
\frac{2}{\frac{1}{2^{\frac{1}{2}}+2^{\frac{3}{4}}+2}+\frac{1}{2^{\frac{1}{2}}-2^{\frac{3}{4}}+2}} &= \frac{2}{\frac{1}{x^2+x^3+2}+\frac{1}{x^2-x^3+2}} \\
&= \frac{2}{\frac{2(x^2+2)}{(x^2+2)^2-x^6}} \\
&= \frac{(x^2+2)^2-x^6}{x^2+2} \\
&= \frac{(\sqrt{2}+2)^2-2\sqrt{2}}{\sqrt{2}+2} \\
&= \frac{6+2\sqrt{2}}{\sqrt{2}+2}\times\frac{2-\sqrt{2}}{2-\sqrt{2}} \\
&= \frac{8-2\sqrt{2}}{2} \\
&= 4-\sqrt{2}.
\end{aligned}$$

7. Answer: (E)

$$\begin{aligned}
x &= \frac{\log(\frac{1}{3})}{\log(\frac{1}{2})}+\frac{\log(\frac{1}{5})}{\log(\frac{1}{4})}+\frac{\log(\frac{1}{7})}{\log(\frac{1}{8})} = \frac{-\log 3}{-\log 2}+\frac{-\log 5}{-\log 4}+\frac{-\log 7}{-\log 8} \\
&= \frac{\log 3+\log 5^{\frac{1}{2}}+\log 7^{\frac{1}{3}}}{\log 2} \\
&= \frac{\log\sqrt{45}+\log 7^{\frac{1}{3}}}{\log 2} < \frac{\log\sqrt{64}+\log 8^{\frac{1}{3}}}{\log 2} = \frac{3\log 2+\log 2}{\log 2} = 4.
\end{aligned}$$

Moreover,

$$\begin{aligned}
2x &= \frac{2(\log 3 + \log 5^{\frac{1}{2}} + \log 7^{\frac{1}{3}})}{\log 2} \\
&= \frac{\log(9 \times 5) + \log(49^{\frac{1}{3}})}{\log 2} > \frac{\log(45 \times 27^{\frac{1}{3}})}{\log 2} \\
&= \frac{\log(45 \times 3)}{\log 2} > \frac{\log(128)}{\log 2} = 7,
\end{aligned}$$

so $x > 3.5$.

8. Answer: (B)

Note that $7^4 - 1 = 2400$, so that $7^{4n} - 1$ is divisible by 100 for any $n \in \mathbb{Z}^+$. Now,

$$\begin{aligned}
7^{5^6} &= 7(7^{5^6-1} - 1 + 1) \\
&= 7(7^{5^6-1} - 1) + 7 \\
&= 7(7^{4n} - 1) + 7,
\end{aligned}$$

where

$$n = \frac{5^6 - 1}{4} \in \mathbb{Z}^+.$$

Since $7(7^{4n} - 1)$ is divisible by 100, its last two digits are 00. It follows that the last two digits of 7^{5^6} are 07.

9. Answer: (A)

$$\begin{aligned}
\frac{\log_x 2011 + \log_y 2011}{\log_{xy} 2011} &= (\frac{\log 2011}{\log x} + \frac{\log 2011}{\log y}) \cdot (\frac{\log xy}{\log 2011}) \\
&= (\frac{1}{\log x} + \frac{1}{\log y}) \cdot (\log x + \log y) \\
&= 2 + \frac{\log x}{\log y} + \frac{\log y}{\log x} \\
&\geq 4 \ \text{(using } AM \geq GM\text{)},
\end{aligned}$$

and the equality is attained when $\log x = \log y$, or equivalently, $x = y$.

10. Answer: (C)

The roots of the equation $x^2 - (c-1)x + c^2 - 7c + 14 = 0$ are a and b, which are real. Thus the discriminant of the equation is non-negative. In other words,

$$(c-1)^2 - 4(c^2 - 7c + 14) = -3c^2 + 26c - 55 = (-3c + 11)(c - 5) \geq 0.$$

So we have $\frac{11}{3} \leq c \leq 5$. Together with the equalities

$$\begin{aligned}
a^2 + b^2 &= (a+b)^2 - 2ab \\
&= (c-1)^2 - 2(c^2 - 7c + 14) \\
&= -c^2 + 12c - 27 = 9 - (c-6)^2,
\end{aligned}$$

we see that maximum value of $a^2 + b^2$ is 8 when $c = 5$.

11. Answer: 400

$$\frac{2011^2 + 2111^2 - 2 \times 2011 \times 2111}{25} = \frac{(2011 - 2111)^2}{25} = \frac{100^2}{25} = 400.$$

12. Answer: 12

Since $6033 = 3 \times 2011$, we have

$$\begin{aligned} n^{6033} < 2011^{2011} &\iff n^{3\times 2011} < 2011^{2011} \\ &\iff n^3 < 2011. \end{aligned}$$

Note that $12^3 = 1728$ and $13^3 = 2197 > 2011$. Thus, the largest possible natural number n satisfying the given inequality is 12.

13. Answer: 14

$$(1 + \sqrt{3})^2 = 1^2 + 3 + 2\sqrt{3} = 4 + 2\sqrt{3}.$$

$$\begin{aligned} (1 + \sqrt{3})^4 = ((1 + \sqrt{3})^2)^2 = (4 + 2\sqrt{3})^2 &= 4^2 + (2\sqrt{3})^2 + 2 \cdot 4 \cdot 2\sqrt{3} \\ &= 16 + 12 + 16\sqrt{3} \\ &= 28 + 16\sqrt{3}. \end{aligned}$$

Hence,

$$\frac{(1 + \sqrt{3})^4}{4} = \frac{28 + 16\sqrt{3}}{4} = 7 + 4\sqrt{3}.$$

Note that $1.7 < \sqrt{3} < 1.8$. Thus,

$$\begin{aligned} & 7 + 4 \times 1.7 < 7 + 4\sqrt{3} < 7 + 4 \times 1.8 \\ \Longrightarrow \quad & 13.8 < 7 + 4\sqrt{3} < 14.2. \end{aligned}$$

Therefore, the integer closest to $\frac{(1+\sqrt{3})^4}{4}$ is 14.

14. Answer: 300

Let P be the intersection of CM and BN, so that P is the centroid of $\triangle ABC$. Then $BP = 2PN$ and $CP = 2PM$. Let $PN = PM = y$, so that $BP = CP = 2y$. Since $\angle BPC = 90^\circ$, we have $BC = 20 = \sqrt{(2y)^2 + (2y)^2}$ and thus $y = \sqrt{50}$. Now

$$AB^2 = 4BM^2 = 4((2y)^2 + y^2) = 20y^2 = 1000.$$

Thus the altitude of $\triangle ABC$ is $\sqrt{AB^2 - 10^2} = 30$. Hence the area of $\triangle ABC$ is $\frac{1}{2} \times 20 \times 30 = 300$.

15. Answer: 8001

Note that $\sqrt{5n}-\sqrt{5n-4}<0.01$ if and only if

$$\sqrt{5n}+\sqrt{5n-4}=\frac{4}{\sqrt{5n}-\sqrt{5n-4}}>400.$$

If $n=8000$, then $\sqrt{5n}+\sqrt{5n-4}=\sqrt{40000}+\sqrt{39996}<400$.

If $n=8001$, then $\sqrt{5n}+\sqrt{5n-4}=\sqrt{40005}+\sqrt{40001}>400$.

So the answer is 8001.

16. Answer: 1006

The series can be paired as

$$(\frac{1}{1+11^{-2011}}+\frac{1}{1+11^{2011}})+(\frac{1}{1+11^{-2009}}+\frac{1}{1+11^{2009}})+\cdots+(\frac{1}{1+11^{-1}}+\frac{1}{1+11^{1}}).$$

Each pair of terms is of the form

$$\frac{1}{1+a^{-1}}+\frac{1}{1+a}=1.$$

There are 1006 pairs of such terms, and thus the sum of the series is 1006.

17. Answer: 54

$$\begin{aligned}
x &= \sin^4\left(\frac{\pi}{8}\right)+\cos^4\left(\frac{\pi}{8}\right)+\sin^4\left(\frac{7\pi}{8}\right)+\cos^4\left(\frac{7\pi}{8}\right)\\
&= \sin^4\left(\frac{\pi}{8}\right)+\cos^4\left(\frac{\pi}{8}\right)+\sin^4\left(\frac{\pi}{8}\right)+\cos^4\left(\frac{\pi}{8}\right)\\
&= 2(\sin^2\left(\frac{\pi}{8}\right)+\cos^2\left(\frac{\pi}{8}\right))^2-4\sin^2\left(\frac{\pi}{8}\right)\cos^2\left(\frac{\pi}{8}\right)=2-\sin^2\left(\frac{\pi}{4}\right)=\frac{3}{2}.
\end{aligned}$$

Thus $36x=54$.

18. Answer: 1005

By the laws of sine and cosine, we have

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c} \quad \text{and} \quad \cos C=\frac{a^2+b^2-c^2}{2ab}.$$

Then

$$\begin{aligned}
\frac{\cot C}{\cot A+\cot B} &= \frac{\cos C}{\sin C}\cdot\frac{1}{\frac{\cos A\sin B+\cos B\sin A}{\sin A\sin B}}\\
&= \frac{\sin A\sin B\cos C}{\sin(A+B)\sin C}\\
&= (\frac{\sin A\sin B}{\sin^2 C})\cos C\\
&= (\frac{(ab/c^2)\sin^2 C}{\sin^2 C})(\frac{a^2+b^2-c^2}{2ab})\\
&= \frac{a^2+b^2-c^2}{2c^2}\\
&= \frac{2011-1}{2}=1005.
\end{aligned}$$

19. Answer: 1011

It is given that there are at least 1000 female participants. Suppose there are more than 1000 female participants. Then we take a group of 1001 female participants, and add any 10 participants to this group of female participants. This will result in a group of 1011 participants with at most 10 male participants, which contradicts the assumption. Therefore, there are exactly 1000 female participants. Hence, the number of male participants is $2011 - 1000 = 1011$.

20. Answer: 1

Substituting $x = \frac{1}{2}, -1, 2$, we get

$$\begin{array}{rcccccl} \frac{1}{4}f(\frac{1}{2}) & + & f(-1) & & & = & \frac{1}{2}, \\ & & f(-1) & + & f(2) & = & 2, \\ f(\frac{1}{2}) & + & & & 4f(2) & = & 8. \end{array}$$

Solving these equations, we get $f(\frac{1}{2}) = 1$. In fact the same method can be used to determine f. Letting $x = z, \frac{z-1}{z}, \frac{1}{1-z}$, we get

$$\begin{array}{rcccccl} z^2 f(z) & + & f\left(\frac{z-1}{z}\right) & & & = & 2z^2, \\ & & \left(\frac{z-1}{z}\right)^2 f\left(\frac{z-1}{z}\right) & + & f\left(\frac{1}{1-z}\right) & = & 2\left(\frac{z-1}{z}\right)^2, \\ f(z) & + & & & \frac{1}{(1-z)^2} f\left(\frac{1}{1-z}\right) & = & \frac{1}{2(1-z)^2}. \end{array}$$

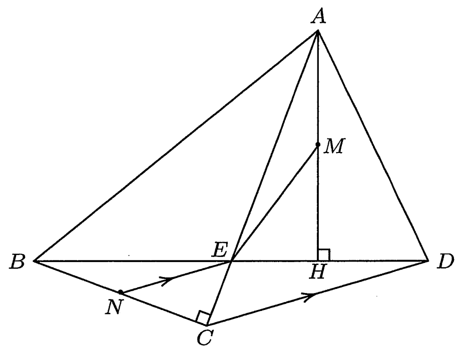
Using Cramer's rule, we can solve this system of linear equations in the unknowns $f(z), f\left(\frac{z-1}{z}\right), f\left(\frac{1}{1-z}\right)$. We obtain

$$f(z) = 1 + \frac{1}{(1-z)^2} - \frac{1}{z^2}.$$

Indeed one can easily check that it satisfies the given functional equation.

21. Answer: 40

Let M be the midpoint of AH and N the midpoint of BC. Then CD is parallel to NE and $CD = 2NE$. Since $\triangle AHE$ is similar to $\triangle BCE$, we have $\triangle MHE$ is similar to $\triangle NCE$. As $ME = \sqrt{15^2 + 20^2} = 25$, we thus have $NE = \frac{EC}{EH} \cdot ME = \frac{12 \times 25}{15} = 20$. Therefore $CD = 2NE = 40$.



22. Answer: 3

Note that

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{211} \Rightarrow xy-211x-211y=0 \Rightarrow (x-211)(y-211)=211^2.$$

Since 211 is a prime number, the factors of 211^2 are 1, 211, 211^2, -1, -211, -211^2. Thus the pairs of integers (x, y) satisfying the last equation are given by:

$$(x-211,\ y-211)=(1,\ 211^2),\ (211,\ 211),\ (211^2,\ 1),\ (-1,\ -211^2),$$
$$(-211,\ -211),\ (-211^2,\ -1).$$

Equivalently, (x, y) are given by

$$(212,\ 211+211^2),\ (422,\ 422),\ (211+211^2,\ 212),\ (210,\ 211-211^2),$$
$$(0,\ 0),\ (211-211^2,\ 210).$$

Note that (0, 0) does not satisfy the first equation. Among the remaining 5 pairs which satisfy the first equation, three of them satisfy the inequality $x \geq y$, and they are given by $(x,\ y)=(422,\ 422),\ (211+211^2,\ 212),\ (210,\ 211-211^2)$.

23. Answer: 93

By long division, we have

$$x^4+ax^2+bx+c=(x^2+3x-1)\cdot(x^2-3x+(a+10))+(b-3a-33)x+(c+a+10).$$

Let m_1, m_2 be the two roots of the equation $x^2+3x-1=0$. Note that $m_1 \neq m_2$, since the discriminant of the above quadratic equation is $3^2-4\cdot1\cdot1\cdot(-1)=13\neq 0$. Since m_1, m_2 also satisfy the equation $x^4+ax^2+bx+c=0$, it follows that m_1 and m_2 also satisfy the equation

$$(b-3a-33)x+(c+a+10)=0.$$

Thus we have

$$(b-3a-33)m_1+(c+a+10)=0,$$

and

$$(b-3a-33)m_2+(c+a+10)=0.$$

Since $m_1 \neq m_2$, it follows that $b-3a-33=0$ and $c+a+10=0$. Hence we have $b=3a+33$ and $c=-a-10$. Thus $a+b+4c+100=a+(3a+33)+4(-a-10)+100=93$.

24. Answer: 1120

Let m and n be positive integers satisfying the given equation. Then $3(n^2-m)=2011n$. Since 2011 is a prime, 3 divides n. By letting $n=3k$, we have $(3k)^2=m+2011k$. This implies that k divides m. Let $m=rk$. Then $9k^2=rk+2011k$ so that $9k=r+2011$. The smallest positive integer r such that $r+2011$ is divisible by 9 is $r=5$. Thus $k=(5+2011)/9=224$. The corresponding values of m and n are $m=1120$ and $n=672$.

25. Answer: 1

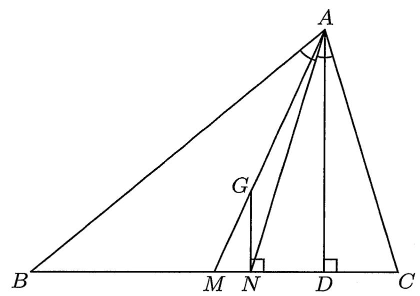
We shall show that the only possible values of a, b, c are $a = b = c = 1$ so that $abc = 1$. From the first equation, we note that $a^2 - b = (x-a)^2$ is a perfect square less than a^2. Thus $a^2 - b \le (a-1)^2$. That is $b \ge 2a - 1$. Likewise $c \ge 2b - 1$ and $a \ge 2c - 1$. Combining these inequalities, we have $a \ge 8a - 7$ or $a \le 1$. Thus $a = 1$. Similarly $b = c = 1$.

26. Answer: 180

Note that
$$A = \frac{\pi}{2} - B - C \le \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4},$$
and $\sin(B - C) \ge 0$. Therefore
$$\begin{aligned}\sin A \cos B \sin C &= \frac{1}{2} \cdot \sin A \cdot [\sin(B+C) - \sin(B-C)] \\ &\le \frac{1}{2} \sin A \sin(B+C) \\ &= \frac{1}{2} \sin A \cos A \\ &= \frac{1}{4} \sin 2A \le \frac{1}{4} \sin \frac{\pi}{2} = \frac{1}{4}.\end{aligned}$$
When $A = \frac{\pi}{4}$ and $B = C = \frac{\pi}{8}$, we have
$$\sin A \cos B \sin C = \sin \frac{\pi}{4} \cos \frac{\pi}{8} \sin \frac{\pi}{8} = \sin \frac{\pi}{4} \cdot \frac{1}{2} \sin \frac{\pi}{4} = \frac{1}{4}.$$
Hence the largest possible value of $720(\sin A)(\cos B)(\sin C)$ is $720 \times \frac{1}{4} = 180$.

27. Answer: 9

Let $BC = a, CA = b$ and $AB = c$. We shall prove that $c + b = a\sqrt{3}$. Thus $c = 5\sqrt{3} \times \sqrt{3} - 6 = 9$. Using the angle bisector theorem, we have $BN/NC = c/a > 1$ so that $BN > BM$. Also $\angle ANC = \angle B + \frac{1}{2}\angle A < \angle C + \frac{1}{2}\angle A < \angle ANB$ so that $\angle ANC$ is acute. This shows that B, M, N, D are in this order, where M is the midpoint of BC and D is the foot of the perpendicular from A onto BC.



First we have $BD = c \cos B = \frac{a^2+c^2-b^2}{2a}$. Using the angle bisector theorem, $BN = \frac{ac}{b+c}$. Therefore, $MN = BN - \frac{a}{2} = \frac{ac}{b+c} - \frac{a}{2} = \frac{a(c-b)}{2(c+b)}$. Also $MD = BD - \frac{a}{2} = \frac{a^2+c^2-b^2}{2a} -$

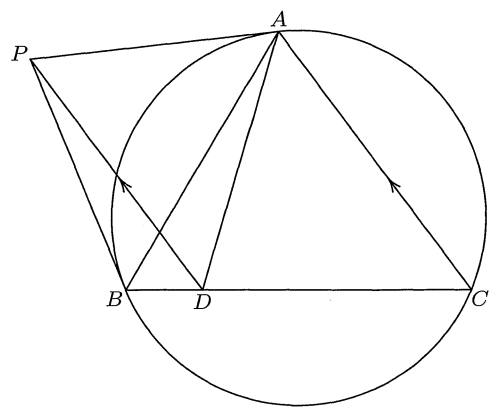
$\frac{a}{2} = \frac{c^2-b^2}{2a}$. Since GN is parallel to AD and G is the centroid of the triangle ABC, we have $MD/MN = 3$. It follows that $c+b = a\sqrt{3}$. Thus, $AB = a\sqrt{3}-b = 15-6 = 9$.

28. Answer: 28

The sum of 55 positive consecutive integers is at least $(55 \times 56)/2 = 1540$. Let the middle number of these consecutive positive integers be x. Then the product $abcd = 55x = 5 \cdot 11 \cdot x$. So we have $55x \geq 1540$ and thus $x \geq 28$. The least value of $a + b + c + d$ is attained when $x = 5(7)$. Thus the answer is $5 + 11 + 5 + 7 = 28$.

29. Answer: 168

First $\angle BDP = \angle BCA = \angle BAP$ so that P, B, D, A are concyclic. Thus $\angle ACD = \angle PBA = \angle PDA = \angle DAC$ so that $DA = DC$.



By cosine rule, $\cos C = 3/5$. Thus $DC = \frac{1}{2}AC/\cos C = 21 \times 5/3 = 35$. Hence $BD = 10$ and $BC = 10 + 35 = 45$. Thus $\text{area}(\triangle ABD) = \frac{10}{45} \times \text{area}(\triangle ABC)$. By Heron's formula, $\text{area}(\triangle ABC) = 756$. Thus $\text{area}(\triangle ABD) = \frac{10}{45} \times 756 = 168$.

30. Answer: 276

Since the number of positive divisors of a is odd, a must be a perfect square. As a is a divisor of $4400 = 2^4 \times 5^2 \times 11$ and a has exactly 9 positive divisors, we see that $a = 2^2 \times 5^2$. Now the least common multiple of a and b is 4400 implies that b must have $2^4 \times 11$ as a divisor. Since $2^4 \times 11$ has exactly 10 positive divisors, we deduce that $b = 2^4 \times 11 = 176$. Hence $a + b = 276$.

31. Answer: 20

First we let ℓ be the line which extends BC in both directions. Let E be the point on ℓ such that AE is perpendicular to ℓ. Similarly, we let F be the point on ℓ such that DF is perpendicular to ℓ. Then, it is easy to see that $BE = AE = \sqrt{6}$, $CF = 2\sqrt{2}$ and $DF = 2\sqrt{6}$. Thus $EF = \sqrt{6} + 4 - 2\sqrt{2} + 2\sqrt{2} = 4 + \sqrt{6}$. Now we let G be the point on DF such that AG is parallel to ℓ. Then $AG = EF = 4 + \sqrt{6}$ and

$DG = DF - GF = DF - AE = 2\sqrt{6} - \sqrt{6} = \sqrt{6}$. So for the right-angled triangle ADG, we have

$$x = AD = \sqrt{AG^2 + DG^2} = \sqrt{(4+\sqrt{6})^2 + (\sqrt{6})^2} = \sqrt{28 + 8\sqrt{6}} = 2 + 2\sqrt{6}.$$

Thus, $x^2 - 4x = (2+2\sqrt{6})^2 - 8 - 8\sqrt{6} = 4 + 24 + 8\sqrt{6} - 8 - 8\sqrt{6} = 20$.

32. Answer: 5

Suppose x and y are positive integers satisfying $x^3 + y^3 - 3xy = p - 1$. That is $(x+y+1)(x^2+y^2-xy-x-y+1) = p$. Since $x, y \geq 1$, we must have $x+y+1 = p$ and $x^2+y^2-xy-x-y+1 = 1$. Suppose $x = y$. Then the second equation gives $x = y = 2$. Thus $p = x+y+1 = 5$. Next we may suppose without loss of generality that $x > y \geq 1$. Thus $x - y \geq 1$. Then the equation $x^2+y^2-xy-x-y+1 = 1$ can be written as $x + y - xy = (x-y)^2 \geq 1$. That is $(x-1)(y-1) \leq 0$. It follows that $x = 1$ or $y = 1$. Since we assume $x > y$, we must have $y = 1$. Then from $x^2+y^2-xy-x-y+1 = 1$, we get $x = 2$. But then $x+y+1 = 4$ is not a prime. Consequently there is no solution in x and y if $x \neq y$. Therefore the only solution is $x = y = 2$ and $p = 5$.

33. Answer: 61

Without loss of generality, we may assume that $a \geq b \geq c$. Let the HCF (or GCD) of a, b and c be d. Then $a = da_1$, $b = db_1$ and $c = dc_1$. Let the LCM of a_1, b_1 and c_1 be m. Thus, $a_1^2 + b_1^2 + c_1^2 = \frac{2011}{d^2}$ and $d + md = 388$ or $1 + m = \frac{388}{d}$. So, $d^2 \mid 2011$ and $d \mid 388$. Note that 2011 is a prime. Thus we must have $d = 1$, and it follows that $a = a_1$, $b = b_1$, $c = c_1$, and thus $a^2 + b^2 + c^2 = 2011$. In particular, $a^2 + b^2 + c^2 < 2025 = 45^2$, so that one has $a, b, c < 45$. Furthermore we have $m = 387 = 3^2 \times 43$. Thus a, b and c can only be 1, 3, 9 or 43, since they must be less than 45. Then it is easy to check that $43^2 + 9^2 + 9^2 = 2011$, and $a = 43$, $b = c = 9$ is the only combination which satisfies the given conditions. Thus we have $a + b + c = 43 + 18 = 61$.

34. Answer: 16

Consider the subset $T = \{1, 2^2, 3^2, 5^2, 7^2, \cdots, 43^2\}$ consisting of the number 1 and the squares of all prime numbers up to 43. Then $T \subseteq S$, $|T| = 15$, and all elements in T are pairwise relatively prime; however, T contains no prime number. Thus $k \geq 16$. Next we show that if A is any subset of S with $|A| = 16$ such that all elements in A are pairwise relatively prime, then A contains a prime number. Suppose to the contrary that A does not contain a prime number. Let $A = \{a_1, a_2, \cdots, a_{16}\}$. We consider two cases:

Case 1. 1 is not in A. Then $a_1, a_2, \cdots, a_{16}$ are composite numbers. Let p_i be the smallest prime factor of $a_i, i = 1, 2, \cdots, 16$. Since $\gcd(a_i, a_j) = 1$ for $i \neq j$, $i, j = 1, 2, \cdots, 16$, we see that the prime numbers $p_1, p_2, \cdots, p_{16}$ are all distinct. By re-ordering $a_1, a_2, \cdots, a_{16}$ if necessary, we may assume that $p_1 < p_2 < \cdots < p_{16}$. In addition, since each a_i is a composite number, it follows that

$$a_1 \geq p_1^2 \geq 2^2,\ a_2 \geq p_2^2 \geq 3^2, \cdots,\ a_{15} \geq p_{15}^2 \geq 47^2 > 2011,$$

which is a contradiction to the given fact that each element of S is less than or equal to 2011.

Case 2. 1 is in A. We may let $a_{16} = 1$. Then $a_1, a_2, \cdots, a_{15}$ are composite numbers. As in Case 1, we have

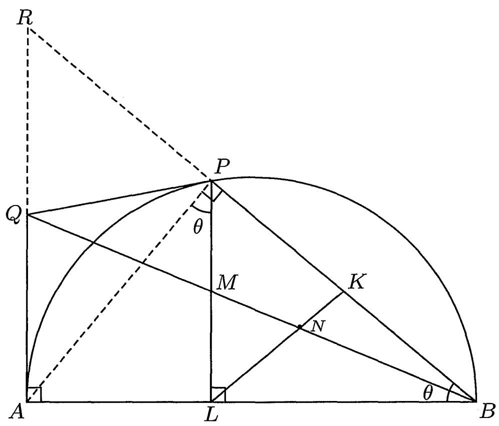
$$a_1 \geq p_1^2 \geq 2^2,\ a_2 \geq p_2^2 \geq 3^2, \cdots,\ a_{15} \geq p_{15}^2 \geq 47^2 > 2011,$$

which is a contradiction.

Thus we have shown that every 16-element subset A of S such that all elements in A are pairwise relatively prime must contain a prime number. Hence the smallest k is 16.

35. Answer: 12

Let the extensions of AQ and BP meet at the point R. Then $\angle PRQ = \angle PAB = \angle QPR$ so that $QP = QR$. Since $QA = QP$, the point Q is the midpoint of AR. As AR is parallel to LP, the triangles ARB and LPB are similar so that M is the midpoint of PL. Therefore, N is the centroid of the triangle PLB, and $3MN = BM$.



Let $\angle ABP = \theta$. Thus $\tan\theta = AR/AB = 2AQ/AB = 5/6$. Then $BL = PB\cos\theta = AB\cos^2\theta$. Also $BM/BL = BQ/BA$ so that $3MN = BM = \frac{BQ}{AB}AB\cos^2\theta = \cos^2\theta(QM + 3MN)$. Solving for MN, we have $MN = \frac{QM}{3\tan^2\theta} = \frac{25}{3\times(5/6)^2} = 12$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2011

(Senior Section, Round 2)

Saturday, 25 June 2011 **0930-1230**

---

1. In the triangle ABC, the altitude at A, the bisector of $\angle B$ and the median at C meet at a common point. Prove that the triangle ABC is equilateral.

2. Determine if there is a set S of 2011 positive integers so that for every pair m, n of distinct elements of S, $|m-n| = (m, n)$. Here (m, n) denotes the greatest common divisor of m and n.

3. Find all positive integers n such that

$$\cos\frac{\pi}{n}\cos\frac{2\pi}{n}\cos\frac{3\pi}{n} = \frac{1}{n+1}.$$

4. Let n and k be positive integers with $n \geq k \geq 2$. For $i = 1, \ldots, n$, let S_i be a nonempty set of consecutive integers such that among any k of them, there are two with nonempty intersection. Prove that there is a set X of $k-1$ integers such that each S_i, $i = 1, \ldots, n$ contains at least one integer in X.

5. Given $x_1, x_2, \cdots, x_n > 0$, $n \geq 5$, show that

$$\frac{x_1x_2}{x_1^2 + x_2^2 + 2x_3x_4} + \frac{x_2x_3}{x_2^2 + x_3^2 + 2x_4x_5} + \cdots + \frac{x_nx_1}{x_n^2 + x_1^2 + 2x_2x_3} \leq \frac{n-1}{2}.$$

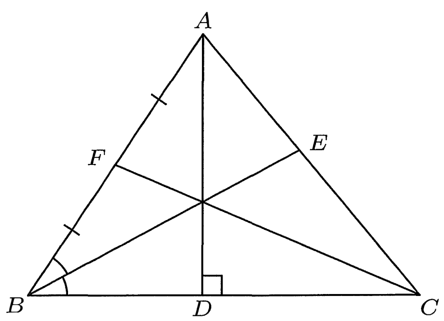
Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2011

(Senior Section, Round 2 solutions)

1. There is an error in this problem. The triangle is not necessarily equilateral. In fact we shall prove that the altitude at A, the bisector of $\angle B$ and the median at C meet at a common point if and only if $\cos B = \frac{a}{a+c}$ where $BC = a$, $CA = b$ and $AB = c$.

Let D, E and F be the points on BC, CA and AB respectively such that AD is the altitude at A, BE is the bisector of $\angle B$ and CF is the median at C. Suppose that AD, BE, CF meet at a common point. The point of concurrence of AD, BE and CF must lie inside the triangle ABC. Since F is the midpoint of AB, by Ceva's theorem $CE : EA = CD : DB$. Using the angle bisector theorem, $CE : EA = a : c$. Thus $CD = a^2/(a+c)$ and $DB = ac/(a+c)$. Thus $\cos B = \frac{BD}{AB} = \frac{a}{a+c}$.



Conversely, if $\cos B = \frac{a}{a+c}$, then $\angle B$ is acute and $BD = c\cos B = ac/(a+c) < a$ so that D is within BC. Thus $DC = a - ac/(a+c) = a^2/(a+c)$. Therefore $BD/DC = c/a$. Consequently $(AF/FB)(BD/DC)(CE/EA) = 1$. By Ceva's theorem, AD, BE and CF are concurrent.

So given a and c, the acute angle B and hence the triangle ABC is determined. If $a \neq c$, then the triangle ABC is not equilateral.

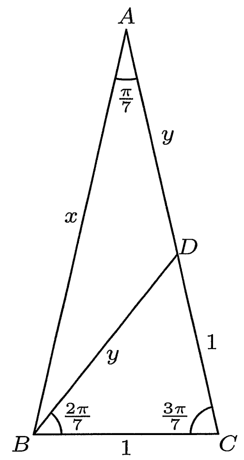
2. Yes, in fact, for any $k \in \mathbb{N}$, there is a set S_k having k elements satisfying the given condition. For $k = 1$, let S_1 be any singleton set. For $k = 2$, let $S_2 = \{2, 3\}$. Suppose that $S_k = \{a_1, \ldots, a_k\}$ satisfies the given conditions. Let

$$\begin{aligned} b_1 &= a_1 a_2 \cdots a_k \\ b_j &= b_1 + a_{j-1}, \quad 2 \le j \le k+1. \end{aligned}$$

Let $S_{k+1} = \{b_1, b_2, \ldots, b_{k+1}\}$. Then the fact that S_{k+1} satisfies the required property can be verified by observing that $|m - n| = (m, n)$ if and only if $(m - n)$ divides m.

3. We shall show that $n = 3$ or 7. Let $f(n) = \cos\frac{\pi}{n}\cos\frac{2\pi}{n}\cos\frac{3\pi}{n}$. One can verify that $f(1) = 1, f(2) = 0, f(3) = \frac{1}{4}, f(4) = 0, f(5) = -\cos^2\frac{2\pi}{5}\cos\frac{\pi}{5} < 0, f(6) = 0$ and $f(8) = \frac{1}{4}$. We shall show that $f(7) = \frac{1}{8}$.

Let ABC be an isosceles triangle with $\angle A = \frac{\pi}{7}$, $\angle B = \angle C = \frac{3\pi}{7}$, $BC = 1$ and $AB = AC = x$. Let D be the point on AC such that $\angle CBD = \frac{2\pi}{7}$. Let $BD = y$. Then the triangles BCD and ADB are isosceles with $BC = CD = 1$ and $AD = BD = y$. Thus $\cos\frac{\pi}{7} = \cos A = \frac{x}{2y}$, $\cos\frac{2\pi}{7} = \cos\angle CBD = \frac{y}{2}$, and $\cos\frac{3\pi}{7} = \cos C = \frac{1}{2x}$. Therefore, $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7} = \frac{1}{8}$.



Lastly, let's show that $f(n) \neq \frac{1}{n+1}$ for $n \geq 9$. For $n \geq 9$, we have $0 < \frac{\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n} < \frac{\pi}{2}$. Since cosine is a decreasing function on $[0, \frac{\pi}{2}]$, we have $f(n)$ is an increasing function of n for $n \geq 9$. Consequently, $f(n) \geq f(9) > \cos^3\frac{3\pi}{9} = \frac{1}{8} > \frac{1}{n+1}$.

4. Let $a_i = \max S_i$. Without loss of generality, assume that $a_1 \leq a_i$ for all i. We shall prove by induction on k. For $k = 2$, since $S_1 \cap S_2 \neq \emptyset$, $a_1 \in S_2$. Therefore $X = \{a_1\}$ works. Now assume that the result is true for $k-1$. Let $\mathbb{I}$ be the collection consisting of S_1 and the sets S_i such that $S_i \cap S_1 \neq \emptyset$ and let $\mathbb{J}$ be the collection of the other sets. Note that a_1 is contained in all the sets in $\mathbb{I}$. If $|\mathbb{J}| < k-1$, then the set X consisting of one integer from each of the sets in $\mathbb{J}$ together with a_1 has the desired property. Otherwise, consider a collection $\mathbb{K}$ of $k-1$ sets in $\mathbb{J}$. $\mathbb{K}$, together with S_1, forms a collection of k sets. Among these there are two that have nonempty intersection. Since S_1 does not intersect any of the sets in $\mathbb{J}$, these two sets must come from $\mathbb{K}$. Thus by the induction hypothesis, there is a set X' of $k-2$ integers such that every set in $\mathbb{J}$ contains one integer in X'. Thus $X = X' \cup \{a_1\}$ has the desired property.

5. Dividing each of the numerator and denominator of LHS by $2x_1x_2$, $2x_2x_3$, ..., writing $a_1 = \frac{x_3x_4}{x_1x_2}$, $a_2 = \frac{x_4x_5}{x_2x_3}$, ..., and noting that $x_i^2 + x_{i+1}^2 \geq 2x_ix_{i+1}$, we get

$$2 \times \text{LHS} \leq \frac{1}{1+a_1} + \frac{1}{1+a_2} + \cdots + \frac{1}{1+a_n}.$$

Note that $a_1a_2\cdots a_n = 1$. It suffices to show that

$$\frac{a_1}{1+a_1}+\frac{a_2}{1+a_2}+\cdots+\frac{a_n}{1+a_n}\geq 1 \tag{$*$}$$

since it is equivalent to

$$\frac{1}{1+a_1}+\frac{1}{1+a_2}+\cdots+\frac{1}{1+a_n}\leq n-1.$$

We shall show that $(*)$ is true for $n \geq 2$. The case $n = 2$ is obvious. We will now prove it by induction. Suppose $(*)$ holds for $n = k$. Now assume $a_1\cdots a_{k+1} = 1$, $a_i > 0$ for all i. To prove the inductive step, it suffices to show that

$$\frac{a_k}{1+a_k}+\frac{a_{k+1}}{1+a_{k+1}}\geq\frac{a_ka_{k+1}}{1+a_ka_{k+1}}.$$

which can be verified directly.

Note: This is an extension of the problem :

$$\frac{x_1^2}{x_1^2+x_2x_3}+\frac{x_2^2}{x_2^2+x_3x_4}+\cdots+\frac{x_{n-1}^2}{x_{n-1}+x_nx_1}+\frac{x_n^2}{x_n^2+x_1x_2}\leq n-1.$$

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2011
(Open Section, Round 1)

Wednesday, 1 June 2011 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Throughout this paper, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. For example, $\lfloor 2.1 \rfloor = 2$, $\lfloor 3.9 \rfloor = 3$ (This notation is used in Questions 7, 9, 19 and 20).

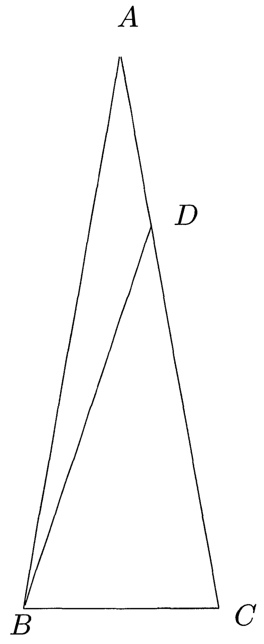
1. A circular coin A is rolled, without sliding, along the circumference of another stationary circular coin B with radius twice the radius of coin A. Let x be the number of degrees that the coin A makes around its centre until it first returns to its initial position. Find the value of x.

2. Three towns X, Y and Z lie on a plane with coordinates $(0,0)$, $(200,0)$ and $(0,300)$ respectively. There are 100, 200 and 300 students in towns X, Y and Z respectively. A school is to be built on a grid point (x, y), where x and y are both integers, such that the overall distance travelled by all the students is minimized. Find the value of $x + y$.

3. Find the last non-zero digit in $30!$.

   (For example, $5! = 120$; the last non-zero digit is 2.)

4. The diagram below shows $\triangle ABC$, which is isoceles with $AB = AC$ and $\angle A = 20°$. The point D lies on AC such that $AD = BC$. The segment BD is constructed as shown. Determine $\angle ABD$ in degrees.



5. Given that $\dfrac{\cos^4 \alpha}{\cos^2 \beta} + \dfrac{\sin^4 \alpha}{\sin^2 \beta} = 1$, evaluate $\dfrac{\cos^4 \beta}{\cos^2 \alpha} + \dfrac{\sin^4 \beta}{\sin^2 \alpha}$.

6. The number 25 is expressed as the sum of positive integers $x_1, x_2, \cdots, x_k$, where $k \leq 25$. What is the maximum value of the product of $x_1, x_2, x_3, \cdots$, and x_k?

7. Let x_0 be the largest (real) root of the equation $x^4 - 16x - 12 = 0$. Evaluate $\lfloor 10x_0 \rfloor$.

8. Let $x_i \in \{\sqrt{2}-1, \sqrt{2}+1\}$, where $i = 1, 2, 3, \cdots, 2012$. Define

$$S = x_1x_2 + x_3x_4 + x_5x_6 + \cdots + x_{2009}x_{2010} + x_{2011}x_{2012}.$$

How many different positive integer values can S attain?

9. Let A be the set of real numbers x satisfying the inequality $x^2 + x - 110 < 0$ and B be the set of real numbers x satisfying the inequality $x^2 + 10x - 96 < 0$. Suppose that the set of integer solutions of the inequality $x^2 + ax + b < 0$ is exactly the set of integers contained in $A \cap B$. Find the maximum value of $\lfloor |a - b| \rfloor$.

10. Given that

$$\begin{aligned} \alpha + \beta + \gamma &= 14 \\ \alpha^2 + \beta^2 + \gamma^2 &= 84 \\ \alpha^3 + \beta^3 + \gamma^3 &= 584, \end{aligned}$$

find $\max\{\alpha, \beta, \gamma\}$.

11. Determine the largest even positive integer which cannot be expressed as the sum of two composite odd positive integers.

12. Let a, b, c be positive integers such that $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$ and $\gcd(a, b, c) = 1$. Suppose $a + b \leq 2011$. Determine the largest possible value of $a + b$.

13. Let $x[n]$ denote $x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}$, where there are n terms of x. What is the minimum value of n such that $9[9] < 3[n]$?

(For example, $3[2] = 3^3 = 27$; $2[3] = 2^{2^2} = 16$.)

14. In the triangle ABC, $\angle B = 90^\circ, \angle C = 20^\circ$, D and E are points on BC such that $\angle ADC = 140^\circ$ and $\angle AEC = 150^\circ$. Suppose $AD = 10$. Find $BD \cdot CE$.

15. Let $S = \{1, 2, 3, \cdots, 65\}$. Find the number of 3-element subsets $\{a_1, a_2, a_3\}$ of S such that $a_i \leq a_{i+1} - (i + 2)$ for $i = 1, 2$.

16. Determine the value of

$$\frac{3}{\sin^2 20^\circ} - \frac{1}{\cos^2 20^\circ} + 64\sin^2 20^\circ.$$

17. A real-valued function f satisfies the relation

$$\mathrm{f}(x^2 + x) + 2\mathrm{f}(x^2 - 3x + 2) = 9x^2 - 15x$$

for all real values of x. Find f(2011).

18. A collection of 2011 circles divide the plane into N regions in such a way that any pair of circles intersects at two points and no point lies on three circles. Find the last four digits of N.

19. If a positive integer N can be expressed as $\lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 3x \rfloor$ for some real numbers x, then we say that N is "visible"; otherwise, we say that N is "invisible". For example, 8 is visible since $8 = \lfloor 1.5 \rfloor + \lfloor 2(1.5) \rfloor + \lfloor 3(1.5) \rfloor$, whereas 10 is invisible. If we arrange all the "invisible" positive integers in increasing order, find the 2011^{th} "invisible" integer.

20. Let A be the sum of all non-negative integers n satisfying

$$\lfloor \frac{n}{27} \rfloor = \lfloor \frac{n}{28} \rfloor.$$

Determine A.

21. A triangle whose angles are $A,\ B,\ C$ satisfies the following conditions

$$\frac{\sin A + \sin B + \sin C}{\cos A + \cos B + \cos C} = \frac{12}{7},$$

and

$$\sin A \sin B \sin C = \frac{12}{25}.$$

Given that $\sin C$ takes on three possible values $s_1,\ s_2$ and s_3, find the value of $100 s_1 s_2 s_3$.

22. Let $x > 1,\ y > 1$ and $z > 1$ be positive integers for which the following equation

$$1! + 2! + 3! + \ldots + x! = y^z$$

is satisfied. Find the largest possible value of $x + y + z$.

23. Let ABC be a non-isosceles acute-angled triangle with circumcentre O, orthocentre H and $\angle C = 41°$. Suppose the bisector of $\angle A$ passes through the midpoint M of OH. Find $\angle HAO$ in degrees.

24. The circle γ_1 centred at O_1 intersects the circle γ_2 centred at O_2 at two points P and Q. The tangent to γ_2 at P intersects γ_1 at the point A and the tangent to γ_1 at P intersects $\gamma_2$at the point B where A and B are distinct from P. Suppose $PQ \cdot O_1O_2 = PO_1 \cdot PO_2$ and $\angle APB$ is acute. Determine the size of $\angle APB$ in degrees.

25. Determine

$$\lim_{n\to\infty} \sum_{i=0}^{n} \frac{1}{\binom{n}{i}}.$$

(Note: Here $\binom{n}{i}$ denotes $\dfrac{n!}{i!(n-i)!}$ for $i = 0, 1, 2, 3, \cdots, n$.)

Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2011
(Open Section, Round 1 Solution)

1. **Answer.** 1080

   Solution. The number of complete revolutions the first coin A has turned through is the sum of two components: the number of revolutions round the stationary coin B if A were *sliding* on B and the number of revolutions round A's own axis (perpendicular to its plane and through its centre) determined by the distance travelled on the circumference of B. Thus, the total number of revolutions is

   $$1 + \frac{2\pi(2r)}{2\pi r} = 3.$$

   Hence the number of degrees $= 3 \times 360 = 1080$. $\square$

2. **Answer.** 300

   Solution. We claim that the school must be built in Z. Suppose the school is to be built at another point A. The change in distance travelled

   $$\begin{aligned} &= 300ZA + 200YA + 100XA - 200YZ - 100XZ \\ &= 100(ZA + AX - ZX) + 200(ZA + AY - ZY) \\ &> 0 \end{aligned}$$

   by triangle inequality. Thus, $\min(x + y) = 0 + 300 = 300$. $\square$

3. **Answer.** 8

   Solution. We first obtain the prime factorization of 30!. Observe that 29 is the largest prime number less than 30. We have

   $$\begin{aligned} &\left\lfloor \frac{30}{2} \right\rfloor + \left\lfloor \frac{30}{2^2} \right\rfloor + \left\lfloor \frac{20}{2^3} \right\rfloor + \left\lfloor \frac{30}{2^4} \right\rfloor = 26 \\ &\left\lfloor \frac{30}{3} \right\rfloor + \left\lfloor \frac{30}{3^2} \right\rfloor + \left\lfloor \frac{30}{3^3} \right\rfloor = 14 \\ &\left\lfloor \frac{30}{5} \right\rfloor + \left\lfloor \frac{30}{5^2} \right\rfloor = 7 \\ &\left\lfloor \frac{30}{7} \right\rfloor = 4 \\ &\left\lfloor \frac{30}{11} \right\rfloor = 2 \\ &\left\lfloor \frac{30}{13} \right\rfloor = 2 \\ &\left\lfloor \frac{30}{17} \right\rfloor = \left\lfloor \frac{30}{19} \right\rfloor = \left\lfloor \frac{30}{23} \right\rfloor = \left\lfloor \frac{30}{29} \right\rfloor = 1. \end{aligned}$$

Thus,

$$\begin{aligned}
30! &= 2^{26} \cdot 3^{14} \cdot 5^7 \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \\
\frac{30!}{10^7} &= 2^{19} \cdot 3^{14} \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \\
&= 6^{14} \cdot 2^5 \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \\
&\equiv 6(2)(1)(1)(9)(7)(9)(3)(9)(\text{mod } 10) \\
&\equiv 2(-1)(-3)(-1)(3)(-1)(\text{mod } 10) \\
&\equiv 8(\text{mod } 10),
\end{aligned}$$

showing that the last non-zero digit is 8.

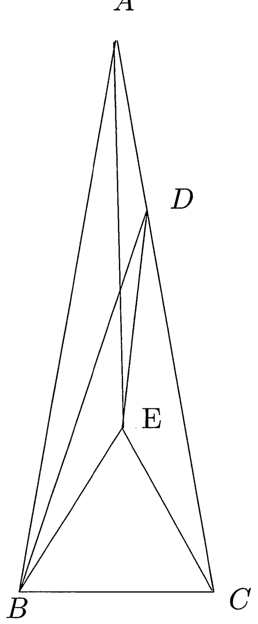
□

4. **Answer.** 10

Solution. Let E be the point inside $\triangle ABC$ such that $\triangle EBC$ is equilateral. Connect A and D to E respectively.

It is clear that $\triangle AEB$ and $\triangle AEC$ are congruent, since $AE = AE$, $AB = AC$ and $BE = CE$. It implies that $\angle BAE = \angle CAE = 10^o$.

Since $AD = BC = BE$, $\angle EBA = \angle DAB = 20^o$ and $AB = BA$, we have $\triangle ABE$ and $\triangle BAD$ are congruent, implying that $\angle ABD = \angle BAE = 10^o$.



□

5. **Answer.** 1

Solution. Since $\dfrac{\cos^4\alpha}{\cos^2\beta} + \dfrac{\sin^4\alpha}{\sin^2\beta} = 1$, set $\cos\theta = \dfrac{\cos^2\alpha}{\cos\beta}$ and $\sin\theta = \dfrac{\sin^2\alpha}{\sin\beta}$. Then

$$\cos(\theta - \alpha) = \cos\theta\cos\alpha + \sin\theta\sin\alpha = \cos^2\alpha + \sin^2\alpha = 1.$$

and so

$$\theta - \alpha = 2k\pi \text{ for some } k \in \mathbb{Z}.$$

Thus $\sin\theta = \sin\alpha$ and $\cos\theta = \cos\alpha$. Consequently,

$$\frac{\cos^4\beta}{\cos^2\alpha} + \frac{\sin^4\beta}{\sin^2\alpha} = \cos^2\beta + \sin^2\beta = 1.$$

$\square$

6. **Answer.** 8748

Solution. Clearly, no x_i should be 1. If $x_i \geq 4$, then splitting it into two factors 2 and $x_i - 2$ will give a product of $2x_i - 4$ which is at least as large as x_i. Further, $3 \times 3 > 2 \times 2 \times 2$, so any three factors of 2 should be replaced by two factors of 3. Thus, split 25 into factors of 3, retaining two 2's, which means $25 = 7 \times 3 + 2 \times 2$. The maximum product is thus $3^7 2^2 = 8748$. $\square$

7. **Answer.** 27

Solution. Since $x^4 - 16x - 12 \equiv x^4 + 4x^2 + 4 - 4(x^2 + 4x + 4) \equiv (x^2 - 2x - 2)(x^2 + 2x + 6)$, we conclude that $x_0 = 1 + \sqrt{3}$ and so $1 + \sqrt{2.89} < x_0 < 1 + \sqrt{3.24}$. Consequently, $\lfloor 10x_0 \rfloor = 27$. $\square$

8. **Answer.** 504

Solution. Note that $(\sqrt{2}-1)^2 = 3 - 2\sqrt{2}$, $(\sqrt{2}+1)^2 = 3 + 2\sqrt{2}$ and $(\sqrt{2}-1)(\sqrt{2}+1) = 1$. There are 1006 pairs of products in S; each pair of the product can be either $3-2\sqrt{2}, 3+2\sqrt{2}$ or 1. Let a be the number of these products with value $3 - 2\sqrt{2}$, b be the number of these products with value $3+2\sqrt{2}$ and c be the number of them with value 1. The $a+b+c = 1006$. Hence

$$S = a(3 - 2\sqrt{2}) + b(3 + 2\sqrt{2}) + c = 3a + 3b + c + 2\sqrt{2}(b - a).$$

For S to be a positive integer, $b = a$ and thus $2a + c = 1006$. Further,

$$S = 6a + c = 6a + 1006 - 2a = 4a + 1006.$$

From $2a + c = 1006$ and that $0 \leq a \leq 503$, it is clear that S can have 504 different positive integer values. $\square$

9. **Answer.** 71

Note that $x^2 + x - 110 = (x - 10)(x + 11)$. Thus the set of real numbers x satisfying the inequality $x^2 + x - 110 < 0$ is $-11 < x < 10$.

Also note that $x^2 + 10x - 96 = (x - 6)(x + 16)$. Thus the set of real numbers x satisfying the inequality $x^2 + 10x - 96 < 0$ is $-16 < x < 6$.

Thus $A = \{x : -11 < x < 10\}$ and $B = \{x : -16 < x < 6\}$, implying that

$$A \cap B = \{x : -11 < x < 6\}.$$

Now let $x^2+ax+b=(x-x_1)(x-x_2)$, where $x_1 \le x_2$. Then the set of integer solutions of $x^2+ax+b<0$ is

$$\{k : k \text{ is an integer}, x_1 < k < x_2\}.$$

By the given condition,

$$\begin{aligned}\{k : k \text{ is an integer}, x_1 < k < x_2\} &= \{k : k \text{ is an integer}, -11 < k < 6\}\\ &= \{-10, -9, \cdots, 5\}.\end{aligned}$$

Thus $-11 \le x_1 < -10$ and $5 < x_2 \le 6$. It implies that $-6 < x_1+x_2 < -4$ and $-66 \le x_1x_2 < -50$.

From $x^2+ax+b=(x-x_1)(x-x_2)$, we have $a=-(x_1+x_2)$ and $b=x_1x_2$. Thus $4<a<6$ and $-66 \le b < -50$. It follows that $54 < a-b < 72$.

Thus $\max\lfloor |a-b| \rfloor \le 71$.

It remains to show that it is possible that $\lfloor |a-b| \rfloor = 71$ for some a and b.

Let $a=5$ and $b=-66$. Then $x^2+ab+b=(x+11)(x-6)$ and the inequality $x^2+ab+b<0$ has solutions $\{x : -11 < x < 6\}$. So the set of integer solutions of $x^2+ab+b<0$ is really the set of integers in $A \cap B$.

Hence $\max\lfloor |a-b| \rfloor = 71$. $\square$

10. **Answer.** 8

Solution. We consider the polynomial

$$P(t) = t^3 + bt^2 + ct + d.$$

Suppose the root of the equation $P(t)=0$ are x, y and z. Then

$$-b = x+y+z = 14,$$

$$c = xy+xz+yz = \frac{1}{2}\Big((x+y+z)^2 - x^2 - y^2 - z^2\Big) = \frac{1}{2}\Big(14^2 - 84\Big) = 56$$

and

$$x^3+y^3+z^3+3d = (x+y+z)(x^2+y^2+z^2-xy-xz-yz).$$

Solving for b, c and d, we get $b=-14$, $c=30$ and $d=-64$. Finally, since $t^3-14t^2+30t-64=0$ implies $t=2$ or $t=4$ or $t=8$, we conclude that $\max\{\alpha, \beta, \gamma\} = 8$. $\square$

11. **Answer.** 38

Solution. Let n be an even positive integer. Then each of the following expresses n as the sum of two odd integers: $n=(n-15)+15, (n-25)+25$ or $(n-35)+35$. Note that at least one of $n-15, n-25, n-35$ is divisible by 3, hence n can be expressed as the sum of two composite odd numbers if $n>38$. Indeed, it can be verified that 38 cannot be expressed as the sum of two composite odd positive integers. $\square$

12. **Answer.** 1936

Solution. We first show that $a+b$ must be a perfect square. The equation $\frac{1}{a}+\frac{1}{b}=\frac{1}{c}$ is equivalent to $\frac{a-c}{c}=\frac{c}{b-c}$. Write $\frac{a-c}{c}=\frac{c}{b-c}=\frac{p}{q}$, where $\gcd(p,q)=1$. From $\frac{a-c}{c}=\frac{p}{q}$, we have $\frac{a}{p+q}=\frac{c}{q}$. Since $\gcd(p,q)=1$, we must have q divides c. Similarly from $\frac{b-c}{c}=\frac{q}{p}$, we have $\frac{b}{p+q}=\frac{c}{p}$. Since $\gcd(p,q)=1$, we must have p divides c. Thus $\gcd(p,q)=1$ implies pq divides c. Therefore $\frac{a}{p(p+q)}=\frac{b}{q(p+q)}=\frac{c}{pq}$ is an integer r. Then r divides a, b and c, so that $r=1$ since $\gcd(a,b,c)=1$. Consequently, $a+b=p(p+q)+q(p+q)=(p+q)^2$.

Next the largest square less than or equal to 2011 is $44^2=1936$. As $1936=1892+44$, and $\frac{1}{1892}+\frac{1}{44}=\frac{1}{43}$, where $\gcd(1892,44,43)=1$, we have $a=1892$, $b=44$ and $c=43$ give the largest value of $a+b$. These values of a,b,c can obtained from the identity $\frac{1}{m^2-m}+\frac{1}{m}=\frac{1}{m-1}$. □

13. **Answer.** 10

Solution. Suppose $9[m]<3[n]$. Note that $9[m]=3^p$ and $3[n]=3^q$ for some integers p and q. Thus, $q\geq p+1$. In particular,

$$2(9[m])<3(9[m])=3^{p+1}\leq 3^q=3[n].$$

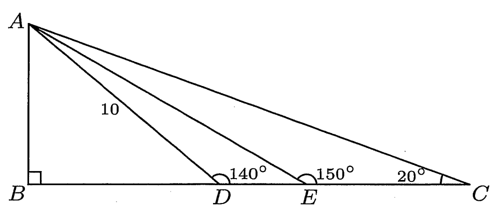
Then we have

$$9[m+1]=(3^2)^{9[m]}=3^{2(9[m])}<3^{3[n]}=3[n+1].$$

Thus, $9[m]<3[n]$ implies $9[m+1]<3[n+1]$. It is clear that $9[2]=81=3^4<3[3]$. Continuing this way, $9[9]<3[10]$. It is also clear that $9[9]>3[9]$, hence the minimum value of n is 10. □

14. **Answer.** 50

Direct calculation gives $\angle DAC=20^\circ$ and $\angle BAD=50^\circ$. Thus $AD=CD=10$. Also $BD=10\sin 50^\circ$. By sine rule applied to the triangle AEC, we have $\frac{CE}{\sin 10^\circ}=\frac{AC}{\sin 150^\circ}=\frac{2\times 10\cos 20^\circ}{\sin 150}=40\cos 20^\circ$. (Note that $AD=DC$.)



Therefore, $BD\cdot CE=400\cos 20^\circ\sin 10^\circ\sin 50^\circ$.

Direct calculation shows that $\cos 20^\circ\sin 10^\circ\sin 50^\circ=\frac{1}{8}$ so that $BD\cdot CE=50$. □

15. **Answer.** 34220

Solution. Note that the condition $a_i\leq a_{i+1}-(i+2)$ for $i=1,2$ is equivalent to that

$$a_1+3\leq a_2,\quad a_2+4\leq a_3.$$

Let A be the set of all 3-element subsets $\{a_1,a_2,a_3\}$ of S such that $a_1+3\leq a_2$ and $a_2+4\leq a_3$.

Let B be the set of all 3-element subsets $\{b_1,b_2,b_3\}$ of the set $\{1,2,\cdots,60\}$.

We shall show that $|A| = |B| = \binom{60}{3} = 34220$ by showing that the mapping ϕ below is a bijection from A to B:

$$\phi : \{a_1, a_2, a_3\} \longrightarrow \{a_1, a_2 - 2, a_3 - 5\}.$$

First, since $\{a_1, a_2, a_3\} \in A$, we have $a_1+3 \le a_2$ and $a_2+4 \le a_3$, and so $a_1 < a_2-2 < a_3-5$, implying that $\{a_1, a_2 - 2, a_3 - 5\} \in B$.

It is clear that ϕ is injective.

It is also surjective, as for any $\{b_1, b_2, b_3\} \in B$ with $b_1 < b_2 < b_3$, we have $\{b_1, b_2+2, b_3+5\} \in A$ and

$$\phi : \{b_1, b_2 + 2, b_3 + 5\} \longrightarrow \{b_1, b_2, b_3\}.$$

Hence ϕ is a bijection and $|A| = |B| = 34220$. □

16. **Answer.** 32

Solution. It is clear that $8(\cos 40^\circ)^3 - 6\cos 40^\circ + 1 = 0$, since $\cos 3A = 4\cos^3 A - 3\cos A$. Observe that

$$\begin{aligned}
& \frac{3}{\sin^2 20^\circ} - \frac{1}{\cos^2 20^\circ} + 64\sin^2 20^\circ \\
= & \frac{6}{1 - \cos 40^\circ} - \frac{2}{1 + \cos 40^\circ} + 32(1 - \cos 40^\circ) \\
= & \frac{8\cos 40^\circ + 4}{1 - (\cos 40^\circ)^2} + 32 - 32\cos 40^\circ \\
= & \frac{8\cos 40^\circ + 4 - 32\cos 40^\circ + 32(\cos 40^\circ)^3}{1 - (\cos 40^\circ)^2} + 32 \\
= & 4 \times \frac{1 - 6\cos 40^\circ + 8(\cos 40^\circ)^3}{1 - (\cos 40^\circ)^2} + 32 \\
= & 32,
\end{aligned}$$

where the last step follows from $8(\cos 40^\circ)^3 - 6\cos 40^\circ + 1 = 0$. □

17. **Answer.** 6029

Solution. Given the original equation

$$\mathrm{f}(x^2 + x) + 2\mathrm{f}(x^2 - 3x + 2) = 9x^2 - 15x,$$

we replace x by $1 - x$ and obtain

$$\mathrm{f}(x^2 - 3x + 2) + 2\mathrm{f}(x^2 + x) = 9(1 - x)^2 - 15(1 - x) = 9x^2 - 3x - 6.$$

Eliminating $\mathrm{f}(x^2 - 3x + 2)$ from the two equations, we obtain

$$3\mathrm{f}(x^2 + x) = 9x^2 + 9x - 12,$$

thereby

$$\mathrm{f}(x^2 + x) = 3x^2 + 3x - 4 = 3(x^2 + x) - 4,$$

hence $\mathrm{f}(2011) = 3(2011) - 4 = 6029$.

□

18. **Answer.** 2112

Solution. We denote the numbers of regions divided by n circles by $P(n)$. We have $P(1) = 2,\ P(2) = 4,\ P(3) = 8,\ P(4) = 14, \ldots$ and from this we notice that

$$\begin{aligned} P(1) &= 2, \\ P(2) &= P(1) + 2, \\ P(3) &= P(2) + 4, \\ P(4) &= P(3) + 6, \\ \ldots & \quad \ldots \\ P(n) &= P(n-1) + 2(n-1). \end{aligned}$$

Summing these equations, we obtain

$$P(n) = 2 + 2 + 4 + \ldots + 2(n-1) = 2 + n(n-1).$$

This formula can be shown by induction on n to hold true.
Base case: $n = 1$ is obvious.
Inductive step: Assume that the formula holds for $n = k \geq 1$, i.e., $P(k) = 2 + k(k-1)$. Consider $k+1$ circles, the $(k+1)$-th circle intersects k other circles at $2k$ points (for each one, it cuts twice), which means that this circle is divided into $2k$ arcs, each of which divides the region it passes into two sub-regions. Therefore, we have in addition $2k$ regions, and so

$$P(k+1) = P(k) + 2k = 2 + k(k-1) + 2k = 2 + k(k+1).$$

The proof by induction is thus complete.
Using this result, put $n = 2011$, the number of regions $N = 2 + 2011 \cdot (2011 - 1) = 4042112$. So, the last 4 digits are 2112. $\square$

19. **Answer.** 6034

Solution. Let n be a positive integer.
If $n \leq x < n + \frac{1}{3}$, then $2n \leq 2x < 2n + \frac{2}{3}$ and $3n \leq 3x < 3n + 1$, giving

$$N = \lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 3x \rfloor = n + 2n + 3n = 6n.$$

If $n + \frac{1}{3} \leq x < n + \frac{1}{2}$, then $2n + \frac{2}{3} \leq 2x < 2n + 1$ and $3n + 1 \leq 3x < 3n + \frac{3}{2}$, giving

$$N = \lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 3x \rfloor = n + 2n + 3n + 1 = 6n + 1.$$

If $n + \frac{1}{2} \leq x < n + \frac{2}{3}$, then $2n + 1 \leq 2x < 2n + \frac{4}{3}$ and $3n + \frac{3}{2} \leq 3x < 3n + \frac{4}{3}$, giving

$$N = \lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 3x \rfloor = n + 2n + 1 + 3n + 1 = 6n + 2.$$

If $n + \frac{2}{3} \leq x < n + 1$, then $2n + \frac{4}{3} \leq 2x < 2n + 2$ and $3n + 2 \leq 3x < 3n + 3$, giving

$$N = \lfloor x \rfloor + \lfloor 2x \rfloor + \lfloor 3x \rfloor = n + 2n + 1 + 3n + 2 = 6n + 3.$$

Thus, "invisible" numbers must be of the form $6n + 4$ and $6n + 5$. The 2011^{th} "invisible" integer is $4 + 6 \times \dfrac{2011 - 1}{2} = 6034$. $\square$

20. **Answer.** 95004

Solution. We shall prove that for any positive integer a, if $f(a)$ denotes the sum of all nonnegative integer solutions to $\lfloor \frac{n}{a} \rfloor = \lfloor \frac{n}{a+1} \rfloor$, then

$$f(a) = \frac{1}{6}a(a^2-1)(a+2).$$

Thus $f(27) = 95004$.

Let n be a solution to $\lfloor \frac{n}{a} \rfloor = \lfloor \frac{n}{a+1} \rfloor$. Write $n = aq + r$, where $0 \le r < a$. Thus $\lfloor \frac{n}{a} \rfloor = q$. Also $n = (a+1)q + r - q$. Since $\lfloor \frac{n}{a+1} \rfloor = q$, we have $0 \le r - q$, that is, $q \le r < a$. Therefore for each $q = 0, 1, \ldots, a-1$, r can be anyone of the values $q, q+1, \ldots, a-1$. Thus

$$\begin{aligned}
A &= \sum_{q=0}^{a-1}\sum_{r=q}^{a-1}(qa+r) \\
&= \sum_{q=0}^{a-1}(a-q)qa + \sum_{q=0}^{a-1}\sum_{r=q}^{a-1} r \\
&= a^2\sum_{q=0}^{a-1} q - a\sum_{q=0}^{a-1} q^2 + \sum_{r=0}^{a-1}\sum_{q=0}^{r} r \\
&= a^2\sum_{q=0}^{a-1} q - a\sum_{q=0}^{a-1} q^2 + \sum_{r=0}^{a-1} r(r+1) \\
&= a^2\sum_{q=0}^{a-1} q - a\sum_{q=0}^{a-1} q^2 + \sum_{r=0}^{a-1} r^2 + \sum_{r=0}^{a-1} r \\
&= (a^2+1)\cdot\frac{1}{2}a(a-1) + (1-a)\cdot\frac{1}{6}a(2a-1)(a-1) \\
&= \frac{1}{6}a(a^2-1)(a+2).
\end{aligned}$$

□

21. **Answer.** 48

By using factor formulae and double angle formulae:

$$\frac{\sin A + \sin B + \sin C}{\cos A + \cos B + \cos C} = \frac{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{12}{7},$$

and

$$\sin A \sin B \sin C = 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{12}{25}.$$

Solving these equations, we obtain

$$\begin{aligned}
\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} &= 0.1 \\
\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} &= 0.6
\end{aligned}$$

Furthermore,

$$\sin\frac{C}{2} = \cos\left(\frac{A+B}{2}\right) = \cos\frac{A}{2}\cos\frac{B}{2} - \sin\frac{A}{2}\sin\frac{B}{2},$$

multiplying both sides by $\sin\frac{C}{2}\cos\frac{C}{2}$, we get

$$\sin^2\frac{C}{2}\cos\frac{C}{2} = 0.6\sin\frac{C}{2} - 0.1\cos\frac{C}{2}.$$

or equivalently,

$$(1-t^2)t = 0.6\sqrt{1-t^2} - 0.1t \iff 11t - 10t^3 = 6\sqrt{1-t^2},$$

where $t = \cos\frac{C}{2}$. This equation solves for $t = \sqrt{\frac{1}{2}},\ \sqrt{\frac{4}{5}},\ \sqrt{\frac{3}{10}}$, and so the corresponding values of $\sin C$ are

$$1,\ 0.8,\ 0.6$$

and hence $100s_1s_2s_3 = 100\cdot 1\cdot 0.8\cdot 0.6 = 48$. □

22. **Answer.** 8

Solution. We first prove that if $x \geq 8$, then $z = 2$. To this end, we observe that the left hand side of the equation $1! + 2! + 3! + \ldots + x!$ is divisible by 3, and hence $3 \mid y^z$. Since 3 is a prime, $3 \mid y$. So, $3^z \mid y^z$ by elementary properties of divisibility.

On the other hand, when $x = 8$,

$$1! + 2! + \ldots + 8! = 46233$$

is divisible by 3^2 but not by 3^3. Now, note that if $n \geq 9$, then we have $3^3 \mid n!$. So, when $x \geq 8$, the left hand side is divisible by 3^2 but not by 3^3. This means that $z = 2$.

We now prove further that when $x \geq 8$, then the given equation has no solutions. To prove this, we observe that $x \geq 8$ implies that

$$1! + 2! + 3! + 4! + \underbrace{5! + \ldots x!}_{\text{divisible by 5}} \equiv 3 \pmod 5.$$

Since we have deduced that $z = 2$, we only have $y^2 \equiv 0,\ 1,\ -1 \pmod 5$. This mismatch now completes the argument that there are no solutions to the equation when $x \geq 8$.

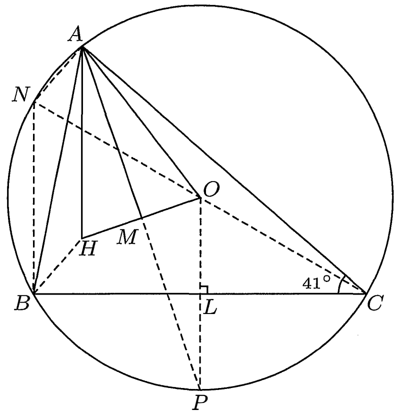
So the search narrows down to $x < 8$. By exhaustion, it is easy to find that there is only one solution:

$$x = y = 3,\ z = 2.$$

Thus, the sum of this only combination must be the largest and is equal to $3 + 3 + 2 = 8$. □

23. **Answer.** 38

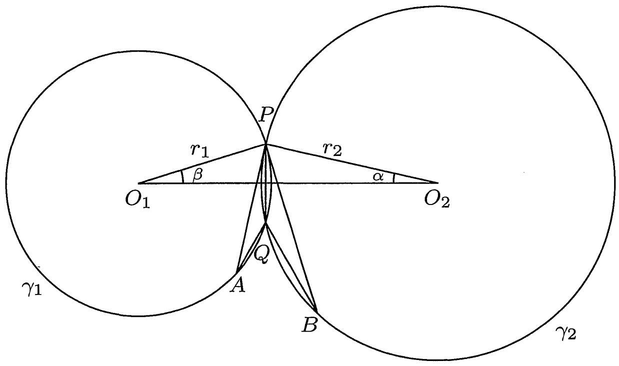
Let P be the midpoint of the arc BC not containing A on the circumcircle of the triangle ABC. Then OP is the perpendicular bisector of BC. Since AM bisects $\angle A$, the points A, M, P are collinear. As both AH and OP are perpendicular to BC, they are parallel. Thus $\angle HAM = \angle OPM = \angle OAM$. Also $\angle HMA = \angle OMP$. Since $HM = OM$, we have the triangles AHM and POM are congruent. Therefore $AH = PO = AO$.



Let L be the midpoint of BC. It is a known fact that $AH = 2OL$. To see this, extend CO meeting the circumcircle of the triangle ABC at the point N. Then $ANBH$ is a parallelogram. Thus $AH = NB = 2OL$. Therefore in the right-angled triangle OLC, $OC = OA = AH = 2OL$. This implies $\angle OCL = 30°$. Since the triangle ABC is acute, the circumcentre O lies inside the triangle. In fact $\angle A = 60°$ and $\angle B = 79°$. Then $\angle OAC = \angle OCA = 41° - 30° = 11°$. Consequently, $\angle HAO = 2\angle OAM = 2 \times (30° - 11°) = 38°$. □

24. **Answer.** 30

Let $PO_1 = r_1$ and $PO_2 = r_2$. First note that O_1O_2 intersects PQ at the midpoint H (not shown in the figure) of PQ perpendicularly. Next observe that $\angle APQ = \angle PBQ = \angle PO_2O_1$, and $\angle BPQ = \angle PAQ = \angle PO_1O_2$. Therefore $\angle APB = \angle APQ + \angle BPQ = \angle PO_2O_1 + \angle PO_1O_2$.



Let $\angle PO_2O_1 = \alpha$ and $\angle PO_1O_2 = \beta$. Then $\sin\alpha = \frac{PQ}{2r_2}, \cos\alpha = \frac{O_2H}{r_2}$ and $\sin\beta = \frac{PQ}{2r_1}, \cos\beta = \frac{O_1H}{r_1}$. Thus $\sin\angle APB = \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \frac{PQ}{2r_2}\cdot\frac{O_1H}{r_1} + \frac{O_2H}{r_2}\cdot\frac{PQ}{2r_1} = \frac{PQ\cdot(O_1H+O_2H)}{2r_1r_2} = \frac{PQ\cdot O_1O_2}{2r_1r_2} = \frac{1}{2}$. Since $\angle APB$ is acute, it is equal to 30°. □

25. **Answer.** 2

Solution. Let

$$a_n = \sum_{i=0}^{n} \binom{n}{i}^{-1}.$$

Assume that $n \geq 3$. It is clear that

$$a_n = 2 + \sum_{i=1}^{n-1} \binom{n}{i}^{-1} > 2.$$

Also note that

$$a_n = 2 + 2/n + \sum_{i=2}^{n-2} \binom{n}{i}^{-1}.$$

Since $\binom{n}{i} \geq \binom{n}{2}$ for all i with $2 \leq i \leq n-2$,

$$a_n \leq 2 + 2/n + (n-3)\binom{n}{2}^{-1} \leq 2 + 2/n + 2/n = 2 + 4/n.$$

So we have show that for all $n \geq 3$,

$$2 < a_n \leq 2 + 4/n.$$

Thus

$$\lim_{n\to\infty} a_n = 2.$$

□

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2011

(Open Section, Round 2)

Saturday, 2 July 2011 **0900-1330**

---

1. In the acute-angled non-isosceles triangle ABC, O is its circumcentre, H is its orthocentre and $AB > AC$. Let Q be a point on AC such that the extension of HQ meets the extension of BC at the point P. Suppose $BD = DP$, where D is the foot of the perpendicular from A onto BC. Prove that $\angle ODQ = 90°$.

2. If 46 squares are colored red in a 9×9 board, show that there is a 2×2 block on the board in which at least 3 of the squares are colored red.

3. Let $x, y, z > 0$ such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < \frac{1}{xyz}$. Show that

$$\frac{2x}{\sqrt{1+x^2}} + \frac{2y}{\sqrt{1+y^2}} + \frac{2z}{\sqrt{1+z^2}} < 3.$$

4. Find all polynomials $P(x)$ with real coefficients such that

$$P(a) \in \mathbb{Z} \quad \text{implies that} \quad a \in \mathbb{Z}.$$

5. Find all pairs of positive integers (m, n) such that

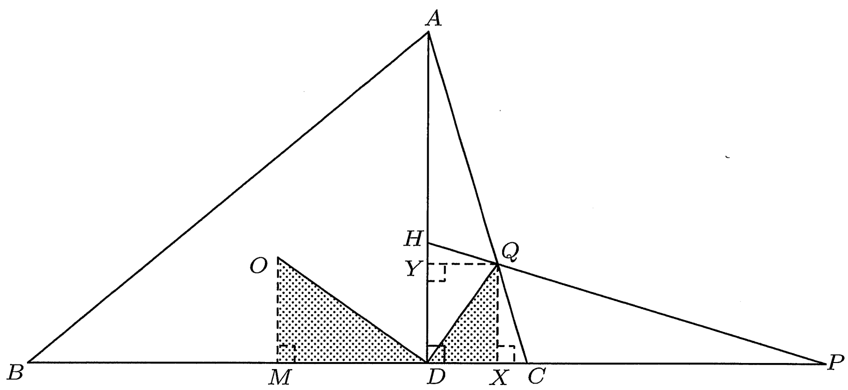
$$m + n - \frac{3mn}{m+n} = \frac{2011}{3}.$$

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2011

(Open Section, Round 2 solutions)

1. Drop perpendiculars OM and QX onto BC, and QY from Q onto AD. First $2DM = DM + BD - BM = BD - (BM - DM) = PD - (CM - DM) = PD - CD = PC$. It is a well-known fact that $2OM = AH$.



Next $\angle CPQ = \angle DBH = \angle HAQ$ so that the triangles CPQ and HAQ are similar. Thus the triangles XPQ and YAQ are similar. Therefore

$$\frac{QX}{DX} = \frac{QX}{QY} = \frac{PC}{AH} = \frac{DM}{OM}.$$

Hence the triangles DXQ and OMD are similar. It follows that $\angle ODQ = 90°$.

2. Suppose that at most 2 squares are colored red in any 2×2 square. Then in any 9×2 block, there are at most 10 red squares. Moreover, if there are 10 red squares, then there must be 5 in each row. This can be seen as follows. There are 8 2×2 blocks. Counting multiplicity, there are altogether 16 red squares. Each red square in the interior is counted twice while each red square at the edge is counted once. If there are 11 red squares, then there are at least 7 red squres in the interior. Thus the total count is at least $4 + 7 \times 2 = 18 > 16$, a contradiction. If there are exactly 10 red squares, then 4 of them must be at the edge and the red squares in each row are not next to each other and hence there 5 in each row.

Now let the number of red squares in row i be r_i. Then $r_i + r_{i+1} \leq 10$, $1 \leq i \leq 8$. Suppose that some $r_i \leq 5$ with i odd. Then

$$(r_1 + r_2) + \cdots + (r_{i-2} + r_{i-1}) + r_i + \cdots + (r_8 + r_9) \leq 4 \times 10 + 5 = 45$$

which leads to a contradiction. On the other hand, suppose that $r_1, r_3, r_5, r_7, r_9 \geq 6$. Then the sum of any 2 consecutive r_i's is ≤ 9. Again we get a contradiction as

$$(r_1 + r_2) + \cdots + (r_7 + r_8) + r_9 \leq 4 \times 9 + 9 = 45.$$

3. Let $r = 1/x, s = 1/y, t = 1/z$. There exists $\alpha < 1$ such that $r + s + t = \alpha^2 rst$ or $\alpha(r + s + t) = \alpha^3 rst$. Let $a = \alpha r$, $b = \alpha s$, $c = \alpha t$. Write $a = \tan A$, $b = \tan B$, $c = \tan C$, then $A + B + C = \pi$. It is clear that

$$\begin{aligned}
\frac{1}{2} \times \text{LHS} &= \frac{1}{\sqrt{1+r^2}} + \frac{1}{\sqrt{1+s^2}} + \frac{1}{\sqrt{1+t^2}} \\
&< \frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \\
&= \cos A + \cos B + \cos B \\
&\leq 3\cos\left(\frac{A+B+C}{3}\right) = \frac{3}{2} = \frac{1}{2} \times \text{RHS}.
\end{aligned}$$

2nd soln: Note that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} < \frac{1}{xyz} \quad \Rightarrow \quad xy + yz + xz < 1.$$

Hence

$$\frac{2x}{\sqrt{1+x^2}} < \frac{2x}{\sqrt{x^2 + xy + xz + yz}} = \frac{2x}{\sqrt{(x+y)(x+z)}}.$$

By AM-GM we have

$$\frac{2x}{\sqrt{(x+y)(x+z)}} \leq \frac{x}{x+y} + \frac{x}{x+z}.$$

Similarly,

$$\frac{2y}{\sqrt{(y+z)(y+x)}} \leq \frac{y}{y+z} + \frac{y}{y+x}, \quad \frac{2z}{\sqrt{(z+x)(z+y)}} \leq \frac{z}{z+x} + \frac{z}{z+y}.$$

The desired inequality then follows by adding up the three inequalities.

4. Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$. Define $Q(x) = P(x+1) - P(x)$. Then $Q(x)$ is of degree $n-1$. We'll prove by contradiction that $|Q(x)| \leq 3$ for all x. This will imply that $n \leq 1$. Assume that $|Q(a)| > 3$ for some $a \in \mathbb{R}$. Then $|P(a+1) - P(a)| > 3$. Thus there are 3 integers between $P(a)$ and $P(a+1)$. Hence there exists three values of $x \in [a, a+1]$ such that $P(x)$ is an integer. Thus there are three integers in $[a, a+1]$, a contradiction. There are two cases:

Case (i) $n = 0$. Here we have $P(x) = c$ where $c \notin \mathbb{Z}$.

Case (ii) $n = 1$. Here $P(x) = sx + t$. There are two integers m, n such that $P(m) = sm + t = 0$ and $P(n) = sn + t = 1$. Thus $s(n - m) = 1$ implying that $1/s \in \mathbb{Z}$ and $sm + t = 0$ implying that $t/s \in \mathbb{Z}$. Letting $1/s = p$ and $t/s = q$, $P(x) = \frac{x}{p} + \frac{q}{p}$ where $p, q \in \mathbb{Z}$ and $p \neq 0$.

5. Answer: $(m, n) = (1144, 377)$ or $(377, 1144)$.

Let m and n be positive integers satisfying the given equation. That is $2011(m + n) = 3(m^2 - mn + n^2)$. Since the equation is symmetric in m and n, we may assume $m \geq n$. If $m = n$, then $m = n = 4022/3$ which is not an integer. So we may further assume $m > n$. Let $p = m+n$ and $q = m-n > 0$. Then $m = (p+q)/2$ and $n = (p-q)/2$, and the equation becomes $8044p = 3(p^2 + 3q^2)$. Since 3 does not divide 8044, it must divide p. By letting $p = 3r$, the above equation reduces to $8044r = 3(3r^2 + q^2)$. From this, 3 must divide r. By letting $r = 3s$, we get $8044s = 27s^2 + q^2$, or equivalently

$$s(8044 - 27s) = q^2. \qquad (*)$$

For s between 1 and $\lfloor 8044/27 \rfloor = 297$, the number $s(8044 - 27s)$ is a square only when $s = 169$. To narrow down the values of s, we proceed as follow.

Let $s = 2^\alpha u$, where α is a nonnegative integer and u is an odd positive integer. Suppose α is odd and $\alpha \geq 3$. Then $(*)$ becomes $2^{\alpha+2}u(2011 - 27 \times 2^{\alpha-2}u) = q^2$ which is a square. Since $\alpha + 2$ is odd, 2 must divide $2011 - 27 \times 2^{\alpha-2}u$ implying 2 divides 2011 which is a contradiction. Next suppose $\alpha = 1$. Then we have $u(2 \times 2011 - 27u) = (q/2)^2$. If u is not a square, then there exists an odd prime factor t of u such that t divides $2 \times 2011 - 27u$. Thus t divides 2×2011 so that t must be 2011 since 2011 is a prime. But then $u \geq t = 2011$ contradicting $2 \times 2011 - 27u > 0$. Therefore u must be a square. This implies that $2 \times 2011 - 27u$ is also a square. Taking mod 4, we have $u \equiv 0$ or $1 \pmod 4$ so that $2 \times 2011 - 27u \equiv 2$ or $3 \pmod 4$ which contradicts the fact that $2 \times 2011 - 27u$ is a square. Thus $\alpha \neq 1$ too. Consequently α must be even. Then dividing both sides of $(*)$ by 2^α, we obtain $u(8044 - 27 \times 2^\alpha u) = q^2/2^\alpha$ which is a square. Now suppose u is not a square. Then there exists an odd prime factor v of u such that v divides $8044 - 27 \times 2^\alpha u$. Then v must divide 8044 so that $v = 2011$. Thus $u \geq v = 2011$. This again contradicts the fact that $8044 - 27 \times 2^\alpha u > 0$. Therefore u is a square. Consequently s is also a square. Write $s = w^2$. Then $(*)$ becomes $w^2(8044 - 27w^2) = q^2 \geq 0$. From this $w \leq \lfloor (8044/27)^{\frac{1}{2}} \rfloor = 17$. A direct verification shows that $8044 - 27w^2$ is a square only when $w = 13$. Thus $s = w^2 = 169$. Then $p = 3r = 9s = 1521$, and by $(*)$ $q = (169 \times (8044 - 27 \times 169))^{\frac{1}{2}} = 767$. Lastly, $m = (p+q)/2 = 1144$ and $n = (p-q)/2 = 377$.