

Engineering Mathematics II



Semester II

Babu Ram

Engineering Mathematics-II for UPTU

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Engineering Mathematics-II for UPTU

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PEARSON

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In the Memory of

MY PARENTS

Smt. Manohari Devi and Sri. Makhan Lal

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Preface

All branches of Engineering, Technology and Science require mathematics as a tool for the description of their contents. Therefore thorough knowledge of various topics in mathematics is essential to pursue study in Engineering, Technology and Science. The aim of this book is to provide the students with sound mathematics skills and their applications. Although the book is designed primarily for use by engineering students, it is also suitable for students pursuing bachelor degrees with mathematics as one of the subject and also for those who prepare for various competitive examinations. The material has been arranged to ensure the suitability of the book for class use and for individual self study. Accordingly, the contents of the book have been divided into seven chapters covering the complete syllabus prescribed for B.Tech. Semester-II of U.P. Technical University, Lucknow. A sufficient number of examples, figures, tables, and exercises have been provided to enable students to develop problem-solving skills. The language used is simple and amicable. Suggestions and feedback on this book are welcome.

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BABU RAM

Syllabus

EAS-203 – MATHEMATICS-II

Syllabi Mapping of *Engineering Mathematics II* for UPTU by Babu Ram

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Unit I: Differential Equations Linear differential equations of n th order with constant coefficients, complementary functions and particular integrals, simultaneous linear differential equations, solution of second order differential equation by changing dependent and independent variables, method of variation of parameters, applications to engineering problems (without derivation).

Refer Chapter 2

Unit II: Series Solution and Special Functions Series solution of ordinary differential equations of 2nd order with variable coefficients (Frobenius Method), Bessel and Legendre equations and their series solutions, properties of Bessel functions and Legendre polynomials.

Refer Chapter 2

Unit III: Laplace Transform Laplace transform, existence theorem, Laplace transform of derivatives and integrals, Inverse Laplace transform, unit step function, Dirac delta function, Laplace transform of periodic functions, convolution theorem, application to solve simple linear and simultaneous differential equations.

Refer Chapters 5, 6 & 7

Unit IV: Fourier Series and Partial Differential Equations Periodic functions, Trigonometric series, Fourier series of period 2π , Eulers formulae, functions having arbitrary period, change of interval, even and odd functions, half range sine and cosine series, harmonic analysis.

Refer Chapter 4

Solution of first order Lagrange's linear partial differential equations, Linear partial differential equations with constant coefficients of 2nd order and their classifications—parabolic, elliptic and hyperbolic with illustrative examples.

Refer Chapter 3

Unit V: Applications of Partial Differential Equations Method of separation of variables for solving partial differential equations, Wave equation up to two-dimensions, Laplace equation in two-dimensions, Heat conduction equations up to two-dimensions, Equations of transmission lines.

Refer Chapter 6

1 Preliminaries

In this chapter, we present some basic mathematical concepts and definitions which shall be frequently used in the forthcoming chapters. The knowledge of these concepts is essential due to their extensive use in the study of Fourier series and various integral transforms. For example, knowledge of real- and complex-valued functions—along with their continuity, differentiability, and integrability, and sequences, series, and special functions—is required time and again in this study.

1.1 SETS AND FUNCTIONS

According to Georg Cantor, a *set* may be viewed as a well-defined collection of objects, called the *elements* or *members* of the set.

The sets are denoted by capital letters such as A , B , and C whereas its elements are denoted by lower-case letters such as a , b , and c . We write $a \in A$ if a is an element of the set A .

EXAMPLE 1.1

- (a) $\mathbb{R} = \{x: x \text{ is a real number}\}$ is called the *set of real numbers*.
- (b) $\mathbb{Z} = \{x: x \text{ is an integer}\}$ represents the set of *all integers*... $-4, -3, -2, -1, 0, 1, 2, 3, 4, \dots$
- (c) $\mathbb{N} = \{x: x \text{ is a positive integer or zero}\}$ represents the set consisting of $0, 1, 2, 3, \dots$
- (d) The set having no elements is represented by ϕ or $\{\}$ and is called the *empty* (or *null* or *void*) set.

Definition 1.1. Let A and B be sets. Then A is called a subset of B , written as $A \subseteq B$ if and only if every element of A is also an element of B .

EXAMPLE 1.2

The set $A = \{6, 5, 8\}$ is a subset of the set $B = \{2, 3, 6, 7, 5, 8\}$.

Definition 1.2. Let A and B be sets. Then A is called a *proper* subset of B if and only if every element of A is in B , but there is at least one element of B that is not in A .

EXAMPLE 1.3

The set $\{5, 6, 8\}$ is a proper subset of the set $\{1, 5, 6, 8, 4\}$.

Definition 1.3. Two sets A and B are called *equal* if every element of A is in B and every element of B is in A . Thus, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Definition 1.4. Suppose we are dealing with sets, all of which are subsets of a set U . Then, the set U is called a *universal set* or a *universe of discourse*.

Definition 1.5. Let A and B be subsets of a universal set U . Then the *union* of A and B , denoted by $A \cup B$, is the set of all elements a in U such that a is in A or a is in B .

Thus,

$$A \cup B = \{a \in U : a \in A \text{ or } a \in B\}.$$

Definition 1.6. Let A and B be subsets of a universal set U . Then, the *intersection* of A and B , denoted by $A \cap B$, is the set of all elements a of U such that $a \in A$ and $a \in B$.

Thus,

$$A \cap B = \{a \in U : a \in A \text{ and } a \in B\}.$$

Definition 1.7. Let A and B be subsets of a universal set U . Then, the *difference* $B - A$ (or the *relative complement of A in B*) is the set of all elements a in U such that $a \in B$ and $a \notin A$.

Thus,

$$B - A = \{a \in U : a \in B \text{ and } a \notin A\}.$$

Definition 1.8. Let A be a subset of a universal set U . Then *complement* of A , denoted by A^c , is the set of all elements a in U such that a is not in A .

Thus,

$$A^c \{a \in U : a \notin A\}.$$

Definition 1.9. Two sets A and B are called *disjoint* if and only if they have no element in common.

Definition 1.10. Let X and Y be arbitrary given sets. By a *function* $f: X \rightarrow Y$ from the set X into Y , we mean a rule which assigns to each member x of X , a unique member $f(x)$ of Y .

The member $f(x)$ is called *image* of x under the function (mapping) f or the *value* of f at x . The set X is called the *domain* of f and the set Y is called the *codomain* of f . The set of elements $f(x)$, $x \in X$ is called the *range* of f . Thus, the range of f is a subset of Y .

If $Y = \mathbb{R}$, the set of real numbers, then f is called the *real-valued* function and if $Y = \mathbb{C}$, the set of complex numbers, then f is called the *complex-valued* function. If $X = \mathbb{C}$, then f is called a *function of complex variables*.

Definition 1.11. Let $f: A \rightarrow B$ be a mapping from the set A into the set B . If $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for every $x_1, x_2 \in A$, then f is called *one-one mapping* or *injective mapping*.

Thus, a function $f: A \rightarrow B$ is injective if and only if the *images of distinct points of A are distinct*, that is, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

EXAMPLE 1.4

Let $\mathbb{Z}^+$ be a set of positive integers and Y be a set of even positive integers. Then, the mapping $f: \mathbb{Z}^+ \rightarrow Y$ defined by $f(x) = 2x$ is injective. In fact, if $x, y \in \mathbb{Z}^+$, then $f(x) = 2x$, $f(y) = 2y$ and so, $f(x) = f(y)$ implies $2x = 2y$ and hence, $x = y$.

Definition 1.12. A function f which is not one-one is called a *many-to-one mapping*.

EXAMPLE 1.5

The function f defined by $f(x) = x^2$, $-\infty < x < \infty$ is not one-one, because 4 is the image of both -2 and 2 .

Definition 1.13. Let $f: X \rightarrow Y$ be a map. If $f(X) = Y$, that is, the range of f is the whole of Y , then f is called a *surjective* or *onto mapping*.

Thus, $f: X \rightarrow Y$ is onto if and only if for every point y in Y there exists at least one point x in X such that $f(x) = y$.

EXAMPLE 1.6

The linear function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = ax + b$, $x \in \mathbb{R}$ is surjective whereas the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \sin x$ is not surjective. In fact, there is no element in $\mathbb{R}$ for which $\sin x = 2$. Thus, the range of f is not equal to $\mathbb{R}$.

Consider $f: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ defined by $f(x) = x^2$, $x \in \mathbb{Z}^+$. Then, if $x, y \in \mathbb{Z}^+$, we note that

$$\begin{aligned} f(x) = f(y) &\Rightarrow x^2 = y^2 \\ &\Rightarrow (\pm x)^2 = (\pm y)^2 \\ &\Rightarrow \pm x = \pm y \\ &\Rightarrow x = y, \text{ since } x, y \in \mathbb{Z}^+. \end{aligned}$$

Hence, f is one-one. Further

$$R(f) = \{1, 4, 9, \dots\},$$

which is a proper subset of $\mathbb{Z}^+$. Thus, f is *not surjective*.

Definition 1.14. A mapping $f: X \rightarrow Y$ is called *bijective* if it is both injective and surjective.

For example, if $X = \{x \in \mathbb{R}, x \neq 0\}$. Then, the mapping $f: X \rightarrow X$ defined by $f(x) = \frac{1}{x}$ is one-one and onto and hence, bijective.

1.2 CONTINUOUS AND PIECEWISE CONTINUOUS FUNCTIONS

Definition 1.15. A function $f: X \rightarrow \mathbb{R}$ is said to be *continuous* at a point $x_0 \in X$, if given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon \quad \text{whenever } |x - x_0| < \delta.$$

Equivalently, we say that f is continuous at x_0 if $\lim_{x \rightarrow x_0} f(x) = f(x_0)$.

The *left-hand limit* of f at the point x_0 is defined by

$$\lim_{\substack{x \rightarrow x_0 \\ x < x_0}} f(x) = f(x_0 - 0),$$

provided the limit exists and is finite.

Similarly, the *right-hand limit* of f at the point x_0 is defined by

$$\lim_{\substack{x \rightarrow x_0 \\ x > x_0}} f(x) = f(x_0 + 0),$$

provided the limit exists and is finite.

At the point of continuity, both the left- and right-hand limit exist and

$$f(x_0 - 0) = f(x_0 + 0) = f(x_0).$$

At a point x_0 of discontinuity, if both $f(x_0 - 0)$ and $f(x_0 + 0)$ exist but are not equal, then x_0 is called a point of *discontinuity of the first kind* or a point of *jump discontinuity*. In such a case, $f(x_0 + 0) - f(x_0 - 0)$ is called the *jump* of the discontinuous function f at the point x_0 .

EXAMPLE 1.7

Consider the function f defined by

$$f(x) = \begin{cases} x^2 & \text{for } x \leq 0 \\ 4x + 3 & \text{for } x > 0. \end{cases}$$

Then,

$$\begin{aligned} f(0-) &= \lim_{x \rightarrow 0-} x^2 = 0 \text{ and} \\ f(0+) &= \lim_{x \rightarrow 0+} 4x + 3 = 3. \end{aligned}$$

Thus, $f(0-) \neq f(0+)$ and so, f has jump discontinuity at $x = 0$. The jump of this discontinuity is

$$f(0+) - f(0-) = 3.$$

If at least one of left- and right-hand limit does not exist at a point x_0 , then x_0 is called a point of *discontinuity of second kind*.

EXAMPLE 1.8

The function f defined by $f(x) = \frac{1}{t-3}$ has a discontinuity at $t = 3$. Since neither the left- nor the right-hand limit exists at $t = 3$, the function has a discontinuity of second kind.

Definition 1.16. A function f (real-valued or complex-valued) is called *piecewise* (or *sectionally*) *continuous* on the interval $[a, b]$ if there exists a partition $a = x_0 < x_1 < \dots < x_n = b$ of $[a, b]$ such that f is continuous in each of the open interval (x_{i-1}, x_i) , $i = 1, 2, \dots, n$ and each of the limit

$f(a+)$, $f(b-)$, $f(x_i+)$, and $f(x_i-)$ exists for $i = 1, 2, \dots, n$.

The function f is called the *piecewise continuous* on $\mathbb{R}$ if f is the piecewise continuous on each subinterval $[a, b]$ of $\mathbb{R}$.

Some of the properties of piecewise continuous functions are listed below:

- A function f continuous on $[a, b]$ is piecewise continuous there.
- The sum, difference, and product of two functions which are piecewise continuous on $[a, b]$ are piecewise continuous on $[a, b]$.
- A function piecewise continuous on $[a, b]$ is *bounded* on $[a, b]$, that is, there exists a positive constant M such that $|f(x)| \leq M$ for all $x \in [a, b]$.
- The definite integral of a piecewise continuous function exists on $[a, b]$ and

$$\int_a^b f(x) dx = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(x) dx.$$

- The indefinite integral $\int_a^t f(x) dx$, $a \leq t \leq b$ exists and is continuous on $[a, b]$.

EXAMPLE 1.9

The function f defined by

$$f(x) = \begin{cases} e^{-x^2/2} & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$$

has a jump discontinuity at $x = 0$ (with jump 1) and is continuous elsewhere. Hence, f is piecewise continuous on $[0, \infty)$. But the function f defined by

$$f(x) = \sin \frac{1}{x}, \quad x \neq 0$$

$$f(0) = 0$$

is not piecewise continuous on $[0, 1]$ because $f(0+)$ does not exist.

1.3 DERIVABILITY OF A FUNCTION AND PIECEWISE SMOOTH FUNCTIONS

Definition 1.17. A function f defined on $[a, b]$ is said to be derivable at x if

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \text{ exists.}$$

In case the limit exists, then this limit is called the *derivative* of f at x and is denoted by $f'(x)$. The limits

$$\lim_{h \rightarrow 0^-} \frac{f(x+h) - f(x)}{h} \quad \text{and} \quad \lim_{h \rightarrow 0^+} \frac{f(x+h) - f(x)}{h}$$

if exist are called, respectively, the *left-* and *right-hand derivative* and are again denoted by $f'_-(x)$ and $f'_+(x)$. Clearly, $f'(x)$ exists if both $f'_-(x)$ and $f'_+(x)$ exist and are equal.

Further, if a function is derivable at a point, then it is continuous at that point. But the converse need not be true. For example, the function f defined by $f(x) = |x|$, $x \in \mathbb{R}$ is continuous at $x = 0$ but it is not derivable at $x = 0$. In fact, $f'_-(x) = -1$ and $f'_+(x) = 1$ for this function.

Definition 1.18. A piecewise continuous function f on the interval $[a, b]$ is called *piecewise smooth* if its derivative f' is piecewise continuous.

A function is called *piecewise smooth on $\mathbb{R}$* if it is piecewise smooth on each interval $[a, b]$ of $\mathbb{R}$. Thus, the graph of a piecewise smooth function is either a continuous curve or a discontinuous curve, which can have a finite number of *corners* (points at which the curve has two distinct tangents).

EXAMPLE 1.10

The function with the graph (Figure 1.1)

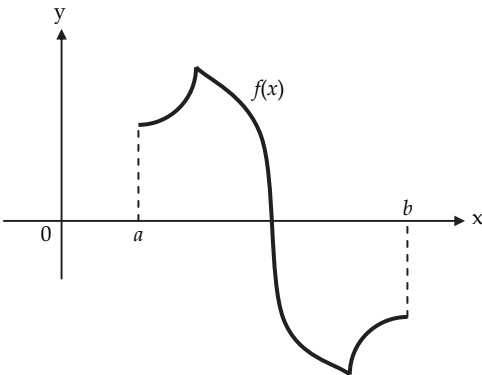


Figure 1.1

is continuous piecewise smooth, whereas the function with graph is discontinuous piecewise smooth (Figure 1.2)

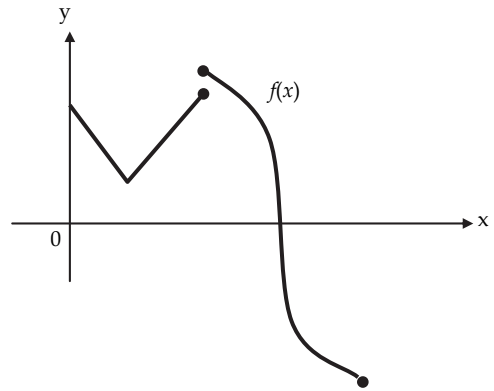


Figure 1.2

1.4 THE RIEMANN INTEGRAL

Let f be a *bounded real function* defined on $[a, b]$ and let

$$a = x_0 \leq x_1 \leq x_2 \leq \dots \leq x_n = b$$

be a partition of $[a, b]$. Corresponding to each partition P of $[a, b]$, let

$$M_i = \text{lub } f(x) \quad (x_{i-1} \leq x \leq x_i),$$

$$m_i = \text{glb } f(x) \quad (x_{i-1} \leq x \leq x_i),$$

$$U(P, f) = \sum_{i=1}^n M_i(x_i - x_{i-1}), \quad \text{and}$$

$$L(P, f) = \sum_{i=1}^n m_i(x_i - x_{i-1}).$$

The $U(P, f)$ and $L(P, f)$ are called *upper Riemann sum* and *lower Riemann sum*, respectively. Further, let

$$\int_a^b f(x) dx = \text{glb } U(P, f) \quad \text{and}$$

$$\int_a^b f(x) dx = \text{lub } L(P, f),$$

where glb and lub are taken over all possible partitions P of $[a, b]$. Then, $\int_a^b f$ is called the upper

Riemann integral and $\int_a^b f$ is called the lower Riemann integral of f over $[a, b]$.

A bounded real function f is said to be *Riemann-integrable* on $[a, b]$ if upper- and lower Riemann integral are equal and then, their common value, denoted by $\int_a^b f(x)dx$, is called the *Riemann-integral* of f .

If a or b is infinite or if f is unbounded for some $x \in [a, b]$, then the integral $\int_a^b f$ is called an *improper integral*.

Definition 1.19. A function f is called *absolutely integrable* on $\mathbb{R}$ if

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty$$

is an improper Riemann integral.

EXAMPLE 1.11

The function $p_a(t)$, known as a *Block function* or *Rectangular pulse function of height 1*, defined by

$$p_a(t) = \begin{cases} 1 & \text{for } |t| \leq \frac{a}{2}, \\ 0 & \text{otherwise} \end{cases},$$

is absolutely integrable.

1.5 THE CAUSAL AND NULL FUNCTION

Definition 1.20. A function f is called *causal* if $f(x) = 0$ for $x < 0$.

For example, f defined by

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ \sin x & \text{for } x > 0 \end{cases}$$

is a *causal sine function*.

A delayed (by $\pi/3$) causal sine function is then

$$g(x) = \begin{cases} 0 & \text{for } x < \pi/3 \\ \sin x & \text{for } x > \pi/3 \end{cases}$$

Similarly, the function H defined by

$$H(t-a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t > a \end{cases}$$

is causal and is called *Heaviside's unit step function*.

Definition 1.21. A function f is said to be a *null function* if for all $t > 0$,

$$\int_0^t f(x) dx = 0.$$

EXAMPLE 1.12

The function f defined by

$$f(t) = \begin{cases} 1 & \text{for } t = 1/2 \\ -1 & \text{for } t = 1 \\ 0 & \text{otherwise} \end{cases}$$

is a null function.

The other examples are

$$(a) N(t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases}$$

(b) a function which is identically zero (which is the only continuous null function).

1.6 FUNCTIONS OF EXPONENTIAL ORDER

Definition 1.22. A function f is said to be of *exponential order* or *exponential growth* α if there exist constants $\alpha \in \mathbb{R}$ and $M > 0$ such that for some $t_0 \geq 0$

$$|f(t)| \leq Me^{\alpha t}, t \geq t_0.$$

In such a case we write $f(t)$ is $O(e^{\alpha t})$.

EXAMPLE 1.13

(a) The function $f(x) = e^{3x} \sin x$ of exponential order $O(e^{3x})$ since with $M = 1$, $|f(x)| = |e^{3x} \sin x| \leq e^{3x}$.

(b) The function $\varepsilon(t)$ defined by

$$\varepsilon(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$

is of exponential order with $M = 1$ and $\alpha = 0$ because $|\varepsilon(t)| \leq 1$.

(c) Every bounded function f is of exponential order with $M = 1$ and $\alpha = 0$ because $|f(t)| \leq M$ for some $M > 0$.

(d) The function f defined by $f(t) = t^2$ is of exponential order 3 because $|t^2| = t^2 < e^{3t}$ for all $t > 0$.

(e) The function f defined by $f(x) = e^{x^3}$ is not of exponential order since $|e^{-\alpha x} e^{x^3}| = e^{x^3 - \alpha x}$ can be made larger than any given constant M by increasing x .

(f) The function f defined by $f(x) = e^{3x^2}$ is not of exponential order.

1.7 PERIODIC FUNCTIONS

Definition 1.23. A function f is called *periodic* if there exists a constant $T > 0$ for which $f(x) = f(x + T)$ for any x in the domain of definition of f .

The smallest value of the constant T for which the above condition holds is called the *fundamental period* (or simply *period*) of the function f .

EXAMPLE 1.14

The functions $\sin t$ and $\cos t$ both have period $T = 2\pi$, whereas $\tan x$ has period $T = \pi$. Similarly, $A \sin(\omega x + \phi)$, where A , ω , and ϕ are constants is periodic with period $\frac{2\pi}{\omega}$. This function is called a *harmonic of amplitude* $|A|$, *angular frequency* ω , and *initial phase* ϕ . In fact,

$$A \sin \left[\omega \left(x + \frac{2\pi}{\omega} \right) + \phi \right] = A \sin [(\omega x + \phi) + 2\pi] \\ = A \sin(\omega x + \phi).$$

If T is a period of the function f , then the numbers $2T$, $3T$, ... are also periods of f . In fact,

$$f(x) = f(x + T) = f(x + 2T) = f(x + 3T) = \dots$$

and also

$$f(x) = f(x - T) = f(x - 2T) = f(x - 3T) = \dots$$

Further, the sum, difference, product, or quotient of two functions of period T is again a function of period T . Also, if periodic function f with period T is integrable on any intervals of length T , then it is integrable on any other interval of the same length and the value of the integrals is the same. Thus for any periodic function f with period T , we have

$$\int_a^{a+T} f(x) dx = \int_b^{b+T} f(x) dx$$

for any a and b .

1.8 EVEN AND ODD FUNCTIONS

Definition 1.24. Let the function f defined either on $\mathbb{R}$ or on some interval be symmetric with respect to the origin of co-ordinates. Then

(a) f is called *even* if

$$f(-t) = f(t) \text{ for all } t.$$

It follows from the definition that the graph of an even function f is symmetric with respect to the y -axis (Figure 1.3).

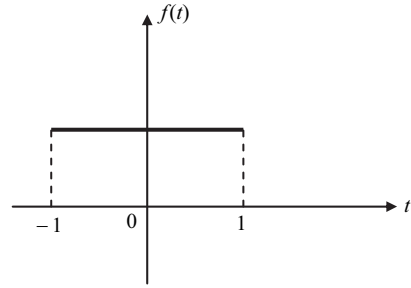


Figure 1.3

Thus interpreting the integral as an area, we have for an even function f ,

$$\int_{-T}^T f(t) dt = 2 \int_0^T f(t) dt \text{ for any } T,$$

provided that f is defined and integrable on $[-T, T]$.

(b) f is called *odd* if

$$f(-t) = -f(t) \text{ for every } t.$$

The definition suggests that the graph of an odd function is symmetric with respect to the origin (Figure 1.4).

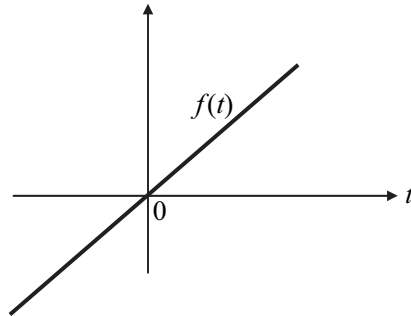


Figure 1.4

Thus, for odd function f , we have

$$\int_{-T}^T f(t) dt = 0 \text{ for any } T,$$

provided f is defined and integrable on $[-T, T]$.

Let f and g be two even functions, then $h = fg$ is even. In fact,

$$h(-x) = (fg)(-x) = f(-x)g(-x) \\ = f(x)g(x) = h(x).$$

Similarly, if f and g both are odd functions, then $h = fg$ is even, since

$$\begin{aligned} h(-x) &= f(-x)g(-x) = [-f(x)] [-g(x)] \\ &= f(x)g(x) = h(x). \end{aligned}$$

But if one of the functions f and g is even and the other is odd, then $h = fg$ is odd. In fact, let f be even and g odd. Then

$$\begin{aligned} h(-x) &= f(-x)g(-x) = [f(x)] [-g(x)] \\ &= -f(x)g(x) = -h(x). \end{aligned}$$

Thus, we have shown that

- (a) The product of two even- or two odd functions is an even function.
- (b) The product of an even function and an odd function is an odd function.

EXAMPLE 1.15

The function $\cos x$ is even, whereas $\sin x$ is odd. Similarly, $f(x) = x^2$ ($-\pi \leq x \leq \pi$) is even and so is $f(x) = |x|$ ($-\pi \leq x \leq \pi$). But $f(x) = x$ ($-\pi < x < \pi$) is odd.

1.9 SEQUENCE AND SERIES

Definition 1.25. By a *sequence* of real numbers, we mean a function $f: \mathbb{N} \rightarrow \mathbb{R}$ from the set of natural numbers into the real line.

Thus, for each $n \in \mathbb{N}$, we have $f(n) = x_n$, $x_n \in \mathbb{R}$. The term $x_n = f(n)$ is called the n th term of the sequence f . The sequence f is generally denoted by

$$f = \{x_1, x_2, \dots, x_n, \dots\}$$

or simply by $\{x_n\}$, (x_n) , or $\langle x_n \rangle$.

EXAMPLE 1.16

$\{\frac{1}{n}\}$ is the sequence $\{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ with n th term $x_n = \frac{1}{n}$.

Definition 1.26. A sequence $\{x_n\}$ is called *increasing* if $x_n \leq x_{n+1}$ for all n and *decreasing* if $x_n \geq x_{n+1}$ for all n .

EXAMPLE 1.17

The sequence $\{x_n\}$, where $x_n = 2n$ is increasing.

Definition 1.27. A sequence $\{x_n\}$ is said to *converge* to the limit x if $\lim_{n \rightarrow \infty} |x_n - x| = 0$.

The limit of a convergent sequence of real numbers is unique.

A sequence $\{x_n\}$ which does not converge to a limit is called *divergent*.

Let $\{x_n\}$ and $\{y_n\}$ be two convergent sequences with limits x and y , respectively. Then

- (a) $\lim_{n \rightarrow \infty} (\alpha x_n + \beta y_n) = \alpha x + \beta y$, $\alpha, \beta \in \mathbb{R}$
- (b) $\lim_{n \rightarrow \infty} x_n y_n = xy$
- (c) $\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \frac{x}{y}$ if $y \neq 0$.

Definition 1.28. A real-valued sequence $\{x_n\}$ is said to be a *Cauchy sequence* if for any $\varepsilon > 0$ there exists a positive integer n_0 such that $|x_n - x_m| < \varepsilon$ for all $n, m \geq n_0$.

Definition 1.29. Let $\{a_n\}$ be a sequence. An expression of the form

$$a_1 + a_2 + a_3 + \dots = \sum_{n=1}^{\infty} a_n$$

is called an *infinite series* or just a *series*.

For the sequence $\{a_n\}$, let

$$s_n = \sum_{k=1}^n a_k = a_1 + a_2 + \dots + a_n.$$

Then s_n is called the *partial sum* of the series $\sum_{n=1}^{\infty} a_n$.

Definition 1.30. A series $\sum_{n=1}^{\infty} a_n$ is called *convergent* if and only if the sequence of its partial sums $\{s_n\}$ converges to some limit, say s . Then, s is called the *sum* of the series $\sum_{n=1}^{\infty} a_n$ and we write $s = \sum_{n=1}^{\infty} a_n$.

EXAMPLE 1.18

Consider the series $\sum_{n=0}^{\infty} x^n$, $x \in \mathbb{R}$. Then

$$s_n = \sum_{k=0}^n x^k = 1 + x + x^2 + \dots + x^n.$$

If $x = 1$, then

$$s_n = 1 + 1 + \dots + 1 = n + 1$$

and so $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} n + 1 = \infty$. Thus the given series, known as *geometric series*, diverges for $x = 1$.

If $x \neq 1$, then

$$s_n = \frac{1 - x^{n+1}}{1 - x}$$

If $|x| < 1$, then $\lim_{n \rightarrow \infty} x^n = 0$ and so the series converges to $\frac{1}{1-x}$, that is,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \text{for } |x| < 1.$$

For $|x| > 1$, the partial sum s_n does not tend to a limit and so the series diverges for these values.

As another example, the series $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$. This series is called *harmonic series*.

Suppose that the series $\sum a_n$ converges and has the sum s . Since $s_n - s_{n-1} = a_n$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \lim_{n \rightarrow \infty} (s_n - s_{n-1}) = \lim_{n \rightarrow \infty} s_n - \lim_{n \rightarrow \infty} s_{n-1} \\ &= s - s = 0. \end{aligned}$$

Therefore it follows that if $\sum a_n$ converges, then $a_n \rightarrow 0$ as $n \rightarrow \infty$.

Definition 1.31. A series $\sum_{n=1}^{\infty} a_n$ is called *absolutely convergent* if $\sum_{n=1}^{\infty} |a_n|$ converges.

An *absolutely convergent series* is convergent. But the converse need not be true. For example, the series $\sum \frac{(-1)^n}{n}$ converges by Leibnitz's Rule but it is not absolutely convergent, because the series $1 + \frac{1}{2} + \frac{1}{3} + \dots$ is divergent.

1.10 SERIES OF FUNCTIONS

Consider the series

$$\sum_{n=1}^{\infty} \frac{\sin nt}{n^2} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

The partial sums of these series are polynomials. The terms of the sequences of partial sums are in fact functions.

Let $\{f_n\}$ be a sequence of functions defined on a set E and suppose that the sequence of numbers $\{f_n(x)\}$ converges for every $x \in E$. Then the function f defined by

$$f(x) = \lim_{n \rightarrow \infty} f_n(x), \quad x \in E$$

is called the *limit function* of $\{f_n\}$ and we say that $\{f_n\}$ converges to f pointwise on E .

Definition 1.32. A sequence of functions $\{f_n\}$ is said to *converge uniformly* on a set E to a function f if for every $\varepsilon > 0$ there exists an integer N such that $n \geq N$ implies

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } x \in E$$

An infinite series of functions $\sum f_n(x)$ is said to *converge uniformly* on E if the sequence of its partial sums converges uniformly on E .

The following result on uniform convergence shall be used to derive *Fourier formulae*.

Theorem 1.1. (Term by Term Integration). If $f_1(x)$, $f_2(x)$, ..., $f_n(x)$, are continuous functions of x in $[a, b]$ and if $\sum f_n(x)$ converges uniformly to $f(x)$ in $[a, b]$, then

$$\begin{aligned} \int_a^b f(x) dx &= \int_a^b f_1(x) dx + \int_a^b f_2(x) dx + \dots \\ &+ \int_a^b f_n(x) dx + \dots \end{aligned}$$

1.11 PARTIAL FRACTION EXPANSION OF A RATIONAL FUNCTION

Definition 1.33. A *rational function* $F(s)$ is a function which has the form

$$F(s) = \frac{P(s)}{Q(s)},$$

where degree of the polynomial $Q(s)$ is greater than the degree of the polynomial $P(s)$, and $P(s)$ and $Q(s)$ have no common factor. The zeros of $Q(s)$ are called *poles* of $F(s)$.

Partial fraction expansion of a rational function will be required to determine inverse Laplace transform and inverse z -function of a rational function in the forthcoming chapters.

To find partial fraction expansion of a rational function $F(s)$, we recall that

- To each linear factor of the form $as + b$ of $Q(s)$, there corresponds a partial fraction of the form $\frac{A}{as+b}$, where A is a constant.
- To each repeated linear factor of the form $(as + b)^n$, there corresponds a partial fraction of the form

$$\frac{A_1}{as+b} + \frac{A_2}{(as+b)^2} + \dots + \frac{A_n}{(as+b)^n},$$

where $A_1, A_2, \dots, A_n$ are constants.

- (c) To each quadratic factor of the form $as^2 + bs + c$ of $Q(s)$, there corresponds a partial fraction of the form $\frac{As+B}{as^2+bs+c}$, where A and B are constants.
- (d) To each repeated quadratic factor of the form $(as^2 + bs + c)^n$ of $Q(s)$, there corresponds a partial fraction of the form

$$\frac{A_1s+B_1}{as^2+bs+c} + \frac{A_2s+B_2}{(as^2+bs+c)^2} + \dots + \frac{A_ns+B_n}{(as^2+bs+c)^n},$$

where $A_1, A_2, \dots, A_n$ are constants.

The constants are determined by clearing fractions and equating the like powers of both sides of the resulting equation.

EXAMPLE 1.19

Resolve into partial fractions:

$$\frac{3s+1}{(s-1)(s^2+1)}$$

Solution. According to the above discussed scheme, we have

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$$

and so

$$3s+1 = A(s^2+1) + (Bs+C)(s-1)$$

Taking $s = 1$ yields $A = 2$. Now taking $s = 0$, we have $1 = A - C = 2 - C$, which gives $C = 1$. Comparing the coefficients of s on both sides of the above equation, we have $3 = -B + C = -B + 1$ and so $B = -2$. Hence

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{2}{s-1} + \frac{-2s+1}{s^2+1}$$

EXAMPLE 1.20

Resolve into partial fraction:

$$\frac{4s+5}{(s-1)^2(s+2)}$$

Solution. Write

$$\frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2}$$

and so

$$4s+5 = A(s-1)(s+2) + B(s+2) + C(s-1)^2.$$

Taking $s = 1$ yields $B = 3$. Taking $s = -2$ yields $C = -\frac{1}{3}$ and equating the coefficients of s^2 on both sides, we get $A = \frac{1}{3}$. Hence

$$\frac{4s+5}{(s-1)^2(s+2)} = \frac{1}{3(s-1)} + \frac{3}{(s-1)^2} - \frac{1}{3(s+2)}$$

1.12 SPECIAL FUNCTIONS

In this section, we present some special functions having applications in science and engineering.

Definition 1.34. (The Gamma Function). The *gamma function* is defined by the improper integral

$$\Gamma(z) = \int_0^{\infty} e^{-u} u^{z-1} du, \quad \operatorname{Re}(z) > 0 \quad (1)$$

Substituting $z = 1$ in (1), we get

$$\begin{aligned} \Gamma(1) &= \int_0^{\infty} e^{-u} du = \lim_{T \rightarrow \infty} \int_0^T e^{-u} du \\ &= \lim_{T \rightarrow \infty} (1 - e^{-T}) = 1. \end{aligned}$$

Further, integration by parts yields

$$\begin{aligned} \Gamma(z+1) &= \int_0^{\infty} e^{-u} u^z du \\ &= 0 + z \int_0^{\infty} e^{-u} u^{z-1} du = z\Gamma(z), \quad (2) \end{aligned}$$

which is *recurrence formula* for $\Gamma(z)$. Using (2), we have

$$\Gamma(2) = 1, \quad \Gamma(3) = 2, \quad \Gamma(4) = 3!$$

and in general,

$$\Gamma(n+1) = n!, \quad n = 0, 1, 2, \dots$$

Thus gamma function is an extension of *factorial function*. The other important properties of gamma function are

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad (3)$$

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi}, \quad 0 < p < 1 \quad (4)$$

Definition 1.35. (The Beta Function). The *beta function* $\beta(m, n)$ is defined by

$$\beta(m, n) = \int_0^1 u^{m-1} (1-u)^{n-1} du, \quad m > 0, n > 0.$$

The beta function has the following properties:

$$(i) \beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad (5)$$

$$(ii) \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta = \frac{1}{2} \beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)} \quad (6)$$

The relation (6) follows from (5). In fact,

$$\beta(m, n) = \int_0^1 u^{m-1} (1-u)^{n-1} du = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

Substituting $u = \sin^2 \theta$, we have

$$\begin{aligned} \beta(m, n) &= 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta \\ &= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \end{aligned}$$

and so (6) follows.

EXAMPLE 1.21

Show that

$$(a) \int_0^{\pi/2} \sin^2 \theta \cos^4 \theta d\theta = \frac{\pi}{32}$$

$$(b) \int_0^{\pi/2} \sin^7 \theta d\theta = \frac{16}{35}$$

$$(c) \int_0^{\pi/2} \frac{d\theta}{\sqrt{\tan \theta}} = \frac{\pi\sqrt{2}}{2}$$

Solution. (a) Comparing the given integral with

$$\int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta,$$

we observe that $m = \frac{3}{2}$, $n = \frac{5}{2}$ and so using (6), we get

$$\begin{aligned} \int_0^{\pi/2} \sin^2 \theta \cos^4 \theta d\theta &= \frac{\Gamma(3/2)\Gamma(5/2)}{2\Gamma[(3/2) + (5/2)]} \\ &= \frac{3\sqrt{\pi}}{3!(16)} \cdot \sqrt{\pi} = \frac{\pi}{32}. \end{aligned}$$

(b) Comparing the given integral with

$$\int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta,$$

we observe that $2m - 1 = 7$ and $2n - 1 = 0$ and so we get $m = 4$ and $n = \frac{1}{2}$. Hence, the application of (6) yields

$$\int_0^{\pi/2} \sin^7 \theta d\theta = \frac{\Gamma(4)\Gamma(1/2)}{2\Gamma(9/2)} = \frac{3!(8)\sqrt{\pi}}{105\sqrt{\pi}} = \frac{16}{35}.$$

(c) We have

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{\tan \theta}} = \int_0^{\pi/2} \sin^{-1/2} \theta \cos^{1/2} \theta d\theta$$

Taking $2m - 1 = -\frac{1}{2}$ and $2n - 1 = \frac{1}{2}$, we get $m = \frac{1}{4}$, $n = \frac{3}{4}$.

Hence, using (6) and the relation (4), we get

$$\begin{aligned} \int_0^{\pi/2} \frac{d\theta}{\sqrt{\tan \theta}} &= \frac{\Gamma(1/4)\Gamma(3/4)}{2\Gamma(1)} = \frac{1}{2} \cdot \frac{\pi}{\sin(\pi/4)} \\ &= \frac{\pi\sqrt{2}}{2}. \end{aligned}$$

Definition 1.36. (Bessel Function). A *Bessel function* of order n is defined by

$$\begin{aligned} J_n(t) &= \frac{t^n}{2^n \Gamma(n+1)} \\ &\times \left[1 - \frac{t^2}{2(n+2)} + \frac{t^4}{2.4(2n+2)(2n+4)} - \dots \right] \quad (7) \end{aligned}$$

The main properties of this function are

$$(a) J_{-n}(t) = (-1)^n J_n(t), \text{ if } n \text{ is a positive integer} \quad (8)$$

$$(b) J_{n+1}(t) = \frac{2n}{t} J_n(t) - J_{n-1}(t) \quad (9)$$

$$(c) \frac{d}{dt} \{t^n J_n(t)\} = t^n J_{n-1}(t) \quad (10)$$

Thus, if $n = 0$, we have

$$J'_0(t) = J_1(t) \text{ [using (a)]}.$$

Definition 1.37. (Error Function and Complementary Error Functions). The *error function* is defined by the integral

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt, \quad (11)$$

where z may be real or complex variable. This function appears in probability theory, heat conduction theory, and mathematical physics. When $z = 0$, $\operatorname{erf}(0) = 0$, and

$$\operatorname{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} dt = \frac{\Gamma(1/2)}{\sqrt{\pi}} = 1.$$

The *complementary error function* is defined by the integral

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt \quad (12)$$

Using the properties of the integrals, we note that

$$\begin{aligned} \operatorname{erfc}(z) &= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2} dt - \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \\ &= \frac{2}{\sqrt{\pi}} \left(\frac{\sqrt{\pi}}{2} \right) - \operatorname{erf}(z) \\ &= 1 - \operatorname{erf}(z) \end{aligned} \quad (13)$$

Definition 1.38. (Heaviside's Unit Step Function). The *Heaviside's unit step function* (also known as *delayed unit step function*) is defined by

$$H(t - a) = \begin{cases} 1 & \text{for } t > a \\ 0 & \text{for } t < a. \end{cases} \quad (14)$$

It delays the output until $t = a$ and then assumes a constant value of 1 unit.

If $a = 0$, then

$$H(x) = \begin{cases} 1 & \text{for } t > 0 \\ 0 & \text{for } t < 0, \end{cases} \quad (15)$$

which is generally called *unit step function*.

The graph of Heaviside's unit step function is shown in Figure 1.5.

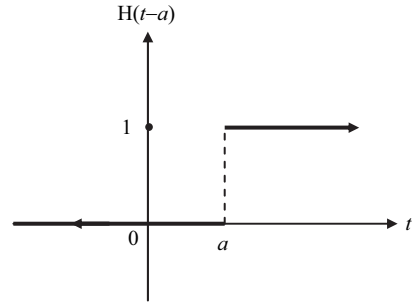


Figure 1.5 Graph of Heaviside's unit step function

This function has a jump discontinuity at $t = a$ with a jump of unit magnitude. The beauty of this function is that it acts like a switch to turn another function on or off at some time. For example, the function

$$g(t) = H(t - a) \cos 2\pi t$$

is zero for $t < a$ but assumes the graph of the cosine function for $t > a$ as shown in Figure 1.6.

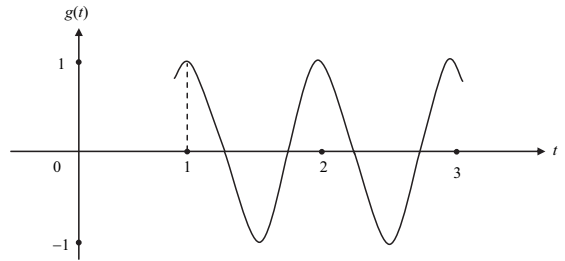


Figure 1.6 Graph of $g(t) = H(t - a) \cos 2\pi t$

Definition 1.39. (Pulse of Unit Height and Duration T). The *pulse of unit height* and duration T is defined by

$$f(t) = \begin{cases} 1 & \text{for } 0 < t < T \\ 0 & \text{for } T < t \end{cases} \quad (16)$$

Definition 1.40. (Sinusoidal Pulse). The *sinusoidal pulse* is defined by

$$f(t) = \begin{cases} \sin at & \text{for } 0 < t < \pi/a \\ 0 & \text{for } \pi/a < t. \end{cases} \quad (17)$$

Definition 1.41. (Rectangle Function). A *rectangle function* f is defined by

$$f(t) = \begin{cases} 1 & \text{for } a < t < b \\ 0 & \text{otherwise.} \end{cases} \quad (18)$$

The graph of this function is shown in Figure 1.7.

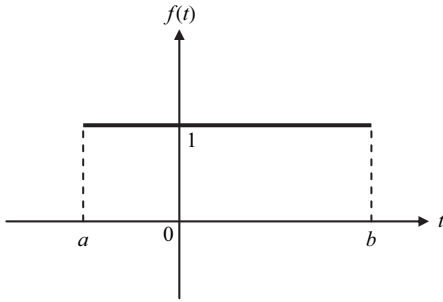


Figure 1.7 Graph of Rectangle Function

In term of Heaviside unit step function, we have

$$f(t) = H(t - a) - H(t - b)$$

If $a = 0$, then rectangle function reduces to pulse of unit height and duration b .

Definition 1.42. (Gate Function). The *gate function* is defined as

$$f_a(t) = \begin{cases} 1 & \text{for } |t| < a \\ 0 & \text{for } |t| > a \end{cases} \quad (19)$$

The graph of the gate function is shown in Figure 1.8.

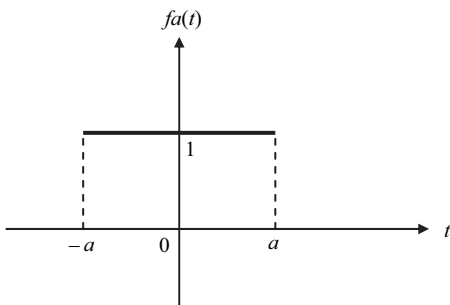


Figure 1.8 Graph of Gate Function

Definition 1.43. (Dirac Delta Function). Consider the function f_ε defined by

$$f_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon} & \text{for } 0 \leq t \leq \varepsilon \\ 0 & \text{for } t > \varepsilon, \end{cases} \quad (20)$$

where $\varepsilon > 0$. The graph of f_ε is shown in Figure 1.9.

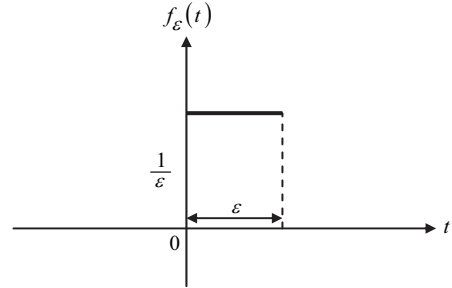


Figure 1.9 Graph of $f_\varepsilon(t)$

We note that as $\varepsilon \rightarrow 0$, the height of the rectangle increases indefinitely and width decreases in such a way that *its area is always equal to 1*.

We further note that $\lim_{\varepsilon \rightarrow 0} f_\varepsilon(t)$ does not exist.

Even then we define a function δ as

$$\delta(t) = \lim_{\varepsilon \rightarrow 0} f_\varepsilon(t). \quad (21)$$

This “generalized function or distribution” $\delta(t)$ is called *Dirac delta function* and has the property

$$\delta(t) = 0 \text{ for } t \neq 0, \quad (22)$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1, \text{ and} \quad (23)$$

$$\int_{-\infty}^{\infty} \delta(t-a)f(t)dt = f(a) \text{ for a continuous function } f. \quad (24)$$

Since $\delta(t-a) = \delta(a-t)$, it follows that $\delta(t)$ is an *even function*.

Definition 1.44. (Signum Function). The *signum function*, denoted by $\text{sgn}(t)$, is defined by

$$\text{sgn}(t) = \begin{cases} 1 & \text{for } t > 0 \\ -1 & \text{for } t < 0. \end{cases} \quad (25)$$

If $H(t)$ is unit step function, then

$$H(t) = \frac{1}{2} [1 + \text{sgn}(t)]$$

and so

$$\text{sgn}(t) = 2H(t) - 1 \quad (26)$$

Definition 1.45. (Saw Tooth Wave Function). The *saw tooth wave function* f with period a is defined by

$$f(t) = \begin{cases} t & \text{for } 0 \leq t < a \\ 0 & \text{for } t \leq 0, \end{cases} \quad (27)$$

$$f(t + a) = f(t).$$

The graph of this function is shown in Figure 1.10.

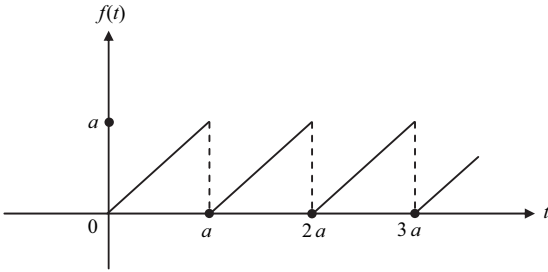


Figure 1.10 Graph of Saw-tooth Function with Period a

The *saw-tooth function with period* 2π is defined as

$$f(t) = \begin{cases} t & \text{for } -\pi < t < \pi \\ 0 & \text{elsewhere.} \end{cases}$$

The graph of this function is shown in Figure 1.11.

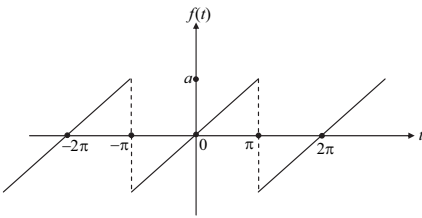


Figure 1.11 Graph of Saw Tooth Function with Period 2π

Definition 1.46. (Triangular Wave Function). The *triangular wave function* f with period $2a$ is defined by

$$f(t) = \begin{cases} t & \text{for } 0 \leq t < a \\ 2a - t & \text{for } a \leq t < 2a, \end{cases} \quad (28)$$

$$f(t + 2a) = f(t).$$

The graph of this function is shown in Figure 1.12.

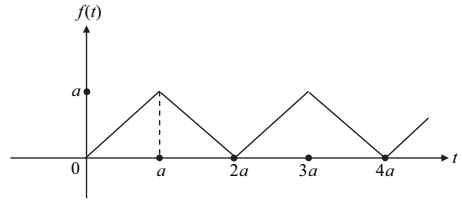


Figure 1.12 Graph of Triangular Wave Function with Period $2a$

Definition 1.47. (Half-wave Rectified Sinusoidal Function). The *half-wave rectified sinusoidal function* f with period 2π is defined by

$$f(t) = \begin{cases} \sin t & \text{for } 0 < t < \pi \\ 0 & \text{for } \pi < t < 2\pi, \end{cases} \quad (29)$$

$$f(t + 2\pi) = f(t).$$

The graph of this function is shown in Figure 1.13.

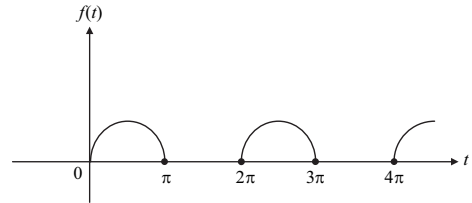


Figure 1.13 Graph of Half-wave Rectified Sine Function

Definition 1.48. (Full Rectified Sine Wave Function). The *full rectified sine wave function* f with period π is defined by

$$f(t) = \begin{cases} \sin t & \text{for } 0 < t < \pi \\ -\sin t & \text{for } \pi < t < 2\pi, \end{cases} \quad (30)$$

$$f(t + \pi) = f(t).$$

or by

$$f(t) = |\sin \omega t| \text{ with period } \pi/\omega.$$

The graph of this function is shown in Figure 1.14.

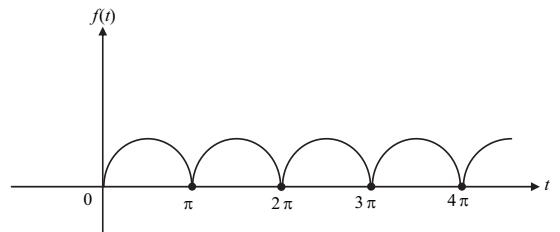


Figure 1.14 Graph of Full Rectifier with Period π

Definition 1.49. (Square Wave Function). The *square wave function* f with period $2a$ is defined by

$$f(t) = \begin{cases} 1 & \text{for } 0 < t < a \\ -1 & \text{for } a < t < 2a \end{cases} \quad (31)$$

$$f(t + 2a) = f(t).$$

In terms of Heaviside's unit step function, the definition converts to

$$f(t) = H(t) - 2H(t - a) + 2H(t - 2a) - 2H(t - 3a) + \dots$$

The graph of this function is shown in Figure 1.15.

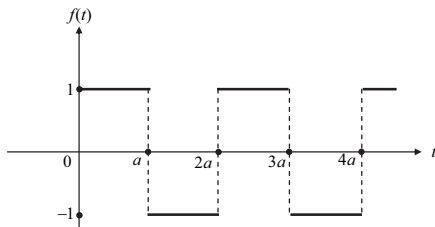


Figure 1.15 Graph of the Square Wave Function

1.13 THE INTEGRAL TRANSFORMS

Definition 1.50. The *integral transform* of a function f is defined by

$$I\{f(x)\} = \int_{x_1}^{x_2} f(x) K(w, x) dx, \quad (32)$$

where $K(w, x)$ is called the *kernel* of the transform and is a known function of w and x .

The function f is called *inverse transform* of $I\{f(x)\}$.

If $K(w, x) = e^{-wx}$, then the integral transform (32) is called *Laplace transform* of f .

If $K(w, x) = e^{-iwx}$, then the integral transform (32) is called *Fourier transform* of f .

If $K(w, x) = x^{w-1}$, then the integral transform (32) is called the *Mellin transform* of f .

If $K(w, x) = \cos nx$, then the integral transform (32) is called the *Fourier cosine transform* of f .

If $K(w, x) = \sin nx$, then the integral transform (32) is called the *Fourier sine transform* of f .

2 Differential Equations

Differential equations play an important role in engineering and science. Many physical laws and relations appear in the form of differential equations. For example, the current I in an LCR circuit is described by the differential equation $LI'' + RI + \frac{1}{C}I = E$, which is derived from Kirchhoff's laws. Similarly, the displacement y of a vibrating mass m on a spring is described by the equation $my'' + ky = 0$. The study of differential equations involves *formation of differential equations*, the *solutions of differential equations*, and the *physical interpretation* of the solution in terms of the given problem.

2.1 DEFINITIONS AND EXAMPLES

Definition 2.1. A *differential equation* is an equation which involves derivatives.

For example,

$$(a) \frac{d^2x}{dt^2} + n^2x = 0, \quad (b) \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

$$(c) \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \quad \text{and} \quad (d) \frac{dy}{dx} = x + 1$$

are differential equations.

We note that a differential equation may have the variables present only in the derivatives. For example in (b), the variables are present only in derivatives. Moreover, a differential equation may have more than one dependent variable. For example,

$$\frac{d\phi}{dt} + \frac{d\psi}{dt} = \phi + \psi$$

has two dependent variables ϕ and ψ and one independent variable t .

Definition 2.2. A differential equation involving derivatives with respect to a single independent variable is called an *ordinary differential equation*. For example,

$$\frac{d^2y}{dx^2} + 3y = 0$$

is an ordinary differential equation.

Definition 2.3. A differential equation involving partial derivatives with respect to two or more independent variables is called a *partial differential equation*.

For example,

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \quad \text{and} \quad \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

are partial differential equations.

Definition 2.4. The order of the highest derivative appearing in a differential equation free from radicals is called the *order* of that differential equation.

For example the, order of the differential equation $y'' + 4y = 0$ is two, the order of the differential equation $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ is two, and order of the differential equation $y = x \frac{dy}{dx} + \frac{x}{dy/dx}$ is one.

Definition 2.5. The degree or exponent of the highest derivative appearing in a differential equation free from radicals and fractions is called the *degree* of the differential equation.

For example, the degree of the differential equation

$$y \frac{dy}{dx} = x \left(\frac{dy}{dx} \right)^2 + 1$$

is two. Similarly, the degree of the differential equation

$$\left(\frac{d^3y}{dx^3} \right)^{2/3} = 1 + 2 \frac{dy}{dx}$$

is two because the given equation can be written as

$$\left(\frac{d^3y}{dx^3} \right)^2 = \left(1 + 2 \frac{dy}{dx} \right)^3.$$

2.2 FORMULATION OF DIFFERENTIAL EQUATION

The derivation of differential equations from physical or other problems is called *modelling*. The modelling involves the successive differentiations and elimination of parameters present in the given system.

EXAMPLE 2.1

Form the differential equation for “free fall” of a stone dropped from the height y under the action of force due to gravity g .

Solution. We know that equation of motion of the free fall is

$$y = \frac{1}{2}gt^2.$$

Differentiating with respect to t , we get

$$\frac{dy}{dt} = gt.$$

Differentiating once more, we get

$$\frac{d^2y}{dt^2} = g,$$

which is the desired differential equation representing the free fall of a stone.

EXAMPLE 2.2

Form the differential equation of simple harmonic motion given by $x = A \cos(\omega t + \phi)$, where A and ϕ are constants.

Solution. To get the required differential equation, we have to differentiate the given relation and eliminate the constants A and ϕ . Differentiating twice, we get

$$\begin{aligned}\frac{dx}{dt} &= -A\omega \sin(\omega t + \phi), \\ \frac{d^2x}{dt^2} &= -A\omega^2 \cos(\omega t + \phi) = -\omega^2 x. \quad (1)\end{aligned}$$

Hence

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

is the differential equation governing simple harmonic motion. Equation (1) shows that acceleration varies as the distance from the origin.

EXAMPLE 2.3

Find the differential equation governing the motion of a particle of mass m sliding down a frictionless curve.

Solution. Velocity of the particle at the starting point $P(x, y)$ is zero since it starts from rest. Let $Q(x, u)$ be some intermediate point during the motion (see Figure 2.1). Let the origin O be

the lowest point of the curve and let the length of the arc OQ be s .

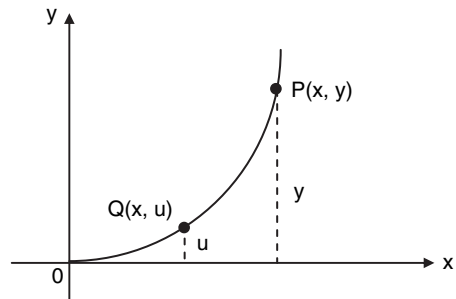


Figure 2.1

By Law of Conservation of energy, we have

$$\begin{aligned}\text{potential energy at P} + \text{kinetic energy at P} \\ = \text{potential energy at Q} + \text{kinetic energy at Q},\end{aligned}$$

and so

$$mgy + 0 = mgu + \frac{1}{2}m\left(\frac{ds}{dt}\right)^2.$$

Hence

$$\left(\frac{ds}{dt}\right)^2 = 2g(y - u)$$

is the required differential equation.

If the duration T_0 of descent is independent of the starting point, the solution of this differential equation comes out to be the equation of a *cycloid*. Thus the shape of the curve is a cycloid. This curve is called *Tautochrone Curve*.

EXAMPLE 2.4

Derive the differential equation governing a mass-spring system.

Solution. Let m be the mass suspended by a spring that is rigidly supported from one end (see Figure 2.2). Let

- (i) rest position be denoted by $x = 0$, downward displacement by $x > 0$ and upward displacement be denoted by $x < 0$.
- (ii) $k > 0$ be spring constant and $a > 0$ be damping constant.

- (iii) $a \frac{dx}{dt}$ be the damping force due to medium (damping force is proportional to the velocity).
 (iv) $f(t)$ be the external impressed force on m .

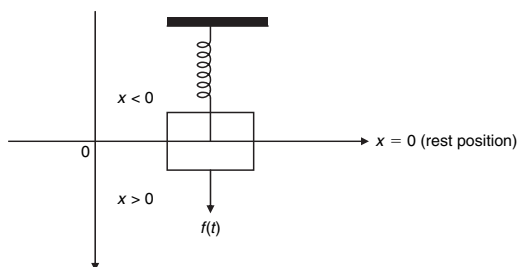


Figure 2.2

Then, by Newton's second law of motion, the sum of force acting on m is $m \frac{d^2x}{dt^2}$ and so

$$m \frac{d^2x}{dt^2} = -kx - a \frac{dx}{dt} + f(t),$$

that is,

$$\frac{d^2x}{dt^2} + \frac{a}{m} \frac{dx}{dt} + \frac{k}{m}x = f(t),$$

which is the required differential equation governing the system.

EXAMPLE 2.5

Find the differential equation of all circles of radius a and centre (h, k) .

Solution. We know that the equation of the circle with radius a and centre (h, k) is

$$(x - h)^2 + (y - k)^2 = a^2 \quad (2)$$

Differentiating twice, we get

$$x - h + (y - k) \frac{dy}{dx} = 0, \quad (3)$$

$$1 + (y - k) \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 = 0 \quad (4)$$

From (4), we have

$$y - k = -\frac{1 + \left(\frac{dy}{dx} \right)^2}{d^2y/dx^2}$$

and then (3) yields

$$x - h = -(y - k) \frac{dy}{dx} = \frac{\frac{dy}{dx} \left[1 + \left(\frac{dy}{dx} \right)^2 \right]}{d^2y/dx^2}.$$

Substituting the values of $x - h$ and $y - k$ in (2), we get

$$\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^3 = a^2 \left(\frac{d^2y}{dx^2} \right)^2,$$

which is the required differential equation. It follows that

$$\frac{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{3/2}}{d^2y/dx^2} = a,$$

and so the radius of curvature of a circle at any point is constant.

EXAMPLE 2.6

Form the differential equation from the equation $xy = Ae^x + B e^{-x}$.

Solution. Differentiating twice, we get

$$\begin{aligned} x \frac{dy}{dx} + y &= Ae^x - B e^{-x} \\ x \frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} &= Ae^x + B e^{-x} = xy. \end{aligned}$$

Hence

$$x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - xy = 0$$

is the required differential equation.

2.3 SOLUTION OF DIFFERENTIAL EQUATION

Definition 2.6. A solution of a differential equation is a functional relation between the variables involved such that this relation and the derivatives obtained from it satisfy the given differential equation.

For example, $x^2 + 4y = 0$ is a solution of the differential equation

$$\left(\frac{dy}{dx} \right)^2 + x \frac{dy}{dx} - y = 0. \quad (5)$$

In fact, differentiating $x^2 + 4y = 0$, we get

$$2x + 4 \frac{dy}{dx} = 0$$

and so $\frac{dy}{dx} = -\frac{x}{2}$. Hence

$$\begin{aligned} \left(\frac{dy}{dx}\right)^2 + x \frac{dy}{dx} - y &= \frac{x^2}{4} + x\left(-\frac{x}{2}\right) - y \\ &= \frac{x^2}{4} - \frac{x^2}{2} - \left(-\frac{x^2}{4}\right) = 0. \end{aligned}$$

Hence $x^2 + 4y = 0$ is a solution of (5).

Definition 2.7. A solution of a differential equation in which the number of arbitrary constants is equal to the order of the differential equation is called the *general* (or *complete*) *solution* of the differential equation.

For example $xy = Ae^x + Be^{-x}$ is the general solution of

$$x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - xy = 0.$$

Definition 2.8. A solution obtained from the general solution of a differential equation by giving particular values to the arbitrary constants is called a *particular solution* of that differential equation.

Definition 2.9. A problem involving a differential equation and one or more supplementary conditions, relating to one value of the independent variable, which the solution of the given differential equation must satisfy is called an *initial-value problem*.

For example,

$$\frac{d^2y}{dt^2} + y = \cos 2t,$$

$$y(0) = 1, \quad y'(0) = -2$$

is an initial-value problem. Similarly, the problem

$$\frac{d^2y}{dx^2} + y = 0,$$

$$y(1) = 3, y'(1) = 4$$

is also an initial-value problem.

Definition 2.10. A problem involving a differential equation and one or more supplementary conditions, relating to more than one values of the independent variable, which the solution of the differential equation must satisfy, is called a *boundary-value problem*.

For example,

$$\frac{d^2y}{dx^2} + y = 0$$

$$y(0) = 2, \quad y\left(\frac{\pi}{2}\right) = 4$$

is a boundary value problem.

2.4 DIFFERENTIAL EQUATIONS OF FIRST ORDER

We consider first the differential equations of first order. Let D be a open connected set in $\mathbb{R}^2$ and let $f: D \rightarrow \mathbb{R}$ be continuous. We discuss the problem of determining solution in D of the first order differential equation

$$\frac{dy}{dx} = f(x, y).$$

Definition 2.11. A real-valued function $f: D \rightarrow \mathbb{R}$ defined on the connected open set D in $\mathbb{R}^2$ is said to satisfy a *Lipschitz condition* in y on D with Lipschitz constant M if and only if

$$|f(x, y_2) - f(x, y_1)| \leq M|y_2 - y_1|$$

for all (x, y_1) and $(x, y_2) \in D$.

Regarding existence of solutions of first order differential equation, we have the following theorems.

Theorem 2.1. (Picard's Existence and Uniqueness Theorem). Let $f: D \rightarrow \mathbb{R}$ be continuous on open connected set D in $\mathbb{R}^2$ and satisfy Lipschitz condition in y on D . Then for every $(x_0, y_0) \in D$, the initial-value problem $\frac{dy}{dx} = f(x, y)$ has a solution passing through (x_0, y_0) .

The solution obtained in Theorem 2.1 is unique. The Lipschitz condition in the hypothesis of Picard's theorem cannot be dropped because continuity of f without this condition will not yield unique solution. For example, consider the equation

$$\frac{dy}{dx} = y^{2/3}, \quad y(0) = 0.$$

Clearly $\phi_1(x) = 0$ is a solution to this equation. Further substituting $y = \sin^3 \theta$, we have $dy = 3 \sin^2 \theta \cos \theta d\theta$ and so

$$\frac{3 \sin^2 \theta \cos \theta d\theta}{dx} = \sin^2 \theta,$$

which yields $3 \cos \theta d\theta = dx$ and $x = 3 \sin \theta$. Thus $x^3 = 27 \sin^3 \theta = 27y$ and hence $y = \frac{x^3}{27}$ is also a

solution. Therefore, the given initial-value problem does not have unique solution. The reason is that it does not satisfy Lipschitz condition.

Theorem 2.2. (Peano's Existence Theorem). Let f be a continuous real-valued function on a non-empty open subset D of the Euclidean space $\mathbb{R}^2$ and let $(x_0, y_0) \in D$. Then there is a positive real number α such that the first order differential equations $\frac{dy}{dx} = f(x, y)$ has a solution ϕ in the interval $[x_0, x_0 + \alpha]$ which satisfies the boundary condition $\phi(x_0) = y_0$.

Clearly, Peano's theorem is merely an existence theorem and not a uniqueness theorem.

We now consider certain basic types of first order differential equations for which an exact solution may be obtained by definite procedures. The most important of these types are separable equations, homogeneous equations, exact equations, and linear equations. The corresponding methods of solution involve various devices. We take these types of differential equations one by one.

2.5 SEPARABLE EQUATIONS

The first order differential equation

$$\frac{dy}{dx} = f(x, y) \quad (6)$$

is separable if f may be expressed as

$$f(x, y) = \frac{M(x)}{N(y)}, \quad (x, y) \in D \quad (7)$$

The function $M(x)$ and $N(y)$ are real-valued functions of x and y , respectively. Thus (6) becomes

$$N(y) \frac{dy}{dx} = M(x) \quad (8)$$

The equation (8) is solved by integrating with respect to x . Thus the solution is

$$\int N(y) dy = \int M(x) dx + C.$$

EXAMPLE 2.7

Solve

$$\frac{dy}{dx} = e^{x+y}, \quad y(1) = 1. \text{ Find } y(-1).$$

Solution. We have

$$\frac{dy}{dx} = \frac{e^x}{e^{-y}}$$

and so

$$e^{-y} dy = e^x dx.$$

Integrating both sides

$$\int e^{-y} dy = \int e^x dx + C \quad \text{or} \quad -e^{-y} = e^x + C.$$

Using initial condition $y(1) = 1$, we get

$$-e^{-1} = e + C$$

and so $C = -\left(\frac{1+e^2}{e}\right)$.

Thus the solution is

$$-e^{-y} = e^x - \left(\frac{1+e^2}{e}\right).$$

Hence $y(-1)$ is given by

$$\begin{aligned} y(-1) &= e^{-1} - \left(\frac{1+e^2}{e}\right) \\ &= \frac{1}{e} - \frac{1+e^2}{e} = -e, \end{aligned}$$

that is

$$-e^{-y} = -e$$

or

$$y = -1.$$

EXAMPLE 2.8

Solve $\frac{dy}{dx} = (4x + y + 1)^2$, $y(0) = 1$.

Solution. Substituting $4x + y + 1 = t$, we get

$$\frac{dy}{dx} = \frac{dt}{dx} - 4$$

Hence, the given equation reduces to

$$\frac{dt}{dx} - 4 = t^2$$

or

$$\frac{dt}{dx} = t^2 + 4$$

or

$$\frac{dt}{t^2 + 4} = dx.$$

Integrating both sides, we have

$$\int \frac{dt}{t^2 + 4} = \int dx + C$$

or

$$\frac{1}{2} \tan^{-1} \frac{t}{2} = x + C$$

or

$$\frac{1}{2} \tan^{-1} \frac{4x + y + 1}{2} = x + C$$

or

$$4x + y + 1 = 2 \tan 2(x + C)$$

Using given initial conditions $x = 0, y = 1$, we get

$$\frac{1}{2} \tan^{-1} 1 = C$$

which gives $C = \frac{\pi}{8}$. Hence the solution is

$$4x + y + 1 = 2 \tan(2x + \pi/4).$$

EXAMPLE 2.9

Solve

$$y - x \frac{dy}{dx} = a \left(y^2 + \frac{dy}{dx} \right).$$

Solution. The given equation can be written as

$$\frac{dy}{dx} (a + x) = y(1 - ay)$$

or

$$\frac{dy}{y(1 - ay)} = \frac{dx}{a + x}.$$

Integrating both sides, we have

$$\log y - \log(1 - ay) = \log(a + x) + C$$

or

$$\log \frac{y}{(a + x)(1 - ay)} = C.$$

Hence

$$y = K(a + x)(1 - ay), \text{ K constant}$$

is the general solution.

EXAMPLE 2.10

Solve

$$x(1 + y^2)dx + y(1 + x^2)dy = 0.$$

Solution. Dividing the given equation throughout by $(1 + x^2)(1 + y^2)$, we get

$$\frac{x}{1 + x^2} dx + \frac{y}{1 + y^2} dy = 0.$$

Integrating both sides, we have

$$\int \frac{x}{1 + x^2} dx + \int \frac{y}{1 + y^2} dy = C$$

or

$$\frac{1}{2} \int \frac{2x}{1 + x^2} dx + \frac{1}{2} \int \frac{2y}{1 + y^2} dy = C$$

or

$$\frac{1}{2} \log(1 + x^2) + \frac{1}{2} \log(1 + y^2) = C$$

or

$$\frac{1}{2} \log(1 + x^2)(1 + y^2) = C$$

or

$$\log(1 + x^2)(1 + y^2) = 2C = \log C$$

Hence

$$(1 + x^2)(1 + y^2) = K \text{ (constant)}$$

is the required solution.

EXAMPLE 2.11

Solve

$$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}.$$

Solution. We have

$$\frac{dy}{dx} = \frac{e^x}{e^y} + \frac{x^2}{e^y}$$

or

$$e^y \frac{dy}{dx} = e^x + x^2$$

or

$$e^y dy = (e^x + x^2) dx = e^x dx + x^2 dx.$$

Integrating both sides, we get

$$e^y = e^x + \frac{x^3}{3} + C \text{ (constant).}$$

EXAMPLE 2.12

Solve

$$16y \frac{dy}{dx} + 9x = 0.$$

Solution. We are given that

$$16y \frac{dy}{dx} + 9x = 0$$

or

$$16y \, dy = -9x \, dx.$$

Integrating both sides, we have

$$16 \frac{y^2}{2} = -9 \frac{x^2}{2} + C$$

or

$$\frac{x^2}{16} + \frac{y^2}{9} = K \text{ (constant),}$$

which is the required solution and represents a family of ellipses.

EXAMPLE 2.13

Solve

$$\frac{dy}{dx} = (x + y)^2.$$

Solution. Substituting $z = x + y$, we get $\frac{dz}{dx} = 1 + \frac{dy}{dx}$. Therefore, the given equation reduces to

$$\frac{dz}{dx} - 1 = z^2$$

or

$$\frac{dz}{dx} = 1 + z^2$$

or

$$\frac{dz}{1 + z^2} = dx.$$

Integrating, we get

$$\tan^{-1} z = x + C.$$

Putting back the value of z , we get

$$\tan^{-1}(x + y) = x + C.$$

Hence

$$x + y = \tan(x + C).$$

EXAMPLE 2.14

Solve

$$(2x - 4y + 5) \frac{dy}{dx} = x - 2y + 3.$$

Solution. We have

$$2(x - 2y + \frac{5}{2}) \frac{dy}{dx} = x - 2y + 3.$$

Substituting $z = x - 2y$, we get $\frac{dz}{dx} = 1 - 2 \frac{dy}{dx}$. The equation becomes

$$(2z + 5) \frac{dz}{dx} = 4z + 11$$

or

$$(4z + 10) \frac{dz}{dx} = 2(4z + 11)$$

or

$$\frac{4z + 10}{4z + 11} dz = 2 \, dx$$

or

$$\frac{4z + 11 - 1}{4z + 11} dz = 2dx$$

or

$$\left(1 - \frac{1}{4z + 11}\right) dz = 2dx$$

Integrating both sides, we have

$$z - \frac{1}{4} \log |4z + 11| = 2x + C.$$

Putting back the value of z , we have

$$4x + 8y + \log |4x - 8y + 11| = C.$$

EXAMPLE 2.15

Solve $(x + 2y)(dx - dy) = dx + dy$.

Solution. We have

$$(x + 2y)(dx - dy) = dx + dy$$

or

$$(x + 2y - 1)dx = (x + 2y + 1)dy$$

or

$$\frac{dy}{dx} = \frac{x + 2y - 1}{x + 2y + 1}.$$

Substituting $x + 2y = z$, we have $1 + 2 \frac{dy}{dx} = \frac{dz}{dx}$ and so

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{dz}{dx} - 1 \right).$$

Hence the above equation becomes

$$\frac{1}{2} \left(\frac{dz}{dx} - 1 \right) = \frac{z-1}{z+1}$$

or

$$\frac{dz}{dx} = \frac{3z-1}{z+1}.$$

Separating the variables, we have

$$\frac{(z+1) dz}{3z-1} = dx$$

or

$$\frac{1}{3} \left(\frac{3z-1+4}{3z-1} \right) dz = dx$$

or

$$\frac{1}{3} \left(1 + \frac{4}{3z-1} \right) dz = dx.$$

Integrating both sides, we get

$$\frac{1}{3} \int \left(1 + \frac{4}{3z-1} \right) dz = \int dx + C$$

or

$$\frac{1}{3} \left(z + \frac{4}{3} \log(3z-1) \right) = x + C$$

or

$$3z + 4 \log(3z-1) = 9x + k \text{ (constant)}$$

or

$$3(y-x) + 2 \log(3x+6y-1) = \frac{k}{2} = K \text{ (constant)}.$$

2.6 HOMOGENEOUS EQUATIONS

Definition 2.12. An expression of the form

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \dots + a_ny^n,$$

in which every term is of the n^{th} degree, is called a *homogeneous function of degree n* .

Definition 2.13. A differential equation of the form

$$\frac{dy}{dx} = \frac{f(x, y)}{\phi(x, y)},$$

where $f(x, y)$ and $\phi(x, y)$ are homogeneous functions of the same degree in x and y is called an *homogeneous equation*.

A homogeneous differential equation can be solved by substituting $y = vx$. Then $\frac{dy}{dx} = v + x \frac{dv}{dx}$. Putting the value of y and $\frac{dy}{dx}$ in the given equation, we get a differential equation in which variables can be separated. Integration then yields the solution in terms of v , which we replace with $\frac{y}{x}$.

EXAMPLE 2.16

Solve

$$(x^2 + y^2)dx - 2xy dy = 0.$$

Solution. We have

$$(x^2 + y^2)dx - 2xy dy = 0$$

and so

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy} \quad (9)$$

This equation is homogeneous in x and y . So, put $y = vx$. We have $\frac{dy}{dx} = v + x \frac{dv}{dx}$. Hence (9) becomes

$$v + x \frac{dv}{dx} = \frac{x^2 + v^2x^2}{2vx^2} = \frac{1 + v^2}{2v}$$

or

$$x \frac{dv}{dx} = \frac{1 + v^2}{2v} - v = \frac{1 - v^2}{2v}$$

or

$$\frac{2v dv}{1 - v^2} = \frac{dx}{x}$$

or

$$\left(\frac{1}{1-v} - \frac{1}{1+v} \right) dv = \frac{dx}{x}.$$

Integrating both sides, we get

$$-\log(1-v) - \log(1+v) = \log x + C$$

or

$$-\log(1-v^2) = \log x + C$$

or

$$\log x + \log(1-v^2) = C$$

or

$$\log x(1 - v^2) = C$$

or

$$\log x \left(1 - \left(\frac{y}{x} \right)^2 \right) = C$$

or

$$x \left(1 - \frac{y^2}{x^2} \right) = C.$$

Hence

$$x^2 - y^2 = Cx$$

is the general solution of the given differential equation.

EXAMPLE 2.17

Solve

$$x \frac{dy}{dx} = y + \sqrt{x^2 + y^2}.$$

Solution. We have

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x},$$

which is homogeneous in x and y . Put $y = vx$ so that

$\frac{dy}{dx} = v + x \frac{dv}{dx}$. Hence the equation takes the form

$$\begin{aligned} v + x \frac{dv}{dx} &= \frac{vx + \sqrt{x^2 + v^2 x^2}}{x} \\ &= v + \sqrt{1 + v^2}. \end{aligned}$$

Thus

$$x \frac{dv}{dx} = \sqrt{1 + v^2}$$

and so

$$\frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}.$$

Integrating both sides, we get

$$\int \frac{dv}{\sqrt{1 + v^2}} = \int \frac{dx}{x} + C$$

or

$$\begin{aligned} \log(v + \sqrt{1 + v^2}) &= \log x + \log C \\ &= \log x C. \end{aligned}$$

Hence

$$v + \sqrt{1 + v^2} = x C.$$

Substituting the value of v , we get

$$\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = x C,$$

which is the required solution.

EXAMPLE 2.18

Solve

$$(y^2 - 2xy)dx = (x^2 - 2xy)dy.$$

Solution. The given differential equation can be expressed by

$$\frac{dy}{dx} = \frac{y^2 - 2xy}{x^2 - 2xy}.$$

Clearly it is a homogeneous equation. Put $y = vx$ so that $\frac{dy}{dx} = v + x \frac{dv}{dx}$. Hence

$$v + x \frac{dv}{dx} = \frac{v^2 x^2 - 2x^2 v}{x^2 - 2x^2 v} = \frac{v^2 - 2v}{1 - 2v}.$$

or

$$\begin{aligned} x \frac{dv}{dx} &= \frac{v^2 - 2v}{1 - 2v} - v \\ &= \frac{v^2 - 2v - v + 2v^2}{(1 - 2v)} \\ &= \frac{3v^2 - 3v}{1 - 2v}. \end{aligned}$$

Thus

$$\frac{1 - 2v}{3v^2 - 3v} dv = \frac{dx}{x}$$

or

$$\frac{1 - 2v}{3(v^2 - v)} = \frac{dx}{x}.$$

Integrating, we get

$$\begin{aligned} -\frac{1}{3} \log(v^2 - v) &= \log x + C \\ -\frac{1}{3} \log \left(\frac{y^2}{x^2} - \frac{y}{x} \right) &= \log x + C \\ -\frac{1}{3} \log \left(\frac{y^2 - xy}{x^2} \right) &= \log x + C \\ \log x^3 \left(\frac{y^2 - xy}{x^2} \right) &= \log C. \end{aligned}$$

Hence

$$x(y^2 - xy) = C.$$

EXAMPLE 2.19

Solve

$$x \frac{dy}{dx} + \frac{y^2}{x} = y.$$

Solution. We have

$$x \frac{dy}{dx} = y - \frac{y^2}{x} = \frac{xy - y^2}{x}$$

or

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2},$$

which is homogeneous in x and y . Putting $y = vx$, we have $\frac{dy}{dx} = v + x \frac{dv}{dx}$. Hence the equation becomes

$$v + x \frac{dv}{dx} = \frac{xy - y^2}{x^2} = \frac{vx^2 - v^2x^2}{x^2} = v - v^2$$

or

$$x \frac{dv}{dx} = -v^2$$

or

$$\frac{dv}{v^2} = -\frac{dx}{x}.$$

Integrating, we get

$$\frac{1}{v} = -\log x + \log C.$$

Putting $v = \frac{y}{x}$, we get

$$\frac{x}{y} = -\log x + C.$$

EXAMPLE 2.20

Solve

$$(x^2 - y^2)dx = 2xy dy.$$

Solution. We have

$$\frac{dy}{dx} = \frac{x^2 - y^2}{2xy}$$

so that the given equation is homogeneous in x and y .

Putting $y = vx$, the equation takes the form

$$v + x \frac{dv}{dx} = \frac{x^2 - v^2x^2}{2vx^2} = \frac{1 - v^2}{2v}.$$

Therefore,

$$x \frac{dv}{dx} = \frac{1 - v^2}{2v} - v = \frac{1 - v^2 - 2v^2}{2v} = \frac{1 - 3v^2}{2v}.$$

Now, variables separation yields

$$\frac{2v}{1 - 3v^2} dv = \frac{dx}{x}.$$

Integrating both sides, we get

$$-\frac{1}{3} \int \frac{-6v}{1 - 3v^2} dv = \int \frac{dx}{x} + C$$

or

$$-\frac{1}{3} \log(1 - 3v^2) = \log x + C$$

or

$$-\log(1 - 3v^2) = \log x^3 + 3C$$

or

$$\log x^3(1 - 3v^2) = -3C$$

or

$$x^3 \left(1 - \frac{3y^2}{x^2}\right) = K$$

or

$$x(x^2 - 3y^2) = K.$$

EXAMPLE 2.21

Solve

$$(1 + e^{x/y} dx + e^{x/y} (1 - \frac{x}{y}) dy = 0.$$

Solution. We have

$$\frac{dx}{dy} = -\frac{e^{x/y}(1 - \frac{x}{y})}{1 + e^{x/y}}.$$

Putting $x = vy$, we have $\frac{dx}{dy} = v + y \frac{dv}{dy}$ and so the above equation reduces to

$$v + y \frac{dv}{dy} = -\frac{e^v(1 - v)}{1 + e^v} = \frac{e^v(v - 1)}{1 + e^v}.$$

Hence

$$y \frac{dv}{dy} = \frac{e^v(v - 1)}{1 + e^v} - v = -\frac{v + e^v}{1 + e^v}$$

and so separation of variable gives

$$\frac{1 + e^v}{v + e^v} dv = -\frac{dy}{y}.$$

Integrating both sides, we get

$$\int \frac{1 + e^v}{v + e^v} dv = -\int \frac{dy}{y} + C$$

or

$$\log(v + e^v) = -\log y + \log k$$

or

$$\log(v + e^v) = \log \frac{k}{y}.$$

Thus

$$v + e^v = \frac{k}{y}.$$

But $v = \frac{x}{y}$. Hence

$$x + ye^{x/y} = k \text{ (constant).}$$

2.7 EQUATIONS REDUCIBLE TO HOMOGENEOUS FORM

Equations of the form

$$\frac{dy}{dx} = \frac{ax + by + C}{a'x + b'y + C'}$$

can be reduced, by substitution, to the homogeneous form and then solved. Two cases arise:

- (i) If $\frac{a}{a'} \neq \frac{b}{b'}$, then the substitution $x = X + h$ and $y = Y + k$, where h and k are suitable constants, makes the given equation homogeneous in X and Y .
- (ii) If $\frac{a}{a'} = \frac{b}{b'}$, then the substitution $ax + by = z$ serves our purpose.

EXAMPLE 2.22

Solve

$$\frac{dy}{dx} = \frac{2x - y + 1}{x + 2y - 3}.$$

Solution. We observe that the condition $\frac{a}{a'} \neq \frac{b}{b'}$ is satisfied in the present case. So, we put

$$x = X + h \quad \text{and} \quad y = Y + k.$$

Therefore, $dx = dX$ and $dy = dY$ and the given equation reduces to

$$\begin{aligned} \frac{dY}{dX} &= \frac{2(X + h) - (Y + k) + 1}{X + h + 2(Y + k) - 3} \\ &= \frac{2X - Y + 2h - k + 1}{X + 2Y + h + 2k - 3}. \end{aligned}$$

We choose h and k such that

$$2h - k + 1 = 0, \text{ and}$$

$$h + 2k - 3 = 0.$$

Solving these two equations, we get $h = \frac{1}{5}$, $k = \frac{7}{5}$. Hence

$$\frac{dY}{dX} = \frac{2X - Y}{X + 2Y},$$

which is homogeneous in X and Y . So put $Y = vX$. Then

$$v + X \frac{dv}{dX} = \frac{2X - vX}{X + 2vX} = \frac{2 - v}{1 + 2v}$$

and so

$$X \frac{dv}{dX} = \frac{2 - v}{1 + 2v} - v = \frac{-2v^2 - 2v + 2}{1 + 2v}.$$

Now separation of variables yields

$$\frac{1 + 2v}{-2(v^2 + v - 1)} dv = \frac{dX}{X}.$$

Integrating both sides, we get

$$-\frac{1}{2} \int \frac{1 + 2v}{v^2 + v - 1} dv = \int \frac{dX}{X} + C$$

or

$$-\frac{1}{2} \log(v^2 + v - 1) = \log X + C$$

or

$$-\log(v^2 + v - 1) = 2 \log X + C = \log X^2 + C$$

or

$$\log X^2(v^2 + v - 1) = \log k$$

or

$$X^2(v^2 + v - 1) = k$$

or

$$X^2 \left(\frac{Y^2}{X^2} + \frac{Y}{X} - 1 \right) = k$$

or

$$Y^2 + YX - X^2 = k$$

or

$$\left(y + \frac{7}{5}\right)^2 + \left(y + \frac{7}{5}\right) \left(x + \frac{1}{5}\right) - \left(x + \frac{1}{5}\right)^2 = k$$

or

$$y^2 + xy - x^2 + \left(\frac{14}{5} + \frac{1}{5}\right)y + \left(-\frac{2}{5} + \frac{7}{5}\right)x = k$$

or

$$y^2 + xy - x^2 + 3y + x = k.$$

EXAMPLE 2.23

Solve

$$\frac{dy}{dx} = \frac{2x + 3y + 4}{4x + 6y + 5}.$$

Solution. We note that the condition $\frac{a}{a'} = \frac{b}{b'}$ is satisfied in the present case. Hence put $2x + 3y = z$ so that $2 + 3\frac{dy}{dx} = \frac{dz}{dx}$. Hence the given equation reduces to

$$\frac{dz}{dx} = \frac{7z + 22}{2z + 5}$$

or

$$\frac{2z + 5}{7z + 22} dz = dx$$

Integrating both sides, we get

$$\int \frac{2z + 5}{7z + 22} dz = \int dx + C$$

or

$$\frac{2}{7}z - \frac{9}{49}\log(7z + 22) = x + C.$$

Substituting $z = 2x + 3y$, we get

$$14(2x + 3y) - 9\log(14x + 21y + 22) = 49x + C$$

or

$$21x - 42y + 9\log(14x + 21y + 22) = C$$

EXAMPLE 2.24

Solve

$$(x + 2y + 1) dx = (2x + 4y + 3) dy.$$

Solution. We have

$$\frac{dy}{dx} = \frac{x + 2y + 1}{2x + 4y + 3} = \frac{ax + by + C}{a'x + b'y + C'}.$$

We observe that $\frac{a}{a'} = \frac{b}{b'} = \frac{1}{2}$. So we put $x + 2y = z$ and have $1 + 2\frac{dy}{dx} = \frac{dz}{dx}$. Then $\frac{dy}{dx} = \frac{dz/dx - 1}{2}$. The given equation now reduces to

$$\frac{(dz/dx) - 1}{2} = \frac{z + 1}{2z + 3}$$

or

$$\frac{dz}{dx} = \frac{2z + 2}{2z + 3} + 1 = \frac{4z + 5}{2z + 3}$$

or

$$\left(\frac{2z + 3}{4z + 5}\right) dz = dx.$$

Integrating both sides, we have

$$\int \frac{2z + 3}{4z + 5} dz = \int dx + C$$

or

$$\int \left[\frac{1}{2} + \frac{1}{2(4z + 5)} \right] dz = \int dx + C$$

or

$$\frac{1}{2}z + \frac{1}{8}\log(4z + 5) = x + C$$

or

$$4z + \log(4z + 5) = 4x + k \text{ (constant)}$$

or

$$4(x + 2y) + \log(4x + 8y + 5) = 8x + k$$

or

$$4(2y - x) + \log(4x + 8y + 5) = k.$$

2.8 LINEAR DIFFERENTIAL EQUATIONS

Definition 2.14. A differential equation is said to be *linear* if the dependent variable and its derivatives occur in the first degree and are not multiplied together.

Thus a linear differential equation is of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P \text{ and } Q \text{ are functions of } x \text{ only}$$

or

$$\frac{dx}{dy} + Px = Q, \text{ where } P \text{ and } Q \text{ are functions of } y \text{ only.}$$

A linear differential equation of the first order is called *Leibnitz's linear equation*.

To solve the linear differential equation

$$\frac{dy}{dx} + Py = Q, \quad (10)$$

we multiply both sides by $e^{\int P dx}$ and get

$$\frac{dy}{dx} e^{\int P dx} + Py e^{\int P dx} = Q e^{\int P dx} \quad (11)$$

But

$$\frac{d}{dx}(y e^{\int P dx}) = \frac{dy}{dx} e^{\int P dx} + P y e^{\int P dx}.$$

Hence (11) reduces to

$$\frac{d}{dx}(y e^{\int P dx}) = Q e^{\int P dx},$$

which an integration yields

$$y e^{\int P dx} = \int Q e^{\int P dx} dx + C,$$

as the required solution.

The factor $e^{\int P dx}$ is called an *integrating factor* (I.F.) of the differential equation.

EXAMPLE 2.25

Solve

$$(x + 2y^3) \frac{dy}{dx} = y.$$

Solution. The given differential equation can be written as

$$y \frac{dx}{dy} = x + 2y^3$$

or

$$y \frac{dx}{dy} - x = 2y^3$$

or

$$\frac{dx}{dy} - \frac{1}{y}x = 2y^2.$$

Comparing with $\frac{dx}{dy} + Px = Q$, we have $P = -\frac{1}{y}$ and $Q = 2y^2$. The integrating factor is

$$\text{I.F.} = e^{\int P dy} = e^{\int -\frac{1}{y} dy} = e^{-\log y} = e^{\log y^{-1}} = y^{-1}.$$

Therefore, the solution of the differential equation is

$$\begin{aligned} xy^{-1} &= \int (2y^2)y^{-1} dy + C \\ &= \int 2y dy + C = y^2 + C. \end{aligned}$$

EXAMPLE 2.26

Solve

$$(1 + y^2) \frac{dx}{dy} = \tan^{-1} y - x.$$

Solution. The given equation can be expressed as

$$\frac{dx}{dy} + \frac{x}{1 + y^2} = \frac{\tan^{-1} y}{1 + y^2}$$

and so is Leibnitz's linear equation in x . Comparing with $\frac{dx}{dy} + Px = Q$, we get $P = \frac{1}{1+y^2}$ and $Q = \frac{\tan^{-1} y}{1+y^2}$. Therefore

$$\text{I.F.} = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}.$$

Hence the solution of the differential equation is

$$\begin{aligned} xe^{\tan^{-1} y} &= \int \frac{\tan^{-1} y}{1 + y^2} e^{\tan^{-1} y} dy + C \\ &= \int t e^t dt + C, \quad \tan^{-1} y = t \\ &= t e^t - e^t + C \quad (\text{integration by parts}) \\ &= (\tan^{-1} y - 1) e^{\tan^{-1} y} + C. \end{aligned}$$

Hence

$$x = \tan^{-1} y - 1 + C e^{-\tan^{-1} y}.$$

EXAMPLE 2.27

Solve

$$\sin 2x \frac{dy}{dx} = y + \tan x.$$

Solution. We have

$$\frac{dy}{dx} - \frac{1}{\sin 2x} y = \frac{\tan x}{2 \cos^2 x \sin x} = \frac{1}{2} \sec^2 x.$$

Thus, the given equation is linear in y . Now

$$\begin{aligned} \text{I.F.} &= e^{\int -\csc 2x dx} = e^{-\frac{1}{2} \log \tan x} \\ &= e^{\log(\tan x)^{-1/2}} = (\tan x)^{-1/2}. \end{aligned}$$

Hence, the solution of the given differential equation is

$$\begin{aligned} y(\tan x)^{-1/2} &= \int \frac{1}{2} \sec^2 x (\tan x)^{-1/2} dx + C \\ &= \frac{1}{2} \int (\sec^2 x) (\tan x)^{-1/2} dx + C \\ &= \frac{1}{2} \frac{(\tan x)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C \\ &= (\tan x)^{1/2} + C, \end{aligned}$$

which can be expressed as

$$y = \tan x + C\sqrt{\tan x}.$$

2.9 EQUATIONS REDUCIBLE TO LINEAR DIFFERENTIAL EQUATIONS

Definition 2.15. An equation of the form

$$\frac{dy}{dx} + Py = Qy^n, \quad (12)$$

where P and Q are functions of x is called *Bernoulli's equation*.

The Bernoulli's equation can be reduced to Leibnitz's differential equation in the following way:

Divide both sides of (12) by y^n to get

$$y^{-n} \frac{dy}{dx} + P y^{1-n} = Q \quad (13)$$

Put $y^{1-n} = z$ to give

$$(1-n)y^{-n} \frac{dy}{dx} = \frac{dz}{dx}$$

or

$$y^{-n} \frac{dy}{dx} = \frac{1}{1-n} \frac{dz}{dx}.$$

Hence (13) reduces to

$$\frac{1}{1-n} \frac{dz}{dx} + Pz = Q$$

or

$$\frac{dz}{dx} + P(1-n)z = Q(1-n),$$

which is Leibnitz's linear equation in z and can be solved by finding the appropriate integrating factor.

EXAMPLE 2.28

Solve

$$\frac{dy}{dx} + x \sin 2y = x^2 \cos^2 y.$$

Solution. Dividing throughout by $\cos^2 y$, we have

$$\sec^2 y \frac{dy}{dx} + \frac{2x \sin y \cos y}{\cos^2 y} = x^2$$

or

$$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3.$$

Putting $\tan y = z$, we have $\sec^2 y \frac{dy}{dx} = \frac{dz}{dx}$. Hence, the given equation reduces to

$$\frac{dz}{dx} + 2xz = x^3, \quad (14)$$

which is Leibnitz-equation in z and x . The integrating factor is given by

$$\text{I.F.} = e^{\int P dx} = e^{\int 2x dx} = e^{x^2}.$$

Hence solution of the equation (14) is

$$\begin{aligned} ze^{x^2} &= \int x^3 \cdot e^{x^2} dx + C = \int x(x^2 e^{x^2}) dx + C \\ &= \frac{1}{2} \int 2x(x^2 e^{x^2}) dx + C = \frac{1}{2} \int te^t dt + C, \quad x^2 = t \\ &= \frac{1}{2}(x^2 - 1)e^{x^2} + C \end{aligned}$$

Putting back the value of z , we get

$$\tan y e^{x^2} = \frac{1}{2}(x^2 - 1) e^{x^2} + C$$

or

$$\tan y = \frac{1}{2}(x^2 - 1) + C e^{-x^2}.$$

EXAMPLE 2.29

Solve

$$\frac{dy}{dx} + y = xy^3.$$

Solution. Dividing throughout by y^3 , we get

$$y^{-3} \frac{dy}{dx} + y^{-2} = x.$$

Put $y^{-2} = z$. Then $\frac{dz}{dx} = -2y^{-3} \frac{dy}{dx}$ and, therefore, the above differential equation reduces to

$$-\frac{1}{2} \frac{dz}{dx} + z = x$$

or

$$\frac{dz}{dx} - 2z = -2x,$$

which is Leibnitz's equation in z . We have

$$\text{I.F.} = e^{-\int 2 dx} = e^{-2x}.$$

Therefore, the solution of the equation in z is

$$z e^{-2x} = \int -2x \cdot e^{-2x} dx + C = \frac{1}{2} e^{-2x} (2x + 1) + C$$

or

$$z = x + \frac{1}{2} + C e^{2x}$$

or

$$y^{-2} = x + \frac{1}{2} + C e^{2x}.$$

EXAMPLE 2.30

Solve

$$x \frac{dy}{dx} + y = x^3 y^6.$$

Solution. Dividing throughout by y^6 , we have

$$y^{-6} \frac{dy}{dx} + \frac{y^{-5}}{x} = x^2.$$

Putting $y^{-5} = z$, we have $-5y^{-6} \frac{dy}{dx} = \frac{dz}{dx}$ and so the preceding equation transfers to

$$-\frac{1}{5} \frac{dz}{dx} + \frac{z}{x} = x^2$$

or

$$\frac{dz}{dx} - \frac{5z}{x} = -5x^2.$$

Now

$$\text{I.F.} = e^{-5 \int \frac{1}{x} dx} = e^{-5 \log x} = x^{-5}.$$

Therefore, the solution the above differential equation in z is

$$z \cdot x^{-5} = \int -5x^2 x^{-5} dx + C$$

or

$$y^{-6} \cdot x^{-5} = -5 \int x^{-3} dx + C = \frac{-5}{-2} x^{-2} + C$$

or

$$1 = (10 + Cx^2)x^3 y^5.$$

2.10 EXACT DIFFERENTIAL EQUATION

Definition 2.16. A differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

is called an *exact differential equation* if there exists a function $U \equiv U(x, y)$ of x and y such that $M(x, y)dx + N(x, y)dy = dU$.

Thus, a differential equation of the form $M(x, y)dx + N(x, y)dy = 0$ is an exact differential equation if $Mdx + Ndy$ is an exact differential.

For example, the differential equation $y^2 dx + 2xy dy = 0$ is an exact equation because it is the total differential of $U(x, y) = xy^2$. In fact the coefficient of dy is $\frac{\partial F}{\partial y}(xy^2) = 2xy$.

The following theorem tells us whether a given differential equation is exact or not.

Theorem 2.3. A necessary and sufficient condition for the differential equation $Mdx + Ndy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

where M and N are functions of x and y having continuous first order derivative at all points in the rectangular domain.

Proof: (1) *Condition is necessary.* Suppose that the differential equation $Mdx + Ndy = 0$ is exact. Then there exists a function $U(x, y)$ such that

$$M dx + N dy = dU.$$

But, in term of partial derivatives, we have

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy.$$

Therefore,

$$M dx + N dy = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy.$$

Equating coefficients of dx and dy , we get

$$M = \frac{\partial U}{\partial x} \quad \text{and} \quad N = \frac{\partial U}{\partial y}.$$

Now

$$\frac{\partial M}{\partial y} = \frac{\partial^2 U}{\partial y \partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 U}{\partial x \partial y}.$$

Since partial derivatives of M and N are continuous, we have

$$\frac{\partial^2 U}{\partial y \partial x} = \frac{\partial^2 U}{\partial x \partial y}.$$

Hence

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

(2) *Condition is sufficient.* Suppose that M and N satisfy

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Let

$$\int M \, dx = U,$$

where y is treated as a constant while integrating M . Then

$$\frac{\partial}{\partial x} \left(\int M \, dx \right) = \frac{\partial U}{\partial x}$$

or

$$M = \frac{\partial U}{\partial x}.$$

Therefore,

$$\frac{\partial M}{\partial y} = \frac{\partial^2 U}{\partial y \, \partial x}$$

and so

$$\frac{\partial M}{\partial y} = \frac{\partial^2 U}{\partial y \, \partial x} = \frac{\partial N}{\partial x}.$$

Also, by continuity of partial derivatives,

$$\frac{\partial^2 U}{\partial y \, \partial x} = \frac{\partial^2 U}{\partial x \, \partial y}.$$

Thus

$$\frac{\partial M}{\partial y} = \frac{\partial^2 U}{\partial y \, \partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 U}{\partial x \, \partial y}.$$

Integrating both sides of $\frac{\partial N}{\partial x} = \frac{\partial^2 U}{\partial x \, \partial y}$ with respect to x , we get

$$N = \frac{\partial U}{\partial y} + f(y).$$

Thus,

$$\begin{aligned} Mdx + Ndy &= \frac{\partial U}{\partial x} dx + \left[\frac{\partial U}{\partial y} + f(y) \right] dy \\ &= \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + f(y) dy \\ &= dU + f(y) dy \\ &= d[U + \int f(y) dy] \end{aligned}$$

Thus, $M \, dx + N \, dy$ is the exact differential of $U + \int f(y) \, dy$ and, hence, the differential equation $M \, dx + N \, dy = 0$ is exact.

2.11 THE SOLUTION OF EXACT DIFFERENTIAL EQUATION

In the proof of Theorem 2.3, we note that if $M \, dx + N \, dy = 0$ is exact, then

$$M \, dx + N \, dy = d(U + \int f(y) dy).$$

Therefore,

$$d(U + \int f(y) dy) = 0$$

or

$$dU + f(y) dy = 0$$

Integrating, we get the required solution as

$$U + \int f(y) dy = C(\text{constant})$$

or

$$\int_{y \text{ constant}} M \, dx + \int f(y) \, dy = C$$

or

$$\int_{y \text{ constant}} M \, dx + \int (\text{terms of } N \text{ not containing } x) dy = C.$$

EXAMPLE 2.31

Solve

$$(2x \cos y + 3x^2 y) dx + (x^3 - x^2 \sin y - y) dy = 0$$

Solution. Comparing with $M \, dx + N \, dy = 0$, we get

$$M = 2x \cos y + 3x^2 y,$$

$$N = x^3 - x^2 \sin y - y.$$

Then

$$\frac{\partial M}{\partial y} = -2x \sin y + 3x^2 = \frac{\partial N}{\partial x}.$$

Hence the given equation is exact. Therefore, the solution of the equation is given by

$$\int_{y \text{ constant}} M \, dx + \int (\text{terms of } N \text{ not containing } x) dy = C$$

or

$$\int_{y \text{ constant}} (2x \cos y + 3x^2 y) dx + \int -y dy = C$$

or

$$x^2 \cos y + x^3 y - \frac{y^2}{2} = C.$$

EXAMPLE 2.32

Solve

$$(2xy + y - \tan y)dx + (x^2 - x \tan^2 y + \sec^2 y)dy = 0.$$

Solution. Comparing with $Mdx + Ndy = 0$, we observe that

$$M = 2xy + y - \tan y, \text{ and}$$

$$N = x^2 - x \tan^2 y + \sec^2 y.$$

Therefore,

$$\frac{\partial M}{\partial y} = 2x - \sec^2 y + 1,$$

$$\frac{\partial N}{\partial x} = 2x - \tan^2 y.$$

Hence, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and so the equation is exact. Its solution is given by

$$\int_{y \text{ constant}} (2xy + y - \tan y) dx + \int \sec^2 y dy = C$$

or

$$x^2 y + xy - x \tan y + \tan y = C.$$

EXAMPLE 2.33

Solve

$$(2xy \cos x^2 - 2xy + 1) dx + (\sin x^2 - x^2) dy = 0.$$

Solution. Comparing with $M dx + N dy = 0$, we note that

$$M = 2xy \cos x^2 - 2xy + 1, N = \sin x^2 - x^2.$$

Then

$$\frac{\partial M}{\partial y} = 2x \cos x^2 - 2x, \frac{\partial N}{\partial x} = 2x \cos x^2 - 2x.$$

Thus $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and, therefore, the given equation is exact. The solution of the given equation is

$$\int_{y \text{ constant}} (2xy \cos x^2 - 2xy + 1) dx + \int 0 dy = C$$

or

$$y \int 2x \cos x^2 dx - 2y \int x dx + \int dx = 0$$

or

$$y \int \cos t dt - x^2 y + x = C, \quad x^2 = t$$

or

$$y \sin x^2 - x^2 y + x = C.$$

EXAMPLE 2.34

Solve

$$\frac{dy}{dx} + \frac{y \cos x + \sin y + y}{\sin x + x \cos y + x} = 0.$$

Solution. The given differential equation is

$$(y \cos x + \sin y + y)dx + (\sin x + x \cos y + x)dy = 0$$

Comparing with $M dx + N dy = 0$, we get

$$M = y \cos x + \sin y + y,$$

$$N = \sin x + x \cos y + x,$$

and so

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1,$$

$$\frac{\partial N}{\partial x} = \cos x + \cos y + 1.$$

Therefore, the given equation is exact and its solution is

$$\int_{y \text{ constant}} (y \cos x + \sin y + y) dx + \int 0 dy = C$$

or

$$y \sin x + x \sin y + xy = C.$$

EXAMPLE 2.35

Solve

$$(\sec x \tan x \tan y - e^x)dx + \sec x \sec^2 y dy = 0.$$

Solution. Comparing with $M dx + N dy = 0$, we have

$$M = \sec x \tan x \tan y - e^x,$$

$$N = \sec x \sec^2 y.$$

Therefore,

$$\frac{\partial M}{\partial y} = \sec x \tan x \sec^2 y,$$

$$\frac{\partial N}{\partial x} = \sec x \tan x \sec^2 y.$$

Hence the equation is exact and its solution is given by

$$\int_{y \text{ constant}} (\sec x \tan x \tan y - e^x) dx + \int 0 dy = C$$

or

$$\tan y \int \sec x \tan x dx - e^x = C$$

or

$$\tan y \sec x - e^x = 0$$

EXAMPLE 2.36

Solve

$$(1 + e^{x/y}) dx + e^{x/y} \left(1 - \frac{x}{y}\right) dy = 0.$$

Solution. Comparing with $M dx + N dy = 0$, we get

$$M = 1 + e^{x/y}, \quad N = e^{x/y} \left(1 - \frac{x}{y}\right),$$

$$\frac{\partial M}{\partial y} = e^{x/y} \left(\frac{-x}{y^2}\right) = -\frac{x}{y^2} e^{x/y},$$

$$\frac{\partial N}{\partial x} = e^{x/y} \left(\frac{1}{y}\right) \left(1 - \frac{x}{y}\right) + e^{x/y} \left(-\frac{1}{y}\right) = -\frac{x}{y^2} e^{x/y}.$$

Hence the given differential equation is exact and its solution is

$$\int_{y \text{ constant}} (1 + e^{x/y}) dx + \int 0 dy = C$$

or

$$x + y e^{x/y} = C.$$

EXAMPLE 2.37

Show that the differential equation

$$(ax + hy + g)dx + (hx + by + f)dy = 0$$

is the differential equation of a family of conics.

Solution. Comparing the given differential equation with $Mdx + Ndy = 0$, we get

$$M = ax + hy + g, \quad N = hx + by + f,$$

$$\frac{\partial M}{\partial y} = h, \quad \frac{\partial N}{\partial x} = h.$$

Hence the given differential equation is exact and its solution is

$$\int_{y \text{ constant}} (ax + hy + g) dx + \int (by + f) dy = C$$

or

$$a \frac{x^2}{2} + hxy + gx + b \frac{y^2}{2} + fy = C$$

or

$$ax^2 + bx^2 + 2hxy + 2gx + 2fy + k = 0,$$

which represents a family of conics.

2.12 EQUATIONS REDUCIBLE TO EXACT EQUATION

Differential equations which are not exact can sometimes be made exact on multiplying by a suitable factor called an *integrating factor*.

The integrating factor for $Mdx + Ndy = 0$ can be found by the following rules:

1. If $Mdx + Ndy = 0$ is a homogeneous equation in x and y , then $\frac{1}{Mx+Ny}$ is an integrating factor, provided $Mx + Ny \neq 0$.
2. If the equation $Mdx + Ndy = 0$ is of the form $f(xy)y dx + \phi(xy)x dy = 0$, then $\frac{1}{Mx-Ny}$ is an integrating factor, provided $Mx - Ny \neq 0$.
3. Let $M dx + N dy = 0$ be a differential equation. If $\frac{(\partial M / \partial y) - (\partial N / \partial x)}{N}$ is a function of x only, say $f(x)$, then $e^{\int f(x) dx}$ is an integrating factor.
4. Let $M dx + N dy = 0$. If $\frac{(\partial N / \partial x) - (\partial M / \partial y)}{M}$ is a function of y only, say $f(y)$, then $e^{\int f(y) dy}$ is an integrating factor.
5. For the equation $x^a y^b (mydx + nx dy) + x^a y^b (m'y dx + n'x dy) = 0$

the integrating factor is $x^h y^k$, where h and k are such that

$$\frac{a+h+1}{m} = \frac{b+k+1}{n},$$

$$\frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}.$$

EXAMPLE 2.38

Solve

$$y dx - x dy + \log x dx = 0.$$

Solution. The given equation is not exact. Dividing by x^2 , we get

$$\frac{y}{x^2} dx - \frac{x}{x^2} dy + \frac{1}{x^2} \log x dx = 0$$

or

$$\frac{y dx - x dy}{x^2} + \frac{1}{x^2} \log x dx = 0$$

or

$$-d\left(\frac{y}{x}\right) + d\left(\int \frac{1}{x^2} \log x dx\right) = 0$$

or

$$d\left(\int \frac{1}{x^2} \log x dx - \frac{y}{x}\right) = 0$$

Thus $\frac{1}{x^2}$ is an integrating factor and on integration, we get the solution as

$$\int \frac{1}{x^2} \log x dx - \frac{y}{x} = C$$

On integration by parts, we have

$$-\frac{\log x}{x} + \int \frac{1}{x^2} dx = C + \frac{y}{x}.$$

or

$$-\frac{1}{x} \log x - \frac{1}{x} = C + \frac{y}{x}.$$

EXAMPLE 2.39

Solve

$$x dx + y dy = \frac{a^2(x dy - y dx)}{x^2 + y^2}.$$

Solution. We know that

$$d\left(\tan^{-1} \frac{y}{x}\right) = \frac{x dy - y dx}{x^2 + y^2}.$$

Therefore, the given differential equation is

$$x dx + y dy - a^2 d\left(\tan^{-1} \frac{y}{x}\right)$$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} - a^2 \tan^{-1} \frac{y}{x} = C$$

or

$$x^2 + y^2 - 2a^2 \tan^{-1} \frac{y}{x} = k \text{ (constant).}$$

EXAMPLE 2.40

Solve

$$x dy - y dx + a(x^2 + y^2) dx = 0.$$

Solution. Dividing throughout by $x^2 + y^2$, we get

$$\frac{x dy}{x^2 + y^2} - \frac{y dx}{x^2 + y^2} + a dx = 0$$

or

$$\frac{x dy - y dx}{x^2 + y^2} + a dx = 0$$

or

$$d\left(\tan^{-1} \frac{y}{x}\right) + a dx = 0.$$

Integrating, we get

$$\tan^{-1} \frac{y}{x} + ax = C.$$

EXAMPLE 2.41

Solve

$$(x^2 y - 2xy^2) dx = (x^3 - 3x^2 y) dy = 0.$$

Solution. The given equation is homogeneous.

Comparing it with $M dx + N dy = 0$, we have

$$M = x^2 y - 2xy^2, N = -x^3 + 3x^2 y,$$

$$\frac{\partial M}{\partial y} = x^2 - 4xy, \frac{\partial N}{\partial x} = -3x^2 + 6xy.$$

Therefore, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ and so the given equation is not exact. Further

$$Mx + Ny = x^3 y - 2x^2 y^2 - x^3 y + 3x^2 y^2 = x^2 y^2 \neq 0.$$

Hence the integrating factor is $\frac{1}{Mx+Ny} = \frac{1}{x^2 y^2}$.

Multiplying the given equation by $\frac{1}{x^2 y^2}$, we have

$$\left(\frac{1}{y} - \frac{2}{x}\right) dx - \left(\frac{x}{y^2} - \frac{3}{y}\right) dy = 0 \quad (15)$$

Since

$$\frac{\partial}{\partial y} \left(\frac{1}{y} - \frac{2}{x} \right) = \frac{\partial}{\partial x} \left(\frac{x}{y^2} - \frac{3}{y} \right) = -\frac{1}{y^2},$$

the equation (15) is exact and so its solution is

$$\int \left(\frac{1}{y} - \frac{2}{x} \right) dx - \int \left(-\frac{3}{y} \right) dy = C$$

y constant

or

$$\frac{x}{y} - 2 \log x + 3 \log y = C.$$

EXAMPLE 2.42

Solve

$$y dx + 2x dy = 0.$$

Solution. The given equation is of type $Mdx + Ndy = 0$ and is homogeneous. Further

$$M = y, N = 2x,$$

$$\frac{\partial M}{\partial y} = 1, \frac{\partial N}{\partial x} = 2, \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

Thus the equation is not exact. But

$$Mx + Ny = xy + 2xy = 3xy \neq 0.$$

Therefore, $\frac{1}{3xy}$ is the integrating factor. Multiplying the given equation throughout by $\frac{1}{3xy}$, we get

$$\frac{1}{3xy} y dx + \frac{2x}{3xy} dy = 0$$

or

$$\frac{1}{3x} dx + \frac{2}{3y} dy = 0.$$

The solution is

$$\frac{1}{3} \int \frac{1}{x} dx + \frac{2}{3} \int \frac{1}{y} dy = C$$

or

$$\frac{1}{3} \log x + \frac{2}{3} \log y = C$$

or

$$\log xy^2 = k = \log p$$

or

$$xy^2 = p \text{ (constant).}$$

EXAMPLE 2.43

Solve

$$x^2 y dx - (x^3 + y^3) dy = 0.$$

Solution. The given equation is homogeneous and comparing with $Mdx + Ndy = 0$, we get

$$M = x^2 y, N = -x^3 - y^3.$$

Then

$$Mx + Ny = x^3 y - x^3 y - y^4 = -y^4 \neq 0.$$

Thus the integrating factor is $\frac{1}{-y^4}$. Multiplying the given differential equation throughout by $-\frac{1}{y^4}$, we get

$$-\frac{1}{y^4} x^2 y dx + \frac{1}{y^4} (x^3 + y^3) dy = 0$$

or

$$-\frac{x^2}{y^3} dx + \left(\frac{x^3}{y^4} + \frac{1}{y} \right) dy = 0,$$

which is exact. Hence the required solution is

$$-\frac{1}{y^3} \int x^2 dx + \int \frac{1}{y} dy = C$$

$$-\frac{x^3}{3y^3} + \log y = C.$$

EXAMPLE 2.44

Solve

$$y(xy + 2x^2 y^2) dx + x(xy - x^2 y^2) dy = 0.$$

Solution. The given differential equation is of the form

$$f(xy)y dx + \phi(xy)x dy = 0.$$

Also comparing with $M dx + N dy = 0$, we get

$$M = y(xy + 2x^2 y^2), N = x(xy - x^2 y^2).$$

Therefore,

$$Mx - Ny = 3x^3 y^3 \neq 0.$$

Thus, $\frac{1}{3x^3 y^3}$ is the integrating factor. Multiplying throughout by $\frac{1}{3x^3 y^3}$, we have

$$\frac{1}{3x^3 y^3} (y(xy + 2x^2 y^2)) dx + \frac{1}{3x^3 y^3} [x(xy - x^2 y^2)] dy = 0.$$

or

$$\left(\frac{1}{3x^2 y} + \frac{2}{3x} \right) dx + \left(\frac{1}{3xy^2} - \frac{1}{3y} \right) x dy = 0.$$

Since

$$\frac{\partial}{\partial y} \left(\frac{1}{3x^2 y} + \frac{2}{3x} \right) = \frac{\partial}{\partial x} \left(\frac{1}{3xy^2} - \frac{1}{3y} \right),$$

the above equation is exact and its solution is

$$\int \left(\frac{1}{3x^2y} + \frac{2}{3x} \right) dx + \int \left(-\frac{1}{3y} \right) dy = C$$

y constant

or

$$-\frac{1}{3xy} + \frac{2}{3} \log x - \frac{1}{3} \log y = C$$

or

$$-\frac{1}{xy} + 2 \log x - \log y = k \text{ (constant).}$$

EXAMPLE 2.45

Solve

$$(1 + xy)y dx + (1 - xy)x dy = 0.$$

Solution. The given differential equation is of the form

$$f(xy) y dx + \phi(xy) x dy = 0.$$

Comparing with $M dx + N dy = 0$, we have

$$M = y + xy^2, \quad N = x - x^2y.$$

Therefore,

$$Mx - Ny = 2x^2y^2 \neq 0.$$

Therefore, the integrating factor is $\frac{1}{2x^2y^2}$. Multiplying the given differential equation throughout by $\frac{1}{2x^2y^2}$, we get

$$\frac{1}{2x^2y^2} (y + xy^2) dx + \frac{1}{2x^2y^2} (x - x^2y) dy = 0$$

or

$$\left(\frac{1}{2x^2y} + \frac{1}{2x} \right) dx + \left(\frac{1}{2xy^2} - \frac{1}{2y} \right) dy = 0.$$

We note that this equation is exact. Hence its solution is

$$\int \left(\frac{1}{2x^2y} + \frac{1}{2x} \right) dx + \int \left(-\frac{1}{2y} \right) dy = C$$

y constant

or

$$\frac{1}{2y} \left(-\frac{1}{x} \right) + \frac{1}{2} \log x - \frac{1}{2} \log y = C$$

or

$$\log \frac{x}{y} - \frac{1}{xy} = k \text{ (constant).}$$

EXAMPLE 2.46

Solve

$$(x^2y^2 + xy + 1)y dx + (x^2y^2 - xy + 1)x dy = 0.$$

Solution. The given differential equation is of the form

$$f(xy)y dx + \phi(xy)x dy = 0.$$

Moreover, comparing the given equation with $M dx + N dy = 0$, we get

$$M = x^2y^3 + xy^2 + y, \quad N = x^3y^2 - x^2y + x.$$

Therefore

$$\begin{aligned} Mx - Ny &= x^3y^3 + x^2y^2 + xy - x^3y^3 + x^2y^2 - xy \\ &= 2x^2y^2 \neq 0. \end{aligned}$$

Therefore, the integrating factor is $\frac{1}{2x^2y^2}$. Multiplying the given differential equation throughout by $\frac{1}{2x^2y^2}$, we get

$$\begin{aligned} \frac{1}{2x^2y^2} (x^2y^3 + xy^2 + y) dx \\ + \frac{1}{2x^2y^2} (x^3y^2 - x^2y + x) dy = 0 \end{aligned}$$

or

$$\left(\frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y} \right) dx + \frac{1}{2} \left(x - \frac{1}{y} + \frac{1}{xy^2} \right) dy = 0,$$

which is exact. Hence the solution of the equation is

$$\int \left(\frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y} \right) dx + \frac{1}{2} \int \left(-\frac{1}{y} \right) dy = C$$

y constant

or

$$\left(\frac{xy}{2} + \frac{1}{2} \log x - \frac{1}{2yx} \right) - \frac{1}{2} \log y = k$$

or

$$xy - \frac{1}{xy} + \log \frac{x}{y} = k \text{ (constant).}$$

EXAMPLE 2.47

Solve

$$y(2xy + 1) dx + x(1 + 2xy - x^3y^3) dy = 0.$$

Solution. The differential equation in question is of the form

$$f(xy)y dx + \phi(xy)x dy = 0.$$

Further comparing the given equation with $M dx + N dy = 0$, we get

$$M = 2xy^2 + y, N = x + 2x^2y - x^4y^3.$$

Therefore,

$$\begin{aligned} Mx - Ny &= 2x^2y^2 + xy - xy - 2x^2y^2 - x^4y^4 \\ &= -x^4y^4. \end{aligned}$$

Thus the integrating factor is $\frac{1}{-x^4y^4}$. Multiplying the given differential equation throughout by $-\frac{1}{x^4y^4}$, we get

$$\begin{aligned} -\frac{1}{x^4y^4}(2xy^2 + y)dx - \frac{1}{x^4y^4}(x + 2x^2y - x^4y^3) dy \\ = 0. \end{aligned}$$

or

$$\left(-\frac{2}{x^3y^2} - \frac{1}{x^4y^3}\right)dx + \left(-\frac{1}{x^3y^4} - \frac{2}{x^2y^3} + \frac{1}{y}\right)dy = 0,$$

which is exact. Hence the solution of the equation is

$$\int \left(-\frac{2}{x^3y^2} - \frac{1}{x^4y^3}\right) dx + \int \frac{1}{y} dy = C$$

or

$$\frac{1}{x^2y^2} + \frac{1}{3x^3y^3} + \log y = C.$$

EXAMPLE 2.48

Solve

$$(x^2 + y^2 + 2x) dx + 2y dy = 0.$$

Solution. Comparing the given equation with $Mdx + Ndy = 0$, we get

$$M = x^2 + y^2 + 2x, N = 2y$$

which gives

$$\frac{\partial M}{\partial y} = 2y, \frac{\partial N}{\partial x} = 0.$$

Thus the equation is not exact. We have

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y}{2y} = 1 = x^0 \text{ (function of } x\text{)}.$$

Therefore, $e^{\int 1 dx} = e^x$ is the integrating factor. Multiplying the given differential equation

throughout by e^x , we get

$$(x^2 + y^2 + 2x)e^x dx + 2ye^x dy = 0,$$

which is exact. Hence the required solution is

$$\int_{y \text{ constant}} (x^2 + y^2 + 2x) e^x dx + \int 0 dy = C,$$

which yield

$$(x^2 + y^2) e^x = C.$$

EXAMPLE 2.49

Solve

$$(xy^2 - e^{x^{1/3}})dx - x^2y dy = 0.$$

Solution. Comparing the given equation with $M dx + N dy = 0$, we get

$$M = xy^2 - e^{x^{1/3}}, N = -x^2y,$$

$$\frac{\partial M}{\partial y} = 2xy, \quad \frac{\partial N}{\partial x} = -2xy.$$

Therefore,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2xy - (-2xy)}{-x^2y} = -\frac{4}{x},$$

which is a function of x only. Hence $e^{\int -\frac{4}{x} dx} = e^{-4 \log x} = \frac{1}{x^4}$ is the integrating factor. Multiplying the equation throughout by $\frac{1}{x^4}$, we get

$$\left(\frac{y^2}{x^3} - \frac{1}{x^4} e^{x^{1/3}}\right) dx - \frac{y}{x^2} dy = 0,$$

which is exact. The required solution is, therefore,

$$\int_{y \text{ constant}} \left(\frac{y^2}{x^3} - \frac{1}{x^4} e^{x^{1/3}}\right) dx = C$$

or

$$-\frac{y^2}{2x^2} + \frac{1}{3} \int -\frac{3}{x^4} e^{x^{1/3}} dx = C$$

or

$$-\frac{y^2}{2x^2} + \frac{1}{3} \int e^t dt = C, \quad t = x^{\frac{1}{3}}.$$

or

$$-\frac{y^2}{2x^2} + \frac{1}{3} e^t = C$$

or

$$-\frac{3y^2}{2x^2} + 2e^{x^{1/3}} = k \text{ (constant).}$$

EXAMPLE 2.50

Solve

$$(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0.$$

Solution. Comparing with $M dx + N dy = 0$, we get

$$\begin{aligned} M &= xy^3 + y, & N &= 2x^2y^2 + 2x + 2y^4, \\ \frac{\partial M}{\partial y} &= 3xy^2 + 1, & \frac{\partial N}{\partial x} &= 4xy^2 + 2. \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the equation is not exact. It is also not homogeneous. It is also not of the form $f(x)y$ $dx + \phi(x)y$ $dy = 0$. We note that

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{xy^2 + 1}{y(xy^2 + 1)} = \frac{1}{y} \text{ (function of } y \text{ alone).}$$

Hence, the integrating factor is $e^{\int \frac{1}{y} dy} = e^{\log y} = y$. Multiplying the given equation throughout by y , we get

$$\begin{aligned} y(xy^3 + y)dx + 2y(x^2y^2 + x + y^4)dy &= 0 \\ (xy^4 + y^2)dx + (2x^2y^3 + 2xy + y^5)dy &= 0, \end{aligned}$$

which is exact. Therefore, its solution is

$$\int (xy^4 + y^2)dx + \int y^5 dy = C$$

y constant

or

$$\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{6} = C.$$

EXAMPLE 2.51

Solve

$$(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0.$$

Solution. The given differential equation is neither homogeneous nor is of the type $f(x)y$ $dx + \phi(x)y$ $dy = 0$. Comparing with $Mdx + Ndy = 0$, we get

$$M = y^4 + 2y, \quad N = xy^3 + 2y^4 - 4x.$$

Therefore,

$$\frac{\partial M}{\partial y} = 4y^3 + 2, \quad \frac{\partial N}{\partial x} = y^3 - 4,$$

which shows that the given equation is not exact. Further,

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-3y^3 - 6}{y^4 + 2y} = \frac{-3(y^2 + 2)}{y(y^3 + 2)} = -\frac{3}{y},$$

which is a function of y alone. Therefore, the integrating factor is $e^{\int -\frac{3}{y} dy} = e^{-3 \log y} = y^{-3} = \frac{1}{y^3}$. Multiplying the given differential equation throughout by $\frac{1}{y^3}$, we get

$$\frac{1}{y^3}(y^4 + 2y)dx + \frac{1}{y^3}(xy^3 + 2y^4 - 4x)dy = 0$$

or

$$\left(y + \frac{2}{y^2}\right)dx + \left(x + 2y - \frac{4x}{y^3}\right)dy = 0,$$

which is exact. The solution of the given differential equation is, therefore,

$$\int \left(y + \frac{2}{y^2}\right) dx + \int 2y dy = 0$$

y constant

or

$$\left(y + \frac{2}{y^2}\right)x + y^2 = C.$$

EXAMPLE 2.52

Solve

$$y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0.$$

Solution. The given equation can be written as

$$xy(ydx + xdy) + x^2y^2(2ydx - xdy) = 0.$$

Comparing it with

$$x^a y^b (mydx + nxdy) + x^{a'} y^{b'} (m'y dx + n'x dy) = 0,$$

we note that $a = b = 1$, $a' = b' = 2$, $m = n = 1$ and $m' = 2$, $n' = -1$. Then the integrating factor is $x^h y^k$, where

$$\frac{a + h + 1}{m} = \frac{b + k + 1}{n}, \quad \frac{a' + h + 1}{m'} = \frac{b' + k + 1}{n'},$$

that is,

$$\frac{h + 2}{1} = \frac{k + 2}{1}, \quad \frac{3 + h}{2} = \frac{3 + k}{-1}$$

or

$$h + 2 = k + 2, \quad -3 - k = 6 + 2k$$

or

$$h - k = 0, \quad h + 2k = -9.$$

Solving for h and k , we get $h = k = -3$. Hence, the integrating factor is $\frac{1}{x^3y^3}$. Multiplying throughout by $\frac{1}{x^3y^3}$, we get

$$\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0,$$

which is exact. Therefore, the solution is

$$\int \left(\frac{1}{x^2y} + \frac{2}{x}\right) dx + \int \left(-\frac{1}{y}\right) dy = C$$

or

$$\frac{1}{y} \left(-\frac{1}{x}\right) + 2 \log x - \log y = C$$

or

$$-\frac{1}{xy} + 2 \log x - \log y = C.$$

EXAMPLE 2.53

Solve

$$(y^2 + 2x^2y) dx + (2x^3 - xy) dy = 0.$$

Solution. The given equation can be written as

$$x^0y^1(y dx - x dy) + x^2y^0(2y dx + 2x dy) = 0.$$

Comparing this with

$$x^a y^b (my dx + nxdy) + x^a y^b (m'y dx + n'xdy) = 0,$$

we get

$$a = 0, \quad b = 1, \quad a' = 2, \quad b' = 0,$$

$$m = 1, \quad n = -1, \quad m' = 2, \quad n' = 2.$$

Then the integrating factor is $x^h y^k$, where

$$\frac{a+h+1}{m} = \frac{b+k+1}{n}, \quad \frac{a'+h+1}{m'} = \frac{b'+k+1}{n'},$$

or

$$\frac{h+1}{1} = \frac{2+k}{-1}, \quad \frac{3+h}{2} = \frac{k+2}{2}$$

or

$$h + k = -3 \quad \text{and} \quad 2h - 2k = -4.$$

Solving for h and k , we get $h = -\frac{5}{2}$, $k = -\frac{1}{2}$. Hence, $\frac{1}{x^{5/2}y^{1/2}}$ is the integrating factor. Multiplying the given differential equation by $\frac{1}{x^{5/2}y^{1/2}}$, we get

$$(x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2})dx + (2x^{1/2}y^{1/2} + 2x^{-3/2}y^{1/2})dy = 0,$$

which is exact. Therefore, the required solution is

$$\int (x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2})dx + \int 0 dy = 0$$

or

$$-\frac{2}{3}y^{3/2}x^{-3/2} + 4y^{1/2}x^{1/2} = C$$

or

$$-y^{3/2}x^{-3/2} + 6x^{1/2}y^{1/2} = C$$

or

$$6\sqrt{xy} - \left(\frac{y}{x}\right)^{3/2} = C.$$

EXAMPLE 2.54

Solve

$$(3xy - 2ay^2) + (x^2 - 2axy) dy = 0.$$

Solution. The given equation can be written as

$$xy^0(3ydx + xdy) + yx^0(-2aydx - 2axy) = 0.$$

Comparing this with

$$x^a y^b (my dx + nxdy) + x^a y^b (m'y dx + n'xdy) = 0,$$

we get

$$a = 1, \quad b = 0, \quad a' = 0, \quad b' = 1,$$

$$m = 3, \quad n = 1, \quad m' = -2a, \quad n' = -2a.$$

Then the integrating factor is $x^h y^k$, where

$$\frac{a+h+1}{m} = \frac{b+k+1}{m}, \quad \frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$$

or

$$\frac{2+h}{3} = \frac{k+1}{1}, \quad \frac{h+1}{-2a} = \frac{k+2}{-2a}$$

or

$$h - 3k = 1, \quad h - k = 1.$$

Thus $h = 1$, $k = 0$. Therefore, the integrating factor is x . Multiplying the given differential equation

throughout by x , we get

$$(3x^2y - 2axy^2)dx + (x^3 - 2ax^2y)dy = 0.$$

Let

$$\text{Then } M = 3x^2y - 2axy^2, \quad N = x^3 - 2ax^2y.$$

$$\frac{\partial M}{\partial y} = 3x^2 - 4axy, \quad \frac{\partial N}{\partial x} = 3x^2 - 4axy.$$

Hence $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and so the transformed equation is exact.

The required solution is

$$\int_{y \text{ constant}} (3x^2y - 2axy^2)dx + \int 0 \, dy = 0$$

or

$$yx^3 - ax^2y^2 = C.$$

EXAMPLE 2.55

Solve

$$(2x^2y^2 + y)dx + (-x^3y + 3x)dy = 0.$$

Solution. Comparing the given differential equation with $Mdx + Ndy = 0$, we get

$$M = 2x^2y^2 + y, \quad N = -x^3y + 3x,$$

$$\frac{\partial M}{\partial y} = 4x^2y + 1, \quad \frac{\partial N}{\partial x} = 3 - 3x^2y.$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the given equation is not exact. However, the given equation can be written in the form

$$x^2y(2ydx - xdy) + x^0y^0(ydx + 3xdy) = 0.$$

Comparing it with

$$x^a y^b (my dx + nx dy) + x^{a'} y^{b'} (m'y dx + n'x dy) = 0,$$

we get

$$a = 2, \quad b = 1, \quad a' = 0, \quad b' = 0,$$

$$m = 2, \quad n = -1, \quad m' = 1, \quad n' = 3.$$

The integrating factor is $x^h y^k$, where

$$\frac{a+h+1}{m} = \frac{b+k+1}{n}, \quad \frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$$

or

$$h + 2k = -7, \quad 3h - k = -2.$$

Solving these equations for h and k , we get $h = -\frac{11}{7}$ and $k = -\frac{19}{7}$. Thus the integrating factor is $x^{-11/7}$

$y^{-19/7}$. Multiplying the given equation throughout by $x^{-11/7} y^{-19/7}$, we get

$$(2x^{3/7}y^{-5/7} + x^{-11/7}y^{-12/7})dx - (x^{10/7}y^{-12/7} - 3x^{-4/7}y^{-10/7})dy = 0.$$

This transformed equation is exact and its solution is

$$\int_{y \text{ constant}} (2x^{3/7}y^{-5/7} + x^{-11/7}y^{-12/7}) dx = C$$

or

$$\frac{7}{5} x^{10/7} y^{-5/7} - \frac{7}{4} x^{-4/7} y^{-12/7} = C.$$

2.13 APPLICATIONS OF FIRST ORDER AND FIRST DEGREE EQUATIONS

The aim of this section is to form differential equations for physical problems like flow of current in electric circuits, Newton law of cooling, heat flow and orthogonal trajectories, and to find their solutions.

(A) Problems Related to Electric Circuits

Consider the RCL circuit shown in the Figure 2.3 and consisting of resistance, capacitor, and inductor connected to a battery.

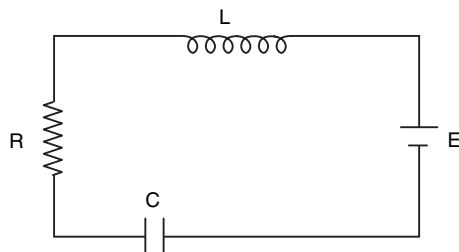


Figure 2.3

We know that the resistance is measured in ohms, capacitance in farads, and inductance in henrys. Let I denote the current flowing through the circuit and Q denote the charge. Since the current is rate of flow of charge, we have $I = \frac{dQ}{dt}$. Also, by Ohm's law, $\frac{V}{I} = R$ (resistance). Therefore, the voltage drop across a resistor R is RI . The voltage drop across the inductor L is $L \frac{dI}{dt}$ and the voltage drop across a capacitor is $\frac{Q}{C}$. If E is the voltage (e.m.f.) of the battery, then by Kirchhoff's law, we have

$$L \frac{dI}{dt} + RI + \frac{Q}{C} = E(t), \quad (16)$$

where L , C , and R are constants. Since $I = \frac{dQ}{dt}$, we have $Q = \int_0^t I(u) du$ and so (16) reduces to

$$L \frac{dI}{dt} + RI + \frac{1}{C} \int_0^t I(u) du = E(t).$$

The forcing function (input function), $E(t)$, is supplied by the battery (voltage source). The system described by the above differential equation is known as *harmonic oscillator*.

The equation (16) can be written as

$$\frac{dI}{dt} + \frac{R}{L}I + \frac{Q}{LC} = \frac{E}{L}(t) \quad (17)$$

EXAMPLE 2.56

Given that $I = 0$ at $t = 0$, find an expression for the current in the LR circuit shown in the Figure 2.4.

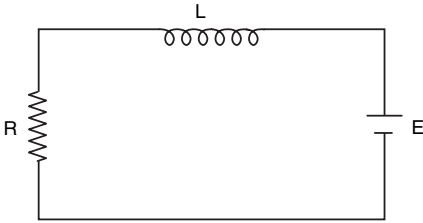


Figure 2.4

Solution. By Kirchhoff's law, we have

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L}, \quad (18)$$

which is Leibnitz's linear equation. Its integrating factor is $e^{\int \frac{R}{L} dt} = e^{Rt/L}$. Hence the solution of (18) is

$$\begin{aligned} I e^{Rt/L} &= \int \frac{E}{L} e^{Rt/L} dt + C = \frac{E}{L} \int e^{Rt/L} dt + C \\ &= \frac{E}{L} \frac{e^{Rt/L}}{R/L} + C = \frac{E}{R} e^{Rt/L} + C. \end{aligned}$$

Thus

$$I = \frac{E}{R} + C e^{-Rt/L}. \quad (19)$$

But $I(0) = 0$, therefore, (19) yields $0 = \frac{E}{R} + C$ and so $C = -\frac{E}{R}$. Hence

$$I = \frac{E}{R} - \frac{E}{R} e^{-Rt/L} = \frac{E}{R} (1 - e^{-Rt/L}).$$

Clearly I increases with time t and attains its maximum value, E/R .

EXAMPLE 2.57

Find the time t when the current reaches half of its theoretical maximum in the circuit of Example 2.56.

Solution. From Example 2.56, we have

$$I = \frac{E}{R} (1 - e^{-Rt/L}).$$

The maximum current is $\frac{E}{R}$. By the requirement of the problem, we must have

$$\frac{1}{2} \frac{E}{R} = \frac{E}{R} (1 - e^{-Rt/L})$$

$$\frac{1}{2} = e^{-Rt/L}$$

or

$$-\frac{Rt}{L} = \log \frac{1}{2} = -\log 2$$

or

$$t = \frac{L}{R} \log 2.$$

EXAMPLE 2.58

In an LR circuit, an e.m.f. of $10 \sin t$ volts is applied. If $I(0) = 0$, find the current in the circuit.

Solution. In an LR circuit the current is governed by the differential equation

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L}.$$

We are given that $E = 10 \sin t$. Therefore,

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{10}{L} \sin t.$$

This is Leibnitz's linear equation with integrating factor as $e^{\int \frac{R}{L} dt} = e^{Rt/L}$. Therefore, its solution is

$$\begin{aligned} I e^{Rt/L} &= \int \frac{10}{L} \sin t e^{Rt/L} dt + C \\ &= \frac{10}{L} \int e^{Rt/L} \sin t dt + C \end{aligned} \quad (20)$$

But we know (using integration by parts) that

$$\begin{aligned} \int e^{at} \sin bt dt &= \frac{e^{at}}{a^2 + b^2} (a \sin bt - b \cos bt) \\ &= \frac{e^{at}}{a^2 + b^2} \sin \left\{ bt - \tan^{-1} \frac{b}{a} \right\}. \end{aligned}$$

Therefore (20) reduces to

$$\begin{aligned} I e^{Rt/L} &= \frac{10}{L} \left[\frac{e^{Rt/L}}{(R^2/L^2) + 1} \left(\frac{R}{L} \sin t - \cos t \right) \right] + C \\ &= \frac{10L^2}{L^2} \left[\frac{e^{Rt/L}}{R^2 + L^2} (R \sin t - L \cos t) \right] + C \\ &= \frac{10}{R^2 + L^2} (R \sin t - L \cos t) + C. \end{aligned}$$

Hence

$$I = \frac{10}{R^2 + L^2} (R \sin t - L \cos t) + C e^{-Rt/L}.$$

Using the initial condition $I(0) = 0$, we get $C = \frac{10L}{R^2 + L^2}$. Hence

$$I = \frac{10}{R^2 + L^2} (R \sin t - L \cos t + L e^{-Rt/L}).$$

EXAMPLE 2.59

If voltage of a battery in an LR circuit is $E_0 \sin t$, find the current I in the circuit under the initial condition $I(0) = 0$.

Solution. Proceeding as in Example 2.58, we get

$$I = \frac{E_0}{R^2 + L^2} e^{-Rt/L} + \frac{E_0 L}{R^2 + L^2} \cos t + \frac{E_0 R}{R^2 + L^2} \sin t.$$

EXAMPLE 2.60

Find the current in the following electric circuit containing condenser C , resistance R , and a battery of e.m.f. $E_0 \sin \omega t$ with the initial condition $I(0) = 0$.

Solution. The RC circuit of the problem is shown in Figure 2.5.

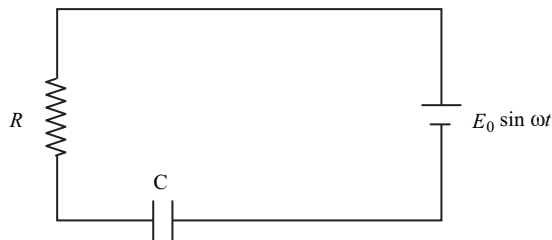


Figure 2.5

Using Kirchhoff's law, we have

$$RI + \frac{1}{C} \int_0^t I(u) du = E_0 \sin \omega t.$$

Differentiating both sides with respect to t , we have

$$R \frac{dI}{dt} + \frac{I}{C} = \omega E_0 \sin \omega t$$

or

$$\frac{dI}{dt} + \frac{I}{RC} = \frac{\omega E_0}{R} \cos \omega t,$$

which is Leibnitz's linear equation. The integrating factor is $e^{\int \frac{1}{RC} dt} = e^{\frac{t}{RC}}$. Therefore, the solution of the above first order equation is

$$\begin{aligned} I e^{\frac{t}{RC}} &= \int \frac{\omega E_0}{R} \cos \omega t \cdot e^{\frac{t}{RC}} dt + C \\ &= \frac{\omega E_0}{R} \int \cos \omega t \cdot e^{\frac{t}{RC}} dt + C \end{aligned}$$

Using $\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$, we have

$$\begin{aligned} I e^{\frac{t}{RC}} &= \frac{\omega E_0}{R} \left[\frac{e^{\frac{t}{RC}}}{\left(\frac{1}{RC}\right)^2 + \omega^2} \left(\frac{1}{RC} \cos \omega t + \omega \sin \omega t \right) \right] + C \\ &= \frac{\omega R^2 C^2 E_0}{R^2 C} \left[\frac{e^{\frac{t}{RC}}}{1 + R^2 \omega^2 C^2} (\cos \omega t + RC \omega \sin \omega t) \right] + C \\ &= \frac{\omega C E_0 e^{\frac{t}{RC}}}{1 + R^2 \omega^2 C^2} (\cos \omega t + RC \omega \sin \omega t) + C. \end{aligned}$$

But $I(0) = 0$ implies $C = -\frac{\omega C E_0}{1 + R^2 \omega^2 C^2}$. Hence

$$I = \frac{\omega C E_0}{1 + R^2 \omega^2 C^2} (\cos \omega t + RC \omega \sin \omega t - e^{-t/RC}).$$

EXAMPLE 2.61

A voltage $E e^{-at}$ is applied at $t = 0$ to a LR circuit. Find the current at any time t .

Solution. The differential equation governing the LR circuit is

$$\frac{dI}{dt} + \frac{R}{L} I = \frac{e.m.f.}{L} = \frac{E e^{-at}}{L}.$$

As in Example 2.58, the integrating factor is $e^{\frac{Rt}{L}}$. Therefore, the solution of the above equation is

$$\begin{aligned} I e^{\frac{Rt}{L}} &= \int \frac{E e^{-at}}{L} e^{\frac{Rt}{L}} dt + C = \frac{E}{L} \int e^{\frac{Rt}{L} - at} dt + C \\ &= \frac{E e^{\frac{Rt}{L} - at}}{L \frac{R}{L} - a} + C = \frac{E}{R - aL} [e^{\frac{Rt}{L} - at}] + C \end{aligned}$$

and so

$$I = \frac{E}{R - aL} e^{-at} + C e^{-\frac{Rt}{L}}.$$

Using the initial condition $I(0) = 0$, we get $C = -\frac{R}{R - aL}$. Hence

$$I = \frac{E}{R - aL} [e^{-at} - e^{-\frac{Rt}{L}}].$$

EXAMPLE 2.62

An RC circuit has an e.m.f. given in volt by $400 \cos 2t$, a resistance of 100 ohms, and a capacitance of 10^{-2} farad. Initially there is no charge on the capacitor. Find the current in the circuit at any time t .

Solution. The equation governing the circuit is

$$RI + \frac{Q}{C} = E.$$

We are given that $R = 100$ ohms, $C = 10^{-2}$ farad and $E = 400 \cos 2t$. Thus, we have

$$I + Q = 4 \cos 2t$$

or

$$\frac{dQ}{dt} + Q = 4 \cos 2t, \text{ since } I = \frac{dQ}{dt}.$$

The integrating factor is $e^{\int 1 dt} = e^t$. Therefore, the solution is

$$\begin{aligned} Q \cdot e^t &= 4 \int \cos 2t e^t dt \\ &= 4 \left[\frac{e^t}{5} [\cos 2t + 2 \sin 2t] \right] + C \\ &= \frac{4}{5} e^t \cos 2t + \frac{8}{5} e^t \sin 2t + C. \end{aligned}$$

Thus

$$Q = \frac{4}{5} \cos 2t + \frac{8}{5} \sin 2t + C e^{-t}.$$

Using the initial condition $Q(0) = 0$, we get $C = -\frac{4}{5}$. Therefore,

$$Q = \frac{4}{5} \cos 2t + \frac{8}{5} \sin 2t - \frac{4}{5} e^{-t}.$$

Hence

$$I = \frac{dQ}{dt} = -\frac{8}{5} \sin 2t + \frac{16}{5} \cos 2t + \frac{4}{5} e^{-t}.$$

(B) Problems Related to Newton's Law of Cooling

Newton's Law of Cooling states that *the time rate of change of the temperature of a body is proportional to the temperature difference between the body and its surrounding medium*.

Let T be the temperature of the body at any time t and T_0 be the temperature of the surrounding at that particular time. Then, according to Newton's Law of Cooling,

$$\frac{dT}{dt} \propto (T - T_0)$$

and so

$$\frac{dT}{dt} = -k(T - T_0), \quad (21)$$

the negative sign with the constant of proportionality is required to make $\frac{dT}{dt}$ negative in cooling process when T is greater than T_0 , and positive in a heating process when T is less than T_0 .

The equation (21) is first order differential equation and can be solved for T .

EXAMPLE 2.63

A body at a temperature of 50°F is placed outdoors, where the temperature is 100°F . If after 5 minutes, the temperature of the body is 60°F , find the time required by the body to reach a temperature of 75°F .

Solution. By Newton's Law of Cooling, we have

$$\frac{dT}{dt} = -k(T - T_0)$$

or

$$\frac{dT}{dt} + k T = k T_0.$$

But $T_0 = 100$ F. Therefore,

$$\frac{dT}{dt} + kT = 100k,$$

which is linear. The integrating factor is $e^{\int k dt} = e^{kt}$. Hence the solution is

$$T \cdot e^{kt} = 100 \int k e^{kt} dt + C = 100e^{kt} + \int C$$

or

$$T = C e^{-kt} + 100.$$

When $t = 0$, $T = 50$, therefore, $C = -50$. Hence

$$T = -50e^{-kt} + 100.$$

Now it is given that $T = 60$ at $t = 5$. Hence

$$60 = -50e^{-5k} + 100 \text{ or } e^{-5k} = \frac{4}{5}.$$

Taking log, we get

$$k = -\frac{1}{5} \log \frac{4}{5} = -\frac{1}{5} (-0.223) = 0.045.$$

Hence $T = -50e^{0.045t} + 100$. When $T = 75$, we get $e^{-0.045t} = \frac{1}{2}$, which yields $-0.045t = \log \frac{1}{2}$ and so $t = 2.4$ minutes.

EXAMPLE 2.64

A body originally at 80°C cools down to 60°C in 20 minutes, the temperature of the air being 40°C . Find the temperature of the body after 40 minutes from the original.

Solution. By Newton's Law of Cooling, we have

$$\frac{dT}{dt} = -k(T - T_0)$$

and so variable separation gives

$$\frac{dT}{T - T_0} = -kdt.$$

Integrating, we have

$$\log(T - T_0) = -kt + \log C$$

or

$$T - T_0 = C e^{-kt} \text{ or } T - 40 = C e^{-kt}.$$

But when $t = 0$, $T = 80^\circ\text{C}$. Therefore, $C = 80 - 40 = 40$ and we have

$$T - 40 = 40e^{-kt}.$$

Now when $t = 20$, $T = 60^\circ$. Therefore

$$20 = 40 e^{-20k} \text{ or } e^{-20k} = \frac{1}{2}$$

or

$$-20k = \log \frac{1}{2}, \text{ which yields } k = \frac{1}{20} \log 2.$$

Hence

$$T - 40 = 40e^{-(\frac{1}{20} \log 2)t}.$$

When $t = 40$, we have

$$T - 40 = 40e^{-2 \log 2}$$

or

$$T = 40 + 40 e^{\log \frac{1}{4}} = 40 + \frac{40}{4} = 50^\circ\text{C}.$$

(C) Problems Relating to Heat Flow

The fundamental principles of heat conduction are:

- (i) Heat always flow from a higher temperature to a lower temperature.
- (ii) The quantity of heat in a body is proportional to its mass and temperature ($Q = mst$), where m is the mass, s is the specific heat, and t is the time.
- (iii) The rate of heat flow across an area is proportional to the area and to the rate of change of temperature with respect to its distance normal to the area.

Let Q be the quantity of heat flow per second across a slab of area A and thickness δx and whose faces are kept at temperature T and $T + \delta T$. Then, by the above principles

$$Q \propto A \frac{dT}{dx}.$$

or

$$Q = -k A \frac{dT}{dx} \quad (22)$$

where k is a constant, called the *coefficient of thermal conductivity* and depends upon the material of the body. Negative sign has been taken since T decreases as x increases. The relation (22) is called the *Fourier's law of conductivity*.

EXAMPLE 2.65

The inner and outer surfaces of a spherical shell are maintained at temperature T_0 and T_1 , respectively. If the inner and outer radii of the shell are r_0 and r_1 , respectively and k is the thermal conductivity, find the amount of heat lost from the shell per unit time. Also find the temperature distribution through the shell.

Solution. We have

$$Q = -k (4\pi x^2) \frac{dT}{dx},$$

where x is the radius. Thus

$$dT = -\frac{Q}{4\pi k} \cdot \frac{dx}{x^2}.$$

Integrating, we get

$$T = -\frac{Q}{4\pi k} \left(-\frac{1}{x}\right) + C = \frac{Q}{4\pi x k} + C \quad (23)$$

Now $T = T_0$ when $x = r_0$ and $T = T_1$ when $x = r_1$. Therefore, we have

$$T_0 = \frac{Q}{4\pi r_0 k} + C \text{ and } T_1 = \frac{Q}{4\pi r_1 k} + C.$$

Subtracting, we get

$$T_0 - T_1 = \frac{Q}{4\pi k} \left(\frac{1}{r_0} - \frac{1}{r_1}\right) = \frac{Q}{4\pi k} \left(\frac{r_1 - r_0}{r_0 r_1}\right)$$

or

$$Q = \frac{4\pi k r_0 r_1 (T_0 - T_1)}{r_1 - r_0}.$$

When $x = r_0$, $T = T_0$, then (23) gives

$$\begin{aligned} T_0 &= \frac{Q}{4\pi r_0 k} + C \\ &= \frac{4\pi k r_0 r_1 (T_0 - T_1)}{4\pi r_0 k (r_1 - r_0)} + C \\ &= \frac{r_1 (T_0 - T_1)}{r_1 - r_0} + C. \end{aligned}$$

Thus

$$\begin{aligned} C &= T_0 - \frac{r_1 (T_0 - T_1)}{r_1 - r_0} \\ &= \frac{T_0 (r_1 - r_0) - r_1 (T_0 - T_1)}{r_1 - r_0} = \frac{r_1 T_1 - r_0 T_0}{r_1 - r_0} \end{aligned}$$

Hence (23) transforms to

$$\begin{aligned} T &= \frac{4\pi k r_0 r_1 (T_0 - T_1)}{4\pi x k (r_1 - r_0)} + \frac{r_1 T_1 - r_0 T_0}{r_1 - r_0} \\ &= \frac{1}{r_1 - r_0} \left[\frac{(T_0 - T_1) r_0 r_1}{x} + r_1 T_1 - r_0 T_0 \right]. \end{aligned}$$

EXAMPLE 2.66

A spherical shell of inner and outer radii 10 cm and 15 cm, respectively, contains steam at 150°C . If the temperature of the outer surface of the shell is 40°C and thermal conductivity $k = 0.0025$, find the temperature half-way through the thickness of the shell under steady state conditions.

Solution. With the notation of Example 2.65, we have

$$\begin{aligned} r_0 &= 10 \text{ cm}, r_1 = 15 \text{ cm}, x = 12.5 \text{ cm}, \\ T_0 &= 150^\circ\text{C}, T_1 = 40^\circ\text{C}. \end{aligned}$$

Hence

$$\begin{aligned} T &= \frac{1}{r_1 - r_0} \left[\frac{(T_0 - T_1) r_0 r_1}{x} + r_1 T_1 - r_0 T_0 \right] \\ &= \frac{1}{5} \left[\frac{(150 - 40) 150}{12.5} + 600 - 1500 \right] \\ &= \frac{1}{5} \left[\frac{16500}{12.5} - 900 \right] = \frac{1}{5} [1320 - 900] = 84^\circ\text{C}. \end{aligned}$$

EXAMPLE 2.67

A long hollow pipe has an inner radius of r_0 cm and outer radius of r_1 cm. The inner surface is kept at a temperature T_0 and the outer surface at the temperature T_1 . If thermal conductivity is k , find the heat lost per second of 1 cm length of the pipe. Also find the temperature distribution through the thickness of the pipe.

Solution. Let Q cal/sec be the constant quantity of heat flowing out radially through the surface of the pipe having radius x cm and length 1 cm. Then the

area of the lateral surface is $2\pi x$. Therefore, by Fourier law,

$$Q = -kA \frac{dT}{dx} = -k(2\pi x) \frac{dT}{dx}$$

and so

$$dT = -\frac{Q}{2\pi k} \cdot \frac{dx}{x}.$$

Integrating, we get

$$T = -\frac{Q}{2\pi k} \log x + C. \quad (24)$$

When $x = r_0$, $T = T_0$, and so

$$T_0 = -\frac{Q}{2\pi k} \log r_0 + C \quad (25)$$

When $x = r_1$, $T = T_1$ and we have

$$T_1 = -\frac{Q}{2\pi k} \log r_1 + C. \quad (26)$$

Subtracting (26) from (25), we have

$$T_0 - T_1 = \frac{Q}{2\pi k} [\log r_0 - \log r_1] = \frac{Q}{2\pi k} \log \frac{r_1}{r_0}.$$

Thus

$$\frac{Q}{\log \frac{r_1}{r_0}} = 2\pi k(T_0 - T_1) \quad (27)$$

which gives the heat lost per second in 1 cm of the pipe. Further subtracting (25) from (24) and using (27), we get

$$\begin{aligned} T - T_0 &= -\frac{Q}{2\pi k} [\log x - \log r_0] = -\frac{Q}{2\pi k} \log \frac{x}{r_0} \\ &= -\frac{2\pi k(T_0 - T_1)}{2\pi k \log \frac{r_1}{r_0}} \log \frac{x}{r_0} \\ &= \frac{(T_1 - T_0)}{\log \frac{r_1}{r_0}} \log \frac{x}{r_0}. \end{aligned}$$

Hence,

$$T = T_0 + \frac{(T_1 - T_0)}{\log \frac{r_1}{r_0}} \log \frac{x}{r_0} \quad (28)$$

which gives the required temperature distribution through the thickness of the pipe.

EXAMPLE 2.68

A pipe 20 cm in diameter contains steam at 150°C and is protected with a covering 5 cm thick. If thermal conductivity is 0.0025 and the temperature of the outer surface of the covering is 40°C , find the

temperature half-way through the covering under steady-state conditions.

Solution. Using the notation of Example 2.67, we have

$$\begin{aligned} r_0 &= 10\text{ cm}, \quad r_1 = 15\text{ cm}, \quad x = 12.5, \\ T_0 &= 150^\circ\text{C}, \quad T_1 = 40^\circ\text{C}, \quad k = 0.0025. \end{aligned}$$

Hence using (28), we have

$$\begin{aligned} T &= 150 + \frac{40 - 150}{\log \frac{15}{10}} \log \frac{12.5}{10} \\ &= 150 - 110 \log \frac{1.25}{1.5} = 89.5^\circ\text{C}. \end{aligned}$$

EXAMPLE 2.69

A long hollow pipe has an inner diameter of 10 cm and outer diameter of 20 cm. The inner surface is kept at 200°C and the outer surface at 50°C . The thermal conductivity is 0.12. How much heat will be lost per second from a portion of 1 cm of the pipe and what is the temperature at a distance of 7.5 cm from the centre of the pipe?

Solution. From Example 2.67, we have

$$\begin{aligned} Q &= \frac{2\pi k(T_0 - T_1)}{\log \frac{r_1}{r_0}} \\ &= \frac{2\pi k(200 - 50)}{\log \frac{10}{5}} = \frac{300\pi k}{\log 2} \\ &= \frac{300\pi(0.12)}{\log 2} = 163\text{ cal/sec}. \end{aligned}$$

Also

$$\begin{aligned} T &= T_0 + \frac{(T_1 - T_0)}{\log \frac{r_1}{r_0}} \log \frac{x}{r_0} \\ &= 200 + \frac{(50 - 200)}{\log 2} \log \frac{7.5}{5} \\ &= 200 - 150 \log \frac{1.5}{2} = 200 - 150(.58) = 113^\circ\text{C}. \end{aligned}$$

(D) Rate Problems

In some problems, the rate at which a quantity changes is a known function of the amount present and/or the time and it is desired to find the quantity itself. Radioactive nuclei decay, population growth, and chemical reactions are some of the phenomenon of this kind.

Let x be the amount of radioactive nuclei present after t years. Then $\frac{dx}{dt}$ represents the rate of decay. Since the nuclei decay at a rate proportional to the amount present, we have

$$\frac{dx}{dt} = kx, \quad (29)$$

where k is constant of proportionality.

The law of chemical reaction states that the *rate of change of chemical reaction is proportional to the amount of substance present at that instant*. Thus, the differential equation (29) governs the chemical reaction of first order.

Moreover, if the rate at which amount of a substance increases or decreases is found to be jointly proportional to two factors, each factor being a linear function of x , then the chemical reaction is said to be of *second order*. For example, if a solution contains two substances whose amounts at the beginning are a and b respectively and if equal amount x of each substance changes in time t , then the amounts of the substance left in the solution at time t are $(a - x)$ and $(b - x)$ and, therefore, we have

$$\frac{dx}{dt} = k(a - x)(b - x).$$

Taking the case of population growth, we assume that the population is a continuous and differentiable function of time. Let x be the number of individuals in a population at time t . Then *rate of change of population is proportional to the number of individuals in it at any time*. Thus equation (29) is valid for population growth also.

EXAMPLE 2.70

If 10% of 50 mg of a radioactive material decays in 2 hours, find the mass of the material left at any time t and the time at which the material has decayed to one-half of its initial mass.

Solution. Let x denote the amount of material present at time t . Then the equation governing the decay is

$$\frac{dx}{dt} = kx.$$

Variable separation gives

$$\frac{dx}{x} = kdt.$$

Integrating, we get

$$\log x = kt + \log C \quad \text{or} \quad x = Ce^{kt}.$$

At $t = 0$, $x = 50$. Therefore, $C = 50$ and so $x = 50e^{kt}$. At $t = 2$, 10% of the mass present is decayed. Thus 5 mg of the substance has been decayed and 45 mg still remains. Therefore, $45 = 50e^{2k}$. Thus

$$k = \frac{1}{2} \log \frac{45}{50} = -0.053.$$

Hence mass of the material left at any time t is

$$x = 50e^{-0.053t}.$$

Further, when half of the material is decayed, we have $x = 25$ mg and so

$$25 = 50e^{-0.053t}$$

or

$$-0.053t = \log \frac{1}{2}$$

or

$$t = 13 \text{ hours.}$$

EXAMPLE 2.71

A tank contains 1000 litres of fresh water. Salt water which contains 150 gm of salt per litre runs into it at the rate of 5 litres/min and well stirred mixture runs out of it at the same rate. When will the tank contain 5000 gm of salt?

Solution. Let x denote the amount of salt in the tank at time t . Then

$$\frac{dx}{dt} = \text{IN} - \text{OUT}.$$

The brine flows at the rate of 5 litres/min and each litre contains 150 gm of salt. Thus

$$\text{IN} = 5 \times 150 = 750 \text{ gm/min}$$

Since the rate of outflow equals the rate of inflow, the tank contains 1000 litres of mixture at any time t . This 1000 litres contains x gm of salt at time t and so the concentration of the salt at time t is $\frac{x}{1000}$ gm/litres. Since mixture flows out at the rate of 5 litres/min, we have

$$\text{OUT} = \frac{x}{1000} \times 5 = \frac{x}{200} \text{ gm/litres.}$$

Thus the differential equation for x becomes

$$\frac{dx}{dt} = 750 - \frac{x}{200}. \quad (30)$$

Since initially there was no salt in the tank, we have the initial condition $x(0) = 0$. The equation (30) is linear and separable. We have in fact

$$\frac{dx}{150000 - x} = \frac{dt}{200}.$$

Integrating, we get

$$-\log(150000 - x) = \frac{t}{200} + C. \quad (31)$$

Using the initial condition $x(0) = 0$, we have

$$C = -\log 150000.$$

Hence (31) yields

$$\frac{t}{200} = \log 150000 - \log(150000 - x)$$

or

$$t = 200 \log \frac{150000}{150000 - x}.$$

If $x = 5000$ gm, then

$$t = 200 \log \frac{150000}{145000} = 200 \log \frac{30}{29} = 6.77 \text{ min.}$$

EXAMPLE 2.72

If the population of a city gets doubled in 2 years and after 3 years the population is 15,000, find the initial population of the city.

Solution. Let x denote the population at any time t and let x_0 be the initial population of the city. Then

$$\frac{dx}{dt} = kx,$$

which has the solution as

$$x = C e^{kt}.$$

At $t = 0$, $x = x_0$. Hence $C = x_0$. Thus

$$x = x_0 e^{kt} \quad (32)$$

But at $t = 2$, $x = 2x_0$. Therefore

$$2x_0 = x_0 e^{2k} \text{ or } e^{2k} = 2$$

or

$$k = \frac{1}{2} \log 2 = 0.347$$

Hence (32) reduces to

$$x = x_0 e^{0.347t}.$$

At $t = 3$, $x = 15,000$ and so

$$15,000 = x_0 e^{(0.347)(3)} = x_0(2.832)$$

Hence

$$x_0 = \frac{15000}{2.832} = 5297.$$

(E) Falling Body Problems

Consider a body of mass m falling under the influence of gravity g and an air resistance, which is proportional to the velocity of the falling body. Newton's second law of motion states that the *net force acting on a body is equal to the time rate of change of the momentum of the body*.

Thus

$$F = \frac{d}{dt}(mv).$$

If m is assumed to be constant, then

$$F = m \frac{dv}{dt}, \quad (33)$$

where F is the net force on the body and v is the velocity of the body at time t .

The falling body is under the action of two forces: (i) Force due to gravity which is given by the weight mg of the body (ii) the force due to resistance of air and that is $-kv$, where $k \geq 0$ is a constant of proportionality. Thus (33) yields

$$mg - kv = m \frac{dv}{dt}$$

or

$$\frac{dv}{dt} + \frac{k}{m} v = g, \quad (34)$$

which is *equation of motion for the falling body*.

If air resistance is negligible, then $k = 0$ and we have

$$\frac{dv}{dt} = g. \quad (35)$$

The differential equation (35) is separable and we have

$$dv = g dt.$$

Integrating, we get

$$v = gt + C.$$

But when $t = 0$, $v = 0$ and so $C = 0$. Hence

$$v = gt. \quad (36)$$

Also velocity is time rate of change of displacement x and so

$$\frac{dx}{dt} = gt \text{ or } dx = gt \, dt.$$

Integrating, we get

$$x = \frac{1}{2}gt^2 + k \text{ (constant).}$$

But at $t = 0$, the displacement is 0. Therefore $k = 0$.

Hence

$$x = \frac{1}{2}gt^2. \quad (37)$$

EXAMPLE 2.73

A body of mass 16 kg is dropped from a height of 625 ft. Assuming that there is no air resistance, find the time required by the body to reach the ground.

Solution. By (37), we have (with $g = 32 \text{ ft/sec}^2$)

$$x = \frac{1}{2}gt^2 = 16t^2,$$

therefore,

$$t^2 = \sqrt{\frac{x}{16}} = \sqrt{\frac{625}{16}} = \frac{25}{4} = 6.25 \text{ sec.}$$

(F) Orthogonal Trajectories

Recall that a curve which cuts every member of a given family of curves according to some definite law is called a *trajectory* of the family.

A curve which cuts every member of a given family of curves at right angles is called *orthogonal trajectories*. Further, two families of curves are said to be orthogonal if every member of either family cuts each member of the other family at right angles.

Consider a one-parameter family of curves in the xy -plane defined by

$$f(x, y, c) = 0, \quad (38)$$

where c denotes the parameter. Differentiating (38), with respect to x and eliminating c between (38) and the resulting equation, we get the differential equation of the family in question. Let the differential equation be

$$F\left(x, y, \frac{dy}{dx}\right) = 0. \quad (39)$$

To obtain the equation of the orthogonal trajectory, we replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ and get the differential equation of orthogonal trajectory as $F\left(x, y, -\frac{dx}{dy}\right)$. Solution of this differential equation will yield the equation of the orthogonal trajectory.

In case of polar curves

$$f(r, \theta, c) = 0. \quad (40)$$

Differentiating (40) and eliminating c between (40) and the resulting equations, we get the differential equation of the family represented by (40). Let the differential equation be

$$F\left(r, \theta, \frac{dr}{d\theta}\right) = 0. \quad (41)$$

Replacing $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$ in (41), we get the differential equation of the orthogonal trajectory as

$$F\left(r, \theta - r^2 \frac{d\theta}{dr}\right) = 0. \quad (42)$$

Solution of (42) will then yield the equation of the required orthogonal trajectory.

EXAMPLE 2.74

Find the orthogonal trajectories of the family of curves $x^2 + y^2 = cx$.

Solution. We have

$$x^2 + y^2 - cx = 0. \quad (43)$$

Differentiating, we get

$$2x + 2y \frac{dy}{dx} = c. \quad (44)$$

Eliminating c between (43) and (44) yields

$$2x + 2y \frac{dy}{dx} = \frac{x^2 + y^2}{x}$$

or

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}. \quad (45)$$

The equation (45) is the differential equation of the family represented by (43). Therefore, the differential equation of the orthogonal trajectory is

$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}. \quad (46)$$

This is an homogeneous equation. Substituting $y = vx$, and separating variables, we get

$$\frac{dx}{x} + \left(-\frac{1}{v} + \frac{2v}{v^2 + 1}\right) dv = 0.$$

Integrating, we get

$$\log x - \log v + \log(v^2 + 1) = C$$

or

$$x(v^2 + 1) = kv, \quad (C = \log k).$$

Substituting $v = \frac{y}{x}$, we get

$$x^2 + y^2 = ky.$$

EXAMPLE 2.75

Find the orthogonal trajectory of the family of the curves $xy = C$.

Solution. The equation of the given family of curves is

$$xy = C. \quad (47)$$

Differentiating, we get

$$x \frac{dy}{dx} + y = 0$$

or

$$\frac{dy}{dx} = -\frac{y}{x}. \quad (48)$$

Therefore, the differential equation of the family of orthogonal trajectory is

$$\frac{dy}{dx} = \frac{x}{y} \quad (49)$$

or

$$x dx - y dy = 0.$$

Integrating, we get

$$x^2 - y^2 = k,$$

which is the equation of orthogonal trajectories called *equipotential lines* (shown in Figure 2.6).

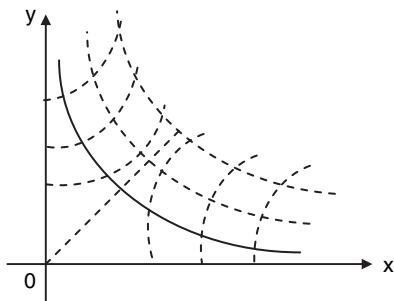


Figure 2.6

EXAMPLE 2.76

Find the orthogonal trajectories of the family of curves $y = ax^2$.

Solution. The given family represented by the equation

$$y = ax^2 \quad (50)$$

is a family of parabolas symmetric about y -axis with vertices at $(0, 0)$. Differentiating with respect to x , we get

$$\frac{dy}{dx} = 2ax. \quad (51)$$

Eliminating a between (50) and (51), we get

$$\frac{dy}{dx} = \frac{2xy}{x^2} = \frac{2y}{x}.$$

Therefore, differential equation of the orthogonal trajectory is

$$\frac{dy}{dx} = -\frac{x}{2y} \quad (52)$$

or

$$2y dy + x dx = 0.$$

Integrating, we get

$$2 \frac{y^2}{2} + \frac{x^2}{2} = C$$

or

$$\frac{x^2}{2} + \frac{y^2}{1} = C. \quad (53)$$

The orthogonal trajectories represented by (53) are ellipses (shown in the Figure 2.7)

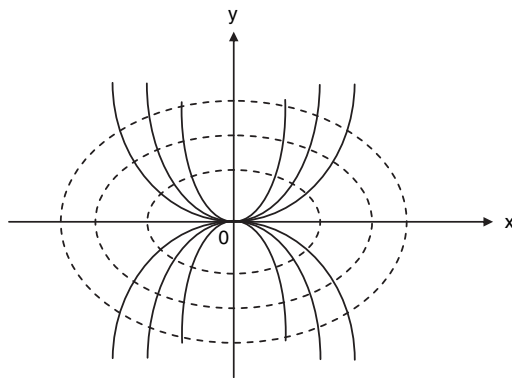


Figure 2.7

EXAMPLE 2.77

Show that the system of confocal and coaxial parabolas is self-orthogonal.

Solution. The equation of the family of confocal parabolas having x -axis as their axis is of the form

$$y^2 = 4a(x + a). \quad (54)$$

Differentiating, we get

$$y \frac{dy}{dx} = 2a$$

or

$$\frac{dy}{dx} = \frac{2a}{y}. \quad (55)$$

From (55), we have $a = \frac{y}{2} \frac{dy}{dx}$. Substituting this value in (54), we get

$$y^2 = 2y \frac{dy}{dx} + \left(x + \frac{1}{2} y \frac{dy}{dx} \right)$$

or

$$y \left(\frac{dy}{dx} \right)^2 + 2x \frac{dy}{dx} - y = 0, \quad (56)$$

which is the differential equation of the given family. Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in (56) we obtain (56) again. Hence, each member of family (54) cuts every other member of the same family orthogonally.

EXAMPLE 2.78

Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2} + \frac{y^2}{b^2 + \lambda} = 1$.

Solution. The equation of the family of the given curve is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2 + \lambda} = 1. \quad (57)$$

Differentiating with respect to x , we get

$$\frac{x}{a^2} + \frac{y}{b^2 + \lambda} \frac{dy}{dx} = 0. \quad (58)$$

From (57) and (58) we have respectively

$$b^2 + \lambda = \frac{a^2 y^2}{a^2 - x^2} \quad \text{and} \quad b^2 + \lambda = -\frac{a^2 y}{x} \frac{dy}{dx}.$$

Hence

$$\frac{a^2 y^2}{a^2 - x^2} = -\frac{a^2 y}{x} \frac{dy}{dx}$$

or

$$\frac{dy}{dx} = \frac{-xy}{a^2 - x^2}.$$

Therefore, differential equation of the orthogonal trajectory is

$$\frac{dy}{dx} = \frac{a^2 - x^2}{xy}$$

or

$$y dy = \frac{a^2 - x^2}{x} dx = \frac{a^2}{x} dx - x dx.$$

Integrating, we get

$$\frac{y^2}{2} = a^2 \log x - \frac{x^2}{2} + C$$

or

$$x^2 + y^2 = 2a^2 \log x + k(\text{constant}),$$

which is the equation of the required orthogonal trajectories.

EXAMPLE 2.79

Find the orthogonal trajectory of the cardioid $r = a(1 - \cos \theta)$.

Solution. The equation of the family of given cardioid is

$$r = a(1 - \cos \theta). \quad (59)$$

Differentiating with respect to θ , we have

$$\frac{dr}{d\theta} = a \sin \theta. \quad (60)$$

Dividing (60) by (59), we get the differential equation of the given family as

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{\sin \theta}{1 - \cos \theta} = \frac{2 \sin \theta/2 \cos \theta/2}{2 \sin^2 \theta/2} = \cot \frac{\theta}{2} \quad (61)$$

Replacing $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$ in (61), we get

$$\frac{1}{r} \left(-r^2 \frac{d\theta}{dr} \right) = \cot \frac{\theta}{2}$$

or

$$\frac{dr}{r} + \tan \frac{\theta}{2} d\theta = 0, \quad (62)$$

which is the equation of the family of orthogonal trajectories. Integrating (62), we get

$$\log r - 2 \log \cos \frac{\theta}{2} = \log C$$

or

$$\log r = \log C + \log \cos^2 \frac{\theta}{2} = \log C \cos^2 \frac{\theta}{2}$$

or

$$r = C \cos^2 \frac{\theta}{2} = C(1 + \cos \theta),$$

which is the equation of orthogonal trajectory of the given family.

EXAMPLE 2.80

Find the orthogonal trajectories of the family of curves $x^2 + y^2 = c^2$

Solution. The equation of the given family of curves is

$$x^2 + y^2 - c^2 = 0 \quad (63)$$

Differentiating (63), we get

$$2x + 2y \frac{dy}{dx} = 0$$

or

$$\frac{dy}{dx} = -\frac{x}{y}, \quad (64)$$

which is the differential equation representing the given curves. Therefore the differential equation of the required family of orthogonal trajectories is

$$\frac{dy}{dx} = \frac{y}{x}$$

or

$$\frac{dy}{y} = \frac{dx}{x} \quad (65)$$

Integrating, we get

$$\log y = \log x + \log k$$

or

$$y = kx,$$

which is the equation of the orthogonal trajectories, which are straight lines through the origin as shown in the Figure 2.8.

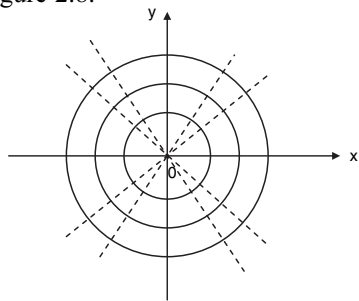


Figure 2.8

EXAMPLE 2.81

Find the orthogonal trajectories of the curves $r^2 = a^2 \cos 2\theta$.

Solution. We are given that

$$r^2 = a^2 \cos 2\theta = a^2(1 - 2 \sin^2 \theta). \quad (66)$$

Differentiating (66) w.r.t. θ , we have

$$2r \frac{dr}{d\theta} = -2a^2 \sin 2\theta. \quad (67)$$

Dividing (67) by (66), we get

$$\frac{2}{r} \frac{dr}{d\theta} = \frac{-2 \sin 2\theta}{\cos 2\theta}.$$

Replacing $\frac{dr}{d\theta}$ with $-r^2 \frac{d\theta}{dr}$, we get

$$-2r \frac{d\theta}{dr} = -\frac{\sin 2\theta}{\cos 2\theta} = -2 \tan 2\theta$$

or

$$\frac{dr}{r} = \cot 2\theta.$$

Integrating, we get

$$\log r = \frac{1}{2} \log \sin 2\theta + \log C$$

or

$$2 \log r = 2 \log C + \log \sin 2\theta$$

or

$$\log r^2 = \log C^2 + \log \sin 2\theta$$

or

$$r^2 = C^2 \sin 2\theta.$$

2.14 LINEAR DIFFERENTIAL EQUATIONS

Definition 2.17. A differential equation in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together is called a *linear differential equation*.

Thus, a linear differential equation of n^{th} order is of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = F(x) \quad (68)$$

where $a_0, a_1, \dots, a_n$ and $F(x)$ are functions of x alone.

If $a_0, a_1, \dots, a_n$ are constants, then the above equation is called a *linear differential equation with constant coefficients*.

If F is identically zero, then the equation (68) reduces to

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = 0 \quad (69)$$

and is called a *homogeneous linear differential equation* of order n .

Definition 2.18. If $f_1, f_2, \dots, f_n$ are n given functions and $c_1, c_2, \dots, c_n$ are n constants, then the expression

$$c_1 f_1 + c_2 f_2 + \dots + c_n f_n$$

is called a linear combination of $f_1, f_2, \dots, f_n$.

Definition 2.19. The set of functions $\{f_1, f_2, \dots, f_n\}$ is said to be linearly independent on $[a, b]$ if the relation

$$c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0$$

for all $x \in [a, b]$ implies that

$$c_1 = c_2 = \dots = c_n = 0.$$

Definition 2.20. The symbol $D = \frac{d}{dx}$ is called a differential operator. Similarly, $D^2 = \frac{d^2}{dx^2}$, $D^3 = \frac{d^3}{dx^3}$, ..., $D^n = \frac{d^n}{dx^n}$ are also regarded as operators. In terms of these symbols, the equation (68) takes the form

$$(a_0 D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n)y = F(x)$$

or

$$f(D)y = F(x),$$

where

$$f(D) = a_0 D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n.$$

We note that

$$D(u + v) = Du + Dv$$

$$D(\lambda u) = \lambda D(u)$$

$$(D^m D^n)(u) = D^{m+n}(u)$$

$$(D^m D^n)(u) = (D^n D^m)(u).$$

Theorem 2.4. Any linear combination of linearly independent solutions of the homogeneous linear differential equation is also a solution (in fact, complete solution) of that equation.

Proof: Let $y_1, y_2, \dots, y_n$ be the solution of the homogeneous linear differential equation

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n)y = 0 \quad (70)$$

Therefore,

$$\left. \begin{aligned} D^n y_1 + a_1 D^{n-1} y_1 + a_2 D^{n-2} y_1 + \dots + a_n y_1 &= 0 \\ D^n y_2 + a_1 D^{n-1} y_2 + a_2 D^{n-2} y_2 + \dots + a_n y_2 &= 0 \\ \dots\dots\dots \\ D^n y_n + a_1 D^{n-1} y_n + a_2 D^{n-2} y_n + \dots + a_n y_n &= 0 \end{aligned} \right\} \quad (71)$$

Let

$$u = c_1 y_1 + c_2 y_2 + \dots + c_n y_n.$$

Then

$$\begin{aligned} D^n u + a_1 D^{n-1} u + a_2 D^{n-2} u + \dots + a_n u \\ &= D^n (c_1 y_1 + c_2 y_2 + \dots + c_n y_n) \\ &\quad + a_1 D^{n-1} (c_1 y_1 + c_2 y_2 + \dots + c_n y_n) \\ &\quad + a_2 D^{n-2} (c_1 y_1 + c_2 y_2 + \dots + c_n y_n) \\ &\quad + \dots + a_n (c_1 y_1 + c_2 y_2 + \dots + c_n y_n) \\ &= c_1 (D^n y_1 + a_1 D^{n-1} y_1 + \dots + a_n y_1) \\ &\quad + c_2 (D^n y_2 + a_1 D^{n-1} y_2 + \dots + a_n y_2) \\ &\quad + \dots + c_n (D^n y_n + a_1 D^{n-1} y_n + \dots + a_n y_n) \\ &= 0 + 0 + \dots + 0 = 0 \text{ using (71).} \end{aligned}$$

Hence, $u = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$ is also a solution of the homogeneous linear differential equation (68). Since this solution contains n arbitrary constants, it is a general or a complete solution of (70).

EXAMPLE 2.82

Show that $c_1 \sin x + c_2 \cos x$ is a solution of $\frac{d^2 y}{dx^2} + y = 0$.

Solution. Let

$$y_1 = \sin x, \quad y_2 = \cos x.$$

Then

$$\begin{aligned} \frac{dy_1}{dx} &= \cos x, & \frac{dy_2}{dx} &= -\sin x \\ \frac{d^2 y_1}{dx^2} &= -\sin x, & \frac{d^2 y_2}{dx^2} &= -\cos x. \end{aligned}$$

We note that

$$\frac{d^2 y_1}{dx^2} + y_1 = -\sin x + \sin x = 0$$

and

$$\frac{d^2 y_2}{dx^2} + y_2 = -\cos x + \cos x = 0.$$

Hence, $\sin x$ and $\cos x$ are solutions of the given equation. These two solutions are linearly independent. Therefore, their linear combination $c_1 \sin x + c_2 \cos x$ is also a solution of the given equation.

Theorem 2.5. If y_1 is a complete solution of the homogeneous equation $f(D)y = 0$ and y_2 is a particular solution containing no arbitrary constants of the differential equation $f(D)y = F(x)$, then $y_1 + y_2$ is the complete solution of the equation $f(D)y = F(x)$.

Proof: Since y_1 is a complete solution of the homogeneous differential equation $f(D)y = 0$, we have

$$f(D)y_1 = 0 \quad (72)$$

Further, since y_2 is a particular solution of linear differential equation $f(D)y = F(x)$, we have

$$f(D)y_2 = F(x). \quad (73)$$

Adding (70) and (71), we get

$$f(D)y_1 + f(D)y_2 = F(x)$$

or

$$f(D)(y_1 + y_2) = F(x).$$

Hence, $y_1 + y_2$ satisfies the equation $f(D)y = F(x)$ and so is the complete solution since it contains n arbitrary constants.

Definition 2.21. Let $f(D)y = F(x)$ be a linear differential equation with constant coefficients. If y_1 is a complete solution of $f(D)y = 0$ and y_2 is a particular solution of $f(D)y = F(x)$, then $y_1 + y_2$ is a complete solution of $f(D)y = F(x)$ and then y_1 is called the *complementary function* and y_2 is called the *particular integral* of the differential equation $f(D)y = F(x)$. Consider the homogeneous differential equation $f(D)y = 0$. Then

$$(D^n + a_1D^{n-1} + a_2D^{n-2} + \dots + a_n)y = 0. \quad (74)$$

Let $y = e^{mx}$ be a solution of (74). Then

$$Dy = me^{mx}, D^2y = m^2e^{mx}, \dots, D^ny = m^ne^{mx}$$

and so (74) transforms to

$$(m^n + a_1m^{n-1} + \dots + a_n)e^{mx} = 0.$$

Since $e^{mx} \neq 0$, we have

$$m^n + a_1m^{n-1} + a_2m^{n-2} + \dots + a_n = 0. \quad (75)$$

It follows, therefore, that if e^{mx} is a solution of $f(D)y = 0$, then equation (75) is satisfied.

The equation (75) is called *auxiliary equation* for the differential equation $f(D)y = 0$.

2.15 SOLUTION OF HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

Consider the homogeneous linear differential equation

$$\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1}\frac{dy}{dx} + a_ny = 0.$$

The symbolic form of this equation is

$$(D^n + a_1D^{n-1} + \dots + a_{n-1}D + a_n)y = 0, \quad (76)$$

where $a_1, a_2, \dots, a_n$ are constants. If $y = e^{mx}$ is a solution of (76), then

$$m^n + a_1m^{n-1} + \dots + a_{n-1}m + a_n = 0. \quad (77)$$

Three cases arise, according as the roots of (77) are *real and distinct*, *real and repeated* or *complex*.

Case I. Distinct Real Roots

Suppose that the auxiliary equation (77) has n distinct roots $m_1, m_2, \dots, m_n$. Therefore, (77) reduces to

$$(m - m_1)(m - m_2) \dots (m - m_n) = 0 \quad (78)$$

Equation (78) will be satisfied by the solutions of the equations

$$(D - m_1)y = 0, (D - m_2)y = 0, \dots,$$

$$(D - m_n)y = 0.$$

We consider $(D - m_1)y = 0$. This can be written as

$$\frac{dy}{dx} - m_1y = 0,$$

which is linear differential equation with integrating factor as e^{-m_1x} . Therefore, its solution is

$$y \cdot e^{-m_1x} = \int 0 \cdot e^{-m_1x} dx + c_1$$

or

$$y = c_1 e^{m_1x}.$$

Similarly,

the solution of $(D - m_2)y = 0$ is $c_2 e^{m_2x}$,

the solution of $(D - m_3)y = 0$ is $c_3 e^{m_3x}$,

$\dots$

$\dots$

the solution of $(D - m_n)y = 0$ is $c_n e^{m_nx}$.

Hence, the complete solution of homogeneous differential equation (76) is

$$y = c_1 e^{m_1x} + c_2 e^{m_2x} + \dots + c_n e^{m_nx}. \quad (79)$$

Case II. Repeated Real Roots

Suppose that the roots m_1 and m_2 of the auxiliary equation are equal. Then the solution (79) becomes

$$\begin{aligned} y &= c_1 e^{m_1x} + c_2 e^{m_1x} + c_3 e^{m_3x} + \dots + c_n e^{m_nx} \\ &= (c_1 + c_2) e^{m_1x} + c_3 e^{m_3x} + \dots + c_n e^{m_nx}. \end{aligned}$$

Since it contains $n - 1$ arbitrary constants, it is not a complete solution of the given differential equation. We shall show that the part of the solution corresponding to equal roots m_1 and m_2 is $(c_1x + c_2)e^{m_1x}$. To prove it, consider the equation

$$(D - m_1)^2 y = 0,$$

that is,

$$(D - m_1)(D - m_1)y = 0.$$

Substituting $(D - m_1)y = U$, the above equation becomes

$$(D - m_1)U = 0$$

or

$$\frac{dU}{dx} - m_1 U = 0$$

or

$$\frac{dU}{U} = m_1 dx.$$

Integrating, we get

$$\log U = m_1 x + \log C_1$$

or

$$\log \frac{U}{C_1} = m_1 x \text{ or } \frac{U}{C_1} = e^{m_1 x}$$

and so

$$U = c_1 e^{m_1 x}.$$

Hence

$$(D - m_1)y = c_1 e^{m_1 x}$$

or

$$\frac{dy}{dx} - m_1 y = c_1 e^{m_1 x},$$

which is again a linear equation with integrating factor $e^{-m_1 x}$. Hence, the solution is

$$\begin{aligned} y \cdot e^{-m_1 x} &= \int c_1 e^{m_1 x} \cdot e^{-m_1 x} + c_2 \\ &= c_1 x + c_2, \end{aligned}$$

which yields

$$y = (c_1 x + c_2)e^{m_1 x}.$$

Hence, the complete solution of the given differential equation is

$$y = (c_1 x + c_2)e^{m_1 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}.$$

Remark 2.1. If three roots of the auxiliary equation are equal, that is, $m_1 = m_2 = m_3$, then the complete

solution will come out to be

$$y = (c_1 x^2 + c_2 x + c_3)e^{m_1 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}.$$

In general, if $m_1 = m_2 = \dots = m_k$, then the complete solution of the differential equation shall be

$$y = (c_1 x^{k-1} + c_2 x^{k-2} + \dots + c_k) + c_{k+1} e^{m_{k+1} x} + \dots + c_n e^{m_n x}.$$

Case III. Conjugate Complex Roots

(a) Suppose that the auxiliary equation has a non-repeated complex root $\alpha + i\beta$. Then, since the coefficients are real, the conjugate complex number $\alpha - i\beta$ is also a non-repeated root. Thus, the solution given in (79) becomes

$$\begin{aligned} y &= c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x} \\ &= e^{\alpha x} (c_1 e^{i\beta x} + c_2 e^{-i\beta x}) + c_3 e^{m_3 x} + \dots + c_n e^{m_n x} \\ &= e^{\alpha x} [c_1 (\cos \beta x + i \sin \beta x) + c_2 (\cos \beta x - i \sin \beta x)] \\ &\quad + c_3 e^{m_3 x} + \dots + c_n e^{m_n x} \\ &= e^{\alpha x} [(c_1 + c_2) \cos \beta x + i(c_1 - c_2) \sin \beta x] \\ &\quad + c_3 e^{m_3 x} + \dots + c_n e^{m_n x} \\ &= e^{\alpha x} [k_1 \cos \beta x + k_2 \sin \beta x] + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}, \end{aligned}$$

where $k_1 = c_1 + c_2$, $k_2 = i(c_1 - c_2)$.

(b) If two pairs of imaginary roots are equal, then

$$m_1 = m_2 = \alpha + i\beta \quad \text{and} \quad m_3 = m_4 = \alpha - i\beta.$$

Using Case II, the complete solution is

$$\begin{aligned} &= e^{\alpha x} [(c_1 x + c_2) \cos \beta x + (c_3 x + c_4) \sin \beta x] \\ &\quad + c_5 e^{m_5 x} + \dots + c_n e^{m_n x}. \end{aligned}$$

EXAMPLE 2.83

Solve

$$\frac{d^3 y}{dx^3} + \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 4y = 0.$$

Solution. The symbolic form of the given equation is

$$(D^3 + D^2 + 4D + 4)y = 0.$$

Therefore its auxiliary equation is

$$m^3 + m^2 + 4m + 4 = 0.$$

By inspection -1 is a root. Therefore, $(m + 1)$ is a factor of $m^3 + m^2 + 4m + 4$. The synthetic division

by $m + 1$ gives

$$\begin{array}{c|cccc} -1 & 1 & 1 & 4 & 4 \\ & & -1 & 0 & -4 \\ \hline & 1 & 0 & 4 & 0 \end{array}$$

Therefore, the auxiliary equation is

$$(m + 1)(m^2 + 4) = 0$$

and so

$$m = -1 \quad \text{and} \quad m = \pm 2i.$$

Hence, the complementary solution is

$$\begin{aligned} y &= c_1 e^{-x} + e^{0x}(c_2 \cos 2x + c_3 \sin 2x) \\ &= c_1 e^{-x} + c_2 \cos 2x + c_3 \sin 2x. \end{aligned}$$

EXAMPLE 2.84

Solve

$$\frac{d^3 y}{dx^3} - 3 \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} - y = 0.$$

Solution. The symbolic form of the given equation is

$$(D^3 - 3D^2 + 3D - 1)y = 0.$$

Therefore, the auxiliary equation is

$$m^3 - 3m^2 + 3m - 1 = 0.$$

By inspection 1 is a root. Then synthetic division yields

$$\begin{array}{c|cccc} 1 & 1 & -3 & 3 & -1 \\ & & 1 & -2 & 1 \\ \hline & 1 & -2 & 1 & 0 \end{array}$$

Therefore, the auxiliary equation is

$$(m - 1)(m^2 - 2m + 1) = 0 \quad \text{or} \quad (m - 1)^3 = 0.$$

Hence the roots are 1, 1, 1 and so the solution of the given equation is

$$y = (c_1 + c_2 x + c_3 x^2)e^x.$$

EXAMPLE 2.85

Find the general solution of

$$\frac{d^4 y}{dx^4} - 5 \frac{d^3 y}{dx^3} + 6 \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} - 8y = 0.$$

Solution. The auxiliary equation is

$$m^4 - 5m^3 + 6m^2 + 4m - 8 = 0,$$

whose roots are 2, 2, 2, and -1 . Hence the general solution is

$$y = (c_1 + c_2 x + c_3 x^2)e^{2x} + c_4 e^{-x}.$$

EXAMPLE 2.86

Solve $\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + 10y = 0$ subject to the conditions $y(0) = 4$, $y'(0) = 1$.

Solution. The symbolic form of the given differential equation is

$$(D^2 - 2D + 10)y = 0.$$

Therefore, the auxiliary equation is

$$m^2 - 2m + 10 = 0,$$

which yields

$$m = 1 \pm 3i.$$

Therefore, the solution is

$$y = e^x(c_1 \cos 3x + c_2 \sin 3x).$$

Now,

$$\begin{aligned} y' &= c_1 [e^x \cos 3x - 3e^x \sin 3x] \\ &\quad + c_2 [e^x \sin 3x + 3e^x \cos 3x] \\ &= e^x \cos 3x(c_1 + 3c_2) + e^x \sin 3x(c_2 - 3c_1). \end{aligned}$$

The initial conditions $y(0) = 4$ and $y'(0) = 1$ yield

$$4 = c_1 \quad \text{and} \quad 1 = c_1 + 3c_2$$

and so $c_1 = 4$ and $c_2 = -1$. Hence the solution is

$$y = e^x(4 \cos 3x - \sin 3x).$$

EXAMPLE 2.87

Solve

$$\frac{d^3 y}{dx^3} + y = 0.$$

Solution. The auxiliary equation is

$$m^3 + 1 = 0$$

or

$$(m + 1)(m^2 - m + 1) = 0.$$

Thus the roots are $-1, \frac{1 \pm \sqrt{3}i}{2}$. Hence, the general solution of the equation is

$$y = c_1 e^{-x} + e^{\frac{1}{2}x} \left(c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x \right).$$

EXAMPLE 2.88

Solve

$$\frac{d^4 y}{dx^4} = m^4 y.$$

Solution. The auxiliary equation for the given differential equation is

$$s^4 - m^4 = 0$$

or

$$(s + m)(s - m)(s^2 + m^2) = 0$$

and so $s = m, -m, \pm mi$. Hence the solution is

$$\begin{aligned} y &= c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx \\ &= c_1 [\cosh mx + \sinh mx] + c_2 [\cosh mx - \sinh mx] \\ &\quad + c_3 \cos mx + c_4 \sin mx \\ &= (c_1 + c_2) \cosh mx + (c_1 - c_2) \sinh mx \\ &\quad + c_3 \cos mx + c_4 \sin mx \\ &= C_1 \cosh mx + C_2 \sinh mx + c_3 \cos mx \\ &\quad + c_4 \sin mx, \end{aligned}$$

where

$$C_1 = c_1 + c_2 \text{ and } C_2 = c_1 - c_2.$$

EXAMPLE 2.89

$$\text{Solve } 4 \frac{d^3 y}{dx^3} + 4 \frac{d^2 y}{dx^2} + \frac{dy}{dx} = 0.$$

Solution. The auxiliary equation for the given differential equation is

$$4m^3 + 4m^2 + m = 0$$

or

$$m(4m^2 + 4m + 1) = 0.$$

Thus the roots are $m = 0, -\frac{1}{2}, \text{ and } -\frac{1}{2}$. Hence the solution is

$$\begin{aligned} y &= c_1 e^{0x} + (c_2 + c_3 x) e^{-x/2} \\ &= c_1 + (c_2 + c_3 x) e^{-x/2}. \end{aligned}$$

2.16 COMPLETE SOLUTION OF LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

We now discuss the methods of finding particular integral of a linear differential equation with

constant coefficients so that complete solution of the equation may be found.

Definition 2.22. $\frac{1}{D}F(x)$ is that function of x which when operated upon by D yields $F(x)$.

Similarly, $\frac{1}{f(D)}F(x)$ is that function of x , free from arbitrary constant, which when operated upon by $f(D)$ yields $F(x)$.

Thus, $\frac{1}{D}$ is the inverse operator of D and $\frac{1}{f(D)}$ is the inverse operator of $f(D)$.

Theorem 2.6. $\frac{1}{D}F(x) = \int F(x) dx$.

Proof: Let

$$\frac{1}{D}F(x) = v. \quad (80)$$

Operating both sides of (80) by D , we get

$$D \cdot \frac{1}{D}F(x) = Dv$$

or

$$F(x) = Dv = \frac{dv}{dx}$$

or

$$dv = F(x)dx.$$

Integrating, we get

$$v = \int F(x) dx,$$

where no constant of integration is added since $\frac{1}{D}F(x)$ contains no constant. Thus,

$$\frac{1}{D}F(x) = \int F(x) dx.$$

Hence, $\frac{1}{D}$ stands for integration.

Theorem 2.7. $\frac{1}{D-\alpha}F(x) = e^{\alpha x} \int F(x)e^{-\alpha x} dx$.

Proof: Let

$$\frac{1}{D-\alpha}F(x) = y. \quad (81)$$

Operating both sides of (81) by $D - \alpha$, we have

$$\begin{aligned} F(x) &= (D - \alpha)y = Dy - \alpha y \\ &= \frac{dy}{dx} - \alpha y. \end{aligned}$$

Therefore,

$$\frac{dy}{dx} - \alpha y = F(x),$$

which is a linear differential equation with integrating factor $e^{-\alpha x}$. Therefore, its solution is

$$y \cdot e^{-\alpha x} \int F(x) e^{-\alpha x} dx$$

or

$$y = e^{\alpha x} \int F(x) e^{-\alpha x} dx$$

or

$$\frac{1}{D - \alpha} F(x) = e^{\alpha x} \int F(x) e^{-\alpha x} dx.$$

Theorem 2.8. $\frac{1}{f(D)}F(x)$ is the particular integral of $f(D)y = F(x)$.

Proof: The given linear differential equation is

$$f(D)y = F(x) \quad (82)$$

Substituting $y = \frac{1}{f(D)}F(x)$ in (82), we have $F(x) = F(x)$, which is true. Hence $y = \frac{1}{f(D)}F(x)$ is a solution of (82).

2.16.1 Standard Cases of Particular Integrals

Consider the linear differential equation

$$f(D)y = F(x).$$

By Theorem 2.8, its particular integral is

$$\text{P.I.} = \frac{1}{f(D)} F(x).$$

Case I. When $F(x) = e^{ax}$

We have

$$f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n.$$

Therefore,

$$\begin{aligned} f(D)e^{ax} &= (D^n + a_1 D^{n-1} + \dots + a_n)e^{ax} \\ &= D^n e^{ax} + a_1 D^{n-1} e^{ax} + \dots + a_{n-1} D e^{ax} + a_n e^{ax} \\ &= a^n e^{ax} + a_1 a^{n-1} e^{ax} + \dots + a_{n-1} a e^{ax} + a_n e^{ax} \\ &= (a^n + a_1 a^{n-1} + \dots + a_{n-1} a + a_n) e^{ax} \\ &= f(a) e^{ax}. \end{aligned}$$

Operating both sides by $\frac{1}{f(D)}$ yields

$$\begin{aligned} e^{ax} &= \frac{1}{f(D)} [f(a) e^{ax}] \\ &= f(a) \frac{1}{f(D)} e^{ax}. \end{aligned}$$

Hence

$$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}, \text{ provided } f(a) \neq 0. \quad (83)$$

If $f(a) = 0$, then $D - a$ is a factor of $f(D)$. So, let

$$f(D) = (D - a)\phi(D). \quad (84)$$

Then

$$\begin{aligned} \frac{1}{f(D)} e^{ax} &= \frac{1}{(D - a)\phi(D)} e^{ax} = \frac{1}{(D - a)} \left[\frac{1}{\phi(D)} e^{ax} \right] \\ &= \frac{1}{D - a} \left[\frac{1}{\phi(a)} e^{ax} \right] \text{ using (83) since } \phi(a) \neq 0 \\ &= \frac{1}{\phi(a)} \left[\frac{1}{D - a} e^{ax} \right] \\ &= \frac{1}{\phi(a)} e^{ax} \int e^{ax} \cdot e^{-ax} dx, \text{ by Theorem 2.7} \\ &= x \frac{e^{ax}}{\phi(a)} \quad (85) \end{aligned}$$

Differentiating (84) with respect to D gives

$$f'(D) = \phi(D) + (D - a)\phi'(D).$$

Putting $D = a$, we get $f'(a) = \phi(a)$. Therefore, (85) reduces to

$$\begin{aligned} \frac{1}{f(D)} e^{ax} &= x \frac{e^{ax}}{\phi(a)} = x \frac{e^{ax}}{\left[\frac{d}{dD} f(D) \right]_{D=a}} \\ &= x \frac{e^{ax}}{f'(a)}, \text{ provided } f'(a) \neq 0. \quad (86) \end{aligned}$$

If $f'(a) = 0$, then the rule can be repeated to give

$$\begin{aligned} \frac{1}{f(D)} e^{ax} &= x^2 \frac{e^{ax}}{f''(a)} e^{ax}, \\ &\text{provided } f''(a) \neq 0 \text{ and so on.} \end{aligned}$$

Case II. When $F(x) = \sin(ax + b)$ or $\cos(ax + b)$ or

We have

$$\begin{aligned} D \sin(ax + b) &= a \cos(ax + b) \\ D^2 \sin(ax + b) &= -a^2 \sin(ax + b) \\ D^3 \sin(ax + b) &= -a^3 \cos(ax + b) \\ D^4 \sin(ax + b) &= a^4 \sin(ax + b) \\ &\vdots \\ &\vdots \end{aligned}$$

We note in general that

$$(D^2)^n \sin(ax + b) = (-a^2)^n \sin(ax + b).$$

Hence

$$f(D^2) \sin(ax + b) = f(-a^2) \sin(ax + b).$$

Operating on both sides by $\frac{1}{f(D^2)}$, we get

$$\sin(ax + b) = f(-a^2) \frac{1}{f(D^2)} \sin(ax + b).$$

Dividing both sides by $f(-a^2)$, we have

$$\frac{1}{f(-a^2)} \sin(ax + b) = \frac{1}{f(D^2)} \sin(ax + b).$$

Hence

$$\frac{1}{f(D^2)} \sin(ax + b) = \frac{1}{f(-a^2)} \sin(ax + b), \quad (87)$$

provided $f(-a^2) \neq 0$.

Similarly,

$$\frac{1}{f(D^2)} \cos(ax + b) = \frac{1}{f(-a^2)} \cos(ax + b), \quad (88)$$

provided $f(-a^2) \neq 0$.

If $f(-a^2) = 0$, then (87) and (88) are not valid.

In such a situation, we proceed as follows:

By Euler's formula

$$e^{i(ax+b)} = \cos(ax + b) + i \sin(ax + b).$$

Thus

$$\frac{1}{f(D^2)} e^{i(ax+b)} = \frac{1}{f(D^2)} [\cos(ax + b) + i \sin(ax + b)]$$

or, by 86,

$$x \cdot \frac{1}{f'(D^2)} e^{i(ax+b)} = \frac{1}{f(D^2)} [\cos(ax + b) + i \sin(ax + b)]$$

$$\begin{aligned} &x \cdot \frac{1}{f'(D^2)} [\cos(ax + b) + i \sin(ax + b)] \\ &= \frac{1}{f(D^2)} [\cos(ax + b) + i \sin(ax + b)] \end{aligned}$$

Equating real and imaginary parts, we have

$$\frac{1}{f(D^2)} \cos(ax + b) = x \cdot \frac{1}{[f'(D^2)]_{D^2=-a^2}} \cos(ax + b) \quad (89)$$

provided that $f'(-a^2) \neq 0$, and

$$\frac{1}{f(D^2)} \sin(ax + b) = x \frac{1}{[f'(D^2)]_{D^2=-a^2}} \sin(ax + b), \quad (90)$$

provided $f'(-a^2) \neq 0$.

If $f'(-a^2) = 0$, then repeating the above process, we have

$$\frac{1}{f(D^2)} \cos(ax + b) = x^2 \frac{1}{[f''(D^2)]_{D^2=-a^2}} \cos(ax + b)$$

provided $f''(-a^2) \neq 0$

and

$$\frac{1}{f(D^2)} \sin(ax + b) = x^2 \frac{1}{[f''(D^2)]_{D^2=-a^2}} \sin(ax + b)$$

provided $f''(-a^2) \neq 0$.

Case III. When $F(x) = x^n$, n being positive integer

Since in this case

$$\text{P.I. } \frac{1}{f(D)} F(x) = \frac{1}{f(D)} x^n,$$

we make the coefficient of the leading term of $f(D)$ unity, take the denominator in numerator and then expand by Binomial theorem. Operate the resulting expansion on x^n .

Case IV. When $F(x) = e^{ax} Q(x)$, where $Q(x)$ is some function of x

Let G is a function of x , we have

$$\begin{aligned} D[e^{ax} G] &= e^{ax} DG + a e^{ax} G = e^{ax} (D+a)G \\ D^2[e^{ax} G] &= e^{ax} (D+a)^2 G \end{aligned}$$

...

...

$$D^n[e^{ax} G] = e^{ax} (D+a)^n G$$

Hence

$$f(D)[e^{ax}G] = e^{ax}f(D+a)G$$

Operating both sides by $\frac{1}{f(D)}$, we get

$$e^{ax}G = \frac{1}{f(D)}[e^{ax}f(D+a)]G$$

Putting $f(D+a)G = Q$, we have $G = \frac{Q}{f(D+a)}$ and so we have

$$e^{ax} \cdot \frac{1}{f(D+a)}Q = \frac{1}{f(D)}(e^{ax}Q)$$

or

$$\frac{1}{f(D)}(e^{ax}Q(x)) = e^{ax} \frac{1}{f(D+a)}Q \quad (91)$$

Case V. When $F(x) = x Q(x)$

Resolving $f(D)$ into linear factors, we have

$$f(D) = (D - m_1)(D - m_2) \dots (D - m_n).$$

Then, using partial fractions and Theorem 2.7, we have

$$\begin{aligned} \text{P.I.} &= \frac{1}{f(D)} F(x) \\ &= \frac{1}{(D - m_1)(D - m_2) \dots (D - m_n)} F(x) \\ &= \left[\frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \dots + \frac{A_n}{D - m_n} \right] F(x) \\ &= A_1 e^{m_1 x} \int F(x) e^{-m_1 x} + A_2 e^{m_2 x} \int F(x) e^{-m_2 x} \\ &\quad + \dots + A_n e^{m_n x} \int F(x) e^{-m_n x}. \end{aligned}$$

EXAMPLE 2.90

Solve $4 \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} - 3y = e^{2x}$.

Solution. The symbolic form of the given differential equation is

$$(4D^2 + 4D - 3)y = e^{2x}$$

and so the auxiliary equation is

$$4m^2 + 4m - 3 = 0.$$

The roots of A.E. are $m = \frac{1}{2}, -\frac{3}{2}$. Therefore, the complementary function is

$$\text{C.F.} = c_1 e^{x/2} + c_2 e^{-3x/2}.$$

Since 2 is not a root of the auxiliary equation, by (82), we have

$$\text{P.I.} = \frac{1}{f(2)} e^{2x} = \frac{1}{4(4) + 4(2) - 2} e^{2x} = \frac{e^{2x}}{21}.$$

Hence the complete solution of the given equation is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{x/2} + c_2 e^{-3x/2} + \frac{e^{2x}}{21}.$$

EXAMPLE 2.91

Solve $\frac{d^2 y}{dx^2} - 5 \frac{dy}{dx} + 6y = e^{3x}$.

Solution. The symbolic form of the equation is

$$(D^2 - 5D + 6)y = e^{3x}.$$

The auxiliary equation is

$$m^2 - 5m + 6 = 0$$

or

$$(m - 3)(m - 2) = 0.$$

Therefore $m = 2, 3$. Then

$$\text{C.F.} = c_1 e^{3x} + c_2 e^{2x}.$$

Since 3 is a root of auxiliary equation, we use (85) and get

$$\text{P.I.} = x \frac{e^{3x}}{\phi(3)} = x \frac{e^{3x}}{(3 - 2)} = x e^{3x}.$$

Hence, the complete solution of the given equation is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{3x} + c_2 e^{2x} + x e^{3x}.$$

Remark 2.2. (a) In the above example, if we use (86), then

$$\begin{aligned} \text{P.I.} &= x \frac{e^{ax}}{\left(\frac{d}{dD} f(D)\right)_{D=a}} \\ &= x \frac{e^{3x}}{[2D - 5]_{D=3}} = x \frac{e^{3x}}{6 - 5} = x e^{3x}. \end{aligned}$$

(b) We can also find the particular integral in the above case by using Theorem 2.7. In fact, we have

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{f(D)} F(x) = \frac{1}{(D-3)(D-2)} e^{3x} \\
 &= \frac{1}{D-3} e^{2x} \int F(x) e^{-2x} dx \\
 &= \frac{1}{D-3} e^{2x} \int e^{3x} e^{-2x} dx = \frac{1}{D-3} e^{3x} \\
 &= e^{3x} \int e^{3x} \cdot e^{-3x} dx = e^{3x} \int e^0 dx = x e^{3x}.
 \end{aligned}$$

EXAMPLE 2.92

Solve

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \cos hx.$$

Solution. The auxiliary equation is

$$m^2 - 3m + 2 = 0,$$

which yields $m = 1, 2$. Therefore,

$$\text{C.F.} = c_1 e^x + c_2 e^{2x}.$$

Now

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

Therefore,

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{f(D)} F(x) = \frac{1}{(D-1)(D-2)} \left[\frac{e^x + e^{-x}}{2} \right] \\
 &= \frac{1}{2} \cdot \frac{1}{D^2 - 3D + 2} e^x + \frac{1}{2} \cdot \frac{1}{D^2 - 3D + 2} e^{-x} \\
 &= \frac{1}{2} \frac{1}{D^2 - 3D + 2} e^x + \frac{1}{2} \frac{1}{1 + 3 + 2} e^{-x} \\
 &= \frac{1}{2} x \frac{1}{\frac{d}{dD}[D^2 - 3D + 2]_{D=1}} e^x + \frac{1}{12} e^{-x} \text{ since } f(1) = 0 \\
 &= \frac{x}{2} \frac{1}{[2D - 3]_{D=1}} e^x + \frac{1}{12} e^{-x} \\
 &= -\frac{x}{2} e^x + \frac{1}{12} e^{-x}.
 \end{aligned}$$

Hence, the complete solution of the equation is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^x + c_2 e^{2x} - \frac{x}{2} e^x + \frac{1}{12} e^{-x}.$$

EXAMPLE 2.93Solve $\frac{d^2y}{dx^2} - 4y = e^x + \sin 2x$.**Solution.** The auxiliary equation for the given differential equation is

$$m^2 - 4 = 0,$$

which yields $m = \pm 2$. Hence

$$\text{C.F.} = c_1 e^{2x} + c_2 e^{-2x}.$$

Now, using (82) and (87), we have

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{D^2 - 4} (e^x + \sin 2x) = \frac{1}{D^2 - 4} e^x + \frac{1}{D^2 - 4} \sin 2x \\
 &= \frac{e^x}{1^2 - 4} + \frac{1}{-4 - 4} \sin 2x = -\frac{1}{3} e^x - \frac{1}{8} \sin 2x.
 \end{aligned}$$

Hence, the complete solution of the given differential equation is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{3} e^x - \frac{1}{8} \sin 2x.$$

EXAMPLE 2.94Solve $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} - y = \cos 2x$.**Solution.** The auxiliary equation is

$$m^3 + m^2 - m - 1 = 0,$$

whose roots are 1, -1, -1. Therefore,

$$\text{C.F.} = c_1 e^x + (c_2 + c_3 x) e^{-x}$$

Further,

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{f(D)} F(x) = \frac{1}{D^3 + D^2 - D - 1} \cos 2x \\
 &= \frac{1}{DD^2 - D^2 - D - 1} \cos 2x \\
 &= \frac{1}{D(-4) + (-4) - D - 1} \cos 2x \\
 &= \frac{1}{-5D - 5} \cos 2x = -\frac{1}{5(D+1)} \cos 2x \\
 &= -\frac{(D-1)}{5(D^2-1)} \cos 2x = -\frac{1}{5} (D-1) \frac{1}{-4-1} \cos 2x \\
 &= \frac{1}{25} (D-1) \cos 2x = \frac{1}{25} (D \cos 2x - \cos 2x) \\
 &= \frac{1}{25} (-2 \sin 2x - \cos 2x) = -\frac{1}{25} (2 \sin 2x + \cos 2x).
 \end{aligned}$$

Hence the complete solution is

$$y = c_1 e^x + (c_2 + c_3 x) e^{-x} - \frac{1}{25} (2 \sin 2x + \cos 2x).$$

EXAMPLE 2.95

Solve $\frac{d^2y}{dx^2} - 4y = x \sin hx$.

Solution. The auxiliary equation is $m^2 - 4 = 0$ and $m = \pm 2$. Therefore,

$$\text{C.F.} = c_1 e^{2x} + c_2 e^{-2x}.$$

Further,

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - 4} x \sinh x = \frac{1}{D^2 - 4} x \left(\frac{e^x - e^{-x}}{2} \right) \\ &= \frac{1}{2} \left[\frac{1}{D^2 - 4} x e^x - \frac{1}{D^2 - 4} x e^{-x} \right] \\ &= \frac{1}{2} \left[e^x \frac{1}{(D+1)^2 - 4} x - e^{-x} \frac{1}{(D-1)^2 - 4} x \right] \\ &= \frac{1}{2} \left[e^x \frac{1}{D^2 + 2D - 3} x - e^{-x} \frac{1}{D^2 - 2D - 3} x \right] \\ &= \frac{1}{2} \left[e^x \frac{1}{-3(1 - \frac{2D}{3} - \frac{D^2}{3})} x - e^{-x} \frac{1}{-3(1 + \frac{2D}{3} - \frac{D^2}{3})} x \right] \\ &= -\frac{1}{6} \left[e^x \left\{ 1 - \left(\frac{2D}{3} + \frac{D^2}{3} \right) \right\}^{-1} x \right. \\ &\quad \left. - e^{-x} \left\{ 1 + \left(\frac{2D}{3} - \frac{D^2}{3} \right) \right\}^{-1} x \right] \\ &= -\frac{1}{6} \left[e^x \left(1 + \frac{2D}{3} + \dots \right) x - e^{-x} \left(1 - \frac{2D}{3} + \dots \right) x \right] \\ &= -\frac{1}{6} \left[e^x \left(x + \frac{2}{3} \right) - e^{-x} \left(x - \frac{2}{3} \right) \right] \\ &= -\frac{x}{3} \left(\frac{e^x - e^{-x}}{2} \right) - \frac{2}{9} \left(\frac{e^x + e^{-x}}{2} \right) \\ &= -\frac{x}{3} \sinh x - \frac{2}{9} \cosh x. \end{aligned}$$

Hence the solution is

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ &= c_1 e^{2x} + c_2 e^{-2x} - \frac{x}{3} \sinh x - \frac{2}{9} \cosh x. \end{aligned}$$

EXAMPLE 2.96

Solve $\frac{d^2y}{dx^2} - 4y = x^2$.

Solution. The auxiliary equation is $m^2 - 4 = 0$ and so $m = \pm 2$. Therefore,

$$\text{C.F.} = c_1 e^{2x} + c_2 e^{-2x}.$$

Further,

$$\begin{aligned} \text{P.I.} &= \frac{1}{f(D)} F(x) = \frac{1}{D^2 - 4} x^2 \\ &= -\frac{1}{4 - D^2} x^2 = -\frac{1}{4(1 - \frac{D^2}{4})} x^2 \\ &= -\frac{1}{4} \left(1 - \frac{D^2}{4} \right)^{-1} x^2 \\ &= -\frac{1}{4} \left[1 + \frac{D^2}{4} + \dots \right] x^2 \\ &= -\frac{1}{4} x^2 - \frac{1}{16} D^2(x^2) \\ &= -\frac{1}{4} x^2 - \frac{1}{16} (2) = -\frac{1}{4} x^2 - \frac{1}{8}. \end{aligned}$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{4} x^2 - \frac{1}{8}.$$

EXAMPLE 2.97

Solve $\frac{d^2y}{dx^2} + 4y = e^x + \sin 3x + x^2$.

Solution. The auxiliary equation is $m^2 + 4 = 0$ and so $m = \pm 2i$. Therefore,

$$\text{C.F.} = c_1 \cos 2x + c_2 \sin 2x.$$

Further,

$$\begin{aligned} \text{P.I.} &= \frac{1}{f(D)} F(x) = \frac{1}{D^2 + 4} (e^x + \sin 3x + x^2) \\ &= \frac{1}{D^2 + 4} e^x + \frac{1}{D^2 + 4} \sin 3x + \frac{1}{D^2 + 4} x^2 \\ &= \frac{1}{5} e^x + \frac{1}{-9 + 4} \sin 3x + \frac{1}{4} \left(1 + \frac{D^2}{4} \right)^{-1} x^2 \\ &= \frac{1}{5} e^x - \frac{1}{5} \sin 3x + \frac{1}{4} \left(1 - \frac{D^2}{4} + \dots \right) x^2 \\ &= \frac{1}{5} e^x - \frac{1}{5} \sin 3x + \frac{x^2}{4} - \frac{1}{16} \cdot 2 \\ &= \frac{1}{5} e^x - \frac{1}{5} \sin 3x + \frac{1}{4} x^2 - \frac{1}{8}. \end{aligned}$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= c_1 \cos 2x + c_2 \sin 2x + \frac{1}{5}e^x - \frac{1}{5}\sin 3x + \frac{1}{4}x^2 - \frac{1}{8}.$$

EXAMPLE 2.98

Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$.

Solution. The auxiliary equation of the given differential equation is

$$m^2 - 2m + 1 = 0,$$

which yields $m = 1, 1$. Hence

$$\text{C.F.} = (c_1 + c_2x)e^x.$$

The particular integral is

$$\begin{aligned} \text{P.I.} &= \frac{1}{f(D)}F(x) = \frac{1}{(D-1)^2}xe^x \sin x \\ &= e^x \frac{1}{(D+1-1)^2}x \sin x = e^x \frac{1}{D^2}x \sin x \\ &= e^x \frac{1}{D} \int x \sin x \, dx = e^x \frac{1}{D}(-x \cos x + \sin x) \\ &= e^x \int (-x \cos x + \sin x) \, dx \\ &= e^x[-x \sin x - \cos x - \cos x] \\ &= -e^x(x \sin x + 2 \cos x). \end{aligned}$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= (c_1 + c_2x)e^x - e^x(x \sin x + 2 \cos x).$$

EXAMPLE 2.99

Solve $(D^2 - 4D + 3)y = \sin 3x \cos 2x$.

Solution. The auxiliary equation is

$$m^2 - 4m + 3 = 0,$$

which yields $m = 3, 1$. Therefore,

$$\text{C.F.} = c_1 e^{3x} + c_2 e^x.$$

Further

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - 4D + 3}[\sin 3x \cos 2x] \\ &= \frac{1}{D^2 - 4D + 3}\left[\frac{1}{2}2 \sin 3x \cos 2x\right] \\ &= \frac{1}{D^2 - 4D + 3}\left[\frac{1}{2}(\sin 5x + \sin x)\right] \\ &= \frac{1}{2} \cdot \frac{1}{D^2 - 4D + 3} \sin 5x + \frac{1}{2} \cdot \frac{1}{D^2 - 4D + 3} \sin x \\ &= \frac{1}{2} \left[\frac{1}{-25 - 4D + 3} \sin 5x + \frac{1}{-1 - 4D + 3} \sin x \right] \\ &= \frac{1}{2} \left[\frac{1}{-22 - 4D} \sin 5x + \frac{1}{2 - 4D} \sin x \right] \\ &= \frac{1}{2} \left[-\frac{1}{2(11 + 2D)} \sin 5x + \frac{1}{2(1 - 2D)} \sin x \right] \\ &= \frac{1}{4} \left[-\frac{11 - 2D}{121 - 4D^2} \sin 5x + \frac{1 + 2D}{1 - 4D^2} \sin x \right] \\ &= \frac{1}{4} \left[-\frac{11 - 2D}{121 - 4(-25)} \sin 5x + \frac{1 + 2D}{1 - 4(-1)} \sin x \right] \\ &= \frac{1}{4} \left[-\frac{11 - 2D}{221} \sin 5x + \frac{1 + 2D}{5} \sin x \right] \\ &= \frac{1}{4} \left[-\frac{1}{221} [11 \sin 5x - 2D \sin 5x] \right. \\ &\quad \left. + \frac{1}{5} (\sin x + 2D \sin x) \right] \\ &= \frac{1}{4} \left[-\frac{11}{221} \sin 5x + \frac{10}{221} \cos 5x + \frac{1}{5} \sin x + \frac{2}{5} \cos x \right] \\ &= -\frac{11}{884} \sin 5x + \frac{10}{884} \cos 5x + \frac{1}{20} \sin x + \frac{1}{10} \cos x. \end{aligned}$$

Hence the complete solution is

$$y = c_1 e^{3x} + c_2 e^x - \frac{11}{884} \sin 5x + \frac{10}{884} \cos 5x$$

$$+ \frac{1}{20} \sin x + \frac{1}{10} \cos x.$$

EXAMPLE 2.100

Solve $(D^2 + 1)y = \operatorname{cosec} x$.

Solution. The auxiliary equation is $m^2 + 1 = 0$, which yields $m = \pm i$. Thus,

$$\text{C.F.} = c_1 \cos x + c_2 \sin x.$$

Now

$$\begin{aligned}\text{P.I.} &= \frac{1}{f(D)} F(x) = \frac{1}{D^2 + 1} \operatorname{cosec} x \\ &= \frac{1}{(D+i)(D-i)} \operatorname{cosec} x \\ &= \frac{1}{2i} \left[\frac{1}{D-i} - \frac{1}{D+i} \right] \operatorname{cosec} x \\ &= \frac{1}{2i} \left[\frac{1}{D-i} \operatorname{cosec} x - \frac{1}{D+i} \operatorname{cosec} x \right].\end{aligned}$$

But, by Theorem 2.7,

$$\begin{aligned}\frac{1}{D-i} \operatorname{cosec} x &= e^{ix} \int \operatorname{cosec} x e^{-ix} dx \\ &= e^{ix} \int \operatorname{cosec} x (\cos x - i \sin x) dx \\ &= e^{ix} \int (\cot x - i) dx \\ &= e^{ix} (\log \sin x - ix).\end{aligned}$$

Similarly,

$$\frac{1}{D+i} \operatorname{cosec} x = e^{-ix} (\log \sin x + ix).$$

Therefore,

$$\begin{aligned}\text{P.I.} &= \frac{1}{2i} [e^{ix} (\log \sin x - ix) - e^{-ix} (\log \sin x + ix)] \\ &= \log \sin x \left(\frac{e^{ix} - e^{-ix}}{2i} \right) - x \left(\frac{e^{ix} + e^{-ix}}{2} \right) \\ &= (\log \sin x) \sin x - x \cos x.\end{aligned}$$

Hence the complete solution is

$$y = c_1 \cos x + c_2 \sin x + \sin x \log \sin x - x \cos x.$$

EXAMPLE 2.101

Solve $\frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} + 6y = e^{-2x} \sin 2x$.

Solution. The auxiliary equation is $m^2 + 5m + 6 = 0$, which yields

$$m = \frac{-5 \pm \sqrt{25 - 24}}{2} = -2, -3.$$

Therefore,

$$\text{C.F.} = c_1 e^{-2x} + c_2 e^{-3x}.$$

Further

$$\begin{aligned}\text{P.I.} &= \frac{1}{f(D)} F(x) \\ &= \frac{1}{D^2 + 5D + 6} e^{-2x} \sin 2x \\ &= e^{-2x} \frac{1}{(D-2)^2 + 5(D-2) + 6} \sin 2x \\ &= e^{-2x} \frac{1}{D^2 + D} \sin 2x \\ &= e^{-2x} \frac{1}{D-4} \sin 2x \\ &= e^{-2x} \cdot \frac{D+4}{D^2-4} \sin 2x \\ &= e^{-2x} \frac{D+4}{-8} \sin 2x \\ &= \frac{e^{-2x}}{-8} [D \sin 2x + 4 \sin 2x] \\ &= \frac{e^{-2x}}{-8} [2 \cos 2x + 4 \sin 2x].\end{aligned}$$

Hence the complete solution is

$$\begin{aligned}y &= \text{C.F.} + \text{P.I.} \\ &= c_1 e^{-2x} + c_2 e^{-3x} - \frac{1}{8} e^{-2x} [2 \cos 2x + 4 \sin 2x].\end{aligned}$$

EXAMPLE 2.102

Solve $\frac{d^3 y}{dx^3} + y = \sin 3x - \cos^2 \frac{x}{2}$.

Solution. The auxiliary equation of the given differential equation is

$$m^3 + 1 = 0$$

or

$$(m+1)(m^2 - m + 1) = 0$$

and so $m = -1, \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$. Therefore,

$$\text{C.F.} = c_1 e^{-x}$$

$$+ e^{x/2} \left[c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x \right].$$

On the other hand,

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{D^3+1} \left[\sin 3x - \frac{1}{2}(1+\cos x) \right] \\
 &= \frac{1}{D^3+1} \sin 3x - \frac{1}{2(D^3+1)}(1+\cos x) \\
 &= \frac{1}{DD^2+1} \sin 3x - \frac{1}{2(1+D^3)}[1+\cos x] \\
 &= \frac{1}{-9D+1} \sin 3x - \frac{1}{2(1+D^3)}[1+\cos x] \\
 &= \frac{1}{1-9D} \sin 3x - \frac{1}{2}(1+D^3)^{-1}(1) - \frac{1}{2(1+D^3)} \cos x \\
 &= \frac{1+9D}{1-81D^2} \sin 3x - \frac{1}{2}(1-D^3+..)(1) \\
 &\quad - \frac{1}{2+2DD^2} \cos x \\
 &= \frac{1+9D}{1-81(-9)} \sin 3x - \frac{1}{2} - \frac{1}{2-2D} \cos x \\
 &= \frac{1+9D}{730} \sin 3x - \frac{1}{2} - \frac{2+2D}{4-4D^2} \cos x \\
 &= \frac{1}{730} (\sin 3x + 27 \cos 3x) - \frac{1}{2} - \frac{2+2D}{8} \cos x \\
 &= \frac{1}{730} (\sin 3x + 27 \cos 3x) - \frac{1}{2} - \frac{1}{4} \cos x + \frac{1}{4} \sin x \\
 &= \frac{1}{730} (\sin 3x + 27 \cos 3x) - \frac{1}{2} - \frac{1}{4} (\cos x - \sin x).
 \end{aligned}$$

Hence the complete solution is

$$\begin{aligned}
 y &= c_1 e^{-x} + e^{x/2} (c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x) \\
 &\quad + \frac{1}{730} (\sin 3x + 27 \cos 3x) - \frac{1}{2} \\
 &\quad - \frac{1}{4} (\cos x - \sin x).
 \end{aligned}$$

EXAMPLE 2.103

Solve $\frac{d^3 y}{dx^3} - 3 \frac{dy}{dx} + 2y = x^2 e^x$.

Solution. The symbolic form of the given equation is

$$(D^3 - 3D + 2)y = x^2 e^x.$$

Its auxiliary equation is

$$m^3 - 3m + 2 = 0,$$

which yields $m = 1, 1, -4$. Therefore,

$$\text{C.F.} = (c_1 + c_2 x)e^x + c_3 e^{-2x}.$$

Further

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{D^3 - 3D + 2} x^2 e^x \\
 &= e^x \frac{1}{(D+1)^3 - 3(D+1) + 2} x^2 \\
 &= e^x \frac{1}{D^3 + 3D^2} x^2 = e^x \frac{1}{3D^2(1 + \frac{D}{3})} x^2 \\
 &= e^x \frac{1}{3D^2} \left[1 - \frac{D}{3} + \left(\frac{D}{3}\right)^2 - \dots \right] x^2 \\
 &= \frac{e^x}{3D^2} \left[1 - \frac{D}{3} + \frac{D^2}{9} - \dots \right] x^2 \\
 &= \frac{e^x}{3D^2} \left(x^2 - \frac{Dx^2}{3} + \frac{D^2 x^2}{9} \right) \\
 &= \frac{e^x}{3D^2} \left(x^2 - \frac{2x}{3} + \frac{2}{9} \right) \\
 &= \frac{e^x}{3D} \left[\int x^2 dx - \frac{2}{3} \int x dx + \frac{2}{9} \int dx \right] \\
 &= \frac{e^x}{3D} \left[\frac{x^3}{3} - \frac{x^2}{3} + \frac{2}{9} x \right] \\
 &= \frac{e^x}{3} \left[\int \frac{x^3}{3} dx - \int \frac{x^2}{3} dx + \frac{2}{9} \int x dx \right] \\
 &= \frac{e^x}{3} \left[\frac{x^4}{12} - \frac{x^3}{9} + \frac{x^2}{9} \right] = \frac{e^x}{108} [3x^4 - 4x^3 + 4x^2] \\
 &= \frac{x^2 e^x}{108} [3x^2 - 4x + 4].
 \end{aligned}$$

Hence complete solution is

$$\begin{aligned}
 y &= \text{C.F.} + \text{P.I.} \\
 &= (c_1 + c_2 x)e^x + c_3 e^{-2x} + \frac{x^2 e^x}{108} [3x^2 - 4x + 4].
 \end{aligned}$$

EXAMPLE 2.104

Solve $(D^2 - 1)y = x \sin 3x + \cos x$.

Solution. The auxiliary equation for the given differential equation is $m^2 - 1 = 0$, and so $m = \pm 1$. Therefore,

$$\text{C.F.} = c_1 e^x + c_2 e^{-x}.$$

Further,

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{D^2 - 1} (x \sin 3x + \cos x) \\
 &= \frac{1}{D^2 - 1} x (\text{I.P. of } e^{3ix}) + \frac{1}{D^2 - 1} \cos x \\
 &= \text{I.P. of } \frac{1}{D^2 - 1} x e^{3ix} + \frac{1}{(-1^2) - 1} \cos x \\
 &= \text{I.P. of } \left[e^{3ix} \frac{1}{(D + 3i)^2 - 1} x \right] - \frac{1}{2} \cos x \\
 &= \text{I.P. of } \left[e^{3ix} \frac{1}{D^2 + 6iD - 10} x \right] - \frac{\cos x}{2} \\
 &= \text{I.P. of } \left[e^{3ix} \frac{1}{-10(1 - \frac{6}{10}iD - \frac{D^2}{10})} x \right] - \frac{1}{2} \cos x \\
 &= -\frac{1}{10} \text{I.P. of } \left[e^{3ix} \left(1 - \left(\frac{3}{5}iD + \frac{D^2}{10} \right) \right)^{-1} x \right] \\
 &\quad - \frac{1}{2} \cos x \\
 &= -\frac{1}{10} \text{I.P. of } \left[e^{3ix} \left(1 + \frac{3}{5}iD + \frac{D^2}{10} + \dots \right) x \right] \\
 &\quad - \frac{1}{2} \cos x \\
 &= -\frac{1}{10} \text{I.P. of } \left[e^{3ix} \left(x + \frac{3}{5}i \right) \right] - \frac{1}{2} \cos x \\
 &= -\frac{1}{10} \text{I.P. of } \left[(\cos 3x + i \sin 3x) \left(x + \frac{3}{5}i \right) \right] \\
 &\quad - \frac{1}{2} \cos x \\
 &= -\frac{1}{10} \text{I.P. of } [x \cos 3x + ix \sin 3x + \frac{3i}{5} \cos 3x \\
 &\quad - \frac{3}{5} \sin 3x] - \frac{1}{2} \cos x \\
 &= -\frac{1}{10} \text{I.P. of } \left[x \cos 3x - \frac{3}{5} \sin 3x \right. \\
 &\quad \left. + i \left(\frac{3}{5} \cos 3x + x \sin 3x \right) \right] - \frac{1}{2} \cos x \\
 &= -\frac{1}{10} \left(\frac{3}{5} \cos 3x + x \sin 3x \right) - \frac{1}{2} \cos x.
 \end{aligned}$$

Hence the complete solution of the given differential equation is

$$\begin{aligned}
 y &= \text{C.F.} + \text{P.I.} = c_1 e^{-x} + c_2 e^{-x} \\
 &\quad - \frac{1}{10} \left(\frac{3}{5} \cos 3x + x \sin 3x \right) - \frac{1}{2} \cos x.
 \end{aligned}$$

2.17 METHOD OF VARIATION OF PARAMETERS TO FIND PARTICULAR INTEGRAL

Definition 2.23. Let $y_1(x)$, $y_2(x)$, ..., $y_n(x)$ be the functions defined on $[a, b]$ such that each function possesses $n-1$ derivatives on $[a, b]$. Then the determinant

$$W(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

is called the *Wronskian* of the set $\{y_1, y_2, \dots, y_n\}$.

If the Wronskian of a set of n functions on $[a, b]$ is non-zero for atleast one point in $[a, b]$, then the set of n functions is linearly independent.

If the Wronskian is identically zero on $[a, b]$ and each of the function is a solution of the same linear differential equation, then the set of functions is linearly dependent.

The method of variation of parameters is applicable to the differential equation of the form

$$\frac{d^2 y}{dx^2} + p \frac{dy}{dx} + qy = F(x), \quad (92)$$

where p , q , and F are functions of x .

Let the complementary function of (92) be $y = c_1 y_1 + c_2 y_2$. Then y_1 and y_2 satisfy the equation

$$\frac{d^2 y}{dx^2} + p \frac{dy}{dx} + qy = 0 \quad (93)$$

Replacing c_1 and c_2 (regarded as parameters) by unknown functions $u(x)$ and $v(x)$, we assume that particular integral of (92) is

$$y = uy_1 + vy_2. \quad (94)$$

Differentiating (94) with respect to x , we get

$$\begin{aligned}
 y' &= uy_1' + xy_2' + u'y_1 + x'y_2 \\
 &= uy_1' + xy_2', \quad (95)
 \end{aligned}$$

under the assumption that

$$u'y_1 + v'y_2 = 0 \quad (96)$$

Differentiating (95) with respect to x , we get

$$y'' = uy_1'' + vy_2'' + u'y_1 + v'y_2. \quad (97)$$

Substituting the values of y , y' , and y'' from (94), (95), and (97) in (92), we have

$$\begin{aligned}
 uy_1'' + vy_2'' + u'y_1 + v'y_2 + p(uy_1' + vy_2') \\
 + q(uy_1 + vy_2) = F(x)
 \end{aligned}$$

or

$$u(y_1'' + py_1' + qy_1) + v(y_2'' + py_2' + qy_2) + u'y_1' + v'y_2' = F(x).$$

Since y_1, y_2 satisfy (93), the above expression reduces to

$$u'y_1' + v'y_2' = F(x). \quad (98)$$

Solving (96) and (98), we get

$$u' = -\frac{y_2 F(x)}{W} \text{ and } v' = \frac{y_1 F(x)}{W},$$

where

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2.$$

Integrating, we have

$$u = -\int \frac{y_2 F(x)}{W} dx, \quad v = \int \frac{y_1 F(x)}{W} dx.$$

Substituting the value of u and v in (94), we get

$$\text{P.I.} = y = -y_1 \int \frac{y_2 F(x)}{W} dx + y_2 \int \frac{y_1 F(x)}{W} dx.$$

EXAMPLE 2.105

Using method of variation of parameters, solve

$$\frac{d^2 y}{dx^2} - 6 \frac{dy}{dx} + 9y = \frac{e^3 x}{x^2}.$$

Solution. The auxiliary equation for the given differential equation is

$$m^2 - 6m + 9 = 0,$$

which yields $m = 3, 3$. Therefore,

$$\text{C.F.} = (c_1 + c_2 x) e^{3x}.$$

Thus, we get

$$y_1 = e^{3x} \text{ and } y_2 = x e^{3x}.$$

The Wronskian of y_1, y_2 is

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} e^{3x} & x e^{3x} \\ 3e^{3x} & (3x+1)e^{3x} \end{vmatrix} \\ &= e^{6x}. \end{aligned}$$

Therefore,

$$\begin{aligned} \text{P.I.} &= -y_1 \int \frac{y_2 F(x)}{W} dx + y_2 \int \frac{y_1 F(x)}{W} dx \\ &= -e^{3x} \int \frac{x e^{3x} \cdot e^{3x}}{x^2 e^{6x}} dx + x e^{3x} \int \frac{e^{3x} \cdot e^{3x}}{x^2 e^{6x}} dx \\ &= -e^{3x} \int \frac{dx}{x} + x e^{3x} \int \frac{1}{x^2} dx \\ &= -e^{3x} \log x + x e^{3x} \left(-\frac{1}{x} \right) = -e^{3x} (\log x + 1). \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} = (c_1 + c_2 x) e^{3x} - e^{3x} (\log x + 1) \\ &= [k + c_2 x - \log x] e^{3x} \text{ where } k = c_1 - 1. \end{aligned}$$

EXAMPLE 2.106

Using method of variation of parameters, solve $\frac{d^2 y}{dx^2} + y = \sec x$.

Solution. The auxiliary equation for the given differential equation is $m^2 + 1 = 0$ and so $m = \pm i$. Thus

$$\text{C.F.} = c_1 \cos x + c_2 \sin x.$$

To find P.I., let

$$y_1 = \cos x \text{ and } y_2 = \sin x.$$

Then

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1.$$

Therefore,

$$\begin{aligned} \text{P.I.} &= -y_1 \int \frac{y_2 F(x)}{W} dx + y_2 \int \frac{y_1 F(x)}{W} dx \\ &= -\cos x \int \frac{\sin x \sec x}{1} dx + \sin x \int \frac{\cos x \sec x}{1} dx \\ &= \cos x \log \cos x + x \sin x. \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} = c_1 \cos x + c_2 \sin x \\ &\quad + \cos x \log \cos x + x \sin x. \end{aligned}$$

EXAMPLE 2.107

Solve the given equation using method of variation of parameters

$$\frac{d^2 y}{dx^2} + y = \operatorname{cosec} x.$$

Solution. The symbolic form of the differential equation is

$$(D^2+1)y = \operatorname{cosec} x.$$

Its auxiliary equation is $m^2 + 1 = 0$ and so $m = \pm i$.
Therefore,

$$\text{C.F.} = c_1 \cos x + c_2 \sin x.$$

To find P.I., let

$$y_1 = \cos x \quad \text{and} \quad y_2 = \sin x.$$

Then Wronskian

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

Therefore,

$$\begin{aligned} \text{P.I.} &= -y_1 \int \frac{y_2 F(x)}{W} dx + y_2 \int \frac{y_1 F(x)}{W} dx \\ &= -\cos x \int \sin x \operatorname{cosec} x dx \\ &\quad + \sin x \int \cos x \operatorname{cosec} x dx \\ &= -\cos x \int dx + \sin x \int \frac{\cos x}{\sin x} dx \\ &= -x \cos x + \sin x \log \sin x. \end{aligned}$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos x + c_2 \sin x - x \cos x + \sin x \log \sin x.$$

EXAMPLE 2.108

Solve the given equation using method of variation of parameters

$$\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} = e^x \sin x.$$

Solution. The auxiliary equation is $m^2 - 2m = 0$ or $m(m-2) = 0$ and so $m = 0, 2$. Hence

$$\text{C.F.} = c_1 + c_2 e^{2x}.$$

Now let

$$y_1 = 1 \quad \text{and} \quad y_2 = e^{2x}.$$

Then the Wronskian of y_1, y_2 is

$$W = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} = 2e^{2x}.$$

Therefore,

$$\begin{aligned} \text{P.I.} &= -y_1 \int \frac{y_2 F(x)}{W} dx + y_2 \int \frac{y_1 F(x)}{W} dx \\ &= -\int \frac{e^{2x} \cdot e^x \sin x}{2e^{2x}} dx + e^{2x} \int \frac{e^x \sin x}{2e^{2x}} dx \\ &= -\frac{1}{2} \int e^x \sin x dx + \frac{e^{2x}}{2} \int e^{-x} \sin x dx \\ &= -\frac{1}{2} e^x \sin x. \end{aligned}$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 + c_2 e^{2x} - \frac{1}{2} e^x \sin x.$$

EXAMPLE 2.109

Solve $y'' - 2y' + 2y = e^x \tan x$.

Solution. The auxiliary equation is $m^2 - 2m + 2 = 0$ and so $m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$. Hence

$$\text{C.F.} = e^x (c_1 \cos x + c_2 \sin x).$$

Let

$$y_1 = e^x \cos x \quad \text{and} \quad y_2 = e^x \sin x.$$

Then the Wronskian of y_1, y_2 is

$$\begin{aligned} W &= \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x (\cos x - \sin x) & e^x (\cos x + \sin x) \end{vmatrix} \\ &= e^{2x}. \end{aligned}$$

Therefore,

$$\begin{aligned} \text{P.I.} &= -y_1 \int \frac{y_2 F(x)}{W} dx + y_2 \int \frac{y_1 F(x)}{W} dx \\ &= -e^x \cos x \int \frac{e^x \sin x e^x \tan x}{e^{2x}} dx \\ &\quad + e^x \sin x \int \frac{e^x \cos x e^x \tan x}{e^{2x}} dx \\ &= -e^x \cos x \int (\sec x - \cos x) dx \\ &\quad + e^x \sin x \int \sin x dx \\ &= -e^{-x} \cos x [\log(\sec x + \tan x) - \sin x] \\ &\quad - e^x \sin x \cos x \\ &= -e^{-x} \cos x \log(\sec x + \tan x). \end{aligned}$$

Hence, the complete solution is

$$y = \text{C.F.} + \text{P.I.} = e^x (c_1 \cos x + c_2 \sin x) - e^{-x} \cos x \log(\sec x + \tan x).$$

EXAMPLE 2.110

Using method of variation of parameters, solve the differential equation

$$\frac{d^2y}{dx^2} + 4y = \tan 2x.$$

Solution. The symbolic form of the given differential equation is

$$(D^2 + 4)y = \tan 2x.$$

Its auxiliary equation is $m^2 + 4 = 0$, which yields $m = \pm 2i$. Thus

$$\text{C.F.} = c_1 \cos 2x + c_2 \sin 2x.$$

To find P.I., let

$$y_1 = \cos 2x \text{ and } y_2 = \sin 2x.$$

Then Wronskian W is

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2 \sin 2x & 2 \cos 2x \end{vmatrix} = 2.$$

Hence

$$\begin{aligned} \text{P.I.} &= -y_1 \int \frac{y_2 F(x)}{W} dx + y_2 \int \frac{y_1 F(x)}{W} dx \\ &= -\frac{\cos 2x}{2} \int \sin 2x \tan 2x dx \\ &\quad + \frac{\sin 2x}{x} \int \cos 2x \tan 2x dx \\ &= -\frac{\cos 2x}{2} \int \frac{\sin^2 2x}{\cos 2x} dx \\ &\quad + \frac{\sin 2x}{2} \int \cos 2x \tan 2x dx \\ &= -\frac{\cos 2x}{2} \int \frac{1 - \cos^2 2x}{\cos 2x} dx \\ &\quad + \frac{\sin 2x}{2} \int \cos 2x \tan 2x dx \\ &= -\frac{1}{2} \cos 2x \int (\sec 2x - \cos 2x) dx \\ &\quad + \frac{1}{2} \sin 2x \int \sin 2x dx \\ &= -\frac{1}{4} \cos 2x [\log(\sec 2x + \tan 2x) - \sin 2x] \\ &\quad - \frac{1}{4} \sin 2x \cos 2x \\ &= -\frac{1}{4} \cos 2x \log(\sec 2x + \tan 2x). \end{aligned}$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} \cos 2x \log(\sec 2x + \tan 2x).$$

EXAMPLE 2.111

Solve the equation using the method of variation of parameters

$$\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + y = e^x \log x.$$

Solution. The auxiliary equation is $m^2 - 2m + 1 = 0$ and so $m = 1, 1$. Thus

$$\text{C.F.} = (c_1 + c_2 x)e^x.$$

To find P.I., let $y_1 = e^x$, $y_2 = xe^x$. Then

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix} = e^{2x}.$$

Therefore,

$$\begin{aligned} \text{P.I.} &= -y_1 \int \frac{y_2 F(x)}{W} dx + y_2 \int \frac{y_1 F(x)}{W} dx \\ &= -e^x \int \frac{x e^x \cdot e^x}{e^{2x}} \log x dx + x e^x \int \frac{e^x \cdot e^x \log x}{e^{2x}} dx \\ &= -e^x \int x \log x dx + x e^x \int \log x dx \\ &= -e^x \left[\frac{x^2}{2} \log x - \int \frac{x^2}{2x} dx \right] \\ &\quad + x e^x \left[x \log x - \int \frac{x}{x} dx \right] \\ &= -e^x \left(\frac{x^2}{2} \log x - \frac{x^2}{4} \right) + x^2 e^x \log x - x^2 e^x \\ &= -e^x \frac{x^2}{2} \log x + \frac{x^2}{4} e^x + x^2 e^x \log x - x^2 e^x \\ &= \frac{1}{2} e^x x^2 \log x - \frac{3}{4} x^2 e^x = \frac{1}{4} x^2 e^x (2 \log x - 3). \end{aligned}$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2 x)e^x + \frac{1}{4} x^2 e^x (2 \log x - 3).$$

2.18 DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

In this section, we shall consider differential equations with variable coefficients which can be reduced to linear differential equations with constant coefficients.

The general form of a linear equation of second order is

$$\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = R,$$

where P , Q and R are functions of x only. For example,

$$\frac{d^2y}{dx^2} + x \frac{dy}{dx} + x^2y = \cos x$$

is a linear equation of second order.

If the coefficients P and Q are constants, then such type of equation can be solved by finding complementary function and particular integral as discussed earlier. But, if P and Q are not constants but variable, then there is no general method to solve such problem. In this section, we discuss three special methods to handle such problems.

(A) Method of Solution by Changing Independent Variable: Let

$$\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = R, \quad (99)$$

be the given linear equation of second order. We change the independent variable x to z by taking then

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx},$$

and

$$\frac{d^2y}{dx^2} = \frac{dy}{dz} \left(\frac{d^2z}{dx^2} \right) + \frac{d^2y}{dz^2} \left[\frac{dz}{dx} \right]^2.$$

Substituting these values in (99), we get

$$\frac{d^2y}{dz^2} \left[\frac{dz}{dx} \right]^2 + \frac{dy}{dz} \left(\frac{d^2z}{dx^2} \right) + P \frac{dy}{dz} \cdot \frac{dz}{dx} + Qy = R$$

or

$$\frac{d^2y}{dz^2} \left[\frac{dz}{dx} \right]^2 + \frac{dy}{dz} \left[\frac{d^2z}{dx^2} + P \frac{dz}{dx} \right] + Qy = R$$

or

$$\frac{d^2y}{dz^2} + \frac{dy}{dz} \left[\frac{\frac{d^2z}{dx^2} + P \frac{dz}{dx}}{\left(\frac{dz}{dx} \right)^2} \right] + \frac{Q}{\left[\frac{dz}{dx} \right]^2} y = \frac{R}{\left[\frac{dz}{dx} \right]^2}$$

or

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1,$$

where

$$P_1 = \frac{\frac{d^2z}{dx^2} + P \frac{dz}{dx}}{\left[\frac{dz}{dx} \right]^2},$$

$$Q_1 = \frac{Q}{\left[\frac{dz}{dx} \right]^2},$$

and

$$R_1 = \frac{R}{\left[\frac{dz}{dx} \right]^2}.$$

Using the functional relation between z and x , it follows that P_1 , Q_1 and R_1 are functions of x .

Choose z so that $P_1 = 0$, that is,

$$\frac{d^2z}{dx^2} + P \frac{dz}{dx} = 0,$$

which yields

$$\frac{dz}{dx} = e^{-\int P dx}$$

or

$$z = \int e^{-\int P dx} dx.$$

If for this value of z , Q_1 becomes constant or a constant divided by z^2 , then the equation (99) can be integrated to find its solution.

EXAMPLE 2.112

Solve $\frac{d^2y}{dx^2} + \cot x \frac{dy}{dx} + 4y \cos ec^2 x = 0$.

Solution. Comparing with the standard form, we get

$$P = \cot x, \quad Q = 4 \cos ec^2 x \text{ and } R = 0.$$

If we choose z such that $P_1 = 0$, then z is given by

$$\begin{aligned} z &= \int e^{-\int P dx} dx \\ &= \int e^{-\int \cot x dx} dx \\ &= \int e^{\log \cos ec x} dx \\ &= \int \cos ec x dx \\ &= \log \tan \frac{x}{2}. \end{aligned}$$

Then

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{4 \cos ec^2 x}{\cos ec^2 x} = 4,$$

and

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{0}{\cos ec^2 x} = 0.$$

Therefore the given equation reduces to

$$\frac{d^2 y}{dz^2} + 4y = 0,$$

or, in symbolic form,

$$(D^2 + 4)y = 0.$$

The auxiliary equation for this differential equation is $m^2 + 4 = 0$, which yields $m = \pm 2i$. Therefore the solution of the given equation is

$$\begin{aligned} y &= c_1 \cos 2z + c_2 \sin 2z \\ &= c_1 \cos \left(2 \log \tan \frac{x}{2} \right) + c_2 \sin \left(2 \log \tan \frac{x}{2} \right). \end{aligned}$$

EXAMPLE 2.113

Solve

$$x \frac{d^2 y}{dx^2} - \frac{dy}{dx} - 4x^3 y = x^5.$$

Solution. Dividing throughout by x , the given differential equation reduces to

$$\frac{d^2 y}{dx^2} - \frac{1}{x} \frac{dy}{dx} - 4x^2 y = x^4.$$

Comparing with the standard form, we have

$$P = -\frac{1}{x}, \quad Q = -4x^2, \quad \text{and} \quad R = x^4.$$

Choose z so that $P_1 = 0$. Then

$$\begin{aligned} z &= \int e^{-\int P dx} dx = \int e^{\int \frac{1}{x} dx} dx \\ &= \int e^{\log x} dx = \int x dx = \frac{x^2}{2}. \end{aligned}$$

Further,

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{-4x^2}{x^2} = -4,$$

and

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{x^4}{x^2} = x^2.$$

Hence the given equation reduces to

$$\frac{d^2 y}{dz^2} - 4y = x^2 = 2z$$

or

$$(D^2 - 4)y = 2z.$$

The auxiliary equation for this differential equation is $m^2 - 4 = 0$ and so $m = \pm 2$. Therefore

$$\text{C.F.} = c_1 e^{2z} + c_2 e^{-2z} = c_1 e^{x^2} + c_2 e^{-x^2}.$$

Now

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - 4} (2z) = \frac{-2}{4} \left(1 - \frac{D^2}{4} \right)^{-1} (z) \\ &= -\frac{1}{2} \left[1 + \frac{D^2}{4} + \dots \right] (z) \\ &= -\frac{z}{2} = -\frac{x^2}{4}. \end{aligned}$$

Hence the complete solution of the given differential equation is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{x^2} + c_2 e^{-x^2} - \frac{1}{4} x^2$$

EXAMPLE 2.114

Solve

$$x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2y = x^3.$$

(This equation is Cauchy-Euler equation and has also been solved in Example 2.121 by taking $t = \log x$).

Solution. Dividing the given equation throughout by x^2 , we get

$$\frac{d^2 y}{dx^2} - \frac{2}{x} \frac{dy}{dx} + \frac{2}{x^2} y = x.$$

Comparing with the standard form, we have

$$P = -\frac{2}{x}, \quad Q = \frac{2}{x^2} \quad \text{and} \quad R = x.$$

Choosing z such that $P_1 = 0$, we have

$$\begin{aligned} z &= \int e^{-\int P dx} dx = \int e^2 \int \frac{1}{x^3} dx \\ &= \int e^{2 \log x} dx = \int x^2 dx = \frac{x^3}{3}. \end{aligned}$$

Therefore the given equation reduces to

$$\frac{d^2 y}{dz^2} + Q_1 y = R_1,$$

where

$$Q_1 = \frac{Q}{\left[\frac{dz}{dx}\right]^2} = \frac{\frac{2}{x^2}}{(x^2)^2} = \frac{2}{x^6} = \frac{2}{9z^2}$$

and

$$R_1 = \frac{R}{\left[\frac{dz}{dx}\right]^2} = \frac{x}{x^4} = \frac{1}{x^3} = \frac{1}{3z}.$$

Hence the equation reduces to

$$\frac{d^2 y}{dx} + \frac{2y}{9z^2} = \frac{1}{3z}$$

or

$$9z^2 \frac{d^2 y}{dz^2} + 2y = 3z.$$

Let $X = \log z$ so that $z = e^X$ and (see Example 2.121), $z^2 \frac{d^2 y}{dz^2} = D(D-1)y$. Thus the equation now reduces to

$$[9D(D-1) + 2]y = 3e^X$$

or

$$(9D^2 - 9D + 2)y = 3e^X$$

The auxiliary equation for this symbolic equation is

$$m^2 - 9m + 2 = 0,$$

which yields $m = \frac{2}{3}, \frac{1}{3}$. Hence the complementary function is given by

$$\begin{aligned} \text{C.F.} &= c_1 e^{\frac{2}{3}X} + c_2 e^{\frac{1}{3}X} \\ &= c_1 e^{\frac{2}{3} \log z} + c_2 e^{\frac{1}{3} \log z} \\ &= c_1 z^{\frac{2}{3}} + c_2 z^{\frac{1}{3}} = c_3 x^2 + c_4 x. \end{aligned}$$

Further,

$$\begin{aligned} \text{P.I.} &= 3 \frac{1}{9D^2 - 9D + 2} e^X \\ &= 3 \frac{1}{9 - 9 + 2} e^X = \frac{3}{2} e^X = \frac{3}{2} e^{\log z} \\ &= \frac{3}{2} z = \frac{3}{2} \left(\frac{x^3}{3} \right) \\ &= \frac{1}{2} x^3. \end{aligned}$$

Hence the complete solution of the given differential equation is

$$y = \text{C.F.} + \text{P.I.} = c_3 x^2 + c_4 x + \frac{1}{2} x^3.$$

(B) Method of Solution by Changing the Dependent Variable

This method is also called the “Method of Removing First Derivative. Let

$$\frac{d^2 y}{dx^2} + P \frac{dy}{dx} + Qy = R \quad (100)$$

be the given linear equation of second order. We change the dependent variable y by taking $y = vz$. Then

$$\begin{aligned} \frac{dy}{dx} &= v \frac{dz}{dx} + z \frac{dv}{dx}, \\ \frac{d^2 y}{dx^2} &= v \frac{d^2 z}{dx^2} + 2 \frac{dv}{dx} \cdot \frac{dz}{dx} + z \frac{d^2 v}{dx^2}. \end{aligned}$$

Therefore equation (100) reduces to

$$\begin{aligned} z \frac{d^2 v}{dx^2} + \left(2 \frac{dz}{dx} + Pz \right) \frac{dv}{dx} + \left(\frac{d^2 z}{dx^2} + P \frac{dz}{dx} + Qz \right) v \\ = R \end{aligned}$$

or

$$\begin{aligned} \frac{d^2 v}{dx^2} + \left(P + 2 \frac{dz}{z dx} \right) \frac{dv}{dx} + \frac{1}{z} \left(\frac{d^2 z}{dx^2} + P \frac{dz}{dx} + Qz \right) v \\ = \frac{R}{z} \end{aligned}$$

or

$$\frac{d^2 v}{dx^2} + P_1 \frac{dv}{dx} + Q_1 v = R_1, \quad (101)$$

where

$$P_1 = P + \frac{2}{z} \frac{dz}{dx},$$

$$Q_1 = \frac{1}{z} \left(\frac{d^2 z}{dx^2} + P \frac{dz}{dx} + Qz \right), \quad R_1 = \frac{R}{z}.$$

By making proper choice of z , any desired value can be assigned to P_1 or Q_1 . In particular, if $P_1 = 0$, then

$$P + \frac{2}{z} \cdot \frac{dz}{dx} = 0$$

or

$$\frac{dz}{dx} = -\frac{Pz}{2}$$

or

$$\frac{dz}{z} = -\frac{1}{2} P dx.$$

Integrating, we get

$$\log z = -\frac{1}{2} \int P dx$$

or

$$z = e^{-\frac{1}{2} \int P dx}.$$

Therefore

$$\frac{dz}{dx} = -\frac{P}{2} e^{-\frac{1}{2} \int P dx}$$

and

$$\frac{d^2 z}{dx^2} = e^{-\frac{1}{2} \int P dx} \left[\frac{P^2}{4} - \frac{1}{2} \frac{dP}{dx} \right].$$

Putting these values in Q_1 , we get

$$Q_1 = Q - \frac{1}{2} \frac{dP}{dx} - \frac{P^2}{4}.$$

Hence the equation (101) reduces to

$$\frac{d^2 v}{dx^2} + \left(Q - \frac{1}{2} \frac{dP}{dx} - \frac{P^2}{4} \right) v = R_1. \quad (102)$$

The equation (102) is called the *normal form* of the given differential equation (100) and we observe that this equation *does not have first derivative*. That is why, the present method is called the “Method of Removing First Derivative”. The normal form (102) can be solved by using already discussed methods.

EXAMPLE 2.115

Solve $\frac{d^2 y}{dx^2} - 2 \tan x \frac{dy}{dx} - 5y = 0$ by the method of removing first derivative.

Solution. Comparing the given equation with standard form, we have

$$P = -2 \tan x, \quad Q = -5, \quad R = 0.$$

Therefore

$$\begin{aligned} Q_1 &= Q - \frac{1}{2} \frac{dP}{dx} - \frac{P^2}{4} = -5 + \sec^2 x - \tan^2 x \\ &= -5 + (1 + \tan^2 x) - \tan^2 x = -4, \end{aligned}$$

and

$$z = e^{-\frac{1}{2} \int P dx} = e^{\int \tan x dx} = \sec x.$$

The normal form of the given equation is

$$\frac{d^2 v}{dx^2} + Q_1 v = R_1$$

or

$$\frac{d^2 v}{dx^2} - 4v = 0$$

or

$$(D^2 - 4)v = 0.$$

The auxiliary equation for this symbolic form is $m^2 - 4 = 0$. Therefore $m = \pm 2$ and so

$$v = c_1 e^{2x} + c_2 e^{-2x}$$

Hence the required solution is

$$y = vz = (c_1 e^{2x} + c_2 e^{-2x}) \sec x.$$

EXAMPLE 2.116

Solve

$$\frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + (4x^2 - 3)y = e^{x^2}.$$

Solution. We have

$$\begin{aligned} Q_1 &= Q - \frac{1}{2} \frac{dP}{dx} - \frac{P^2}{4} \\ &= 4x^2 - 3 - \frac{1}{2}(-4) - 4x^2 = -1 \end{aligned}$$

and

$$z = e^{-\frac{1}{2} \int P dx} = e^2 \int x dx = e^{x^2}.$$

The normal form of the given equation is

$$\frac{d^2 v}{dx^2} - v = \frac{e^{x^2}}{z} = \frac{e^{x^2}}{e^{x^2}} = 1$$

or

$$(D^2 - 1)v = 1.$$

Therefore A.E. is $m^2 - 1 = 0$, which yields $m = \pm 1$.
Thus

$$\text{C.F.} = c_1 e^x + c_2 e^{-x}.$$

Moreover

$$\text{P.I.} = -1.$$

Hence

$$v = \text{C.F.} + \text{P.I.} = c_1 e^x + c_2 e^{-x} - 1.$$

But $y = vz$. Therefore

$$y = (c_1 e^x + c_2 e^{-x} - 1)e^{x^2}.$$

(C) Method of Undetermined Coefficients

This method is used to find Particular integral of the differential equation $F(D) = X$, where the input (forcing) function X consists of the sum of the terms, each of which possesses a finite number of essentially different derivatives. A trial solution consisting of terms in X and their finite derivatives is considered. Putting the values of the derivatives of the trial solution in $f(D)$ and comparing the coefficient on both sides of $f(D) = X$, the P.I. can be found. Obviously the method fails if X consists of terms like $\sec x$ and $\tan x$ having infinite number of different derivatives. Further, if any term in the trial solution is a part of the complimentary function, then that terms should be multiplied by x and then tried.

EXAMPLE 2.117

Solve $\frac{d^2 y}{dx^2} + y = e^x + \sin x$.

Solution. The symbolic form of the given differential equation is

$$(D^2 + 1)y = e^x + \sin x.$$

The auxiliary equation is $m^2 + 1 = 0$. Therefore $m = \pm i$ and so

$$\text{C.F.} = c_1 \cos x + c_2 \sin x.$$

The forcing function consists of terms e^x and $\sin x$. Their derivatives are e^x and $\cos x$. So consider the trial solution.

$$y = a e^x + b x \sin x + c x \cos x.$$

Then

$$\frac{dy}{dx} = a e^x + b(x \cos x + \sin x) + c(\cos x - x \sin x)$$

$$\frac{d^2 y}{dx^2} = a e^x + b[\cos x - x \sin x + \cos x]$$

$$+ c[-\sin x - \sin x - x \cos x]$$

$$= a e^x + 2b \cos x - b x \sin x - 2c \sin x - c x \cos x$$

$$= a e^x + (2b - cx) \cos x - (bx + 2c) \sin x.$$

Substituting the values of $\frac{d^2 y}{dx^2}$ and y in the given equation, we get

$$2a e^x + 2b \cos x - 2c \sin x = e^x + \sin x.$$

Comparing corresponding coefficients, we have

$$2a = 1, 2b = 0 \text{ and } 2c = -1.$$

Thus $a = \frac{1}{2}, b = 0, c = -\frac{1}{2}$ and so

$$\text{P.I.} = \frac{1}{2} e^x - \frac{1}{2} x \cos x.$$

Hence the solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{2} e^x - \frac{1}{2} x \cos x.$$

EXAMPLE 2.118

Solve $\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 4y = e^x \sin x$.

Solution. The symbolic form of the given differential equation is

$$m^2 - 4m + 4 = 0$$

which yields $m = 2, 2$. Hence

$$\text{C.F.} = (c_1 + c_2 x)e^{2x}.$$

The forcing function is $e^x \sin x$. Its derivative is $e^x \cos x + e^x \sin x$. Therefore, we consider

$$y = a e^x \sin x + b e^x \cos x$$

as the trial solution. Then

$$\begin{aligned}\frac{dy}{dx} &= a(e^x \cos x + e^x \sin x) + b(e^x \cos x - e^x \sin x) \\ &= (a+b)e^x \cos x + (a-b)e^x \sin x, \\ \frac{d^2y}{dx^2} &= (a+b)[e^x \cos x - e^x \sin x] \\ &\quad + (a-b)[e^x \cos x + e^x \sin x] \\ &= 2ae^x \cos x - 2be^x \sin x.\end{aligned}$$

Substituting the values of $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$ and y in the given differential equation, we get

$$\begin{aligned}2ae^x \cos x - 2be^x \sin x - 4(a+b)e^x \cos x \\ - 4(a-b)e^x \sin x \\ + 4ae^x \sin x + 4be^x \cos x = e^x \sin x\end{aligned}$$

or

$$-2ae^x \cos x + 2be^x \sin x = e^x \sin x.$$

Comparing coefficients, we get $2b = 1$ or $b = \frac{1}{2}$.
Hence

$$\text{P.I.} = \frac{1}{2}e^x \cos x$$

and so the complete solution of the given differential equation is

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2x)e^{2x} + \frac{1}{2}e^x \cos x.$$

(D) Method of Reduction of Order

This method is used to find the complete solution of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$, where P , Q and R are function of x only, and when part of complementary function is known. So, let u , a function of x , be a part of the complementary function of the above differential equation. Then

$$\frac{d^2u}{dx^2} + P\frac{du}{dx} + QU = 0. \quad (103)$$

Let $y = uv$ be the complete solution of the given differential equation, where v is also a function of x . Then

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx}, \\ \frac{d^2y}{dx^2} &= u\frac{d^2v}{dx^2} + 2\frac{du}{dx} \cdot \frac{dv}{dx} + v\frac{d^2u}{dx^2}.\end{aligned}$$

Substituting these values of $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$ and y in the given equation, we get

$$u\frac{d^2v}{dx^2} + 2\frac{du}{dx} \cdot \frac{dv}{dx} + v\frac{d^2u}{dx^2} + P\left(u\frac{dv}{dx} + \frac{du}{dx}\right) + Quv = R$$

or

$$u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + Pu\right) \cdot \frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)v = R$$

or, using (103)

$$u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + Pu\right) \frac{dv}{dx} = R$$

or, division by u yields,

$$\frac{d^2v}{dx^2} + \left(\frac{2}{u}\frac{du}{dx} + P\right) \frac{dv}{dx} = \frac{R}{u}$$

or, taking $\frac{dv}{dx} = z$, we get

$$\frac{dz}{dx} + \left(\frac{2}{u}\frac{du}{dx} + P\right)z = \frac{R}{u}, \quad (104)$$

which is a first order differential equation in z and x .

The integration factor for (104) is

$$\text{I.F.} = e^{\int \left(\frac{2}{u}\frac{du}{dx} + P\right)dx} = e^{\left(\int \frac{2}{u}dx + \int Pdx\right)} = u^2 e^{\int Pdx}.$$

Therefore, the solution of (104) is

$$zu^2 e^{\int Pdx} = \int \frac{R}{u} \left(u^2 e^{\int Pdx}\right) dx + c_1$$

or

$$z = \frac{1}{u^2 e^{\int Pdx}} \left[\int \frac{R}{u} \left(u^2 e^{\int Pdx}\right) dx + c_1 \right]$$

or

$$\frac{dv}{dx} = \frac{1}{u^2} e^{-\int Pdx} \left[\int Ru e^{\int Pdx} dx + c_1 \right]. \quad (105)$$

Integrating (105) with respect to x , we get

$$v = \int \frac{1}{u^2} e^{-\int Pdx} \left[\int Ru e^{\int Pdx} dx + c_1 \right] dx + c_2,$$

where c_1 and c_2 are constants of integration. Hence the complete solution of the given differential equation is

$y = uv$

$$= u \left\{ \int \frac{1}{u^2} e^{-\int Pdx} \left[\int Ru e^{\int Pdx} dx + c_1 \right] dx \right\} + c_2 u.$$

EXAMPLE 2.119

Find the complete solution of

$$\frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + (4x^2 - 2)y = 0,$$

if $y = e^{x^2}$ is an integral included in the complementary solution.

Solution. Let $y = uv$, where $u = e^{x^2}$ be the complete solution of the given equation.

Then

$$\frac{d^2v}{dx^2} + \left[P + \frac{2}{u} \frac{du}{dx} \right] \frac{dv}{dx} = 0,$$

where

$$P = -4x, \quad Q = 4x^2 - 2 \text{ and } R = 0.$$

Thus

$$\frac{d^2v}{dx^2} + \left[-4x + \frac{2}{e^{x^2}} (2xe^{x^2}) \right] \frac{dv}{dx} = 0$$

or

$$\frac{d^2v}{dx^2} + [4x - 4x] \frac{dv}{dx} = 0$$

or

$$\frac{d^2v}{dx^2} = 0.$$

Integrating, we get

$$\frac{dv}{dx} = c_1.$$

Integrating once more, we get

$$v = c_1x + c_2.$$

Hence the complete solution is

$$y = uv = e^{x^2}[c_1x + c_2].$$

EXAMPLE 2.120

Solve $\sin^2 x \frac{d^2y}{dx^2} = 2y$, given that $y = \cot x$ is an integral included in the complementary function.

Solution. The given equation is

$$\frac{d^2y}{dx^2} - (2 \cos ec^2 x)y = 0.$$

Comparing with

$$\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = R,$$

we get

$$P = 0, \quad Q = -2 \cos ec^2 x, \quad R = 0.$$

Therefore, putting $y = uv = (\cot x)v$, the reduced equation is

$$\frac{d^2v}{dx^2} + \left[P + \frac{2}{u} \frac{du}{dx} \right] \frac{dv}{dx} = 0,$$

that is,

$$\frac{d^2v}{dx^2} + \left[\frac{2}{\cot x} (-\cos ec^2 x) \right] \frac{dv}{dx} = 0$$

or

$$\cot x \frac{d^2v}{dx^2} - 2 \cos ec^2 x \frac{dv}{dx} = 0. \quad (106)$$

Let $\frac{dv}{dx} = z$. Then (106) reduces to

$$\cot x \frac{dz}{dx} - (2 \cos ec^2 x)z = 0$$

or

$$\frac{dz}{z} = 2 \frac{\cos ec^2 x}{\cot x} dx. \quad (107)$$

Integrating (107), we get

$$\log z = -2 \log \cot x + \log c_1$$

or

$$\log z + \log \cot^2 x = \log c_1$$

or

$$z = c_1 \tan^2 x$$

or

$$\frac{dv}{dx} = c_1 \tan^2 x = c_1 (\sec^2 x - 1). \quad (108)$$

Integrating (108), we get

$$v = c_1 (\tan x - x) + c_2,$$

where c_1 and c_2 are constants of integration.

Hence the complete solution is

$$\begin{aligned} y &= uv = \cot x [c_1 (\tan x - x) + c_2] \\ &= c_1 (1 - x \cot x) + c_2 \cot x. \end{aligned}$$

(E) Cauchy-Euler Homogeneous Linear Equation

Consider the following differential equation with variable coefficients:

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} x \frac{dy}{dx} + a_n = F(x), \quad (109)$$

where a_i are constant and F is a function of x . This equation is known as *Cauchy-Euler homogeneous linear equation* (or *equidimensional equation*). The Cauchy-Euler homogeneous linear equation can be reduced to linear differential equation with constant coefficients by putting $x = e^t$ or $t = \log x$. Then

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x}$$

or

$$x \frac{dy}{dx} = \frac{dy}{dt} = Dy.$$

Now

$$\begin{aligned} \frac{d^2 y}{dx^2} &= \frac{d}{dx} \left(\frac{1}{x} \frac{dy}{dt} \right) = -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x} \frac{d^2 y}{dt^2} \cdot \frac{dt}{dx} \\ &= -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x^2} \frac{d^2 y}{dt^2} \end{aligned}$$

and so

$$x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dt^2} - \frac{dy}{dt} = D_2 y - Dy = D(D-1)y.$$

Similarly,

$$x^3 \frac{d^3 y}{dx^3} = D(D-1)(D-2)y$$

and so on. Putting these values in (109), we obtain a linear differential equation with constant coefficients which can be solved by using the methods discussed already.

EXAMPLE 2.121

Solve $x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2y = x^3$.

Solution. This is a Cauchy-Euler equation. Putting $x = e^t$ or $t = \log x$, we have $x \frac{dy}{dx} = Dy$, $x^2 \frac{d^2 y}{dx^2} = D(D-1)y$. Hence the given equation transforms to

$$(D(D-1) - 2D + 2)y = e^{3t}$$

or

$$(D^2 - 3D + 2)y = e^{3t},$$

which is a linear differential equation with constant coefficient. The auxiliary equation is $m^2 - 3m + 2 = 0$

and so $m = 1, 2$. Therefore

$$\text{C.F.} = c_1 e^t + c_2 e^{2t}.$$

The particular integral is

$$\begin{aligned} \text{P.I.} &= \frac{1}{f(D)} F(x) = \frac{1}{D^2 - 3D + 2} e^{3t} \\ &= \frac{1}{9 - 9 + 2} e^{3t} = \frac{1}{2} e^{3t}. \end{aligned}$$

Hence the complete solution is

$$y = c_1 e^t + c_2 e^{2t} + \frac{1}{2} e^{3t}$$

Returning back to the variable x , we have

$$y = c_1 x + c_2 x^2 + \frac{1}{2} x^3.$$

EXAMPLE 2.122

Solve the Cauchy-Euler equation

$$x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = \log x.$$

Solution. Putting $x = e^t$, we have

$$x \frac{dy}{dx} = Dy \text{ and } x^2 \frac{d^2 y}{dx^2} = D(D-1)y$$

and so the equation transforms to

$$(D(D-1) - D + 1)y = t$$

or

$$(D-1)^2 y = t.$$

The complementary function is

$$\text{C.F.} = (c_1 + c_2 t)e^t.$$

Now

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D-1)^2} t = (1-D)^{-2} t \\ &= (1+2D+\dots)t = t+2. \end{aligned}$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2 t)e^t + t + 2.$$

Returning back to x , we get

$$y = (c_1 + c_2 \log x)x + \log x + 2.$$

EXAMPLE 2.123

Solve Cauchy-Euler equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = \log x \sin(\log x).$$

Solution. Putting $x = e^t$, this equation transforms to

$$(D^2 + 1)y = t \sin t.$$

The complementary function is

$$\text{C.F.} = c_1 \cos t + c_2 \sin t.$$

Further

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 + 1} t \sin t = \text{I.P. of } \frac{1}{D^2 + 1} t e^{it} \\ &= \text{I.P. of } e^{it} \frac{1}{(D + i)^2 + 1} \\ &= \text{I.P. of } e^{it} \frac{1}{2iD(1 + \frac{D}{2i})} t \\ &= \text{I.P. of } e^{it} \cdot \frac{1}{2i} \cdot \frac{1}{D} \left(1 - \frac{iD}{2}\right)^{-1} t \\ &= \text{I.P. of } e^{it} \left(-\frac{it^2}{4} + \frac{t}{4}\right) \\ &= \text{I.P. of } (\cos t + i \sin t) \left(\frac{-it^2}{4} + \frac{t}{4}\right) \\ &= -\frac{t^2}{4} \cos t + \frac{t}{4} \sin t. \end{aligned}$$

Therefore, the complete solution is

$$y = c_1 \cos t + c_2 \sin t - \frac{t^2}{4} \cos t + \frac{t}{4} \sin t.$$

Returning back to x , we get

$$\begin{aligned} y &= c_1 \cos(\log x) + c_2 \sin(\log x) - \frac{1}{4}(\log x)^2 \cos(\log x) \\ &\quad + \frac{1}{4} \log x \sin(\log x). \end{aligned}$$

EXAMPLE 2.124

Solve the Cauchy-Euler equation

$$x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + 2y = x \log x.$$

Solution. Putting $x = e^t$, the equation reduces to

$$(D(D - 1) - D + 2)y = t e^t$$

or

$$(D^2 - 2D + 2)y = t e^t$$

The C.F. for this equation is

$$\text{C.F.} = e^t (c_1 \cos t + c_2 \sin t).$$

The particular integral is

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - 2D + 2} t e^t \\ &= e^t \frac{1}{(D + 1)^2 - 2(D + 1) + 2} t \\ &= e^t \frac{1}{D^2 + 1} t = e^t (1 - D^2)^{-1} t = e^t (t - 0) = t e^t. \end{aligned}$$

Therefore, the complete solution is

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} = e^t (c_1 \cos t + c_2 \sin t) + t e^t \\ &= x (c_1 \cos(\log x) + c_2 \sin(\log x) + x \log x). \end{aligned}$$

EXAMPLE 2.125

Solve $x^2 \frac{d^2 y}{dx^2} - 2y = x^2 + \frac{1}{x}$.

Solution. Putting $x = e^t$, the given equation reduces to $(D(D - 1) - 2)y = e^{2t} + \frac{1}{e^t}$.

The auxiliary equation is $m^2 - m - 2 = 0$ and so $m = 2, -1$. Therefore,

$$\text{C.F.} = c_1 e^{2t} + c_2 e^{-t}.$$

Moreover,

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - D - 2} (e^{2t} + \frac{1}{e^t}) \\ &= \frac{1}{D^2 - D - 2} e^{2t} + \frac{1}{D^2 - D - 2} (e^{-t}) \\ &= t \frac{1}{[2D - 1]_{D=2}} e^{2t} + t \frac{1}{(2D - 1)_{D=-1}} e^{-t} \\ &= \frac{t}{3} e^{2t} - \frac{1}{3} t e^{-t}. \end{aligned}$$

Thus, the complete solution is

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} = c_1 e^{2t} + c_2 e^{-t} + \frac{t}{3} e^{2t} - \frac{t}{3} e^{-t} \\ &= c_1 x^2 + \frac{c_2}{x} + \frac{1}{3} \left(x^2 - \frac{1}{x}\right) \log x. \end{aligned}$$

EXAMPLE 2.126

Solve the Cauchy-Euler equation

$$x^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} - 20y = (x + 1)^2.$$

Solution. Putting $x = e^t$, the given equation transforms into

$$(D(D-1) + 2D - 20)y = e^{2t} + 2e^t + 1$$

or

$$(D^2 + D - 20)y = e^{2t} + 2e^t + 1.$$

The auxiliary equation is $m^2 + m - 20 = 0$ and so $m = -5, 4$. Therefore,

$$\text{C.F.} = c_1 e^{-5t} + c_2 e^{4t}.$$

Now

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 + D - 20} e^{2t} + \frac{2}{D^2 - D - 20} e^t \\ &\quad + \frac{1}{D^2 - D - 20} x^{0t} \\ &= -\frac{1}{14} e^{2t} + \frac{2}{-20} e^t - \frac{1}{20}. \end{aligned}$$

Thus the complete solution is

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} = c_1 e^{-5t} + c_2 e^{4t} - \frac{1}{14} e^{2t} - \frac{1}{10} e^t - \frac{1}{20} \\ &= c_1 x^{-5} + c_2 x^4 - \frac{1}{14} x^2 - \frac{x}{10} - \frac{1}{20} \end{aligned}$$

(F) Legendre's Linear Equation

An equation of the form

$$(ax+b)^n \frac{d^n y}{dx^n} + a_1(ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n y = F(x) \quad (110)$$

where a_n are constants and F is a function of x , is called *Legendre's linear equation*.

To reduce the Legendre's equation to a linear differential equation with constant coefficient, we put $ax + b = e^t$ or $t = \log(ax + b)$. Then

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{a}{ax+b} \frac{dy}{dt}$$

or

$$(ax+b) \frac{dy}{dx} = aDy.$$

Further

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{a}{ax+b} \frac{dy}{dt} \right) = \frac{a^2}{(ax+b)^2} \left(\frac{d^2 y}{dt^2} - \frac{dy}{dt} \right)$$

and so

$$(ax+b)^2 \frac{d^2 y}{dx^2} = a^2(D^2 - D)y = a^2 D(D-1)y.$$

Similarly,

$$(ax+b)^3 \frac{d^3 y}{dx^3} = a^3 D(D-1)(D-2)y$$

and so on.

Putting these values in (110), we get a linear differential equation with constant coefficients which can be solved by usual methods.

EXAMPLE 2.127

Solve $(2x+3)^2 \frac{d^2 y}{dx^2} - 2(2x+3) \frac{dy}{dx} - 12y = 6x$.

Solution. Putting $2x+3 = e^t$ or $t = \log(2x+3)$, the given equation reduces to

$$(4(D^2 - D) - 4D - 12)y = 6 \frac{e^t - 3}{2} = 3e^t - 9$$

or

$$(4D^2 - 8D - 12)y = 3e^t - 9.$$

The auxiliary equation is

$$4m^2 - 8m - 12 = 0,$$

which yields $m = 3, -1$. Therefore,

$$\text{C.F.} = c_1 e^{-t} + c_2 e^{3t}.$$

Now

$$\text{P.I.} = \frac{1}{4D^2 - 8D - 12} (3e^t - 9) = \frac{3}{-16} e^t + \frac{3}{4}.$$

Hence the complete solution is

$$\begin{aligned} y &= c_1 e^{-t} + c_2 e^{3t} - \frac{3}{16} e^t + \frac{3}{4} \\ &= \frac{c_1}{(2x+3)} + c_2 (2x+3)^3 - \frac{3}{16} (2x+3) + \frac{3}{4}. \end{aligned}$$

EXAMPLE 2.128

Solve Legendre's equation

$$(1+x)^2 \frac{d^2 y}{dx^2} + (1+x) \frac{dy}{dx} + y = 4 \cos \log(1+x).$$

Solution. Putting $x+1 = e^t$ or $t = \log(x+1)$, the given equations transforms to $(1^2(D^2 - D) + 1D + 1)y = 4 \cos t$

or

$$(D^2+1)y = 4\cos t.$$

The auxiliary equation is $m^2 + 1 = 0$ and so $m = \pm i$. Therefore,

$$\text{C.F.} = c_1 \cos t + c_2 \sin t.$$

The particular integral is

$$\begin{aligned} \text{P.I.} &= \frac{4}{D^2+1} \cos t = t \frac{4(2D)}{[4D^2]_{D^2=-1}} \cos t \\ &= \frac{8t}{-4} D \cos t = -2t(-\sin t) = 2t \sin t. \end{aligned}$$

Thus, the complete solution is

$$\begin{aligned} y &= c_1 \cos t + c_2 \sin t + 2t \sin t \\ &= c_1 \cos(\log(x+1)) + c_2 \sin(\log(x+1)) \\ &\quad + 2\log(x+1) \sin(\log(x+1)). \end{aligned}$$

EXAMPLE 2.129

Solve the Legendre's equation

$$(1+x)^2 \frac{d^2 y}{dx^2} + (1+x) \frac{dy}{dx} + y = \sin(2\log(1+x)).$$

Solution. As in Example 2.128, putting $x+1 = e^t$, the given equation reduces to

$$(D^2+1)y = \sin(2t).$$

The complementary function is

$$\text{C.F.} = c_1 \cos t + c_2 \sin t$$

Further,

$$\text{P.I.} = \frac{1}{D^2+1} \sin[2t] = \frac{1}{-4+1} \sin 2t = -\frac{1}{3} \sin 2t.$$

Therefore, the complete solution is

$$\begin{aligned} y &= c_1 \cos t + c_2 \sin t - \frac{1}{3} \sin 2t \\ &= c_1 \cos(\log(x+1)) + c_2 \sin(\log(x+1)) \\ &\quad - \frac{1}{3} \sin(2\log(x+1)). \end{aligned}$$

EXAMPLE 2.130

Solve $(1+x)^2 \frac{d^2 y}{dx^2} + (1+x) \frac{dy}{dx} + y = 2 \sin(\log(x+1))$.

Solution. As in Example 2.128, putting $x+1 = e^t$, the given equation transforms to

$$(D^2+1)y = 2 \sin t.$$

Its complementary function is given by

$$\text{C.F.} = c_1 \cos t + c_2 \sin t.$$

Its particular integral is given by

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2+1} (2 \sin t) = \frac{2}{D^2+1} \sin t \\ &= \frac{2t}{[2D]_{D^2=-1}} \sin t = \frac{D \sin t}{-1} = -t \cos t. \end{aligned}$$

Therefore, the complete solution is

$$\begin{aligned} y &= c_1 \cos t + c_2 \sin t - t \cos t \\ &= c_1 \cos(\log(x+1)) + c_2 \sin(\log(x+1)) \\ &\quad - \log(x+1) \cos(\log(x+1)). \end{aligned}$$

2.19 SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Differential equations, in which there are two or more dependent variables and a single independent variable are called *simultaneous linear equations*. The aim of this section is to solve a system of linear differential equations with constant coefficients. The solution is obtained by eliminating all but one of the dependent variables and then solving the resultant equations by usual methods.

EXAMPLE 2.131

Solve the simultaneous equations

$$\frac{dx}{dt} = 7x - y, \quad \frac{dy}{dt} = 2x + 5y.$$

Solution. In symbolic form, we have

$$(D-7)x + y = 0 \quad (111)$$

$$(D-5)y - 2x = 0 \quad (112)$$

Multiplying (111) by $(D-5)$ and subtracting (112) from it, we get

$$(D-5)(D-7)x + 2x = 0$$

or

$$(D^2 - 12D + 35 + 2)x = 0$$

or

$$(D^2 - 12D + 37)x = 0.$$

Its auxiliary equation is $m^2 - 12m + 37 = 0$, which yields $m = 6 \pm i$. Therefore, its complete solution is $x = e^{6t}(c_1 \cos t + c_2 \sin t)$.

Now

$$\begin{aligned}\frac{dx}{dt} &= \frac{d}{dt}(c_1 e^{6t} \cos t) + \frac{d}{dt}(c_2 e^{6t} \sin t) \\ &= c_1 [6e^{6t} \cos t - e^{6t} \sin t] + c_2 [6e^{6t} \sin t + e^{6t} \cos t] \\ &= e^{6t} [(6c_1 + c_2) \cos t + (6c_2 - c_1) \sin t]\end{aligned}$$

Putting the values of x and $\frac{dx}{dt}$ in (111), we get

$$\begin{aligned}e^{6t} [(6c_1 + c_2) \cos t + (6c_2 - c_1) \sin t] \\ - 7e^{6t} [c_1 \cos t + c_2 \sin t] + y = 0.\end{aligned}$$

Therefore,

$$\begin{aligned}y &= 7e^{6t} (c_1 \cos t + c_2 \sin t) \\ &\quad - e^{6t} [(6c_1 + c_2) \cos t + (6c_2 - c_1) \sin t] \\ &= e^{6t} [(c_1 - c_2) \cos t + (c_1 + c_2) \sin t].\end{aligned}$$

Hence the solution is

$$\begin{aligned}x &= e^{6t} (c_1 \cos t + c_2 \sin t) \\ y &= e^{6t} [(c_1 - c_2) \cos t + (c_1 + c_2) \sin t].\end{aligned}$$

EXAMPLE 2.132

Solve $\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = \sin t$, $\frac{dx}{dy} + x - 3y = 0$.

Solution. The given system of equations are

$$(D^2 + D - 2)y = \sin t \quad (113)$$

$$(D + 1)x - 3y = 0 \quad (114)$$

The auxiliary equation for (113) is

$$m^2 + m - 2 = 0$$

and so $m = -2, 1$. Therefore,

$$\text{C.F.} = c_1 e^t + c_2 e^{-2t}.$$

$$\begin{aligned}\text{P.I.} &= \frac{1}{D^2 + D - 2} \sin t = \frac{1}{D - 3} \sin t \\ &= \frac{D + 3}{D^2 - 9} \sin t = -\frac{1}{10} (D + 3 \sin t) \\ &= -\frac{3}{10} \sin t - \frac{1}{10} \cos t.\end{aligned}$$

Therefore, the complete solution of (113) is

$$y = c_1 e^t + c_2 e^{-2t} - \frac{1}{10} (\cos t + 3 \sin t).$$

Putting this value of y in (114), we get

$$\frac{dx}{dt} + x = 3[c_1 e^t + c_2 e^{-2t} - \frac{1}{10} (\cos t + 3 \sin t)]$$

The integrating factor is $e^{\int 1 dt} = e^t$. Therefore,

$$x \cdot e^t = 3 \int e^t [c_1 e^t + c_2 e^{-2t} - \frac{1}{10} (\cos t + 3 \sin t)] dt,$$

which yields

$$x = \frac{3}{2} c_1 e^{2t} - 3c_2 e^{-t} - \frac{3}{10} e^t (\cos t - 2 \sin t).$$

Therefore, the solution is

$$x = \frac{3}{2} c_1 e^{2t} - 3c_2 e^{-t} - \frac{3}{10} e^t (\cos t - 2 \sin t)$$

$$y = c_1 e^t + c_2 e^{-2t} - \frac{1}{10} (\cos t + 3 \sin t).$$

EXAMPLE 2.133

A mechanical system with two degrees of freedom satisfies the equation

$$2 \frac{d^2x}{dt^2} + 3 \frac{dy}{dt} = 4, 2 \frac{d^2y}{dt^2} - 3 \frac{dx}{dt} = 0$$

under the condition that $x, y, \frac{dx}{dt}, \frac{dy}{dt}$ all vanish at $t = 0$. Find x and y .

Solution. The given equations are

$$2D^2x + 3Dy = 4$$

$$2D^2y - 3Dx = 0 \quad (115)$$

or

$$4D^3x + 6D^2y = 2D(4) = 0$$

$$-9Dx + 6D^2y = 0.$$

Subtracting, we get

$$4D^3x + 9Dx = 0$$

or

$$(4D^3 + 9D)x = 0.$$

Auxiliary equation is

$$4m^3 + 9m = 0, \text{ and so } m = 0, \pm \frac{3}{2}i.$$

Hence

$$x = c_1 + c_2 \cos \frac{3}{2}t + c_3 \sin \frac{3}{2}t.$$

At $t = 0, x = 0$. Therefore, $0 = c_1 + c_2$ or $c_1 = -c_2$.

Also

$$\frac{dx}{dt} = -\frac{3}{2} c_2 \sin \frac{3}{2}t + \frac{3}{2} c_3 \cos \frac{3}{2}t$$

At $t = 0$, $\frac{dx}{dt} = 0$ and so $0 = \frac{3}{2}c_3$ or $c_3 = 0$. Thus

$$x = c_1 - c_1 \cos \frac{3}{2}t.$$

Therefore,

$$\frac{dx}{dt} = \frac{3}{2}c_1 \sin \frac{3}{2}t \text{ and } \frac{d^2x}{dt^2} = \frac{9}{4}c_1 \cos \frac{3}{2}t.$$

Putting this value of $\frac{dx}{dt}$ in (115), we get

$$2D^2y - \frac{9}{2}c_1 \sin \frac{3}{2}t = 0 \text{ or } D^2y = \frac{9}{4}c_1 \sin \frac{3}{2}t.$$

Integrating, we get

$$Dy = -\frac{3}{2}c_1 \cos \frac{3}{2}t + k.$$

Using initial condition $\frac{dy}{dt} = 0$ at $t = 0$, we get $k = \frac{3}{2}c_1$. Therefore,

$$Dy = -\frac{3}{2}c_1 \cos \frac{3}{2}t + \frac{3}{2}c_1.$$

Integrating again, we have

$$y = -c_1 \sin \frac{3}{2}t + \frac{3}{2}c_1t + k.$$

When $t = 0$, $y = 0$. So we have $k = 0$. Hence

$$y = -c_1 \sin \frac{3}{2}t + \frac{3}{2}c_1t.$$

Further, putting the value of $\frac{d^2x}{dt^2}$ and $\frac{dy}{dt}$ in the first of the given equations, we get $c_1 = \frac{8}{9}$. Hence

$$x = \frac{8}{9} \left(1 - \cos \frac{3}{2}t \right), \quad y = \frac{4}{3}t - \frac{8}{9} \sin \frac{3}{2}t.$$

2.20 APPLICATIONS OF LINEAR DIFFERENTIAL EQUATIONS

Linear differential equations play an important role in the analysis of electrical, mechanical and other linear systems. Some of the applications of these equations are discussed below.

(A) Electrical Circuits

Consider an LCR circuit consisting of inductance L , capacitor C , and resistance R . Then, using Kirchhoff's law [see (16)], the equation governing the flow of

current in the circuit is

$$L \frac{dI}{dt} + RI + \frac{Q}{C} = E(t),$$

where I is the current flowing in the circuit, Q is the charge, and E is the e.m.f. of the battery. Since $I = \frac{dQ}{dt}$, the above equation reduces to

$$L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = E(t) \quad (116)$$

If we consider LC circuit without having e.m.f. source, then the differential equation describing the circuit is

$$L \frac{d^2Q}{dt^2} + \frac{Q}{C} = 0$$

or

$$\frac{d^2Q}{dt^2} + \frac{Q}{LC} = 0$$

or

$$\frac{d^2Q}{dt^2} + \mu^2 Q = 0, \mu^2 = \frac{1}{LC}.$$

This equation represents *free electrical oscillations* of the current having period

$$T = \frac{2\pi}{\mu} = 2\pi\sqrt{LC}.$$

EXAMPLE 2.134

In an LCR circuit, an inductance L of one henry, resistance of 6 ohm, and a condenser of $1/9$ farad have been connected through a battery of e.m.f. $E = \sin t$. If $I = Q = 0$ at $t = 0$, find charge Q and current I .

Solution. The differential equation for the given circuit is

$$L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = E(t).$$

Here $L = 1$, $R = 6$, $C = \frac{1}{9}$, $E(t) = \sin t$. Thus, we have

$$\frac{d^2Q}{dt^2} + 6 \frac{dQ}{dt} + 9Q = \sin t$$

subject to $Q(0) = 0$, $Q'(0) = 0 = I(0)$. The auxiliary equation for this differential equation is $m^2 + 6m + 9 = 0$ and so $m = -3, -3$. Thus

$$\text{C.F.} = (c_1 + c_2t)e^{-3t}.$$

Now

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 + 6D + 9} \sin t = \frac{1}{6D + 8} \sin t \\ &= \frac{6D - 8}{36D^2 - 64} \sin t = \frac{6D - 8}{-100} \sin t \\ &= -\frac{1}{100} [6D \sin t - 8 \sin t] \\ &= -\frac{6}{100} \cos t + \frac{8}{100} \sin t. \end{aligned}$$

Hence the complete solution is

$$Q = (c_1 + c_2 t)e^{-3t} - \frac{6}{100} \cos t + \frac{8}{100} \sin t.$$

Now $Q(0) = 0$ gives $0 = c_1 - \frac{6}{100}$ and so $c_1 = \frac{6}{100}$. Also

$$\begin{aligned} \frac{dQ}{dt} &= -3c_1 e^{-3t} + c_2(e^{-3t} - 3te^{-3t}) \\ &\quad + \frac{6}{100} \sin t + \frac{8}{100} \cos t. \end{aligned}$$

Therefore $\frac{dQ}{dt} = 0$ at $t = 0$ yields

$$0 = -3c_1 + c_2 + \frac{8}{100} = -\frac{18}{100} + c_2 + \frac{8}{100}$$

and so $c_2 = \frac{1}{10}$. Hence

$$\begin{aligned} Q &= \left(\frac{6}{100} + \frac{t}{10} \right) e^{-3t} - \frac{6}{100} \cos t + \frac{8}{100} \sin t \\ &= \frac{e^{-3t}}{50} (5t + 3) - \frac{3}{50} \cos t + \frac{2}{25} \sin t. \end{aligned}$$

Since $I = \frac{dQ}{dt}$, we have

$$\begin{aligned} I &= \frac{5e^{-3t}}{50} - \frac{3}{50} (5t + 3)e^{-3t} + \frac{3}{50} \sin t + \frac{2}{25} \cos t \\ &= -\frac{e^{-3t}}{50} (15t + 4) + \frac{3}{50} \sin t + \frac{2}{25} \cos t. \end{aligned}$$

EXAMPLE 2.135

Find the frequency of *free vibrations* in a closed electrical circuit with inductance L and capacity C in series.

Solution. Since there is no applied e.m.f., the differential equation governing this LC circuit is

$$L \frac{d^2 Q}{dt^2} + \frac{Q}{C} = 0$$

or

$$\frac{d^2 Q}{dt^2} = -\frac{Q}{LC} = -\omega^2 Q,$$

where $\omega^2 = \frac{1}{LC}$. Thus the equation represents oscillatory current with period.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{LC}.$$

Then

$$\begin{aligned} \text{Frequency} &= \frac{1}{T} = \frac{1}{2\pi\sqrt{LC}} \text{ per second} \\ &= \frac{60}{2\pi\sqrt{LC}} = \frac{30}{\pi\sqrt{LC}} \text{ per minute.} \end{aligned}$$

EXAMPLE 2.136

The differential equation for a circuit in which self-inductance and capacitance neutralize each other is

$$L \frac{d^2 i}{dt^2} + \frac{i}{C} = 0.$$

Find the current i as a function of t , given that I is maximum current and $i = 0$ when $t = 0$.

Solution. We have

$$\frac{d^2 i}{dt^2} + \frac{i}{LC} = 0.$$

The auxiliary equation is $m^2 + \frac{1}{LC} = 0$ and so $m = \pm \frac{i}{\sqrt{LC}}$. Hence the solution is

$$i = c_1 \cos \frac{t}{\sqrt{LC}} + c_2 \sin \frac{t}{\sqrt{LC}}.$$

Since $i = 0$ at $t = 0$, we have $c_1 = 0$ and so

$$i = c_2 \sin \frac{t}{\sqrt{LC}}.$$

For maximum current I , we have $I = c_2 \max \sin \frac{t}{\sqrt{LC}} = c_2$. Hence $i = I \sin \frac{t}{\sqrt{LC}}$.

EXAMPLE 2.137

An LCR circuit with battery e.m.f. $E \sin pt$ is tuned to resonance so that $p^2 = \frac{1}{LC}$. Show that for small value of $\frac{R}{L}$, the current in the circuit at time t is given by $\frac{Et}{2L} \sin pt$.

Solution. The differential equation governing the LCR circuit is $L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = E(t) = E \sin pt$. The auxiliary equation is

$$m^2 + \frac{R}{L}m + \frac{1}{LC} = 0,$$

which yields

$$\begin{aligned}
 m &= \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2} \\
 &= \frac{-\frac{R}{L} \pm \sqrt{-\frac{4}{LC}}}{2} \text{ since } \frac{R}{L} \text{ is small} \\
 &= -\frac{R}{2L} \pm \frac{1}{\sqrt{LC}} i \\
 &= -\frac{R}{2L} \pm pi \text{ since } p^2 = \frac{1}{LC}.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \text{C.F.} &= e^{-\frac{Rt}{2L}}(c_1 \cos pt + c_2 \sin pt) \\
 &= \left(1 - \frac{Rt}{2L}\right)(c_1 \cos pt + c_2 \sin pt) \\
 &\quad \text{rejecting higher power of } \frac{R}{L}.
 \end{aligned}$$

Further

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{LD^2 + RD + \frac{1}{C}}(E \sin pt) \\
 &= \frac{E}{-Lp^2 + RD + \frac{1}{C}} \sin pt \\
 &= \frac{E}{-\frac{L}{LC} + \frac{1}{C} + RD} \sin pt \\
 &= \frac{E}{RD} \sin pt = \frac{E}{R} \frac{D}{D^2} \sin pt \\
 &= -\frac{E}{Rp^2} D \sin pt \\
 &= -\frac{E}{Rp^2} p \cos pt = -\frac{E}{Rp} \cos pt.
 \end{aligned}$$

Thus, the complete solution is

$$q = \left(1 - \frac{Rt}{2L}\right)(c_1 \cos pt + c_2 \sin pt) - \frac{E}{Rp} \cos pt.$$

Using the initial condition $q = 0$ for $t = 0$, we get

$$0 = c_1 - \frac{E}{Rp} \text{ or } c_1 = \frac{E}{Rp}.$$

Also

$$\begin{aligned}
 i &= \frac{dq}{dt} = \left(1 - \frac{Rt}{2L}\right)(-c_1 \sin pt + c_2 \cos pt)p \\
 &\quad - \frac{R}{2L}(c_1 \cos pt + c_2 \sin pt) + \frac{E}{R} \sin pt.
 \end{aligned}$$

Using the initial condition $i = 0$ for $t = 0$, we get

$$\begin{aligned}
 0 &= pc_2 - \frac{Rc_1}{2L} \\
 &= pc_2 - \frac{R}{2L} \left(\frac{E}{Rp}\right) \\
 &= pc_2 - \frac{RE}{2Lp}
 \end{aligned}$$

and so $c_2 = \frac{E}{2Lp^2}$. Hence, the solution is

$$\begin{aligned}
 i &= \left(1 - \frac{Rt}{2L}\right) \left(-\frac{E}{Rp} \sin pt + \frac{2}{2Lp^2} \cos pt\right)p \\
 &\quad - \frac{R}{2L} \left(\frac{E}{Rp} \cos pt + \frac{E}{2Lp^2} \sin pt\right) + \frac{E}{R} \sin pt \\
 &= \frac{Et}{2L} \sin pt
 \end{aligned}$$

2.21 MASS-SPRING SYSTEM

In Example 2.4, we have seen that the differential equation governing a Mass-spring system is

$$m \frac{d^2x}{dt^2} + a \frac{dx}{dt} + kx = f(t)$$

where m is the mass, $a \frac{dx}{dt}$ the damping force due to the medium, k is spring constant, and x represents the displacement of the mass.

This is exactly the same differential equation which occurs in LCR electric circuits. When $a = 0$, the motion is called *undamped* whereas if $a \neq 0$, the motion is called *damped*. If $f(t) = 0$, then the motion is called *forced*.

EXAMPLE 2.138

A mass of 10 kg is attached to a spring having spring constant 140 N/m. The mass is started in motion from the equilibrium position with a velocity of 1 m/sec in the upward direction and with an applied external force $f(t) = 5 \sin t$. Find the subsequent motion of the mass if the force due to air resistance is $-90x'$ N.

Solution. The describing differential equation is

$$\frac{d^2x}{dt^2} + \frac{a}{m} \frac{dx}{dt} + \frac{k}{m} x = \frac{1}{m} f(t).$$

Here $a = 90$, $k = 140$, and $m = 10$. Therefore, we have

$$\frac{d^2x}{dt^2} + 9 \frac{dx}{dt} + 14x = \frac{1}{2} \sin t.$$

The auxiliary equation is $m^2 + 9m + 14 = 0$, and so $m = -2, -7$. Therefore,

$$\text{C.F.} = c_1 e^{-2t} + c_2 e^{-7t}.$$

Further

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 + 9D + 14} \left(\frac{1}{2} \sin t \right) = \frac{1}{2(-1 + 9D + 14)} \sin t \\ &= \frac{1}{2} \frac{1}{(9D + 13)} \sin t = \frac{1}{2} \frac{9D - 13}{(-81 - 169)} \sin t \\ &= -\frac{1}{500} (-13 \sin t + 9 \cos t) \\ &= \frac{13}{500} \sin t - \frac{9}{500} \cos t. \end{aligned}$$

Hence, the complete solution is

$$x = c_1 e^{-2t} + c_2 e^{-7t} + \frac{13}{500} \sin t - \frac{9}{500} \cos t.$$

Using the initial condition $x(0) = 0$, we get $0 = c_1 + c_2 - \frac{9}{500}$ and so $c_1 + c_2 = \frac{9}{500}$. Since $\frac{dx}{dt}(0) = -1$ (initial velocity in upper direction), we get

$$-1 = -2c_1 - 7c_2 + \frac{13}{500} \text{ or } 2c_1 + 7c_2 = \frac{513}{500}.$$

Solving for c_1 and c_2 , we get $c_1 = -\frac{90}{500}$, $c_2 = \frac{99}{500}$. Hence

$$x = \frac{1}{500} (-90e^{-2t} + 99e^{-7t} + 13 \sin t - 9 \cos t).$$

EXAMPLE 2.139

If in a mass spring system, mass = 4kg, spring constant = 64, $f(t) = 8 \sin 4t$, and if there is no air resistance and initial velocity, then find the subsequent motion of the weight. Show that the resonance occurs in this case.

Solution. The governing equation is

$$\frac{d^2x}{dt^2} + \frac{a}{m} \frac{dx}{dt} + \frac{k}{m} x = \frac{1}{m} f(t).$$

Therefore, we have

$$\frac{d^2x}{dt^2} + 16x = 2 \sin 4t.$$

The auxiliary equation is $m^2 + 16 = 0$ and so $m = \pm 4i$. Therefore,

$$\text{C.F.} = c_1 \cos 4t + c_2 \sin 4t.$$

Now

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 + 16} (2 \sin 4t) = \frac{2t}{2D} (\sin 4t) \\ &= \frac{t D(\sin 4t)}{-16} = -\frac{t}{16} (4 \cos 4t) \\ &= -\frac{t}{4} \cos 4t. \end{aligned}$$

Hence the complete solution is

$$x = c_1 \cos 4t + c_2 \sin 4t - \frac{t}{4} \cos 4t.$$

Using initial condition $x(0) = 0$, we have $0 = c_1$. Differentiating w.r.t. t , we get

$$\begin{aligned} \frac{dx}{dt} &= -4c_1 \sin 4t + 4c_2 \cos 4t \\ &\quad - \frac{1}{4} [\cos 4t - 4t \sin 4t]. \end{aligned}$$

Now $\frac{dx}{dt} = 0$ at $t = 0$. Therefore, $0 = 4c_2 - \frac{1}{4}$ which gives $c_2 = \frac{1}{16}$. Hence

$$x = \frac{1}{16} \sin 4t - \frac{t}{4} \cos 4t.$$

We observe that $x(t) \rightarrow \infty$ as $t \rightarrow \infty$ due to the presence of the term $t \cos 4t$. This term is called a *secular term*. The presence of secular term causes resonance because the solution becomes unbounded.

2.22 SIMPLE PENDULUM

The system in which a heavy particle (bob) is attached to one end of a light inextensible string, the other end of which is fixed, and oscillates under the action of gravity force in a vertical plane is called a *simple pendulum*. To describe its motion, let m be the mass of the particle, l be the length of the string, and O be the fixed point of the string (Figure 2.9).

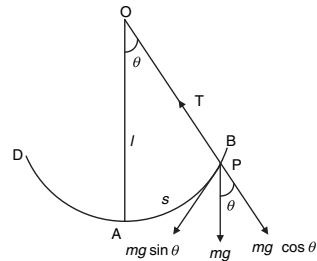


Figure 2.9

Let P be the position of the heavy particle at any time t and let $\angle AOP = \theta$, where OA is vertical line through O. Then the force acting on the bob are

(a) weight mg acting vertically downward

(b) tension T in the string.

Resolving mg , we note that tension is balanced by $mg \cos \theta$. The equation of motion along the tangent is

$$m \frac{d^2 s}{dt^2} = -mg \sin \theta$$

or

$$\frac{d^2}{dt^2} (l\theta) = -g \sin \theta$$

or

$$\begin{aligned} \frac{d^2 \theta}{dt^2} &= -\frac{g}{l} \sin \theta \\ &= -\frac{g}{l} \left[\theta - \frac{\theta^3}{3!} + \dots \right] \text{ (using expansion for } \sin \theta) \\ &= -\frac{g\theta}{l} \text{ to the first approximation.} \end{aligned}$$

Thus the differential equation describing the motion of bob is

$$\frac{d^2 \theta}{dt^2} + \omega^2 \theta = 0,$$

where $\omega^2 = \frac{g}{l}$. The auxiliary equation of this equation is

$$m^2 + \omega^2 = 0$$

and so $m = \pm \omega i$. Therefore, the solution is

$$\begin{aligned} \theta &= c_1 \cos \omega t + c_2 \sin \omega t \\ &= c_1 \cos \sqrt{\frac{g}{l}} t + c_2 \sin \sqrt{\frac{g}{l}} t. \end{aligned}$$

The motion in case of simple pendulum is *simple harmonic motion* where time period

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{l}{g}}.$$

The motion of the bob from one extreme position to the other extreme position on the other side of A is called a *beat* or a *swing*. Therefore,

$$\text{Time for one swing} = \frac{1}{2} T = \pi \sqrt{\frac{l}{g}}.$$

The pendulum which beats once every second is called a *second's pendulum*. Therefore, the number of beats is a second pendulum in one day is equal to the number of seconds in a day. Thus, a second pendulum beats 86,400 times a day.

Now, since, the time for 1 beat in a second's pendulum is 1 sec, we have

$$1 = \pi \sqrt{\frac{l}{g}} = \pi \sqrt{\frac{l}{981}}$$

and so $l = 99.4$ cm is the length of second's pendulum.

EXAMPLE 2.140

The differential equation of a simple pendulum is $\frac{d^2 x}{dt^2} + \omega_0^2 x = F_0 \sin nt$, where ω_0 and F_0 are constants. If initially $\frac{dx}{dt} = 0$, determine the motion when $\omega_0 = n$

Solution. We have

$$\frac{d^2 x}{dt^2} + n^2 x = F_0 \sin nt \text{ since, } \omega_0 = n.$$

The auxiliary equation is $m^2 + n^2 = 0$ and so $m = \pm ni$. Therefore,

$$\text{C.F.} = c_1 \cos nt + c_2 \sin nt.$$

Now

$$\begin{aligned} \text{P.I.} &= \frac{F_0}{D^2 + n^2} \sin nt = \frac{F_0 t}{2D} \sin nt \\ &= \frac{F_0 t}{2} \int \sin nt \, dt = -\frac{F_0 t}{2n} \cos nt. \end{aligned}$$

Thus, the complete solution is

$$x = c_1 \cos nt + c_2 \sin nt - \frac{F_0 t}{2n} \cos nt.$$

Initially $x = 0$ for $t = 0$, so $c_1 = 0$.

Also

$$\begin{aligned} \frac{dx}{dt} &= -nc_1 \sin nt + nc_2 \cos nt \\ &\quad - \frac{F_0}{2n} [\cos nt - nt \sin nt]. \end{aligned}$$

But for $t = 0$, $\frac{dx}{dt} = 0$. Therefore, $0 = nc_2 - \frac{F_0}{2n}$ and so $c_2 = \frac{F_0}{2n^2}$.

Hence

$$\begin{aligned} x &= \frac{F_0}{2n^2} \sin nt - \frac{F_0 t}{2n} \cos nt \\ &= \frac{F_0}{2n^2} (\sin nt - nt \cos nt). \end{aligned}$$

2.23 SOLUTION IN SERIES

The method of series solution of differential equations is applied to obtain solutions of *linear differential equations with variable coefficients*.

Consider the differential equation

$$P_0(x) \frac{d^2y}{dx^2} + P_1(x) \frac{dy}{dx} + P_2(x)y = 0, \quad (117)$$

where $P_0(x)$, $P_1(x)$, and $P_2(x)$ are polynomials in x . This equation can be written as

$$\frac{d^2y}{dx^2} + \frac{P_1(x)}{P_0(x)} \frac{dy}{dx} + \frac{P_2(x)}{P_0(x)} y = 0. \quad (118)$$

The point $x = a$ is called an *ordinary point* of the equation (117) or (118) if the functions $\frac{P_1(x)}{P_0(x)}$ and $\frac{P_2(x)}{P_0(x)}$ are analytic at $x = a$. In other words, $x = a$ is an ordinary point of (117) if $P_0(a) \neq 0$.

If either (or both) of $\frac{P_1(x)}{P_0(x)}$ or $\frac{P_2(x)}{P_0(x)}$ is (are) not analytic at $x = a$, then $x = a$ is called a *singular point* of the equation (117) or (118). Thus $x = a$ is a singular point of (117) if $P_0(a) = 0$.

Further, let

$$Q_1(x) = (x - a) \frac{P_1(x)}{P_0(x)},$$

$$Q_2(x) = (x - a)^2 \frac{P_2(x)}{P_0(x)}.$$

If Q_1 and Q_2 are both analytic at $x = a$, then $x = a$ is called a *regular singular point* of (117) otherwise it is called *irregular point* of (117).

For example, consider the equation

$$9x(1 - x) \frac{d^2y}{dx^2} - 12 \frac{dy}{dx} + 4y = 0.$$

We have

$$\frac{P_1(x)}{P_0(x)} = \frac{-12}{9x(1 - x)} \text{ and } \frac{P_2(x)}{P_0(x)} = \frac{4}{9x(1 - x)},$$

which are not analytic at $x = 0$ and $x = 1$. Hence $x = 0$ and $x = 1$ are singular points of the given equation. Further, at $x = 0$,

$$Q_1(x) = \frac{-12}{9(1 - x)}, \quad Q_2(x) = \frac{4}{-(1 - x)},$$

which are analytic at $x = 0$. Thus $x = 0$ is a regular singular point.

For $x = 1$, we have

$$Q_1(x) = \frac{12}{9x}, \quad Q_2(x) = \frac{-4}{9x},$$

which are analytic if $x = 1$. Hence $x = 1$ is also regular.

2.23.1 Solution About Ordinary Point

If $x = a$ is an ordinary point of the differential equation (117), then its every solution can be expressed in the form

$$y = a_0 + a_1(x - a) + a_2(x - a)^2 + a_3(x - a)^3 + \dots, \quad (119)$$

where the power series converges in some interval $|x - a| < R$ about a . Thus the series may be differentiated term by term on this interval and we have

$$\frac{dy}{dx} = a_1 + 2a_2(x - a) + 3a_3(x - a)^2 + \dots,$$

$$\frac{d^2y}{dx^2} = 2a_2 + 6a_3(x - a) + \dots$$

Substituting the values of y , $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in (117), we get an equation of the type

$$c_0 + c_1(x - a) + c_2(x - a)^2 + \dots = 0, \quad (120)$$

where the coefficients c_0 , c_1 , and c_2 are functions of a . Then (120) will be valid for all x in $|x - a| < R$ if all $c_1, c_2, \dots$ are zero. Thus

$$c_0 = c_1 = c_2 = \dots = 0. \quad (121)$$

The coefficients a_i of (119) are obtained from (121). In case (120) is expressed in powers of x , then equating to zero the coefficients of the various powers of x will determine $a_2, a_3, a_4, \dots$ in terms of a_0 and a_1 . The relation obtained by equating to zero the coefficient of x^n is called the *recurrence relation*.

2.23.2 Solution About Singular Point (Forbenious Method)

If $x = a$ is a regular singular point of (117), then the equation has at least one non-trivial solution of the form

$$y = (x - a)^m [a_0 + a_1(x - a) + a_2(x - a)^2 + \dots], \quad (122)$$

where m is a definite constant (real or complex) and the series on the right converges at every point of the interval of convergence with centre a . Differentiating (122) twice, we get

$$\frac{dy}{dx} = ma_0(x - a)^{m-1} + (m + 1)a_1(x - a)^m + \dots$$

$$\frac{d^2y}{dx^2} = m(m - 1)a_0(x - a)^{m-2} + m(m + 1)a_1(x - a)^{m-1} + \dots$$

Substituting the values of y , $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (117), we get an equation of the form

$$c_0(x-a)^{m+k} + c_1(x-a)^{m+k+1} + c_2(x-a)^{m+k+2} + \dots = 0, \quad (123)$$

where k is an integer and the coefficients c_i are functions of m and a_i . In order that (123) be valid in $|x-a| < R$, we must have

$$c_0 = c_1 = c_2 = \dots = 0. \quad (124)$$

On equating to zero the coefficient c_0 in (123), we get a quadratic equation in m , called *indicial equation*, which gives the value of m . The two roots m_1 and m_2 of the indicial equation are called *exponents of the differential equation* (117). The coefficients $a_1, a_2, a_3, \dots$ are obtained in terms of a_0 from $c_1 = c_2 = \dots = 0$. Putting the values of $a_1, a_2, \dots$ in (122), the solution of (117) is obtained.

If $m_1 - m_2 \neq 0$ or a positive integer, then the complete solution of equation (117) is

$$y = c_1(y)_{m_1} + c_2(y)_{m_2}.$$

If $m_1 - m_2 = 0$, that is, the roots of indicial equation are equal, then the two independent solutions are obtained by substituting the value of m in y and $\frac{\partial y}{\partial m}$. Thus, in this case,

$$y = c_1(y)_{m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_1}$$

If $m_1 - m_2$ is a positive integers making a coefficient of y infinite when $m = m_2$, then the form of y is modified by replacing a_0 by $k(m - m_2)$. Two independent solutions of the differential equation (117) are then

$$y = c_1(y)_{m_2} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_2}$$

If $m_1 - m_2$ is a positive integer making a coefficient of y indeterminate when $m = m_2$, then the complete solution of (117) is

$$y = c_1(y)_{m_2}.$$

EXAMPLE 2.141

Find the power series solution of the equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

in powers of x , that is, about $x = 0$.

Solution. Let

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$$

Differentiating twice we get

$$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots + na_nx^{n-1} + \dots$$

$$\frac{d^2y}{dx^2} = 2a_2 + 6a_3x + \dots + n(n-1)a_nx^{n-2} + \dots$$

Substituting the values of y , $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given differential equation, we get

$$\begin{aligned} (1-x^2)[2a_2 + 6a_3x + \dots + n(n-1)a_nx^{n-2} + \dots] \\ - 2x[a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1} + \dots] \\ + 2(a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots) = 0 \end{aligned}$$

or

$$\begin{aligned} 2(a_2 + a_0) + (6a_3 - 2a_1)x \\ + (12a_4 - 2a_2 - 4a_2 + 2a_2)x^2 \\ + \dots + [(n+2)(n+1)a_{n+2} \\ - n(n-1)a_n - 2na_n + 2a_n]x^n + \dots = 0 \end{aligned}$$

or

$$\begin{aligned} 2(a_2 + a_0) + 6a_3x + (12a_4 - 4a_2)x^2 \\ + \dots + [(n^2 + 3n + 2)a_{n+2} - (n^2 - n)a_n \\ + 2a_n(1 - n)]x^n + \dots = 0. \end{aligned}$$

Equating to zero the coefficients of the various powers of x , we get

$$a_2 = -a_0, \quad a_3 = 0, \quad a_4 = -\frac{1}{3}a_0, \dots$$

$$a_{n+2}[n^2 + 3n + 2] + a_n(-n^2 - n + 2) = 0.$$

Taking $n = 3, 4, \dots$

$$20a_5 - 10a_3 = 0 \Rightarrow a_5 = 0$$

$$30a_6 - 18a_4 = 0 \Rightarrow a_6 = \frac{18}{30}a_4 = -\frac{1}{5}a_0$$

$\dots$

$\dots$

Therefore,

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \dots$$

$$= a_0 + a_1x - a_0x^2 - \frac{1}{3}a_0x^4 - \frac{1}{5}a_0x^6 + \dots$$

$$= a_0\left(1 - x^2 - \frac{1}{3}x^4 - \frac{1}{5}x^6 - \dots\right) + a_1x.$$

EXAMPLE 2.142

Find the solution in series of the equation

$$\frac{d^2y}{dx^2} + x \frac{dy}{dx} + x^2y = 0$$

about $x = 0$.

Solution. The point $x = 0$ is a regular point of the given differential equation. So, let the required solution be

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots + a_nx^n + \dots$$

Then differentiating twice, we get

$$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots + na_nx^{n-1} + \dots$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 \\ &+ \dots + n(n-1)a_nx^{n-2} + \dots \end{aligned}$$

Substituting the values of y , $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in the given equation, we get

$$\begin{aligned} &[2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 \\ &+ \dots + n(n-1)a_nx^{n-2} + \dots] \\ &+ x[a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 \\ &+ \dots + na_nx^{n-1} + \dots] \\ &+ x^2[a_0 + a_1x + a_2x^2 + a_3x^3 \\ &+ a_4x^4 + \dots + a_nx^n + \dots] = 0 \end{aligned}$$

or

$$\begin{aligned} &2a_2 + (6a_3 + a_1)x + (12a_4 + 2a_2 + a_0)x^2 \\ &+ (20a_5 + 3a_3 + a_1)x^3 + (30a_6 + 4a_4 + a_2)x^4 \\ &+ (42a_7 + 5a_5 + a_3)x^5 + \dots \\ &+ \dots [(n+2)(n+1)a_{n+2} + na_n + a_{n-2}]x^n + \dots = 0. \end{aligned}$$

Equating to zero the coefficients of the various powers of x , we get

$$a_2 = 0, a_3 = -\frac{1}{6}a_1,$$

$$12a_4 + 2a_2 + a_0 = 0, \text{ which yields } a_4 = -\frac{1}{12}a_0$$

$$20a_5 + 3a_3 + a_1 = 0 \text{ which yields } a_5 = -\frac{1}{40}a_1$$

$$30a_6 + 4a_4 + a_2 = 0 \text{ which yields } a_6 = \frac{1}{90}a_0, \text{ and so on.}$$

Hence

$$\begin{aligned} y &= a_0 + a_1x - \frac{1}{6}a_1x^3 - \frac{1}{12}a_0x^4 \\ &- \frac{1}{40}a_1x^5 + \frac{1}{90}a_0x^6 \dots \\ &= a_0 \left(1 - \frac{1}{12}x^4 + \frac{1}{90}x^6 - \dots \right) \\ &+ a_1 \left(x - \frac{1}{6}x^3 - \frac{1}{40}x^5 - \dots \right). \end{aligned}$$

EXAMPLE 2.143

Find power series solution of the equation

$$\frac{d^2y}{dx^2} + xy = 0$$

in powers of x , that is, about $x = 0$.

Solution. We note that $x = 0$ is a regular point of the given equation. Therefore, its solution is of the form

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + \dots + a_nx^n + \dots$$

Differentiating twice successively, we get

$$\begin{aligned} \frac{dy}{dx} &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 \\ &+ 6a_6x^5 + \dots + na_nx^{n-1} + \dots \end{aligned}$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 \\ &+ 30a_6x^4 + \dots + n(n-1)a_nx^{n-2} + \dots \end{aligned}$$

Putting the values of y , $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation, we get

$$\begin{aligned} &2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 \\ &+ \dots + n(n-1)a_nx^{n-2} + \dots \\ &+ x[a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots] = 0 \end{aligned}$$

or

$$\begin{aligned} &2a_2 + (6a_3 + a_0)x + (12a_4 + a_1)x^2 + (20a_5 + a_2)x^3 \\ &+ \dots + [(n+2)(n+1)a_{n+2} + a_{n+1}]x^n + \dots \end{aligned}$$

Equating to zero the coefficients of various powers of x , we get

$$a_2 = 0, 6a_3 + a_0 = 0 \text{ which yields } a_3 = -\frac{1}{6}a_0$$

$$12a_4 + a_1 = 0 \text{ which yields } a_4 = -\frac{1}{12}a_1$$

$$20a_5 + a_2 = 0 \text{ which yields } a_5 = 0$$

$$(n+2)(n+1)a_{n+2} + a_{n-1} = 0.$$

Putting $n = 4, 5, 6, 7, \dots$, we get

$$30a_6 + a_3 = 0 \text{ which yields } a_6 = -\frac{a_3}{30} = \frac{1}{180}a_0$$

$$42a_7 + a_4 = 0 \text{ which yields } a_7 = -\frac{a_4}{42} = \frac{1}{504}a_1,$$

and so on.

Hence

$$y = a_0 \left(1 - \frac{1}{6}x^3 + \frac{1}{180}x^6 - \dots \right) + a_1 \left(x - \frac{1}{12}x^4 + \frac{1}{504}x^7 - \dots \right).$$

EXAMPLE 2.144

Find series solution of the differential equation

$$x \frac{d^2 y}{dx^2} + \frac{dy}{dx} - y = 0$$

about $x = 0$.

Solution. The point $x = 0$ is a regular singular point of the given equation. So, let

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \dots, \quad a_0 \neq 0.$$

Differentiating twice in succession, we get

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1}$$

and

$$\frac{d^2 y}{dx^2} = \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m-2}.$$

Substituting the values of y , $\frac{dy}{dx}$ and $\frac{d^2 y}{dx^2}$ in the given equation, we get

$$x \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m-2} + \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1} - \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

or

$$\sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m-1} + \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1} - \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

or

$$\sum_{n=0}^{\infty} (n+m)^2 a_n x^{n+m-1} - \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

or

$$\sum_{n=-1}^{\infty} (n+m+1)^2 a_{n+1} x^{n+m} - \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

or

$$m^2 a_0 x^{m-1} + \sum_{n=0}^{\infty} (n+m+1)^2 a_{n+1} x^{n+m} - \sum_{n=0}^{\infty} a_n x^{n+m} = 0$$

or

$$m^2 a_0 x^{m-1} + \sum_{n=0}^{\infty} x^{n+m} \times [(n+m+1)^2 a_{n+1} - a_n] = 0 \quad (125)$$

Therefore, the indicial equation is $m^2 = 0$, which yields $m = 0, 0$. Equating to zero other coefficients in (125), we get

$$(n+m+1)^2 a_{n+1} = a_n, \quad n \geq 0.$$

Therefore,

$$a_1 = \frac{1}{(m+1)^2} a_0,$$

$$a_2 = \frac{1}{(m+2)^2} a_1 = \frac{1}{(m+1)^2 (m+2)^2} a_0,$$

$$a_3 = \frac{1}{(m+3)^2} a_2 = \frac{1}{(m+1)^2 (m+2)^2 (m+3)^2} a_0,$$

and so on. Thus

$$\begin{aligned} y &= a_0 x^m + \frac{1}{(m+1)^2} a_0 x^{m+1} + \frac{1}{(m+1)^2 (m+2)^2} a_0 x^{m+2} \\ &\quad + \frac{1}{(m+1)^2 (m+2)^2 (m+3)^2} a_0 x^{m+3} + \dots \\ &= a_0 x^m \left[1 + \frac{1}{(m+1)^2} x + \frac{1}{(m+1)^2 (m+2)^2} x^2 \right. \\ &\quad \left. + \frac{1}{(m+1)^2 (m+2)^2 (m+3)^2} x^3 + \dots \right]. \end{aligned}$$

Putting $m = 0$, we get one solution of the given differential equation as

$$y_1 = c_1 \left[1 + x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \dots \right].$$

Further,

$$\begin{aligned} \frac{\partial y}{\partial m} = & a_0 x^m \log x \left[1 + \frac{1}{(m+1)^2} x + \frac{1}{(m+1)^2(m+2)^2} x^2 \right. \\ & \left. + \frac{1}{(m+1)^2(m+2)^2(m+3)^2} x^3 + \dots \right] \\ & + a_0 x^m \left[-\frac{2x}{(m+1)^3} - \frac{1}{(m+1)^2(m+2)^2} \right. \\ & \left. \times \left(\frac{2}{(m+1)} + \frac{2}{(m+2)^2} \right) x^2 + \dots \right]. \end{aligned}$$

Therefore,

$$\begin{aligned} \left(\frac{\partial y}{\partial m} \right)_{m=0} = & a_0 \log \left[1 + x + \frac{1}{4} x^2 + \frac{1}{36} x^3 + \dots \right] \\ & + a_0 \left[-2x - \frac{1}{4} (2+1) x^2 + \dots \right]. \end{aligned}$$

Hence the solution of the given equations is

$$\begin{aligned} y = & c_1 (y)_{m=0} + c_2 \left(\frac{\partial y}{\partial m} \right)_{m=0} \\ = & (c_1 + c_2 \log x) \left[1 + x + \frac{1}{4} x^2 + \frac{1}{36} x^3 + \dots \right] \\ & - 2c_2 \left[x + \frac{1}{4} \left(1 + \frac{1}{2} \right) x^2 + \dots \right]. \end{aligned}$$

EXAMPLE 2.145

Find the series solution near $x = 0$ of the differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + x^2 y = 0.$$

(This equation can also be written as $x \frac{d^2 y}{dx^2} + \frac{dy}{dx} + xy = 0$ and is known as *Bessel's Equation of order zero*.)

Solution. The point $x = 0$ is a regular singular point of the given equation. So, let

$$y = \sum_{n=0}^{\infty} a_n x^{n+m}.$$

Differentiating twice in succession, we get

$$\begin{aligned} \frac{dy}{dx} &= \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1} \\ \frac{d^2 y}{dx^2} &= \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m-2}. \end{aligned}$$

Putting the values of y , $\frac{dy}{dx}$ and $\frac{d^2 y}{dx^2}$ in the given equation, we get

$$\begin{aligned} & x \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m-2} \\ & + \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1} + x \sum_{n=0}^{\infty} a_n x^{n+m} = 0 \end{aligned}$$

or

$$\begin{aligned} & \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m-1} \\ & + \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1} + \sum_{n=0}^{\infty} a_n x^{n+m+1} = 0 \end{aligned}$$

or

$$\sum_{n=0}^{\infty} (n+m)^2 a_n x^{n+m-1} + \sum_{n=0}^{\infty} a_n x^{n+m+1} = 0$$

or

$$\sum_{n=-1}^{\infty} (n+m+1)^2 a_{n+1} x^{n+m} + \sum_{n=0}^{\infty} a_n x^{n+m+1} = 0$$

or

$$\begin{aligned} & m^2 a_0 x^{m-1} + \sum_{n=0}^{\infty} (n+m+1)^2 a_{n+1} x^{n+m} \\ & + \sum_{n=0}^{\infty} a_n x^{n+m+1} = 0. \end{aligned}$$

Therefore, the indicial equation is $m^2 = 0$, which yields $m = 0, 0$. Equating to zero the coefficients of powers of x^m , x^{m+1} , x^{m+2} , ..., we get

$$\begin{aligned} (n+m+1)^2 a_1 &= 0 \text{ which yields } a_1 = 0, \\ (m+2)^2 a_2 + a_0 &= 0 \text{ and so } a_2 = -\frac{a_0}{(m+2)^2}, \\ (m+3)^2 a_3 + a_1 &= 0 \text{ and so } a_3 = 0, \\ (m+4)^2 a_4 + a_2 &= 0 \text{ and so } a_4 = \frac{-a_2}{(m+4)^2} \\ &= \frac{a_0}{(m+2)^2(m+4)^2}, \end{aligned}$$

and so on. Hence,

$$y = a_0 x^m - \frac{a_0}{(m+2)^2} x^{m+2} + \frac{a_0}{(m+2)^2(m+4)^2} x^{m+4} - \dots$$

$$= a_0 x^m \left[1 - \frac{1}{(m+2)^2} x^2 + \frac{1}{(m+2)^2(m+4)^2} x^4 - \dots \right].$$

Further,

$$\frac{\partial y}{\partial m} = a_0 x^m \log x \left[1 - \frac{1}{(m+2)^2} x^2 + \frac{1}{(m+2)^2(m+4)^2} x^4 - \dots \right]$$

$$+ a_0 x^m \left[\frac{2x^2}{(m+2)^3} - \frac{x^4}{(m+2)^2(m+4)^2} \right]$$

$$\times \left(\frac{2}{m+2} + \frac{2}{m+4} \right) + \dots \Big].$$

Therefore,

$$\left(\frac{\partial y}{\partial m} \right)_{m=0} = a_0 \log x \left[1 - \frac{1}{4} x^2 + \frac{1}{64} x^4 - \dots \right]$$

$$+ a_0 \left[\frac{x^2}{4} - \frac{3x^4}{2 \cdot 4 \cdot 16} + \dots \right].$$

Hence the solution of the given equation is

$$y = c_1(y)_{m=0} + c_2 \left(\frac{\partial y}{\partial m} \right)_{m=0}$$

$$= c_1 \left[1 - \frac{1}{2^2} x^2 + \frac{1}{2^2 \cdot 4^2} x^4 - \frac{1}{2^2 \cdot 4^2 \cdot 6^2} x^6 + \dots \right]$$

$$+ c_2 \log x \left[1 - \frac{1}{2^2} x^2 + \frac{1}{2^2 \cdot 4^2} x^4 - \frac{1}{2^2 \cdot 4^2 \cdot 6^2} x^6 + \dots \right]$$

$$+ a_0 \left[\frac{1}{2^2} x^2 - \frac{1}{2^2 \cdot 4^2} \left(1 + \frac{1}{2} \right) x^4 + \frac{1}{2^2 \cdot 4^2 \cdot 6^2} \left(1 + \frac{1}{2} + \frac{1}{3} \right) x^6 - \dots \right]$$

$$= (c_1 + c_2 \log x) \left[1 - \frac{1}{2^2} x^2 + \frac{1}{2^2 \cdot 4^2} x^4 - \frac{1}{2^2 \cdot 4^2 \cdot 6^2} x^6 + \dots \right]$$

$$+ a_0 \left[\frac{1}{2^2} x^2 - \frac{1}{2^2 \cdot 4^2} \left(1 + \frac{1}{2} \right) x^4 + \frac{1}{2^2 \cdot 4^2 \cdot 6^2} \left(1 + \frac{1}{2} + \frac{1}{3} \right) x^6 + \dots \right].$$

The solution

$$c_1(y)_{m=0} = c_1 \left[1 - \frac{1}{2^2} x^2 + \frac{1}{2^2 \cdot 4^2} x^4 - \frac{1}{2^2 \cdot 4^2 \cdot 6^2} x^6 + \dots \right]$$

is called *Bessel function of the first kind of order zero* and is represented by $J_0(x)$, where as the solution.

$$c_2 \left(\frac{\partial y}{\partial m} \right)_{m=0} = c_2 \log x \left[1 - \frac{1}{2^2} x^2 + \frac{1}{2^2 \cdot 4^2} x^4 - \frac{1}{2^2 \cdot 4^2 \cdot 6^2} x^6 - \dots \right]$$

$$+ a_0 \left[\frac{1}{2^2} x^2 - \frac{1}{2^2 \cdot 4^2} \left(1 + \frac{1}{2} \right) x^4 + \frac{1}{2^2 \cdot 4^2 \cdot 6^2} \left(1 + \frac{1}{2} + \frac{1}{3} \right) x^6 - \dots \right]$$

is called *Neumann function* or *Bessel function of second kind of order zero* and is denoted by $Y_0(x)$.

EXAMPLE 2.146

Find series solution about $x = 0$ of the differential equations

$$2x(1-x) \frac{d^2 y}{dx^2} + (1-x) \frac{dy}{dx} + 3y = 0.$$

Solution. Evidently $x = 0$ is a regular singular point of the given equation. So, let

$$y = \sum_{n=0}^{\infty} a_n x^{n+m}, \quad a_n \neq 0.$$

Differentiating twice in succession, we get

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1},$$

$$\frac{d^2 y}{dx^2} = \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m-2}.$$

Substituting the values of y , $\frac{dy}{dx}$, and $\frac{d^2 y}{dx^2}$ in the given equation, we get

$$2x(1-x) \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m-2}$$

$$+ (1-x) \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1} + 3 \sum_{n=0}^{\infty} a_n x^{n+m} = 0.$$

or

$$\begin{aligned}
& 2 \sum_{n=0}^{\infty} (n+m)(n+m-1)a_n x^{n+m-1} \\
& - 2 \sum_{n=0}^{\infty} (n+m)(n+m-1)a_n x^{n+m} \\
& + \sum_{n=0}^{\infty} (n+m)a_n x^{n+m-1} - \sum_{n=0}^{\infty} (n+m)a_n x^{n+m} \\
& + 3 \sum_{n=0}^{\infty} a_n x^{n+m} = 0
\end{aligned}$$

or

$$\begin{aligned}
& \sum_{n=0}^{\infty} (n+m)(2n+2m-1)a_n x^{n+m-1} \\
& - \sum_{n=0}^{\infty} [(n+m)(2n+2m-1) - 3]a_n x^{n+m} = 0
\end{aligned}$$

or

$$\begin{aligned}
& \sum_{n=-1}^{\infty} (n+m+1)(2n+2m+1)a_{n+1} x^{n+m} \\
& - \sum_{n=0}^{\infty} [(n+m)(2n+2m-1) - 3]a_n x^{n+m} = 0
\end{aligned}$$

or

$$\begin{aligned}
& m(2m-1)a_0 x^{m-1} \\
& + \sum_{n=0}^{\infty} (n+m+1)(2n+2m+1)a_{n+1} x^{n+m} \\
& - \sum_{n=0}^{\infty} [(n+m)(2n+2m-1) - 3]a_n x^{n+m} = 0.
\end{aligned}$$

Therefore, the indicial equation is

$$m(2m-1) = 0, \text{ which yields } m = 0, \frac{1}{2}.$$

Equating to zero the other coefficients, we get

$$\begin{aligned}
& (n+m+1)(2n+2m+1)a_{n+1} \\
& = [(n+m)(2n+2m-1) - 3]a_n \\
& = [2n^2 + 2m^2 + 4nm - m - n - 3]a_n
\end{aligned}$$

or

$$a_{n+1} = \frac{2n^2 + 2m^2 + 4nm - m - n - 3}{(n+m+1)(2n+2m+1)} a_n.$$

For $m = 0$, we get

$$\begin{aligned}
a_{n+1} &= \frac{2n^2 - n - 3}{(n+1)(2n+1)} a_n = \frac{(n+1)(2n-3)}{(n+1)(2n+1)} a_n \\
&= \frac{2n-3}{2n+1} a_n, n \geq 0.
\end{aligned}$$

Putting $n = 0, 1, 2, \dots$, we have

$$\begin{aligned}
a_1 &= -3a_0, a_2 = -\frac{1}{3}a_1 = a_0, \\
a_3 &= \frac{1}{5}a_2 = \frac{1}{5}a_0, a_4 = \frac{3}{7}a_3 = \frac{3}{35}a_0, \text{ and so on.}
\end{aligned}$$

Therefore, the solution for $m = 0$ is

$$\begin{aligned}
y_1 &= a_0 x^0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \\
&= a_0 - 3a_0 x + a_0 x^2 + \frac{1}{5}a_0 x^3 + \frac{3}{35}a_0 x^4 + \dots \\
&= a_0 \left[1 - 3x + x^2 + \frac{1}{5}x^3 + \frac{3}{35}x^4 + \dots \right].
\end{aligned}$$

For $m = \frac{1}{2}$, we have

$$\begin{aligned}
a_{n+1} &= \frac{2n^2 + 2m^2 + 4mn - m - n - 3}{(n+m+1)(2n+2m+1)} a_n \\
&= \frac{2n^2 + \frac{1}{2} + 2n - \frac{1}{2} - n - 3}{(n+\frac{3}{2})(2n+2)} a_n \\
&= \frac{2n^2 + n - 3}{(2n+3)(n+1)} a_n \\
&= \frac{(2n+3)(n-1)}{(2n+3)(n+1)} a_n = \frac{n-1}{n+1} a_n.
\end{aligned}$$

Putting $n = 0, 1, 2, \dots$, we get

$$\begin{aligned}
a_1 &= -a_0, \quad a_2 = 0, \\
a_3 &= \frac{1}{4}a_2 = 0, \quad a_4 = \frac{1}{2}a_3 = 0,
\end{aligned}$$

and so on. Therefore, the solution corresponding to $m = \frac{1}{2}$ is

$$\begin{aligned}
y_2 &= a_0 x^{\frac{1}{2}} + a_1 x^{\frac{3}{2}} + a_2 x^{\frac{5}{2}} + a_3 x^{\frac{7}{2}} + \dots \\
&= a_0 x^{\frac{1}{2}} - a_0 x^{\frac{3}{2}} = a_0 x^{\frac{1}{2}}(1-x).
\end{aligned}$$

Hence, the general solution of the given equation is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left[1 - 3x + x^2 + \frac{1}{5}x^3 + \frac{3}{35}x^4 + \dots \right] \\
&\quad + c_2 x^{\frac{1}{2}}(1-x).
\end{aligned}$$

EXAMPLE 2.147

Find series solution about $x = 0$ of the differential equation

$$x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - \left(x^2 + \frac{5}{4}\right)y = 0.$$

Solution. The point $x = 0$ is a regular singular point of the given equation. So, let

$$y = \sum_{n=0}^{\infty} a_n x^{n+m}, \quad a_0 \neq 0.$$

Differentiating twice in succession, we have

$$\begin{aligned} \frac{dy}{dx} &= \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1}, \\ \frac{d^2 y}{dx^2} &= \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m-2}. \end{aligned}$$

Putting the values of y , $\frac{dy}{dx}$, and $\frac{d^2 y}{dx^2}$ in the given differential equation, we have

$$\begin{aligned} \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m} - \sum_{n=0}^{\infty} (n+m) a_n x^{n+m} \\ - \sum_{n=0}^{\infty} a_n x^{n+m+2} - \frac{5}{4} \sum_{n=0}^{\infty} a_n x^{n+m} = 0 \end{aligned}$$

or

$$\begin{aligned} \sum_{n=0}^{\infty} \left[(n+m)(n+m-1) - (n+m) - \frac{5}{4} \right] a_n x^{n+m} \\ - \sum_{n=0}^{\infty} a_n x^{n+m+2} = 0 \end{aligned}$$

or

$$\begin{aligned} \sum_{n=0}^{\infty} \left[(n+m)(n+m-2) - \frac{5}{4} \right] a_n x^{n+m} \\ - \sum_{n=2}^{\infty} a_{n-2} x^{n+m} = 0 \end{aligned}$$

or

$$\begin{aligned} \left[m(m-2) - \frac{5}{4} \right] a_0 x^m + \left[(m+1)(m-1) - \frac{5}{4} \right] a_1 x^{m+1} \\ - \sum_{n=2}^{\infty} \left[\left((n+m)(n+m-2) - \frac{5}{4} \right) a_n - a_{n-2} \right] x^{n+m} = 0. \end{aligned}$$

Therefore, the indicial equation is

$$m(m-2) - \frac{5}{4} = 0, \text{ which yields } m = \frac{5}{2}, -\frac{1}{2}.$$

Equating to zero the other coefficients, we get

$$\left[(m+1)(m-1) - \frac{5}{4} \right] a_1 = 0 \text{ and so } a_1 = 0$$

and

$$\left[(n+m)(n+m-2) - \frac{5}{4} \right] a_n = a_{n-2} \text{ for } n \geq 2.$$

For $m = -\frac{1}{2}$, we get

$$\left[\left(n - \frac{1}{2} \right) \left(n - \frac{5}{2} \right) - \frac{5}{4} \right] a_n = a_{n-2}, \quad n \geq 2 \quad (126)$$

or

$$a_n = \frac{1}{n(n-3)} a_{n-2}, \quad n \geq 2, \quad n \neq 3 \quad (127)$$

Hence, for $m = -\frac{1}{2}$, we have, from (127), $a_2 = -\frac{1}{2} a_0$. Putting $n = 3$ in (126), we have

$$\left[\left(3 - \frac{1}{2} \right) \left(3 - \frac{5}{2} \right) - \frac{5}{4} \right] a_3 = a_1 \quad \text{or} \quad 0 \cdot a_3 = a_1$$

and so a_3 may be any constant. Further, from (127), we have

$$\begin{aligned} a_4 &= -\frac{1}{8} a_0, \quad a_5 = \frac{a_3}{10}, \quad a_6 = \frac{-a_0}{144}, \\ a_7 &= \frac{a_3}{280}, \text{ and so on.} \end{aligned}$$

Hence for $m = -\frac{1}{2}$, the required solution is

$$\begin{aligned} y &= a_0 x^{-\frac{1}{2}} \left(1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{144} - \dots \right) \\ &+ a_3 x^{-\frac{1}{2}} \left(x^3 + \frac{x^5}{10} + \frac{x^7}{280} + \dots \right). \quad (128) \end{aligned}$$

Since this solution contains two constants a_0 and a_3 and $a_0 \neq 0$, this is general solution of the given differential equation. By taking $m = \frac{5}{2}$, the solution is

$$y = x^{\frac{5}{2}} \left(1 + \frac{x^2}{10} + \frac{x^4}{280} + \dots \right).$$

Hence, (128) is general solution.

EXAMPLE 2.148

Find series solution about $x = 0$ of the differential equation

$$9x(1-x) \frac{d^2 y}{dx^2} - 12 \frac{dy}{dx} + 4y = 0.$$

Solution. The point $x = 0$ is a regular singular point or of the given equation. Let

$$y = \sum_{n=0}^{\infty} a_n x^{n+m}, \quad a_0 \neq 0.$$

Differentiation of y with respect to x yields

$$\begin{aligned} \frac{dy}{dx} &= \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1}, \\ \frac{d^2y}{dx^2} &= \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m-2}. \end{aligned}$$

Substituting the values of y , $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in the given equation, we get

$$\begin{aligned} &9 \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m-1} \\ &- 9 \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m} \\ &- 12 \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1} + 4 \sum_{n=0}^{\infty} a_n x^{n+m} = 0 \end{aligned}$$

or

$$\begin{aligned} &\sum_{n=0}^{\infty} (n+m)(9n+9m-21) a_n x^{n+m-1} \\ &- \sum_{n=0}^{\infty} [9(n+m)(n+m-1) - 4] a_n x^{n+m} = 0 \end{aligned}$$

or

$$\begin{aligned} &\sum_{n=-1}^{\infty} (n+m+1)(9n+9m-12) a_{n+1} x^{n+m} \\ &- \sum_{n=0}^{\infty} [9(n+m)(n+m-1) - 4] a_n x^{n+m} = 0 \end{aligned}$$

or

$$\begin{aligned} &m(9m-21) a_0 x^{m-1} + \sum_{n=0}^{\infty} [(n+m+1)(9n+9m-12) a_{n+1} \\ &- (9(n+m)(n+m-1) - 4) a_n] x^{n+m} = 0. \end{aligned}$$

Therefore, the indicial equation is

$$m(9m-21) = 0, \text{ which yields } m = 0, \frac{7}{3}.$$

We note that the roots are distinct and their difference is not an integer. Equating other coefficients of the powers of x to zero, we get

$$\begin{aligned} &(n+m+1)(9n+9m-12) a_{n+1} \\ &= [9(n+m)(n+m-1) - 4] a_n \end{aligned}$$

$$a_{n+1} = \frac{9(n+m)(n+m-1) - 4}{(n+m+1)(9n+9m-12)} a_n, \quad n \geq 0.$$

For $m = 0$, we get

$$a_{n+1} = \frac{3m+1}{3m+3} a_n, \quad n \geq 0.$$

Putting $n = 0, 1, 2, 3, \dots$, we obtain

$$\begin{aligned} a_1 &= \frac{1}{3} a_0, \quad a_2 = \frac{2}{3} a_1 = \frac{2}{9} a_0, \\ a_3 &= \frac{7}{9} a_2 = \frac{14}{81} a_0, \quad a_4 = \frac{5}{6} a_3 = \frac{35}{243} a_0, \end{aligned}$$

and so on. Thus, the solution corresponding to $m = 0$ is

$$y_1 = a_0 \left[1 + \frac{x}{3} + \frac{2}{9} x^2 + \frac{14}{81} x^3 + \frac{35}{243} x^4 + \dots \right].$$

For $m = -\frac{7}{3}$, we have

$$a_{n+1} = \frac{9(n-\frac{7}{3})(n-\frac{7}{3}-1)}{3(n-\frac{7}{3}+1)(3n-7-4)} = \frac{3n-6}{3n-4} a_n, \quad n \geq 0.$$

Putting $n = 0, 1, 2, 3, \dots$, we get

$$\begin{aligned} a_1 &= \frac{3}{2} a_0, \quad a_2 = 3a_1 = \frac{9}{2} a_0, \\ a_3 &= 0, \quad a_4 = 0, \quad a_5 = 0, \dots \end{aligned}$$

Thus, the solution corresponding to $m = -\frac{7}{3}$ is

$$y_2 = a_0 x^{\frac{7}{3}} \left(1 + \frac{3}{2} x + \frac{9}{2} x^2 \right).$$

Hence, the general solution of the given differential equation is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left[1 + \frac{x}{3} + \frac{2}{9} x^2 + \frac{14}{81} x^3 + \frac{35}{243} x^4 + \dots \right] \\ &\quad + c_2 x^{\frac{7}{3}} \left(1 + \frac{3}{2} x + \frac{9}{2} x^2 \right). \end{aligned}$$

EXAMPLE 2.149

Find series solution about $x = 0$ of the differential equation

$$(1+x^2) \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = 0.$$

Solution. Since $x = 0$ is a regular point of the given equation, so let

$$y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$$

Then,

$$\begin{aligned}\frac{dy}{dx} &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots + na_nx^{n-1} + \dots \\ \frac{d^2y}{dx^2} &= 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 \\ &\quad + \dots + n(n-1)a_nx^{n-2} + \dots\end{aligned}$$

Substituting the values of y , $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in the given equation, we get

$$\begin{aligned}(1+x^2)[2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 \\ + \dots + n(n-1)a_nx^{n-2}] \\ + x[a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots + na_nx^{n-1} + \dots] \\ - [a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots] = 0.\end{aligned}$$

Equating to zero the coefficients of various powers of x , we get

$$2a_2 - a_0 = 0 \text{ and so } a_2 = \frac{a_0}{2},$$

$$6a_3 + a_1 - a_1 = 0 \text{ and so } a_3 = 0,$$

$$2a_2 + 12a_4 + 2a_2 - a_2 = 0 \text{ and so}$$

$$a_4 = -\frac{1}{4}a_2 = -\frac{1}{8}a_0,$$

$$6a_3 + 20a_5 + 3a_3 + a_3 = 0 \text{ and so } a_5 = 0,$$

and in general,

$$n(n-1)a_n + (n+2)(n+1)a_{n+2} + na_n - na_n = 0$$

or

$$a_{n+2} = \frac{n(n+1)}{(n+1)(n+2)}a_n.$$

Putting $n = 4, 5, \dots$, we get

$$a_6 = \frac{12}{30}a_4 = \frac{2}{5}a_4 = -\frac{1}{20}a_0$$

and so on. Thus, the required solution is

$$y = a_0 \left(1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{20}x^6 - \dots \right) + a_1x.$$

2.24 BESSEL'S EQUATION AND BESSEL'S FUNCTION

The equation

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0,$$

where n is a non-negative real number, is called Bessel's equation of order n . It occurs in problems related to vibrations, electric fields, heat conduction, etc. For $n = 0$, we have already found its solution in Example 2.136. Now we find series solution of Bessel's equation of order n . This equation can be written as

$$\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left(1 - \frac{n^2}{x^2} \right) = 0.$$

Since $\frac{1}{x}$ and $\left(1 - \frac{n^2}{x^2} \right)$ are not analytic at 0, it follows that 0 is a singular point of the given equation. But $x\left(\frac{1}{x}\right)$ and $x^2\left(1 - \frac{n^2}{x^2}\right)$ are analytic at 0. Therefore, $x = 0$ is a regular singular point of the equation. So, let

$$y = \sum_{m=0}^{\infty} a_m x^{m+r}, \quad a_m \neq 0$$

be the series solution of the equation about $x = 0$. Differentiating twice in succession, we get

$$\frac{dy}{dx} = \sum_{m=0}^{\infty} (m+r)a_m x^{m+r-1}$$

$$\frac{d^2y}{dx^2} = \sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r-2}.$$

Putting the values of y , $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in the given differential equation, we get

$$\begin{aligned}\sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r} + \sum_{m=0}^{\infty} (m+r)a_m x^{m+r} \\ + \sum_{m=0}^{\infty} a_m x^{m+r+2} - n^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0\end{aligned}$$

or

$$\sum_{m=0}^{\infty} [(m+r)^2 - n^2]a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} = 0$$

or

$$\sum_{m=0}^{\infty} [(m+r)^2 - n^2]a_m x^{m+r} + \sum_{m=2}^{\infty} a_{m-2} x^{m+r} = 0$$

or

$$\begin{aligned}(r^2 - n^2)a_0 x^r + [(r+1)^2 - n^2]a_1 x^{r+1} \\ + \sum_{m=2}^{\infty} [(m+r)^2 - n^2]a_m + a_{m-2} x^{m+r} = 0.\end{aligned}$$

Equating to zero the coefficient of lower power of x , we get the indicial equation as

$$r^2 - n^2 = 0, \text{ which yields } r = n, -n.$$

Equating other coefficients to zero, we get

$$[(r+1)^2 - n^2]a_1 = 0 \text{ or } a_1 = 0$$

and

$$[(m+r)^2 - n^2]a_m + a_{m-2} = 0,$$

which yields

$$a_m = \frac{a_{m-2}}{(m+r)^2 - n^2}, \quad m \geq 2.$$

Putting $m = 2, 3, 4, 5, 6, \dots$, we get

$$a_3 = a_5 = a_7 = \dots = 0$$

and

$$a_2 = -\frac{a_0}{(r+2)^2 - n^2}$$

$$a_4 = \frac{(-1)^2 a_0}{\{(r+2)^2 - n^2\} \{(r+4)^2 - n^2\}}$$

$$a_6 = \frac{(-1)^3 a_0}{\{(r+2)^2 - n^2\} \{(r+4)^2 - n^2\} \{(r+6)^2 - n^2\}}$$

and so on. For $r = n$, we have

$$a_1 = 0, \quad a_2 = -\frac{a_0}{(n+2)^2 - n^2} = -\frac{a_0}{4(n+1)},$$

$$a_4 = \frac{a_0}{4^2 2! (n+1)(n+2)},$$

$$a_6 = \frac{-a_0}{4^3 3! (n+1)(n+2)(n+3)}, \text{ and so on,}$$

where as $a_2 = a_5 = a_7 = \dots = 0$. Thus the solution corresponding to $r = n$ is

$$y_1 = a_0 x^n \left[1 - \frac{1}{4(n+1)} x^2 + \frac{1}{4^2 2! (n+1)(n+2)} x^4 - \frac{1}{4^3 3! (n+1)(n+2)(n+3)} x^6 + \dots \right]. \quad (129)$$

Similarly, for $r = -n$, the solution is

$$y_2 = a_0 x^{-n} \left[1 - \frac{1}{4(1-n)} x^2 + \frac{1}{4^2 2! (1-n)(2-n)} x^4 - \frac{1}{4^3 3! (1-n)(2-n)(3-n)} x^6 + \dots \right]. \quad (130)$$

We observe that $y_1 = y_2$ for $n = 0$. Further, y_1 is meaningless if n is a negative integer and y_2 is

meaningless if n is a positive integer. Hence if n is non-zero and non-integer, then the general solution of the Bessel's equation of order n is

$$y = c_1 y_1 + c_2 y_2.$$

But y_1 can be expressed as

$$\begin{aligned} y_1 &= a_0 x^n \Gamma(n+1) \left[\frac{1}{\Gamma(n+1)} - \frac{1}{2^2 (n+1) \Gamma(n+1)} x^2 \right. \\ &\quad + \frac{1}{2^4 2! (n+1)(n+2) \Gamma(n+1)} x^4 \\ &\quad \left. + \frac{1}{2^6 3! (n+1)(n+2)(n+3) \Gamma(n+1)} x^6 + \dots \right] \\ &= a_0 x^n \Gamma(n+1) \left[\frac{1}{\Gamma(n+1)} - \frac{1}{2^2 \Gamma(n+2)} x^2 \right. \\ &\quad \left. + \frac{1}{2^4 2! \Gamma(n+3)} x^4 + \frac{1}{2^6 3! \Gamma(n+4)} x^6 + \dots \right] \\ &= a_0 x^n \Gamma(n+1) \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m! \Gamma(n+m+1)} \\ &= a_0 2^n \Gamma(n+1) \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(n+m+1)} \left(\frac{x}{2}\right)^{2m+n} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(n+m+1)} \left(\frac{x}{2}\right)^{2m+n}, \end{aligned} \quad (131)$$

where $a_0 = \frac{1}{2^n \Gamma(n+1)}$. The solution (131) is called the *Bessel's function of the first kind of order n* and is denoted by $J_n(x)$. Thus,

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(n+m+1)} \left(\frac{x}{2}\right)^{2m+n}. \quad (132)$$

Replacing n by $-n$ in $J_n(x)$, we get

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(-n+m+1)} \left(\frac{x}{2}\right)^{2m-n}, \quad (133)$$

which is called *Bessel's function of the first kind of order $-n$* . Thus, the complete solution of the Bessel's equation of order n may be expressed as

$$y = c_1 J_n(x) + c_2 J_{-n}(x), \quad (134)$$

whose n is not an integer.

When n is an integer, let $y = u(x)J_n(x)$ be a solution of the Bessel's equation of order n . Then

$$\begin{aligned}\frac{dy}{dx} &= u'(x)J_n(x) + u(x)J_n'(x) \\ \frac{d^2y}{dx^2} &= u''(x)J_n(x) + J_n'(x)u'(x) + u'(x)J_n'(x) + u(x)J_n''(x) \\ &= u''(x)J_n(x) + 2u'(x)J_n'(x) + u(x)J_n''(x).\end{aligned}$$

Putting the values of y , $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in the given equation, we get

$$\begin{aligned}u(x)[x^2J_n''(x) + xJ_n'(x) + (x^2 - n^2)J_n(x)] \\ + x^2u''(x)J_n(x) + 2x^2u'(x)J_n'(x) + xu'(x)J_n(x) = 0.\end{aligned}$$

Since $J_n(x)$ is a solution of the given equation, we have

$$x^2J_n''(x) + xJ_n'(x) + (x^2 - n^2)J_n(x) = 0.$$

Therefore, the above expression reduces to

$$x^2u''(x)J_n(x) + 2x^2u'(x)J_n'(x) + xu'(x)J_n(x) = 0$$

or

$$\frac{u''(x)}{u'(x)} + 2\frac{J_n'(x)}{J_n(x)} + \frac{1}{x} = 0$$

or

$$\frac{d}{dx}(\log u'(x)) + 2\frac{d}{dx}(\log J_n(x)) + \frac{d}{dx}(\log x) = 0$$

or

$$\frac{d}{dx}[\log(xu'(x)J_n^2(x))] = 0.$$

Integrating this expression, we get

$$\log xu'(x)J_n^2(x) = \log B \text{ so that } xu'(x)J_n^2(x) = B.$$

Thus

$$u'(x) = \frac{B}{xJ_n^2(x)} \text{ and so } u(x) = B \int \frac{dx}{xJ_n^2(x)} + A.$$

Hence, the complete solution is

$$\begin{aligned}y = u(x)J_n(x) &= \left[A + B \int \frac{dx}{xJ_n^2(x)} \right] J_n(x) \\ &= AJ_n(x) + BJ_n(x) \int \frac{dx}{xJ_n^2(x)} = AJ_n(x) + BY_n(x),\end{aligned}$$

where

$$Y_n(x) = J_n(x) \int \frac{dx}{xJ_n^2(x)}$$

is called the Bessel function of the second kind of order n or the Neumann function.

EXAMPLE 2.150

Show that

$$J_{-n}(x) = (-1)^n J_n(x).$$

Solution. We have

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(n+m+1)} \left(\frac{x}{2}\right)^{2m+n}$$

and

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(-n+m+1)} \left(\frac{x}{2}\right)^{2m-n}.$$

But for positive integer n ,

$$\Gamma(n+m+1) = (n+m)! \text{ and}$$

$$\Gamma(-n+m+1) = (m-n)!$$

Therefore,

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(n+m)!} \left(\frac{x}{2}\right)^{2m+n}$$

and

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m-n)!} \left(\frac{x}{2}\right)^{2m-n}.$$

Since, $(-n)!$ is infinite for $n > 0$, we have

$$\begin{aligned}J_{-n}(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m-n)!} \left(\frac{x}{2}\right)^{2m-n} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^{m+n}}{m!(n+m)!} \left(\frac{x}{2}\right)^{2m+n}, \text{ by changing } m \text{ to } m+n \\ &= (-1)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(n+m)!} \left(\frac{x}{2}\right)^{2m+n} = (-1)^n J_n(x).\end{aligned}$$

EXAMPLE 2.151

Show that

$$(i) \frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$$

$$(ii) \frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$$

$$(iii) J_n(x) = \frac{x}{2n} [J_{n-1}(x) + J_{n+1}(x)]$$

$$(iv) J_n'(x) = \frac{1}{2} [J_{n-1}(x) - J_{n+1}(x)].$$

These results are known as *recurrence formulae for the Bessel's function* $J_n(x)$.

Solution. (i) We know that

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(n+m+1)} \left(\frac{x}{2}\right)^{2m+n}.$$

Therefore,

$$x^n J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(n+m+1)} \frac{x^{2(m+n)}}{2^{2m+n}}$$

and so

$$\begin{aligned} \frac{d}{dx}(x^n J_n(x)) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(n+m+1)} \cdot \frac{x^{2(m+n)-1}}{2^{2m+n}} \\ &= x^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(n-1+m+1)} \left(\frac{x}{2}\right)^{n-1+2m} \\ &= x^n J_{n-1}(x). \end{aligned}$$

(ii) Multiplying the expression for $J_n(x)$ by x^{-n} throughout and differentiating, we get (ii)

(iii) Part (i) implies

$$x^n J'_n(x) + nx^{n-1} J_n(x) = x^n J_{n-1}(x)$$

and so

$$J'_n(x) + \frac{n}{x} J_n(x) = J_{n-1}(x). \quad (135)$$

Similarly, part (ii) yields

$$-J'_n(x) + \frac{n}{x} J_n(x) = J_{n+1}(x). \quad (136)$$

Adding (135) and (136), we obtain

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

or

$$J_n(x) = \frac{x}{2n} [J_{n-1}(x) + J_{n+1}(x)]$$

or

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x).$$

(iv) Subtracting (136) from (135) yields

$$2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)$$

or

$$J'_n(x) = \frac{1}{2} [J_{n-1}(x) - J_{n+1}(x)].$$

EXAMPLE 2.152

Show that $e^{\frac{1}{2}x(z-\frac{1}{z})}$ is the generating function of the Bessel's functions.

Solution. We have

$$\begin{aligned} e^{\frac{1}{2}x(z-\frac{1}{z})} &= e^{\frac{1}{2}xz} \cdot e^{-\frac{1}{2}xz^{-1}} \\ &= \left[1 + \left(\frac{1}{2}x\right)z + \left(\frac{1}{2}x\right)^2 \frac{z^2}{2!} \right. \\ &\quad \left. + \dots + \frac{\left(\frac{1}{2}x\right)^r z^r}{r!} + \dots \right] \\ &\quad \times \left[1 + \left(-\frac{1}{2}x\right)(z^{-1}) + \left(-\frac{1}{2}x\right)^2 \frac{z^{-2}}{2!} \right. \\ &\quad \left. + \dots + \frac{\left(-\frac{1}{2}x\right)^r z^{-r}}{r!} \right]. \end{aligned}$$

The coefficient of z^n in this expansion is

$$\begin{aligned} &\frac{\left(\frac{1}{2}x\right)^n}{n!} + \frac{\left(\frac{1}{2}x\right)^{n+1} \left(-\frac{1}{2}x\right)^2}{(n+1)!} + \frac{\left(\frac{1}{2}x\right)^{n+2} \left(-\frac{1}{2}x\right)^2}{(n+2)! 2!} + \dots \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m! (m+n)!} \left(\frac{1}{2}x\right)^{2m+n} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(n+m+1)} \left(\frac{x}{2}\right)^{2m+n} = J_n(x). \end{aligned}$$

Thus

$$e^{\frac{1}{2}x(z-\frac{1}{z})} = \sum_{n=-\infty}^{\infty} z^n J_n(x).$$

Hence, $e^{\frac{1}{2}x(z-\frac{1}{z})}$ is the generating function of the Bessel's function.

EXAMPLE 2.153

Show that

$$(i) J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

$$(ii) J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

$$(iii) \int x J_0^2(x) dx = \frac{1}{2} x^2 [J_0^2(x) + J_1^2(x)].$$

Solution. (i) Putting $n = \frac{1}{2}$ in the expression for $J_n(x)$, we have

$$\begin{aligned} J_{\frac{1}{2}}(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(\frac{1}{2} + m + 1)} \left(\frac{x}{2}\right)^{2m+\frac{1}{2}} \\ &= \left(\frac{x}{2}\right)^{\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \frac{3}{2})} \left(\frac{x}{2}\right)^{2m} \\ &= \left(\frac{x}{2}\right)^{\frac{1}{2}} \left[\frac{1}{\Gamma(\frac{3}{2})} - \frac{1}{\Gamma(\frac{5}{2})} \left(\frac{x}{2}\right)^2 + \frac{1}{2! \Gamma(\frac{7}{2})} \left(\frac{x}{2}\right)^4 - \dots \right] \\ &= \left(\frac{x}{2}\right)^{\frac{1}{2}} \left[\frac{1}{\frac{1}{2} \Gamma(\frac{1}{2})} - \frac{1}{\frac{3}{2} \Gamma(\frac{1}{2})} \left(\frac{x}{2}\right)^2 \right. \\ &\quad \left. + \frac{1}{2 \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma(\frac{1}{2})} \left(\frac{x}{2}\right)^4 - \dots \right] \\ &= \left(\frac{x}{2}\right)^{\frac{1}{2}} \frac{2}{\Gamma(\frac{1}{2})} \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots \right] \\ &= \frac{\sqrt{2}}{\sqrt{x} \sqrt{\pi}} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right] = \sqrt{\frac{2}{\pi x}} \sin x. \end{aligned}$$

(ii) Putting $n = \frac{1}{2}$ in the expression for $J_{-n}(x)$ and proceeding as in part (i), we obtain part (ii).

(iii) We have

$$\begin{aligned} &\int x J_0^2(x) dx \\ &= \frac{x^2}{2} J_0^2(x) - \int \frac{x^2}{2} J_0(x) J_0'(x) dx \text{ (integration by parts)} \\ &= \frac{x^2}{2} J_0^2(x) + \int x^2 J_0(x) J_1(x) dx \text{ since } J_0'(x) = -J_1(x) \\ &= \frac{x^2}{2} J_0^2(x) + \int x J_1(x) \cdot \frac{d}{dx}(x J_1(x)) dx \text{ since} \\ &\quad \frac{d}{dx}(x J_1(x)) = x J_0(x) \\ &= \frac{x^2}{2} J_0^2(x) + \frac{1}{2} (x J_1(x))^2 = \frac{x^2}{2} [J_0^2(x) + J_1^2(x)]. \end{aligned}$$

EXAMPLE 2.154

Show that $J_n(\lambda_i x)$ is a solution of

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (\lambda_i^2 x^2 - n^2) y = 0.$$

Solution. We know that $J_n(z)$ is a solution of the Bessel's equation

$$z^2 \frac{d^2 y}{dz^2} + z \frac{dy}{dz} + (z^2 - n^2) y = 0. \quad (137)$$

Put $z = \lambda_i x$. Then

$$\frac{dy}{dz} = \frac{dy}{dx} \cdot \frac{dx}{dz} = \frac{1}{\lambda_i} \frac{dy}{dx}$$

and

$$\frac{d^2 y}{dz^2} = \frac{1}{\lambda_i^2} \frac{d^2 y}{dx^2}.$$

Putting the values of z , $\frac{dy}{dz}$, and $\frac{d^2 y}{dz^2}$ in (137), we get

$$\lambda_i^2 x^2 \left(\frac{1}{\lambda_i^2} \frac{d^2 y}{dx^2} \right) + \lambda_i x \left(\frac{1}{\lambda_i} \frac{dy}{dx} \right) + (\lambda_i^2 x^2 - n^2) y = 0.$$

or

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (\lambda_i^2 x^2 - n^2) y = 0. \quad (138)$$

Hence $J_n(\lambda_i x)$ is a solution of (138).

EXAMPLE 2.155

If $\lambda_1, \lambda_2, \dots$ are the roots of $J_n(ax) = 0$, $a \in \mathbb{R}$, show that the system

$$J_n(\lambda_1 x), J_n(\lambda_2 x), J_n(\lambda_3 x), \dots$$

of Bessel's functions is an orthogonal system, with weight function x , on $[0, a]$, $a \in \mathbb{R}$.

Solution. Let

$$u_i = J_n(\lambda_i x) \text{ and } u_j = J_n(\lambda_j x), \quad i \neq j.$$

Then, by Example 2.145, we have

$$x^2 u_i'' + x u_i' + (\lambda_i^2 x^2 - n^2) u_i = 0 \quad (139)$$

and

$$x^2 u_j'' + x u_j' + (\lambda_j^2 x^2 - n^2) u_j = 0. \quad (140)$$

Multiplying (139) by u_j and (140) by u_i and subtracting, we get

$$x^2 (u_j u_i'' - u_i u_j'') + x (u_j u_i' - u_i u_j') + x^2 u_i u_j (\lambda_i^2 - \lambda_j^2) = 0$$

or

$$x (u_j u_i'' - u_i u_j'') + (u_j u_i' - u_i u_j') = x (\lambda_j^2 - \lambda_i^2) u_i u_j$$

or

$$x \frac{d}{dx} (u_j u'_i - u_i u'_j) + (u_j u'_i - u_i u'_j) = x (\lambda_j^2 - \lambda_i^2) u_i u_j$$

or

$$\frac{d}{dx} [x (u_j u'_i - u_i u'_j)] = x (\lambda_j^2 - \lambda_i^2) u_i u_j. \quad (141)$$

Integrating (141) with respect to x in $[0, a]$, we get

$$\left[x (u_j u'_i - u_i u'_j) \right]_0^a = (\lambda_j^2 - \lambda_i^2) \int_0^a x u_i u_j dx$$

or

$$\begin{aligned} & a [\lambda_i J_n(\lambda_j a) J'_n(\lambda_i a) - \lambda_j J_n(\lambda_i a) J'_n(\lambda_j a)] \\ &= (\lambda_j^2 - \lambda_i^2) \int_0^a x u_i u_j dx. \end{aligned} \quad (142)$$

Since $J_n(\lambda_j a) = J_n(\lambda_i a) = 0$, it follows that

$$(\lambda_j^2 - \lambda_i^2) \int_0^a x u_i u_j = 0, \quad \lambda_i \neq \lambda_j$$

or

$$\int_0^a x J_n(\lambda_i x) J_n(\lambda_j x) dx = 0, \quad i \neq j.$$

Hence the system $\{J_n(\lambda_i x)\}$ forms an orthogonal system with weight x .

Remark. From (142), we have

$$\begin{aligned} & \int_0^a x J_n(\lambda_i x) J_n(\lambda_j x) dx \\ &= \frac{a [(\lambda_i J_n(\lambda_j a) J'_n(\lambda_i a) - \lambda_j J_n(\lambda_i a) J'_n(\lambda_j a))]}{\lambda_j^2 - \lambda_i^2}. \end{aligned}$$

If $\lambda_i = \lambda_j$, then right hand side of the above expression is of $\frac{0}{0}$ form. We assume that λ_i is a root of $J_n(ax) = 0$ and $\lambda_j \rightarrow \lambda_i$. Then we have

$$\lim_{\lambda_j \rightarrow \lambda_i} \int_0^a x J_n(\lambda_i x) J_n(\lambda_j x) dx = \lim_{\lambda_j \rightarrow \lambda_i} \frac{a \lambda_i J_n(\lambda_j a) J'_n(\lambda_i a)}{\lambda_j^2 - \lambda_i^2}$$

Or, using L Hospital's Rule,

$$\begin{aligned} \int_0^a x J_n^2(\lambda_i x) dx &= \lim_{\lambda_j \rightarrow \lambda_i} \frac{a^2 \lambda'_i J_n(\lambda_j a) J'_n(\lambda_i a)}{2 \lambda_j} \\ &= \frac{a^2}{2} [J'_n(\lambda_i a)]^2 = \frac{a^2}{2} J_{n+1}^2(\lambda_i a). \end{aligned}$$

Hence

$$\frac{2}{a^2 J_{n+1}^2(\lambda_i a)} \int_0^a x J_n^2(\lambda_i x) dx = 1.$$

EXAMPLE 2.156

Show that $x^n J_n(x)$ is a solution of the differential equation

$$x \frac{d^2 y}{dx^2} + (1 - 2n) \frac{dy}{dx} + xy = 0.$$

Solution. Let $y = x^n J_n(x)$. Then

$$\frac{dy}{dx} = \frac{d}{dx} (x^n J_n(x)) = x^n J_{n-1}, \text{ and}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} (x^n J_{n-1}) = x^n J'_{n-1} + n x^{n-1} J_{n-1}.$$

Putting the values of y , $\frac{dy}{dx}$, and $\frac{d^2 y}{dx^2}$ in the given equation, we get

$$\begin{aligned} & x \frac{d^2 y}{dx^2} + (1 - 2n) \frac{dy}{dx} + xy \\ &= x^{n+1} \left[J'_{n-1} - \frac{n-1}{x} J_{n-1} \right] + x^{n+1} J_n \\ &= x^{n+1} (-J_n) + x^{n+1} J_n = 0. \end{aligned}$$

EXAMPLE 2.157

Express $J_4(x)$ in term of $J_0(x)$ and $J_1(x)$.

Solution. We know that

$$J_n(x) = \frac{x}{2n} [J_{n-1}'(x) + J_{n+1}'(x)] \text{ (recurrence formula)}$$

or

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x).$$

Putting $n = 1, 2$, and 3 , we get

$$J_2(x) = \frac{2}{x}J_1(x) - J_0(x)$$

$$\begin{aligned} J_3(x) &= \frac{4}{x}J_2(x) - J_1(x) \\ &= \frac{4}{x} \left[\frac{2}{x}J_1(x) - J_0(x) \right] - J_1(x) \\ &= \frac{8}{x^2}J_1(x) - \frac{4}{x}J_0(x) - J_1(x) \\ &= \left(\frac{8}{x^2} - 1 \right) J_1(x) - \frac{4}{x}J_0(x), \end{aligned}$$

$$\begin{aligned} J_4(x) &= \frac{6}{x}J_3(x) - J_2(x) \\ &= \frac{6}{x} \left[\left(\frac{8}{x} - 1 \right) J_1(x) - \frac{4}{x}J_0(x) \right] \\ &\quad - \left[\frac{2}{x}J_1(x) - J_0(x) \right] \\ &= \frac{48}{x^3}J_1(x) - \frac{6}{x}J_1(x) - \frac{24}{x^2}J_0(x) - \frac{2}{x}J_1(x) + J_0(x) \\ &= \left(\frac{48}{x^3} - \frac{8}{x} \right) J_1(x) + \left(1 - \frac{24}{x^2} \right) J_0(x). \end{aligned}$$

EXAMPLE 2.158

Solve

$$x^2 \frac{d^2 z}{dx^2} + x \frac{dz}{dx} + (x^2 - 64)z = 0.$$

Solution. The given equation is evidently Bessel's equation of order 8, which is an integer. Therefore, its general solution is

$$z = c_1 J_8(x) + c_2 Y_8(x),$$

where c_1 and c_2 are arbitrary constants.

EXAMPLE 2.159

Solve the differential equation

$$x \frac{d^2 y}{dx^2} + \frac{dy}{dx} + \frac{1}{4}y = 0.$$

Solution. Let $z = \sqrt{x}$. Then

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{2\sqrt{x}}, \text{ and}$$

$$\begin{aligned} \frac{d^2 y}{dx^2} &= \frac{dy}{dz} \cdot \frac{1}{2} \left(-\frac{1}{2}x^{-\frac{3}{2}} \right) + \frac{1}{2\sqrt{x}} \left(\frac{d^2 y}{dz^2} \cdot \frac{1}{2\sqrt{x}} \right) \\ &= \frac{1}{4x} \frac{d^2 y}{dz^2} - \frac{1}{4x^{\frac{3}{2}}} \frac{dy}{dz} = \frac{1}{4z^2} \frac{d^2 y}{dz^2} - \frac{1}{4z^3} \frac{dy}{dz}. \end{aligned}$$

Putting these values of the partial derivatives in the given equation, we get

$$z^2 \left[\frac{1}{4z^2} \frac{d^2 y}{dz^2} - \frac{1}{4z^3} \frac{dy}{dz} \right] + \frac{1}{2z} \frac{dy}{dz} + \frac{1}{4}y = 0$$

or

$$\frac{d^2 y}{dz^2} - \frac{1}{z} \frac{dy}{dz} + \frac{2}{z} \frac{dy}{dz} + y = 0$$

or

$$z \frac{d^2 y}{dz^2} + \frac{dy}{dz} + zy = 0$$

or

$$z^2 \frac{d^2 y}{dz^2} + z \frac{dy}{dz} + (z^2 - 0^2)y = 0,$$

which is Bessel's equation of order 0. Therefore, its general solution is

$$y = c_1 J_0(z) + c_2 Y_0(z) = c_1 J_0(\sqrt{x}) + c_2 Y_0(\sqrt{x}).$$

EXAMPLE 2.160

Show that

$$\begin{aligned} e^{\frac{x}{2}(z - \frac{1}{z})} &= J_0(x) + \left(z - \frac{1}{z} \right) J_1(x) + \left(z^2 + \frac{1}{z^2} \right) J_2(x) \\ &\quad + \left(z^3 - \frac{1}{z^3} \right) J_3(x) + \dots \end{aligned}$$

Hence or otherwise show that

$$J_0(x) = \frac{1}{\pi} \int_0^\pi \cos(x \cos \theta) d\theta.$$

Solution. We know that

$$e^{\frac{x}{2}(z - \frac{1}{z})} = \sum_{n=-\infty}^{\infty} z^n J_n(x) \text{ (generating function)}$$

$$= J_0 + \sum_{n=1}^{\infty} (J_n z^n + J_{-n} z^{-n})$$

$$= J_0 + \sum_{n=1}^{\infty} (J_n z^n + (-1)^n J_n z^{-n})$$

$$= J_0 + \sum_{n=1}^{\infty} [z^n + (-1)^n z^{-n}] J_n$$

$$= J_0 + \left(z - \frac{1}{z} \right) J_1 + (z^2 + z^{-2}) J_2$$

$$+ (z^3 - z^{-3}) J_3 + \dots \quad (143)$$

Putting $z = e^{i\phi}$ we get $z - \frac{1}{z} = 2i \sin \phi$, $z^2 + \frac{1}{z^2} = 2 \cos 2\phi$, and so on. Therefore, (143) reduces to

$$e^{i(x \sin \phi)} = J_0 + (2i \sin \phi)J_1 + (2 \cos 2\phi)J_2 + \dots$$

or

$$\begin{aligned} \cos(x \sin \phi) + i \sin(x \sin \phi) \\ = J_0 + (2i \sin \phi)J_1 + (2 \cos 2\phi)J_2 + \dots \end{aligned}$$

Separating real and imaginary parts, we get

$$\begin{aligned} \cos(x \sin \phi) = J_0 + (2 \cos 2\phi)J_2 \\ + (2 \cos 4\phi)J_4 + \dots \end{aligned} \quad (144)$$

and

$$\begin{aligned} \sin(x \sin \phi) = (2 \sin \phi)J_1 + (2 \sin 3\phi)J_3 \\ + (2 \sin 5\phi)J_5 + \dots, \end{aligned} \quad (145)$$

which are called *Jacobi's series*.

Putting $\phi = \frac{\pi}{2} - \theta$ in (144), we get

$$\cos(x \cos \theta) = J_0 - (2 \cos 2\theta)J_2 + \dots$$

Therefore,

$$\begin{aligned} \int_0^\pi \cos(x \cos \theta) d\theta &= \int_0^\pi J_0 d\theta \\ &- \int_0^\pi (2 \cos 2\theta)J_2 d\theta + \dots = \pi J_0. \end{aligned}$$

Hence

$$J_0 = \frac{1}{\pi} \int_0^\pi \cos(x \cos \theta) d\theta.$$

EXAMPLE 2.161

Solve the differential equation

$$x \frac{d^2 y}{dx^2} + \frac{dy}{dx} - ixy = 0.$$

Solution. The given equation can be written as

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - i x^2 y = 0.$$

Putting $z = \sqrt{-i} x$, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dz} \cdot \frac{dz}{dx} = \sqrt{-i} \frac{dy}{dz}, \\ \frac{d^2 y}{dx^2} &= -i \frac{d^2 y}{dz^2}. \end{aligned}$$

Thus, the given equation reduces to

$$z^2 \frac{d^2 y}{dz^2} + z \frac{dy}{dz} + (z^2 - 0)y = 0,$$

which is Bessel's equation of order zero. Its solution is

$$\begin{aligned} y &= J_0(z) = J_0(\sqrt{-i} x) = J_0\left(\frac{3}{i^2} x\right) \\ &= 1 - \frac{i^3 x^2}{2^2} + \frac{i^6 x^4}{2^4 (2!)^2} - \frac{i^9 x^6}{2^6 (3!)^2} + \frac{i^{12} x^8}{2^8 (4!)^2} - \dots \\ &= \left[1 - \frac{x^4}{2^2 \cdot 4^2} + \frac{x^8}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \dots \right] \\ &\quad + i \left[\frac{x^2}{2^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \frac{x^{10}}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10^2} - \dots \right] \\ &= \text{ber } x + i \text{bei } x, \end{aligned}$$

where

$$\text{ber } x = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{x^{4m}}{2^2 \cdot 4^2 \cdot 6^2 \dots (4m)^2}$$

is called *Bessel's real (or ber) function* and

$$\text{bei } x = - \sum_{m=1}^{\infty} (-1)^m \frac{x^{4m-2}}{2^2 \cdot 4^2 \cdot 6^2 \dots (4m-2)^2}$$

called *Bessel's imaginary (or bei) function*.

2.25 FOURIER-BESSEL EXPANSION OF A CONTINUOUS FUNCTION

We have proved in Example 2.155 that Bessel's functions $J_n(\lambda x)$ form an orthogonal set with weight x . This property allows us to expand a continuous function f in Fourier-Bessel series in a range 0 to a . So, let

$$\begin{aligned} f(x) &= c_1 J_n(\lambda_1 x) + c_2 J_n(\lambda_2 x) + \dots \\ &\quad + c_n J_n(\lambda_n x), \end{aligned} \quad (146)$$

where λ_i are the roots of the equation $J_n(\lambda_a) = 0$. To determine the coefficients c_i , we multiply both sides of the equation (146) by $x J_n(\lambda_i x)$ and integrate with respect to x between the limits 0 to a . Using orthogonal property and the remark cited above,

we have

$$\begin{aligned}\int_0^a xf(x)J_n(\lambda_i x)dx &= c_i \int_0^a xJ_n(\lambda_i x)dx \\ &= c_i \left[\frac{a^2}{2} J_{n+1}^2(\lambda_i a) \right].\end{aligned}$$

Hence

$$c_i = \frac{2}{a^2 J_{n+1}^2(\lambda_i a)} \int_0^a xf(x)J_n(\lambda_i x)dx.$$

Substituting the values of c_i in the expression (146), we get the Fourier-Bessel series of f .

EXAMPLE 2.162

Expand $f(x) = x^2$ ($0 < x < a$) in Fourier-Bessel series in $J_2(\lambda_n x)$ if $\lambda_n a$ are the positive roots of $J_2(x) = 0$.

Solution. Let the Fourier-Bessel series of the function $f(x) = x^2$ ($0 < x < a$) be

$$x^2 = \sum_{n=1}^{\infty} c_n J_2(\lambda_n x).$$

We know that

$$\begin{aligned}c_i &= \frac{2}{a^2 J_{n+1}^2(\lambda_i a)} \int_0^a xf(x)J_n(\lambda_i x)dx \\ &= \frac{2}{a^2 J_3^2(\lambda_i a)} \int_0^a x^3 J_2(\lambda_i x)dx \\ &= \frac{2}{a^2 J_3^2(\lambda_i a)} \left[\frac{x^3 J_3(\lambda_i x)}{\lambda_i} \right]_0^a \\ &= \frac{2a}{\lambda_i J_3(\lambda_i a)}.\end{aligned}$$

Hence the Fourier-Bessel Series for $f(x) = x^2$ is

$$x^2 = 2a \sum_{n=1}^{\infty} \frac{J_2(\lambda_n x)}{\lambda_n J_3(\lambda_n a)}.$$

EXAMPLE 2.163

Expand $f(x) = 1$ in Fourier-Bessel series in $J_0(\lambda_n x)$ if $\lambda_1, \lambda_2, \dots, \lambda_n$ are the roots of $J_0(x) = 0$.

Solution. Let the Fourier-Bessel series of $f(x) = 1$ be

$$1 = \sum_{n=1}^{\infty} c_n J_0(\lambda_n x).$$

Since $f(x) = 1$ and $a = 1$, the coefficients c_i are given by

$$\begin{aligned}c_i &= \frac{2}{J_1^2(\lambda_i)} \int_0^1 x J_0(\lambda_i x) dx \\ &= \frac{2}{J_1^2(\lambda_i)} \left[\frac{x J_1(\lambda_i x)}{\lambda_i} \right]_0^1 \\ &= \frac{2}{\lambda_i J_1(\lambda_i)}.\end{aligned}$$

Hence the required Fourier-Bessel series is

$$1 = 2 \sum_{n=1}^{\infty} \frac{J_0(\lambda_n x)}{\lambda_n J_1(\lambda_n)}.$$

2.26 LEGENDRE'S EQUATION AND LEGENDRE'S POLYNOMIAL

The differential equation

$$(1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0 \quad (147)$$

is called *Legendre's equation of order n* , where n is a real number. The equation can be written in the form

$$\frac{d^2 y}{dx^2} - \frac{2x}{1-x^2} \frac{dy}{dx} + \frac{n(n+1)}{1-x^2} y = 0.$$

Since $-\frac{2x}{1-x^2}$ and $\frac{n(n+1)}{1-x^2}$ are analytic at 0, the point $x = 0$ is a regular point of the Legendre's equation. So let

$$y = \sum_{m=0}^{\infty} a_m x^m$$

be the power series solution of the given Legendre's equation about $x = 0$. Then

$$\frac{dy}{dx} = \sum_{m=1}^{\infty} m a_m x^{m-1}, \text{ and}$$

$$\frac{d^2 y}{dx^2} = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}.$$

Putting the values of y , $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$ in the given equation, we get

$$(1-x^2) \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - 2x \sum_{m=1}^{\infty} m a_m x^{m-1} + n(n+1) \sum_{m=0}^{\infty} a_m x^m = 0$$

or

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m - 2 \sum_{m=1}^{\infty} m a_m x^m + n(n+1) \sum_{m=0}^{\infty} a_m x^m = 0$$

or

$$\sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m - \sum_{m=2}^{\infty} m(m-1)a_m x^m - 2 \sum_{m=1}^{\infty} m a_m x^m + n(n+1) \sum_{m=0}^{\infty} a_m x^m = 0$$

or

$$[2a_2 + n(n+1)a_0] + [6a_3 - 2a_1 + n(n+1)a_1]x + \sum_{m=2}^{\infty} [(m+2)(m+1)a_{m+2} + (n-m)(n+m+1)a_m]x^m = 0.$$

Equating to zero the coefficient of powers of x , we get

$$2a_2 + n(n+1)a_0 = 0, \text{ which yields}$$

$$a_2 = -\frac{n(n+1)}{2!}a_0,$$

$$6a_3 - 2a_1 + n(n+1)a_1 = 0, \text{ which gives}$$

$$a_3 = -\frac{(n-1)(n+2)}{3!}a_1,$$

and in general

$$a_{m+2} = -\frac{(n-m)(n+m+1)}{(m+1)(m+2)}a_m, \quad m \geq 2.$$

Putting $m = 0, 1$, we observe that a_2 and a_3 are the same as found above. Therefore,

$$a_{m+2} = -\frac{(n-m)(n+m+1)}{(m+1)(m+2)}a_m, \quad m \geq 0,$$

which is called the *recurrence solution of the Legendre's equation*. Putting $m = 2, 3, \dots$ we have

$$a_4 = \frac{(n-2)n(n+1)(n+3)}{4!}a_0,$$

$$a_5 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}a_1$$

and so on. Hence

$$\begin{aligned} y &= a_0 + a_1x - \frac{n(n+1)}{2!}a_0x^2 - \frac{(n-1)(n+2)}{3!}a_0x^3 \\ &\quad + \frac{(n-2)n(n+1)(n+3)}{4!}a_0x^4 \\ &\quad + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}a_1x^5 + \dots \\ &= a_0 \left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 + \dots \right] \\ &\quad + a_1 \left[x - \frac{(n-1)(n+2)}{3!}x^3 \right. \\ &\quad \left. + \frac{(n-3)(n-1)(n+2)(n+4)}{4!}x^5 + \dots \right] \\ &= a_0y_1(x) + a_1y_2(x), \text{ say.} \end{aligned}$$

Thus, y_1 contains only even powers of x while y_2 contains odd powers of x . Also y_1 and y_2 are linearly independent.

If n is even, then $y_1(x)$ is a polynomial of degree n and if n is odd, then $y_2(x)$ is a polynomial of degree n . Moreover if $a_n = \frac{(2n)!}{2^n(n!)^2}$, then

$$P_n(x) = \begin{cases} a_0 + a_2x^2 + \dots + a_nx^n & \text{if } n \text{ is even} \\ a_1x + a_3x^3 + \dots + a_nx^n & \text{if } n \text{ is odd} \end{cases}$$

is called the *Legendre polynomial of degree n* .

We note that

$$P_0(x) = 1, P_1(x) = x,$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1), P_3(x) = \frac{1}{2}(5x^2 - 3x),$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \text{ and so on.}$$

Evidently, the value of the polynomial $P_n(x)$, $n = 0, 1, 2, \dots$ is 1 for $x = 1$.

EXAMPLE 2.164

Show that

$$P_n(x) = \frac{1}{n!} \frac{d^n}{dx^n} (x^2 - 1)^n \text{ (Rodrigue's formula).}$$

Solution. Putting $u = (x^2 - 1)^n$, we have

$$\frac{du}{dx} = 2nx(x^2 - 1)^{n-1} = \frac{2nxu}{x^2 - 1}$$

and so

$$(1 - x^2) \frac{du}{dx} + 2nxu = 0. \quad (148)$$

Differentiating (148), $(n + 1)$ times by Leibnitz's theorem, we get

$$(1 - x^2)u_{n+2} - 2(n+1)xu_{n+1} - n(n+1)u_n + 2n[xu_{n+1} + (n+1)u_n] = 0$$

or

$$(1 - x^2)u_{n+2} - 2xu_{n+1} + n(n+1)u_n = 0$$

or

$$(1 - x^2) \frac{d^2u_n}{dx^2} - 2x \frac{du_n}{dx} + n(n+1)u_n = 0.$$

Thus, $u_n = \frac{d^n}{dx^n} (x^2 - 1)^n$ satisfies the Legendre's equation

$$(1 - x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0.$$

Since the solution of Legendre's equation is $P_n(x)$, we have

$$P_n(x) = cu_n = c \frac{d^n}{dx^n} (x^2 - 1)^n. \quad (149)$$

Putting $x = 1$ in (149), we get

$$1 = c \left[\frac{d^n}{dx^n} (x-1)^n (x+1)^n \right]_{x=1} \\ = c[n! (x+1)^n + \text{term containing } x-1$$

along with its powers] $_{x=1}$

$$= c n! 2^n,$$

and so $c = \frac{1}{n! 2^n}$. Thus,

$$P_n(x) = \frac{1}{n! 2^n} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

EXAMPLE 2.165

Show that

$$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x), \quad |x| < 1, \quad |t| < \frac{1}{3}.$$

[Thus $(1 - 2xt + t^2)^{-\frac{1}{2}}$ is a *generating function of the Legendre's polynomials*.]

Solution. We have, by binomial expansion,

$$\begin{aligned} & [1 - (2xt - t^2)]^{\frac{1}{2}} \\ &= 1 + \frac{1}{2}(2xt - t^2) + \frac{1.3}{2^2 \cdot 2!}(2xt - t^2)^2 + \dots \\ & \quad + \frac{1.3 \dots (2n-5)}{2^{n-2}(n-2)!}(2xt - t^2)^{n-2} \\ & \quad + \frac{1.3 \dots (2n-3)}{2^{n-1}(n-1)!}(2xt - t^2)^{n-1} \\ & \quad + \frac{1.3 \dots (2n-1)}{2^n \cdot n!}(2xt - t^2)^n + \dots \\ &= 1 + xt + \left(-\frac{1}{2} + \frac{3}{8} \cdot 4x^2\right)t^2 + \dots \\ & \quad + \left[\frac{1.3 \dots (2n-1)}{2^n \cdot n!}(2x)^n - \frac{1.3 \dots (2n-3)}{2^{n-1}(n-1)!}(n-1)(2x)^{n-2}\right. \\ & \quad \left.+ \frac{1.3 \dots (2n-5)(n-2)(n-3)}{2^{n-2}(n-2)!} \cdot \frac{1.2}{1.2}(2x)^{n-4} + \dots\right]t^n + \dots \\ &= 1 + xt + \frac{1}{2}(3x^2 - 1)t^2 + \dots \\ & \quad + \frac{1.3 \dots (2n-1)}{n!} \left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} \right. \\ & \quad \left. + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} + \dots \right]t^n + \dots \\ &= P_0(x) + tP_1(x) + t^2P_2(x) + \dots + t^n P_n(x) + \dots \\ &= \sum_{n=0}^{\infty} t^n P_n(x). \end{aligned}$$

Hence, $(1 - 2xt + t^2)^{-\frac{1}{2}}$ generates Legendre polynomials $P_n(x)$.

EXAMPLE 2.166

Express the polynomial $x^3 + 2x^2 - x - 3$ in terms of Legendre's polynomials.

Solution. We know that

$$P_0(x) = 1, \quad P_1(x) = x,$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \text{ and so } x^2 = \frac{2}{3}P_2(x) + \frac{1}{3},$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \text{ and so } x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x).$$

Hence,

$$\begin{aligned}
 & x^3 + 2x^2 - x - 3 \\
 &= \left[\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x) \right] + 2 \left[\frac{2}{3}P_2(x) + \frac{1}{3} \right] \\
 &\quad - P_1(x) - 3P_0(x) \\
 &= \frac{2}{5}P_3(x) + \frac{4}{3}P_2(x) - \frac{2}{5}P_1(x) - \frac{7}{3}P_0(x).
 \end{aligned}$$

EXAMPLE 2.167

Show that Legendre polynomials are *orthogonal* on the interval $[-1, 1]$.

Solution. Let P_m and P_n be Legendre polynomials for $m \neq n$. We want to show that

$$\int_{-1}^1 P_m(x)P_n(x)dx = 0, \quad m \neq n.$$

Since P_m and P_n satisfy Legendre's equation, we have

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0 \quad (150)$$

$$(1-x^2)P_m''(x) - 2xP_m'(x) + m(m+1)P_m(x) = 0. \quad (151)$$

Multiplying (150) by $P_n(x)$ and (151) by $P_m(x)$ and subtracting, we get

$$\begin{aligned}
 & (1-x^2)[P_n(x)P_m''(x) - P_m(x)P_n''(x)] \\
 & \quad - 2x[P_n(x)P_m'(x) - P_m(x)P_n'(x)] \\
 &= (n^2 - m^2 + n - m)P_m(x)P_n(x)
 \end{aligned}$$

or

$$\begin{aligned}
 & \frac{d}{dx} [(1-x^2)(P_n(x)P_m'(x) - P_m(x)P_n'(x))] \\
 &= (n-m)(n+m+1)P_m(x)P_n(x).
 \end{aligned}$$

Integration on the interval $[-1, 1]$ yields

$$\begin{aligned}
 & [(1-x^2)(P_n(x)P_m'(x) - P_m(x)P_n'(x))]_{-1}^1 \\
 &= (n-m)(n+m+1) \int_{-1}^1 P_m(x)P_n(x)dx
 \end{aligned}$$

or

$$0 = (n-m)(n+m+1) \int_{-1}^1 P_m(x)P_n(x)dx$$

or

$$\int_{-1}^1 P_m(x)P_n(x)dx = 0, \quad m \neq n.$$

Hence, $\{P_n(x)\}$ forms an orthogonal system.

EXAMPLE 2.168

Show that $\left\{ \sqrt{\frac{2n+1}{2}} P_n(x) \right\}$ forms an *orthonormal* system.

Solution. The system of polynomials $\left\{ \sqrt{\frac{2n+1}{2}} P_n(x) \right\}$ will form an orthonormal system if

$$\frac{2n+1}{2} \int_{-1}^1 P_m(x)P_n(x)dx = \begin{cases} 0 & \text{for } m \neq n \\ 1 & \text{for } m = n. \end{cases}$$

By Example 2.167, it follows that

$$\frac{2n+1}{2} \int_{-1}^1 P_m(x)P_n(x)dx = 0, \quad m \neq n.$$

It remains to show that $\int_{-1}^1 P_n^2(x)dx = \frac{2}{2n+1}$.

In this direction, we have by generating functions,

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{m=0}^{\infty} t^m P_m(x)$$

and

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x).$$

Multiplying these two expressions, we get

$$(1-2xt+t^2)^{-1} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} t^{m+n} P_m(x)P_n(x).$$

Integrating w.r.t. x on $[-1, 1]$, we have

$$\begin{aligned}
 & \int_{-1}^1 (1-2xt+t^2)^{-1} dx \\
 &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \int_{-1}^1 t^{m+n} P_m(x)P_n(x)dx.
 \end{aligned}$$

Since $\int_{-1}^1 P_m(x)P_n(x)dx = 0$ for $m \neq n$, the above expression reduces to

$$\int_{-1}^1 (1 - 2xt + t^2)^{-1} dx = \sum_{n=0}^{\infty} \int_{-1}^1 t^{2n} P_n^2(x) dx.$$

Thus

$$\begin{aligned} \sum_{n=0}^{\infty} \int_{-1}^1 t^{2n} P_n^2(x) dx &= \left[-\frac{1}{2t} \log(1 + t^2 - 2xt) \right]_{x=-1}^1 \\ &= -\frac{1}{2t} \log \frac{(1-t)^2}{(1+t)^2} = \frac{1}{t} \log \frac{1+t}{1-t} \\ &= \frac{1}{t} \left[2 \left(t + \frac{t^3}{3} + \frac{t^5}{5} + \dots \right) \right] \\ &= 2 \left(1 + \frac{t^2}{3} + \frac{t^4}{5} + \dots + \frac{t^{2n}}{2n+1} + \dots \right). \end{aligned}$$

Comparing the coefficients of t^{2n} , we have

$$\int_{-1}^1 P_n^2(x) dx = \frac{2}{2n+1}$$

or

$$\frac{2n+1}{2} \int_{-1}^1 P_n^2(x) dx = 1.$$

EXAMPLE 2.169

Using Rodrigue's formula, find the polynomial expression for $P_4(x)$.

Solution. The Rodrigue's formula is

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

Therefore,

$$P_4(x) = \frac{1}{2^4 4!} \frac{d^4}{dx^4} (x^2 - 1)^4 = \frac{1}{384} \frac{d^4}{dx^4} (x^2 - 1)^4.$$

So let

$$y = (x^2 - 1)^4 = x^8 - 4x^6 + 6x^4 - 4x^2 + 1.$$

Then,

$$\begin{aligned} \frac{dy}{dx} &= 8x^7 - 24x^5 + 24x^3 - 8x \\ \frac{d^2y}{dx^2} &= 56x^6 - 120x^4 + 72x^2 - 8 \\ \frac{d^3y}{dx^3} &= 336x^5 - 480x^3 + 144x \\ \frac{d^4y}{dx^4} &= 1680x^4 - 1440x^2 + 144. \end{aligned}$$

Therefore,

$$\begin{aligned} P_4(x) &= \frac{1}{384} [1680x^4 - 1440x^2 + 144] \\ &= \frac{1}{8} [35x^4 - 30x^2 + 3]. \end{aligned}$$

EXAMPLE 2.170

Establish the following recurrence relations for Legendre polynomial.

- (i) $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$
- (ii) $nP_n(x) = xP'_n(x) - P'_{n-1}(x)$
- (iii) $(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$
- (iv) $(n+1)P_n(x) = P'_{n+1}(x) - xP'_n(x)$
- (v) $(1-x^2)P'_n(x) = n(P_{n-1}(x) - xP_n(x))$
- (vi) $(1-x^2)P'_n(x) = (n+1)(xP_n(x) - P_{n+1}(x))$

Solution. (i) The generating formula for $P_n(x)$ is

$$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x).$$

Differentiating w.r.t. t , we get

$$-\frac{1}{2}(1 - 2xt + t^2)^{-\frac{3}{2}}(-2x + 2t) = \sum_{n=0}^{\infty} nP_n(x)t^{n-1}$$

or

$$(x-t)(1 - 2xt + t^2)^{-\frac{1}{2}} = (1 - 2xt + t^2) \sum_{n=0}^{\infty} nP_n(x)t^{n-1}$$

or

$$(x-t) \sum_{n=0}^{\infty} t^n P_n(x) = (1 - 2xt + t^2) \sum_{n=0}^{\infty} nP_n(x)t^{n-1}.$$

Comparing coefficients of t^n , we get

$$\begin{aligned} xP_n(x) - P_{n-1}(x) &= (n+1)P_{n+1}(x) - 2nP_n(x) \\ &\quad + (n-1)P_{n-1}(x) \end{aligned}$$

or

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

(ii) Differentiating the generating formula w.r.t. x , we get

$$-\frac{1}{2}(1 - 2xt + t^2)^{-\frac{3}{2}}(-2t) = \sum_{n=0}^{\infty} t^n P'_n(x)$$

or

$$t(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P'_n(x). \quad (152)$$

On the other hand, differentiation of the generating formula w.r.t. t yields

$$(x - t)(1 - 2tx + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} nt^{n-1} P_n(x). \quad (153)$$

Dividing (153) by (152), we get

$$\frac{x - t}{t} = \frac{\sum_{n=0}^{\infty} nt^{n-1} P_n(x)}{\sum_{n=0}^{\infty} t^n P'_n(x)}$$

or

$$\sum_{n=0}^{\infty} nt^n P_n(x) = (x - t) \sum_{n=0}^{\infty} t^n P'_n(x).$$

Comparing the coefficients of t^n on both sides, we have

$$nP_n(x) = xP'_n(x) - P'_{n-1}(x).$$

(iii) From (i), we have

$$(n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) - nP_{n-1}(x).$$

Differentiating w.r.t. x , we get

$$\begin{aligned} (n + 1)P'_{n+1}(x) &= (2n + 1)P_n(x) \\ &+ (2n + 1)xP'_n(x) - nP'_{n-1}(x). \end{aligned} \quad (154)$$

But from part (ii)

$$xP'_n(x) = nP_n(x) + P'_{n-1}(x).$$

Therefore, (154) reduces to

$$\begin{aligned} (n + 1)P'_{n+1}(x) &= (2n + 1)P_n(x) \\ &+ (2n + 1)x[nP_n(x) + P'_{n-1}(x)] - nP'_{n-1}(x) \end{aligned}$$

or

$$(2n + 1)P_n(x) = P'_{n+1} - P'_{n-1}.$$

(iv) From part (ii), we have

$$(2n + 1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$$

and from part (ii), we have

$$nP_n(x) = xP'_n(x) - P'_{n-1}(x).$$

Subtracting, we obtain

$$(n + 1)P_n(x) = P'_{n+1}(x) - xP'_n(x).$$

(v) From part (iv), we have

$$(n + 1)P_n(x) = P'_{n+1}(x) - xP'_n(x).$$

Replacing n by $n - 1$, we get

$$nP_{n-1}(x) = P'_n(x) - xP'_{n-1}(x). \quad (155)$$

Also, from part (ii), we have

$$nP_n(x) = xP'_n(x) - P'_{n-1}(x)$$

or

$$nxP_n(x) = x^2P'_n(x) - xP'_{n-1}(x). \quad (156)$$

Subtracting (156) from (155), we get

$$nP_{n-1}(x) - nxP_n(x) = (1 - x^2)P'_n(x)$$

or

$$(1 - x^2)P'_n(x) = n[P_{n-1}(x) - xP_n(x)].$$

(vi) From (155), we have

$$nxP_{n-1}(x) = xP'_n(x) - x^2P'_{n-1}(x) \quad (157)$$

and from part (ii), we have

$$nP_n(x) = xP'_n(x) - P'_{n-1}(x). \quad (158)$$

Subtracting (158) from (157), we obtain

$$n[xP_{n-1}(x) - P_n(x)] = (1 - x^2)P'_{n-1}(x).$$

Replacing n by $n + 1$, we get

$$(1 - x^2)P'_n(x) = (n + 1)[xP_n(x) - P_{n+1}(x)].$$

EXAMPLE 2.171

Show that if n is odd, then

$$\begin{aligned} P'_n(x) &= (2n - 1)P_{n-1}(x) + (2n - 5)P_{n-3}(x) \\ &+ (2n - 9)P_{n-5}(x) + \dots + 3P_1(x). \end{aligned}$$

Solution. We have

$$(2n + 1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x) \quad (\text{recurrence formula}).$$

Changing n to $(n - 1)$, $(n - 3)$, $(n - 5)$, ..., we get

$$(2n - 1)P_{n-1}(x) = P'_n(x) - P'_{n-2}(x)$$

$$(2n - 5)P_{n-3}(x) = P'_{n-2}(x) - P'_{n-4}(x)$$

$$(2n - 9)P_{n-5}(x) = P'_{n-4}(x) - P'_{n-6}(x)$$

...

...

$$3P_1(x) = P'_2(x) - P'_0(x) \text{ if } n \text{ is odd.}$$

Adding these equations, we get

$$\begin{aligned} & (2n-1)P_{n-1}(x) + (2n-5)P_{n-3}(x) \\ & + (2n-9)P_{n-5}(x) + \dots + 3P_1(x) \\ & = P'_n(x) - P'_0(x) = P'_n(x) - 0 \\ & = P'_n(x), \text{ since } P_0(x) = 1. \end{aligned}$$

EXAMPLE 2.172

Show that

$$\int_{-1}^1 xP_n(x)P_{n-1}(x)dx = \frac{2n}{4n^2-1}.$$

Solution. We know that

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x).$$

Changing n to $n-1$, we get

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$$

or

$$xP_{n-1}(x) = \frac{1}{2n-1} [nP_n(x) + (n-1)P_{n-2}(x)].$$

Therefore,

$$\begin{aligned} & \int_{-1}^1 xP_n(x)P_{n-1}(x)dx \\ & = \frac{1}{2n-1} \int_{-1}^1 [nP_n(x) + (n-1)P_{n-2}(x)]P_n(x)dx \\ & = \frac{n}{2n-1} \int_{-1}^1 P_n^2(x)dx + \frac{n-1}{2n-1} \int_{-1}^1 P_n(x)P_{n-2}(x)dx \\ & = \frac{1}{2n-1} \int_{-1}^1 P_n^2(x)dx + 0 \text{ using orthogonal property} \\ & = \frac{n}{2n-1} \left[\frac{2}{2n+1} \right] = \frac{2n}{4n^2-1}. \end{aligned}$$

EXAMPLE 2.173

Show that

- (i) $P_n(-x) = (-1)^n P_n(x)$
- (ii) $P_n(1) = 1$
- (iii) $P_n(-1) = (-1)^n$.

Solution. (i) Generating function formula is

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x).$$

Therefore,

$$\begin{aligned} \sum_{n=0}^{\infty} t^n P_n(-x) &= (1+2xt+t^2)^{-\frac{1}{2}} \\ &= [1-2x(-t)+t^2]^{-\frac{1}{2}} \\ &= \sum_{n=0}^{\infty} (-t)^n P_n(x) = \sum_{n=0}^{\infty} (-1)^n t^n P_n(x) \end{aligned}$$

and so $P_n(-x) = (-1)^n P_n(x)$.

(ii) We have

$$\begin{aligned} \sum_{n=0}^{\infty} P_n(1)t^n &= (1-2t+t^2)^{-\frac{1}{2}} = (1-t)^{-1} \\ &= 1+t+t^2+\dots+t^n+\dots \\ &= \sum_{n=0}^{\infty} t^n. \end{aligned}$$

Hence $P_n(1) = 1$.

(iii) We note that

$$\begin{aligned} \sum_{n=0}^{\infty} t^n P_n(-1) &= (1+2t+t^2)^{-\frac{1}{2}} \\ &= (1+t)^{-1} = \sum_{n=0}^{\infty} (-1)^n t^n. \end{aligned}$$

Hence $P_n(-1) = (-1)^n$.

Second Method. From part (i)

$$P_n(-x) = (-1)^n P_n(x).$$

Therefore,

$$P_n(-1) = (-1)^n P_n(1) = (-1)^n \text{ using (ii).}$$

2.27 FOURIER-LEGENDRE EXPANSION OF A FUNCTION

We have established in Examples 2.167 and 2.168 that $\sqrt{\frac{2n+1}{2}}\{P_n(x)\}$ constitutes an orthonormal system. Therefore, as in the case of Fourier expansion, the expansion in terms of $P_n(x)$ of a given function $f(x)$ is possible. Let

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x). \quad (159)$$

Multiplying both sides by $P_n(x)$ and integrating both sides by $[-1, 1]$, we get

$$\int_{-1}^1 f(x)P_n(x)dx = c_n \int_{-1}^1 P_n^2(x)dx = c_n \left(\frac{2}{2n+1} \right)$$

and so

$$c_n = \frac{2n+1}{2} \int_{-1}^1 f(x)P_n(x)dx.$$

The coefficients c_n are called Fourier–Legendre coefficients and the expansion (159) is called Fourier–Legendre expansion of $f(x)$.

EXAMPLE 2.174

Expand

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 < x < 1. \end{cases}$$

in Fourier–Legendre series.

Solution. We have

$$c_n = \frac{2n+1}{2} \int_{-1}^1 f(x)P_n(x)dx = \frac{2n+1}{2} \int_{-1}^1 P_n(x)dx.$$

Therefore,

$$c_0 = \frac{1}{2} \int_0^1 P_0(x)dx = \frac{1}{2} \int_0^1 dx = \frac{1}{2},$$

$$c_1 = \frac{3}{2} \int_0^1 P_1(x)dx = \frac{3}{2} \int_0^1 xdx = \frac{3}{2} \left(\frac{1}{2} \right) = \frac{3}{4},$$

$$c_2 = \frac{5}{2} \int_0^1 P_2(x)dx = \frac{5}{2} \int_0^1 \frac{3x^2-1}{2}dx = \frac{5}{4} [x^3-x]_0^1 = 0,$$

$$\begin{aligned} c_3 &= \frac{7}{2} \int_0^1 P_3(x)dx = \frac{7}{2} \int_0^1 \frac{5x^3-3x}{2}dx \\ &= \frac{7}{4} \left[5\frac{x^4}{4} - 3\frac{x^2}{2} \right]_0^1 = -\frac{7}{16}, \end{aligned}$$

$$c_4 = \frac{9}{2} \int_0^1 P_4(x)dx = \frac{9}{2} \int_0^1 \frac{35x^4-30x^2+3}{8}dx = \frac{27}{16},$$

and so on. Hence, Fourier–Legendre expansion of the given function is

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n P_n(x) \\ &= \frac{1}{2} P_0(x) + \frac{3}{4} P_1(x) - \frac{7}{16} P_3(x) + \frac{27}{16} P_4(x) \\ &\quad + \dots \end{aligned}$$

2.28 MISCELLANEOUS EXAMPLES

EXAMPLE 2.175

Solve the differential equation

$$x dy - y dx = (x^2 + y^2) dx$$

Solution. We have

$$x dy - y dx = (x^2 + y^2) dx$$

or

$$\frac{x dy - y dx}{x^2 + y^2} = dx$$

or

$$d\left(\tan^{-1} \frac{y}{x}\right) = dx.$$

Integrating, we get

$$\tan^{-1} \frac{y}{x} = x + C.$$

EXAMPLE 2.176

Solve $(1 + y^2)dx = (e^{\tan^{-1}y} - x)dy$.

Solution. The given equation can be written as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{\tan^{-1}y}}{1+y^2},$$

which is of the form

$$\frac{dx}{dy} + Px = Q,$$

where P and Q are function of y only. The integrating factor is

$$I.F = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1}y}.$$

Therefore the solution of the differential equation is

$$\begin{aligned} x e^{\tan^{-1}y} &= \int \frac{e^{\tan^{-1}y}}{1+y^2} \cdot e^{\tan^{-1}y} dy + c \\ &= \int \frac{e^{2\tan^{-1}y}}{1+y^2} dy + c = \frac{e^{2\tan^{-1}y}}{2} + c \end{aligned}$$

or

$$2xe^{\tan^{-1}y} = e^{2\tan^{-1}y} + c_1.$$

EXAMPLE 2.177

Solve the initial value problem $e^x (\cos y \, dx - \sin y \, dy) = 0$, $y(0) = 0$.

Solution. The given equation is of the form $Mdx + Ndy = 0$, where

$$M = e^x \cos y \quad \text{and} \quad N = -e^x \sin y.$$

Therefore

$$\frac{\partial M}{\partial y} = -e^x \sin y \quad \text{and} \quad \frac{\partial N}{\partial x} = -e^x \sin y.$$

Hence the given equation is exact. The solution of the given differential equation is

$$\int_{y \text{ constant}} M dx + \int (\text{terms in } N \text{ not containing } x) dy = k$$

or

$$e^x \cos y = k.$$

Using the initial condition $y(0) = 0$, we get $k = e^0 \cos 0 = 1$. Hence the required solution is

$$e^x \cos y = 1.$$

EXAMPLE 2.178

Solve the following equation

$$(xy^2 - e^{\frac{1}{x^3}})dx - x^2y \, dy = 0.$$

Solution. The equation here is

$$(xy^2 - e^{\frac{1}{x^3}})dx - x^2y \, dy = 0$$

Comparing it with $Mdx + Ndy = 0$, we have

$$M = xy^2 - e^{\frac{1}{x^3}} \quad \text{and} \quad N = -x^2y$$

Therefore

$$\frac{\partial M}{\partial y} = 2xy \quad \text{and} \quad \frac{\partial N}{\partial x} = -2xy.$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the given equation is not exact. But

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2xy - (-2xy)}{-x^2y} = -\frac{4}{x},$$

which is a function of x only. Hence $e^{\int \frac{-4}{x} dx} = e^{-4 \log x} = \frac{1}{x^4}$ is the integrating factor. So multiplying the given equation throughout by $\frac{1}{x^4}$, we get

$$\left(\frac{y^2}{x^3} - \frac{1}{x^4} e^{\frac{1}{x^3}} \right) dx - \frac{y}{x^2} dy = 0,$$

which is an exact equation. Therefore the required solution is

$$\int_{y \text{ constant}} \left(\frac{y^2}{x^3} - \frac{1}{x^4} e^{\frac{1}{x^3}} \right) dx = c$$

y constant
or

$$-\frac{y^2}{2x^2} + \int \frac{1}{3} e^{\frac{1}{x^3}} \left(\frac{-3}{x^4} \right) dx = c$$

or

$$-\frac{y^2}{2x^2} + \frac{1}{3} e^{\frac{1}{x^3}} = c.$$

EXAMPLE 2.179

Solve the differential equation

$$(y \cos x + 1) dx + \sin x \, dy = 0.$$

Solution. The given differential equation is of the form $Mdx + Ndy = 0$, where

$$M = y \cos x + 1, \quad N = \sin x.$$

Now

$$\frac{\partial M}{\partial y} = \cos x, \quad \frac{\partial N}{\partial x} = \cos x.$$

Therefore the given equation is exact and its solution is

$$\int_{y \text{ constant}} (y \cos x + 1) dx + \int 0 dy = C$$

or

$$y \sin x + x = C.$$

EXAMPLE 2.180

If the air is maintained at 30°C and the temperature of the body cools from 80°C to 60°C in 12 minutes, find the temperature of the body after 24 minutes.

Solution. Let T be the temperature of the body at any instant t . By Newton's law of cooling, we have

$$\frac{dT}{dt} = k(T - T_0)$$

and so variable separation yields

$$\frac{dT}{T - T_0} = -k dt.$$

Integrating, we get

$$\log(T - T_0) = -kt + \log C$$

or

$$T - T_0 = C e^{-kt}$$

or

$$T - 30 = C e^{-kt}.$$

But when $t = 0$, $T = 80^\circ$. Therefore, $C = 80 - 30 = 50$, and we have

$$T - 30 = 50 e^{-kt}.$$

Now when $t = 12$, $T = 60^\circ$. Therefore,

$$60 - 30 = 50 e^{-12k}$$

or

$$30 = 50 e^{-12k}$$

or

$$-12k = \log \frac{3}{5}, \text{ which yields } k = \frac{1}{12} \log \frac{5}{3}.$$

Hence,

$$T = 30 + 50 e^{-\left(\frac{1}{12} \log \frac{5}{3}\right) t}$$

When $t = 24$, we have

$$\begin{aligned} T &= 30 + 50 e^{-2 \log \frac{5}{3}} \\ &= 30 + 50 e^{\log \frac{9}{25}} \\ &= 30 + 50 \left(\frac{9}{25} \right) = 30 + 18 = 48^\circ \text{C}. \end{aligned}$$

EXAMPLE 2.181

Suppose that an object is heated to 300° and allowed to cool in a room whose air temperature is 80°. If after 10 minutes the temperature of the object is 250°, what will be its temperature after 20 minutes?

Solution. By Newton's law of cooling, we have

$$\frac{dT}{dt} = -k(T - T_0),$$

or

$$\frac{dT}{dt} = -k dt.$$

Integrating, we get

$$\log(T - T_0) = -kt + \log C$$

or

$$T - T_0 = C e^{-kt} \quad (160)$$

Equation (160) gives

$$T - 80 = C e^{-kt}$$

When $t = 0$, $T = 300^\circ$. Therefore, $C = 300 - 80 = 220^\circ$.

Thus

$$T = 80 + 220 e^{-kt}.$$

But when $t = 10$, $T = 250^\circ$. Therefore

$$250 = 80 + 220 e^{-10k},$$

which yields

$$e^{-10k} = \frac{250 - 80}{220} = \frac{17}{22}$$

or

$$k = \frac{1}{10} \log \frac{29}{7} = 0.02578.$$

Hence,

$$T = 80 + 220 e^{-0.2578t}.$$

Therefore, temperature of the body after 20 minutes is

$$T = 80 + 220 e^{-5.156}.$$

EXAMPLE 2.182

A bacterial culture growing exponentially increases 200 to 500 grms in the period from 6 AM to 9 AM. How many gram will it be at noon?

Solution. The governing differential equation of this problem is

$$\frac{dx}{dt} = kx,$$

which has the solution as

$$x = C e^{kt}. \quad (161)$$

When $t = 0$, $x = 200$. Therefore (161) gives $C = 200$. Thus

$$x = 200 e^{kt}. \quad (162)$$

When $t = 3$ hours, $x = 500$. Therefore (162) yields

$$500 = 200 e^{3k}$$

or

$$e^{3k} = \frac{5}{2}. \quad (163)$$

At noon $t = 6$ hour, using (163), we have

$$x = 200 e^{6k} = 200(e^{3k})^2 = 200\left(\frac{5}{2}\right)^2 = 1250 \text{ gms.}$$

EXAMPLE 2.183

Find the orthogonal trajectories of the family $\frac{2a}{r} = 1 - \cos \theta$.

Solution. We have

$$\frac{2a}{r} = 1 - \cos \theta,$$

or

$$r = \frac{2a}{1 - \cos \theta}. \quad (164)$$

Differentiating (164) with respect to θ , we get

$$\begin{aligned} \frac{dr}{d\theta} &= \frac{(1 - \cos \theta)(0) - 2a(\sin \theta)}{(1 - \cos \theta)^2} \\ &= -\frac{2a \sin \theta}{(1 - \cos \theta)^2}. \end{aligned} \quad (165)$$

Dividing (165) by (164), we get

$$\begin{aligned} \frac{1}{r} \frac{dr}{d\theta} &= -\frac{\sin \theta}{1 - \cos \theta} = -\frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \sin^2 \frac{\theta}{2}} \\ &= -\cot \frac{\theta}{2}. \end{aligned} \quad (166)$$

Replacing $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$ in (166), we get

$$\frac{1}{r} \left(-r^2 \frac{d\theta}{dr} \right) = -\cot \frac{\theta}{2}$$

or

$$r \frac{d\theta}{dr} = \cot \frac{\theta}{2}$$

or

$$\frac{dr}{r} = \tan \frac{\theta}{2} d\theta = \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} d\theta.$$

Integrating, we get

$$\log r = -2 \log \cos \frac{\theta}{2} + \log c$$

or

$$\log r + 2 \log \cos \frac{\theta}{2} = \log c$$

or

$$\log r + \log \cos^2 \frac{\theta}{2} = \log c$$

or

$$r \cos^2 \frac{\theta}{2} = c$$

or

$$\frac{r}{2} (1 + \cos \theta) = c$$

or

$$r = \frac{2c}{1 + \cos \theta},$$

which is the required orthogonal trajectory

EXAMPLE 2.184

Solve

$$(D^3 - 6D^2 + 11D - 6)y = e^{-2x} + e^{-3x}$$

Solution. The auxiliary equation for the given non-homogeneous equation is

$$m^3 - 6m^2 + 11m - 6 = 0$$

or

$$(m - 1)(m - 2)(m - 3) = 0,$$

which yields $m = 1, 2, 3$. Therefore

$$\text{C.F.} = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}.$$

The particular integral is given by

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D-1)(D-2)(D-3)} e^{-2x} \\ &\quad + \frac{1}{(D-1)(D-2)(D-3)} e^{-3x} \\ &= \frac{1}{(-2-1)(-2-2)(-2-3)} e^{-2x} \\ &\quad + \frac{1}{(-3-1)(-3-2)(-3-3)} e^{-3x} \\ &= -\frac{1}{60} e^{-2x} - \frac{1}{120} e^{-3x}. \end{aligned}$$

Hence the complete solution of the given differential equation is

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ &= C_1 e^x + C_2 e^{2x} + C_3 e^{3x} - \frac{1}{60} e^{-2x} - \frac{1}{120} e^{-3x}. \end{aligned}$$

EXAMPLE 2.185

Solve $(D^2 + 4D + 13)y = e^{2x}$

Solution. The auxiliary equation for the given differential equation is

$$m^2 - 4m + 13 = 0,$$

which yields $m = \frac{4 \pm \sqrt{16-52}}{2} = 2 \pm 3i$. Therefore

$$\text{C.F.} = (c_1 \cos 3x + c_2 \sin 3x)e^{2x}.$$

The particular integral is given by

$$\begin{aligned} \text{P.I.} &= \frac{1}{f(D)} F(x) = \frac{1}{D^2 - 4D + 13} e^{2x} \\ &= \frac{1}{2^2 - 4(2) + 13} e^{2x} = \frac{1}{9} e^{2x}. \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ &= (c_1 \cos 3x + c_2 \sin 3x)e^{2x} + \frac{1}{9} e^{2x}. \end{aligned}$$

EXAMPLE 2.186

Solve $(D^2 + 1)y = \sin x \sin 2x + e^x x^2$.

Solution. The auxiliary equation for the given differential equation is

$$m^2 + 1 = 0, \text{ which yields } m = \pm 2i.$$

Therefore

$$\text{C.F.} = c_1 \cos x + c_2 \sin x.$$

The particular integral is given by

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 + 1} \sin x \sin 2x + \frac{1}{D^2 + 1} e^x x^2 \\ &= \frac{1}{D^2 + 1} \left[\frac{1}{2} (\cos x - \cos 3x) \right] + \frac{1}{D^2 + 1} e^x x^2 \\ &= \frac{1}{2(D^2 + 1)} \cos x - \frac{1}{2(D^2 + 1)} \cos 3x \\ &\quad + e^x \frac{1}{(D+1)^2 + 1} x^2 \\ &= \frac{x}{4} \sin x - \frac{1}{2(-9+1)} \cos 3x \\ &\quad + \frac{e^x}{2} \left(1 + D + \frac{D^2}{2} \right)^{-1} x^2 \\ &= \frac{x}{4} \sin x + \frac{1}{16} \cos 3x + \frac{1}{2} e^x (x^2 - 2x + 1). \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ &= c_1 \cos x + c_2 \sin x + \frac{1}{4} x \sin x + \frac{1}{16} \cos 3x \\ &\quad + \frac{1}{2} e^x (x^2 - 2x + 1). \end{aligned}$$

EXAMPLE 2.187

Solve $(D^2 + 1)y = \sin^2 x$.

Solution. We have

$$(D^2 + 1)y = \sin^2 x.$$

The auxiliary equation is $m^2 + 1 = 0$, which yields $m = \pm i$. Therefore

$$\text{C.F.} = c_1 \cos x + c_2 \sin x.$$

Now

$$\begin{aligned}\text{P.I.} &= \frac{1}{f(D)} F(x) = \frac{1}{D^2 + 1} \left[\frac{1 - \cos 2x}{2} \right] \\ &= \frac{1}{2(D^2 + 1)} [1 - \cos 2x] \\ &= \frac{1}{2} - \frac{1}{2(D^2 + 1)} \cos 2x \\ &= \frac{1}{2} - \frac{1}{2(-4 + 1)} \cos 2x = \frac{1}{2} + \frac{1}{6} \cos 2x.\end{aligned}$$

Hence the solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{2} + \frac{1}{6} \cos 2x.$$

EXAMPLE 2.188

Find the particular integral of $(D^2 - 2D + 1)y = \cos h x$.

Solution. The required particular integral is

$$\begin{aligned}\text{P.I.} &= \frac{1}{D^2 - 2D + 1} \cosh x \\ &= \frac{1}{D^2 - 2D + 1} \left[\frac{e^x + e^{-x}}{2} \right] \\ &= \frac{1}{2} \cdot \frac{1}{D^2 - 2D + 1} e^x + \frac{1}{2} \cdot \frac{1}{D^2 - 2D + 1} e^{-x} \\ &= \frac{1}{2} x \cdot \frac{1}{2D - 2} e^x + \frac{1}{8} e^{-x} \\ &= \frac{1}{2} x^2 \cdot \frac{1}{2} e^x + \frac{1}{8} e^{-x} \\ &= \frac{1}{4} x^2 e^x + \frac{1}{8} e^{-x}.\end{aligned}$$

EXAMPLE 2.189

Solve the differential equation

$$\frac{d^2 y}{dx^2} + 4y = x \sin x.$$

Solution. The auxiliary equation for the given differential equation is $m^2 + 4 = 0$, which yields $m = \pm 2i$. Therefore

$$\text{C.F.} = c_1 \cos 2x + c_2 \sin 2x.$$

Further,

$$\begin{aligned}\text{P.I.} &= \frac{1}{D^2 + 4} (x \sin x) \\ &= \frac{1}{D^2 + 4} x [\text{I.P. of } e^{ix}] = \text{I.P. of } \frac{1}{D^2 + 4} x e^{ix}\end{aligned}$$

$$\begin{aligned}&= \text{I.P. of } \left[e^{ix} \frac{1}{(D+i)^2 + 4} x \right] \\ &= \text{I.P. of } \left[e^{ix} \frac{1}{D^2 + 2iD - 1 + 4} x \right] \\ &= \text{I.P. of } \left[e^{ix} \frac{1}{D^2 + 2iD - 1 + 3} x \right] \\ &= \text{I.P. of } \left[e^{ix} \frac{1}{3(1 + \frac{2}{3}iD + \frac{D^2}{3})} x \right] \\ &= \frac{1}{3} \text{I.P. of } \left[e^{ix} \left(1 + \frac{2}{3}iD + \frac{D^2}{3} \right)^{-1} x \right] \\ &= \frac{1}{3} \text{I.P. of } \left\{ e^{ix} \left[1 - \frac{2}{3}iD - \frac{D^2}{3} + \left(\frac{2}{3}iD + \frac{D^2}{3} \right)^2 - \dots \right] x \right\} \\ &= \frac{1}{3} \text{I.P. of } \left\{ e^{ix} \left(x - \frac{2}{3}i \right) \right\} \\ &= \frac{1}{3} \text{I.P. of } \left\{ (\cos x + i \sin x) \left(x - \frac{2}{3}i \right) \right\} \\ &= \frac{1}{3} \text{I.P. of } \left\{ x \cos x + ix \sin x - \frac{2}{3}i \cos x + \frac{2}{3} \sin x \right\} \\ &= \frac{1}{3} \text{I.P. of } \left\{ x \cos x + \frac{2}{3} \sin x + i \left(x \sin x - \frac{2}{3} \cos x \right) \right\} \\ &= \frac{1}{3} \left[x \sin x - \frac{2}{3} \cos x \right].\end{aligned}$$

Hence the complete solution of the given differential equation is

$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3} \left[x \sin x - \frac{2}{3} \cos x \right].$$

EXAMPLE 2.190

Solve

$$\frac{d^2 y}{dx^2} - 3 \frac{dy}{dx} + 2y = x e^{3x} + \sin 2x.$$

Solution. The auxiliary equation for the given differential equation is $m^2 - 3m + 2 = 0$, which yields $m = 1, 2$. Hence

$$\text{C.F.} = c_1 e^x + c_2 e^{2x}.$$

Further,

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{f(D)} F(x) = \frac{1}{D^2 - 3D + 2} [xe^{3x} + \sin 2x] \\
 &= \frac{1}{D^2 - 3D + 2} (xe^{3x}) + \frac{1}{D^2 - 3D + 2} (\sin 2x) \\
 &= e^{3x} \frac{1}{(D+3)^2 - 3(D+3) + 2} x \\
 &\quad + \frac{1}{-4 - 3D + 2} (\sin 2x) \\
 &= e^{3x} \frac{1}{D^2 + 3D + 2} x \\
 &\quad - \frac{2 - 3D}{(2 + 3D)(2 - 3D)} (\sin 2x) \\
 &= \frac{1}{2} e^{3x} \left(1 + \frac{D^2 + 3D}{2} \right)^{-1} x - \frac{2 - 3D}{4 - 9D^2} (\sin 2x) \\
 &= \frac{1}{2} e^{3x} \left(1 - \frac{3D}{2} - \frac{D^2}{2} + \dots \right) x \\
 &\quad - \frac{2 - 3D}{4 + 36} (\sin 2x) \\
 &= \frac{1}{2} e^{3x} \left(x - \frac{3}{2} \right) - \frac{1}{40} [2 \sin 2x - 3D(\sin 2x)] \\
 &= \frac{1}{2} x e^{3x} - \frac{3}{4} e^{3x} - \frac{1}{20} \sin 2x + \frac{6}{40} \cos 2x \\
 &= \frac{1}{4} e^{3x} (2x - 3) + \frac{1}{20} (3 \cos 2x - \sin 2x).
 \end{aligned}$$

Hence the complete solution of the given differential equation is

$$\begin{aligned}
 y &= c_1 e^x + c_2 e^{2x} + \frac{1}{4} e^{3x} (2x - 3) \\
 &\quad + \frac{1}{20} (3 \cos 2x - \sin 2x).
 \end{aligned}$$

EXAMPLE 2.191

Solve $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 13x = 0$ with $x(0) = 0$, $\frac{dx(0)}{dt} = 2$.

Solution. The auxiliary equation for the given equation is

$$m^2 - 4m + 13 = 0,$$

which yield $m = 2 \pm 3i$. Therefore the general solution is

$$x = e^{2t} [c_1 \cos 3t + c_2 \sin 3t].$$

When $t = 0$ we have $x = 0$. Therefore $c_1 = 0$.

Further,

$$\begin{aligned}
 \frac{dx}{dt} &= e^{2t} [-3c_1 \sin 3t + 3c_2 \cos 3t] \\
 &\quad + 2e^{2t} [c_1 \cos 3t + c_2 \sin 3t]
 \end{aligned}$$

When $t = 0$, $\frac{dx}{dt} = 2$ and so we have

$$2 = 3c_2 + 2c_1 \Rightarrow c_2 = \frac{2}{3}.$$

Hence

$$x = \frac{2}{3} e^{2t} \sin 3t.$$

EXAMPLE 2.192

Solve $(x^2 D^2 + 4xD + 2)y = x \log x$.

Solution. The given Cauchy-Euler equation is

$$x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = x \log x.$$

Proceeding as in Example 2.124, we put $x = e^t$ and reduce the equation to the form

$$[D(D-1) + 4D + 2]y = t e^t.$$

The C.F. for this equation is

$$\text{C.F.} = c_1 e^{-t} + c_2 e^{-2t}.$$

The particular integral is

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{D^2 + 3D + 2} t e^t \\
 &= e^t \frac{1}{(D+1)^2 + 3(D+1) + 2} t \\
 &= e^t \frac{1}{D^2 + 5D + 6} t \\
 &= e^t \frac{1}{6(1 + \frac{5}{6}D + \frac{1}{6}D^2)} t \\
 &= e^t \frac{1}{6} \left(1 + \frac{5}{6}D + \frac{1}{6}D^2 \right)^{-1} t \\
 &= \frac{1}{6} e^t \left[1 - \frac{5}{6}D - \frac{1}{6}D^2 + \dots \right] t \\
 &= \frac{1}{6} e^t \left[t - \frac{5}{6} \right] = \frac{1}{6} t e^t - \frac{5}{36} e^t.
 \end{aligned}$$

Hence the solution is

$$y = c_1 e^{-t} + c_2 e^{-2t} + \frac{1}{6} t e^t - \frac{5}{36} e^t$$

$$= \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{1}{6} x \log x - \frac{5}{6} x.$$

EXAMPLE 2.193

Convert $(x^3 D^2 + 3x^2 D + 5x)y = 2$ into a differential equation with constant co-efficients.

Solution. Given equation can be written as

$$(x^2 D^2 + 3xD + 5)y = \frac{2}{x},$$

which is Cauchy's homogeneous linear difference equation, Let $x = e^t$. Then, we have $x \frac{dy}{dx} = Dy$ and $x^2 \frac{d^2 y}{dx^2} = D(D-1)y$, where $D = \frac{d}{dt}$. Hence given equation reduces to $[D(D-1) + 3D + 5]y = 2e^{-t}$ or $(D^2 + 2D + 5)y = 2e^{-t}$.

EXAMPLE 2.194

Solve $(xD^2 + D)y \equiv 0$.

Solution. The given differential equation with variable coefficients is

$$x D^2 y + Dy = 0.$$

Multiplying throughout by x , we get

$$x^2 D^2 y + x Dy = 0,$$

which is Cauchy-Euler equation. Putting $x = e^t$ so that $t = \log x$, we have

$$x \frac{dy}{dx} = Dy \text{ and } x^2 \frac{d^2 y}{dx^2} = D(D-1)y.$$

Consequently, the given equation transforms to $[D(D-1) + D]y = 0$ or $D^2 y = 0$.

Integrating twice, we get

$$y = at + b = a \log x + b.$$

EXAMPLE 2.195

Solve $(x^2 D^2 - 3x D + 4)y = x^2$ given that $y(1) = 1$ and $y'(1) = 0$.

Solution. We are given the differential equation

$$(x^2 D^2 - 3x D + 4)y = x^2,$$

which is a Cauchy-Euler homogeneous equation. Putting $x = e^t$ so that $t = \log x$, we have

$$x \frac{dy}{dx} = Dy \text{ and } x^2 \frac{d^2 y}{dx^2} = [D(D-1)]y.$$

Therefore the given equation transforms to

$$[D(D-1) - 3D + 4]y = e^{2t}$$

or

$$[D^2 - 4D + 4]y = e^{2t} \quad (167)$$

The auxiliary equation for the equation (167) is $m^2 - 4m + 4 = 0$, which yields $m = 2, 2$.

Therefore

$$\text{C.F.} = (c_1 + c_2 t)e^{2t}.$$

Further

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - 4D + 4} e^{2t} \\ &= t \frac{1}{2D - 4} e^{2t} = t^2 \cdot \frac{1}{2} e^{2t} \\ &= \frac{1}{2} t^2 e^{2t}. \end{aligned}$$

Hence the required solution is

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ &= (c_1 + c_2 \log x) x^2 + \frac{x^2}{2} (\log x)^2. \end{aligned}$$

EXAMPLE 2.196

Solve the differential equations:

$$(x^2 D^2 - xD - 3)y = x^2 (\log x)^2.$$

Solution. The given equation

$$x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - 3y = x^2 (\log x)^2$$

is a Cauchy-Euler homogeneous linear equation. So put $x = e^t$ so that $t = \log x$. Then

$$x \frac{dy}{dx} = Dy \text{ and } x^2 \frac{d^2 y}{dx^2} = D(D-1)y,$$

and the given equation transforms to

$$[D(D-1) - D - 3]y = t^2 e^{2t}$$

or

$$(D^2 - 2D - 3)y = t^2 e^{2t}.$$

The roots of the auxiliary equation $m^2 - 2m - 3 = 0$ are $m = 3$ and $m = -1$. Therefore

$$\text{C.F.} = c_1 e^{3t} + c_2 e^{-t}.$$

Now

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - 2D - 3} (t^2 e^{2t}) \\ &= e^{2t} \frac{1}{[(D+2)^2 - 2(D+2) - 3]} t^2 \\ &= e^{2t} \frac{1}{D^2 + 2D - 3} t^2 = -\frac{e^{2t}}{3} \frac{1}{(1 - \frac{2}{3}D - \frac{D^2}{3})} t^2 \\ &= -\frac{1}{3} e^{2t} \left[1 - \left(\frac{2D}{3} + \frac{D^2}{3} \right) \right]^{-1} t^2 \\ &= -\frac{1}{3} e^{2t} \left[1 + \frac{2D}{3} + \frac{D^2}{3} + \frac{4D^2}{9} + \dots \right] t^2 \\ &= -\frac{1}{3} e^{2t} \left[1 + \frac{2D}{3} + \frac{7D^2}{9} + \dots \right] t^2 \\ &= -\frac{1}{3} e^{2t} \left[t^2 + \frac{2}{3} D(t^2) + \frac{7}{9} D^2(t^2) \right] \\ &= -\frac{1}{3} e^{2t} \left[t^2 + \frac{4}{3} t + \frac{14}{9} \right]. \end{aligned}$$

Therefore the complete solution is

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ &= c_1 e^{3t} + c_2 e^{-t} - \frac{1}{3} e^{2t} \left[t^2 + \frac{4}{3} t + \frac{14}{9} \right] \\ &= c_1 x^3 + \frac{c_2}{x} - \frac{x^2}{3} \left[(\log x)^2 + \frac{4}{3} \log x + \frac{14}{9} \right]. \end{aligned}$$

EXAMPLE 2.197

Solve the differential equation

$$\frac{x^2 d^2 y}{dx^2} - 2x \frac{dy}{dx} - 4y = x^2 + 2 \log x.$$

Solution. The given Cauchy-Euler homogeneous linear equation is

$$x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} - 4y = x^2 + 2 \log x.$$

Substitute $x = e^t$ so that $t = \log x$. Then

$$x \frac{dy}{dx} = Dy \text{ and } \frac{d^2 y}{dx^2} = D(D-1)y.$$

Therefore the given equation reduces to

$$D(D-1)y - 2Dy - 4y = e^{2t} + 2t$$

or

$$(D^2 - 3D - 4)y = e^{2t} + 2t.$$

The characteristic equation for the equation is

$$m^2 - 3m - 4 = 0, \text{ which yields } m = 4, -1.$$

Thus

$$\text{C.F.} = c_1 e^{4t} + c_2 e^{-t}.$$

The particular integral is given by

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - 3D + 4} e^{2t} + \frac{2t}{D^2 - 3D + 4} e^{-t}. \\ &= \frac{e^{2t}}{4 - 6 - 4} - \frac{2t}{4(1 - \frac{D-3D}{4})} \\ &= -\frac{e^{2t}}{6} - \frac{1}{2} \left(1 - \frac{D^2 - 3D}{4} \right)^{-1} t \\ &= -\frac{1}{6} e^{2t} - \frac{1}{2} \left(1 + \frac{D^2 - 3D}{4} + \dots \right) t \\ &= -\frac{1}{6} e^{2t} - \frac{1}{2} t - \frac{1}{2} \left[\frac{D^2}{4} - \frac{3D}{4} \right] t \\ &= -\frac{1}{6} e^{2t} - \frac{1}{2} t + \frac{3}{8}. \end{aligned}$$

Hence the complete solution of the given differential equation is

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{6} e^{2t} - \frac{1}{2} t + \frac{3}{8} \\ &= c_1 x^4 + \frac{c_2}{x} - \frac{x^2}{6} - \frac{1}{2} \log x + \frac{3}{8}. \end{aligned}$$

EXAMPLE 2.198

Solve

$$(2x+3)^2 \frac{d^2 y}{dx^2} + 5(2x+3) \frac{dy}{dx} + y = 4x.$$

Solution. We have

$$(2x+3)^2 \frac{d^2y}{dx^2} + 5(2x+3) \frac{dy}{dx} + y = 4x,$$

which is Legendre's linear equation. So putting $2x+3 = e^t$ or $t = \log(2x+3)$, the given equation reduces to

$$[4(D^2 - D) + 10D + 1]y = 4 \frac{e^t - 3}{2}$$

or

$$(4D^2 + 6D + 1)y = 2e^t - 6.,$$

The auxiliary equation is

$$4m^2 + 6m + 1 = 0,$$

which yields

$$m = \frac{-6 \pm \sqrt{36 - 16}}{8} = \frac{-6 \pm 2\sqrt{5}}{8} = \frac{-3 \pm \sqrt{5}}{4}.$$

Therefore

$$\text{C.F.} = c_1 e^{\left(\frac{-3+\sqrt{5}}{4}\right)t} + c_2 e^{\left(\frac{-3-\sqrt{5}}{4}\right)t},$$

Further,

$$\begin{aligned} \text{P.I.} &= \frac{1}{4D^2 + 6D + 1} (2e^t - 6) = \frac{2}{11} e^t - 6 \\ &= \frac{2}{11} (2x+3) - 6. \end{aligned}$$

Hence the solution is

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ &= c_1 (2x+3)^{\frac{-3+\sqrt{5}}{4}} + c_2 (2x+3)^{\frac{-3-\sqrt{5}}{4}} \\ &\quad + \frac{2}{11} (2x+3) - 6. \end{aligned}$$

EXAMPLE 2.199

Solve by using the method of variation of parameters $(D^2 + 9)y = \cot 3x$.

Solution. The auxiliary equation is $m^2 + 9 = 0$, which yields $m = \pm 3i$. Therefore

$$\text{C.F.} = c_1 \cos 3x + c_2 \sin 3x.$$

To find P.I., let $F(x) = \cot 3x$, $y_1 = \cos 3x$ and $y_2 = \sin 3x$.

Then Wronskian

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 3x & \sin 3x \\ -3 \sin 3x & 3 \cos 3x \end{vmatrix} = 3.$$

Therefore

$$\begin{aligned} \text{P.I.} &= -y_1 \int \frac{y_2 F(x)}{W} dx + y_2 \int \frac{y_1 F(x)}{W} dx \\ &= -\frac{1}{3} \cos 3x \int \sin 3x \cot 3x dx \\ &\quad + \frac{1}{3} \sin 3x \int \cos 3x \cot 3x dx \\ &= -\frac{1}{3} \cos 3x \int \cos 3x dx \\ &\quad + \frac{1}{3} \sin 3x \int \frac{\cos^2 3x}{\sin 3x} dx \\ &= -\frac{1}{9} \cos 3x \sin 3x \\ &\quad + \frac{1}{3} \sin 3x \left[\int \frac{1 - \sin^2 3x}{\sin 3x} dx \right] \\ &= -\frac{1}{9} \cos 3x \sin 3x \\ &\quad + \frac{1}{3} \sin 3x \int (\operatorname{cosec} 3x - \sin 3x) dx \\ &= -\frac{1}{9} \cos 3x \sin 3x \\ &\quad + \frac{1}{3} \sin 3x \left[\frac{1}{3} \log(\operatorname{cosec} 3x - \cot 3x) + \frac{\cos 3x}{3} \right] \\ &= \frac{1}{9} \sin 3x [\log(\operatorname{cosec} 3x - \cot 3x)] \end{aligned}$$

EXAMPLE 2.200

Solve $(D^2 + a^2)y = \tan ax$ by the method of variation of parameters.

Solution. We have $(D^2 + a^2)y = \tan ax$. It's auxiliary equation is $m^2 + a^2 = 0$, which yields $m = \pm ai$. Therefore

$$\text{C.F.} = c_1 \cos ax + c_2 \sin ax.$$

To find particular integral, let

$$y_1 = \cos ax \text{ and } y_2 = \sin ax.$$

Then

$$y_1' = -a \sin ax \text{ and } y_2' = a \cos ax.$$

Thus the wronskian W is given by

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a \sin ax & a \cos ax \end{vmatrix} = a.$$

Hence

$$\begin{aligned} \text{P.I.} &= -y_1 \int \frac{y_2 F(x)}{W} dx + y_2 \int \frac{y_1 F(x)}{W} dx \\ &= \frac{-\cos ax}{a} \int \sin ax \tan ax dx + \frac{\sin ax}{a} \int \cos ax \tan ax dx \\ &= \frac{-\cos ax}{a} \int \frac{\sin^2 ax}{\cos ax} dx + \frac{\sin ax}{a} \int \cos ax \tan ax dx \\ &= \frac{-\cos ax}{a} \int \frac{1 - \cos^2 ax}{\cos ax} dx + \frac{\sin ax}{a} \int \cos ax \tan ax dx \\ &= \frac{-\cos ax}{a} \int (\sec ax - \cos ax) dx + \frac{\sin ax}{a} \int \sin ax dx \\ &= \frac{-\cos ax}{a^2} [\log(\sec ax + \tan ax) - \sin ax] \\ &\quad - \frac{1}{a^2} \sin ax \cos ax \\ &= \frac{-\cos ax}{a^2} [\log(\sec ax + \tan ax)]. \end{aligned}$$

Hence the complete solution of the given differential equation is

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ &= c_1 \cos ax + c_2 \sin ax \\ &\quad - \frac{\cos ax}{a^2} [\log(\sec ax + \tan ax)] \end{aligned}$$

EXAMPLE 2.201

Solve the differential equation by the method of variations of parameters:

$$y'' + y = \sec^2 x$$

Solution. The given differential equation is

$$y'' + y = \sec^2 x.$$

The symbolic form of the equation is

$$(D^2 + 1)y = \sec^2 x.$$

The auxiliary equation is $m^2 + 1 = 0$, which yields $m = \pm i$. Therefore

$$\text{C.F.} = c_1 \cos x + c_2 \sin x.$$

To find particular integral, let

$$y_1 = \cos x \text{ and } y_2 = \sin x.$$

Then

$$y_1' = -\sin x \text{ and } y_2' = \cos x.$$

The wronskian W is given by

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

Therefore

$$\begin{aligned} \text{P.I.} &= -y_1 \int \frac{y_2 F(x)}{W} dx + y_2 \int \frac{y_1 F(x)}{W} dx \\ &= -\cos x \int \sin x \sec^2 x dx \\ &\quad + \sin x \int \cos x \sec^2 x dx \\ &= -\cos x \int \sec x \tan x dx \\ &\quad + \sin x \int \sec x \cos x dx \\ &= -\cos x \sec x + \sin x \log(\sec x + \tan x) \\ &= -1 + \sin x \log(\sec x + \tan x). \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ &= c_1 \cos x + c_2 \sin x - 1 + \sin x \log(\sec x + \tan x). \end{aligned}$$

EXAMPLE 2.202

Find the general solution of the equation $y'' + 16y = 32 \sec 2x$, using method of variation of parameters.

Solution. The symbolic form of the given differential equation is

$$(D^2 + 16)y = 32 \sec 2x.$$

The roots of the auxiliary equation $m^2 + 16 = 0$ are $m = \pm 4i$. Therefore the complementary function is

$$\text{C.F.} = c_1 \cos 4x + c_2 \sin 4x.$$

To find the particular integral, let $F(x) = 32 \sec 2x$, $y_1 = \cos 4x$, $y_2 = \sin 4x$. Then the wronskian W is given by

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 4x & \sin 4x \\ -4 \sin 4x & 4 \cos 4x \end{vmatrix} = 4.$$

By the method of variation of parameters, we have

$$\begin{aligned}
 \text{P.I.} &= -y_1 \int \frac{y_2 F(x)}{w} dx + y_2 \int \frac{y_1 F(x)}{w} dx \\
 &= -\cos 4x \int \frac{\sin 4x(32 \sec 2x)}{4} dx \\
 &\quad + \sin 4x \int \frac{\cos 4x(32 \sec 2x)}{4} dx \\
 &= -8 \cos 4x \int \sec 2x(2 \sin 2x \cos 2x) dx \\
 &\quad + 8 \sin 4x \int (2 \cos^2 2x - 1) \sec 2x dx \\
 &= -16 \cos 4x \int \sin 2x dx + 8 \sin 4x \\
 &\quad \times \int (\cos 2x - \sec 2x) dx \\
 &= -16 \cos 4x \left(\frac{-\cos 2x}{2} \right) \\
 &\quad + 8 \sin 4x \left[\sin 2x - \frac{1}{2} \log (\sec 2x + \tan 2x) \right] \\
 &= 8 \cos 4x \cos 2x + 8 \sin 4x \sin 2x \\
 &\quad - 4 \sin 4x \log (\sec 2x + \tan 2x) \\
 &= 8 \cos (4x - 2x) \\
 &\quad - 4 \sin 4x \log (\sec 2x + \tan 2x) \\
 &= 8 \cos 2x - 4 \sin 4x \log (\sec 2x + \tan 2x).
 \end{aligned}$$

Hence the required solution is

$$\begin{aligned}
 y &= \text{C.F.} + \text{P.I.} \\
 &= c_1 \cos 4x + c_2 \sin 4x + 8 \cos 2x \\
 &\quad - 4 \sin 4x \log (\sec 2x + \tan 2x)
 \end{aligned}$$

EXAMPLE 2.203

(a) Solve $\frac{d^2 y}{dx^2} + \frac{dy}{dx} - 2y = \frac{1}{1-e^x}$ by using the method of variation of parameters.

(b) Solve $\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + y = e^{-x} \log x$ by using the method of variation of parameters.

Solution. The symbolic form of the given differential equation is

$$(D^2 + D - 2)y = \frac{1}{1 - e^x}.$$

The corresponding auxiliary equation is

$$m^2 + m - 2 = 0,$$

which yields $m = 1, -2$. Hence

$$\text{C.F.} = c_1 e^x + c_2 e^{-2x}.$$

Let

$$\begin{aligned}
 y_1 &= e^x, \quad y_2 = e^{-2x} \text{ (so that) } y'_1 = e^x \text{ and} \\
 y'_2 &= -2e^{-2x}.
 \end{aligned}$$

Let $F(x) = \frac{1}{1-e^x}$. Then Wronskian W is given by

$$\begin{aligned}
 W &= \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} e^x & e^{-2x} \\ e^x & -2e^{-2x} \end{vmatrix} \\
 &= -3e^x e^{-2x}.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \text{P.I.} &= -y_1 \int \frac{y_2 F(x)}{W} dx + y_2 \int \frac{y_1 F(x)}{W} dx \\
 &= -e^x \int \frac{e^{-2x}}{-3e^x e^{-2x}(1 - e^x)} dx \\
 &\quad + e^{-2x} \int \frac{e^x}{-3e^x e^{-2x}(1 - e^x)} dx \\
 &= \frac{1}{3} e^x \int \frac{e^{-x}}{1 - e^x} dx - \frac{1}{3} e^{-2x} \int \frac{e^{2x}}{1 - e^x} dx.
 \end{aligned}$$

But

$$\begin{aligned}
 \int \frac{e^{-x}}{1 - e^x} dx &= \int \frac{1}{z^2(1 - z)} dz, \quad e^x = z \\
 &= \int \left[\frac{1}{z} + \frac{1}{z^2} + \frac{1}{1 - z} \right] dz \\
 &= \log z - \frac{1}{z} - \log(1 - z) \\
 &= \log \frac{z}{1 - z} - \frac{1}{z} = \log \frac{e^x}{1 - e^x} - e^{-x}
 \end{aligned}$$

and

$$\begin{aligned}
 \int \frac{e^{2x}}{1 - e^x} dx &= \frac{1}{2} \int \frac{dz}{1 - z} = -\frac{1}{2} \log(1 - z) \\
 &= -\frac{1}{2} \log(1 - e^x).
 \end{aligned}$$

Therefore

$$\text{P.I.} = \frac{1}{3} e^x \left[\log \frac{e^x}{1 - e^x} - e^{-x} \right] + \frac{1}{6} e^{-2x} \log(1 - e^x).$$

Hence the complete solution is

$$y = c_1 e^x + c_2 e^{-2x} + \frac{1}{3} e^x \left[\log \frac{e^x}{1 - e^x} - e^{-x} \right] + \frac{1}{6} e^{-2x} \log(1 - e^x).$$

(b) The auxiliary equation for the given differential equation is

$$m^2 + 2m + 1 = 0,$$

Which yields $m = -1, -1$. Thus C.F. = $(c_1 + c_2 x)e^{-x}$.

To find P. I, let

$$y_1 = e^{-x} \quad \text{and} \quad y_2 = x e^{-x}.$$

Then

$$W = \begin{vmatrix} e^{-x} & x e^{-x} \\ -e^{-x} & -x e^{-x} + e^{-x} \end{vmatrix} = e^{-2x}.$$

Therefore

$$\begin{aligned} \text{P.I.} &= -y_1 \int \frac{y_2 F(x)}{W} dx + y_2 \int \frac{y_1 F(x)}{W} dx \\ &= -e^{-x} \int \frac{x e^{-x} \cdot e^{-x} \log x}{e^{-2x}} dx \\ &\quad + x e^{-x} \int \frac{e^{-x} \cdot e^{-x} \log x}{e^{-2x}} dx \\ &= -e^{-x} \int x \log x dx + x e^{-x} \int \log x dx \\ &= -e^{-x} \left[\frac{x^2}{2} \log x - \frac{x^2}{4} \right] + x e^{-x} [x \log x - x]. \end{aligned}$$

EXAMPLE 2.204

Solve $(D^2 - 3DD' + 2D'^2)z = \sin x \cos y$.

Solution. Replacing D by m and D' by 1, the auxiliary equation is

$$m^2 - 3m + 2 = 0,$$

which yields $m = 1, 2$. Therefore

$$\text{C.F.} = \phi_1(y + x) + \phi_2(y + 2x).$$

Further,

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - 3DD' + 2D'^2} \sin x \cos y \\ &= \frac{1}{D^2 - 3DD' + 2D'^2} \left\{ \frac{1}{2} [\sin(x+y) + \sin(x-y)] \right\} \\ &= \frac{1}{2} \left[\frac{1}{D^2 - 3DD' + 2D'^2} \sin(x+y) \right. \\ &\quad \left. + \frac{1}{D^2 - 3DD' + 2D'^2} \sin(x-y) \right] \\ &= \frac{1}{2} \left[\frac{1}{(D-2D')(D-D')} \sin(x+y) \right. \\ &\quad \left. + \frac{1}{2} \left[\frac{1}{(D-2D')(D-D')} \sin(x-y) \right] \right] \\ &= \frac{1}{2} \left[\frac{1}{(1-2)(D-D')} \int \cos u du, \text{ where } u = x+y \right] \\ &\quad + \frac{1}{2} \left[\frac{1}{(D-2D')(1+1)} \int \sin u du, \text{ where } u = x-y \right] \\ &= -\frac{1}{2(D-D')} \sin(x+y) - \frac{1}{4(D-2D')} \cos(x-y) \\ &= -\frac{1}{2} \frac{x^1}{1^2 \cdot 1!} \sin(x+y) - \frac{1}{4} \frac{x^1}{1 \cdot 1!} \cos(x-y) \\ &= -\frac{1}{2} x \sin(x+y) - \frac{1}{4} x \cos(x-y). \end{aligned}$$

EXAMPLE 2.205

A simple pendulum of length l is oscillating through a small angle θ in a medium in which the resistance is proportional to velocity. Find the differential equation of its motion. Discuss the motion and find the period of oscillation.

Solution. Please refer to figure 2.9 of article 2.22. We have $\angle AOP = \theta$, $AP = s = l\theta$, $OA = l$. The resistance is proportional to the velocity and so it is $\lambda \frac{ds}{dt}$, where λ is a constant. Therefore the equation of motion along the tangent is

$$m \frac{d^2 s}{dt^2} = -mg \sin \theta - \lambda \frac{ds}{dt}.$$

or

$$\frac{d^2(l\theta)}{dt^2} + \frac{\lambda}{m} \frac{d(l\theta)}{dt} + g \sin \theta = 0$$

or

$$\frac{d^2\theta}{dt^2} + \frac{\lambda}{m} \frac{d\theta}{dt} + \frac{g}{l} \sin \theta = 0$$

Thus, to the first approximation of $\sin \theta$, we have

$$\frac{d^2\theta}{dt^2} + 2k \frac{d\theta}{dt} + \frac{g\theta}{l} = 0, \quad (168)$$

where $\frac{\lambda}{m} = 2k$. The differential equation (168) describes the motion of the bob. Its auxiliary equation is

$$m^2 - 2km + \frac{g}{l} = 0,$$

whose roots are $m = -k \pm \sqrt{k^2 - \omega^2}$, where $\omega^2 = \frac{g}{l}$. Since the oscillatory motion of the bob is possible only if $k < \omega$, the roots are $-k \pm i\sqrt{\omega^2 - k^2}$. Therefore the solution of differential equation (166) is

$$\theta = e^{-kt} \left[C_1 \cos \left(\sqrt{\omega^2 - k^2} \right) t + C_2 \sin \left(\sqrt{\omega^2 - k^2} \right) t \right],$$

which gives a vibratory motion of period

$$\frac{2\pi}{\sqrt{\omega^2 - k^2}}.$$

EXAMPLE 2.206

(a). Solve the simultaneous equations $\frac{dx}{dt} - y = t$ and $\frac{dy}{dt} + x = t^2$.

(b). Solve the following simultaneous equations $\frac{dx}{dt} = 3x + 2y$ and $\frac{dy}{dt} = 5x + 3y$.

(c). Solve the following simultaneous equations $\frac{dx}{dt} + 5x - 2y = t$, $\frac{dy}{dt} + 2x + y = 0$.

Solution. (a). We have

$$Dx - y - t = 0, \quad (169)$$

$$x + Dy - t^2 = 0. \quad (170)$$

Multiplying (169) by D and adding to (170), we get

$$D^2x - Dt - t^2 = 0$$

or

$$D^2x - 1 - t^2 = 0$$

or

$$\frac{D^2x}{dt^2} = 1 + t^2.$$

Therefore

$$\frac{dx}{dt} = t + \frac{t^3}{3} + c_1$$

and

$$x = \frac{t^2}{2} + \frac{t^4}{12} + c_1x + c_2.$$

Putting the value of $\frac{dx}{dt}$ in (167), we get

$$t = \frac{t^3}{3} + c_1 - y - t = 0$$

or

$$y = \frac{t^3}{3} + c_1.$$

Thus the required solution is

$$x = \frac{t^2}{2} + \frac{t^4}{12} + c_1x + c_2,$$

$$y = \frac{t^3}{3} + c_1.$$

(b) The given equations are

$$\frac{dx}{dt} = 3x + 2y \text{ and } \frac{dy}{dt} = 5x + 3y.$$

In symbolic form, we have

$$\begin{aligned} (D - 3)x - 2y &= 0, \\ -5x + (D - 3)y &= 0 \end{aligned} \quad (171)$$

or

$$(D - 3)^2x - 2(D - 3)y = 0 \quad (172)$$

$$-10x + 2(D - 3)y = 0 \quad (173)$$

Adding (172) and (173), we get

$$[(D - 3)^2x - 10]x = 0$$

or

$$(D^2 - 6D - 1)x = 0. \quad (174)$$

The auxiliary equation of (174) is

$$m^2 - 6m - 1 = 0,$$

which gives $m = 3 \pm \sqrt{10}$. Therefore

$$x = e^{3t} [c_1 \cosh \sqrt{10}t + c_2 \sin \sqrt{10}t]. \quad (175)$$

Form equation (171), we have (using (175))

$$\begin{aligned} 2y &= \frac{dx}{dt} - 3x \\ &= e^{3t} \sqrt{10} [c_1 \sinh \sqrt{10}t + c_2 \cosh \sqrt{10}t] \\ &\quad + 3e^{3t} \sqrt{10} [c_1 \cosh \sqrt{10}t + c_2 \sinh \sqrt{10}t] \\ &\quad - 3e^{3t} \sqrt{10} [c_1 \cosh \sqrt{10}t + c_2 \sinh \sqrt{10}t] \\ &= e^{3t} [c_1 \sinh \sqrt{10}t + c_2 \cosh \sqrt{10}t]. \end{aligned}$$

Hence

$$y = \frac{1}{2} [e^{3t} \sqrt{10} [c_1 \sinh \sqrt{10}t + c_2 \cosh \sqrt{10}t]].$$

(c) The symbolic forms of the equations are

$$(D+5)x - 2y = t \quad (176)$$

$$2x + (D+1)y = 0 \quad (177)$$

Operating (176) by $(D+1)$ and multiplying (177) by 2 and then adding, we get

$$(D^2 + 6D + 9)x = 1 + t.$$

The auxiliary equation is

$$m^2 + 6D + 9 = 0$$

and so $m = -3, -3$ and hence

$$\text{C.F.} = (c_1 + c_2t)e^{-3t}$$

Further,

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D+3)^2} (1+t) = \frac{1}{9} \left(1 + \frac{D}{3}\right)^{-2} (1+t) \\ &= \frac{1}{9} \left[1 - \frac{2D}{3} + \dots\right] (1+t) = \frac{1}{9} \left(t + \frac{1}{3}\right). \end{aligned}$$

Hence

$$\begin{aligned} x &= \text{C.F.} + \text{P.I.} \\ &= (c_1 + c_2t)e^{-3t} + \frac{1}{9} \left(t + \frac{1}{3}\right). \end{aligned} \quad (178)$$

Therefore

$$\frac{dx}{dt} = -3(c_1 + c_2t)e^{-3t} + c_2e^{-3t} + \frac{1}{9}.$$

Putting in the first (given) equation, we get

$$y = (c_1 + c_2t)e^{-3t} + \frac{1}{2}c_2e^{-3t} - \frac{2}{9}t + \frac{4}{27}. \quad (179)$$

Hence the solution is given by (178) and (179).

EXAMPLE 2.207

Solve the following simultaneous equations

$$\frac{dx}{dt} + 2x + 3y = 0, \quad 3x + \frac{dy}{dt} + 2y = 2e^t$$

Solution. In symbolic representation, we have

$$(D+2)x + 3y = 0 \quad (180)$$

and

$$3x + (D+2)y = 2e^t. \quad (181)$$

Multiplying (180) by $(D+2)$ and (181) by 3 and subtracting, we get

$$(D+2)^2x - 9x = -6e^t$$

or

$$(D^2 + 4D - 5)x = -6e^t \quad (182)$$

The auxiliary equation for (182) is $m^2 + 4m - 5 = 0$, whose roots are 1 and -5 . Therefore

$$\text{C.F.} = c_1e^t + c_2e^{-5t}.$$

Further,

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 + 4D - 5} (-6e^t) \\ &= t \frac{1}{2D + 4} (-6e^t) = \frac{t}{6} (-6e^t) \\ &= -te^t. \end{aligned}$$

Hence the complete solution of (182) is

$$x = c_1e^t + c_2e^{-5t} - te^t \quad (183)$$

Differentiating (183) with respect to t , we have

$$\frac{dx}{dt} = c_1e^t - 5c_2e^{-5t} - [e^t + te^t].$$

Putting the value of $\frac{dx}{dt}$ in the equation $\frac{dx}{dt} + 2x + 3y = 0$, we get

$$\begin{aligned} 3y &= -\frac{dx}{dt} - 2x = -[c_1e^t - 5c_2e^{-5t} - e^t - te^t] \\ &\quad - 2[c_1e^t + c_2e^{-5t} - te^t] \\ &= -3c_1e^t + 4c_2e^{-5t} + e^t + 3te^t \end{aligned}$$

or

$$y = \frac{4}{3}c_2e^{-5t} - c_1e^t + \frac{1}{3}e^t + te^t \quad (184)$$

Expressions (183) and (184) represent the required solution.

EXAMPLE 2.208

(a) Show that $\int_{-1}^1 (1-x^2)[P'_n(x)]^2 dx = \frac{2n(n+1)}{2n+1}$.

(b) Show that $\int_{-1}^1 f(x)P_n(x)dx = \frac{(-1)^n}{2^n n!} \int_{-1}^1 (x^2-1)^n f^{(n)}(x)dx$.

(c) Show that $J_3(x) + 3J'_0(x) + 4J'''_0(x) = 0$

(d) $P_{2n}(0) = \frac{(-1)^2(2n)!}{2^{2n}(n!)^2}$.

Solution. (a) We have (see solved Exercise 49, chapter 2)

$$\frac{d}{dx}[(1-x^2)P'_n(x)]P_n(x) + n(n+1)P_n(x) = 0$$

or

$$n(n+1)P_n(x) = -\frac{d}{dx}[(1-x^2)P'_n(x)]P_n(x). \quad (185)$$

Multiplying both sides of (185) by $P_n(x)$ and integrating between the limits -1 and 1 , we get

$$\begin{aligned} n(n+1) \int_{-1}^1 P_n^2(x)dx &= - \int_{-1}^1 [(1-x^2)P'_n(x)]' P_n(x)dx \\ &= \{-P_n(x)[(1-x^2)P'_n(x)]\}_{-1}^1 \\ &\quad + \int_{-1}^1 P'_n(x)(1-x^2)P'_n(x)dx \\ &= \int_{-1}^1 (1-x^2)[P'_n(x)]^2 dx. \end{aligned} \quad (186)$$

But

$$\int_{-1}^1 P_n^2(x)dx = \frac{2}{2n+1}.$$

Therefore (186) implies

$$\int_{-1}^1 (1-x^2)(P'_n(x))^2 dx = \frac{2n(n+1)}{2n+1}.$$

(b) By Rodrigue's formula, we have

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n.$$

Therefore integrating by parts, we have

$$\begin{aligned} \int_{-1}^1 f(x)P_n(x)dx &= \frac{1}{2^n n!} \int_{-1}^1 f(x) \frac{d^n}{dx^n} (x^2-1)^n dx \\ &= \frac{1}{2^n n!} \left[\left\{ f(x) \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \right\}_{-1}^1 \right. \\ &\quad \left. - \int_{-1}^1 f'(x) \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n dx \right] \\ &= \frac{(-1)}{2^n n!} \int_{-1}^1 f'(x) \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n dx \\ &= \frac{(-1)^2}{2^n n!} \int_{-1}^1 f''(x) \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n dx \end{aligned}$$

(using integration by parts second time)

$$\begin{aligned} &\dots\dots\dots \\ &\dots\dots\dots \\ &\dots\dots\dots \\ &= \frac{(-1)^n}{2^n n!} \int_{-1}^1 f^{(n)}(x)(x^2-1)^n dx \end{aligned}$$

(using integration by parts n th time)

(c) We know that

$$\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$$

(Please see Example 2.151 (ii))

Putting $n = 0$, we have

$$J'_0(x) = -J_1(x).$$

Therefore

$$\begin{aligned} J''_0(x) &= -J'_1(x) \\ &= -\frac{1}{2}[J_0 - J_2] \text{ (Please see Example 2.151 (iv))} \end{aligned}$$

and

$$\begin{aligned}
 J_0'''(x) &= -\frac{1}{2}[J_0'(x) - J_2'(x)] \\
 &= -\frac{1}{2}J_0'(x) + \frac{1}{2}\left[\frac{1}{2}\{J_1(x) - J_0(x)\}\right] \\
 &= -\frac{1}{2}J_0'(x) + \frac{1}{4}J_1(x) - \frac{1}{4}J_3(x) \\
 &= -\frac{1}{2}J_0'(x) + \frac{1}{4}[-J_0'(x)] - \frac{1}{4}J_3(x) \\
 &= -\frac{3}{4}J_0'(x) - \frac{1}{4}J_3(x).
 \end{aligned}$$

Hence

$$4J_0'''(x) + 3J_0'(x) + J_3(x) = 0.$$

(d) Since $(1 - 2xt + t^2)^{-\frac{1}{2}}$ is generating function of the Legendre's polynomial, we have

$$(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum t^n P_n(x)$$

or

$$\begin{aligned}
 \sum t^n P_n(0) &= 1 - \frac{1}{2}t^2 + \frac{1.3}{2.4}t^4 \\
 &\quad - \dots (-1)^r \frac{1.3.5 \dots (2r-1)}{2.4.6 \dots (2r)} t^{2r} + \dots
 \end{aligned}$$

Equating coefficients of t^{2n} on both sides, we get

$$\begin{aligned}
 P_{2n}(0) &= (-1)^n \frac{1.3.5 \dots (2n-1)}{2.4.6 \dots (2n)} \\
 &= (-1)^n \frac{1.2.3.4 \dots (2n-1)2^n}{[2.4.6 \dots (2n)]^2} \\
 &= (-1)^n \frac{(2n)!}{[2^n(1.2 \dots n)]^2} = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}.
 \end{aligned}$$

EXAMPLE 2.209

Solve the simultaneous equations

$$\frac{dx}{dt} + y = \sin t, \quad \frac{dy}{dt} + x = \cos t,$$

given that $x = 2, y = 0$ when $t = 0$

Solution. We have

$$\frac{dx}{dy} + y = \sin t \text{ and } \frac{dy}{dt} + x = \cos t,$$

under the condition that $x = 2, y = 0$ when $t = 0$. In symbolic form, we have

$$Dx + y = \sin t \quad (187)$$

$$Dy + x = \cos t. \quad (188)$$

Multiplying (187) by D and subtracting (188) from the resultant equation, we have

$$D^2x - x = \cos t - \cos t = 0.$$

Its Auxiliary equation is

$$m^2 - 1 = 0, \text{ which yields } m = \pm 1.$$

Hence

$$x = c_1 e^t + c_2 e^{-t}. \quad (189)$$

When $t = 0$, we have $x = 2$. Therefore

$$c_1 + c_2 = 2 \quad (190)$$

Now putting the value of x from (189) in (187), we get

$$D(c_1 e^t + c_2 e^{-t}) + y = \sin t$$

or

$$c_1 e^t - c_2 e^{-t} + y = \sin t.$$

or

$$y = -c_1 e^t + c_2 e^{-t} + \sin t \quad (191)$$

When $t = 0, y = 0$. Therefore

$$0 = -c_1 + c_2 \quad (192)$$

Solving (190) and (192), we have $c_1 = 1, c_2 = 1$. Hence the required solution is

$$x = e^t + e^{-t}, \quad y = -e^t + e^{-t} + \sin t.$$

EXAMPLE 2.210

Solve the following differential equation in series:

$$2x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + (x - 5)y = 0.$$

Solution. Comparing the given equation with

$$\frac{d^2y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = 0,$$

we get

$$P(x) = -\frac{1}{2x}, \quad Q(x) = \frac{x-5}{2x^2}.$$

At $x = 0$, both $P(x)$ and $Q(x)$ are not analytic. Thus $x = 0$ is a singular point. Further

$$xP(x) = -\frac{1}{2} \text{ and } x^2Q(x) = \frac{x-5}{2},$$

which are analytic at $x = 0$. Hence $x = 0$ is a regular singular point. So, let

$$y = \sum_{n=0}^{\infty} a_n x^{n+m}, \quad a_n \neq 0.$$

Differentiating twice, we get

$$\begin{aligned} \frac{dy}{dx} &= \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1} \\ \frac{d^2y}{dx^2} &= \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m-2}. \end{aligned}$$

Substituting the values of y , $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation, we have

$$\begin{aligned} &\sum_{n=0}^{\infty} 2(n+m)(n+m-1) a_n x^{n+m} \\ &- \sum_{n=0}^{\infty} (n+m) a_n x^{n+m} \\ &+ \sum_{n=0}^{\infty} a_n x^{n+m+1} - \sum_{n=0}^{\infty} 5 a_n x^{n+m} = 0 \end{aligned}$$

or

$$\begin{aligned} &\sum_{n=0}^{\infty} [2(n+m)(n+m-1) - (n+m) - 5] a_n x^{n+m} \\ &+ \sum_{n=0}^{\infty} a_n x^{n+m+1} = 0 \end{aligned}$$

or

$$\begin{aligned} &\sum_{n=0}^{\infty} [2n^2 + 2m^2 + 4mn - 3n - 3m - 5] a_n x^{n+m} \\ &+ \sum_{n=0}^{\infty} a_n x^{n+m+1} = 0 \end{aligned}$$

or

$$\begin{aligned} &\sum_{n=0}^{\infty} [2n^2 + 2m^2 + 4mn - 3n - 3m - 5] a_n x^{n+m} \\ &+ \sum_{n=0}^{\infty} a_n x^{n+m+1} = 0 \end{aligned}$$

or

$$\begin{aligned} &\sum_{n=0}^{\infty} [2n^2 + 2m^2 + 4mn - 3n - 3m - 5] a_n x^{n+m} \\ &+ \sum_{n=1}^{\infty} a_{n-1} x^{n+m} = 0 \end{aligned}$$

or

$$\begin{aligned} &(2m^2 - 3m - 5) a_0 x^m \\ &+ \sum_{n=1}^{\infty} [(2n^2 + 2m^2 + 4mn - 3n - 3m - 5) a_n \\ &+ a_{n-1}] x^{m+n} = 0. \end{aligned}$$

Therefore the indicial equation is

$$2m^2 - 3m - 5 = 0, \text{ which yields } m = \frac{5}{2}, -1.$$

Equating to zero the other coefficients, we have

$$\begin{aligned} &(2n^2 + 2m^2 + 4mn - 3n - 3m - 5) a_n \\ &= -a_{n-1} \text{ for } n \geq 1 \end{aligned} \quad (193)$$

For $m = \frac{5}{2}$, we have from (193),

$$(2n^2 + 7n) a_n = -a_{n-1}, \quad n \geq 1$$

or

$$a_n = -\frac{a_{n-1}}{2n^2 + 7n}, \quad n \geq 1.$$

Thus, putting $n = 1, 2, 3, \dots$ we get

$$\begin{aligned} a_1 &= -\frac{a_0}{9} \\ a_2 &= \frac{a_0}{198} \\ a_3 &= -\frac{a_0}{7722} \end{aligned}$$

and so on. Therefore

$$y_1 = a_0 x^{\frac{5}{2}} \left[1 - \frac{x}{9} + \frac{x^2}{198} - \frac{x^3}{7722} + \dots \right].$$

For $m = -1$, the relation (193) yields

$$(2n^2 - 7n) a_n = -a_{n-1}, \quad n \geq 1$$

or

$$a_n = \frac{-a_{n-1}}{2n^2 - 7n}, \quad n \geq 1.$$

Therefore

$$\begin{aligned}a_1 &= -\frac{a_0}{-5} = \frac{a_0}{5} \\a_2 &= -\frac{a_1}{-6} = \frac{a_0}{30} \\a_3 &= -\frac{a_2}{-3} = \frac{a_0}{90},\end{aligned}$$

and so on. Hence

$$y_2 = a_0 x^{-1} \left[1 + \frac{x}{5} + \frac{x^2}{30} + \frac{x^3}{90} + \dots \right]$$

Therefore the general solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\&= A x^{\frac{5}{2}} \left[1 + \frac{x}{9} + \frac{x^2}{198} - \frac{x^3}{7722} + \dots \right] \\&\quad + B x^{-1} \left[1 + \frac{x}{5} + \frac{x^2}{30} + \frac{x^3}{90} + \dots \right].\end{aligned}$$

EXAMPLE 2.211

For Bessel's polynomial $J_n(x)$ of degree n , show that

$$\begin{aligned}\text{(a)} \quad J'_2(x) &= \left(1 - \frac{4}{x^2}\right) J_1(x) + \frac{2}{x} J_0(x) \\ \text{(b)} \quad J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + \dots) &= 1.\end{aligned}$$

Solution. (a) We know that

$$J'_n(x) = \frac{1}{2} [J_{n-1}(x) - J_{n+1}(x)].$$

Therefore

$$J'_2(x) = \frac{1}{2} [J_1(x) - J_3(x)]. \quad (194)$$

Also

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x).$$

Therefore

$$\begin{aligned}J_3(x) &= \frac{4}{x} J_2(x) - J_1(x) \\&= \frac{4}{x} \left[\frac{2}{x} J_1(x) - J_0(x) \right] - J_1(x) \\&= \left(\frac{8}{x^2} - 1 \right) J_1(x) - \frac{4}{x} J_0(x).\end{aligned}$$

Putting this value in (194), we get

$$J'_2(x) = \left(1 - \frac{4}{x^2}\right) J_1(x) + \frac{2}{x} J_0(x).$$

(b) From example 2.160, the Jacobi's series are
 $J_0 + 2J_2 \cos 2\phi + 2J_4 \cos 4\phi + \dots = \cos(x \sin \phi)$
 and

$$\begin{aligned}2J_1 \sin \phi + 2J_3 \sin 3\phi + 2J_5 \sin 5\phi + \dots \\= \sin(x \sin \phi).\end{aligned}$$

Squaring these equations and integrating with respect to ϕ in the interval $(0, \pi)$, we get

$$\begin{aligned}\pi J_0^2 + 2\pi J_2^2 + 2\pi J_4^2 + \dots \\= \int_0^\pi \cos^2(x \sin \phi) d\phi\end{aligned} \quad (195)$$

and

$$\begin{aligned}2\pi J_1^2 + 2\pi J_3^2 + 2\pi J_5^2 + \dots \\= \int_0^\pi \sin^2(x \sin \phi) d\phi\end{aligned} \quad (196)$$

Adding (195) and (196) we get

$$\pi J_0^2 + 2\pi J_1^2 + 2\pi J_2^2 + \dots = \int_0^\pi d\phi = \pi.$$

Hence

$$J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \dots = 1.$$

EXERCISES

1. Form differential equation from the following equations

$$\text{(a)} \quad y = a e^{2x} + b e^{-3x} + c e^x$$

$$\text{Ans. } x \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} - xy = 0$$

$$\text{(b)} \quad x = A \cos(nt + \alpha) \quad \text{Ans. } \frac{d^2 x}{dt^2} + n^2 x = 0$$

2. Solve the separable equations

$$\text{(a)} \quad (x-4)y^4 dx - x^3(y^3 - 3) dy = 0.$$

$$\text{Ans. } -\frac{1}{x} + \frac{2}{x^2} + \frac{1}{y} - \frac{1}{y^3} = c$$

$$\text{(b)} \quad \frac{dy}{dx} = e^{2x-3y} + 4x^2 e^{-3y}$$

$$\text{Ans. } 3e^x - 2e^{3y} + 8x^3 = c$$

$$\text{(c)} \quad \frac{dy}{dx} - x \tan(y-x) = 1.$$

$$\text{Ans. } \log \sin(y-x) = \frac{1}{2}x^2 + c$$

$$(d) (x - y)^2 \frac{dy}{dx} = a^2.$$

$$\text{Ans. } a \log \left(\frac{x-y-a}{x-y+a} \right) = 2y + k$$

$$(e) \frac{dx}{dt} = x^2 - 2x + 2$$

$$\text{Ans. } x = 1 + \tan(t + c)$$

$$(f) x \frac{dy}{dx} + \cot y = 0 \text{ if } y(\sqrt{2}) = \pi/4.$$

$$\text{Ans. } x = 2 \cos y$$

3. Solve the homogeneous equations :

$$(a) \frac{dy}{dx} = \frac{y}{x} + \sin \frac{y}{x} \quad \text{Ans. } 2x \tan^{-1}(cx)$$

$$(b) ydx - xdy = \sqrt{x^2 + y^2} dx$$

$$\text{Ans. } y + \sqrt{x^2 + y^2} = c$$

$$(c) 2xy \frac{dy}{dx} = 3y^2 + x^2 \quad \text{Ans. } x^2 + y^2 = cx^3$$

$$(d) y' = \frac{y+x}{x} \quad \text{Ans. } y = x \log |cx|$$

4. Reduce the following equations to homogeneous form and solve them

$$(a) (2x + y - 3)dy = (x + 2y - 3)dx$$

$$\text{Ans. } (y - x)^3 = k(y + x - 2)$$

$$(b) (x + 2y)(dx - dy) = dx + dy$$

$$\text{Ans. } \log(x + y + \frac{1}{3}) + \frac{3}{2}(y - x) + k$$

$$(c) \frac{dy}{dx} + \frac{ax+by+g}{hx+by+f} = 0.$$

$$\text{Ans. } ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

5. Solve the following linear equations :

$$(a) x^2 \frac{dy}{dx} = 3x^2 - 2yx + 1$$

$$\text{Ans. } y = x + \frac{1}{x} + \frac{c}{x^2}.$$

$$(b) (x+1) \frac{dy}{dx} - y = e^{3x}(x+1)^2.$$

$$\text{Ans. } y = (x+1) \left(\frac{1}{3} e^{3x} + c \right)$$

$$(c) \frac{dy}{dx} = -\frac{x+y \cos x}{1+\sin x}$$

$$\text{Ans. } y(1 + \sin x) = c - \frac{x^2}{2}$$

$$(d) y \log y dx + (x - \log y) dy = 0$$

$$\text{Ans. } x = \frac{1}{2} \log y + \frac{c}{\log y}$$

$$(e) x \log x \frac{dy}{dx} + y = \log x^2. \text{ Ans. } y = \log x + \frac{c}{\log x}$$

6. Solve the following equations

$$(a) (x^3 y^2 + xy) dx = dy.$$

$$\text{Ans. } \frac{1}{y} = x^2 - 2 + ce^{-x^2/2}$$

$$(b) \frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$$

$$\text{Ans. } \sin y = (1+x)(e^x + c)$$

$$(c) \frac{dz}{dx} + \frac{z}{x} \log z = \frac{z}{x} (\log z)^2$$

$$\text{Ans. } \frac{1}{\log z} = 1 + cx$$

$$(d) \frac{dy}{dx} + y \tan x = y^3 \sec x$$

$$\text{Ans. } \cos^2 x = y^2(c + 2 \sin x)$$

7. Solve the exact equations

$$(a) [y(1 + \frac{1}{x}) + \cos y]dx + (x + \log x - x \sin y)dy = 0$$

$$\text{Ans. } y(x + \log x) + x \cos y = 0$$

$$(b) [5x^4 + 3x^2 y^2 - 2xy^3]dx = (2x^3 y - 3x^2 y^2 - 5y^4)dy = 0$$

$$\text{Ans. } x^5 + x^3 y^2 - x^2 y^3 - y^5 = 0$$

$$(c) \frac{2x}{y^3} dx + \frac{y^2 - 3x^2}{y^4} dy = 0$$

$$\text{Ans. } x^2 - y^2 = cy^3$$

$$(d) (x^2 + y^2 - a^2)x dx + (x^2 - y^2 - b^2)y dy = 0.$$

$$\text{Ans. } x^4 + 2x^2 y^2 - y^4 - 2a^2 x^2 - 2b^2 y^2 = c$$

8. Solve the following equations which are reducible to exact equations:

$$(a) (y^2 + 2x^2 y) dx + (2x^3 - xy) dy = 0.$$

$$\text{Ans. } 6\sqrt{xy} - \left(\frac{y}{x}\right)^{3/2} = c$$

$$(b) (x^2 + y^2 + 1) dx - 2xy dy = 0.$$

$$\text{Ans. } x^2 - \frac{1}{x} - \frac{y^2}{x} = c$$

$$(c) (xy e^{x/y} + y^2) dx - x^2 e^{x/y} dy = 0$$

$$\text{Ans. } 3 \log x - 2 \log y + \frac{y}{x} = c$$

$$(d) x dy - y dx + a(x^2 + y^2) dx = 0$$

$$\text{Ans. } ax + \tan^{-1} \frac{y}{x} = c$$

$$(e) x dy - y dx = xy^2 dx \quad \text{Ans. } \frac{x^2}{2} + \frac{x}{y} = c$$

$$(f) \frac{dy}{dx} = \frac{x^3 + y^3}{xy^2} \quad \text{Ans. } 3 \log x - \left(\frac{y}{x}\right)^3 = c$$

9. In RC circuit containing steady voltage V , find the change at any time t . Also derive expression for the current flowing through the circuit.

$$\text{Ans. } Q = CV(1 - e^{-\frac{t}{RC}}) \text{ and } I = \frac{V}{R} e^{-\frac{t}{RC}}$$

10. An RL circuit has an e.m.f. of $3 \sin 2t$ volts, a resistance of 10 ohms, an inductance of 0.5 henry, and an initial current of 6 amp. Find the current in the circuit at any time t .

$$\text{Ans. } I = \frac{609}{101} e^{-20t} + \frac{30}{101} \sin 2t - \frac{3}{101} \cos 2t$$

11. If the air is maintained at 30°C and the temperature of the body cools from 80°C to 60°C in 12 min, find the temperature of the body after 24 min.

$$\text{Ans. } 48^\circ\text{C}$$

12. A pipe 20 cm in diameter contains steam at 200°C . It is covered by a layer of insulation 6 cm thick and thermal conductivity 0.0003. If the temperature of the outer surface is 30°C , find the heat loss per hour from 2 metre length of the pipe.

$$\text{Ans. } 490,000 \text{ cal.}$$

13. A steam pipe 20 cm in diameter is covered with an insulating sheath 5 cm thick, the conductivity of which is 0.00018. If the pipe has the constant temperature 100°C , and outer surface of the sheath is kept at 30°C , find the temperature of the sheath as a function of the distance x from the axis of the pipe. How much heat is lost per hour through a section 1 metre long.

Ans. $T = 497.5 - 172.6 \log x$ and
 $Q = 70300 \text{ cal.}$

14. A tank contains 5000 litres of fresh water. Salt water which contains 100 gm of salt per litre flows into it at the rate of 10 l/min and the mixture kept uniform by stirring, runs out at the same rate. When will the tank contain 2,00,000 gm of salt? How long will it take for the quantity of salt in the tank to increase from 1,50,000 gm to 2,50,000 gm?

Ans. 4 h 2.52 min, 2 h 48.23 min

15. The amount x of a substance present in a certain chemical reaction at time t is given by $\frac{dx}{dt} + \frac{x}{10} = 2 - 1.5e^{-\frac{t}{10}}$. If at $t = 0$, $x = 0.5$, find x at $t = 10$.

Ans. $20 - \frac{69}{2e}$

16. Radium decomposes at a rate proportional to the amount present. If a fraction p of the original amount disappears in 1 year, how much will it remain at the end of 21 years?

Hint: use $x = x_0 e^{kt}$. Using given data $k = \log \frac{1}{p}$

Ans. $(1 - \frac{1}{p})^{21}$ times the original amount

17. Find the orthogonal trajectories of the family of curves $r = a(1 + \cos \theta)$.

Ans. $r = c(1 - \cos \theta)$

18. Find the orthogonal trajectories of the family of parabolas $y^2 = 4ax$.

Ans. $2x^2 + y^2 = c$

19. Find the orthogonal trajectories of the family of curves $ay^2 = x^3$.

Ans. $3y^2 + 2x^2 = c^2$

20. Solve the following differential equation

(a) $\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 14\frac{d^2y}{dx^2} - 20\frac{dy}{dx} + 25y = 0$

Hint: Roots of A.E. are $1 + 2i, 1 - 2i, 1 + 2i, 1 - 2i$.

Ans. $y = e^x[(c_1 + c_2x) \sin 2x + (c_3 + c_4x) \cos 2x]$

(b) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0$, $y(0) = -3$, $y'(0) = -1$.

Ans. $y = e^{3x}(2 \sin 4x - 3 \cos 4x)$

(c) $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0$.

Ans. $y = (c_1 + c_2x) \cos 2x + (c_3 + c_4x) \sin 2x$

(d) $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0$

Ans. $x = (c_1 + c_2t) e^{-3t}$

(e) $\frac{d^2y}{dx^2} + (a + b)\frac{dy}{dx} + aby = 0$

Ans. $y = c_1 e^{-ax} + c_2 e^{-bx}$

(f) $(D^3 + D^2 + 4D + 4)y = 0$

Ans. $y = c_1 e^{-x} + c_2 \cos 2x + c_3 \sin 2x$

- (g) $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 2y = 0$. Are the solutions linearly independent?

Ans. $y = c_1 e^x + c_2 e^x$, Wronskian

$$W(e^x, e^{2x}) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x} \neq 0,$$

Therefore, e^x, e^{2x} are linearly independent.

21. Solve the following differential equations:

(a) $\frac{d^2y}{dx^2} + a^2y = \tan ax$

Ans. $y = c_1 \cos ax + c_2 \sin ax$
 $-\frac{1}{a^2} \cos ax \log (\sec ax + \tan ax)$

(b) $(D^2 - 3D + 2)y = 6e^{-3x} + \sin 2x$.

Ans. $y = c_1 e^x + c_2 e^{2x}$
 $+\frac{3}{10}e^{-3x} + \frac{1}{20}(3 \cos 2x - \sin 2x)$

(c) $\frac{d^2y}{dx^2} - 4y = x^2 + 2x$.

Ans. $y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{4}(x^2 + \frac{1}{2})$

(d) $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$.

Ans. $y = (c_1 + c_2x)e^{2x}$
 $-e^{2x}[4x \cos 2x + (2x^2 - 3) \sin 2x]$

(e) $(D^2 - 1)y = x \sin x + (1 + x^2)e^x$.

Ans. $y = c_1 e^x + c_2 e^{-x} - \frac{1}{2}(x \sin x + \cos x)$
 $+\frac{1}{12}xe^x(2x^2 - 3x + 9)$

(f) $(D - 1)^2(D + 1)^2y = \sin^2 \frac{x}{2} + e^x + x$.

Ans. $y = (c_1 + c_2x)e^x + (c_3 + c_4x)e^{-x}$
 $+\frac{1}{2} - \frac{1}{8} \cos x + e^x + x$

(g) $(D^2 + 4)y = e^x + \sin 2x$.

Ans. $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{5}e^x - \frac{x}{4} \cos 2x$

$$(h) \frac{d^3y}{dx^3} + 3 \frac{dy}{dx} + 2y = 4 \cos^2 x.$$

$$\text{Ans. } y = c_1 e^{-x} + c_2 e^{-2x} + 1 + \frac{1}{10} (3 \sin 2x - \cos 2x)$$

$$(i) \frac{d^3y}{dx^3} - 3 \frac{dy}{dx} + 2y = x e^{3x} + \sin 2x.$$

$$\text{Ans. } y = c_1 e^x + c_2 e^{2x} + \frac{1}{4} e^{3x} (2x - 3) + \frac{1}{20} (3 \cos 2x - \sin 2x)$$

$$(j) \frac{d^3y}{dx^3} + 2 \frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 e^{2x} + \sin^2 x.$$

$$\text{Ans. } y = c_1 + (c_2 + c_3 x) e^{-x} + \frac{e^{2x}}{18} (x^2 - \frac{7}{8} x + \frac{11}{6}) + \frac{1}{100} (3 \sin 2x + 4 \cos 2x)$$

$$(k) \frac{d^2y}{dx^2} + 3 \frac{dy}{dx} + 2y = e^{e^x}.$$

$$\text{Ans. } y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} e^{e^x}$$

$$(l) (D^2 - D)y = 2x + 1 + 4 \cos x + 2e^x.$$

$$\text{Ans. } y = c_1 + c_2 e^x + c_3 e^{-x} + x e^x - (x^2 + x) - 2 \sin x$$

22. Solve the following differential equations of second order by changing the independent variable.

$$(i) x^4 \frac{d^2y}{dx^2} + 2x^3 \frac{dy}{dx} + 4y = 0$$

$$\text{Ans. } y = c_1 \cos \frac{2}{x} + c_2 \sin \left(\frac{2}{x} \right)$$

$$(ii) x^6 \frac{d^2y}{dx^2} + 3x^5 \frac{dy}{dx} + a^2 y = \frac{1}{x^2}$$

$$\text{Ans. } y = c_1 \cos \frac{a}{2x^2} - c_2 \sin \frac{a}{2x^2} + \frac{1}{a^2 x^2}$$

$$(iii) x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = \log x$$

$$\text{Ans. } y = (c_1 + c_2 \log x)x + \log x + 2$$

$$(iv) \cos x \frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} - (2 \cos^3 x)y = 2 \cos^5 x$$

$$\text{Ans. } y = c_1 e^{\sqrt{2} \sin x} + c_2 e^{-\sqrt{2} \sin x} + \sin^2 x$$

23. Solve the following differential equation of second order by changing the dependent variable.

$$(i) \frac{d^2y}{dx^2} + 2n \cot nx \frac{dy}{dx} + (m^2 - n^2)y = 0$$

$$\text{Ans. } y = \frac{1}{\sin nx} [c_1 \cos mx + c_2 \sin mx]$$

$$(ii) x^2 \frac{d^2y}{dx^2} - 2y = x^2 + \frac{1}{x}$$

$$\text{Ans. } y = c_1 x^2 + \frac{c_2}{x} + \frac{1}{3} \left(x^2 - \frac{1}{x} \right) \log x$$

$$(iii) \cos^2 x \frac{d^2y}{dx^2} - (2 \sin x \cos x) \frac{dy}{dx} + (\cos^2 x)y = 0$$

$$\text{Ans. } y = [c_1 \cos(\sqrt{2}x) + c_2 \sin(\sqrt{2}x)] \sec x$$

$$(iv) \left[\frac{d^2y}{dx^2} + y \right] \cot x + 2 \left[y \tan x + \frac{dy}{dx} \right] = 0$$

$$\text{Ans. } y = (c_1 + c_2 x) \cos x$$

24. Solve by method of undetermined coefficients.

$$(i) \frac{d^2y}{dx^2} + y = 2e^x + \cos x$$

$$\text{Ans. } y = c_1 \cos x + c_2 \sin x + e^x + \frac{1}{2} + x \sin x$$

$$(ii) \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + y = x^2 e^x$$

$$\text{Ans. } y = (c_1 + c_2 x) e^x + \frac{x^4}{12} e^x$$

$$(iii) \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + y = x - e^x$$

$$\text{Ans. } y = (c_1 + c_2 x) e^{-x} + x - 2 - \frac{e^x}{4}$$

25. Solved by method of reduction of order.

$$(i) x^2 \frac{d^2y}{dx^2} - 2x(1+x) \frac{dy}{dx} + 2(1+x)y = x^3 \text{ if } y = x \text{ is a solution of the C.F.}$$

$$\text{Ans. } y = x \left[-\frac{x}{2} + \frac{c_1}{2} e^{2x} + c_2 \right]$$

$$(ii) (1-x^2) \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = x(1-x^2)^{\frac{3}{2}} \text{ if } y = x \text{ is a part of C.F.}$$

$$\text{Ans. } y = \frac{-x(1-x^2)^{\frac{3}{2}}}{9} - c_1 \left[\sqrt{1-x^2} + x \sin^{-1} x \right] + c_2 x$$

$$(iii) \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + y = x - e^x$$

$$\text{Ans. } y = (c_1 + c_2 x) e^{-x} + x - 2 - \frac{e^x}{4}$$

26. Solve the following equations using the method of variation of parameters:

$$(a) \frac{d^2y}{dx^2} + a^2 y = \sec ax.$$

$$\text{Ans. } y = c_1 \cos ax + c_2 \sin ax + \frac{1}{a^2} \cos ax \log(\cos ax) + \frac{1}{a} x \sin ax$$

$$(b) \frac{d^2y}{dx^2} + y = x \sin x.$$

$$\text{Ans. } y = c_1 \cos x + c_2 \sin x + \frac{x}{4} \sin x - \frac{x^2}{4} \cos x$$

$$(c) \frac{d^2y}{dx^2} + 4y = \tan 2x.$$

$$\text{Ans. } y = c_1 \cos 2x + c_2 \sin 2x$$

$$(d) \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2y = e^x \tan x.$$

$$\text{Ans. } y = e^x (c_1 \cos x + c_2 \sin x) - e^x \cos x \log(\sec x + \tan x)$$

$$(e) (D^2 + 1)y = \frac{1}{1 + \sin x}.$$

$$\text{Ans. } y = c_1 \cos x + c_2 \sin x + \sin x \log(1 + \sin x) - x \cos x - 1$$

$$(f) (D^2 + 1)y = x \sin x.$$

$$\text{Ans. } y = c_1 \cos x + c_2 \sin x + \frac{x}{4} \sin x - \frac{x^2}{4} \cos x$$

27. Solve the following Cauchy-Euler equations:

$$(a) x^3 \frac{d^3 y}{dx^3} + 2x^2 \frac{d^2 y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right).$$

$$\text{Ans. } y = \frac{c_1}{x} + x[c_2 \cos(\log x) + c_3 \sin(\log x)]$$

$$(b) x^2 \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + 5y = x^2 \sin(\log x)$$

$$\text{Ans. } y = x^2[c_1 \cos(\log x) + c_2 \sin(\log x)] - \frac{1}{2} x^2 \log x \cos(\log x)$$

$$(c) x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} - 4y = x^2 + 2 \log x$$

$$\text{Ans. } y = \frac{c_1}{x} + c_2 x^4 - \frac{x^6}{6} - \frac{1}{2} \log x + \frac{3}{8}$$

$$(d) x^2 \frac{d^3 y}{dx^3} - 4x \frac{d^2 y}{dx^2} + 6 \frac{dy}{dx} = 4$$

Hint: Write the given equation in the form

$$x^3 \frac{d^3 y}{dx^3} - 4x^2 \frac{d^2 y}{dx^2} + 6x \frac{dy}{dx} = 4x$$

$$\text{Ans. } y = c_1 + c_2 x^3 + c_3 x^4 + \frac{2}{3} x$$

$$(e) x^2 \frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^2$$

$$\text{Ans. } y = \frac{1}{x}(c_1 + c_2 \log x) + \frac{1}{x} \log \frac{x}{1-x}$$

$$(f) x^2 \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + y = \log \frac{\sin(\log x) + 1}{x}$$

$$\begin{aligned} \text{Ans. } y = & c_1 x^{\sqrt{3}+2} + c_2 x^{2-\sqrt{3}} \\ & + \frac{1}{61x} \log x [5 \sin(\log x) + 6 \cos(\log x)] \\ & + \frac{2}{61} [21 \sin(\log x) + 191 \cos(\log x)] \\ & + \frac{1}{6x} (1 + \log x) \end{aligned}$$

$$(g) x^3 \frac{d^3 y}{dx^3} + 3x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + 8y = 65 \cos(\log x)$$

$$\begin{aligned} \text{Ans. } y = & c_1 x^{-2} + x[c_2 \cos \sqrt{3}(\log x) \\ & + c_3 \sin(\sqrt{3} \log x)] \\ & + 8 \cos(\log x) - \sin(\log x) \end{aligned}$$

28. Solve the following Legendre's linear equations:

$$(a) (3x+2) \frac{d^2 y}{dx^2} + 3(3x+2) \frac{dy}{dx} - 36y = 3x^2 + 4x + 1.$$

$$\begin{aligned} \text{Ans. } y = & c_1 (3x+2)^2 + c_2 (3x+2)^{-2} \\ & + \frac{1}{108} [(3x+2)^2 \log(3x+2) + 1 \end{aligned}$$

$$(b) (x+1)^2 \frac{d^2 y}{dx^2} + (x+1) \frac{dy}{dx} = (2x+3)(2x+4).$$

$$\begin{aligned} \text{Ans. } y = & c_1 + c_2 \log(x+1) \\ & + [\log(x+1)]^2 + x^2 + 8x \end{aligned}$$

$$(c) (1+2x)^2 \frac{d^2 y}{dx^2} - 6(1+2x) \frac{dy}{dx} + 16y = 8(1+2x)^2$$

$$\text{Ans. } (1+2x)^2 [c_1 + c_2 \log(1+2x) + \log(1+2x)]$$

29. Solve the following simultaneous equations:

$$(a) \frac{d^2 x}{dt^2} + 4x + 5y = t^2, \quad \frac{d^2 y}{dt^2} + 5x + 4y = t + 1$$

$$\begin{aligned} \text{Ans. } x = & c_1 e^t + c_2 e^{-t} + c_3 \cos 3t + c_4 \sin 3t \\ & - \frac{1}{9} (4t^2 - 5t + \frac{37}{9}), \\ y = & -c_1 e^t - c_2 e^{-t} + c_3 \cos 3t \\ & + c_4 \sin 3t + \frac{1}{9} (5t^2 - 4t + \frac{44}{9}) \end{aligned}$$

$$(b) \frac{d^2 y}{dt^2} + \frac{dy}{dt} - 2y = \sin t, \quad \frac{dx}{dt} + x - 3y = 0.$$

$$\begin{aligned} \text{Ans. } x = & \frac{3}{2} c_1 e^{2t} - 3c_2 e^{-t} \\ & - \frac{3}{10} e^t (\cos t - 2 \sin t), \\ y = & c_1 e^t + c_2 e^{-2t} - \frac{1}{10} (\cos t + 3 \sin t) \end{aligned}$$

$$(c) \frac{dx}{dt} + 2x + 3y = 0, \quad \frac{dy}{dt} + 3x + 2y = 2e^{2t}.$$

$$\begin{aligned} \text{Ans. } x = & c_1 e^t + c_2 e^{-5t} + \frac{6}{7} e^{2t}, \\ y = & c_2 e^{-5t} - c_1 e^t + \frac{8}{7} e^{2t} \end{aligned}$$

$$(d) \frac{dx}{dt} + y = \sin t, \quad \frac{dy}{dt} + x = \cos t \text{ subject to } x = 2 \text{ and } y = 0 \text{ when } t = 0.$$

$$\text{Ans. } x = e^{-t} + e^t, \quad y = e^{-t} - e^t + \sin t$$

$$(e) 2 \frac{d^2 x}{dt^2} + 3 \frac{dy}{dt} = 4, \quad 2 \frac{d^2 y}{dt^2} - 3 \frac{dx}{dt} = 0 \text{ subject to } x(0) = y(0) = 0, \quad x'(0) = y'(0) = 0.$$

$$\text{Ans. } x = \frac{8}{9} (1 - \cos \frac{3t}{2}), \quad y = \frac{4}{3} t - \frac{8}{9} \sin \frac{3t}{2}$$

30. An uncharged condenser of capacity C is charged by applying an e.m.f. $E \sin \frac{t}{\sqrt{LC}}$ through leads of self inductance L and negligible resistance. Then find the charge at the condenser plate at time t .

$$\text{Ans. } \frac{EC}{2} \left[\sin \frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}} \cos \frac{t}{\sqrt{LC}} \right]$$

31. A circuit consists of an inductance L and a capacitor of capacity C in series. An alternating e.m.f. $E \sin nt$ is applied to the circuit at time $t = 0$, the initial current and charge on the condenser being zero. Prove that the current at time t is

$\frac{nE}{L(n^2 - \omega^2)} (\cos \omega t - \cos nt)$, where $CL \omega^2 = 1$.
If $n = \omega$, show that current at time t is $\frac{Et \sin \omega t}{2L}$.

Hint: Solve $L \frac{d^2 Q}{dt^2} + \frac{Q}{C} = E \sin nt$ under the conditions $I(0) = 0$, $Q(0) = 0$. Then use $I = \frac{dQ}{dt}$.

32. A pendulum of length l hangs against a wall at an angle θ to the horizontal. Show that the time of complete oscillation is $2\pi \sqrt{\frac{l}{g \sin \theta}}$.
33. A simple pendulum has a period T . When the length of the string is increased by a small fraction $\frac{1}{n}$ of its original length, the period is T' . Show that (approximately) $\frac{1}{n} = \frac{2(T' - T)}{T}$.
34. How many seconds a clock would lose per day if the length of its pendulum were increased in the ratio 900:901.

Ans. 48 s/day

Solution about regular points

35. Solve the following differential equations about $x = 0$.

(i) $\frac{d^2 y}{dx^2} + x^2 y = 0$

Ans. $y = a_0 \left[x - \frac{x^4}{4.3} + \frac{x^8}{8.7.4.3} - \frac{x^{12}}{12.11.8.7.4.3} + \dots \right]$
 $+ a_1 \left[x - \frac{x^5}{5.4} + \frac{x^9}{9.8.5.4} - \frac{x^{13}}{13.12.9.8.5.4} + \dots \right]$

(ii) $(x^2 + 1) \frac{d^2 y}{dx^2} + xy \frac{dy}{dx} = xy = 0$

Ans. $y = a_0 \left(1 - \frac{1}{6}x^3 - \frac{3}{40}x^5 + \dots \right)$
 $+ a_1 \left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{3}{40}x^5 + \dots \right)$

(iii) $\frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (2x^2 + 1)y = 0$

Ans. $y = a_0 \left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4 + \dots \right)$
 $+ a_1 \left(x - \frac{x^3}{3} - \frac{x^5}{30} - \dots \right)$

(iv) $\frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (3x + 2)y = 0$

Ans. $y = a_0 \left(1 - x^2 - \frac{x^3}{2} + \frac{x^4}{3} + \frac{11x^5}{40} + \dots \right)$
 $+ a_1 \left(x - \frac{x^3}{2} - \frac{x^4}{4} + \frac{x^5}{8} + \dots \right)$

36. Find the power series solution in the powers of $(x - 1)$ of the initial value problem.

$x \frac{d^2 y}{dx^2} + \frac{dy}{dx} + 2y = 0$, $y(1) = 1$, $y'(1) = 2$.

Hint: $x = 1$ is the ordinary point. So consider

$y = \sum_{n=0}^{\infty} a_n (x - 1)^n$, find $\frac{dy}{dx}$ and $\frac{d^2 y}{dx^2}$, put in the given equation. Use the initial conditions in finding a_0 and a_1 by putting $x = 1$ in y and $\frac{dy}{dx}$. Other coefficients to be found by equating to zero the powers of $(x - 1)$.

Ans. $y = 1 + 2(x - 1) - 2(x - 1)^2$
 $+ \frac{2}{3}(x - 1)^3 - \frac{1}{6}(x - 1)^4 + \dots$

Solution about regular singular point

37. Solve the following differential equations about $x = 0$

(i) $x(1 - x) \frac{d^2 y}{dx^2} + (1 - x) \frac{dy}{dx} - y = 0$

Ans. $y = a_0(1 + \log x)(1 + x + \frac{2x^2}{4}$
 $+ \frac{2.5}{4.9}x^3 + \dots) + a_0(-2x - x^2 - \dots)$

(ii) $x(1 - x) \frac{d^2 y}{dx^2} - (1 + 3x) \frac{dy}{dx} - y = 0$

Ans. $y = (A + B \log x)(1.2x^2 + 2.3x^3$
 $+ 3.4x^4 + \dots)$
 $+ B(-1 + x + 2x^2 + 11x^3 + \dots)$

(iii) $2x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - 1)y = 0$

Ans. $y = c_1 x \left(1 - \frac{x^2}{14} + \frac{x^4}{616} + \dots \right)$
 $+ c_2 \sqrt{x} \left(1 - \frac{x^2}{2} + \frac{x^4}{40} + \dots \right)$

(iv) $(2x + x^3) \frac{d^2 y}{dx^2} - \frac{dy}{dx} - 6xy = 0$

Ans. $y = c_1 \left(1 + 3x^2 + \frac{3x^4}{5} + \dots \right)$
 $+ c_2 x^{\frac{3}{2}} \left(1 + \frac{3x^2}{8} - \frac{3x^4}{128} + \dots \right)$

(v) $4x \frac{d^2 y}{dx^2} + 2(1 - x) \frac{dy}{dx} - y = 0$

Ans. $y = c_1 \left(1 + \frac{x}{2!} + \frac{x^2}{2^2.2!} + \frac{x^3}{2^3.3!} + \dots \right)$
 $+ c_2 x^{\frac{1}{2}} \left(1 + \frac{x}{1.3} + \frac{x^2}{1.3.5} + \frac{x^3}{1.3.5.7} + \dots \right)$

(vi) $x \frac{d^2 y}{dx^2} + \frac{dy}{dx} + 2y = 0$

Ans. $y = (c_1 + c_2 \log x) \left(1 - 2x + \frac{2x^2}{(2!)^2} - \dots \right)$
 $+ c_2(4x - 3x^2 + \dots)$

Bessel's Equation and Bessel Function

38. Write the general solution of the differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + \left(x^2 - \frac{9}{16}\right)y = 0 \text{ in terms of } J_n(x).$$

$$\text{Ans. } y = c_1 J_{\frac{3}{4}}(x) + c_2 J_{-\frac{3}{4}}(x).$$

39. Express $J_5(x)$ in terms of $J_0(x)$ and $J_1(x)$.

$$\text{Ans. } J_5(x) = \left(\frac{384}{x^4} - \frac{72}{x^2} - 1\right)J_1(x) + \left(\frac{12}{x} - \frac{192}{x^3}\right)J_0(x)$$

40. Solve $x \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + \frac{1}{2}xy = 0$ in terms of Bessel's function.

$$\text{Ans. } y = \sqrt{x} \left[c_1 J_{\frac{1}{2}}\left(\frac{x}{\sqrt{2}}\right) + c_2 J_{-\frac{1}{2}}\left(\frac{x}{\sqrt{2}}\right) \right]$$

41. Show that $\int J_3(x) dx = -J_2(x) - \frac{2}{x}J_0(x) + c$

42. Show that $J_0^2(x) + 2J_1^2(x) + 2J_2^2(x) + 2J_3^2(x) + \dots = 1$.

Hint: Squaring and integrating the Jacobi series

$$\cos(x \sin \phi) = J_0 + 2J_2 \cos 2\phi + 2J_4 \cos 4\phi + \dots, \quad \text{and}$$

$$\sin(x \sin \phi) = 2J_1 \sin \phi + 2J_3 \sin 3\phi + 2J_5 \sin 5\phi + \dots,$$

we get

$$\int_0^\pi \cos^2(x \sin \phi) d\phi = \pi(J_0^2 + 2J_2^2 + 2J_4^2 + \dots)$$

$$\int_0^\pi \sin^2(x \sin \phi) d\phi = \pi(2J_1^2 + 2J_3^2 + 2J_5^2 + \dots).$$

Adding these expressions, we get the required result

43. Show that

$$(i) \cos x = J_0 - 2J_1 + 2J_4 + \dots$$

$$= J_0 + 2 \sum_{n=1}^{\infty} (-1)^n J_{2n},$$

$$(ii) \sin x = 2J_1 - 2J_3 + 2J_5 + \dots$$

$$= 2 \sum_{n=1}^{\infty} (-1)^n J_{2n+1}$$

Hint: Put $\phi = \frac{\pi}{2}$ in the Jacobi series

$$\cos(x \sin \phi) = J_0 + 2J_2 \cos 2\phi + 2J_4 \cos 4\phi + \dots$$

and

$$\begin{aligned} \sin(x \sin \phi) &= 2J_1 \sin \phi + 2J_3 \sin 3\phi \\ &\quad + 2J_5 \sin 5\phi + \dots \end{aligned}$$

44. Show that

$$\frac{d}{dx} [J_n^2(x)] = \frac{x}{2n} [J_{n-1}^2(x) - J_{n+1}^2(x)].$$

Legendre Equation and Legendre Polynomial

45. Express in the terms of Legendre polynomial
(i) $4x^3 - 2x^2 - 3x + 8$

$$\text{Ans. } \frac{8}{5}P_3(x) - \frac{4}{3}P_2(x) - \frac{3}{5}P_1(x) + \frac{22}{5}P_0(x)$$

$$(ii) x^4 + 3x^3 - x^2 + 5x - 2$$

$$\text{Ans. } \frac{8}{35}P_4(x) + \frac{6}{5}P_3(x) - \frac{2}{21}P_2(x) + \frac{34}{5}P_1(x) - \frac{224}{105}P_0(x)$$

$$(iii) 4x^2 - 3x + 2$$

$$\text{Ans. } \frac{8}{3}P_2(x) - 3P_1(x) + \frac{10}{3}P_0(x)$$

46. Show that

$$P_n(x) = P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x)$$

Hint: Use recurrence formulae

$$(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x) \quad (i)$$

$$nP_n(x) = xP'_n(x) - P'_{n-1}(x) \quad (ii)$$

$$\text{From (i), } P_n(x) = P'_{n+1}(x) - P'_{n-1}(x) - 2nP_n(x)$$

$$= P'_{n+1}(x) - P'_{n-1}(x) - 2xP'_n(x)$$

$$+ 2P'_{n-1}(x)$$

$$= P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x)$$

$$47. \text{ Show that } \int_{-1}^1 P_n(x) dx = \begin{cases} 2 & \text{if } n = 0 \\ 0 & \text{if } n \geq 1. \end{cases}$$

48. Show that

$$\int_{-1}^1 xP_n(x)P'_n(x) dx = \frac{2n}{2n+1}.$$

Hint: Use integration by parts and

$$\int_{-1}^1 P_n^2(x) dx = \frac{2}{2n+1}.$$

49. Using Rodrigue's formula, show that $P_n(x)$ satisfies the differential equation

$$\frac{d}{dx} [(1-x^2) \frac{d}{dx} P_n(x)] + n(n+1)P_n(x) = 0.$$

Hint: Let $y = (x^2 - 1)^n$. Find $\frac{dy}{dx}$ and differentiate once more to get $(x^2 - 1) \frac{d^2 y}{dx^2} + 2xy_1 = 2n(xy_1 + n)$.

Then use Leibnitz's theorem to get

$$(1-x^2)y_{n+2} - 2xy_{n+1} + n(n+1)y_n = 0$$

or

$$[(1-x^2)y_{n+1}] + n(n+1)y_n = 0$$

or

$$\begin{aligned} \frac{d}{dx} \left[(1-x^2) \frac{d}{dx} \left(\frac{d^n}{dx^n} (x^2-1)^n \right) \right] \\ + n(n+1) \frac{d^n}{dx^n} (x^2-1)^n = 0 \end{aligned}$$

50. Expand

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 < x < 1 \end{cases}$$

in Fourier–Legendre series.

$$\begin{aligned} \text{Ans. } f(x) &= \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) \\ &+ \frac{5}{16}P_2(x) - \frac{3}{32}P_3(x) + \dots \end{aligned}$$

3

Partial Differential Equations

A differential equation which involves partial derivatives with respect to two or more independent variables is called a *partial differential equation*. Such equations appear in the description of physical processes in applied sciences and engineering. The equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ (two-dimensional Laplace equation),}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \text{ (wave equation),}$$

and

$$\frac{\partial^2 u}{\partial x^2} = k \frac{\partial u}{\partial t} \text{ (heat conduction equation)}$$

are all partial differential equations.

The order of the highest partial derivative occurring in a partial differential equation is called the *order* of that partial differential equation. The degree of the highest partial derivative appearing in the partial differential equation free from radicals and fractions is called the *degree* of that partial differential equation. For example,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x^2 + y^2$$

is a partial differential equation of order two and degree one.

3.1 FORMULATION OF PARTIAL DIFFERENTIAL EQUATION

Partial differential equation can be formulated in two ways:

- (i) By the elimination of arbitrary constants from a given relation involving dependent and independent variables: If the number of arbitrary constants equals the number of independent

variables, then the partial differential equation formed is of first order. On the other hand, if the number of arbitrary constants is more than the number of independent variables, then the partial differential equation formed will be of second or higher order.

- (ii) By the elimination of arbitrary functions from the given functional relations: In this case, the order of the differential equation is the same as the number of arbitrary functions involved in the equation.

The following examples will illustrate the procedure of formation of partial differential equation. We shall use the following notation for the derivatives:

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}.$$

EXAMPLE 3.1

Form the partial differential equations from the relations

$$(i) \quad z = ax + a^2 y^2 + b,$$

$$(ii) \quad 2z = (ax + y)^2 + b.$$

Solution. (i) Differentiating z with respect to x and y , we get

$$\frac{\partial z}{\partial x} = a, \quad \frac{\partial z}{\partial y} = 2a^2 y.$$

Eliminating a between these relations, we have

$$\frac{\partial z}{\partial y} = 2 \left(\frac{\partial z}{\partial x} \right)^2 y,$$

or

$$q = 2p^2 y,$$

which is a partial differential equation of first order.

- (ii) Differentiating with respect to x , we get

$$\frac{\partial z}{\partial x} = a(ax + y).$$

Differentiating with respect to y , we get

$$\frac{\partial z}{\partial y} = ax + y.$$

Therefore, we have

$$\frac{\partial z}{\partial x} = a \frac{\partial z}{\partial y} = \frac{1}{x} \left(\frac{\partial z}{\partial y} - y \right) \frac{\partial z}{\partial y}$$

or

$$x \frac{\partial z}{\partial x} = \left(\frac{\partial z}{\partial y} - y \right) \frac{\partial z}{\partial y},$$

or, in notations

$$px = (q - y)q$$

or

$$px + qy = q^2,$$

which is the required partial differential equation.

EXAMPLE 3.2

Eliminate the arbitrary function f from the relation

$$z = y^2 + 2f\left(\frac{1}{x} + \log y\right).$$

Solution. The given relation is

$$z = y^2 + 2f\left(\frac{1}{x} + \log y\right).$$

Differentiating z partially with respect to x , we get

$$\frac{\partial z}{\partial x} = 2f'\left(\frac{1}{x} + \log y\right) \left(-\frac{1}{x^2}\right)$$

or

$$-px^2 = 2f'\left(\frac{1}{x} + \log y\right). \quad (1)$$

Differentiating z with respect to y , we get

$$\frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right) \left(\frac{1}{y}\right)$$

or

$$q = 2y + 2f'\left(\frac{1}{x} + \log y\right) \frac{1}{y}$$

or

$$qy - 2y^2 = 2f'\left(\frac{1}{x} + \log y\right). \quad (2)$$

The relations (1) and (2) yield

$$-px^2 = qy - 2y^2$$

or

$$px^2 + qy = 2y^2,$$

which is a partial differential equation of first order.

EXAMPLE 3.3

Eliminate arbitrary function f from the relation

$$f(x^2 + y^2, z - xy) = 0.$$

Solution. We are given that

$$f(x^2 + y^2, z - xy) = 0.$$

Let

$$x^2 + y^2 = u, \quad z - xy = v.$$

Then the given relation becomes

$$f(u, v) = 0.$$

Differentiating with respect to x , we get

$$\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} = 0$$

or

$$f_u(2x) + f_v\left(\frac{\partial z}{\partial x} - y\right). \quad (3)$$

Differentiating $f(u, v) = 0$ with respect to y , we get

$$\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} = 0$$

or

$$f_u(2y) + f_v\left(\frac{\partial z}{\partial y} - x\right) = 0. \quad (4)$$

Eliminating f_u and f_v from (3) and (4), we get

$$\begin{vmatrix} 2x & p - y \\ 2y & q - x \end{vmatrix} = 0$$

or

$$2x(q - x) - 2y(p - y) = 0$$

or

$$2xq - 2x^2 - 2yp + 2y^2 = 0$$

or

$$py - qx = y^2 - x^2.$$

EXAMPLE 3.4

Eliminate the function f from the relation

$$f(xy + z^2, x + y + z) = 0.$$

Solution. The given relation is

$$f(xy + z^2, x + y + z) = 0.$$

We put

$$xy + z^2 = u, \quad x + y + z = v,$$

so that the given equation becomes

$$f(u, v) = 0.$$

Differentiating with respect to x , we get

$$\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} = 0$$

or

$$f_u \left(y + 2z \frac{\partial z}{\partial x} \right) + f_v \left(1 + \frac{\partial z}{\partial x} \right) = 0 \quad (5)$$

Differentiating f with respect to y , we have

$$\frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} = 0$$

or

$$f_u \left(x + 2z \frac{\partial z}{\partial y} \right) + f_v \left(1 + \frac{\partial z}{\partial y} \right) = 0 \quad (6)$$

Eliminating f_u and f_v from (5) and (6), we have

$$\begin{vmatrix} y + 2pz & 1 + p \\ x + 2qz & 1 + q \end{vmatrix} = 0$$

or

$$(y + 2pz)(1 + q) - (1 + p)(x + 2qz) = 0$$

or

$$y + 2pz + yq + 2pqz - [x + 2qz + px + 2pqz] = 0$$

or

$$p(2z + 2qz - 2qz - x) + q(y - 2z) = x - y$$

or

$$p(x - 2z) + q(2z - y) = y - x.$$

EXAMPLE 3.5

Form partial differential equation from the relation

$$(i) \quad xyz = \phi(x + y + z)$$

$$(ii) \quad z = f(x^2 - y^2).$$

Solution. (i) Put $x + y + z = u$, $xyz = v$. Then the relation becomes

$$\phi(u) - v = 0.$$

Differentiating with respect to x , we get

$$\frac{\partial \phi}{\partial u} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x}$$

or

$$\frac{\partial \phi}{\partial u} \left(1 + \frac{\partial z}{\partial x} \right) = y \left(z + x \frac{\partial z}{\partial x} \right),$$

that is,

$$\frac{\partial \phi}{\partial u} (1 + p) = y(z + xp). \quad (7)$$

Differentiating with respect to y , we get

$$\frac{\partial \phi}{\partial u} (1 + q) = x(z + yq). \quad (8)$$

From (7) and (8), we have

$$\frac{1 + p}{1 + q} = \frac{y(z + xp)}{x(z + yq)}$$

or

$$x(y - z)p + y(z - x)q = z(x - y).$$

(ii) Differentiating z with respect to x , we have

$$\frac{\partial z}{\partial x} = f'(x^2 - y^2)2x$$

or

$$p = f'(x^2 - y^2)2x. \quad (9)$$

Differentiating z with respect to y , we have

$$\frac{\partial z}{\partial y} = f'(x^2 - y^2)(-2y)$$

or

$$q = f'(x^2 - y^2)(-2y). \quad (10)$$

Dividing (9) by (10), we get

$$\frac{p}{q} = -\frac{x}{y} \text{ or } py + qx = 0.$$

EXAMPLE 3.6

Form the partial differential equations from the relations

$$(i) \quad z = f(x + it) + g(x - it)$$

$$(ii) \quad z = f_1(y + 2x) + f_2(y - 3x).$$

Solution. (i) Differentiating z with respect to x , we have

$$p = \frac{\partial z}{\partial x} = f'(x + it) + g'(x - it). \quad (11)$$

$$r = \frac{\partial^2 z}{\partial x^2} = f''(x + it) + g''(x - it).$$

Differentiating z with respect to t , we have

$$q = \frac{\partial z}{\partial t} = if'(x + it) - ig'(x - it) \quad (12)$$

$$t = \frac{\partial^2 z}{\partial t^2} = i^2 f''(x + it) + i^2 g''(x - it).$$

Adding (11) and (12), we get

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \text{ or } r + t = 0,$$

which is partial differential equation of second order and whose solution is $z = f(x + it) + g(x - it)$.

(ii) Differentiating z with respect to x , we get

$$\begin{aligned}\frac{\partial z}{\partial x} &= 2f'_1(y + 2x) - 3f'_2(y - 3x), \\ \frac{\partial^2 z}{\partial x^2} &= 4f''_1(y + 2x) + 9f''_2(y - 3x).\end{aligned}$$

Differentiating z with respect to y , we get

$$\begin{aligned}\frac{\partial z}{\partial y} &= f'_1(y + 2x) + f'_2(y - 3x), \\ \frac{\partial^2 z}{\partial y^2} &= f''_1(y + 2x) + f''_2(y - 3x).\end{aligned}$$

Further,

$$\frac{\partial^2 z}{\partial x \partial y} = 2f''_1(y + 2x) - 3f''_2(y - 3x).$$

Therefore,

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6 \frac{\partial^2 z}{\partial y^2} = 0.$$

EXAMPLE 3.7

Form partial differential equation from the relation

$$z = yf(x) + xg(y).$$

Solution. We have

$$\begin{aligned}\frac{\partial z}{\partial x} &= yf'(x) + g(y), \\ \frac{\partial z}{\partial y} &= f(x) + xg'(y), \\ \frac{\partial^2 z}{\partial x \partial y} &= f'(x) + g'(y).\end{aligned}$$

Then

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} - z = xyf'(x) + xyg'(y) = xy \frac{\partial^2 z}{\partial x \partial y}.$$

Hence

$$xy \frac{\partial^2 z}{\partial x \partial y} = x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} - z,$$

which is the required partial differential equation.

EXAMPLE 3.8

Form partial differential equation from the relation

$$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

Solution. Differentiating z partially with respect to x and y , respectively, we get

$$\begin{aligned}2 \frac{\partial z}{\partial x} &= \frac{2x}{a^2} \quad \text{which yields} \quad \frac{1}{a^2} = \frac{p}{x} \\ \text{and} \quad 2 \frac{\partial z}{\partial y} &= \frac{2y}{b^2} \quad \text{which yields} \quad \frac{1}{b^2} = \frac{q}{y}.\end{aligned}$$

Substituting for $\frac{1}{a^2}$ and $\frac{1}{b^2}$ in the given equation, we get

$$2z = px + qy.$$

3.2 SOLUTIONS OF A PARTIAL DIFFERENTIAL EQUATION

A functional relation between x , y , and z is called a *solution* of the first order partial differential equation $f(x, y, z, p, q) = 0$ if this relation along with the derivatives $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$ satisfies that partial differential equation.

A solution such as $\phi(x, y, z, a, b) = 0$ of the equation $f(x, y, z, p, q) = 0$ which contains two arbitrary constants is called a *complete solution* or *complete integral* of that equation.

A solution obtained from the complete solution $\phi(x, y, z, a, b) = 0$ by giving particular values to the constants a and b is called a *particular integral* or a *particular solution* of the partial differential equation $f(x, y, z, p, q) = 0$.

We will now discuss the methods to find solutions of partial differential equation of first order.

(A) Direct Integration Method

This method is applicable to the partial differential equations which contain only one partial derivative. We should keep in mind that *the constant of integration will be arbitrary function of the variable kept constant*.

EXAMPLE 3.9

Solve

$$\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x + 3y).$$

Solution. Integrating the given equation twice with respect to x , we get

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{2} \sin(2x + 3y) + f(y), \quad \text{and}$$

$$\frac{\partial z}{\partial y} = -\frac{1}{4} \cos(2x + 3y) + xf(y) + \phi(y).$$

Integrating now with respect to y , we get the required solution as

$$z = -\frac{1}{12} \sin(2x + 3y) + xf_1(y) + \phi_1(y)\psi(x),$$

where $f_1(y)$ and $\phi_1(y)$ are integral of $f(y)$ and $\phi(y)$ with respect to y .

EXAMPLE 3.10

Solve

$$\frac{\partial^2 z}{\partial x^2} + z = 0$$

subject to the condition that when $x = 0$, $z = e^y$ and $\frac{\partial z}{\partial x} = 1$.

Solution. If z was a function of x alone, then since the roots of the auxiliary equation would have been imaginary, the solution would have been

$$z = c_1 \cos x + c_2 \sin x,$$

where c_1, c_2 are constants. But here z is a function of x and y , therefore, c_1 and c_2 are functions of y . Therefore, the solution of the given equation is

$$z = f(y) \cos x + \phi(y) \sin x.$$

Then

$$\frac{\partial z}{\partial x} = -f(y) \sin x + \phi(y) \cos x.$$

As $x = 0$ implies $z = e^y$ and also since we have $\frac{\partial z}{\partial x} = 1$ for $x = 0$, then,

$$e^y = z = f(y) \text{ and } 1 = \phi(y).$$

Hence

$$z = \sin x + e^y \cos x.$$

EXAMPLE 3.11

Solve

$$\frac{\partial^2 z}{\partial x^2} = a^2 z,$$

given that when $x = 0$, $\frac{\partial z}{\partial x} = a \sin y$ and $\frac{\partial z}{\partial y} = 0$.

Solution. If z were a function of x only, then the solution would have been $z = c_1 e^{ax} + c_2 e^{-ax}$. But z is a function of x and y . Therefore, c_1 and c_2 should be a function of y . Thus, the solution is

$$z = f_1(y)e^{ax} + f_2(y)e^{-ax}.$$

Therefore, using the condition that $\frac{\partial z}{\partial x} = a \sin y$ for $x = 0$, we have

$$a \sin y = a[f_1(y) - f_2(y)]$$

and so

$$f_1(y) = \sin y + f_2(y).$$

Thus

$$z = (\sin y + f_2(y))e^{ax} + f_2(y)e^{-ax}.$$

Now

$$\frac{\partial z}{\partial y} = [\cos y + f_2'(y)]e^{ax} + f_2'(y)e^{-ax}.$$

Now using the condition that $\frac{\partial z}{\partial y} = 0$ for $x = 0$, we have

$$0 = \cos y + f_2'(y) + f_2'(y)$$

and so

$$f_2'(y) = -\frac{\cos y}{2}.$$

Hence integrating, we get

$$f_2(y) = -\frac{1}{2} \sin y.$$

Thus

$$f_1(y) = \sin y - \frac{1}{2} \sin y = \frac{1}{2} \sin y$$

and so

$$z = \frac{1}{2} \sin y e^{ax} - \frac{1}{2} \sin y e^{-ax}$$

or

$$z = \sin y \sinh ax.$$

EXAMPLE 3.12

Solve

$$\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y,$$

given that $\frac{\partial z}{\partial y} = -2 \sin y$ when $x = 0$, and $z = 0$ when y is an odd multiple of $\pi/2$.

Solution. We are given that

$$\frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y.$$

Therefore, integrating with respect to x , keeping y constant, we have

$$\frac{\partial z}{\partial y} = -\cos x \sin y + f(y).$$

When $x = 0$, $\frac{\partial z}{\partial y} = -2 \sin y$. Therefore, $-2 \sin y = -\sin y + f(y)$ and so $f(y) = \sin y$. Thus

$$\frac{\partial z}{\partial y} = -\cos x \sin y - \sin y.$$

Integrating with respect to y , keeping x constant, we get

$$z = \cos x \cos y + \cos y + \phi(x).$$

Now when y is an odd multiple of $\pi/2$, $z = 0$. So $\phi(x) = 0$. Hence

$$z = \cos x \cos y + \cos y,$$

which is the required solution.

EXAMPLE 3.13

Solve

$$\frac{\partial^2 z}{\partial y^2} = z,$$

given that when $y = 0$, $z = e^x$ and $\frac{\partial z}{\partial y} = e^{-x}$.

Solution. The solution is

$$z = \phi_1(x)e^y + \phi_2(x)e^{-y}.$$

Now when $y = 0$, $z = e^x$. Therefore,

$$e^x = \phi_1(x) + \phi_2(x).$$

Further

$$\frac{\partial z}{\partial y} = \phi_1(x)e^y - \phi_2(x)e^{-y}.$$

When $y = 0$, $\frac{\partial z}{\partial y} = e^{-x}$ and so

$$e^{-x} = \phi_1(x) - \phi_2(x).$$

Adding the above expressions for e^x and e^{-x} , we get

$$2\phi_1(x) = e^x + e^{-x} \text{ or } \phi_1(x) = \frac{e^x + e^{-x}}{2} = \cosh x$$

Substituting this value of $\phi_1(x)$ in $e^x = \phi_1(x) + \phi_2(x)$, we get $\phi_2(x) = \sinh x$. Hence

$$z = \cosh x e^y + \sinh x e^{-y}.$$

(B) Lagrange's Method

The partial differential equation of the form $Pp + Qq = R$, where P , Q , and R are functions of x , y , z and $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, is known as *Lagrange's linear equation*. Regarding the solution of this equation, we have the following theorem.

Theorem 3.1. The general solution of a linear partial differential equation

$$Pp + Qq = R \quad (13)$$

is of the form

$$f(u, v) = 0, \quad (14)$$

where $f(u, v)$ is an arbitrary function of

$$\left. \begin{aligned} u(x, y, z) &= c_1 \\ \text{and} \\ v(x, y, z) &= c_2 \end{aligned} \right\} \quad (15)$$

which form a solution of

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \quad (16)$$

Proof: Since $u(x, y, z) = c_1$ is a solution of (16),

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0$$

and

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

must be compatible. Therefore,

$$Pu_x + Qu_y + Ru_z = 0. \quad (17)$$

Similarly

$$Pv_x + Qv_y + Rv_z = 0. \quad (18)$$

Solving (17) and (18) for P , Q , and R , we get

$$\frac{P}{u_y v_z - u_z v_y} = \frac{Q}{u_z v_x - u_x v_z} = \frac{R}{u_x v_y - v_x u_y}$$

or

$$\frac{P}{\frac{\partial(u,v)}{\partial(y,z)}} = \frac{Q}{\frac{\partial(u,v)}{\partial(x,z)}} = \frac{R}{\frac{\partial(u,v)}{\partial(x,y)}}. \quad (19)$$

Differentiating (14) w.r.t. x and y , we get

$$\frac{\partial f}{\partial u} (u_x + pu_z) + \frac{\partial f}{\partial v} (v_x + pv_z) = 0, \quad (20)$$

and

$$\frac{\partial f}{\partial u} (u_y + qu_z) + \frac{\partial f}{\partial v} (v_y + qv_z) = 0. \quad (21)$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (20) and (21), we get

$$\begin{vmatrix} u_x + pu_z & v_x + pv_z \\ u_y + qu_z & v_y + qv_z \end{vmatrix} = 0.$$

Simplifying we get

$$p \frac{\partial(u, v)}{\partial(y, z)} + q \frac{\partial(u, v)}{\partial(z, x)} = \frac{\partial(u, v)}{\partial(x, y)}. \quad (22)$$

Now equation (19) and (22) yield

$$Pp + Qq = R.$$

The equations (16) are called *auxiliary equations* or *subsidiary equations*.

EXAMPLE 3.14

Solve

$$(x^2 - yz)p + (y^2 - zx)q = z^2 - xy.$$

Solution. The given Lagranges' differential equation is

$$(x^2 - yz)p + (y^2 - zx)q = z^2 - xy.$$

The corresponding auxiliary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}. \quad (23)$$

Therefore,

$$\frac{dx - dy}{(x - y)(x + y + z)} = \frac{dy - dz}{(y - z)(x + y + z)}$$

or

$$\frac{dx - dy}{x - y} = \frac{dy - dz}{y - z}.$$

Integrating we get

$$\log(x - y) = \log(y - z) + \log a$$

or

$$\log\left(\frac{x - y}{y - z}\right) = \log a$$

or

$$\frac{x - y}{b - z} = a \quad (24)$$

Using x, y, z as multiplier, each fraction of (23) is equal to

$$\begin{aligned} & \frac{xdx + ydy + zdz}{x^3 + y^3 + z^3 - 3xyz} \\ &= \frac{xdx + ydy + zdz}{(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)}. \end{aligned} \quad (25)$$

Also each fraction of (23) is equal to

$$\frac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx}. \quad (26)$$

From (25) and (26), we have

$$\frac{xdx + ydy + zdz}{x + y + z} = dx + dy + dz$$

or

$$xdx + ydy + zdz = (x + y + z)d(x + y + z).$$

Integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{(x + y + z)^2}{2} + c$$

or

$$x^2 + y^2 + z^2 = (x + y + z)^2 + b$$

or

$$xy + yz + zx = b. \quad (27)$$

From (24) and (27), the general solution is

$$f\left(\frac{x - y}{y - z}, xy + yz + zx\right) = 0,$$

where f is an arbitrary function.

EXAMPLE 3.15

Solve

$$(mz - ny)p + (nx - lz)q = ly - mx.$$

Solution. The auxiliary equations are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}. \quad (28)$$

Using multipliers x, y , and z , each fraction of (28) is equal to

$$\frac{xdx + ydy + zdz}{0}.$$

Therefore, $xdx + ydy + zdz = 0$, which on integration gives

$$x^2 + y^2 + z^2 = a. \quad (29)$$

Now using multipliers l, m , and n , each fraction of (28) is equal to

$$\frac{ldx + mdx + ndz}{0}$$

and so $l dx + m dy + n dz = 0$, which on integration gives

$$lx + my + nz = b. \quad (30)$$

Hence, from (29) and (30), the required solution is

$$x^2 + y^2 + z^2 = f(lx + my + nz),$$

where f is an arbitrary function.

EXAMPLE 3.16

Solve

$$(x^2 - y^2 - z^2)p + 2xyz = 2xz.$$

Solution. The auxiliary equations are

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}. \quad (31)$$

The last two members yield

$$\frac{dy}{y} = \frac{dz}{z},$$

which on integration yields

$$\log y = \log z + \log a$$

or

$$\log \frac{y}{z} = \log a \text{ or } \frac{y}{z} = a. \quad (32)$$

Using x, y, z as multipliers, we get each fraction of (31) equal to

$$\frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)}.$$

Therefore,

$$\frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)} = \frac{dz}{2xz}$$

or

$$\frac{2xdx + 2ydy + 2zdz}{x^2 + y^2 + z^2} = \frac{dz}{z}.$$

Integrating this equation, we get

$$\log(x^2 + y^2 + z^2) = \log z + \log b$$

or

$$\frac{x^2 + y^2 + z^2}{z} = b. \quad (33)$$

Therefore, by (32) and (33), the general solution is

$$\frac{x^2 + y^2 + z^2}{z} = f\left(\frac{y}{z}\right)$$

or

$$x^2 + y^2 + z^2 = zf\left(\frac{y}{z}\right),$$

where f is an arbitrary function.

EXAMPLE 3.17

Solve

$$z(xp - yq) = y^2 - x^2.$$

Solution. The given equation is

$$xzp - yzq = y^2 - x^2.$$

The auxiliary equations are

$$\frac{dx}{xz} = \frac{dy}{-yz} = \frac{dz}{y^2 - x^2} = \frac{xdx + ydy + zdz}{0}. \quad (34)$$

The first and second member of (34) imply

$$\frac{dx}{x} = \frac{dy}{-y},$$

which on integration yields

$$\log x + \log y = \log c$$

or

$$xy = c, \text{ where } c \text{ is a constant of integration.}$$

(35)

The last member of (34) implies

$$xdx + ydy + zdz = 0,$$

which on integration yields

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = a$$

or

$$x^2 + y^2 + z^2 = 2a = b. \quad (36)$$

Therefore, the general solution of the given partial differential equation is

$$f(xy, x^2 + y^2 + z^2) = 0,$$

where f is an arbitrary function.

EXAMPLE 3.18

Solve

$$(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx.$$

Solution. The corresponding auxiliary equations are

$$\begin{aligned} \frac{dx}{z^2 - 2yz - y^2} &= \frac{dy}{xy + zx} = \frac{dz}{xy - zx} \\ &= \frac{xdx + ydy + zdz}{0} \end{aligned} \quad (37)$$

The second and third members of (37) gives

$$\frac{dy}{y + z} = \frac{dz}{y - z}$$

or

$$(y - z)dy = (y + z)dz$$

or

$$ydy - (zdy + ydz) - zdz = 0$$

or

$$ydy - d(yz) - zdz = 0.$$

Integrating we get

$$y^2 - 2yz - z^2 = a. \quad (38)$$

The last member of (37) gives

$$xdx + ydy + zdz = 0$$

Integrating we get

$$x^2 + y^2 + z^2 = b \quad (39)$$

Therefore (38) and (39) yield the general solution of the differential equation as

$$f(x^2 + y^2 + z^2, y^2 - 2yz - z^2) = 0.$$

EXAMPLE 3.19

Solve

$$p \tan x + q \tan y = \tan z.$$

Solution. The corresponding auxiliary equations are

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}. \quad (40)$$

Integrating the first two members of (40), we have

$$\log \sin x = \log \sin y = \log a$$

or

$$\frac{\sin x}{\sin y} = a. \quad (41)$$

Similarly, the last two members of (40) yield

$$\frac{\sin y}{\sin z} = b. \quad (42)$$

Therefore, (41) and (42) yield the general solution of the given equation as

$$f\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0,$$

where f is an arbitrary function.**EXAMPLE 3.20**

Solve

$$x(y-z)p + y(z-x)q = z(x-y).$$

Solution. The auxiliary equations for the given equation are

$$\begin{aligned} \frac{dx}{xy - xz} &= \frac{dy}{yz - yx} = \frac{dz}{zx - zy} \\ &= \frac{dx + dy + dz}{0}. \end{aligned} \quad (43)$$

The last member of (43) yields

$$dx + dy + dz = 0,$$

which on integration gives

$$x + y + z = a. \quad (44)$$

Also, using $\frac{1}{x}$, $\frac{1}{y}$, and $\frac{1}{z}$ as the multipliers, we get

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0,$$

which on integration yields

$$\log x + \log y + \log z = \log b$$

or

$$xyz = b, \text{ where } b \text{ is a constant of integration.} \quad (45)$$

From (44) and (45), the general solution of the given partial differential equation is

$$f(x + y + z, xyz) = 0,$$

where f is an arbitrary function.**EXAMPLE 3.21**

Solve

$$\left(\frac{1}{z} - \frac{1}{y}\right)p + \left(\frac{1}{x} - \frac{1}{z}\right)q = \frac{1}{y} - \frac{1}{x}.$$

Solution. The corresponding auxiliary equations are

$$\frac{dx}{\frac{1}{z} - \frac{1}{y}} = \frac{dy}{\frac{1}{x} - \frac{1}{z}} = \frac{dz}{\frac{1}{y} - \frac{1}{x}}. \quad (46)$$

We note that each member of (46) is equal to

$$\frac{dx + dy + dz}{0}.$$

Thus

$$dx + dy + dz = 0,$$

which on integration gives

$$x + y + z = a. \quad (47)$$

Further using $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$ as the multipliers, each member of (46) is equal to

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

and so

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0.$$

Integrating, we get

$$\log x + \log y + \log z = \log b$$

or

$$xyz = b, \text{ where } b \text{ is a constant of integration.}$$

(48)

Combining (47) and (48), the general solution of the equation is

$$f(x + y + z, xyz) = 0,$$

where f is an arbitrary function.**EXAMPLE 3.22**

Solve

$$p\sqrt{x} + q\sqrt{y} = \sqrt{z}.$$

Solution. The auxiliary equations are

$$\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}.$$

On integration, the first two members of auxiliary equation give

$$\sqrt{x} - \sqrt{y} = c_1.$$

Similarly, on integration, the last two members give

$$\sqrt{y} - \sqrt{z} = c_2.$$

Therefore, the required solution is

$$f(\sqrt{x} - \sqrt{y}, \sqrt{y} - \sqrt{z}) = 0,$$

where f is an arbitrary function.

EXAMPLE 3.23

Solve $xp + yq = 3z$.

Solution. The auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{3z}$$

The first two members, on integration, yield

$$\log x = \log y + \log c \text{ or } \frac{x}{y} = c.$$

The first and third members on integration yield

$$\log x = \frac{1}{3} \log z + \log b$$

or

$$\log x^3 = \log z + \log b^3$$

or

$$\frac{x^3}{z} = a.$$

Hence, the required general solution is

$$f\left(\frac{x}{y}, \frac{x^3}{z}\right) = 0,$$

where f is an arbitrary function.

EXAMPLE 3.24

Solve

$$(z - y)p + (x - z)q = y - x.$$

Solution. The corresponding auxiliary equations are

$$\frac{dx}{z - y} = \frac{dy}{x - z} = \frac{dz}{y - x} = \frac{dx + dy + dz}{0}. \quad (49)$$

The last member of (49) yields

$$dx + dy + dz = 0,$$

which on integration, gives

$$x + y + z = a. \quad (50)$$

Further using x, y, z as the multipliers, each fraction of (49) is equal to

$$\frac{xdx + ydy + zdz}{0}.$$

Thus,

$$xdx + ydy + zdz = 0,$$

which on integration gives

$$x^2 + y^2 + z^2 = b. \quad (51)$$

From (50) and (51), the general solution of the given equation is

$$x^2 + y^2 + z^2 = f(x + y + z),$$

where f is an arbitrary function.

EXAMPLE 3.25

Solve

$$x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2).$$

Solution. The corresponding auxiliary equations are

$$\begin{aligned} \frac{dx}{x(y^2 - z^2)} &= \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)} \\ &= \frac{xdx + ydy + zdz}{0} \end{aligned} \quad (52)$$

The last member of (52) implies

$$xdx + ydy + zdz = 0,$$

which on integration gives

$$x^2 + y^2 + z^2 = a. \quad (53)$$

On the other hand, using $\frac{1}{x}, \frac{1}{y}$, and $\frac{1}{z}$ as the multipliers, each member of (52) is equal to

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0},$$

and so

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0.$$

Integrating, we have

$$\log x + \log y + \log z = \log b$$

or

$$xyz = b. \quad (54)$$

Hence (53) and (54) provide the general solution as

$$f(x^2 + y^2 + z^2, xyz) = 0,$$

where f is an arbitrary function.

3.3 NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

A partial differential equation, which involves first order partial derivatives p and q with degree higher than one and the products of p and q , is called as a *non-linear partial differential equation of the first order*. A general method for solving non-linear partial differential equation is due to Charpit and is known, after his name, as Charpit's method.

3.4 CHARPIT'S METHOD

Consider the partial differential equation

$$f(x, y, z, p, q) = 0, \quad (55)$$

which may be non-linear in p and q . The fundamental idea in this method is to find another *first order* partial differential equation.

$$g(x, y, z, p, q, a) = 0, \quad (56)$$

where a is constant, so that

(i) equations (55) and (56) can be solved for p and q to give

$$p = p(x, y, z, a) \text{ and } q = q(x, y, z, a)$$

(ii) the equation

$$dz = p dx + q dy \quad (57)$$

is integrable.

The main step is to find the function g . After this, we can solve it for p and q along with (55). Substituting the values of p and q in (57) and then integrating, we get the required solution. Since this integral will contain two arbitrary constants a and b , it will be a complete integral of the partial differential equation (55). The condition (i) will be satisfied if

$$J = \frac{\partial(f, g)}{\partial(p, q)} = \begin{vmatrix} \frac{\partial f}{\partial p} & \frac{\partial g}{\partial p} \\ \frac{\partial f}{\partial q} & \frac{\partial g}{\partial q} \end{vmatrix} \neq 0 \quad (58)$$

Assuming that (58) holds, the equation (57) exists and the condition (ii) will be satisfied if

$$p \left(\frac{\partial q}{\partial z} \right) + q \left(-\frac{\partial p}{\partial z} \right) - \left(\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x} \right) = 0$$

or

$$p \frac{\partial q}{\partial z} + \frac{\partial q}{\partial x} = q \frac{\partial p}{\partial z} + \frac{\partial p}{\partial y}. \quad (59)$$

Putting the values of p and q as functions of x, y , and z in (55) and (56) and differentiating with respect

to x , we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial f}{\partial q} \frac{\partial q}{\partial x} = 0$$

and

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial g}{\partial q} \frac{\partial q}{\partial x} = 0.$$

Therefore,

$$\left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial q} - \frac{\partial f}{\partial q} \frac{\partial g}{\partial x} \right) \frac{\partial q}{\partial x} = \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial x}$$

or

$$\left. \begin{aligned} \frac{\partial q}{\partial x} &= \frac{1}{J} \left\{ \frac{\partial f}{\partial x} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} \right\} \\ \text{Similarly,} \\ \frac{\partial p}{\partial y} &= \frac{1}{J} \left\{ -\frac{\partial f}{\partial y} \frac{\partial g}{\partial q} + \frac{\partial f}{\partial q} \frac{\partial g}{\partial y} \right\} \\ \frac{\partial p}{\partial z} &= \frac{1}{J} \left\{ -\frac{\partial f}{\partial z} \frac{\partial g}{\partial q} + \frac{\partial f}{\partial q} \frac{\partial g}{\partial z} \right\} \\ \frac{\partial q}{\partial z} &= \frac{1}{J} \left\{ -\frac{\partial f}{\partial z} \frac{\partial g}{\partial p} + \frac{\partial f}{\partial p} \frac{\partial g}{\partial z} \right\} \end{aligned} \right\} \quad (60)$$

Substituting from (60) into the equation (59), we have

$$\begin{aligned} & \frac{1}{J} \left[p \left(\frac{\partial f}{\partial z} \frac{\partial g}{\partial q} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial z} \right) + \left(\frac{\partial f}{\partial x} \frac{\partial f}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} \right) \right] \\ &= \frac{1}{J} \left[q \left(-\frac{\partial f}{\partial z} \frac{\partial g}{\partial q} + \frac{\partial f}{\partial q} \frac{\partial g}{\partial z} \right) + \left(-\frac{\partial f}{\partial y} \frac{\partial g}{\partial q} - \frac{\partial f}{\partial q} \frac{\partial g}{\partial y} \right) \right] \end{aligned}$$

or

$$\begin{aligned} & \left(-\frac{\partial f}{\partial p} \right) \frac{\partial g}{\partial x} + \left(-\frac{\partial f}{\partial q} \right) \frac{\partial g}{\partial y} + \left(-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q} \right) \frac{\partial g}{\partial z} \\ &+ \left(p \frac{\partial f}{\partial z} + \frac{\partial f}{\partial x} \right) + \left(q \frac{\partial f}{\partial z} + \frac{\partial f}{\partial y} \right) \frac{\partial f}{\partial q} = 0. \quad (61) \end{aligned}$$

The equation (61) is linear in the variables x, y, z, p, q and has the auxiliary equations

$$\begin{aligned} \frac{dx}{-\frac{\partial f}{\partial p}} &= \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}} \\ &= \frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}} \quad (62) \end{aligned}$$

Any of the solutions of equations (62) satisfies equation (61). If such a solution involves p or q it can be taken for (56). Thus, once equation (56) has been found, the problem reduces to solve (55) and (56) for p and q and then to integrate the equation (57).

EXAMPLE 3.26

Solve by Charpit's method

$$px + qy = pq.$$

Solution. We have

$$f = px + qy - pq = 0. \quad (63)$$

Since

$$\frac{\partial f}{\partial x} = p, \quad \frac{\partial f}{\partial y} = q, \quad \frac{\partial f}{\partial p} = x - q, \quad \frac{\partial f}{\partial q} = y - p,$$

the Charpit's auxiliary equations are

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x}} = \frac{dq}{\frac{\partial f}{\partial y}}$$

or

$$\frac{dx}{q - x} = \frac{dy}{p - y} = \frac{dp}{p} = \frac{dq}{q}.$$

Taking the last pair, we have

$$\frac{dp}{p} = \frac{dq}{q}.$$

Integrating this equation, we get

$$\log p = \log q + \log a$$

or

$$\frac{p}{q} = a.$$

Substituting $p = aq$ is (63), we get

$$aqx + qy - aq^2 = 0$$

or

$$ax + y - aq = 0$$

or

$$q = \frac{ax + y}{a}.$$

Therefore,

$$p = ax + y.$$

Now

$$\begin{aligned} dz &= p dx + q dy \\ &= (ax + y) dx + \frac{ax + y}{a} dy \end{aligned}$$

or

$$\begin{aligned} adz &= a(ax + y) dx + (ax + y) dy \\ &= (ax + y)(adx + dy). \end{aligned}$$

Integrating, we get

$$az = \frac{(ax + y)^2}{2} + c,$$

which is the required solution of the equation.

EXAMPLE 3.27

Solve by Charpit's method

$$z^2 = pqxy.$$

Solution. We have

$$z^2 - pqxy = 0. \quad (65)$$

Therefore,

$$\frac{\partial f}{\partial x} = -pqy, \quad \frac{\partial f}{\partial y} = -pqx,$$

$$\frac{\partial f}{\partial z} = 2z, \quad \frac{\partial f}{\partial p} = -qxy, \quad \frac{\partial f}{\partial q} = -pxy$$

and, thus, the Charpit's auxiliary equations are

$$\begin{aligned} \frac{dx}{qxy} &= \frac{dy}{pxy} = \frac{dz}{pqxy + pqxy} = \frac{dp}{-pqy + 2z} \\ &= \frac{dq}{-pqx + 2zq} \end{aligned}$$

or

$$\begin{aligned} \frac{dx}{qxy} &= \frac{dy}{pxy} = \frac{dz}{2pqxy} = \frac{dp}{p(2z - qy)} \\ &= \frac{dq}{q(2z - px)}. \end{aligned} \quad (66)$$

(64) From (66), we get

$$\frac{\frac{dp}{p} - \frac{dq}{q}}{px - qy} = \frac{\frac{dy}{y} - \frac{dx}{x}}{px - qy}$$

and so

$$\frac{dp}{p} + \frac{dx}{x} = \frac{dy}{y} + \frac{dq}{q}.$$

On Integrating we get

$$\log p + \log x = \log y + \log q + \log a^2$$

or

$$px = a^2 qy. \quad (67)$$

Substituting this value of px in (65), we get

$$z^2 = a^2 q^2 y^2 \text{ or } q = \frac{z}{ay}$$

and then (67) implies $p = \frac{az}{x}$. Therefore, $dz = p dx + q dy$ becomes

$$dz = \frac{az}{x} dx + \frac{z}{ay} dy$$

or

$$\frac{dz}{z} = a \frac{dx}{x} + \frac{1}{a} \frac{dy}{y}.$$

On integrating we get

$$\begin{aligned}\log z &= a \log x + \frac{1}{a} \log y + \log b \\ &= \log x^a + \log y^{1/a} + \log b = \log bx^a y^{1/a}.\end{aligned}$$

or

$$z = bx^a y^{1/a}.$$

EXAMPLE 3.28

Solve by Charpit's method

$$(p^2 + q^2)y = qz.$$

Solution. The given equation is

$$f \equiv (p^2 + q^2)y - qz = 0.$$

We have

$$\begin{aligned}\frac{\partial f}{\partial x} &= 0, \quad \frac{\partial f}{\partial y} = p^2 + q^2, \quad \frac{\partial f}{\partial z} = -q, \\ \frac{\partial f}{\partial p} &= 2py, \quad \frac{\partial f}{\partial q} = 2qy - z.\end{aligned}$$

Therefore, Charpit's auxiliary equations are

$$\frac{dx}{-2py} = \frac{dy}{z - 2qy} = \frac{dz}{-qz} = \frac{dp}{-pq} = \frac{dq}{p^2}.$$

From last two members, we have

$$p^2 dp = -pq \, dq$$

or

$$pdp = -q dq$$

and so integration implies

$$p^2 + q^2 = a^2. \quad (68)$$

Substituting this value of $p^2 + q^2$ in the given equation, we have

$$a^2 y = qz$$

and so

$$q = \frac{a^2 y}{z},$$

which with the help of (68) implies

$$p = \sqrt{a^2 - q^2} = \sqrt{a^2 - \frac{a^4 y^2}{z^2}} = \frac{a}{z} \sqrt{z^2 - a^2 y^2}.$$

Therefore,

$$\begin{aligned}dz &= p dx + q dy \\ &= \frac{a}{z} \sqrt{z^2 - a^2 y^2} dx + \frac{a^2 y}{z} dy.\end{aligned}$$

or

$$z dz = a \sqrt{z^2 - a^2 y^2} dx + a^2 y dy$$

or

$$\frac{z dz - a^2 y dy}{\sqrt{z^2 - a^2 y^2}} = a dx$$

or

$$\frac{\frac{1}{2} d(z^2 - a^2 y^2)}{\sqrt{z^2 - a^2 y^2}} = a dx.$$

Integrating, we get

$$\sqrt{z^2 - a^2 y^2} = ax + b$$

or

$$z^2 = (ax + b)^2 + a^2 y^2.$$

EXAMPLE 3.29

Solve by Charpit's method

$$(p^2 + q^2)x = pz$$

Solution. The given equation is

$$f \equiv (p^2 + q^2)x - pz = 0.$$

We have

$$\begin{aligned}\frac{\partial f}{\partial x} &= p^2 + q^2, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial z} = -p, \\ \frac{\partial f}{\partial p} &= 2px - z, \quad \frac{\partial f}{\partial q} = 2qx.\end{aligned}$$

Therefore, Charpit's auxiliary equations are

$$\begin{aligned}\frac{dx}{z - 2px} &= \frac{dy}{-2qx} = \frac{dz}{p(z - 2px) - 2q^2 x} \\ &= \frac{dp}{p^2 + q^2 - p^2} = \frac{dq}{-pq}\end{aligned}$$

or

$$\frac{dx}{z - 2px} = \frac{dy}{-2qx} = \frac{dz}{-pz} = \frac{dp}{q^2} = \frac{d^2}{-pq}.$$

The last two members give

$$-pq \, dp = q^2 dq$$

or

$$-pdp = qdq,$$

which on integration gives

$$p^2 + q^2 = a^2. \quad (69)$$

Substituting this in the given equation, we get

$$a^2x = pz \text{ or } p = \frac{a^2x}{z}.$$

Substituting this value of p in (69), we have

$$q = \sqrt{a^2 - p^2} = \sqrt{a^2 - \frac{a^4x^2}{z^2}} = \frac{a}{z} \sqrt{z^2 - a^2x^2}.$$

Therefore,

$$dz = p dx + q dy = \frac{a^2x}{z} dx + \frac{a}{z} \sqrt{z^2 - a^2x^2} dy$$

or

$$z dz = a^2x dx + a \sqrt{z^2 - a^2x^2} dy$$

or

$$\frac{z dz - a^2x dx}{\sqrt{z^2 - a^2x^2}} = a dy$$

or

$$\frac{\frac{1}{2} d(z^2 - a^2x^2)}{\sqrt{z^2 - a^2x^2}} = a dy.$$

On integrating, we get

$$\sqrt{z^2 - a^2x^2} = ay + b$$

or

$$z^2 = a^2x^2 + (ay + b)^2.$$

EXAMPLE 3.30

Solve by Charpit's method

$$p = (z + qy)^2.$$

Solution. Given

$$f \equiv p - (z + qy)^2 = 0$$

or

$$p - z^2 - q^2y^2 - 2yzq = 0.$$

Thus

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = -2q^2y - 2zq, \quad \frac{\partial f}{\partial z} = -2z - 2yq$$

$$\frac{\partial f}{\partial p} = 1, \quad \frac{\partial f}{\partial q} = -2qy^2 - 2yz.$$

Therefore, Charpit's auxiliary equations are

$$\begin{aligned} \frac{dx}{-1} &= \frac{dy}{2y(z + qy)} = \frac{dz}{-p + 2qy(z + qy)} \\ &= \frac{dp}{-2p(z + qy)} = \frac{dq}{-4q(z + qy)}. \end{aligned}$$

From second and fifth members, we get

$$\frac{dy}{y} = \frac{dq}{-2q} \text{ and so } q = \frac{a}{y^2}.$$

Putting this value of q in the given equation, we get

$$p = \left(z + \frac{a}{y}\right)^2.$$

Then $dz = p dx + q dy$ becomes

$$dz = \left(z + \frac{a}{y}\right)^2 dx + \frac{a}{y^2} dy$$

or

$$dx = \frac{d\left(z + \frac{1}{y}\right)}{\left(z + \frac{a}{y}\right)^2}.$$

Hence,

$$x + \frac{1}{z + \frac{a}{y}} = b$$

or

$$x + \frac{y}{a + yz} = b,$$

is the required solution.

EXAMPLE 3.31

Solve by Charpit's method $z = p^2x + q^2y$.

Solution. The given equation is

$$f \equiv z - p^2x - q^2y = 0.$$

Thus

$$\frac{\partial f}{\partial x} = -p^2, \quad \frac{\partial f}{\partial y} = -q^2, \quad \frac{\partial f}{\partial z} = 1$$

$$\frac{\partial f}{\partial p} = -2px, \quad \frac{\partial f}{\partial q} = -2qy.$$

Therefore, Charpit's auxiliary equations are

$$\frac{dx}{2px} = \frac{dy}{2qy} = \frac{dz}{2p^2x + 2q^2y} = \frac{dp}{-p^2 + p} = \frac{dq}{-q^2 + q}$$

or

$$\begin{aligned} \frac{dx}{2px} &= \frac{dy}{2qy} = \frac{dz}{2(p^2x + q^2y)} = \frac{dp}{p(1 - p)} \\ &= \frac{dq}{q(1 - q)}. \end{aligned}$$

It follows that

$$\frac{p^2dx + 2pxdp}{p^2x} = \frac{q^2dy + 2qy dq}{q^2y}$$

or

$$\frac{dx}{x} + \frac{2dp}{p} = \frac{dy}{y} + \frac{2dq}{q},$$

which yields

$$\log x + 2 \log p = \log y + 2 \log q + \log a$$

or

$$xp^2 = aq^2y, \text{ where } a \text{ is constant.} \quad (70)$$

Putting the value of xp^2 in the given equation, we get

$$z = aq^2y + q^2y = q^2(y + ay)$$

or

$$q = \sqrt{\frac{z}{y + ay}}$$

and then (70) yields $p = \sqrt{\frac{az}{x+ax}}$. Then $dz = p dx + q dy$ becomes

$$dz = \sqrt{\frac{az}{x+ax}} dx + \sqrt{\frac{z}{y+ay}} dy$$

or

$$\sqrt{\frac{1+a}{z}} dz = \sqrt{\frac{a}{x}} dx + \sqrt{\frac{1}{y}} dy.$$

On integrating, we get

$$\sqrt{(1+a)z} = \sqrt{ax} + \sqrt{y} + b.$$

EXAMPLE 3.32

Solve by Charpit's method

$$q + xp = p^2.$$

Solution. We are given that

$$f \equiv q + xp - p^2 = 0.$$

Thus,

$$\begin{aligned} \frac{\partial f}{\partial x} &= p, & \frac{\partial f}{\partial y} &= 0, & \frac{\partial f}{\partial z} &= 0, \\ \frac{\partial f}{\partial p} &= x - 2p, & \frac{\partial f}{\partial q} &= 1. \end{aligned}$$

Therefore, Charpit's auxiliary equations are

$$\begin{aligned} \frac{dx}{-x+2p} &= \frac{dy}{-1} \\ &= \frac{dz}{2p^2 - px - q} = \frac{dp}{p} = \frac{dq}{0}. \end{aligned} \quad (71)$$

From second and fourth members of (71), we have

$$-\frac{dy}{1} = \frac{dp}{p},$$

which yields

$$-y = \log p + \log c = \log pc$$

and so $p = ae^{-y}$. Putting this value of p in the given equation, we get $q = a^2 e^{-2y} - axe^{-y}$. Then $dz = p dx + q dy$ becomes

$$dz = ae^{-y} dx + (a^2 e^{-2y} - axe^{-y}) dy.$$

On integrating, we get

$$\begin{aligned} z &= axe^{-y} + \frac{a^2 e^{-2y}}{-2} - \frac{axe^{-y}}{-1} + b \\ &= 2axe^{-y} - \frac{a^2}{2} e^{-2y} + b. \end{aligned}$$

EXAMPLE 3.33

Using Charpit's method, find the complete integral of the equation $xp + 3yq = 2(z - x^2 q^2)$.

Solution. We have

$$f \equiv xp + 3yq - 2z + 2x^2 q^2 = 0.$$

Therefore

$$\frac{\partial f}{\partial x} = p + 4xq^2, \quad \frac{\partial f}{\partial y} = 3q, \quad \frac{\partial f}{\partial z} = -2,$$

$$\frac{\partial f}{\partial p} = x, \quad \frac{\partial f}{\partial q} = 3y + 4x^2 q.$$

Thus, the Charpit's auxiliary equations are

$$\begin{aligned} \frac{dx}{-x} &= \frac{dy}{-3y - 4x^2 q} = \frac{dz}{-px - q(3y + 4x^2 q)} \\ &= \frac{dp}{p + 4xq^2 - 2p} = \frac{dq}{3q - 2q} \end{aligned}$$

or

$$\begin{aligned} \frac{dx}{-x} &= \frac{dy}{-3y - 4x^2 q} = \frac{dz}{-px - q(3y + 4x^2 q)} \\ &= \frac{dp}{4xq^2 - q} = \frac{dq}{q}. \end{aligned}$$

Thus

$$\frac{dq}{q} = \frac{dx}{-x},$$

which yields $qx = a$ (constant). Putting this value of q in the given equation, we have

$$p = \frac{2(z - x^2 q^2)}{x} - \frac{3yq}{x} = \frac{2(z - a^2)}{x} - \frac{3ya}{x^2}.$$

Therefore, $dz = p dx + q dy$ gives

$$dz = \left[\frac{2(z - a^2)}{x} - \frac{3ya}{x^2} \right] dx + \frac{a}{x} dy$$

or

$$\begin{aligned} x^2 dz &= [2x(z - a^2) - 3ya] dx + ax dy \\ &= 2x(z - a^2) dy - 3ya dx + ax dy \end{aligned}$$

or

$$x^2 dz - 2x(z - a^2) dx = -3ya dx + ax dy$$

or

$$x^4 d\left(\frac{z - a^2}{x^2}\right) - 3ya dx + ax dy$$

or

$$d\left(\frac{z - a^2}{x^2}\right) - \frac{3ay}{x^4} dx + \frac{a}{x^3} dy = d\left(\frac{ay}{x^3}\right).$$

On integrating, we have the required solution as

$$\frac{z - a^2}{x^2} = \frac{ay}{x^3} + b$$

or

$$z = a\left(a + \frac{y}{x}\right) + bx^2,$$

where a and b are arbitrary constants.

EXAMPLE 3.34

Solve by Charpit's method $pxy + pq + qy = yz$.

Solution. The given equation is

$$f \equiv pxy + pq + qy - yz = 0.$$

Then

$$\begin{aligned} \frac{\partial f}{\partial x} &= py, \quad \frac{\partial f}{\partial y} = px + q - z, \quad \frac{\partial f}{\partial z} = -y \\ \frac{\partial f}{\partial p} &= xy + q, \quad \frac{\partial f}{\partial q} = p + y. \end{aligned}$$

Therefore, Charpit's auxiliary equations are

$$\begin{aligned} \frac{dx}{-xy - q} &= \frac{dy}{-y - p} = \frac{dz}{p((xy + q) - (p + y)q)} \\ &= \frac{dp}{py - pq} = \frac{dq}{(px + q - z) - yq} \end{aligned}$$

From the fourth member of the auxiliary equations, we have

$$dp = 0 \text{ or } p = a \text{ (constant).}$$

Putting this value in the given equation, we have

$$axy + aq + qy = yz,$$

or

$$q(a + y) = yz - axy$$

or

$$q = \frac{yz - axy}{a + y}.$$

Then $dz = p dx + q dy$ becomes

$$dz = adx + \frac{yz - axy}{a + y} dy = adx + \frac{y(z - ax)}{a + y} dy$$

or

$$dz - adx = \frac{y(z - ax)}{a + y} dy$$

or

$$\begin{aligned} \frac{dz - adx}{z - ax} &= \frac{y}{a + y} dy = \frac{y - a + a}{a + y} dy \\ &= dy - \frac{a}{a + y} dy. \end{aligned}$$

On integrating, we get

$$\log(z - ax) = y - a \log(a + y) + b.$$

3.5 SOME STANDARD FORMS OF NON-LINEAR EQUATIONS

We have observed that the general method of finding solution is usually much longer. However, there are a few standard forms of the partial differential equation $f(x, y, z, p, q) = 0$ which can be solved by very short methods. We discuss these standard forms one-by-one.

(A) Equations of the Form $f(p, q) = 0$

Equation of this standard form involves only p and q and not x , y , and z explicitly. The Charpit's auxiliary equations for such equations are

$$\frac{dp}{0} = \frac{dq}{0}.$$

Therefore, $dp = 0$ and so $p = a$ (constant). Putting this value of p in $f(p, q) = 0$ yields $f(a, q) = 0$, which gives $q = b$. Now $f(p, q) = 0$ implies $f(a, b) = 0$ and so $b = \phi(a)$. Thus, $dz = p dx + q dy$ reduces to

$$dz = adx + \phi(a)dy.$$

Integrating, we get

$$z = ax + \phi(a)y = c,$$

where a , b , and c are constants and a and b are connected by the relation $f(a, b) = 0$.

EXAMPLE 3.35

Solve

(i) $p^2 + q^2 = 1$

(ii) $p + q = pq$.

Solution. (i) The equation is of the form $f(p, q) = 0$. Therefore, its solution is

$$z = ax + by + c,$$

where $a^2 + b^2 = 1$. Thus $b^2 = 1 - a^2$ or $b = \sqrt{1 - a^2}$. Hence the solution is

$$z = ax + \sqrt{1 - a^2}y + c.$$

(ii) The given equation is of the form $f(p, q) = 0$ and so its solution is $z = ax + by + c$, where $a + b = ab$, that is, $b(a - 1) = a$ or $b = \frac{a}{a - 1}$.

Thus

$$z = ax + \frac{a}{a - 1}y + c.$$

EXAMPLE 3.36Solve $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1$.

Solution. First we reduce the given equation to a standard form $f(p, q) = 0$. Put

$$x + y = U^2, (x - y) = V^2. \quad (72)$$

Then

$$\begin{aligned} p &= \frac{\partial z}{\partial x} = \frac{\partial z}{\partial U} \frac{\partial U}{\partial x} + \frac{\partial z}{\partial V} \frac{\partial V}{\partial x} \\ &= \frac{\partial z}{\partial U} \left(\frac{1}{2U} \right) + \frac{\partial z}{\partial V} \left(\frac{1}{2V} \right), \\ q &= \frac{\partial z}{\partial y} = \frac{\partial z}{\partial U} \frac{\partial U}{\partial y} + \frac{\partial z}{\partial V} \frac{\partial V}{\partial y} \\ &= \frac{\partial z}{\partial U} \left(\frac{1}{2U} \right) + \frac{\partial z}{\partial V} \left(-\frac{1}{2V} \right). \end{aligned}$$

Therefore,

$$\left. \begin{aligned} p + q &= \frac{1}{U} \frac{\partial z}{\partial U}, \\ p - q &= \frac{1}{V} \frac{\partial z}{\partial V} \end{aligned} \right\} \quad (73)$$

Putting the values of $x + y$, $x - y$, $p + q$ and $p - q$ from (72) and (73) in the given equation, we get

$$\left(\frac{\partial z}{\partial U} \right)^2 + \left(\frac{\partial z}{\partial V} \right)^2 = 1.$$

Therefore, we get equation of the form $f(p, q) = 0$. Hence the solution is

$$z = aU + bV + c,$$

where $a^2 + b^2 = 1$. Thus $b = \sqrt{1 - a^2}$. Hence the solution is

$$\begin{aligned} z &= aU + \sqrt{1 - a^2}V + c \\ &= a\sqrt{x + y} + \sqrt{1 - a^2}\sqrt{x - y} + c. \end{aligned}$$

EXAMPLE 3.37Solve $x^2 p^2 + y^2 q^2 = z^2$.

Solution. The given equation can be written as

$$\left(\frac{px}{z} \right)^2 + \left(\frac{qy}{z} \right)^2 = 1.$$

We convert it to the form $f(p, q) = 0$. To do so, put

$$\frac{dx}{x} = dX, \frac{dy}{y} = dY, \frac{dz}{z} = dZ,$$

that is,

$$X = \log x, Y = \log y, \text{ and } Z = \log z.$$

Then

$$\begin{aligned} \frac{\partial Z}{\partial X} &= \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial X} = \frac{1}{z} \cdot \frac{x}{1} \cdot \frac{\partial z}{\partial x} = \frac{x}{z} \frac{\partial z}{\partial x} = \frac{px}{z}, \\ \frac{\partial Z}{\partial Y} &= \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial Y} = \frac{1}{z} \cdot \frac{y}{1} \cdot \frac{\partial z}{\partial y} = \frac{y}{z} \frac{\partial z}{\partial y} = \frac{qy}{z}. \end{aligned}$$

Hence the given equation reduces to

$$\left(\frac{\partial Z}{\partial X} \right)^2 + \left(\frac{\partial Z}{\partial Y} \right)^2 = 1.$$

Therefore, its solution is

$$Z = aX + bY + c,$$

where $a^2 + b^2 = 1$ or $b = \sqrt{1 - a^2}$. Thus

$$Z = aX + \sqrt{1 - a^2}Y + c.$$

Returning back to x, y, z , we get

$$\log z = a \log x + \sqrt{1 - a^2} \log y + c.$$

EXAMPLE 3.38Solve $(x - y)(px - qy) = (p - q)^2$.

Solution. We convert first the given equation to the form $f(p, q) = 0$. To this end, let us put

$$x + y = U, xy = V.$$

Then

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial U} \frac{\partial U}{\partial x} + \frac{\partial z}{\partial V} \frac{\partial V}{\partial x} = \frac{\partial z}{\partial U} + y \frac{\partial z}{\partial V}$$

and

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial U} \cdot \frac{\partial U}{\partial y} + \frac{\partial z}{\partial V} \cdot \frac{\partial V}{\partial y} = \frac{\partial z}{\partial U} + x \frac{\partial z}{\partial V}.$$

Then

$$\begin{aligned} px - qy &= \left[x \frac{\partial z}{\partial U} + xy \frac{\partial z}{\partial V} \right] - \left[y \frac{\partial z}{\partial U} + xy \frac{\partial z}{\partial V} \right] \\ &= (x - y) \frac{\partial z}{\partial U} \end{aligned}$$

and

$$p - q = (y - x) \frac{\partial z}{\partial V}$$

or

$$(p - q)^2 = (x - y)^2 \left(\frac{\partial z}{\partial V} \right)^2.$$

Thus the given equation reduces to

$$\frac{\partial z}{\partial U} = \left(\frac{\partial z}{\partial V} \right)^2.$$

Hence the solution is

$$\begin{aligned} z &= aU + bV + c, \text{ where } a = b^2 \\ &= aU + \sqrt{a}V + c \\ &= a(x + y) + \sqrt{a}xy + c. \end{aligned}$$

(B) Equation of the form $f(z, p, q) = 0$

These types of equations do not contain x and y . In this case, the Charpit's auxiliary equations are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}.$$

Since $f_x = f_y = 0$, we have

$$\frac{dp}{p} = \frac{dq}{q},$$

and so $p = aq$ (or $q = ap$). Putting this value in the given equation, we find p and q . Putting these values of p and q in $dz = p dx + q dy$ and integrating, the required solution is obtained.

EXAMPLE 3.39

Solve $p(1 + q) = qz$.

Solution. The given equation is of the form $f(z, p, q) = 0$. Therefore, Charpit's auxiliary equations give

$p = aq$. Putting this value of p in the given equation, we get

$$aq(1 + q) = qz \text{ or } a(1 + q) = z$$

and so $q = \frac{z-a}{a}$. Hence $p = z - a$. Therefore, $dz = p dx + q dy$ becomes

$$dz = (z - a)dx + \left(\frac{z - a}{a} \right) dy$$

or

$$adz = a(z - a)dx + (z - a)dy$$

or

$$\frac{a dz}{z - a} = adx + dy.$$

On integrating, we get

$$a \log(z - a) = ax + y + \log b$$

or

$$\log(z - a)^a = b e^{ax+y}.$$

EXAMPLE 3.40

Solve $z = p^2 + q^2$.

Solution. The equation is of the form $f(z, p, q) = 0$. So taking $p = aq$, we have $z = a^2 q^2 + q^2 = q^2(a^2 + 1)$ and so $q = \sqrt{\frac{z}{a^2+1}}$. Then $p = \frac{a\sqrt{z}}{\sqrt{a^2+1}}$. Therefore,

$$dz = p dx + q dy = \frac{a\sqrt{z}}{\sqrt{a^2+1}} dx + \frac{\sqrt{z}}{\sqrt{a^2+1}} dy$$

or

$$\sqrt{a^2+1} \frac{dz}{\sqrt{z}} = adx + dy$$

or

$$2\sqrt{a^2+1}\sqrt{z} = ax + y + b$$

or

$$4(a^2+1)z = (ax + y + b)^2.$$

EXAMPLE 3.41

Solve the equation $q^2 = z^2 p^2(1 - p^2)$.

Solution. The given equation is of the type $f(z, p, q) = 0$. So, by Charpit's auxiliary equations, we have $q = ap$. Putting this value in the given equation, we have

$$a^2 p^2 = z^2 p^2(1 - p^2)$$

or

$$p^2 = \frac{z^2 - a^2}{z^2}$$

or

$$p = \frac{\sqrt{z^2 - a^2}}{z}$$

and so $q = \frac{a\sqrt{z^2 - a^2}}{z}$. Thus

$$dz = p dx + q dy = \frac{\sqrt{z^2 - a^2}}{z} dx + \frac{a\sqrt{z^2 - a^2}}{z} dy$$

or

$$\frac{z dz}{\sqrt{z^2 - a^2}} = dx + a dy$$

or

$$d(\sqrt{z^2 - a^2}) = dx + a dy.$$

On integrating, we get

$$\begin{aligned}\sqrt{z^2 - a^2} &= x + ay + c \text{ or} \\ z^2 - a^2 &= (x + ay + b)^2.\end{aligned}$$

EXAMPLE 3.42

Solve $z^2(p^2 + q^2 + 1) = a^2$.

Solution. The given equation is of the form $f(z, p, q) = 0$. So, Charpit's auxiliary equations yield $q = ap$. Substituting this value in the given equation, we have

$$z^2(p^2 + a^2p^2 + 1) = a^2$$

or

$$p^2(1 + a^2) = \frac{a^2}{z^2} - 1 = \frac{a^2 - z^2}{z^2}$$

or

$$p = \sqrt{\frac{a^2 - z^2}{z^2(1 + a^2)}} = \frac{1}{z} \sqrt{\frac{a^2 - z^2}{1 + a^2}}$$

and so $q = \frac{a}{z} \sqrt{\frac{a^2 - z^2}{1 + a^2}}$. Thus

$$dz = p dx + q dy = \frac{1}{z} \sqrt{\frac{a^2 - z^2}{1 + a^2}} dx + \frac{a}{z} \sqrt{\frac{a^2 - z^2}{1 + a^2}} dy$$

or

$$\frac{z dz}{\sqrt{a^2 - z^2}} = \frac{1}{\sqrt{1 + a^2}} dx + \frac{a}{\sqrt{1 + a^2}} dy$$

or

$$-d(\sqrt{a^2 - z^2}) = \frac{1}{\sqrt{1 + a^2}} dx + \frac{a}{\sqrt{1 + a^2}} dy.$$

On integrating, we get

$$-\sqrt{a^2 - z^2} = \frac{1}{\sqrt{1 + a^2}} x + \frac{a}{\sqrt{1 + a^2}} y + b$$

or

$$(-\sqrt{a^2 - z^2})(\sqrt{1 + a^2}) = x + ay + c$$

or

$$(a^2 - z^2)(1 + a^2) = (x + ay + c)^2.$$

EXAMPLE 3.43

Solve $z^2(p^2 + q^2 + 1) = 0$.

Solution. The given equation is of the type $f(z, p, q) = 0$ and so $q = ap$. Thus the given equation implies

$$z^2(p^2 + a^2p^2 + 1) = 1$$

or

$$p^2(1 + a^2) = \frac{1}{z^2} - 1 = \frac{1 - z^2}{z^2}$$

or

$$p = \frac{\sqrt{(1 - z^2)}}{z\sqrt{1 + a^2}}.$$

Then $q = \frac{a}{z} \sqrt{\frac{1 - z^2}{1 + a^2}}$. Therefore, $dz = p dx + q dy$ becomes

$$dz = \frac{1}{z} \sqrt{\frac{1 - z^2}{1 + a^2}} dx + \frac{a}{z} \sqrt{\frac{1 - z^2}{1 + a^2}} dy$$

or

$$\sqrt{1 + a^2} \frac{z dz}{\sqrt{1 - z^2}} = dx + a dy$$

or

$$\sqrt{1 + a^2} d\sqrt{1 - z^2} = dx + a dy.$$

On integrating, we get

$$\sqrt{1 + a^2} (\sqrt{1 - z^2}) = x + ay + b$$

or

$$(1 + a^2)(1 - z^2) = (x + ay + b)^2.$$

EXAMPLE 3.44

Solve $zpq = p + q$.

Solution. The given equation is of the type $f(z, p, q) = 0$. So $q = ap$ and putting this value of q

in the given equation, we get

$$zap^2 = ap + p$$

and so $p = \frac{a+1}{az}$. Then $q = \frac{a+1}{z}$. Hence

$$dz = \frac{a+1}{az} dx + \frac{a+1}{z} dy$$

or

$$zdz = \frac{a+1}{a} dx + (a+1) dy.$$

On integrating, we get

$$\begin{aligned} \frac{z^2}{2} &= \frac{a+1}{a}x + (a+1)y + c \text{ or} \\ z^2 &= \frac{2}{a}(a+1)x + 2(a+1)y + b. \end{aligned}$$

EXAMPLE 3.45

Solve $z^2(p^2x^2 + q^2) = 1$.

Solution. We reduce the given equation to the form $f(z, p, q) = 0$ by putting

$$\frac{dx}{x} = dX,$$

so that $X = \log x$. Then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{1}{x},$$

which yields

$$px = \frac{\partial z}{\partial X}.$$

Therefore, the given equation becomes

$$z^2 \left[\left(\frac{\partial z}{\partial X} \right)^2 + q^2 \right] = 1, \quad (74)$$

which is of the form $f(z, P, q) = 0$, where $P = \frac{\partial z}{\partial X}$. Therefore, we have $q = aP$. Putting in (74), we get

$$z^2[P^2 + a^2P^2] = 1$$

or

$$P^2(1 + a^2) = \frac{1}{z^2}$$

or

$$P = \frac{1}{z\sqrt{1+a^2}}.$$

Then $q = \frac{a}{z\sqrt{1+a^2}}$ and so the equation $dz = P dX + q dy$ becomes

$$dz = \frac{1}{z\sqrt{1+a^2}} dX + \frac{a}{z\sqrt{1+a^2}} dy$$

or

$$z(\sqrt{1+a^2})dz = dX + a dy.$$

Integrating, we get the required solution as

$$\sqrt{1+a^2} \frac{z^2}{2} = X + ay + b$$

or

$$\begin{aligned} \sqrt{1+a^2} z^2 &= 2X + 2ay + 2b \\ &= 2(\log x + ab) + c. \end{aligned}$$

(C) Separable Equations

The equations of the form $f(x, p) = g(y, q)$, in which z is absent and the terms containing x and p can be separated from those containing y and q . The Charpit's auxiliary equations in this case becomes

$$\frac{dx}{f_p} = \frac{dy}{-g_q} = \frac{dz}{pf_p - qf_q} = \frac{dp}{-f_x} = \frac{dq}{g_y}.$$

Therefore,

$$f_x dx + f_p dp = 0,$$

which implies that f is constant, that is, $f(x, p) = a$. Therefore, $g(y, q) = a$. Hence solving for p , we get $p = \phi(x)$ and solving for q , we have $q = \psi(y)$. Therefore, $dz = p dx + q dy$ gives

$$dz = \phi(x)dx + \psi(y)dy$$

and so

$$z = \int \phi(x) dx + \int \psi(y) dy + c.$$

EXAMPLE 3.46

Solve $p^2 - q^2 = x - y$.

Solution. The given equation can be separated as $p^2 - x = q^2 - y$ and, therefore, by Charpit's auxiliary equations,

$$p^2 - x = a = q^2 - y.$$

Thus

$$p = \sqrt{a+x} \quad \text{and} \quad q = \sqrt{a+y}.$$

Thus $dz = p dx + q dy$ reduces to

$$dz = \sqrt{a+x} dx + \sqrt{a+y} dy$$

On integrating, we get the solution as

$$z = \frac{2}{3}(a+x)^{3/2} + \frac{2}{3}(a+y)^{3/2} + b.$$

EXAMPLE 3.47

Solve $z(p^2 - q^2) = x - y$.

Solution. The given equation can be written as

$$(\sqrt{z}p)^2 - (\sqrt{z}q)^2 = x - y.$$

Substitute

$$\sqrt{z} dz = dZ, \quad \text{so that } Z = \frac{2}{3}z^{3/2}.$$

Then

$$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} = \sqrt{z} \frac{\partial z}{\partial x} = \sqrt{z} p,$$

$$\frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial y} = \sqrt{z} \frac{\partial z}{\partial y} = \sqrt{z} q.$$

Therefore, the given equation transforms to

$$\left(\frac{\partial Z}{\partial x}\right)^2 - \left(\frac{\partial Z}{\partial y}\right)^2 = x - y$$

or

$$P^2 - Q^2 = x - y,$$

where $P = \frac{\partial Z}{\partial x}$, $Q = \frac{\partial Z}{\partial y}$. Thus

$$P^2 - x = Q^2 - y = a,$$

which yields

$$P = \sqrt{a+x}, \quad Q = \sqrt{a+y}.$$

Thus $dz = p dx + q dy$ transforms to

$$dZ = P dx + Q dy = \sqrt{a+x} dx + \sqrt{a+y} dy.$$

On integrating, we get

$$Z = \frac{2}{3}(a+x)^{3/2} + \frac{2}{3}(a+y)^{3/2} + c$$

or

$$\frac{2}{3}z^{3/2} = \frac{2}{3}(a+x)^{3/2} + \frac{2}{3}(a+y)^{3/2} + c$$

or

$$z^{3/2} = (a+x)^{3/2} + (a+y)^{3/2} + b.$$

EXAMPLE 3.48

Solve $p^2 + q^2 = x + y$.

Solution. The given equation is separable and we have

$$p^2 - x = y - q^2 = a.$$

Thus

$$p = \sqrt{a+x} \text{ and then } q = \sqrt{y-a}.$$

Therefore, $dz = p dx + q dy$ takes the form

$$dz = \sqrt{a+x} dx + \sqrt{y-a} dy.$$

Integrating, we get the required solution as

$$z = \frac{2}{3}(a+x)^{3/2} + \frac{2}{3}(y-a)^{3/2} + b.$$

EXAMPLE 3.49

Solve $p + q = \sin x + \sin y$.

Solution. The given equation is separable, that is, of the form $f(x, p) = g(y, q)$. Thus

$$p - \sin x = \sin y - q = a$$

and so

$$p = a + \sin x,$$

$$q = \sin y - a.$$

Thus, $dz = p dx + q dy$ takes the form

$$dz = (a + \sin x)dx + (\sin y - a)dy.$$

On integrating, we get

$$\begin{aligned} z &= ax - \cos x - \cos y - ay + b \\ &= a(x - y) - (\cos x + \cos y) + b. \end{aligned}$$

EXAMPLE 3.50

Solve $z^2 (p^2 + q^2) = x^2 + y^2$.

Solution. The given equation can be written as

$$(zp)^2 + (zq)^2 = x^2 + y^2.$$

Put $Z = \frac{1}{2}z^2$. Then

$$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial x} = z \frac{\partial z}{\partial x} = zp,$$

$$\frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z} \cdot \frac{\partial z}{\partial y} = z \frac{\partial z}{\partial y} = zq.$$

Hence the given equation becomes

$$\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 = x^2 + y^2$$

or

$$P^2 + Q^2 = x^2 + y^2,$$

where $P = \frac{\partial Z}{\partial x}$ and $Q = \frac{\partial Z}{\partial y}$. This equation is of the form $f(x, P) = g(y, Q)$. Therefore,

$$P^2 - x^2 = y^2 - Q^2 = a,$$

which yields

$$P = \sqrt{a+x^2}, \quad Q = \sqrt{y^2-a}.$$

Thus the equation $dZ = P dx + Q dy$ becomes

$$dZ = \sqrt{a+x^2} dx + \sqrt{y^2-a} dy.$$

On integrating, we get

$$Z = \frac{1}{2}x\sqrt{x^2+a} + \frac{a}{2}\log(x + \sqrt{x^2+a}) \\ + \frac{1}{2}y\sqrt{y^2-a} - \frac{a}{2}\log(y + \sqrt{y^2-a}) + b$$

or

$$z^2 = x\sqrt{x^2+a} + a\log(x + \sqrt{x^2+a}) \\ + y\sqrt{y^2-a} - a\log(y + \sqrt{y^2-a}) + 2b \\ = x\sqrt{x^2+a} + y\sqrt{y^2-a} + a\log\frac{x + \sqrt{x^2+a}}{y + \sqrt{y^2-a}} + 2b.$$

(D) Clairut's Equation

Any first order partial differential equation of the form

$$z = px + qy + f(p, q)$$

is called a *Clairut's equation*.

For such equations, the Charpit's auxiliary equations are

$$\frac{dx}{p+f_p} = \frac{dy}{y+f_q} = \frac{dz}{px+qy+pf_p+qf_q} = \frac{dp}{0} = \frac{dq}{0}.$$

The last two members yield

$$p = a \quad \text{and} \quad q = b.$$

Therefore $dz = p dx + q dy$ implies

$$dz = a dx + b dy,$$

which on integration gives the required solution as

$$z = ax + by + c, \quad \text{where } c = f(a, b).$$

EXAMPLE 3.51

Solve $(pq - p - q)(z - px - qy) = pq$.

Solution. The given equation can be written as

$$pqz - p^2xq - pq^2y - pz + p^2x + pqy - qz$$

$$+ pqx + q^2y = pq$$

$$z(pq - p - q) = -px(p + q - pq)$$

$$- qy(p + q - pq) + pq$$

or

$$z = px + qy + \frac{pq}{pq - p - q},$$

which is Clairut's equation. Therefore, replacing p by a and q by b , we get the required solution as

$$z = ax + by + \frac{ab}{ab - a - b}.$$

EXAMPLE 3.52

Solve $pqz = p^2(xq + p^2) + q^2(yq + q^2)$.

Solution. Dividing throughout by pq , we get

$$z = \frac{p}{q}(xq + p^2) + \frac{q}{p}(yp + q^2)$$

$$= px + qy + \frac{p^3}{q} + \frac{q^3}{p}.$$

Thus, the given equation is Clairut's equation. So, replacing p by a and q by b , we get the solution

$$z = ax + by + \frac{a^3}{b} + \frac{b^3}{a}$$

or

$$z = ax + by + \frac{a^4 + b^4}{ab}.$$

EXAMPLE 3.53

Solve $z = px + qy + \sqrt{1 + p^2 + q^2}$.

Solution. It is a Clairut's equation and, therefore, replacing p by a and q by b , we get the solution

$$z = ax + by + \sqrt{1 + a^2 + b^2}.$$

EXAMPLE 3.54

Solve $4xyz = pq + 2px^2y + 2qxy^2$.

Solution. Dividing throughout by xy , we get

$$4z = 2px + 2qy + \frac{pq}{xy}.$$

Putting $x^2 = X$ and $y^2 = Y$, we get

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = 2x \frac{\partial z}{\partial X}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y} = 2y \frac{\partial z}{\partial Y}.$$

Therefore, the equation transforms to

$$4z = 4x^2 \frac{\partial z}{\partial Y} + 4y^2 \frac{\partial z}{\partial X} + 4 \frac{\partial z}{\partial X} \cdot \frac{\partial z}{\partial Y}$$

or

$$z = XP + YQ + PQ,$$

where

$$P = \frac{\partial z}{\partial X}, \quad Q = \frac{\partial z}{\partial Y}.$$

This equation is Clairut's equation and so the solution is

$$z = aX + bY + ab = ax^2 + by^2 + ab.$$

EXAMPLE 3.55

Solve $pq(px + qy - z)^3 = 1$.

Solution. The given equation is

$$pq(px + qy - z)^3 = 1.$$

Therefore,

$$(px + qy - z)^3 = \frac{1}{pq}$$

or

$$px + qy - z = \frac{1}{(pq)^{1/3}}$$

or

$$z = px + qy - \frac{1}{(pq)^{1/3}},$$

which is Clairut's equation. Hence its solution is

$$z = ax + by - \frac{1}{(ab)^{1/3}},$$

EXAMPLE 3.56

Solve $(px + qy - z)^2 = 1 + p^2 + q^2$.

Solution. We have

$$(px + qy - z)^2 = 1 + p^2 + q^2.$$

Therefore

$$px + qy - z = \pm \sqrt{1 + p^2 + q^2}.$$

or

$$z = px + qy \pm \sqrt{1 + p^2 + q^2},$$

which is Clariut's equations. Hence its solution is

$$z = ax + by \pm \sqrt{1 + a^2 + b^2}.$$

3.6 LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

A linear partial differential equation of order n is an equation of the form

$$\begin{aligned} & \left(a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \dots + a_n \frac{\partial^n z}{\partial y^n} \right) \\ & + \left(b_0 \frac{\partial^{n-1} z}{\partial x^{n-1}} + b_1 \frac{\partial^{n-1} z}{\partial x^{n-2} \partial y} + \dots + b_{n-1} \frac{\partial^{n-1} z}{\partial y^{n-1}} \right) \\ & + \dots + \left(m_0 \frac{\partial z}{\partial x} + m_1 \frac{\partial z}{\partial y} \right) + nz = f(xy), \end{aligned} \quad (75)$$

where the coefficients $a_0, a_1, \dots, b_0, b_1, \dots, m_0, m_1, \dots$, are constants or functions of x and y . In case these coefficients are all constants, then such a partial differential equation is called a *linear partial differential equation with constant coefficients*. On the other hand, if these coefficients are function of x or y or both, then the above equation turned out to be *linear partial differential equation with variable coefficients*. In case, the orders of all partial derivatives involved in (75) are same, then it is called a *homogeneous linear partial differential equation*. Similarly, if the orders of the partial derivatives are distinct, then (75) is called *non-homogeneous linear partial differential equation*. For Example

(i) the equations

$$2 \frac{\partial^2 z}{\partial x^2} + 7 \frac{\partial^2 z}{\partial x \partial y} + 5 \frac{\partial^2 z}{\partial y^2} = \sin x$$

and

$$\frac{\partial^3 z}{\partial x^3} - 3 \frac{\partial^3 z}{\partial x^2 \partial y} + 4 \frac{\partial^3 z}{\partial y^3} = e^{x+2y}$$

are homogeneous linear partial differential equations with constant coefficients, whereas the equations

$$\frac{\partial z}{\partial x} - \frac{\partial^2 z}{\partial y^2} = e^x$$

and

$$\begin{aligned} & \frac{\partial^2 z}{\partial x^2} + 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} + 4 \frac{\partial z}{\partial y} + 3 \frac{\partial z}{\partial y} + z \\ & = \sin(x + y) \end{aligned}$$

are non-homogeneous linear partial differential equations with constant coefficients.

Similarly, the equation

$$x^2 \frac{\partial^2 z}{\partial x^2} + y \frac{\partial^2 z}{\partial x \partial y} + x \frac{\partial^2 z}{\partial y^2} + x = e^x$$

is linear partial differential equation with variable coefficients.

(A) Homogeneous Linear Partial Differential Equations with Constant Coefficients

Consider the homogeneous linear partial differential equation

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \dots + a_n \frac{\partial^n z}{\partial y^n} = f(x, y)$$

with constant coefficients $a_0, a_1, \dots, a_n$. Let

- (i) D stands for $\frac{\partial}{\partial x}$
- (ii) D' stands for $\frac{\partial}{\partial y}$
- (iii) DD' stands for $\frac{\partial^2}{\partial x \partial y}$
- (iv) D^2 stands for $\frac{\partial^2}{\partial x^2}$
- (v) D'^2 stands for $\frac{\partial^2}{\partial y^2}$

and so on. Then the above equation transforms to

$$(a_0 D^n + a_1 D^{n-1} D' + \dots + a_n D'^n)z = f(x, y).$$

or

$$\phi(D, D')z = f(x, y). \quad (76)$$

As, in the case of ordinary linear differential equation with constant coefficients, the complete solution of (76) consists of two parts:

- (a) Complementary function (C.F), which is the solution of $\phi(D, D')z = 0$ and shall have n arbitrary functions for partial differential equation of order n .
- (b) Particular integral (P.I), which is the particular solution of $\phi(D, D')z = f(x, y)$.

The complete solution of (76) is then given by $z = \text{C.F.} + \text{P.I.}$

Theorem 3.2. Let $\phi(D, D')z = 0$ be a linear partial differential equation with constant coefficients. If $u_1, u_2, \dots, u_m$ are m solutions of $\phi(D, D')z = 0$, then their linear combination $\sum_{i=1}^m e_i u_i$ is also a solution of $\phi(D, D')z = 0$.

Proof: Since $u_i, 1 \leq i \leq m$ is a solution of $\phi(D, D')z = 0$, we have $\phi(D, D')u_i = 0$ for $1 \leq i \leq m$. Therefore

$$\begin{aligned} \phi(D, D') \left(\sum_{i=1}^m c_i u_i \right) &= \sum_{i=1}^m \phi(D, D')(c_i u_i) \\ &= \sum_{i=1}^m c_i \phi(D, D')u_i \\ &= \sum_{i=1}^m c_i (0) = 0 \end{aligned}$$

and hence $\sum_{i=1}^m c_i u_i$ is also a solution of $\phi(D, D')z = 0$.

Theorem 3.3. Let $\phi(D, D')z = f(x, y)$ be a linear partial differential equation with constant coefficients. If u is a solution of $\phi(D, D')z = 0$ and v is a solution of $\phi(D, D')z = f(x, y)$, then $u + v$ is a solution of $\phi(D, D')z = f(x, y)$.

Proof: Since u is a solution of $\phi(D, D')z = 0$, we have

$$\phi(D, D')u = 0. \quad (77)$$

Similarly, since v is a solution of $\phi(D, D')z = f(x, y)$, we have

$$\phi(D, D')v = f(x, y) \quad (78)$$

Then, by (77) and (78), we get

$$\begin{aligned} \phi(D, D')(u + v) &= \phi(D, D')u + \phi(D, D')v \\ &= 0 + f(x, y) = f(x, y). \end{aligned}$$

Hence $u + v$ is a solution of $\phi(D, D')z = f(x, y)$.

(B) Complementary Function of Second Order Linear Homogeneous Partial Differential Equation

Consider the second order linear homogeneous partial differential equation

$$a_0 \frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = 0. \quad (79)$$

The symbolic form of equation (79) is

$$(a_0 D^2 + a_1 DD' + \dots + a_2 D'^2)z = 0. \quad (80)$$

The auxiliary equation (A.E.) for equation (80) is

$$a_0 m^2 + a_1 m + a_2 = 0, \quad (81)$$

which is obtained by putting $D = m$ and $D' = 1$ in the symbolic form (80). The auxiliary equation (81),

being a quadratic equation, has two roots, say m_1 and m_2 , which may be *real and distinct* or *real and repeated*.

Case I. Let the roots of the auxiliary equation (81) be real and distinct. Then equation (79) may be written as

$$a_0(D - m_1D')(D - m_2D')z = 0. \quad (82)$$

Now, for $i = 1, 2$, the solution of $(D - mD')z = 0$ is also a solution of (82) and hence of (79). This yields

$$\frac{\partial z}{\partial x} - m_i \frac{\partial z}{\partial y} = 0$$

or

$$p - m_i q = 0, \quad (83)$$

which is Lagrange's linear equation of first order. The auxiliary equations of (83) are

$$\frac{dx}{1} = \frac{dy}{-m_i} = \frac{dz}{0},$$

which yield

$$dy + m_i dx = 0 \quad \text{and} \quad dz = 0.$$

Then, integration yields

$$y + m_i x = c_1 \quad \text{and} \quad z = c_2.$$

Hence

$$z = f_i(y + m_i x),$$

where f_i are arbitrary functions. Thus

$$z = f_1(y + m_1 x) \quad \text{and} \quad z = f_2(y + m_2 x)$$

are solutions of the partial differential equation (79). Since linear combination of solutions of partial differential equation is also a solution, the solution of (79) is

$$z = f_1(y + m_1 x) + f_2(y + m_2 x).$$

Proceeding in a similar way, the solution for the homogeneous linear partial differential equation of order n with constant coefficients (for distinct roots of A.E.) is

$$z = f_1(y + m_1 x) + f_2(y + m_2 x) + \dots + f_n(y + m_n x).$$

Case II. Let the roots of the auxiliary equation be real and repeated. Thus for partial differential equation (79), we have $m_1 = m_2$. Then equation (80) can be written as

$$a_0(D - m_1D')^2 z = 0$$

or

$$(D - m_1D')^2 z = 0. \quad (84)$$

Put $(D - m_1D')z = u$, then (84) becomes

$$(D - m_1D')u = 0.$$

Therefore, as in case 1, its solution is

$$u = f_1(y + m_1 x)$$

so that

$$(D - m_1D')z = f_1(y + m_1 x)$$

or

$$\frac{\partial z}{\partial x} - m_1 \frac{\partial z}{\partial y} = f_1(y + m_1 x)$$

or

$$p - m_1 q = f_1(y + m_1 x),$$

which is a Lagrange's linear equation with auxiliary equation

$$\frac{dx}{1} = \frac{dy}{-m_1} = \frac{dz}{f_1(y + m_1 x)}.$$

Thus

$$dy + m_1 dx = 0 \quad \text{and} \quad dx = \frac{dz}{f_1(y + m_1 x)}$$

or

$$dy + m_1 dx = 0 \quad \text{and} \quad dz = f_1(y + m_1 x) dx. \quad (85)$$

Integration of first member of (85) yields

$$y + m_1 x = c_1. \quad (86)$$

Using (86) in the second member of (85), we have

$$dz = f_1(c_1) dx,$$

which, on integration, yields

$$\begin{aligned} z &= f_1(c_1)x + c_2 \\ &= f_1(c_1)x + f_2(c_1), \quad \text{putting } c_2 = f_2(c_1) \\ &= x f_1(y + m_1 x) + f_2(y + m_1 x). \end{aligned}$$

Thus, the solution of (79), in this case, is

$$z = f_2(y + m_1 x) + x f_1(y + m_1 x).$$

Proceeding similarly, it can be shown that if the root m_1 of homogeneous linear partial differential equation of order n is repeated r times, then the

complementary function of that partial differential equation is given by

$$z = f_1(y + m_1x) + xf_2(y + m_1x) + x^2f_3(y + m_1x) + \dots + x^{n-1}f_r(y + m_1x),$$

where $f_1, f_2, \dots, f_r$ are arbitrary functions.

Remark 3.1. If $a_0 = 0$ in (79), then we have

$$(a_1DD' + a_2D^2)z = 0$$

or

$$D'(a_1D + a_2D')z = 0.$$

Therefore the solution of $D'z = 0$ is also a solution of (79). The solution of $D'z = 0$, that is, solution of $\frac{\partial z}{\partial y} = 0$ is $z = \phi(x)$, where ϕ is an arbitrary function.

EXAMPLE 3.57

Solve $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = 0$.

Solution: The symbolic form of the given homogeneous linear partial differential equation of order 2 is

$$r^2 - 4s + 4t = 0,$$

or

$$(D^2 - 4DD' + 4D'^2)z = 0.$$

Putting $D = m$, and $D' = 1$, we get the auxiliary equation for the given equation as

$$m^2 - 4m + 4 = 0 \text{ or } (m - 2)^2 = 0,$$

which yields $m = 2, 2$. Thus the general solution of the given equation is

$$z = f_1(y + 2x) + xf_2(y + 2x).$$

EXAMPLE 3.58

Solve $(D^3D'^2 + D^2D'^3)z = 0$.

Solution. We have

$$(D^3D'^2 + D^2D'^3)z = 0$$

or

$$D'^2D^2(D + D')z = 0.$$

The part of the general solution (see Remark above) corresponding to the factor D'^2 is $f_1(x) + yf_2(x)$.

Further, the auxiliary equation of $D^2(D + D')z = 0$ is

$$m^2(m + 1) = 0,$$

which yields $m = 0, 0, -1$. Therefore part of the general solution corresponding to the factor $D^2(D + D')$ is

$$f_3(y + 0x) + xf_4(y + 0x) + f_5(y - x).$$

Hence the general solution of the given partial differential equation is

$$z = f_1(x) + yf_2(y) + f_3(y) + xf_4(y) + f_5(y - x),$$

where f_1, f_2, f_3, f_4, f_5 are arbitrary functions.

(C) Particular Integral (P.I.) of $\phi(D, D')z = f(x, y)$

Since

$$\phi(D, D') \left[\frac{1}{\phi(D, D')} f(x, y) \right] = f(x, y),$$

the operator $\frac{1}{\phi(D, D')}$ is called the *inverse operator* of the operator $\phi(D, D')$. Further, $\frac{1}{\phi(D, D')} f(x, y)$ is a particular solution of the equation $\phi(D, D')z = f(x, y)$, because

$$\phi(D, D') \left[\frac{1}{\phi(D, D')} f(x, y) \right] = f(x, y).$$

We now discuss four different cases for finding particular integral according to different forms of $f(x, y)$.

Case I. When $f(x, y) = x^m y^n$, where m and n are constants. In this case, the particular integral $\frac{1}{\phi(D, D')} f(x, y)$ is obtained by expanding $\frac{1}{\phi(D, D')}$ into an infinite series in ascending powers of either D or D' . The particular integral obtained by using D and D' may not be identical. So, any of the two particular integrals may be used. However, if $m < n$, then it is advised to expand $\frac{1}{\phi(D, D')}$ in ascending powers of D while for $n < m$, we should expand $\frac{1}{\phi(D, D')}$ in ascending power of D' .

Case II. When $f(x, y) = \sin(ax + by)$ or $\cos(ax + by)$.
In this case

$$\text{P.I.} = \frac{1}{\phi(D^2, DD', D'^2)} [\sin(ax + by) \text{ or } \cos(ax + by)].$$

Replace $D^2 by -a^2$, DD' by $-ab$ and $D'^2 by -b^2$ so that

$$\text{P.I.} = \frac{1}{\phi(-a^2, -ab, -b^2)} [\sin(ax + by)] \text{ or } \cos(ax + by).$$

In case $\phi(a, b) = 0$ then $bD - aD'$ must be a factor of $\phi(D, D')$. If $bD - aD'$ is repeated r times, then the value of $\frac{1}{(bD - aD')^r} f(ax + by) = \frac{x^r}{b^r r!} f(ax + by)$.

Case III. When $f(x, y) = e^{ax+by}$. We note that

$$\begin{aligned} D[e^{ax+by}] &= a e^{ax+by}, \\ D'[e^{ax+by}] &= b e^{ax+by}, \\ D^2[e^{ax+by}] &= a^2 e^{ax+by}, \\ DD'[e^{ax+by}] &= ab e^{ax+by}, \\ D'^2[e^{ax+by}] &= b^2 e^{ax+by}. \end{aligned}$$

Therefore

$$\begin{aligned} (D^2 + a_1 DD' + a_2 D'^2) e^{ax+by} \\ = (a^2 + a_1 ab + a_2 b^2) e^{ax+by} \end{aligned}$$

or

$$\phi(D, D') e^{ax+by} = \phi(a, b) e^{ax+by},$$

or

$$\frac{\phi(D, D') e^{ax+by}}{\phi(a, b)} = f(x, y).$$

Operating $\frac{1}{\phi(D, D')}$ on both sides, we get

$$\frac{1}{\phi(a, b)} e^{ax+by} = \frac{1}{\phi(D, D')} f(x, y) = \text{P.I.}$$

Hence, for $f(x, y) = e^{ax+by}$, we have

$$\text{P.I.} = \frac{1}{\phi(a, b)} e^{ax+by}$$

Case IV. When $f(x, y)$ is any function of x and y , specially other than the above discussed three cases, then resolve $\frac{1}{\phi(D, D')}$ into partial fractions, treating

$\phi(D, D')$ as a function of D alone and operate each partial fraction on $f(x, y)$ as

$$\frac{1}{(D - m D')} f(x, y) = \int f(x, c - mx) dx,$$

that is, replacing y by $c - mx$ in $f(x, y)$ and integrate it with respect to x . After integration replace c by $y + mx$. Repeat the process with the second factor and so on.

EXAMPLE 3.59

$$\text{Solve } \frac{\partial^2 z}{\partial x^2} - 4 \frac{\partial z}{\partial x \partial y} + 4 \frac{\partial^2 z}{\partial y^2} = e^{2x+y}$$

Solution. From Example 3.57, the complementary function for this partial differential equation is

$$\text{C.F.} = f_1(y + 2x) + x f_2(y + 2x).$$

Further,

$$\text{P.I.} = \frac{1}{\phi(D, D')} f(x, y) = \frac{1}{(D - 2D')^2} e^{2x+y}.$$

Now if we put $D = 2$ and $D' = 1$, the rule III fails. Therefore we use rule IV and have (taking $y = c - mx = c - 2x$) from $(D - 2D')u = e^{2x+y}$, the solution

$$\begin{aligned} u &= \int f(x, c - mx) dx = \int e^{2x+c-2x} dx \\ &= \int e^c dx = x e^c = x e^{2x+y}. \end{aligned}$$

Now from $(D - 2D')z = u = x e^{2x+y}$, the solution is

$$z = \int x e^{2x+c-2x} dx = \frac{1}{2} x^2 e^c = \frac{1}{2} x^2 e^{2x+y}.$$

Hence the complete solution of the given equation is

$$z = f_1(y + 2x) + x f_2(y + 2x) + \frac{1}{2} x^2 e^{2x+y}.$$

EXAMPLE 3.60

$$\text{Solve } \frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y.$$

Solution. The symbolic form of the given linear homogeneous partial differential equation of order 2 is

$$(D^2 - 2 DD')z = \sin x \cos 2y.$$

The auxiliary equation for this equation is

$$m^2 - 2m = 0,$$

which yields $m = 0, 2$. Therefore

$$\begin{aligned}\text{C.F.} &= f_1(y + 0) + f_2(y + 2x) \\ &= f_1(y) + f_2(y + 2x).\end{aligned}$$

Further,

$$\begin{aligned}\text{P.I.} &= \frac{1}{\phi(D, D')} f(x, y) \\ &= \frac{1}{(D^2 - 2DD')} [\sin x \cos 2y] \\ &= \frac{1}{D^2 - 2DD'} \left[\frac{1}{2} \{ \sin(x + 2y) + \sin(x - 2y) \} \right] \\ &= \frac{1}{2} \cdot \frac{1}{D^2 - 2DD'} [\sin(x + 2y)] \\ &\quad + \frac{1}{2} \cdot \frac{1}{D^2 - 2DD'} [\sin(x - 2y)] \\ &= \frac{1}{2} \left(\frac{1}{-1 + 4} \right) \sin(x + 2y) + \frac{1}{2} \left(\frac{1}{-1 - 4} \right) \\ &\quad \times \sin(x - 2y) \\ &= \frac{1}{6} \sin(x + 2y) - \frac{1}{10} \sin(x - 2y).\end{aligned}$$

Hence the complete solution is

$$\begin{aligned}z &= \text{C.F.} + \text{P.I.} = f_1(y) + f_2(y + 2x) \\ &\quad + \frac{1}{6} \sin(x + 2y) - \frac{1}{10} \sin(x - 2y).\end{aligned}$$

EXAMPLE 3.61

Solve the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6 \frac{\partial^2 z}{\partial y^2} = \cos(2x + y).$$

Solution. The symbolic form of the given homogeneous linear partial differential equation of order 2 is

$$(D^2 + DD' - 6D'^2)z = \cos(2x + y).$$

The auxiliary equation for the equation is

$$m^2 + m - 6 = 0,$$

which yields $m = 2, -3$. Therefore

$$\text{C.F.} = f_1(y + 2x) + f_2(y - 3x).$$

To find particular integral, we have

$$\text{P.I.} = \frac{1}{D^2 + DD' - 6D'^2} \cos(2x + y).$$

But putting $D^2 = -4$, $DD' = -2$ and $D'^2 = -1$, we get

$$\frac{1}{D^2 + DD' - 6D'^2} = \frac{1}{-4 - 2 + 6} = \frac{1}{0} = \infty$$

and so particular integral can not be found by the rule for $\sin(ax + by)$ and $\cos(ax + by)$. Now

$$\begin{aligned}\frac{1}{D^2 + DD' - 6D'^2} &= \frac{1}{(D - 2D')(D + 3D')} \\ &= \frac{1}{5(D - 2D')} - \frac{1}{5(D + 3D')}\end{aligned}$$

and so (by case IV),

$$\begin{aligned}\text{P.I.} &= \frac{1}{5(D - 2D')} \cos(2x + y) \\ &\quad - \frac{1}{5(D + 3D')} \cos(2x + y) \\ &= \frac{1}{5} \int \cos(2x + c_1 - 2x) dx - \frac{1}{5} \int \cos(2x \\ &\quad + c_2 + 3x) dx \\ &= \frac{1}{5} \int (\cos c_1) dx - \frac{1}{5} \int \cos(5x + c_2) dx \\ &= \frac{1}{5} x \cos c_1 - \frac{1}{25} \sin(5x + c_2) \\ &= \frac{1}{5} x \cos(2x + y) - \frac{1}{25} \sin(y + 2x).\end{aligned}$$

Hence the complete solution of the given equation is

$$\begin{aligned}z &= \text{C.F.} + \text{P.I.} \\ &= f_1(y + 2x) + f_2(y - 3x) + \frac{1}{5} x \cos(2x + y) \\ &\quad - \frac{1}{25} \sin(y + 2x).\end{aligned}$$

EXAMPLE 3.62

Solve the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x^2 y.$$

Solution. The symbolic form of the given equation is Then

$$(D^2 - D^2)z = x^2y.$$

Its auxiliary equation is

$$m^2 - 1 = 0,$$

which yields $m = 1, -1$. Therefore

$$\text{C.F.} = f_1(y + x) + f_2(y - x).$$

Further,

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - D^2} x^2y = \frac{1}{D^2} \left(1 - \frac{D^2}{D^2}\right)^{-1} (x^2y) \\ &= \frac{1}{D^2} \left(1 + \frac{D^2}{D^2} + \dots\right) (x^2y) \\ &= \frac{1}{D^2} \left[x^2y + \frac{1}{D^2} \{D^2(x^2y)\} + \dots\right] \\ &= \frac{1}{D^2} \left[x^2y + \frac{1}{D^2} \{x^2(0)\}\right] = \frac{1}{D^2} (x^2y) \\ &= \frac{1}{D} \left(\frac{x^3}{3}y\right) = \frac{x^4y}{12}. \end{aligned}$$

Hence the complete solution of the given partial differential equation is

$$z = f_1(y + x) + f_2(y - x) + \frac{x^4y}{12}.$$

EXAMPLE 3.63

Solve the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6 \frac{\partial^2 z}{\partial y^2} = y \cos x.$$

Solution.

As in Example 3.61, we have

$$\text{C.F.} = f_1(y + 2x) + f_2(y - 3x)$$

Further,

$$\text{P.I.} = \frac{1}{(D - 2D')(D + 3D')} (y \cos x).$$

Let

$$(D + 3D')u = y \cos x.$$

$$u = \int (c + 3x) \cos x, \text{ since } y = c - mx = c + 3x$$

$$= (c + 3x) \sin x - \int 3 \sin x dx$$

$$= (c + 3x) \sin x + 3 \cos x = y \sin x + 3 \cos x.$$

Now $(D - 2D')z = u = y \sin x + 3 \cos x$ yields

$$z = \int [(c - 2x) \sin x + 3 \cos x] dx, \text{ because}$$

$$\begin{aligned} y &= c - mx = c - 2x \\ &= (c - 2x)(-\cos x) - (-2)(-\sin x) + 3 \sin x \\ &= (y + 2x - 2x)(-\cos x) - 2 \sin x + 3 \sin x \\ &= -y \cos x + \sin x. \end{aligned}$$

Hence the complete solution is

$$z = f_1(y + 2x) + f_2(y - 3x) + \sin x - y \cos x.$$

EXAMPLE 3.64

Solve the partial differential equation

$$\frac{\partial^3 z}{\partial x^3} - 3 \frac{\partial^3 z}{\partial x^2 y} + 4 \frac{\partial^3 z}{\partial y^3} = e^{x+2y}.$$

Solution. The symbolic form of the given equation is

$$(D^3 - 3D^2D' + 4D'^3)z = e^{x+2y}.$$

Its auxiliary equation is

$$m^3 - 3m^2 + 4 = 0,$$

which yields $m = -1, 2, 2$. Therefore

$$\text{C.F.} = f_1(y - x) + f_2(y + 2x) + xf_3(y + 2x).$$

Also,

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^3 - 3D^2D' + 4D'^3} (e^{x+2y}) \\ &= \frac{1}{1 - 6 + 32} e^{x+2y} = \frac{1}{27} e^{x+2y}. \end{aligned}$$

Hence the complete solution of the given linear homogenous partial differential equation of order 2 is

$$\begin{aligned} z &= f_1(y - x) + f_2(y + 2x) + xf_3(y + 2x) \\ &\quad + \frac{1}{27} e^{x+2y}. \end{aligned}$$

EXAMPLE 3.65

Solve the partial differential equation

$$(D^3 - 4D^2D' + 4DD'^2)z = \cos(2x + y).$$

Solution. The auxiliary equation for the given partial differential equation is

$$m^3 - 4m^2 + 4m = 0 \quad \text{or} \quad m(m^2 - 4m + 4) = 0,$$

which yields $m = 0, 2, 2$. Hence

$$\text{C.F.} = f_1(y) + f_2(y + 2x) + xf_3(y + 2x).$$

Further,

$$\text{P.I.} = \frac{1}{\phi(D, D')} \cos(2x + y).$$

But, taking $D^2 = -4$, $DD' = -2$, $D'^2 = -1$, we have

$$\frac{1}{\phi(D, D')} = \frac{1}{D^3 - 4D^2D' + 4DD'^2} = \frac{1}{0}.$$

Therefore the rule fails. But then

$$\begin{aligned} \text{P.I.} &= \frac{1}{D(D - 2D')^2} \cos(2x + y) \\ &= \frac{1}{(D - 2D')^2} \left[\frac{1}{D} \cos(2x + y) \right] \\ &= \frac{1}{(D - 2D')^2} \int \cos(2x + y) dx \\ &= 2 \frac{1}{(D - 2D')^2} \sin(2x + y). \end{aligned}$$

Now since $\frac{1}{D - 2D'} = \frac{1}{0}$ for $D = 2$, $D' = 1$ and the factor $D - 2D'$ is repeated twice, we have

$$\text{P.I.} = \frac{1}{2} \left[\frac{x^2}{1^2 2!} \sin(2x + y) \right] = \frac{x^2}{4} \sin(2x + y).$$

Hence the complete solution of the given partial differential equation is

$$\begin{aligned} z &= f_1(y) + f_2(y + 2x) + xf_3(y + 2x) \\ &\quad + \frac{x^2}{4} \sin(2x + y). \end{aligned}$$

EXAMPLE 3.66

Solve the partial Differential equations

$$(i) \quad \frac{\partial^3 z}{\partial x^3} + \frac{\partial^3 z}{\partial x^2 \partial y} + 4 \frac{\partial^2 z}{\partial x \partial y^2} = 2 \sin(3x + 2y).$$

$$(ii) \quad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \cos mx \cdot \cos ny.$$

$$(iii) \quad \frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial y} = \sin x \cdot \cos 2y.$$

Solution (i). As in the above Example, the complementary function of the given equation is

$$\text{C.F.} = f_1(y) + f_2(y + 2x) + xf_3(y + 2x).$$

Further,

$$f(x, y) = 2 \sin(3x + 2y),$$

$$\phi(D, D') = D^3 - 4D^2D' + 4DD'^2,$$

$$\phi(a, b) = 27 - 4.9.2 + 4.3.4 = 3 \neq 0.$$

Therefore

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^3 - 4D^2D' + 4DD'^2} [2 \sin(3x + 2y)] \\ &= \frac{1}{D(D^2 - 4DD' + 4D'^2)} [2 \sin(3x + 2y)] \\ &= \frac{1}{D(-9 + 24 - 16)} [2 \sin(3x + 2y)] \\ &= -\frac{2}{D} \sin(3x + 2y) \\ &= -2 \int \sin(3x + 2y) dx \\ &= \frac{2}{3} \cos(3x + 2y). \end{aligned}$$

Hence the complete solution of the given partial differential equation is

$$\begin{aligned} z &= f_1(y) + f_2(y + 2x) + xf_3(y + 2x) \\ &\quad + \frac{2}{3} \cos(3x + 2y) \end{aligned}$$

(ii) The symbolic form of the equation is

$$(D^2 + D'^2)z = \cos mx \cos ny.$$

The auxiliary equation is

$$m^2 + 1 = 0, \text{ which yields } m = \pm i.$$

Therefore

$$\text{C.F.} = f_1(y + ix) + f_2(y - ix).$$

Further,

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{D^2 + D'^2} \cos mx \cos ny \\
 &= \frac{1}{D^2 + D'^2} \left[\frac{1}{2} \{ \cos(mx + ny) + \cos(mx - ny) \} \right] \\
 &= \frac{1}{2} \left[\frac{1}{D^2 + D'^2} \cos(mx + ny) \right. \\
 &\quad \left. + \frac{1}{D^2 + D'^2} \cos(mx - ny) \right] \\
 &= \frac{1}{2} \left[\frac{1}{m^2 + n^2} \iint \cos u du dv \right. \\
 &\quad \left. + \frac{1}{m^2 + n^2} \iint \cos v dv dv \right],
 \end{aligned}$$

where $mx + ny = u$ and $mx - ny = v$

$$\begin{aligned}
 &= \frac{1}{2} \left[\frac{1}{m^2 + n^2} (-\cos u) + \frac{1}{m^2 + n^2} (-\cos v) \right] \\
 &= -\frac{1}{2(m^2 + n^2)} [\cos u + \cos v] \\
 &= \frac{-1}{2(m^2 + n^2)} [\cos(mx + ny) + \cos(mx - ny)] \\
 &= \frac{-1}{m^2 + n^2} \cos mx \cos ny.
 \end{aligned}$$

Hence the solution is

$$\begin{aligned}
 z &= \text{C.F.} + \text{P.I.} \\
 &= f_1(y + ix) + f_2(y - ix) \\
 &\quad - \frac{1}{m^2 + n^2} \cos mx \cos ny.
 \end{aligned}$$

(iii) The symbolic form of the given equation is

$$(D^2 - 2DD')z = \sin x \cos 2y.$$

The auxiliary equation is

$$m^2 - 2m = 0, \text{ which yields } m = 0, 2.$$

Therefore

$$\text{C.F.} = f_1(y) + f_2(y + 2x),$$

where f_1 and f_2 are arbitrary functions. Further,

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{D^2 - 2DD'} \cdot \sin x \cos 2y \\
 &= \frac{1}{D^2 - 2DD'} \cdot \frac{1}{2} [\sin(x + 2y) + \sin(x - 2y)] \\
 &= \frac{1}{2} \left[\frac{1}{D^2 - 2DD'} \sin(x + 2y) \right. \\
 &\quad \left. + \frac{1}{D^2 - 2DD'} \sin(x - 2y) \right] \\
 &= \frac{1}{2} \left[\frac{1}{1^2 - 2(1)(2)} \iint \sin u du dv \right. \\
 &\quad \left. + \frac{1}{1^2 - 2(1)(2)} \iint \sin v dv dv \right],
 \end{aligned}$$

where $x + 2y = u$, $x - 2y = v$

$$\begin{aligned}
 &= \frac{1}{2} \left[\frac{1}{3} \sin u - \frac{1}{5} \sin v \right] \\
 &= \frac{1}{6} \sin(x + 2y) - \frac{1}{10} \sin(x - 2y).
 \end{aligned}$$

Hence the complete solution of the given partial differential equation is

$$\begin{aligned}
 z &= \text{C.F.} + \text{P.I.} = f_1(y) + f_2(y + x) \\
 &\quad + \frac{1}{6} \sin(x + 2y) - \frac{1}{10} \sin(x - 2y).
 \end{aligned}$$

EXAMPLE 3.67

Solve the partial differential equation

$$\frac{\partial^3 z}{\partial x^3} + \frac{\partial^3 z}{\partial x^2 \partial y} - \frac{\partial^3 z}{\partial x \partial y^2} - \frac{\partial^3 z}{\partial y^3} = e^x \cos 2y.$$

Solution. The symbolic form of the given partial differential equation is

$$(D^3 + D^2 D' - D D'^2 - D'^3)z = e^x \cos 2y.$$

The auxiliary equation is

$$m^3 + m^2 - m - 1 = 0,$$

whose roots are 1, -1, -1. Therefore

$$\text{C.F.} = f_1(y + x) + f_2(y - x) + x f_3(y - x).$$

Now

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D-D')(D+D')^2} e^x \cos 2y \\ &= \frac{1}{(D+D')^2} \left[\frac{1}{D-D'} e^x \cos 2y \right]. \quad (87) \end{aligned}$$

But integration by parts yields

$$\begin{aligned} \frac{1}{D-D'} e^x \cos 2y &= \int e^x \cos 2(c-x) dx, \\ &\quad y = c - mx = c - x \\ &= \frac{e^x}{5} [\cos 2(c-x) - 2 \sin 2(c-x)] \\ &= \frac{e^x}{5} [\cos 2y - 2 \sin 2y]. \end{aligned}$$

Therefore (87) reduces to

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D+D')^2} \left[\frac{e^x}{5} \{\cos 2y - 2 \sin 2y\} \right] \\ &= \frac{1}{D+D'} \left[\frac{1}{D+D'} \left\{ \frac{e^x}{5} (\cos 2y - 2 \sin 2y) \right\} \right]. \quad (88) \end{aligned}$$

But integration by parts yields

$$\begin{aligned} \frac{1}{D+D'} e^x \cos 2y &= \int e^x \cos 2(c+x) dx, \\ &\quad y = c + x \\ &= \frac{e^x}{5} [\cos 2(c+x) \\ &\quad + 2 \sin 2(c+x)]. \\ &= \frac{e^x}{5} [\cos 2y + 2 \sin 2y] \quad (89) \end{aligned}$$

and similarly,

$$\begin{aligned} \frac{1}{D+D'} e^x \sin 2y &= \int e^x \sin 2(c+x) dx \\ &= \frac{e^x}{5} [\sin 2y - 2 \cos 2y]. \end{aligned}$$

Hence (88) reduces to

$$\begin{aligned} \text{P.I.} &= \frac{1}{D+D'} \left[\frac{e^x}{25} (\cos 2y + 2 \sin 2y - 2 \sin 2y \right. \\ &\quad \left. + 4 \cos 2y) \right] \\ &= \frac{1}{D+D'} \left[\frac{e^x}{5} \cos 2y \right] = \frac{e^x}{25} [\cos 2y + 2 \sin 2y], \end{aligned}$$

using (89) again.

Hence the complete solution of the given partial differential equation is

$$\begin{aligned} z &= \text{C.F.} + \text{P.I.} \\ &= f_1(y+x) + f_2(y-x) + xf_3(y-x) + \frac{e^x}{25} [\cos 2y \\ &\quad + 2 \sin 2y]. \end{aligned}$$

EXAMPLE 3.68

Solve the partial differential equation

$$\frac{\partial^3 z}{\partial x^3} - 2 \frac{\partial^3 z}{\partial x^2 \partial y} = x^2 y.$$

Solution. The symbolic form of the given equation is

$$(D^3 - 2D^2 D') z = x^2 y.$$

The auxiliary equation is

$$m^3 - 2m^2 = 0 \quad \text{or} \quad m^2(m-2) = 0,$$

which yields $m = 0, 0, 2$. Therefore

$$\text{C.F.} = f_1(y) + xf_2(y) + f_3(y+2x).$$

Further

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2(D-2D')} x^2 y = \frac{1}{D^3(1-\frac{2D'}{D})} x^2 y \\ &= \frac{1}{D^3} \left[1 + \frac{2D'}{D} + \frac{4D'^2}{D^2} + \dots \right] x^2 y \\ &= \frac{1}{D^3} \left[x^2 y + 2 \frac{2}{D} x^2 \right] = \frac{1}{D^3} \left[x^2 y + 2 \frac{x^3}{3} \right] \\ &= \frac{1}{D^3} x^2 y + \frac{2}{3D^3} x^3 = \frac{x^5 y}{60} + \frac{x^6}{180}. \end{aligned}$$

Hence the complete solution of the given differential equation is

$$z = \text{C.F.} + \text{P.I.}$$

$$= f_1(y) + x f_2(y) + f_3(y + 2x) + \frac{x^5 y}{60} + \frac{x^6}{180}.$$

(D) Solution of Non-homogeneous Linear Partial Differential Equation

As pointed out earlier, a partial differential equation $\phi(D, D')z = f(x, y)$ is called *non-homogeneous* if all the terms of $\phi(D, D')$ are not of the same order.

The complete solution of such equation also consists of two parts—complementary function (C.F.) and particular integral (P.I.). *The particular integral of such equation can be found using the rules of homogeneous linear partial differential equations.* The complementary function of such equation can be found by the following two methods:

Method I. Let $\phi(D, D')$ be factorized as

$$\phi(D, D') = (a_1 D + b_1 D' + c_1)(a_2 D + b_2 D' + c_2) \times \dots (a_n D + b_n D' + c_n),$$

where a_i, b_i , and c_i are constants. The solution of the equation

$$a_i D + b_i D' + c_i = 0 \quad (90)$$

will be a solution of $\phi(D, D')z = 0$. The Lagrange's auxiliary equations for (90) are

$$\frac{dx}{a_i} = \frac{dy}{b_i} = \frac{dz}{-c_i z}.$$

Thus

$$b_i dx - a_i dy = 0$$

and

$$\frac{dz}{z} = -\frac{c_i}{a_i} dx \quad \text{or} \quad \frac{dz}{z} = -\frac{c_i}{b_i} dy.$$

Integrating these equations, we get

$$b_i x - a_i y = k_1 \quad (\text{constant})$$

and

$$\log z = -\frac{c_i}{a_i} x + \log k_2 \quad \text{if} \quad a_i \neq 0$$

or

$$\log z = -\frac{c_i}{b_i} y + \log k_3 \quad \text{if} \quad b_i \neq 0,$$

that is,

$$b_i x - a_i y = k_1$$

and

$$z = k_2 e^{-\frac{c_i}{a_i} x} \quad \text{if} \quad a_i \neq 0 \quad \text{or} \quad z = k_3 e^{-\frac{c_i}{b_i} y} \quad \text{if} \quad b_i \neq 0.$$

Hence the general solution of (90) is

$$z = e^{-\frac{c_i}{a_i} x} f_i(b_i x - a_i y) \quad \text{if} \quad a_i \neq 0$$

or

$$z = e^{-\frac{c_i}{b_i} y} f_i(b_i x - a_i y) \quad \text{if} \quad b_i \neq 0,$$

where f_i is an arbitrary function.

The solution corresponding to the other factors can be found similarly. The complementary function of the given equation is the sum of all solutions obtained as above.

In case of repeated factors, the solution of $(a_i D + b_i D' + c_i)^n$ shall be

$$z = e^{-\frac{c_i}{a_i} x} \sum_{i=1}^n x^{i-1} f_i(b_i x - a_i y) \quad \text{if} \quad a_i \neq 0$$

or

$$z = e^{-\frac{c_i}{b_i} y} \sum_{i=1}^n y^{i-1} f_i(b_i x - a_i y) \quad \text{if} \quad b_i \neq 0.$$

Method II. Suppose that $z = e^{ax+by}$ is a trial solution of $\phi(D, D')z = 0$. Then

$$D(e^{ax+by}) = \frac{\partial}{\partial x}(e^{ax+by}) = a e^{ax+by}$$

and

$$D'(e^{ax+by}) = \frac{\partial}{\partial y}(e^{ax+by}) = b e^{ax+by}.$$

Therefore $\phi(D, D')z = 0$ implies

$$\phi(a, b)e^{ax+by} = 0.$$

Since $e^{ax+by} \neq 0$, we have

$$\phi(a, b) = 0. \quad (91)$$

Suppose that solving (91) for b in terms of a , we get

$$b = f_1(a), f_2(a), \dots$$

Then

$$\text{C.F.} = \sum c_i e^{ax} f_i(a)y.$$

EXAMPLE 3.69

Solve

$$(D - D' - 1)(D - D' - 2)z = e^{2x-y}.$$

Solution. The factors of $\phi(D, D')$ in the given non-homogeneous partial differential are already of the type $a_i D + b_i D' + c_i$. Therefore we use method I in this case. In this case $a_1 = 1, b_1 = -1, c_1 = -1$ and $a_2 = 1, b_2 = -1, c_2 = -2$.

Therefore

$$\begin{aligned} \text{C.F.} &= e^{-\frac{c_1}{a_1}x} f_1(x+y) + e^{-\frac{c_2}{a_2}x} f_2(x+y) \\ &= e^x f_1(y+x) + e^{2x} f_2(y+x). \end{aligned}$$

Moreover,

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D - D' - 1)(D - D' - 2)} e^{2x-y} \\ &= \frac{1}{(2)(1)} e^{2x-y}, \text{ replacing } D \text{ by } 2 \text{ and } D' \text{ by } -1 \\ &= \frac{1}{2} e^{2x-y}. \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} z &= \text{C.F.} + \text{P.I.} \\ &= e^x f_1(y+x) + e^{2x} f_2(y+x) + \frac{1}{2} e^{2x-y}. \end{aligned}$$

EXAMPLE 3.70

Solve the non-homogeneous partial differential equation $(D^2 - DD' + D' - 1)z = \cos(x + 2y)$.

Solution. We have

$$\begin{aligned} \phi(D, D') &= D^2 - DD' + D' - 1 \\ &= (D - 1)(D - D' + 1). \end{aligned}$$

Thus $a_1 = 1, b_1 = 0, c_1 = -1$ and $a_2 = 1, b_2 = -1, c_2 = 1$. Therefore

$$\text{C.F.} = e^x f_1(y) + e^{-x} f_2(y+x).$$

More over,

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - DD' + D' - 1} \cos(x + 2y) \\ &= \frac{1}{-1 + 2 + D' - 1} \cos(x + 2y) \\ &= \frac{1}{D'} \cos(x + 2y) = \int \cos(x + 2y) dy \\ &= \frac{1}{2} \sin(x + 2y) \end{aligned}$$

Hence the complete equation is

$$z = e^x f_1(y) + e^{-x} f_2(y+x) + \frac{1}{2} \sin(x + 2y).$$

EXAMPLE 3.71

Solve the partial differential equation

$$(D^3 - 3DD' + D' + 4)z = e^{2x+y}.$$

Solution. We note that $\phi(D, D') = D^3 - 3DD' + D' + 4$ is irreducible (does not have linear factor). Therefore we use method II in this case. We have

$$\phi(a, b) = a^3 - 3ab + 4 = 0$$

Being a cubic in a , it has three values of a in terms of b . Let these values be $f_1(b), f_2(b)$ and $f_3(b)$. Then

$$\text{C.F.} = c_1 e^{f_1(b)x+y} + c_2 e^{f_2(b)x+by} + c_3 e^{f_3(b)x+by}.$$

Further,

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^3 - 3DD' + D' + 4} e^{2x+y} \\ &= \frac{1}{8 - 6 + 1 + 4} e^{2x+y} = \frac{1}{7} e^{2x+y}. \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} z &= \text{C.F.} + \text{P.I.} \\ &= c_1 e^{f_1(b)x+by} + c_2 e^{f_2(b)x+by} + c_3 e^{f_3(b)x+by} \\ &\quad + \frac{1}{7} e^{2x+y}. \end{aligned}$$

EXAMPLE 3.72

Solve $(2D^4 - 3D^2D' + D^2)z = 0$.

Solution. We have

$$\begin{aligned} 2D^4 - 3D^2D' + D^2 &= (2D^2 - D')(D^2 - D') \\ &= \phi_1(D, D')\phi_2(D, D'). \end{aligned}$$

Since $2D^2 - D'$ is irreducible, we have

$$\phi_1(a, b) = 2a^2 - b = 0$$

or

$$b = 2a^2.$$

Therefore contribution to C.F. due to $\phi_1(D, D')z = 0$ is

$$c_1 e^{ax+2a^2y}.$$

Again, since $D^2 - D'$ is irreducible, we have

$$\phi_2(a', b') = a'^2 - b' = 0 \quad \text{or} \quad b' = a'^2.$$

Therefore contribution to C.F. due to $\phi_2(D, D')z = 0$ is

$$c_2 e^{a'x + a'^2 y}.$$

Hence the complete solution is

$$z = c_1 e^{ax + 2a^2 y} + c_2 e^{a'x + a'^2 y}.$$

3.7 CLASSIFICATION OF SECOND ORDER LINEAR PARTIAL DIFFERENTIAL EQUATIONS

Consider the second order linear partial differential equation

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + F(u) = 0, \quad (92)$$

where A, B, C, D, E, and F are real constants. The equation (92) is said to be

- (i) hyperbolic if $B^2 - 4AC > 0$
- (ii) parabolic if $B^2 - 4AC = 0$
- (iii) elliptic if $B^2 - 4AC < 0$.

For example, the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \quad (\text{special case of wave equation})$$

is hyperbolic, since $A = 1$, $B = 0$, $C = -1$, and $B^2 - 4AC = 4 > 0$. This equation is satisfied by small transverse displacement of the points of a vibrating string.

On the other hand, the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(two-dimensional Laplace equation)

is *elliptic* since $A = 1$, $B = 0$, $C = 1$, and $B^2 - 4AC = -4 < 0$. This equation is satisfied by the steady temperature at points of a thin rectangular plate. Similarly, the *heat equation*

$$\frac{\partial u}{\partial t} = c \frac{\partial^2 u}{\partial x^2}$$

is *parabolic*.

3.8 THE METHOD OF SEPARATION OF VARIABLES

The method of separation of variables is a very powerful method for obtaining solutions for certain

problems involving partial differential equations. Problems those are of great physical interest can be solved by this method. For example, wave equation, heat equation, and Laplace equation can be solved by this method.

This method involves a solution which breaks up into a product of functions, each of which contains only one of the variables. If the partial differential equation involves n independent variables $x_1, x_2, \dots, x_n$, we first assume that the equation possesses product solution of the form $X_1 X_2 \dots X_n$, where X_i is a function of only x_i ($i = 1, 2, \dots, n$). This basic assumption will produce ordinary differential equations, one in each of the unknown functions X_i ($i = 1, 2, \dots, n$). We solve these n ordinary differential equations, which may also involve initial or boundary conditions. The solution of these n equations will produce particular solutions of the form $X_1 X_2 \dots X_n$ satisfying some supplementary conditions of the original problem. Then these particular solutions are combined by superposition rule to produce a solution of the problem.

EXAMPLE 3.73

Solve $\frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$.

Solution. Let

$$z = X(x) Y(y), \quad (93)$$

be a trial solution of the given partial differential equation. Then the given equation reduces to

$$X''(x)Y(y) - 2X'(x)Y(y) + X(x)Y'(y) = 0$$

Separating the variables, we get

$$\frac{X''(x) - 2X'(x)}{X(x)} = -\frac{Y'(y)}{Y(y)}. \quad (94)$$

The left-hand side is a function of x only, whereas the right-hand side is a function of y only. Since x and y are different variables, equality in (94) can occur only if the left-hand side and right-hand side are both equal to a constant, say a . Thus, we get two ordinary differential equations

$$X''(x) - 2X'(x) - aX'(x) = 0 \quad (95)$$

and

$$Y'(y) + aY(y) = 0. \quad (96)$$

The auxiliary equation for (95) is

$$m^2 - 2m - a = 0,$$

which yields $m = 1 \pm \sqrt{1+a}$. Therefore, the solution of (94) is

$$X(x) = c_1 e^{(1+\sqrt{1+a})x} + c_2 e^{(1-\sqrt{1+a})x}.$$

The auxiliary equation for (96) is

$$m + a = 0$$

and so $m = -a$. Therefore, the solution of (96) is

$$Y(y) = c_3 e^{-ay}.$$

Substituting the value of $X(x)$ and $Y(y)$ into (93), we get

$$\begin{aligned} z = X(x) Y(y) &= [c_1 e^{(1+\sqrt{1+a})x} \\ &+ c_2 e^{(1-\sqrt{1+a})x}] c_3 e^{-ay} \\ &= [c_4 e^{(1+\sqrt{1+a})x} + c_5 e^{(1-\sqrt{1+a})x}] e^{-ay}, \end{aligned}$$

which is the required solution of the given differential equation.

EXAMPLE 3.74

Solve $\frac{\partial u}{\partial x} = 2 \frac{\partial u}{\partial t} + u$, subject to the condition

$$u(x, 0) = 6e^{-3x}.$$

Solution. The given equation involves two independent variables x and t . So let

$$u(x, t) = X(x)T(t). \quad (97)$$

Then the given equation transforms to

$$X'(x)T(t) = 2X(x)T'(t) + X(x)T(t)$$

or

$$X'(x)T(t) = (2T'(t) + T(t))X(x).$$

Separating variables, we obtain

$$\frac{X'(x)}{X(x)} = \frac{2T'(t) + T(t)}{T(t)}. \quad (98)$$

Since x and t are different variables, the left-hand side and right-hand side are both equal to a constant, say a . Therefore, we get two ordinary differential equations

$$\frac{X'(x)}{X(x)} = a \quad (99)$$

and

$$2T'(t) + T(t) - aT(t) = 0. \quad (100)$$

The solution of (99) is

$$\log X(x) = ax + \log c_1$$

or

$$\log \frac{X(x)}{c_1} = ax$$

or

$$X(x) = c_1 e^{ax} \quad (101)$$

Equation (100) can be written as

$$2T'(t) = T(t)(a - 1)$$

or

$$\frac{T'(t)}{T(t)} = \frac{1}{2}(a - 1)$$

and so its solution is

$$\log T(t) = \frac{1}{2}(a - 1)t + \log c_2$$

or

$$\log \frac{T(t)}{c_2} = \frac{1}{2}(a - 1)t$$

or

$$T(t) = c_2 e^{\frac{1}{2}(a-1)t} \quad (102)$$

Putting the values of $X(x)$ and $T(t)$ obtained from (101) and (102) in (97), the solution is

$$u(x, t) = X(x)T(t) = c_1 c_2 e^{ax} \cdot e^{\frac{1}{2}(a-1)t}.$$

Using the initial condition $u(x, 0) = 6e^{-3x}$, we get

$$6e^{-3x} = c_1 c_2 e^{ax}.$$

So

$$c_1 c_2 = 6 \quad \text{and} \quad a = -3.$$

Hence, the solution of the given partial differential equation is

$$u(x, t) = 6e^{-3x} e^{-2t} = 6e^{-(3x+2t)}.$$

EXAMPLE 3.75

Solve $4 \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$ subject to the condition that $u(0, y) = 3e^{-y} - 5e^{-5y}$.

Solution. The given equation involves two variables x and y . So, let

$$u(x, y) = X(x)Y(y). \quad (103)$$

Then the given equation reduces to

$$4X'(x)Y(y) + X(x)Y'(y) = 3X(x)Y(y)$$

or equivalently,

$$4 \frac{X'(x)}{X(x)} = \frac{3Y(y) - Y'(y)}{Y(y)}.$$

Since x and y are independent variables, the left-hand side and right-hand side are both equal to some constant, say a . Thus, we get two differential equations

$$4 \frac{X'(x)}{X(x)} = a, \quad (104)$$

and

$$\frac{3Y(y) - Y'(y)}{Y(y)} = a \quad (105)$$

The solution of (104) is

$$\log X(x) = \frac{a}{4}x + \log c_1$$

or

$$\frac{X(x)}{c_1} = e^{\frac{a}{4}x}$$

or

$$X(x) = c_1 e^{\frac{a}{4}x}. \quad (106)$$

The equation (105) can be written as

$$\frac{Y'(y)}{Y(y)} = 3 - a$$

and so its solution is

$$\log Y(y) = (3 - a)y + \log c_2$$

or

$$Y(y) = c_2 e^{(3-a)y}. \quad (107)$$

Using (105) and (107), the solution (103) reduce to

$$u(x, y) = c_1 c_2 e^{\frac{a}{4}x} \cdot e^{(3-a)y}.$$

Therefore, using the initial condition $u(0, y) = 3e^{-y} - e^{-5y}$, we get

$$3e^{-y} - e^{-5y} = c_1 c_2 e^{(3-a)y}.$$

Now

$$3e^{-y} - e^{-5y} = c_1 c_2 e^{-(a-3)y} + 0 \cdot e^{-5y}$$

gives one value of set as

$$c_1 c_2 = 3 \quad \text{and} \quad a - 3 = 1,$$

that is,

$$c_1 c_2 = 3 \quad \text{and} \quad a = 4.$$

The other set of values is given by

$$3e^{-y} - e^{-5y} = 0 \cdot e^{-y} + c_1 c_2 e^{-(a-3)y}$$

and that set is

$$c_1 c_2 = -1 \quad \text{and} \quad a - 3 = 5,$$

that is,

$$c_1 c_2 = -1 \quad \text{and} \quad a = 8,$$

Thus we get two solutions

$$u_1(x, y) = 3e^{\frac{4}{4}x} \cdot e^{(3-4)y} = 3e^x \cdot e^{-y} = 3e^{x-y}$$

and

$$u_2(x, y) = -1 \cdot e^{\frac{8}{4}x} \cdot e^{(3-8)y} = -e^{2x} \cdot e^{-5y} = -e^{2x-5y}.$$

Hence the required solution is

$$\begin{aligned} u(x, y) &= u_1(x, y) + u_2(x, y) \\ &= 3e^{x-y} - e^{2x-5y}. \end{aligned}$$

3.9 BASIC PARTIAL DIFFERENTIAL EQUATIONS

(A) One-Dimensional Heat Equation

Consider the flow of heat through a homogeneous metallic thin bar with area of cross-section A and with insulated sides so that the heat flows only in one direction perpendicular to an end of the bar. We take this end as origin and direction of heat flow as positive x -axis. The temperature u at any point of the bar depends on the distance x of the point from the fixed end, taken as the origin, and time t . We further assume that the temperature at all points of same cross-section is same.

The quantity of heat flow per second across any area of cross-section A is proportional to the area A and the rate of change of temperature with respect to distance x , normal to A . Thus, if, Q_1 is the quantity of heat that flows across the cross-section A , then

$$Q_1 = -k A \left(\frac{\partial u}{\partial x} \right)_x \text{ per second,}$$

where k is the coefficient of conductivity.

We have taken negative sign on the right side because as x increases, u decreases.

Similarly, the quantity Q_2 of heat that flows per second across the cross-section at a distance $x + \delta x$ is given by

$$Q_2 = -k A \left(\frac{\partial u}{\partial x} \right)_{x+\delta x} \text{ per second.}$$

Hence the amount of heat retained per second by the slab of thickness δx is

$$Q_1 - Q_2 = kA \left[\left(\frac{\partial u}{\partial x} \right)_{x+\delta x} - \left(\frac{\partial u}{\partial x} \right)_x \right].$$

But the rate of change of heat in a solid is $s\rho A \delta x \frac{\partial u}{\partial t}$, where s is specific heat of the material, ρ is the density, $A\delta x$ is the volume, and $\frac{\partial u}{\partial t}$ is rate of change of temperature with time. Hence

$$s\rho A \delta x \frac{\partial u}{\partial t} = kA \left[\left(\frac{\partial u}{\partial x} \right)_{x+\delta x} - \left(\frac{\partial u}{\partial x} \right)_x \right]$$

or

$$\frac{\partial u}{\partial t} = \frac{k}{s\rho} \left[\left(\frac{\partial u}{\partial x} \right)_{x+\delta x} - \left(\frac{\partial u}{\partial x} \right)_x \right] / \delta x.$$

Taking limit as $\delta x \rightarrow 0$, we get

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad (108)$$

where $c^2 = \frac{k}{s\rho}$ is called *diffusivity* of the bar material. Equation (108) is called *heat flow equation* or *heat conduction equation* or *one-dimensional heat equation*.

We now solve *one-dimensional heat equation* using method of separation of variables.

EXAMPLE 3.76

Solve one dimensional *heat equation*

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad x \in (0, L),$$

with initial condition $u(x, 0) = f(x)$ and the boundary conditions $u(0, t) = 0$, $u(L, t) = 0$, $t \geq 0$.

Solution. The boundary conditions show that the temperature at both ends of the rod is kept at zero. We shall use the method of separation of variables. So let,

$$u(x, t) = T(t)X(x).$$

Then, the given heat equation reduces to

$$T'(t)X(x) = k X''(x)T(t),$$

which gives

$$\frac{T'(t)}{T(t)} = k \frac{X''(x)}{X(x)}. \quad (109)$$

The left-hand side of (109) is a function of t only whereas the right-hand side is a function of x only. Since t and x are independent variables, equality in

(109) can occur only if the left-hand side and right-hand side are both equal to a constant, say C . Thus, we have

$$\frac{T'(t)}{T(t)} = k \frac{X''(x)}{X(x)} = C \text{ (constant of separation).}$$

The boundary conditions imply that

$$u(0, t) = T(t)X(0) = 0 \quad (110)$$

and

$$u(L, t) = T(t)X(L) = 0. \quad (111)$$

Since we do not want trivial solution $T(t) = 0$, the relations (110) and (111) yield $X(0) = 0$ and $X(L) = 0$.

If $C = 0$, then $X''(x) = 0$ and so $X(x) = ax + b$. Therefore, $X(0) = 0$ implies $b = 0$ and $X(L) = 0$ implies $aL = 0$. Thus $a = b = 0$ and as such $X(x)$ is a trivial solution. Hence $C = 0$ is discarded.

Let $C \neq 0$, then the characteristic equation of $X''(x) - \frac{C}{k} X(x) = 0$ is $s^2 - \frac{C}{k} = 0$ which yields two roots s_1 and s_2 such that $s_2 = -s_1$. The general solution is, therefore,

$$X(x) = \alpha e^{s_1 x} + \beta e^{-s_1 x} \quad (112)$$

The boundary condition $X(0) = 0$ implies $\alpha + \beta = 0$ and so $\beta = -\alpha$.

The boundary condition $X(L) = 0$ implies $\alpha(e^{s_1 L} - e^{-s_1 L}) = 0$ (since $\beta = -\alpha$).

Now α cannot be zero, because $\alpha = 0$ gives $\beta = 0$ and so (112) has trivial solution. Hence $e^{s_1 L} - e^{-s_1 L} = 0$, which yields $e^{2s_1 L} = 1$ and so $s_1 = \frac{i n \pi}{L}$, where $n \neq 0$ is an integer. Thus $\frac{C}{k} = s_1^2$ yields $C = \frac{-k n^2 \pi^2}{L^2}$. Hence (111) reduces to

$$X_n(x) = \alpha \left[e^{\frac{i n \pi x}{L}} - e^{-\frac{i n \pi x}{L}} \right] = 2\alpha \left[\sin \frac{n \pi x}{L} \right].$$

Also the equation $T'(t) - CT(t) = 0$ has characteristic equation as $s - C = 0$ and so $s = C$. So the fundamental solution is

$$T_n(t) = \alpha_0 e^{-Ct} = \alpha_0 e^{-\frac{k n^2 \pi^2}{L^2} t} = \alpha_0 e^{-\left(\frac{n \pi}{L}\right)^2 k t}.$$

Hence the fundamental solution of the heat equation is given by

$$\begin{aligned} u_n(x, t) &= T_n(t)X_n(x) \\ &= A_n e^{-\left(\frac{n \pi}{L}\right)^2 k t} \sin \frac{n \pi x}{L} \end{aligned}$$

for $n = 1, 2, \dots$; $A_n = 2\alpha\alpha_0$.

By the principle of superposition (principle of adding all solutions for $n = 1, 2, 3, \dots$), we have

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{L}\right)^2 kt} \sin\left(\frac{n\pi x}{L}\right).$$

Using initial condition $u(x, 0) = f(x)$, we have

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right), \quad 0 \leq x \leq L. \quad (113)$$

Thus the coefficients A_n are nothing but the Fourier sine coefficients of the function $f(x)$ with respect to the system $\sin \frac{n\pi x}{L}$, that is,

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Substituting the value of A_n in $u(x, t)$ shall yield the solution of the given heat equation.

For example, if $u(x, 0) = x$, $0 < x < 2\pi$, then

$$A_n = \frac{1}{\pi} \int_0^{2\pi} x \sin \frac{nx}{2} dx = -\frac{4}{n} (-1)^n,$$

and so in that case the solution of the heat equation becomes

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} -\frac{4}{n} (-1)^n e^{-n^2 \pi^2 kt/L^2} \sin\left(\frac{n\pi x}{L}\right), \\ &= \sum_{n=1}^{\infty} -\frac{4}{n} (-1)^n e^{-n^2 kt/4} \sin\left(\frac{nx}{2}\right) \end{aligned}$$

since $L = 2\pi$

EXAMPLE 3.77

Solve

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 2\pi$$

with the condition

$$u(x, 0) = x^2, \quad (0, t) = u(2\pi, t) = 0$$

Solution. From Example 3.76, we have

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} A_n e^{-n^2 \pi^2 kt/L^2} \sin \frac{n\pi x}{L}, \quad L = 2\pi \\ &= \sum_{n=1}^{\infty} A_n e^{-n^2 kt/L^2} \sin \frac{nx}{2}, \end{aligned}$$

where

$$\begin{aligned} A_n &= \frac{1}{\pi} \int_0^{2\pi} x^2 \sin \frac{nx}{2} dx \\ &= \frac{-2}{\pi n} \left[x^2 \frac{nx}{2} \right]_0^{2\pi} + \frac{4}{\pi n} \int_0^{2\pi} x \cos \frac{nx}{2} dx \\ &= -\frac{8\pi}{n} (-1)^n + \frac{8}{\pi n^2} \left[x \sin \frac{nx}{2} \right]_0^{2\pi} \\ &\quad - \frac{8}{\pi n^2} \int_0^{2\pi} \sin \frac{nx}{2} dx \\ &= \frac{8}{n} (-1)^n \left[\frac{2}{\pi n^2} - \pi \right]. \end{aligned}$$

Hence

$$u(x, t) = \sum_{n=1}^{\infty} \frac{8}{n} (-1)^n \left[\frac{2}{\pi n^2} - \pi \right] e^{-n^2 kt/4} \sin \frac{nx}{2}.$$

EXAMPLE 3.78

Solve the heat conduction equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0$$

with the boundary condition $u_x(0, t) = 0$, $u_x(L, t) = 0$, $t \geq 0$ and the initial condition $u(x, 0) = f(x)$, $0 \leq x \leq L$.

Solution. The boundary conditions show that the ends of the rod are insulated. As in Example 3.76, we have

$$\frac{T'(t)}{T(t)} = k \frac{X''(x)}{X(x)} = C \quad (114)$$

Differentiating $u(x, t) = T(t) X(x)$ with respect to x , we have

$$u_x(x, t) = T(t) X'(x).$$

Therefore, the boundary value conditions yield

$$u_x(0, t) = T(t) X'(0) = 0$$

and

$$u_x(L, t) = T(t) X'(L) = 0,$$

and so

$$X'(0) = X'(L) = 0.$$

Now if $C = 0$, then (114) implies that $X''(x) = 0$ and so $X(x) = ax + b$. Therefore, $X'(x) = a$ and so

$X'(0) = a = 0$ and so $X(x) = b$. Therefore, $C = 0$ is an eigenvalue with eigenfunction a constant.

If $C \neq 0$, then the characteristic equation of $X''(x) - \frac{C}{k}X(x) = 0$ is $s^2 - \frac{C}{k} = 0$ which yield, two roots s_1 and s_2 with $s_2 = -s_1$. Therefore, the general solution is

$$X(x) = \alpha e^{s_1 x} + \beta e^{-s_1 x} \quad (115)$$

We have now from (115),

$$X'(x) = \alpha s_1 e^{s_1 x} - \beta e^{-s_1 x}.$$

So the boundary condition $X'(0) = 0$ and $X'(L) = 0$ implies

$$\alpha - \beta = 0 \text{ yielding } \alpha = \beta,$$

and

$$\alpha s_1 (e^{s_1 L} - e^{-s_1 L}) = 0.$$

If $\alpha = 0$, then $\beta = 0$, and so we will have a trivial solution. Therefore, $e^{s_1 L} - e^{-s_1 L} = 0$, which gives $e^{2s_1 L} = 1$ and so $s_1 = \frac{i n \pi}{L}$. Then $\frac{C}{k} = s_1^2$ implies $C = -\frac{k n^2 \pi^2}{L^2}$. Hence (115) becomes

$$\begin{aligned} X_n(x) &= \alpha [e^{i n \pi x / L} + e^{-i n \pi x / L}] \\ &= 2\alpha \cos \frac{n \pi x}{L}. \end{aligned}$$

Moreover (see Example 3.76), $T'(t) - CT(t) = 0$ has general solution as

$$T_n(t) = \alpha_0 e^{-n^2 \pi^2 k t / L^2}.$$

Thus, the general solution of the heat equation becomes

$$\begin{aligned} u_n(x, t) &= T_n(t) X_n(x) \\ &= \frac{A_0}{2} + A_n e^{-\frac{n^2 \pi^2 k t}{L^2}} \cos \frac{n \pi x}{L}. \end{aligned}$$

Now superposition of general solutions yield

$$u(x, t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-\frac{n^2 \pi^2 k t}{L^2}} \cos \frac{n \pi x}{L}. \quad (116)$$

The initial condition $u(x, 0) = f(x)$, $0 \leq x \leq L$ gives

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \frac{n \pi x}{L}, \quad 0 \leq x \leq L.$$

Thus, the Fourier coefficients A_n of $f(x)$ with respect to the system $\cos \frac{n \pi x}{L}$ are given by

$$A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n \pi x}{L} dx, \quad n = 0, 1, 2, \dots$$

Substituting the value of A_n in (116) will yield the solution of the given heat conduction equation.

EXAMPLE 3.79

An insulated rod of length l has its ends A and B maintained at 0°C and 100°C , respectively, until steady state condition prevails. If B is suddenly reduced to 0°C and maintained at 0°C , find the temperature at a distance x from A at time t .

Solution. The heat flow equation is

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

Prior to the sudden change of temperature at the end B, the temperature u depends only upon x and not on t . Hence the equation of heat flow is $\frac{\partial^2 u}{\partial x^2} = 0$, whose solution is $u = ax + b$. Since $u = 0$ for $x = 0$ and $u = 100$ for $x = l$, we get $0 = b$ and $100 = al$ or $a = \frac{100}{l}$. Thus $u(x) = \frac{100}{l}x$. Hence $u(x) = \frac{100}{l}x$ gives the temperature at $t = 0$, that is, $u(x, 0) = \frac{100}{l}x$ is the initial condition. The boundary conditions for the subsequent flow are $u(0, t) = 0$ and $u(l, t) = 0$ for all values of t . Therefore, by Example 3.76, the solution of the problem is

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi}{l})^2 c^2 t} \sin \frac{n \pi x}{l},$$

where

$$A_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n \pi x}{l} dx.$$

Here $f(x) = \frac{100x}{l}$. Therefore,

$$\begin{aligned} A_n &= \frac{2}{l} \int_0^l \frac{100}{l} x \sin \frac{n \pi x}{l} dx \\ &= \frac{200}{l^2} \int_0^l x \sin \frac{n \pi x}{l} dx \\ &= \frac{200}{l^2} \left(-\frac{l^2}{n \pi} \cos n \pi \right), \text{ using integration by parts} \\ &= \frac{200}{n \pi} (-1)^{n+1}, \text{ since } \cos n \pi = (-1)^n. \end{aligned}$$

Hence

$$u(x, t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} e^{-\frac{c^2 n^2 \pi^2 t}{l^2}}.$$

EXAMPLE 3.80

A bar 10 cm long with insulated sides has its ends A and B maintained at temperature 50°C and 100°C, respectively, until steady state condition prevails. The temperature at A is suddenly raised to 90°C and at the same time lowered to 60°C at B. Find the temperature distribution in the bar at time t .

Solution. Prior to sudden change, the temperature distribution is described by $\frac{\partial^2 u}{\partial x^2} = 0$, whose solution is

$$u = ax + b.$$

Since $u = 50^\circ\text{C}$ for $x = 0$ and $u = 100^\circ\text{C}$ for $x = 10$ cm, we get $50 = b$ and $100 = 10a + b = 10a + 50$ and so $a = 5$. Hence the initial temperature distribution in the rod is $u(x, 0) = 5x + 50$. Similarly, taking $u = 90^\circ\text{C}$ for $x = 0$ and $u = 60^\circ\text{C}$ for $x = 10$ the final temperature distribution in the rod is $u_s(x, 0) = -3x + 90$. We want to find the temperature distribution during the intermediate period, measuring time from the instant when the end temperature was changed. Let

$$u(x, t) = u_s(x) + u_t(x, t),$$

where $u_s(x)$ is the steady state temperature and $u_t(x, t)$ is the transient temperature distribution which decreases as time increases. Thus

$$u_s(x) = -3x + 90,$$

and $u_t(x, t)$ satisfies the heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

Hence the solution is

$$u(x, t) = -3x + 90 + \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} e^{-\frac{c^2 n^2 \pi^2 t}{l^2}}.$$

Here $l = 10$ and A_n are determined by

$$A_n = \frac{2}{10} \int_0^{10} f(x) \sin \frac{n\pi x}{10} dx,$$

where $f(x) = u_t(x, 0) = u(x, 0) - u_s(x, 0) = 5x + 50 - (-3x + 90) = 8x - 40$.

Thus

$$\begin{aligned} A_n &= \frac{1}{5} \int_0^{10} (8x - 40) \sin \frac{n\pi x}{10} dx \\ &= \frac{1}{5} \left[(8x - 40) \left(-\frac{10}{n\pi} \cos \frac{n\pi x}{10} \right) \right. \\ &\quad \left. - 8 \left(-\frac{100}{n^2 \pi^2} \sin \frac{n\pi x}{10} \right) \right]_0^{10} \\ &= \frac{1}{5} \left[-\frac{400}{n\pi} \cos n\pi - \frac{400}{n\pi} \right] \\ &= -\frac{80}{n\pi} (\cos n\pi + 1) \\ &= \begin{cases} 0 & \text{if } n \text{ is odd} \\ -\frac{160}{n\pi} & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} u(x, t) &= -3x + 90 - \frac{160}{\pi} \sum_{\text{even}} \frac{1}{n} e^{-\frac{c^2 n^2 \pi^2 t}{100}} \sin \frac{n\pi x}{10} \\ &= -3x + 90 - \frac{160}{\pi} \sum_{m=1}^{\infty} \frac{1}{2m} e^{-\frac{c^2 m^2 \pi^2 t}{25}} \sin \frac{m\pi x}{5} \\ &= -3x + 90 - \frac{80}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} e^{-\frac{c^2 m^2 \pi^2 t}{25}} \sin \frac{m\pi x}{5}. \end{aligned}$$

EXAMPLE 3.81

A rod of length l with insulated side is initially at a uniform temperature u_0 . Its ends are suddenly cooled at 0°C and kept at that temperature. Find the temperature function $u(x, t)$.

Solution. We have to solve the heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

under the initial condition $u(x, 0) = u_0$, $u(0, t) = 0$, $u(l, t) = 0$ because both ends are kept at zero temperature. Therefore, by Example 3.76 the solution is

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-\frac{c^2 n^2 \pi^2 t}{l^2}} \sin \frac{n\pi x}{l},$$

where

$$\begin{aligned} A_n &= \frac{2}{l} \int_0^l u(x, 0) \sin \frac{n\pi x}{l} dx \\ &= \frac{2}{l} \int_0^l u_0 \sin \frac{n\pi x}{l} dx \\ &= \begin{cases} 0 & \text{for even } n \\ \frac{4u_0}{n\pi} & \text{for odd } n. \end{cases} \end{aligned}$$

Hence the temperature distribution is given by

$$\begin{aligned} u(x, t) &= \frac{4u_0}{\pi} \sum_{\text{odd } n} \frac{1}{n} e^{-\frac{c^2 n^2 \pi^2 t}{l^2}} \cdot \sin \frac{n\pi x}{l} \\ &= \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} e^{-\frac{c^2 (2n-1)^2 \pi^2 t}{l^2}} \cdot \sin \frac{(2n-1)\pi x}{l}. \end{aligned}$$

EXAMPLE 3.82

Find the temperature $u(x, t)$ in a bar which is perfectly insulated laterally, whose ends are kept at temperature 0°C and whose initial temperature is $f(x) = x(100 - x)$, given that its length is 10 cm, constant cross-section of area 1 cm^2 , density 10.6 gm/cm^3 , thermal conductivity $1.04 \text{ cal/cm deg sec}$, and specific heat 0.056 cal/gm deg .

Solution. As in Example 3.76, the temperature distribution is given by

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-\frac{c^2 n^2 \pi^2 t}{l^2}} \cdot \sin \frac{n\pi x}{l}.$$

Here $l = 10 \text{ cm}$, $c^2 = \frac{k}{s\rho} = 1.75$ and A_n are given by

$$\begin{aligned} A_n &= \frac{2}{l} \int_0^l u(x, 0) \sin \frac{n\pi x}{l} dx \\ &= \frac{1}{5} \int_0^{100} x(100 - x) \sin \frac{n\pi x}{l} dx \\ &= \frac{400}{n^3 \pi^3} [1 - (-1)^n] \\ &= \begin{cases} 0 & \text{for even } n \\ \frac{800}{n^3 \pi^3} & \text{for odd } n. \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} u(x, y) &= \frac{800}{\pi^3} \sum_{\text{odd } n} \frac{1}{n^3} e^{-0.0175 n^2 \pi^2 t} \cdot \sin \frac{n\pi x}{10} \\ &= \frac{800}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} e^{-0.0175 (2n-1)^2 \pi^2 t} \\ &\quad \sin \frac{(2n-1)\pi x}{10}. \end{aligned}$$

(B) One-dimensional Wave Equation

Consider a uniform elastic string of length L stretched tightly with its ends fixed on the x -axis at $x = 0$ and $x = L$. Let for each x in the interval $0 < x < L$, the string is displaced into the xy -plane and let for each x , the displacement from the x -axis be given by $f(x)$, where f is a function of x (Fig. 3.1).

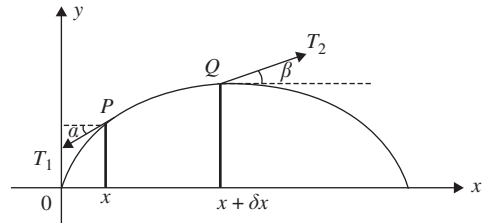


Figure 3.1

We assume that the string perfectly flexible, is of constant linear density ρ and of constant tension T at all times. We also assume that the motion takes place entirely in the xy -plane and that each point on the string moves on a straight line perpendicular to the x -axis as the string vibrates. The displacement y at each point of the string and the slope $\frac{\partial y}{\partial x}$ are small compared to the length L . Further, no external force acts upon the string during motion and angle between the string and the x -axis at each point is sufficiently small.

Let m be the mass per unit length of the string and let PQ be an element of length δs of the string. Then mass of PQ is $m \delta s$. The tensions T_1 and T_2 at P and Q are respectively tangential to the curve (Figure 3.1) making angles α and β with the horizontal direction.

Since there is no motion in the horizontal direction, we have

$$T_1 \cos \alpha = T_2 \cos \beta = \text{constant} = T, \text{ say.} \quad (117)$$

By Newton's Second Law of Motion, the equation of motion in the vertical direction is

$$m\delta s \frac{\partial^2 y}{\partial t^2} = T_2 \sin \beta - T_1 \sin \alpha$$

or

$$\frac{m\delta s \frac{\partial^2 y}{\partial t^2}}{T} = \frac{T_2 \sin \beta}{T} - \frac{T_1 \sin \alpha}{T}$$

Using (117), we get

$$\begin{aligned} \frac{m\delta s \frac{\partial^2 y}{\partial t^2}}{T} &= \frac{T_2 \sin \beta}{T_2 \cos \beta} - \frac{T_1 \sin \alpha}{T_1 \cos \alpha} \\ &= \tan \beta - \tan \alpha \end{aligned}$$

or

$$\begin{aligned} \frac{\partial^2 y}{\partial t^2} &= \frac{T}{m\delta s} [\tan \beta - \tan \alpha] \\ &= \frac{T}{m\delta x} [\tan \beta - \tan \alpha], \delta s \approx \delta x \\ &= \frac{T}{m} \left[\frac{\left(\frac{\partial y}{\partial x} \right)_{x+\delta x} - \left(\frac{\partial y}{\partial x} \right)_x}{\delta x} \right] \\ &= \frac{T}{m} \frac{\partial^2 y}{\partial x^2} \text{ in the limit as } \delta x \rightarrow 0. \end{aligned}$$

Hence

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \quad (118)$$

where $a^2 = T/m$ is a constant related to tension in the vibrating string of length L having fixed ends. The boundary conditions are

$$y(0, t) = y(L, t) = 0, \quad t \geq 0,$$

and initial conditions are

$$\begin{aligned} y(x, 0) &= f(x), \quad 0 \leq x \leq L \\ y_x(x, 0) &= 0, \quad 0 \leq x \leq L. \end{aligned}$$

The equation (118) is called *one-dimensional wave equation*. We find its solution in the following Example 3.83.

EXAMPLE 3.83

Solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0,$$

where a is a constant related to tension in the vibrating string of length L having fixed ends. The boundary conditions and initial conditions are

$$u(0, t) = u(L, t) = 0, \quad t \geq 0$$

$$u(x, 0) = f(x), \quad 0 \leq x \leq L$$

$$u_t(x, 0) = 0, \quad 0 \leq x \leq L.$$

Solution. Let

$$u(x, t) = T(t)X(x)$$

Then the wave equation takes the form

$$T''(t)X(x) = a^2 X''(x)T(t)$$

or

$$\begin{aligned} \frac{T''(t)}{T(t)} &= a^2 \frac{X''(x)}{X(x)} = C \\ &\text{(constant of separation).} \end{aligned}$$

The boundary conditions $u(0, t) = u(L, t) = 0, t \geq 0$ imply $X(0) = 0$ and $X(L) = 0$. Further, the condition $u_t(x, 0) = 0$ gives $T'(0)X(x) = 0, 0 \leq x \leq L$. Therefore, $T'(0) = 0$.

The auxiliary equation for $X(x)$ is $s^2 - \frac{C}{a^2} = 0$ which yields $s_2 = -s_1$ as the two roots. Therefore, the fundamental (general) solution is

$$X(x) = \alpha e^{s_1 x} + \beta e^{-s_1 x}. \quad (119)$$

The boundary condition $X(0) = 0$ and $X(L) = 0$ gives $\alpha + \beta = 0$, that is, $\beta = -\alpha$ and

$$0 = \alpha e^{s_1 L} + \beta e^{-s_1 L} = \alpha(e^{s_1 L} - e^{-s_1 L}).$$

Since $\alpha = 0$ implies $\beta = 0$ and the solution then becomes trivial, so $\alpha \neq 0$ and thus $e^{s_1 L} = e^{-s_1 L}$ and so $e^{2s_1 L} = 1$. Thus $s_1 = \frac{i n \pi}{L}$, $n \neq 0$ being an integer. Therefore, $s_1^2 = \frac{C}{a^2}$ yields $C = -\frac{a^2 n^2 \pi^2}{L^2}$. The fundamental solution (119) now takes the form

$$\begin{aligned} X_n(x) &= \alpha [e^{i n \pi x / L} - e^{-i n \pi x / L}] \\ &= 2i \alpha \sin \frac{n \pi x}{L}. \end{aligned}$$

Further, characteristic equation for $T''(t) - CT(t) = 0$ is $s^2 - C = 0$, that is, $s^2 + \frac{a^2 n^2 \pi^2}{L^2} = 0$. Thus the fundamental solution is

$$T_n(t) = \alpha \cos\left(\frac{n \pi a t}{L}\right) + \beta \sin\left(\frac{n \pi a t}{L}\right).$$

We have

$$T'(t) = \frac{n \pi a}{L} \left(-\alpha \sin \frac{n \pi a t}{L} + \beta \cos \frac{n \pi a t}{L} \right).$$

The condition $T'(0) = 0$ yields $\beta = 0$ and so

$$T_n(t) = \alpha \cos\left(\frac{n\pi\alpha t}{L}\right), \quad n = 1, 2, 3, \dots$$

Hence the fundamental solution for the wave equation is

$$\begin{aligned} u_n(x, t) &= T_n(t)X_n(x) \\ &= A_n \cos \frac{n\pi\alpha t}{L} \sin \frac{n\pi x}{L}, \quad n = 1, 2, \dots; \quad A_n = 2\alpha\alpha_0. \end{aligned}$$

Superposition of the fundamental solutions implies

$$u(x, t) = \sum_{n=1}^{\infty} A_n \cos \frac{n\pi\alpha t}{L} \sin \frac{n\pi x}{L}. \quad (120)$$

Using initial condition $u(x, 0) = f(x)$, we have

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}, \quad 0 \leq x \leq L.$$

The Fourier coefficients A_n of $f(x)$ with respect to the system $\sin \frac{n\pi x}{L}$ is given by

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

Putting the values of A_n in (120), we get the solution of the wave equation.

EXAMPLE 3.84

A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape defined by $f(x) = \mu x(l - x)$, where μ is a constant and then released. Find the displacement of any point x of the string at any time $t > 0$.

Solution. By Example 3.83, we have

$$y(x, t) = \sum A_n \cos \frac{n\pi\alpha t}{l} \sin \frac{n\pi x}{l},$$

where

$$A_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx.$$

Here $f(x) = y(x, 0) = \mu x(l - x)$. Hence integrating by part, we have

$$\begin{aligned} A_n &= \frac{2}{l} \int_0^l \mu x(l - x) \sin \frac{n\pi x}{l} dx \\ &= \frac{2\mu}{l} \left[-\frac{2l^3}{n^3\pi^3} \cos n\pi + \frac{2l^3}{n^3\pi^3} \right] \\ &= \frac{4\mu}{n^3\pi^3} l^2 [1 - (-1)^n] \\ &= \begin{cases} \frac{8\mu}{n^3\pi^3} l^2 & \text{for odd } n \\ 0 & \text{for even } n \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} y(x, t) &= \frac{8\mu l^2}{\pi^3} \sum_{\text{odd } n} \frac{1}{n^3} \cos \frac{n\pi\alpha t}{l} \sin \frac{n\pi x}{l} \\ &= \frac{8\mu l^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \cos \frac{(2n-1)\pi\alpha t}{l} \\ &\quad \times \sin \frac{(2n-1)\pi x}{l}. \end{aligned}$$

EXAMPLE 3.85

A string is stretched and fastened to two points, l distant apart. Motion is started by displacing the string in the form $y = a \sin \frac{\pi x}{l}$ from which it is released at time $t = 0$. Show that the displacement of any point of the string at a distance x from one end at any time t is given by

$$y(x, t) = a \sin \frac{\pi x}{l} \cos \frac{\pi c t}{l}.$$

Solution. By Example 3.83,

$$y(x, t) = \sum_{n=1}^{\infty} A_n \cos \frac{n\pi c t}{l} \sin \frac{n\pi x}{l},$$

where

$$A_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx,$$

and $f(x) = u(x, 0)$. Here

$$u(x, 0) = a \sin \frac{\pi x}{l}.$$

Therefore,

$$\begin{aligned} A_n &= \frac{2}{l} \int_0^l a \sin \frac{\pi x}{l} \sin \frac{n\pi x}{l} dx \\ &= \frac{2a}{l} \int_0^l \sin \frac{\pi x}{l} \sin \frac{n\pi x}{l} dx, \end{aligned}$$

which vanishes for all values of n except $n = 1$. Therefore,

$$\begin{aligned} A_1 &= \frac{2a}{l} \int_0^l \sin^2 \frac{\pi x}{l} dx = \frac{a}{l} \int_0^l (1 - \cos \frac{2\pi x}{l}) dx \\ &= \frac{a}{l} \left[x - \frac{l}{2\pi} \sin \frac{2\pi x}{l} \right]_0^l = \frac{a}{l} (l) = a. \end{aligned}$$

Hence

$$y(x, t) = A_1 \cos \frac{\pi ct}{l} \sin \frac{\pi x}{l} = a \cos \frac{\pi ct}{l} \sin \frac{\pi x}{l}.$$

EXAMPLE 3.86

Show that the solution of the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ can be expressed in the form

$$y(x, t) = \phi(x + ct) + \psi(x - ct). \quad (121)$$

If $y(x, 0) = f(x)$ and $\frac{\partial y}{\partial t}(x, 0) = 0$, show that $y(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)]$ [The solution (121) is called the *D'Alembert's solution of the wave equation*].

Solution. Put

$$u = x + ct \quad \text{and} \quad v = x - ct$$

so that $\frac{\partial u}{\partial x} = 1$ and $\frac{\partial v}{\partial x} = 1$, $\frac{\partial u}{\partial t} = c$, $\frac{\partial v}{\partial t} = -c$.

Then y becomes a function of u and v and we have

$$\text{and} \quad \frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial y}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}$$

$$\begin{aligned} \frac{\partial^2 y}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \right) \\ &= \frac{\partial}{\partial u} \left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \right) + \frac{\partial}{\partial v} \left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \right) \\ &= \frac{\partial^2 y}{\partial u^2} + 2 \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}, \\ &\quad \text{supposing } \frac{\partial^2 y}{\partial u \partial v} = \frac{\partial^2 y}{\partial v \partial u}. \end{aligned}$$

Similarly,

$$\frac{\partial^2 y}{\partial t^2} = c^2 \left[\frac{\partial^2 y}{\partial u^2} - 2 \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \right].$$

Substituting these values of $\frac{\partial^2 y}{\partial x^2}$ and $\frac{\partial^2 y}{\partial t^2}$ in the wave equation, we get

$$\frac{\partial^2 y}{\partial u \partial v} = 0.$$

Integrating with respect to v , we get

$$\frac{\partial y}{\partial u} = f(u), \text{ where } f \text{ is an arbitrary function of } u.$$

Now integrating w.r.t u , we have

$$y = \int f(u) du + \psi(v), \text{ where } \psi(v)$$

is an arbitrary function of v

$$= \phi(u) + \psi(v), \text{ say,}$$

since $\int f(u) du$ is a function of u only.

Thus

$$y = \phi(x + ct) + \psi(x - ct). \quad (122)$$

Now let initially $y(x, 0) = f(x)$ and $\frac{\partial y}{\partial t}(x, 0) = 0$. Differentiating (122) w.r.t. t , we get

$$\frac{\partial y}{\partial t} = c\phi'(x + ct) - c\psi'(x - ct).$$

But at $t = 0$, $\phi'(x) = \psi'(x)$, and $y(x, 0) = \phi(x) + \psi(x) = f(x)$. Thus, we have $\phi(x) = \psi(x) + k$. Therefore,

$$2\psi(x) + k = f(x)$$

or

$$\psi(x) = \frac{1}{2}[f(x) - k] \text{ and then } \phi(x) = \frac{1}{2}[f(x) + k].$$

Hence the solution becomes

$$\begin{aligned} y(x, t) &= \frac{1}{2}[f(x + ct) + k] + \frac{1}{2}[f(x - ct) - k] \\ &= \frac{1}{2}[f(x + ct) + f(x - ct)]. \end{aligned}$$

EXAMPLE 3.87

Find the deflection of a vibrating string of unit length having fixed ends with initial velocity zero and initial deflection $f(x) = k(\sin x - \sin 2x)$.

Solution. We are given that

$$y(x, 0) = k(\sin x - \sin 2x) = f(x).$$

By D'Alembert method, the solution is

$$\begin{aligned} y(x, t) &= \frac{1}{2} [f(x + ct) + f(x - ct)] \\ &= \frac{1}{2} [k\{\sin(x + ct) - \sin 2(x + ct)\} \\ &\quad + k\{\sin(x - ct) - \sin 2(x - ct)\}] \\ &= k[\sin x \cos ct - \sin 2x \cos 2ct] \end{aligned}$$

Since $y(x, 0) = k(\sin x - \sin 2x)$; we have

$$\begin{aligned} \frac{\partial y}{\partial t}(x, 0) &= k[-c \sin x \sin ct + 2c \sin 2x \sin 2ct]_{t=0} \\ &= 0. \end{aligned}$$

Thus the given boundary conditions are satisfied.

(C) Two-Dimensional Heat Equation

Consider the flow of heat in a metal plate in the XOY plane. Suppose that the temperature at any point of the plate depends on x , y , and t (time) and not on the z -coordinate. Then this type of flow of heat is known as *two-dimensional heat flow*. It lies entirely in XOY plane and is zero along the normal to the plane XOY. The equation governing this type of flow is

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),$$

where $c^2 = \frac{k}{s\rho}$, s is specific heat and ρ is the density of the metal plate.

In steady state, u is independent of time and so $\frac{\partial u}{\partial t} = 0$. Therefore, the above equation transforms to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (123)$$

which is known as *two-dimensional Laplace equation*.

We now find the solution of two dimensional Laplace equation by separation of variable method. Since there are two independent variables, let

$$u(x, y) = X(x)Y(y). \quad (124)$$

Then using (124), the equation (123) reduces to

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

Therefore, separation of variables yields

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = k(\text{separation parameter}).$$

Hence we get two differential equations

$$\frac{X''(x)}{X(x)} - kX(x) = 0 \quad (125)$$

and

$$Y''(y) + kY(y) = 0. \quad (126)$$

Now the following three cases are to be considered:

Case I. If $k > 0$, then $k = p^2$, where p is real. In this case the equations (125) and (126) take the form

$$\frac{d^2 X}{dx^2} - p^2 X = 0 \quad (127)$$

and

$$\frac{d^2 Y}{dy^2} + p^2 Y = 0. \quad (128)$$

The auxiliary equation for (127) is $m^2 - p^2 = 0$ and so $m = \pm p$. Thus the solution of (127) is

$$X(x) = c_1 e^{px} + c_2 e^{-px}.$$

The auxiliary equation for (128) is $m^2 + p^2 = 0$ and so $m = \pm pi$. Hence the solution of (128) is

$$Y(y) = c_3 \cos py + c_4 \sin py.$$

Thus, the solution of the Laplace equations in this case is

$$\begin{aligned} u(x, y) &= X(x)Y(y) \\ &= (c_1 e^{px} + c_2 e^{-px}) \\ &\quad \times (c_3 \cos py + c_4 \sin py). \end{aligned} \quad (129)$$

Case II. If $k = 0$, then equations (125) and (126) reduce to

$$\frac{d^2 X}{dx^2} = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} = 0,$$

whose solutions are

$$X(x) = c_5 x + c_6 \quad \text{and} \quad Y(y) = c_7 y + c_8.$$

Thus the solution of the Laplace equation is of the form

$$u(x, y) = (c_5 x + c_6) (c_7 y + c_8). \quad (130)$$

Case III. Let $k < 0$. Then $k = -p^2$. Then equations (125) and (126) reduce to

$$\frac{d^2 X}{dx^2} + p^2 X = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} - p^2 Y = 0,$$

whose solutions are, respectively

$$X(x) = c_9 \cos px + c_{10} \sin px$$

and

$$Y(y) = c_{11} e^{py} + c_{12} e^{-py}.$$

Thus the solution of the Laplace equation in this case is

$$u(x, y) = (c_9 \cos px + c_{10} \sin px) (c_{11} e^{py} + c_{12} e^{-py}). \quad (131)$$

In all these cases c_i , $i = 1, 2, \dots, 12$ are constant of integration and are calculated using the boundary conditions.

EXAMPLE 3.88

Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, subject to the conditions $u(x, 0) = 0$, $u(x, a) = 0$, $u(x, y) \rightarrow 0$ as $x \rightarrow \infty$ when $x \geq 0$ and $0 \leq y \leq a$.

Solution. The suitable method of separation of variables in this case is

$$u(x, y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py).$$

Since $u(x, y) \rightarrow 0$ as $x \rightarrow \infty$, we have $c_1 = 0$ for all y . Therefore,

$$u(x, y) = c_2 e^{-px} [c_3 \cos py + c_4 \sin py].$$

Since $u(x, 0) = 0$, we obtain $0 = c_2 c_3 e^{-px}$ and so $c_3 = 0$ since $c_2 \neq 0 \neq e^{-px}$ for all x . Therefore,

$$u(x, y) = c_2 c_4 e^{-px} \sin py.$$

Now using $u(x, a) = 0$, we get $0 = c_2 c_4 e^{-px} \sin pa$ and so $\sin pa = 0$, that is, $\sin pa = \sin n\pi$. Hence $pa = n\pi$ or $p = \frac{n\pi}{a}$, $n = 0, \pm 1, \pm 2, \dots$. Therefore,

$$u_n(x, y) = A_n e^{-\frac{n\pi x}{a}} \sin \frac{n\pi y}{a},$$

$$n = 0, \pm 1, \pm 2, \dots$$

Using principles of superposition, the solution of the wave equation is

$$u(x, y) = \sum A_n e^{-\frac{n\pi x}{a}} \sin \frac{n\pi y}{a}, \quad A_n \text{ constant.}$$

EXAMPLE 3.89

A rectangular plate with insulated surface is 10 cm wide and so long compared to its width that it may be considered in length without introducing an appreciable error. If the temperature of the short edge $y = 0$ is given by $u = 20$ for $0 \leq x \leq 5$ and $u = 20(10 - x)$ for $5 \leq x \leq 10$ and the two long edges $x = 0$, $x = 10$ as well as the other short edge are kept at 0°C , show that the temperature u at any point (x, y) is given by

$$u = \frac{800}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} e^{-\frac{(2n-1)\pi y}{10}} \cdot \sin \frac{(2n-1)\pi x}{10}.$$

Solution. To find the temperature $u(x, t)$, we have to solve the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

subject to the conditions

$$u(0, y) = 0, \quad y \geq 0$$

$$u(10, y) = 0, \quad y \geq 0$$

$$u(x, \infty) = 0, \quad 0 \leq x \leq 10$$

$$u(x, 0) = \begin{cases} 20x & 0 \leq x \leq 5 \\ 20(10 - x) & 5 \leq x \leq 10. \end{cases}$$

We know that three possible solutions of Laplace equations are

$$(i) \quad u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$$

$$(ii) \quad u = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$$

$$(iii) \quad u = (c_1 x + c_2)(c_3 x + c_4).$$

Now solution (i) is not suitable because for $x = 0$, $u \neq 0$. The solution (iii) is also not suitable because it does not satisfy $u(x, \infty) = 0$ in $0 \leq x \leq 10$. Thus only (ii) is possible. Since $u(0, y) = 0$, (ii) yields

$$0 = c_1(c_3 e^{py} + c_4 e^{-py})$$

and so $c_1 = 0$. Thus

$$u(x, y) = c_2 \sin px (c_3 e^{py} + c_4 e^{-py}).$$

Since $u(10, y) = 0$, we have

$$0 = c_2 \sin 10 p (c_3 e^{py} + c_4 e^{-py})$$

and $\sin 10 p = 0 = \sin n\pi$. Thus $p = \frac{n\pi}{10}$, $n = 0, \pm 1, \pm 2, \dots$. Also $u(x, \infty) = 0$. Therefore $c_3 = 0$. Hence

the solution is

$$\begin{aligned} u_n(x, y) &= c_2 c_4 e^{-\frac{n\pi y}{10}} \sin \frac{n\pi x}{10} \\ &= A_n e^{-\frac{n\pi y}{10}} \sin \frac{n\pi x}{10}. \end{aligned}$$

Hence the general solution is

$$u(x, y) = \sum_{n=1}^{\infty} A_n e^{-\frac{n\pi y}{10}} \sin \frac{n\pi x}{10}$$

Now

$$u(x, 0) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{10}.$$

Therefore, A_n is Fourier sine coefficient of $u(x, 0)$.
Thus

$$\begin{aligned} A_n &= \frac{2}{l} \int_0^l u(x, 0) \sin \frac{n\pi x}{10} dx \\ &= \frac{1}{5} \left(\int_0^5 20 \sin \frac{n\pi x}{10} dx + \int_5^{10} 20(10-x) \sin \frac{n\pi x}{10} dx \right) \\ &= \frac{800}{n^2 \pi^2} \sin \frac{n\pi}{2} \\ &= \begin{cases} 0 & \text{for even } n \\ \frac{800}{n^2 \pi^2} (-1)^{\frac{n-1}{2}} & \text{for odd } n. \end{cases} \end{aligned}$$

Hence the required solution is

$$\begin{aligned} u(x, y) &= \frac{800}{\pi^2} \sum_{\text{odd } n} \frac{(-1)^{\frac{n-1}{2}}}{n^2} e^{-\frac{n\pi y}{10}} \sin \frac{n\pi x}{10} \\ &= \frac{800}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} e^{-\frac{(2n-1)\pi y}{10}} \sin \frac{(2n-1)\pi x}{10}. \end{aligned}$$

EXAMPLE 3.90

Solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

subject to the conditions $u(0, y) = u(l, y) = u(x, 0) = 0$,
 $u(x, a) = \sin \frac{n\pi x}{l}$.

Solution. As per the given conditions, out of the three solutions of the Laplace equation, the solution

$$u(x, y) = (c_1 \cos px + c_2 \sin px) (c_3 e^{py} + c_4 e^{-py})$$

is suitable. The boundary condition $u(0, y) = 0$ implies $c_1 = 0$. Therefore,

$$u(x, y) = c_2 \sin px (c_3 e^{py} + c_4 e^{-py}).$$

Now using the condition $u(l, y) = 0$ gives

$$0 = c_2 \sin pl (c_3 e^{py} + c_4 e^{-py})$$

and so

$$\sin pl = 0 = \sin n\pi, \quad n = 0, \pm 1, \pm 2, \dots$$

giving $p = \frac{n\pi}{l}$. Thus the solution becomes

$$u(x, y) = c_2 \sin \frac{n\pi x}{l} (c_3 e^{n\pi y/l} + c_4 e^{-n\pi y/l}).$$

Now the use of condition $u(x, 0) = 0$ gives

$$0 = c_2 \sin \frac{n\pi x}{l} (c_3 + c_4)$$

and so $c_3 + c_4 = 0$ or $c_4 = -c_3$. Thus the solution reduces to

$$\begin{aligned} u(x, y) &= c_2 c_3 \sin \frac{n\pi x}{l} (e^{n\pi y/l} - e^{-n\pi y/l}) \\ &= A_n \sin \frac{n\pi x}{l} \sinh \frac{n\pi y}{l}. \end{aligned}$$

Now the last condition $u(x, a) = \sin \frac{n\pi x}{l}$ yields

$$\sin \frac{n\pi x}{l} = A_n \sin \frac{n\pi x}{l} \sinh \frac{n\pi a}{l}$$

or

$$A_n = \frac{1}{\sinh \frac{n\pi a}{l}}.$$

Hence

$$u(x, y) = \frac{\sin n\pi x}{l} \cdot \frac{\sin n\pi y/l}{\sin n\pi a/l}.$$

EXAMPLE 3.91

Solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \leq x \leq a, \quad 0 \leq y \leq b$$

subject to the conditions

$$u(0, y) = u(a, y) = u(x, b) = 0, \quad u(x, 0) = x(a - x).$$

Solution. Three possible solutions to the given equation are

- (i) $u(x, y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$
- (ii) $u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$
- (iii) $(c_1 x + c_2)(c_3 y + c_4).$

Since $u \neq 0$ for $x = 0$, the solution (i) does not satisfy $u(0, y) = 0$. The solution (iii) does not satisfy $u(x, 0) = a(a - x)$. Hence solution (ii) is suitable. Now $u(0, y) = 0$ implies $0 = c_1(c_3 e^{py} + c_4 e^{-py})$ and so $c_1 = 0$. Thus

$$u(x, y) = c_2 \sin px (c_3 e^{py} + c_4 e^{-py}).$$

Now $u(a, y) = 0$ implies $0 = c_2 \sin pa (c_3 e^{py} + c_4 e^{-py})$ and so $\sin pa = 0 = \sin n\pi$, $n = 0, \pm 1, \pm 2, \dots$. Thus $p = \frac{n\pi}{a}$, $n = 0, \pm 1, \pm 2, \dots$. Further $u(x, b) = 0$ implies

$$0 = c_2 \sin \frac{n\pi x}{a} (c_3 e^{n\pi b/a} + c_4 e^{-n\pi b/a})$$

and so

$$\text{either } c_4 = \frac{c_3 e^{n\pi b/a}}{e^{-n\pi b/a}} \text{ or } c_3 = -\frac{c_4 e^{-n\pi b/a}}{e^{n\pi b/a}}.$$

For the first case,

$$\begin{aligned} u_1(x, y) &= c_2 \sin \frac{n\pi x}{a} \left(c_3 e^{\frac{n\pi y}{a}} - \frac{c_3 e^{n\pi b/a}}{e^{-n\pi b/a}} e^{-n\pi y/a} \right) \\ &= c_2 c_3 \sin \frac{n\pi x}{a} \left(e^{\frac{n\pi y}{a}} - \frac{e^{n\pi b/a}}{e^{-n\pi b/a}} e^{-\frac{n\pi y}{a}} \right) \\ &= 2c_2 c_3 \sin \frac{n\pi x}{a} \left[\frac{e^{\frac{n\pi y}{a}} e^{-\frac{n\pi b}{a}} - e^{\frac{n\pi b}{a}} e^{-\frac{n\pi y}{a}}}{2e^{-\frac{n\pi b}{a}}} \right] \\ &= 2c_2 c_3 \sin \frac{n\pi x}{a} \frac{(\sinh \frac{n\pi}{a} (b - y))}{e^{-n\pi b/a}} \end{aligned}$$

Similarly, for the second case,

$$u_2(x, y) = 2c_2 c_4 \sin \frac{n\pi x}{a} \left[-\frac{\sinh \frac{n\pi}{a} (b - y)}{e^{n\pi b/a}} \right].$$

Therefore, the solution is

$$\begin{aligned} u(x, y) &= 2c_2 c_3 u_1(x, y) - 2c_2 c_4 u_2(x, y) \\ &= A_n \sin \frac{n\pi x}{a} \frac{\sinh \frac{n\pi}{a} (b - y)}{\sinh \frac{n\pi b}{a}}. \end{aligned}$$

But $u(x, 0) = a - x$. Thus

$$x(a - x) = \sum A_n \frac{\sin n\pi x}{a}.$$

Therefore,

$$\begin{aligned} A_n &= \frac{2}{a} \int_0^a x(a - x) \sin \frac{n\pi x}{a} dx \\ &= \frac{2}{a} \left[a(ax - x^2) \left(-\frac{\cos \frac{n\pi x}{a}}{n\pi} \right) \right. \\ &\quad \left. - (a - 2x)a^2 \left(-\frac{\sin \frac{n\pi x}{a}}{n^2 \pi^2} \right) \right. \\ &\quad \left. + (-2)a^3 \left(\frac{\cos \frac{n\pi x}{a}}{n^3 \pi^3} \right) \right]_0^a \\ &= \frac{2}{a} \left[0 - 0 - \frac{2a^3}{n^3 \pi^3} \cos \frac{n\pi a}{a} + \frac{2a^3}{n^3 \pi^3} \right] \\ &= \frac{4a^2}{n^3 \pi^3} [1 - (-1)^n] \\ &= \begin{cases} 0 & \text{for even } n \\ \frac{8a^2}{n^3 \pi^2} & \text{for odd } n. \end{cases} \end{aligned}$$

Hence the solution becomes

$$\begin{aligned} u(x, y) &= \frac{8a^2}{\pi^3} \sum_{\text{odd } n} \frac{1}{n^3} \sin \frac{n\pi x}{a} \cdot \frac{\sinh n\pi(b - y)/a}{\sinh n\pi b/a} \\ &= \frac{8a^2}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin \frac{(2n+1)\pi x}{a} \\ &\quad \times \frac{\sinh \frac{(2n+1)\pi}{a} (b - y)}{\sinh \frac{(2n+1)\pi b}{a}}. \end{aligned}$$

EXAMPLE 3.92

A rectangular plate with insulated surface is 8 cm wide and so long compared to its width that it may be considered infinite in length. If the temperature along one short edge $y = 0$ is given by $u(x, 0) = 100 \sin \frac{\pi x}{8}$, $0 < x < 8$ while two long edges $x = 0$ and $x = 8$ as well as the other short edges are kept at 0°C . Find the steady state temperature.

Solution. The partial differential equation governing the problem is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

along with the boundary conditions

$$u(0, y) = 0, u(8, y) = 0, u(x, y) = 0 \text{ as } y \rightarrow \infty,$$

$$u(x, 0) = 100 \sin \frac{\pi x}{8}.$$

The solution

$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$$

is the suitable solution under the given conditions.

The use of initial condition $u(0, y) = 0$ yields

$$0 = c_1(c_3 e^{py} + c_4 e^{-py})$$

and so $c_1 = 0$. Thus the solution becomes

$$u(x, y) = c_2 \sin px (c_3 e^{py} + c_4 e^{-py}).$$

The condition $u(8, y) = 0$ implies

$$0 = c_2 \sin 8p (c_3 e^{py} + c_4 e^{-py})$$

and so $\sin 8p = 0$. Therefore, $\sin 8p = \sin n\pi$ and so $p = \frac{n\pi}{8}$, $n = 0, \pm 1, \pm 2, \dots$. The solution becomes

$$u(x, y) = c_2 \sin \frac{n\pi x}{8} (c_3 e^{n\pi y/8} + c_4 e^{-n\pi y/8}).$$

The condition $u(x, y) = 0$ as $y \rightarrow \infty$ implies $c_3 = 0$. Hence

$$u(x, y) = c_2 c_4 e^{-\frac{n\pi y}{8}} \sin \frac{n\pi x}{8} = A_n e^{-\frac{n\pi y}{8}} \sin \frac{n\pi x}{8}.$$

But $u(x, 0) = 100 \sin \frac{\pi x}{8}$. Therefore,

$$100 \sin \frac{\pi x}{8} = A_n \sin \frac{n\pi x}{8}$$

and so $A_n = 100$, $n = 1$. Hence the solution is

$$u(x, y) = 100 e^{-\frac{\pi y}{8}} \sin \frac{\pi x}{8}.$$

(D) Two-Dimensional Wave Equation

Consider the vibrations of a tightly stretched membrane, such as membrane of drum. We assume that the membrane is uniform and that the tension per unit length in the membrane is same at every point of it in all directions. Let m be the mass per unit area and T be the tension per unit length of the membrane. Let u be the displacement of an element $\delta x \delta y$ of the membrane perpendicular to the xy -plane. Then the forces acting on the membrane are:

- (i) The forces $T \delta y$ tangential to the membrane on its opposite edges of length δy acting respectively at angles α and β to the horizontal.

- (ii) The forces $T \delta x$ acting on the opposite edges of length δx (Fig. 3.2).

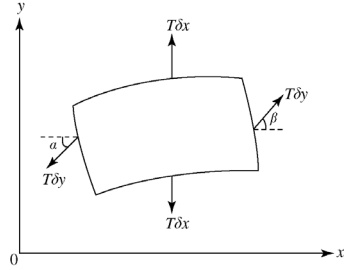


Figure 3.2

The angles α and β , being small, the vertical component of forces corresponding to (i) is

$$(T \delta y) \sin \beta - (T \delta y) \sin \alpha = T \delta y (\tan \beta - \tan \alpha)$$

$$= T \delta y \left[\left(\frac{\partial u}{\partial x} \right)_{x+\delta x} - \left(\frac{\partial u}{\partial x} \right)_{x-1} \right]$$

$$= T \delta y \delta x \left[\frac{\left(\frac{\partial u}{\partial x} \right)_{x+\delta x} - \left(\frac{\partial u}{\partial x} \right)_x}{\delta x} \right]$$

$$= T \delta x \delta y \frac{\partial^2 u}{\partial x^2} \text{ in the limit as } \delta x \rightarrow 0.$$

Similarly, the vertical component of forces corresponding to (ii) is $T \delta x \delta y \frac{\partial^2 u}{\partial y^2}$. Hence the equation of motion of the element $\delta x \delta y$ is

$$m \delta x \delta y \frac{\partial^2 u}{\partial t^2} = T \delta x \delta y \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

or

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (132)$$

where $c^2 = \frac{T}{m}$. The equation (132) is called **Two-Dimensional Wave Equation**.

We now find the solution of the equation (132) by separation of variables method. So let

$$u(x, y, t) = X(x)Y(y)T(t)$$

be a trial solution of (132). Then

$$\frac{\partial^2 u}{\partial x^2} = X''YT, \quad \frac{\partial^2 u}{\partial y^2} = XY''T, \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2} = XYT''.$$

Substituting these values of second order derivatives in the equation (132), we have

$$\frac{1}{c^2} XYT'' = X''YT + XY''T$$

Dividing throughout by XYT , we get

$$\frac{1}{c^2} \frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y}.$$

Since x , y , and t are independent variables, the above equation is valid if each term in this is constant. Choosing the constants as $-k^2$, $-l^2$, and $-(k^2 + l^2)$, we get

$$\frac{d^2 X}{dx^2} + k^2 X = 0, \quad (133)$$

$$\frac{d^2 Y}{dy^2} + l^2 Y = 0, \quad (134)$$

$$\frac{d^2 T}{dt^2} + (k^2 + l^2) T = 0, \quad (135)$$

The solutions to the homogenous equations (133) (134) (135) are respectively

$$X = c_1 \cos kx + c_2 \sin kx,$$

$$Y = c_3 \cos ly + c_4 \sin ly,$$

$$T = c_5 \cos \sqrt{k^2 + l^2} t + c_6 \sin \sqrt{k^2 + l^2} t.$$

Hence the solution of the two-dimensional wave equation (131) is

$$\begin{aligned} u(x, y, t) &= (c_1 \cos kx + c_2 \sin kx) \\ &\quad \times (c_3 \cos ly + c_4 \sin ly) \\ &\quad \times (c_5 \cos \sqrt{k^2 + l^2} t + c_6 \sin \sqrt{k^2 + l^2} t) \end{aligned}$$

EXAMPLE 3.93

Find the deflection $u(x, y, t)$ of the square membrane with $a = b = c = 1$ if the initial velocity is zero and the initial deflection $f(x) = A \sin \pi x \sin 2\pi y$.

Solution. We know that the vibrations of a membrane are governed by two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

The membrane is stretched between the lines

$$X = 0, x = 1, y = 0, \text{ and } y = 1.$$

Therefore the boundary conditions are

$$u(0, y, t) = 0, u(1, y, t) = 0,$$

$$u(x, 0, t) = 0, \text{ and } u(x, 1, t) = 0.$$

The initial conditions are

$$\begin{aligned} u(x, y, 0) &= f(x, y) \\ &= A \sin \pi x \sin 2\pi y \text{ and } \left(\frac{\partial u}{\partial t} \right)_{t=0} = 0. \end{aligned}$$

We know that the solution of two-dimensional wave equation is

$$\begin{aligned} u(x, y, t) &= (c_1 \cos kx + c_2 \sin kx) \\ &\quad (c_3 \cos ly + c_4 \sin ly) \\ &\quad (c_5 \cos \sqrt{k^2 + l^2} t + c_6 \sin \sqrt{k^2 + l^2} t) \end{aligned}$$

It is given that $c = 1$. Therefore

$$\begin{aligned} u(x, y, t) &= (c_1 \cos kx + c_2 \sin kx) \\ &\quad \times (c_3 \cos ly + c_4 \sin ly) \\ &\quad \times (c_5 \cos \sqrt{k^2 + l^2} t + c_6 \sin \sqrt{k^2 + l^2} t). \end{aligned}$$

Using the boundary condition $u(0, y, t) = 0$, we get

$$\begin{aligned} 0 &= c_1 (c_3 \cos ly + c_4 \sin ly) \\ &\quad \times (c_5 \cos \sqrt{k^2 + l^2} t + c_6 \sin \sqrt{k^2 + l^2} t). \end{aligned}$$

Thus $c_1 = 0$ and the solution reduces to

$$\begin{aligned} u(x, y, t) &= c_2 \sin kx (c_3 \cos ly + c_4 \sin ly) \\ &\quad \times (c_5 \cos \sqrt{k^2 + l^2} t + c_6 \sin \sqrt{k^2 + l^2} t) \end{aligned} \quad (136)$$

Now the boundary condition $u(1, y, t) = 0$ gives

$$\begin{aligned} 0 &= c_2 \sin k (c_3 \cos ly + c_4 \sin ly) \\ &\quad \times (c_5 \cos \sqrt{k^2 + l^2} t + c_6 \sin \sqrt{k^2 + l^2} t), \end{aligned}$$

and so $\sin k = 0$, which implies $k = m\pi$. Hence (136) reduces to

$$\begin{aligned} u(x, y, t) &= c_2 \sin m\pi x (c_3 \cos ly + c_4 \sin ly) \\ &\quad \times (c_5 \cos \sqrt{k^2 + l^2} t + c_6 \sin \sqrt{k^2 + l^2} t). \end{aligned} \quad (137)$$

The boundary condition $u(x, 0, t) = 0$ implies

$$\begin{aligned} 0 &= c_2 \sin m\pi x \left[c_3 \left(c_5 \cos \sqrt{k^2 + l^2} t \right. \right. \\ &\quad \left. \left. + c_6 \sin \sqrt{k^2 + l^2} t \right) \right] \end{aligned}$$

and so $c_3 = 0$. Thus (137) reduces to

$$\begin{aligned} u(x, y, t) &= c_2 c_4 \sin m\pi x \sin ly \\ &\quad \times (c_5 \cos \sqrt{k^2 + l^2} t + c_6 \sin \sqrt{k^2 + l^2} t). \end{aligned} \quad (138)$$

Lastly, the boundary condition $u(x, 1, t) = 0$ yields

$$\begin{aligned} 0 &= c_2 c_4 \sin m\pi x \sin l \\ &\quad \times (c_5 \cos \sqrt{k^2 + l^2} t + c_6 \sin \sqrt{k^2 + l^2} t) \end{aligned}$$

and so $\sin l = 0$ or $l = n\pi$. Therefore (138) reduces to

$$\begin{aligned} u(x, y, t) &= c_2 c_4 \sin m\pi x \sin n\pi y \\ &\quad \times \left(c_5 \cos \sqrt{k^2 + l^2} t + c_6 \sin \sqrt{k^2 + l^2} t \right) \\ &= \sin m\pi x \sin n\pi y (A_{mn} \cos pt + B_{mn} \sin pt), \end{aligned}$$

$$\text{where } p = \sqrt{k^2 + l^2} \quad (139)$$

Therefore superposition of general solutions yields

$$\begin{aligned} u(x, y, t) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin m\pi x \sin n\pi y \\ &\quad \times (A_{mn} \cos pt + B_{mn} \sin pt). \quad (140) \end{aligned}$$

The application of initial condition $\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0$ implies $B_{mn} = 0$ and use of the initial condition $f(x, y) = A \sin \pi x \sin 2\pi y$ at $t = 0$ yields

$$A \sin \pi x \sin 2\pi y = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin m\pi x \sin n\pi y. \quad (141)$$

The right hand side of (141) is a Double Fourier Series of $A \sin \pi x \sin 2\pi y$. The Fourier coefficients are given by

$$A_{mn} = \frac{2}{1} \cdot \frac{2}{1} \int_0^1 \int_0^1 A \sin \pi x \sin 2\pi y [\sin mx \sin n\pi y] dx dy.$$

We note that

$$A_{m1} = A_{m3} = A_{m4} = \dots = 0,$$

$$A_{22} = A_{32} = A_{42} = \dots = 0,$$

$$A_{m2} = 4A \int_0^1 \int_0^1 \sin \pi x \sin m\pi x \sin^2(2\pi y) dx dy$$

$$= 2A \int_0^1 \int_0^1 \sin \pi x \sin m\pi x [1 - \cos 4y] dx dy$$

$$= 2A \int_0^1 \sin \pi x \sin m\pi x dx = 0, \text{ and}$$

$$A_{12} = 2A \int_0^1 \sin^2 \pi x dx = A \int_0^1 (1 - \cos 2\pi x) dx = A.$$

Hence the solution (140) becomes

$$\begin{aligned} u(x, y, t) &= A \sin \pi x \sin 2\pi y \cos pt, \\ &\quad \text{since } m = 1 \text{ and } n = 2 \\ &= A \cos \left(\pi \sqrt{5} t \right) \sin \pi x \sin 2\pi y, \end{aligned}$$

since

$$\begin{aligned} p &= \sqrt{k^2 + l^2} = \sqrt{m^2 \pi^2 + n^2 \pi^2} = \pi \sqrt{m^2 + n^2} \\ &= \pi \sqrt{1^2 + 2^2} = \pi \sqrt{5}. \end{aligned}$$

3.10 SOLUTIONS OF LAPLACE EQUATION

(A) Laplace Equation in Two Dimensions

(1) **Cartesian Form:** Please see Article 3.9 (C).

(2) **Polar Form:** Putting $x = r \cos \theta$, $y = r \sin \theta$ in $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, the Laplace equation in polar coordinates is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0. \quad (142)$$

Let $u(r = 0) = R(r) F(\theta)$ be trial solution of (142). Then

$$\begin{aligned} \frac{\partial u}{\partial r} &= F \frac{dR}{dr}, \\ \frac{\partial^2 u}{\partial r^2} &= F \frac{d^2 R}{dr^2}, \\ \frac{\partial^2 u}{\partial \theta^2} &= R \frac{d^2 F}{d\theta^2}. \end{aligned}$$

Therefore equation (142) reduces to

$$r^2 F \frac{d^2 R}{dr^2} + r F \frac{dR}{dr} + R \frac{d^2 F}{d\theta^2} = 0.$$

Separation of variables yield

$$\frac{r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr}}{R} = - \frac{\frac{d^2 F}{d\theta^2}}{F} = k \text{ (constant)}$$

Thus, we get two equations

$$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} - kR = 0 \quad (143)$$

and

$$\frac{d^2 F}{d\theta^2} + kF = 0. \quad (144)$$

The equation (143) is a homogeneous linear differential equation. So, substituting $r = e^z$ the equation (143) reduces to

$$\frac{d^2 R}{dz^2} - kR = 0. \quad (145)$$

Case I. $k = p^2$ (positive)

The auxiliary equation of (145) is $m^2 - p^2 = 0$ and so $m = \pm p$. Thus the solution of (145) is

$$R = c_1 e^{pz} + c_2 e^{-pz} \quad (146)$$

The auxiliary equation of (144) is $m^2 + p^2 = 0$ and so $m = ip$. Thus the solution of (144) is

$$F = c_3 \cos p\theta + c_4 \sin p\theta \quad (147)$$

Case II. $k = -p^2$ (negative)

In this case the auxiliary equation for (145) is $m^2 + p^2 = 0$ and so $m = \pm ip$. Thus the solution of (145) is

$$\begin{aligned} R &= c_1 \cos pz + c_2 \sin pz \\ &= c_1 \cos(p \log r) + c_2 \sin(p \log r). \end{aligned} \quad (148)$$

The auxiliary equation of (144) is $m^2 - p^2 = 0$, which yields $m = \pm p$. Thus solution of (144) is

$$F = c_3 e^{p\theta} + c_4 e^{-p\theta} \quad (149)$$

Case III. $k = 0$

In this case the solutions of (145) and (144) are respectively

$$R = c_1 z + c_2 = c_1 \log r + c_2 \quad (150)$$

and

$$F = c_3 \theta + c_4. \quad (151)$$

Hence the three possible solution of the Laplace equation (142) are

$$u = (c_1 r^p + c_2 r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta), \quad (152)$$

$$\begin{aligned} u &= [c_1 \cos(p \log r) \\ &\quad + c_2 \sin(p \log r)](c_3 e^{p\theta} + c_4 e^{-p\theta}), \end{aligned} \quad (153)$$

$$u = (c_1 \log r + c_2)(c_3 \theta + c_4). \quad (154)$$

Out of these three solutions, the solution consistent with the boundary conditions is considered. Generally, the solution (152) is considered to be most suitable.

(B) Laplace Equation in Three Dimensions

(1) **Cartesian Form:** The Laplace equation in three dimensions is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \quad (155)$$

Let $u(x, y, z) = X(x) Y(y) Z(z)$ be the trial solution. Then (155) reduces to

$$X''YZ + XY''Z + XYZ'' = 0$$

Dividing throughout by XYZ , we get

$$\frac{X''}{x} + \frac{Y''}{y} + \frac{Z''}{z} = 0,$$

that is

$$\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0 \quad (156)$$

Since x, y, z are independent, (156) is possible only if

$$\frac{1}{X} \frac{d^2 X}{dx^2}, \frac{1}{Y} \frac{d^2 Y}{dy^2} \text{ and } \frac{1}{Z} \frac{d^2 Z}{dz^2} \text{ are all constant.}$$

Let

$$\frac{1}{X} \frac{d^2 X}{dx^2} = k^2, \frac{1}{Y} \frac{d^2 Y}{dy^2} = l^2 \text{ and } \frac{1}{Z} \frac{d^2 Z}{dz^2} = -(k^2 + l^2).$$

Thus, we get three equations

$$\frac{d^2 X}{dx^2} - k^2 X = 0 \quad (157)$$

$$\frac{d^2 Y}{dy^2} - l^2 Y = 0 \quad (158)$$

and

$$\frac{d^2 Z}{dz^2} + (k^2 + l^2) Z = 0 \quad (159)$$

The solution of (157), (158), and (159) are respectively

$$X = c_1 e^{kx} + c_2 e^{-kx}$$

$$Y = c_3 e^{ly} + c_4 e^{-ly} \text{ and}$$

$$Z = c_5 \cos \sqrt{k^2 + l^2} z + c_6 \sin \sqrt{k^2 + l^2} z.$$

Hence the solution of the three dimensional Laplace equation in Cartesian form is

$$u = (c_1 e^{kx} + c_2 e^{-kx})(c_3 e^{ly} + c_4 e^{-ly}) \\ \times (c_5 z \cos \sqrt{k^2 + l^2} + c_6 z \sin \sqrt{k^2 + l^2}).$$

The choice of the constant depends on the initial and boundary conditions.

(C) Laplace Equation in Cylindrical Co-ordinates

Substituting $x = r \cos \theta$, $y = r \sin \theta$ and $z = z$ in the Cartesian form of three dimensional Laplace equation, we get

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0. \quad (160)$$

Let $u(r, \theta, z) = R(r) F(\theta) Z(z)$ be the trial solution of (160). Then (160) takes the form

$$R'' F Z + \frac{1}{r} R' F Z + \frac{1}{r^2} R F'' Z + R F Z'' = 0.$$

Dividing throughout by $R F Z$, we get

$$\frac{R''}{R} + \frac{1}{rR} R' + \frac{F''}{r^2 F} + \frac{Z''}{Z} = 0$$

or

$$\frac{1}{R} \left[\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right] + \frac{1}{r^2 F} \frac{d^2 F}{d\theta^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0. \quad (161)$$

Taking

$$\frac{d^2 F}{d\theta^2} = -n^2 F \text{ and } \frac{d^2 Z}{dz^2} = k^2 Z, \quad (162)$$

we have

$$\frac{1}{R} \left[\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right] - \frac{n^2}{r^2} + k^2 = 0$$

or

$$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} + (k^2 r^2 - n^2) = 0. \quad (163)$$

The solution of the Bessel's equation (163) is

$$R = c_1 J_n(kr) + c_2 Y_n(kr).$$

The solution of (162) are

$$F = c_3 \cos n\theta + c_4 \sin n\theta,$$

$$Z = c_5 e^{kz} + c_6 e^{-kz}.$$

Thus the solution of the Laplace equation (160) is

$$u = [c_1 J_n(kr) + c_2 Y_n(kr)](c_3 \cos n\theta + c_4 \sin n\theta) \\ \times (c_5 e^{kz} + c_6 e^{-kz}),$$

which is called **cylindrical harmonic**.

(D) Laplace Equation in Spherical Co-ordinates

Substituting $x = r \sin \theta \cos \Phi$, $y = r \sin \theta \sin \Phi$ and $z = r \cos \theta$ in the three dimensional Cartesian form of the Laplace equation, we get

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = 0. \quad (164)$$

Let $u(r, \theta, \phi) = R(r) G(\theta) H(\phi)$ be a trial solution of (164). Then (164) takes the form

$$R'' G H + \frac{2}{r} R' G H + \frac{1}{r^2} R G'' H + \frac{\cot \theta}{r^2} R G' H \\ + \frac{1}{r^2 \sin^2 \theta} R G H'' = 0.$$

Dividing this equation throughout by $R G H$, we have

$$\frac{1}{R} \left[r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} \right] \\ + \frac{1}{G} \left[\frac{d^2 G}{d\theta^2} + \cot \theta \frac{dG}{d\theta} \right] + \frac{1}{H \sin^2 \theta} \frac{d^2 H}{d\phi^2} = 0. \quad (165)$$

Taking

$$\frac{1}{R} \left[r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} \right] = n(n+1) \quad (166)$$

and

$$\frac{1}{H} \frac{d^2 H}{d\phi^2} = -m^2, \quad (167)$$

the equation (165) reduces to

$$\frac{d^2 G}{d\theta^2} + \cot \theta \frac{dG}{d\theta} + [n(n+1) - m^2 \operatorname{cosec}^2 \theta] G = 0.$$

The solution of this Legendre's Equation is

$$G = c_1 P_n^m(\cos \theta) + c_2 Q_n^m(\cos \theta).$$

Further, the solution of equation (167) is

$$H = c_3 \cos m\phi + c_4 \sin m\phi.$$

Taking $R = r^k$ in (166), we get

$$k(k-1) + 2k = n(n+1)$$

or

$$(k^2 - n^2) + (k - n) = 0$$

or

$$(k - n)(k + n + 1) = 0$$

and so $k = n, -n - 1$. Therefore

$$R = c_5 r^n + c_6 r^{-n-1}.$$

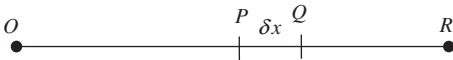
Hence the general solution of the Laplace Equation (164) is

$$u = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} [c_1 P_n^m(\cos \theta) + c_2 Q_n^m(\cos \theta)] \\ \times (c_3 \cos m\phi + c_4 \sin m\phi)(c_5 r^n + c_6 r^{-n-1}),$$

which is called **spherical harmonic**.

3.11 TELEPHONE EQUATIONS OF A TRANSMISSION LINE

Let O be the transmitting end and R the receiving end of a transmission cable carrying an electric current with resistance R ohms/km, inductance L henries/km, capacitance C farads/km and leakance G mhos/km. Let P be a point on the cable at a distance x km from the transmitting end O and let $v(x, t)$ and $i(x, t)$ be respectively the instantaneous voltage and current at the point P at any time t . Consider a small segment PQ of length δx of the cables.



The voltage drop across PQ is the sum of the voltage drop due to the resistance R and the voltage drop due to the inductance L . Thus the voltage drop across the section PQ is given by

$$-\delta v = R i \delta x + L \delta x \frac{\partial i}{\partial t}$$

or

$$-\frac{\delta v}{\delta x} = R i + L \frac{\partial i}{\partial t}.$$

Taking limit as $\delta x \rightarrow 0$, we have

$$-\frac{\partial v}{\partial x} = R i + L \frac{\partial i}{\partial t}$$

or

$$\frac{\partial v}{\partial x} + \left(R + L \frac{\partial}{\partial t}\right) i = 0. \quad (168)$$

On the other hand, the current loss across the segment PQ is the sum of current loss due to capacitance and the current loss due to leakance. Therefore

$$-\delta i = C \frac{\partial v}{\partial t} \delta x + G v \delta x$$

or

$$-\frac{\delta i}{\delta x} = C \frac{\partial v}{\partial t} + G v.$$

Taking limit as $\delta x \rightarrow 0$, we have

$$-\frac{\partial i}{\partial x} = C \frac{\partial v}{\partial t} + G v$$

or

$$\frac{\partial i}{\partial x} + \left(C \frac{\partial}{\partial t} + G\right) v = 0. \quad (169)$$

Differentiating (168) partially with respect to x , we get

$$\frac{\partial^2 v}{\partial x^2} + R \frac{\partial i}{\partial x} + L \frac{\partial^2 i}{\partial t \partial x} = 0. \quad (170)$$

Operating (169) by $(R + L \frac{\partial}{\partial t})$, we have

$$R \frac{\partial i}{\partial x} + L \frac{\partial^2 i}{\partial t \partial x} + R \left(C \frac{\partial}{\partial t} + G\right) v \\ + L C \frac{\partial^2 v}{\partial t^2} + L G \frac{\partial v}{\partial t} = 0. \quad (171)$$

Subtracting (171) from (170), we have

$$\frac{\partial^2 v}{\partial x^2} = L C \frac{\partial^2 v}{\partial t^2} + (L G + R C) \frac{\partial v}{\partial t} + R G v. \quad (172)$$

Similarly, operating (168) by $(C \frac{\partial}{\partial t} + G)$, we get

$$C \frac{\partial^2 v}{\partial t \partial x} + RC \frac{\partial i}{\partial t} + LC \frac{\partial^2 i}{\partial t^2} + G \frac{\partial v}{\partial x} + RGi + GL \frac{\partial i}{\partial t} = 0. \quad (173)$$

Differentiating (169) partially with respect to x , we have

$$\frac{\partial^2 i}{\partial x^2} + C \frac{\partial^2 v}{\partial t \partial x} + G \frac{\partial v}{\partial x} = 0. \quad (174)$$

Subtracting (174) from (173), we get

$$\frac{\partial^2 i}{\partial x^2} = LC \frac{\partial^2 i}{\partial t^2} + (LG + RC) \frac{\partial i}{\partial t} + RGi = 0. \quad (175)$$

We observe that the equation (175) can be obtained by replacing v in equation (172) by i . The equations (172) and (175) are called *telephone equations*.

If $R = G = 0$, then the telephone equations reduce to

$$\frac{\partial^2 v}{\partial x^2} = LC \frac{\partial^2 v}{\partial t^2}$$

and

$$\frac{\partial^2 i}{\partial x^2} = LC \frac{\partial^2 i}{\partial t^2},$$

which are called *radio equations*.

If $L = G = 0$, then the telephone equations reduce to

$$\frac{\partial^2 v}{\partial x^2} = RC \frac{\partial v}{\partial t},$$

and

$$\frac{\partial^2 i}{\partial x^2} = RC \frac{\partial i}{\partial t},$$

which are called *telegraph equations*. Setting $c^2 = \frac{1}{RC}$, we note that the telegraph equations are similar to the heat equation $\frac{\partial v}{\partial t} = c^2 \frac{\partial^2 v}{\partial x^2}$.

EXAMPLE 3.94

A transmission line 1000 kilometers long is initially under steady state condition with potential 1300 volts at the sending edge ($x = 0$) and 1200 volts at the receiving end ($x = 1000$). The terminal end of the line is suddenly grounded, but the potential at

the source is kept at 1300 volts. If inductance and leakage are negligible, determine the potential $v(x, t)$.

Solution. When inductance L and leakage G are negligible, then the partial differential equation governing the given problem is

$$\frac{\partial^2 v}{\partial x^2} = RC \frac{\partial v}{\partial t}$$

or

$$\frac{\partial v}{\partial t} = \frac{1}{RC} \frac{\partial^2 v}{\partial x^2}, \quad (176)$$

where R and C represent resistance and capacitance. The equation (176) is of *one dimensional heat equation* type and is known as *telegraph equation*. Since the transmission line is initially under steady state, therefore $\frac{\partial v}{\partial t} = 0$ and so (176) reduces to

$$\frac{\partial^2 v}{\partial x^2} = 0,$$

which yields (on integration)

$$v = c_1 x + c_2. \quad (177)$$

Now

- (i) When $x = 0$, $v = 1300$ volts, therefore (176) yields $c_2 = 1300$.

When $x = 1000$, $v = 1200$ volts and so

$$1200 = 1000 c_1 + c_2 = 1000 c_1 + 1300$$

and so $c_1 = -\frac{1}{10}$. Hence (176) implies

$$v = 1300 - \frac{x}{10} \text{ at } t = 0. \quad (178)$$

- (ii) When the terminal end is grounded, we have $v = 1300$ for $x = 0$ and so (177) yields $c_2 = 1300$. When $x = 1000$, $v = 0$ and so $0 = 1300 + 1000 c_1$, which implies $c_1 = -\frac{13}{10}$. Thus $v_g = 1300 - \frac{13}{10}x$ (when terminal is grounded).

If $v_t(x, t)$ is the transient part of the solution, then

$$v(x, t) = v_g + v_t(x, t), \quad (179)$$

where (see Example 3.76)

$$v_t(x, t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{1000}\right) e^{\frac{-x^2 \pi^2 t}{RC(1000)^2}}.$$

Thus (179) reduces to

$$v(x, t) = 1300 - \frac{13x}{10} + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{1000}\right) e^{\frac{-n^2 \pi^2 t}{RC(1000)^2}}.$$

When $t = 0$, we have

$$v(x, 0) = 1300 - \frac{13x}{10} + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{1000}\right)$$

or, using (178), we have

$$1300 - \frac{x}{10} = 1300 - \frac{13x}{10} + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{1000}\right)$$

or

$$\frac{6x}{5} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{1000}\right).$$

Thus b_n is Fourier (sine) coefficient of the function $\frac{6x}{5}$ and as such

$$\begin{aligned} b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{1000} dx \\ &= \frac{2}{1000} \int_0^{1000} \frac{6x}{5} \sin \frac{n\pi x}{1000} dx \\ &= \frac{-6}{2500} \left[-\frac{(1000)^2}{n\pi} \cos n\pi \right] = \frac{2400}{n\pi} (-1)^n. \end{aligned}$$

Hence the required solution is

$$\begin{aligned} v(x, t) &= 1300 - \frac{13x}{10} \\ &+ \frac{2400}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\left(\frac{n\pi x}{1000}\right) e^{\frac{-n^2 \pi^2 t}{RC(1000)^2}}. \end{aligned}$$

EXAMPLE 3.95

Find a solution of radio equation $\frac{\partial^2 v}{\partial x^2} = LC \frac{\partial^2 v}{\partial t^2}$, when a periodic e.m.f $V_0 \cos pt$ is applied at the end and $x = 0$ of the line.

Solution. The given radio equation is

$$\frac{\partial^2 v}{\partial x^2} = LC \frac{\partial^2 v}{\partial t^2}.$$

We are given that

$$v(0, t) = V_0 \cos pt = f(t).$$

By D'Alembert method, the solution is

$$\begin{aligned} v(x, t) &= \frac{1}{2} \left[f(t + x\sqrt{LC}) + f(t - x\sqrt{LC}) \right] \\ &= \frac{1}{2} \left[V_0 \cos p(t + x\sqrt{LC}) \right. \\ &\quad \left. + V_0 \cos p(t - x\sqrt{LC}) \right] \\ &= \frac{V_0}{2} \left[\cos pt \cos px\sqrt{LC} \right. \\ &\quad \left. - \sin pt \sin px\sqrt{LC} \right. \\ &\quad \left. + \cos pt \cos px\sqrt{LC} + \sin pt \sin px\sqrt{LC} \right] \\ &= V_0 \cos pt \cos px\sqrt{LC}. \end{aligned}$$

Since the solution implies $v(0, t) = V_0 \cos pt$ and

$$\begin{aligned} \frac{\partial v}{\partial x}(0, t) &= \left[V_0 \cos pt (-p\sqrt{LC} \sin px\sqrt{LC}) \right]_{x=0} \\ &= 0, \end{aligned}$$

the boundary conditions for the D'Alembert's method are satisfied.

EXAMPLE 3.96

Neglecting the resistance R and the leakance G in a transmission line l km. long, obtain the voltage drop $v(x, t)$, t seconds after the ends are suddenly grounded taking the initial conditions

$$i(x, 0) = i_0 \text{ and } v(x, 0) = v_0 \sin \frac{\pi x}{l}.$$

Solution. The problem is governed by the radio equation

$$\frac{\partial^2 v}{\partial x^2} = LC \frac{\partial^2 v}{\partial t^2}.$$

Since the ends are suddenly earthed, we have

$$v(0, t) = v(l, t) = 0.$$

Also, when $t = 0$, we have $i = i_0$ (constant). Therefore $\frac{\partial i}{\partial t} = 0$, which implies $-C \frac{\partial v}{\partial t} = 0$ and so $\left(\frac{\partial v}{\partial t}\right)_{t=0} = 0$.

Let $v = X(x) T(t)$ be a trial solution. Then the radio equation reduces to

$$T(t) \frac{d^2 X(x)}{dx^2} = LC X(x) \frac{d^2 T(t)}{dt^2}$$

or

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = \frac{LC}{T(t)} \frac{d^2 T(t)}{dt^2} = -p^2, \text{ say.}$$

Thus we get two equations

$$(D^2 + p^2)X(x) = 0 \text{ and } \left(D^2 + \frac{p^2}{LC}\right)T(t) = 0.$$

The solution of these equations are

$$X(x) = c_1 \cos px + c_2 \sin px$$

and

$$T(t) = c_3 \cos \frac{p}{\sqrt{LC}}t + c_4 \sin \frac{p}{\sqrt{LC}}t.$$

Hence solution of the radio equation is

$$v(x, t) = (c_1 \cos px + c_2 \sin px) \times \left(c_3 \cos \frac{p}{\sqrt{LC}}t + c_4 \sin \frac{p}{\sqrt{LC}}t\right).$$

The initial condition $v(0, t) = 0$ implies

$$0 = c_1 \left(c_3 \cos \frac{p}{\sqrt{LC}}t + c_4 \sin \frac{p}{\sqrt{LC}}t\right)$$

and so $c_1 = 0$. Therefore

$$v(x, t) = c_2 \sin px \left(c_3 \cos \frac{p}{\sqrt{LC}}t + c_4 \sin \frac{p}{\sqrt{LC}}t\right).$$

The initial condition $v(l, t) = 0$ implies

$$0 = c_2 \sin pl \left(c_3 \cos \frac{p}{\sqrt{LC}}t + c_4 \sin \frac{p}{\sqrt{LC}}t\right)$$

and so $\sin pl = 0$ or $p = \frac{n\pi}{l}$. Hence the solution takes the form

$$v(x, t) = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{n\pi t}{l\sqrt{LC}} + c_4 \sin \frac{n\pi t}{l\sqrt{LC}}\right).$$

Differentiating partially with respect to t , we get

$$\begin{aligned} \frac{\partial v}{\partial t} = c_2 \sin \frac{n\pi x}{l} & \left[\frac{-n\pi c_3}{l\sqrt{LC}} \sin \frac{n\pi t}{l\sqrt{LC}} \right. \\ & \left. + \frac{n\pi c_4}{l\sqrt{LC}} \cos \frac{n\pi t}{l\sqrt{LC}} \right]. \end{aligned}$$

Now the initial condition $\left(\frac{\partial v}{\partial t}\right)_{t=0} = 0$ implies $c_4 = 0$. Hence the solution reduces to

$$\begin{aligned} v(x, t) &= -\frac{n\pi}{l\sqrt{LC}} c_2 c_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{l\sqrt{LC}} \\ &= A_n \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{l\sqrt{LC}}, \\ A_n &= -\frac{n\pi}{l\sqrt{LC}} c_2 c_3. \end{aligned}$$

Therefore the general solution is

$$v(x, t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{l\sqrt{LC}}.$$

But $v(x, 0) = v_0 \sin \frac{\pi x}{l}$. Therefore

$$v_0 \sin \frac{\pi x}{l} = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}.$$

Hence $A_1 = v_0$, $A_2 = A_3 = \dots = 0$ and the required solution is

$$v(x, t) = v_0 \sin \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}}.$$

3.12 MISCELLANEOUS EXAMPLE

EXAMPLE 3.97

Form the partial differential equation by eliminating the arbitrary constants 'a' and 'b' from the equation $(x - a)^2 + (y - b)^2 + z^2 = 1$.

Solution. The given equation is $(x - a)^2 + (y - b)^2 + z^2 = 1$.

Therefore

$$z^2 = 1 - (x - a)^2 - (y - b)^2.$$

Differentiating partially with respect to x , we get

$$2z \frac{\partial z}{\partial x} = -2(x - a),$$

and so

$$(x - a) = -z \frac{\partial z}{\partial x}. \quad (180)$$

Similarly, differentiating partially with respect to y , we have $2z \frac{\partial z}{\partial y} = -2(y - b)$, and so

$$(y - b) = -z \frac{\partial z}{\partial y} \quad (181)$$

Putting the values of $(x - a)$ and $(y - b)$ from (180) and (181) in the given equation, we get

$$z^2 \left(\frac{\partial z}{\partial x} \right)^2 + z^2 \left(\frac{\partial z}{\partial y} \right)^2 + z^2 = 1$$

or

$$z^2(p^2 + q^2 + 1) = 1.$$

EXAMPLE 3.98

Form the PDE by eliminating the arbitrary functions from the equation $z = xf\left(\frac{y}{x}\right) + yg(x)$.

Solution. We have $z = xf\left(\frac{y}{x}\right) + yg(x)$. Therefore

$$\begin{aligned} \frac{\partial z}{\partial x} &= f\left(\frac{y}{x}\right) + xf'\left(\frac{y}{x}\right) \left(\frac{-y}{x^2}\right) + yg'(x) \\ &= f\left(\frac{y}{x}\right) - \frac{y}{x} f'\left(\frac{y}{x}\right) + yg'(x), \end{aligned} \quad (182)$$

$$\begin{aligned} \frac{\partial z}{\partial y} &= xf'\left(\frac{y}{x}\right) \cdot \frac{1}{x} + g(x) \\ &= f'\left(\frac{y}{x}\right) + g(x). \end{aligned} \quad (183)$$

From (182) and (183), we have

$$x \frac{\partial z}{\partial x} = xf\left(\frac{y}{x}\right) - yf'\left(\frac{y}{x}\right) + xyg'(x)$$

and

$$y \frac{\partial z}{\partial y} = yf'\left(\frac{y}{x}\right) + yg(x).$$

Therefore

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} - z = xyg'(x). \quad (184)$$

Also

$$\frac{\partial^2 z}{\partial x \partial y} = f''\left(\frac{y}{x}\right) \left(\frac{-y}{x^2}\right) + g'(x), \quad (185)$$

$$\frac{\partial^2 z}{\partial y^2} = f''\left(\frac{y}{x}\right) \left(\frac{1}{x}\right). \quad (186)$$

From (186), we have $f''\left(\frac{y}{x}\right) = x \frac{\partial^2 z}{\partial y^2}$. Putting this value in (185), we get

$$g'(x) = \frac{\partial^2 z}{\partial x \partial y} + \frac{y}{x} \frac{\partial^2 z}{\partial y^2}.$$

Hence (184) reduces to

$$\begin{aligned} x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} - z &= xy \left[\frac{\partial^2 z}{\partial x \partial y} + \frac{y}{x} \frac{\partial^2 z}{\partial y^2} \right] \\ &= xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} \end{aligned}$$

EXAMPLE 3.99

Solve: $x(1 + y)p = y(1 + x)q$.

Solution. The given partial differential equation can be written as $\frac{xp}{1+x} = \frac{yq}{1+y}$, which is of the form $f(x, p) = g(y, q)$ (separable equation). Therefore $\frac{xp}{1+x} = \frac{yq}{1+y} = a$, which yields

$$p = \frac{a(1+x)}{x} \quad \text{and} \quad q = \frac{a(1+y)}{y}.$$

Then $dz = p dx + q dy$ reduces to

$$dz = \frac{a(1+x)}{x} dx + \frac{a(1+y)}{y} dy.$$

Integrating we get

$$z = a[\log x + x] + a[\log y + y] + b$$

or

$$z = a[\log x + \log y] + ax + ay + b$$

or

$$z = a[\log xy + x + y] + b.$$

EXAMPLE 3.100

Solve $\frac{\partial^2 z}{\partial x^2} = xy$.

Solution. It is given that $\frac{\partial^2 z}{\partial x^2} = xy$.

Integrating with respect to x , we get

$$\frac{\partial z}{\partial x} = \frac{x^2 y}{2} + f(y).$$

Integrating again, we have

$$z = \frac{x^3 y}{6} + xf(y) + g(y).$$

EXAMPLE 3.101

Solve the partial differential equation:

$$x^2(y - z)p + y^2(z - x)q = z^2(x - y)$$

Solution. The given partial differential equation is

$$x^2(y - 3)p + y^2(3 - x)q = z^2(x - y).$$

It is of the form $Pp + Qq = R$. The auxiliary equation for the Lagrange's method is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R},$$

that is,

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} \quad (187)$$

Using multipliers $\frac{1}{x}$, $\frac{1}{y}$ and $\frac{1}{z}$, each fraction of (187) is equivalent to

$$\frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

and so

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0 \quad (188)$$

Integrating (188), we get

$$\log x + \log y + \log z = \log a$$

or

$$\log xyz = \log a$$

or

$$xyz = a \quad (189)$$

Now using multipliers $\frac{1}{x^2}$, $\frac{1}{y^2}$, $\frac{1}{z^2}$, each fraction of (187) is equivalent to

$$\frac{\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2}}{0} \quad (190)$$

Integrating (190), we get

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = b \quad (191)$$

Hence, from (189) and (191), the required solution is

$$f\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz\right) = 0.$$

EXAMPLE 3.102

Solve the partial differential equation

$$x^2p + y^2q = (x+y)z$$

Solution. We have

$$x^2p + y^2q = (x+y)z.$$

The auxiliary equation for Lagrange's method are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{x+y}. \quad (192)$$

Form (192), we get

$$\frac{dx - dy}{x^2 - y^2} = \frac{dz}{x+y},$$

or

$$\frac{dx - dy}{x - y} = \frac{dz}{1}. \quad (193)$$

Integrating (193), we get

$$\log(x - y) = z + \log c. \quad (194)$$

Again, from (192), we have

$$\frac{dx}{x^2} = \frac{dy}{y^2}.$$

Integrating both sides, we get

$$\int \frac{dx}{x^2} = \int \frac{dy}{y^2}$$

or

$$\frac{1}{x} = \frac{1}{y} + D$$

or

$$\frac{1}{x} - \frac{1}{y} = D$$

or

$$\frac{x - y}{xy} = -D. \quad (195)$$

From (194) and (195), the required solution of the given partial differential equation is

$$f\left(\frac{x-y}{e^z}, \frac{x-y}{xy}\right)$$

EXAMPLE 3.103

Solve $(2z - y)p + (x + z)q + 2x + y = 0$.

Solution. We have

$$(2z - y)p + (x + z)q + 2x + y = 0.$$

The auxiliary equations are

$$\frac{dx}{2z - y} = \frac{dy}{x + z} = -\frac{dz}{2x + y} \quad (196)$$

Using multipliers x, y and z , each fraction of (196) is equal to $\frac{xdx+yd y+zd z}{0}$. Therefore $xdx + ydy + zdz = 0$. Integrating, we get

$$x^2 + y^2 + z^2 = a \quad (197)$$

Using multipliers $1, -2$ and -1 , each fraction of (196) is equal to $\frac{dx-2dy-dz}{0}$, and so $dx - 2dy - dz = 0$, which on integration yields

$$x - 2y - z = b \quad (198)$$

Hence, from (197) and (198), the required solution is

$$x^2 + y^2 + z^2 = f(x - 2y - z),$$

where f is an arbitrary function.

EXAMPLE 3.104

Solve $z = px + qy + p^2q^2$ and find the complete and singular solutions.

Solution. We have $z = px + qy + p^2q^2$, which is a Clairaut's equation. Hence its complete integral is

$$z = ax + by + a^2b^2 \quad (\text{See 3.5D}) \quad (199)$$

The singular integral is obtained by eliminating a and b between (199) and the equations

$$0 = x - \frac{\partial f}{\partial a} = x - 2ab^2, \quad (200)$$

$$0 = y - \frac{\partial f}{\partial b} = y - 2a^2b. \quad (201)$$

From (199), (200) and (201), we have

$$\frac{x}{z} = \frac{2ab^2}{z} = \frac{2ab^2}{2a^2b^2 + 2a^2b^2 + a^2b^2} = \frac{2ab^2}{5a^2b^2} = \frac{2}{5a}$$

and

$$\frac{y}{z} = \frac{2a^2b}{5a^2b^2} = \frac{2}{5b}.$$

Therefore $a = \frac{2z}{5x}$ and $b = \frac{2z}{5y}$. Putting these values in (199), we get

$$z = \frac{2z}{5} + \frac{2z}{5} + \left(\frac{4z^2}{25xy}\right)^2 = \frac{4z}{5} + \frac{16z^4}{625x^2y^2}$$

or

$$1 = \frac{4}{5} + \frac{16z^3}{625x^2y^2}$$

or

$$16z^3 = 105x^2y^2.$$

EXAMPLE 3.105

Find the temperature in a thin metal rod of length L , with both the ends insulated (so that there is no passage of heat through the ends) and with initial temperature in the rod $\sin(\pi x/L)$.

Solution. As shown is Example 3.78, since the ends of the rod of length L are insulated, the solution is

$$U(x, t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2 kt}{L^2}} \cos \frac{n\pi x}{L},$$

where

$$A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, \dots$$

Here $f(x) = \sin \frac{\pi x}{L}$. Therefore

$$\begin{aligned} A_n &= \frac{2}{L} \int_0^L \sin \frac{\pi x}{L} \cos \frac{n\pi x}{L} dx \\ &= \frac{2}{L} \int_0^L \frac{1}{2} \left[\sin \frac{(n+1)\pi x}{L} - \sin \frac{(n-1)\pi x}{L} \right] dx \\ &= \frac{1}{L} \left[-\frac{\cos(n+1)\pi \frac{x}{L}}{(n+1)\pi} + \frac{\cos(n-1)\pi \frac{x}{L}}{(n-1)\pi} \right]_0^L \\ &= \frac{1}{L} \left[-\frac{\cos(n+1)\pi}{(n+1)\pi} + \frac{\cos(n-1)\pi}{(n-1)\pi} \right] \\ &\quad + \frac{1}{(n+1)\pi} - \frac{1}{(n-1)\pi} \\ &= \frac{1}{L} \left[\frac{(-1)^{n+2}}{(n+1)\pi} + \frac{(-1)^{n-1}}{(n-1)\pi} + \frac{1}{(n+1)\pi} - \frac{1}{(n-1)\pi} \right] \\ &= \frac{1}{\pi L} \left[\frac{(-1)^n}{(n+1)} - \frac{(-1)^n}{(n-1)} + \frac{1}{(n+1)} - \frac{1}{(n-1)} \right] \\ &= \frac{-2}{\pi L} [1 + (-1)^n]. \end{aligned}$$

Putting $n = 0$, we have $A_0 = -\frac{4}{\pi L}$. Hence

$$u(x, t) = -\frac{2}{\pi L} - \frac{2}{\pi L} \sum_{n=1}^{\infty} (1 + (-1)^n) e^{-\frac{n^2 \pi^2 k t}{L^2}} \cos \frac{n \pi x}{L}.$$

EXAMPLE 3.106

The initial temperature along the length of an infinite bar is given by

$$u(x, 0) = \begin{cases} 2, & |x| < 1 \\ 0, & |x| > 1. \end{cases}$$

If the temp $u(x, t)$ satisfies

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, -\infty < x < \infty, t > 0,$$

find the temperature at any pt. of the bar at any instant.

Solution. See Example 18.39 or 18.40. (See *Engineering Mathematics* by the author) We take $k = 1$, $a = 1$ and $f(x) = 2$ for $|x| < 1$.

Ans. $\frac{2}{\pi} \int_0^{\infty} e^{-\omega^2 t} \frac{\sin(1+x)\omega + \sin(1-x)\omega}{\omega} d\omega.$

EXAMPLE 3.107

Solve

$$\log \left[\frac{\partial^2 z}{\partial x \partial y} \right] = x + y$$

Solution. We have

$$\log \left[\frac{\partial^2 z}{\partial x \partial y} \right] = x + y$$

or

$$\frac{\partial^2 z}{\partial x \partial y} = e^{x+y}.$$

Integrating w.r.t. x , we get

$$\frac{\partial z}{\partial y} = e^{x+y} + f(y).$$

Integrating w.r.t. y , we have

$$z = e^{x+y} + \int f(y) dy + g(x).$$

EXAMPLE 3.108

A string is stretched between two fixed points at a distant $2l$ apart and the points of the string are given initial velocities

$$v = \begin{cases} \frac{cx}{l}, & 0 < x < l \\ \frac{c(2l-x)}{l}, & l < x < 2l \end{cases},$$

where x being the distance from an end point, Find the displacement of the string at any time.

Solution. The given problem is governed by one-dimensional wave equation $\frac{\partial^2 y}{\partial x^2} = c^2 \frac{\partial^2 y}{\partial t^2}$, under the conditions

$$y(0, t) = y(2l, t) = 0, t \geq 0$$

$$y(x, 0) = 0$$

$$\left(\frac{\partial y}{\partial t} \right)_{t=0} = \begin{cases} \frac{cx}{l}, & 0 \leq x < l \\ \frac{c(2l-x)}{l}, & l < x < 2l \end{cases}$$

Let

$$y(x, t) = T(t) X(x)$$

be a trial solution of the given partial differential equation. Then the wave equation takes the form

$$T''(t)X(x) = c^2 X''(x) T(t).$$

Separation of variables yields $\frac{T''(x)}{T(t)} = c^2 \frac{X''(x)}{X(x)} = k$ (constant of separation). The boundary conditions $y(0, t) = y(2l, t) = 0, t \geq 0$ imply $X(0) = 0$ and $X(2l) = 0$. The auxiliary equation for $X''(x) - \frac{k}{c^2} X(x) = 0$ is $s^2 - \frac{k}{c^2} = 0$, which yields s_1 and $-s_1$ as the two roots. Therefore the fundamental solution is

$$X(x) = \alpha e^{s_1 x} + \beta e^{-s_1 x}. \quad (202)$$

The boundary conditions $X(0) = 0 = X(2l)$ imply $\alpha + \beta = 0$ or $\beta = -\alpha$ and $0 = \alpha e^{s_1(2l)} + \beta e^{-s_1(2l)} = \alpha(e^{2s_1 l} - e^{-2s_1 l})$. Since $\alpha = 0$ implies $\beta = 0$ and since the solution in that case becomes trivial, it follows that $\alpha \neq 0$ and hence $e^{2s_1 l} = e^{-2s_1 l}$ or $e^{4s_1 l} = 1$ or $s_1 = \frac{in\pi}{2l}$, when $n \neq 0$ is an integer. Also, s_1 being the root, we have $s_1^2 = \frac{k}{c^2}$. Therefore $k = -\frac{c^2 n^2 \pi^2}{4l^2}$.

The solution (202) now reduces to

$$\begin{aligned} X_n(x) &= \alpha \left[e^{\frac{i n \pi x}{2l}} - e^{-\frac{i n \pi x}{2l}} \right] \\ &= 2\alpha \sin \frac{n \pi x}{2l} \end{aligned} \quad (203)$$

On the other hand, the characteristic equation for $T''(t) - kT(t) = 0$ is $s^2 - k = 0$, that is $s^2 + \frac{c^2 n^2 \pi^2}{4l^2} = 0$. Therefore the fundamental solution in this case is

$$T_n(t) = \gamma \cos\left(\frac{n\pi c t}{2l}\right) + \delta \sin\left(\frac{n\pi c t}{2l}\right). \quad (204)$$

From (203) and (204), the fundamental solution of the given wave equation is

$$\begin{aligned} y_n(x, t) &= T_n(t) X_n(x) \\ &= 2\alpha \left[\gamma \cos\left(\frac{n\pi c t}{2l}\right) + \delta \sin\left(\frac{n\pi c t}{2l}\right) \right] \\ &\quad \times \sin \frac{n\pi x}{2l}. \end{aligned}$$

Since the string was initially at rest, $y(x, 0) = 0$. This implies $y_n(x, t) = B_n \sin\left(\frac{n\pi c t}{2l}\right) \sin\left(\frac{n\pi x}{2l}\right)$. Now the superposition of the fundamental solutions implies

$$y(x, t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi c t}{2l}\right) \sin\left(\frac{n\pi x}{2l}\right) \quad (205)$$

Differentiating (205) partially with respect to t , we get

$$\begin{aligned} \frac{\partial y}{\partial t} &= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{2l}\right) \left[\cos\left(\frac{n\pi c t}{2l}\right) \left(\frac{n\pi c}{2l}\right) \right] \\ &= \frac{\pi c}{2l} \sum_{n=1}^{\infty} n B_n \cos\left(\frac{n\pi c t}{2l}\right) \sin\left(\frac{n\pi x}{2l}\right) \end{aligned} \quad (206)$$

But, at $t = 0$, we are given that

$$\frac{\partial y}{\partial t} = \begin{cases} \frac{cx}{l}, & 0 < x < l \\ \frac{c(2l-x)}{l}, & l < x < 2l. \end{cases}$$

Therefore

$$\frac{\pi c}{2l} \sum_{n=1}^{\infty} n B_n \sin\left(\frac{n\pi x}{2l}\right) = \begin{cases} \frac{cx}{l}, & 0 < x < l \\ \frac{c(2l-x)}{l}, & l < x < 2l. \end{cases}$$

The series on the left hand side is a Fourier sine series for $0 < x < 2l$. Therefore

$$\begin{aligned} \frac{\pi c}{2l} n B_n &= \frac{2}{2l} \int_0^{2l} f(x) \sin\left(\frac{n\pi x}{2l}\right) dx \\ &= \frac{c}{l^2} \left[\int_0^l x \sin\left(\frac{n\pi x}{2l}\right) dx \right. \\ &\quad \left. + \int_l^{2l} (2l-x) \sin\left(\frac{n\pi x}{2l}\right) dx \right] \\ &= \frac{c}{l^2} \left\{ \left[x \left(-\frac{\cos\left(\frac{n\pi x}{2l}\right)}{\pi^x/2l} \right) + \frac{\sin\left(\frac{n\pi x}{2l}\right)}{n^2 \pi^2 / (4l^2)} \right]_0^l \right. \\ &\quad \left. + \left[(2l-x) \left(-\frac{\cos\left(\frac{n\pi x}{2l}\right)}{n\pi/2l} \right) - \frac{\sin\left(\frac{n\pi x}{2l}\right)}{\frac{n^2 \pi^2}{4l^2}} \right]_l^{2l} \right\} \\ &= \frac{c}{l^2} \left[\frac{8l^2 \sin\left(\frac{n\pi}{2}\right)}{n^2 \pi^2} - 4l^2 \frac{\sin n\pi}{n^2 \pi^2} \right] \\ &= \frac{4c}{n^2 \pi^2} \left[2 \sin \frac{n\pi}{2} - \sin n\pi \right] \\ &= \frac{8c}{n^2 \pi^2} \sin \frac{n\pi}{2} = \pm \frac{8c}{n^2 \pi^2}. \end{aligned}$$

Therefore

$$B_n = \frac{16lc}{cn^3 \pi^3},$$

and the required solution is

$$y(x, t) = \frac{16lc}{c\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin\left(\frac{n\pi c t}{2l}\right) \sin\left(\frac{n\pi x}{2l}\right).$$

EXAMPLE 3.109

A semi circular plate of radius a centimeter has insulated faces and heat flows in plane curves. The bounding diameter is kept at 0°C and the semi-circumference is maintained at a temperature given by

$$u(a, \theta) = \begin{cases} \frac{k\theta}{\pi}, & 0 < \theta \leq \pi/2 \\ \frac{k(\pi-\theta)}{\pi}, & \pi/2 \leq \theta < \pi \end{cases}$$

Find the steady-state temperature distribution $u(r, \theta)$.

Solution. We want to solve the Laplace Equation

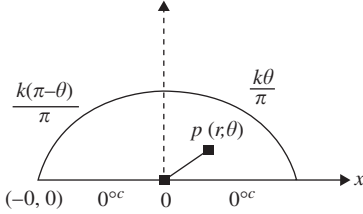
$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0, \quad (207)$$

subject to the conditions

$$u(r, 0) = 0, \quad u(r, \pi) = 0, \quad u(0, \theta) = 0,$$

and

$$u(a, \theta) = \begin{cases} \frac{k\theta}{\pi}, & 0 < \theta \leq \frac{\pi}{2} \\ \frac{k(\pi-\theta)}{\pi}, & \frac{\pi}{2} \leq \theta < \pi. \end{cases}$$



The solution to the equation (207) is

$$u(r, \theta) = (c_1 r^p + c_2 r^{-p}) (c_3 \cos p\theta + c_4 \sin p\theta) \quad (208)$$

Using the boundary condition $u(r, 0) = 0$, we get $c_3 = 0$. Thus

$$u(r, \theta) = (c_1 r^p + c_2 r^{-p}) c_4 \sin p\theta. \quad (209)$$

Now the condition $u(0, \theta) = 0$ implies

$$0 = c_2 \text{ since } r \rightarrow 0 \text{ implies } u = 0.$$

Thus (209) transforms to

$$u(r, \theta) = c_1 c_4 r^p \sin p\theta = b_n r^p \sin p\theta, \text{ say.}$$

Now $u(r, \pi) = 0$ implies

$$0 = b_n r^p \sin p\pi,$$

which yields $p\pi = n\pi$, that is, $p = n$ (integer). Hence the solution is

$$u_n(r, \theta) = b_n r^n \sin n\theta. \quad (210)$$

The superposition of the solutions implies

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta. \quad (211)$$

Thus

$$u(a, \theta) = \sum_{n=1}^{\infty} b_n a^n \sin n\theta \quad (212)$$

The series on the right hand side of (212) is the half-range sine series of $u(a, \theta)$. Therefore

$$\begin{aligned} a^n b_n &= \frac{2}{\pi} \left[\int_0^{\pi/2} \frac{k\theta}{\pi} \sin n\theta d\theta + \int_{\pi/2}^{\pi} \frac{k(\pi-\theta)}{\pi} \sin n\theta d\theta \right] \\ &= \frac{2k}{\pi} \left[0 \left(\frac{-\cos n\theta}{n} \right) + \frac{\sin n\theta}{n^2} \right]_0^{\pi/2} \\ &\quad + \left[(\pi-\theta) \left(\frac{-\cos n\theta}{n} \right) - \frac{\sin n\theta}{n^2} \right]_{\pi/2}^{\pi} \\ &= \frac{2k}{n} \left[\frac{-\pi}{2n} \cos \frac{n\pi}{2} + \frac{\sin \frac{n\pi}{2}}{n^2} \right. \\ &\quad \left. + \frac{\pi}{2n} \cos \frac{n\pi}{2} + \frac{\sin \frac{n\pi}{2}}{n^2} \right] \\ &= \frac{4k}{\pi n^2} \sin \frac{n\pi}{2}. \end{aligned}$$

Thus

$$b_n = \frac{4k}{\pi a^n n^2} \sin \frac{n\pi}{2}.$$

Hence the solution of the Laplace equation is $u(r, \theta) = \frac{4k}{\pi} \sum \frac{1}{n^2} \left(\frac{r}{a} \right)^n \sin \frac{n\pi}{2} \sin n\theta$.

EXAMPLE 3.110

Use the method of separation of variables to solve the equation $\frac{\partial^2 V}{\partial x^2} = \frac{\partial V}{\partial t}$.

Solution. Let $V(x, t) = T(t)X(x)$ be the possible solution of $\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}$. Then the given heat equation reduces to

$$T'(t)X(x) = X''(x)T(t).$$

Separation of variable yields

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}. \quad (213)$$

The left hand side of (213) is a function of t only, where as the right hand side is a function of x only. Since x and t are independent variables, (213) is valid only if each side is equal to some constant C , called the constant of separation. Thus

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = C.$$

Three cases arise:

Case I. When C is positive, Let $c = k^2$. Then

For $\frac{T'(t)}{T(t)} = c = k^2$	For $\frac{X''(x)}{X(x)} = c = k^2$
We have $T'(t) - k^2 T(t) = 0.$ The auxiliary equation is $m - k^2 = 0$ and so $m = k^2$ and $\text{C.F.} = c_1 e^{k^2 t}.$ Hence $T(t) = c_1 e^{k^2 t}.$	We have $X'(x) - k^2 X(x) = 0.$ The auxiliary equation is $m^2 - k^2 = 0$ and so $m = \pm k$. Hence $\text{C.F.} = c_2 e^{kx} + c_3 e^{-kx}$ and thus the solution is $X(x) = c_2 e^{kx} + c_3 e^{-kx}.$

Case II. Where C is negative, let $c = -k^2$. Then

For $\frac{T'(t)}{T(t)} = -k^2$	For $\frac{X''(x)}{X(x)} = -k^2$
We have $T'(t) + k^2 T(t) = 0.$ The auxiliary equation is $m + k^2 = 0,$ Which yields $m = -k^2$. Thus the solution is $T(t) = c_1 e^{-k^2 t}$	We have $X''(x) + k^2 X(x) = 0.$ The auxiliary equation is $m + k^2 = 0,$ which yields $m = \pm ki$. Hence the solution is $X(x) = c_2 \cos kx + c_3 \sin kx.$

Case III. Where $C = 0$. Then

$\frac{T'(t)}{T(t)} = 0$ and so $T'(t) = 0$, which implies $T(t) = c_1.$	$\frac{X''(x)}{X(x)} = 0$ and so $X''(x) = 0$, which implies $X(x) = c_2 x + c_3.$

The above three cases suggest that the possible solutions of the given heat equation are:

- $V(x, t) = c_1 e^{k^2 t} (c_2 e^{kx} + c_3 e^{-kx})$
- $V(x, t) = c_1 e^{-k^2 t} (c_2 \cos kx + c_3 \sin kx)$
- $V(x, t) = c_1 (c_2 x + c_3).$

EXAMPLE 3.111

Solve the following equations by method of separation of variables:

- $\frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$,
given that $u = 0$ when $t = 0$ and $\frac{\partial u}{\partial t} = 0$ when $x = 0$.
- $\frac{\partial u}{\partial x} = 4 \frac{\partial u}{\partial y}$, $u(0, y) = 8 e^{-3y}$.

Solution. (a). Let

$$u = X(x)T(t)$$

be a trial solution of the given differential equation. Then

$$\begin{aligned}\frac{\partial u}{\partial t} &= X(x)T'(t), \\ \frac{\partial^2 u}{\partial x \partial t} &= T'(t)X'(x).\end{aligned}$$

Therefore the given equation transforms to

$$T'(t)X'(x) = e^{-t} \cos x.$$

The separation of variable yields

$$e^t T'(t) = \frac{\cos x}{X'(x)} = a \text{ (constant)}$$

Then

$$e^t T'(t) = a$$

or

$$dT = ae^{-t} dt \quad (214)$$

Integration of (214) yields

$$T = -ae^{-t} + c_1 \quad (215)$$

Further,

$$X'(x) = \frac{1}{a} \cos x \, dx \quad (216)$$

Integrating (216), we get

$$X = \frac{1}{a} \sin x + c_2 \quad (217)$$

From (215) and (217), we get

$$u(x, t) = \left(\frac{1}{a} \sin x + c_2 \right) (-ae^{-t} + c_1) \quad (218)$$

Since u is zero when $t = 0$, we have

$$0 = \left(\frac{1}{a} \sin x + c_2 \right) (-a + c_1)$$

This is possible if $(-a + c_1) = 0$ or if $c_1 = a$.

Further, the solution (218) implies

$$\frac{\partial u}{\partial t} = \left(\frac{1}{a} \sin x + c_2 \right) (ae^{-t}). \quad (219)$$

Using the condition that $\frac{\partial u}{\partial t} = 0$ when, $x = 0$, we get from (219) that $0 = c_2 a e^{-t}$, which implies $c_2 = 0$.

Substituting the values c_1 and c_2 in the solution (218), we have

$$u(x, t) = \frac{1}{a} \sin x [-ae^{-t} + a] = \sin x (1 - e^{-t}).$$

(b) The given partial differential equation is

$$\frac{\partial u}{\partial x} = 4 \frac{\partial u}{\partial y}, \quad u(0, y) = 8e^{-3y} \quad (220)$$

Let

$$u = X(x)Y(y)$$

be a trial solution of the given partial differential equation. Then (220) implies

$$X'(x)Y(y) = 4X(x)Y'(y)$$

Separating the variables, we get

$$\frac{X'(x)}{X(x)} = 4 \frac{Y'(y)}{Y(y)} = a \text{ (constant)}$$

Thus we get two ordinary differential equations

$$\frac{X'(x)}{X(x)} = a \quad (221)$$

and

$$4 \frac{Y'(y)}{Y(y)} = a \quad (222)$$

Integrating (221), we get

$$\log X = ax + \log c_1$$

or

$$X = c_1 e^{ax} \quad (223)$$

Integrating (222), we have

$$\log Y = \frac{a}{4}y + \log c_2$$

or

$$Y = c_2 e^{\frac{ay}{4}} \quad (224)$$

Therefore

$$\begin{aligned}u &= X(x)Y(y) = c_1 c_2 e^{ax} e^{\frac{ay}{4}} \\ &= c_1 c_2 e^{ax + \frac{ay}{4}} = c_1 c_2 e^{\frac{4ax + ay}{4}}\end{aligned}$$

But $u(0, y) = 8e^{-3y}$. Therefore

$$8e^{-3y} = c_1 c_2 e^{\frac{ay}{4}},$$

which implies $c_1 c_2 = 8$ and $\frac{a}{4} = -3$. Thus $a = -12$.

Hence the solution is

$$u(x, y) = 8e^{\frac{-48x - 12y}{4}} = 8e^{-12x - 3y}.$$

EXERCISES

1. Form the partial differential equation from the following:

(a) $z = ax + by + a^2 + b^2$

Ans. $z = px + qy + p^2 + q^2$

(b) $z = (a + x^2)(b + y^2)$

Ans. $pq = 4xyz$

(c) $(x - a)^2 + (y - b)^2 + z^2 = c^2$

Ans. $z^2(p^2 + q^2 + 1) = c^2$

2. Find the differential equation of all planes which are at a constant distance 'a' from the origin.

Ans. $z = px + qy + a \sqrt{1 + p^2 + q^2}$

3. Form partial differential equation by eliminating the arbitrary functions from the following:

(a) $z = f_1(x)f_2(y)$

Ans. $z \frac{\partial^2 z}{\partial x \partial y} - \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = 0$

(b) $z = f\left(\frac{y}{x}\right)$

Ans. $px + qy = 0$

(c) $z = x f_1(x + t) + f_2(x + c)$

Ans. $\frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial t} + \frac{\partial^2 z}{\partial t^2} = 0$

(d) $z = f(x) + e^y g(x)$

Ans. $\frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial y}$

4. Solve

(a) $\frac{\partial^2 z}{\partial x^2} + z = 0$ given that when $x = 0$, $z = e^y$, and $\frac{\partial z}{\partial x} = 1$.

Ans. $z = e^y \cos x + \sin x$

(b) $\frac{\partial^2 z}{\partial y^2} = \sin(xy)$

Ans. $z = -\frac{1}{x^2} \sin(xy) + y f(x) + \phi(x)$

(c) $\frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$

Ans. $u = -e^{-t} \sin x + f(x) + g(t)$

5. Solve the Lagrange's linear equations:

(a) $\frac{y^2 z}{x} p + xzq = y^2$

Ans. $\phi(x^3 - y^3, x^2 - z^2) = 0$

(b) $x^2(y - z)p + y^2(z - x)q = z^2(x - y)$

Ans. $f\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}, xyz\right) = 0$

(c) $x^2 p + y^2 q = (x + y)z$

Ans. $f\left(\frac{xy}{z}, \frac{1}{y} - \frac{1}{x}\right) = 0$

(d) $y^2 p - xyq = x(z - 2y)$

Ans. $f(x^2 + y^2, yz - y^2) = 0$

(e) $xp + yq = 3z$

Ans. $f\left(\frac{x}{y}, \frac{x^3}{z}\right) = 0$

(f) $(y - z)p + (x - y)q = z - x$

Ans. $f(x + y + z, \frac{x^2}{2} + yz) = 0$

(g) $p - q = \log(x + y)$

Ans. $x \log(x + y) - z = f(x + y)$

(h) $px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$

Ans. $f\left(\frac{y}{z}, \frac{z}{x} - \frac{a^2 z^2}{x} + x^2\right) = 0$

6. Solve the following equations by Charpit's method:

(a) $2z + p^2 + qy + 2y^2 = 0$

Ans. $y^2[(x - a)^2 + y^2 + 2z] = b$

(b) $2(z + xp + yp) = yp^2$

Ans. $z = \frac{ax}{y^3} - \frac{a^2}{4y^3} + \frac{b}{y}$

(c) $2zx - px^2 - 2pxy + pq = 0$

Ans. $z = ay + b(x^2 - a)$

(d) $1 + p^2 = qz$

Ans. $\frac{z}{2} \pm \left[\frac{z}{2} \sqrt{z^2 - 4a^2} - 2a^2 \log(z + \sqrt{z^2 - 4a^2}) \right] = 2ax + 2y + b]$

(e) $p(q^2 + 1) + (b - z)q = 0$

Ans. $q\sqrt{z - a - b} = \sqrt{ax} + \frac{1}{\sqrt{a}}y + c$

7. Solve

(a) $(y - x)(qx - px) = (p - q)^2$

Hint: Convert to $f(P, Q) = 0$ form

Ans. $z = a(x + y) + \sqrt{axy} + c$

(b) $p^2 + p = q^2$

Ans. $z = ax + \sqrt{a^2 + ay} + c$

(c) $z^2 \left(\frac{p^2}{x^2} + \frac{q^2}{y^2} \right) = 1$

Ans. $z = ax^2 + \sqrt{1 - a^2}y^2 + c$

(d) $p^2 x + q^2 y = z$ **Hint:** Put $\frac{dx}{\sqrt{x}} = dX$, $\frac{dy}{\sqrt{y}} = dY$, $\frac{dz}{\sqrt{z}} = dZ$

Ans. $2\sqrt{z} = 2a\sqrt{x} + 2\sqrt{y}\sqrt{1 - a^2} + c$

(e) $\sqrt{p} + \sqrt{q} = 1$

Ans. $z = ax + (1 - \sqrt{a})^2 y + c$

(f) $pq = x^m y^n z^{2l}$ **Hint:** Put $x^m dx = dX$, $y^n dy = dY$, $z^{-l} dz = dZ$

Ans. $\frac{z^{-l}}{1-l} = a \frac{x^{m+1}}{m+1} + \frac{y^{n+1}}{n+1} + c$

8. Solve:

(a) $p + q = z$

Ans. $(1 + a) \log z = z + ay + b$

(b) $z^2(p^2 z^2 + q^2) = 1$

Ans. $(z^2 + a^2)^3 = 9(x + ay + b)^2$

- (c) $z^2 = 1 + p^2 + q^2$
Ans. $z = \cosh \frac{x+ay+b}{\sqrt{a^2+1}}$
- (d) $p^2 = qz$
Ans. $z = b e^{ax+a^2y}$
- (e) $z = pq + p + q = 0$
Ans. $z = ax - ay/(1+a) + b$
9. Solve:
 (a) $p - x^2 = q + y^2$
Ans. $z = \frac{1}{3}(x^3 - y^3) + a(x+y) + b$
- (b) $q = xyp^2$
Ans. $z = 2\sqrt{ax} + \frac{1}{2}ay^2 + b$
- (c) $pq = x^4 y^3 z^4$
Ans. $z = \frac{-1}{c\frac{x^5}{5} + \frac{y^4}{4c} + b}$
- (d) $\sqrt{p} + \sqrt{q} = 2x$
Ans. $z = \frac{1}{6}(2x+a)^3 + a^2y + b$
- (e) $p^2 + q^2 = z^2 (x+y)$
Ans. $\log z = \frac{2}{3}(a+x)^{3/2} + \frac{2}{3}(y-a)^{3/2} + c$
10. Solve :
 (a) $z = px + qy + \sin(p+q)$
Ans. $z = ax + by + \sin(a+b)$
- (b) $z = px + qy - 2\sqrt{pq}$
Ans. $z = ax + by - 2\sqrt{ab}$
11. Solve the following by the method of separation of variables
 (a) $\frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$, $u(0) = 0$, $\frac{\partial u}{\partial t} = 0$ at $t = 0$.
Ans. $u = (1 - e^{-t}) \sin x$
- (b) $3 \frac{\partial u}{\partial x} + 2 \frac{\partial u}{\partial y} = 0$, $u(x, 0) = 4e^{-x}$.
Ans. $3e^{-5x-3y} + 2e^{-3x-2y}$
- (c) $\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} - 2u = 0$, $u(0, y) = 0$, $\frac{\partial u}{\partial x}(0, y) = 1 + e^{-3y}$.
Ans. $u = \frac{1}{\sqrt{2}} \sin \sqrt{2x} + e^{-3y} \sin x$
- (d) $\frac{\partial^2 x^2}{\partial x^2} = h^2 \frac{\partial u}{\partial t}$, $u(0, t) = u(l, t) = 0$,
 $u(x, 0) = \sin \frac{\pi x}{l}$
Ans. $u(x, t) = e^{-\frac{h^2 \pi^2 t}{l^2}} \sin \frac{\pi x}{l}$
12. Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with boundary $u(x, 0) = 3 \sin nx$, $u(0, t) = 0$, $u(l, t) = 0$, where $0 < x < l$, $t > 0$.
Ans. $u(x, t) = 3 \sum_{n=1}^{\infty} e^{-n^2 \pi^2 t} \cdot \sin(n\pi x)$
13. Solve $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$, $u_x(0, l) = 0$, $u(x, y) \rightarrow$ finite number as $t \rightarrow \infty$, $u(0, l) = lx - x^2$.
Ans. $u(x, t) = \frac{1}{6}l^2 - \frac{l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \frac{2n\pi x}{l} e^{-\frac{4\pi^2 n^2 k t}{l^2}}$
14. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points a velocity of $\lambda x(l-x)$, find the displacement of the string at any distance x from one end at any time t .
Ans. $y(x, t) = \frac{8\lambda l^3}{c\pi^4} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^4} \sin \frac{(2m-1)\pi x}{l} \sin \frac{(2m-1)\pi ct}{l}$
15. Solve the wave equation $\frac{\partial^2 y}{\partial t^2} = 4 \frac{\partial^2 y}{\partial x^2}$ subject to the conditions
 $y(0, t) = 0$, $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 3 \sin(2\pi x) - 2 \sin(5\pi x)$
Ans. $y(x, t) = \frac{1}{20\pi} [15 \sin(4\pi t) \sin(2\pi x) - 4 \sin(5\pi x) \sin(10\pi t)]$
16. A tightly stretched string with fixed points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3 \left(\frac{\pi x}{l}\right)$. If it is released from rest from this position, find the displacement $y(x, t)$ at any time t .
Ans. $y(x, t) = \frac{y_0}{4} \left(3 \sin \frac{\pi x}{l} \cos \frac{n\pi c}{l} - \sin \frac{3\pi x}{l} \cos \frac{3n\pi c}{l}\right)$
17. An infinitely long plate uniform plate is bounded by two parallel edges and an end at right angle to them. The breadth is π . This end is maintained at a temperature u_0 at all points and the other edges are at zero temperature. Determine the temperature at any point of the plate in a steady state.
Ans. $u(x, y) = \frac{440}{\pi} \left[e^{-y} \sin x + \frac{1}{3}e^{-3y} \sin 3x + \frac{1}{5}e^{-5y} \sin 5x + \dots\right]$
18. A homogeneous rod of conducting material of length 100 cm has its ends kept at zero temperature and the initial temperature is
 $u(x, 0) = \begin{cases} x, & 0 \leq x \leq 50 \\ 100 - x, & 50 \leq x \leq 100. \end{cases}$
 Find the temperature $u(x, t)$ at any time t .
Ans.
 $u(x, t) = \frac{400}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^2} e^{-\frac{(2n+1)^2 c^2 \pi^2 t}{100^2}} \cdot \sin \frac{(2n+1)\pi x}{100}$
19. $\frac{\partial^2 y}{\partial t^2} - a^2 \frac{\partial^2 y}{\partial x^2} = E \sin pt$.
Hint: Roots of A.E are $m = \pm a$. Therefore C.F. = $f_1(x-at) + f_2(x+at)$
 P.I. = $\frac{1}{D^2 - 2D^2} E \sin pt = -\frac{E}{p^2} \sin pt$
Ans. $y = f_1(x-at) + f_2(x+at) = \frac{E}{p^2} \sin pt$

20. $(D^2 - DD' - 2D'^2)z = e^x(y - 1)$

Ans. $z = f_1(y - x) + f_2(y + 2x) + y e^x$

21. $(D^2 - DD')z = \cos x \cos 2y$

Hint: Convert $\cos x \cos 2y$ into $\frac{1}{2}[\cos(x + 2y) + \cos(x - 2y)]$ and then use standard method to find P.I.

Ans. $z = f_1(y) + f_2(y + x) - \frac{1}{6}\cos(x - 2y) + \frac{1}{2}\cos(x + 2y)$

22. $(D^3 - 7DD'^2 - 6D'^3)z = x^2 + xy^2 + y^3 + \cos(x - y)$

Ans. $f_1(y - x) + f_2(y - 2x) + f_3(y + 3x) + \frac{5}{72}x^6 + \frac{1}{60}x^5 + \frac{7}{50}x^5y + \frac{1}{24}x^4x^2 + \frac{1}{6}x^3y^3 + \frac{x}{4}\cos(x - y).$

23. $(D^2 - 2DD' + D'^2)z = \sin x$

Ans. $z = f_1(y + x) + xf_2(y + x) - \sin x.$

24. $(D^2 - D'^2)z = \tan^3 x \tan y - \tan^3 y \tan x$

Ans. $z = f_1(y + x) + f_2(y - x) + \frac{1}{2}\tan x \tan y.$

25. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = e^{x+2y}$

Ans. $z = f_1(y + x) + f_2(y - x) - \frac{1}{3}e^{x+y}$

Solve the following non-homogeneous partial differential equations

26. $(D^2 - DD' + D')z = x^2 + y^2$

Ans.

$z = f_1(y) + e^{-x}f_2(y + x) + \frac{x^3}{3} - x^2 + xy^2 + 6x$

27. $(D^2 + D + D')z = 0$

Ans. $z = c e^{ax - (a^2 + a)y}$, where a and c are constants.

28. $(2DD' + D'^2 - 3D')z = 3\cos(3x - 2y)$

Ans. $x = f_1(x) + e^{3y}f_2(2y - x) + \frac{3}{50}[3\sin(3x - 2y) + 4\cos(3x - 2y)]$

29. $r - s + p = 1$

Ans. $z = f_1(y) + e^{-x}f_2(y + x) + x$

30. $(D^2 - D' - 1)z = x^2y$

Ans. $z = c_1 e^{ax + (a^2 - 1)y} - x^2y + x^2 - 2y + 8$, where a and c are constants.

31. A tightly stretched unit square membrane starts vibrating from rest and its initial displacement $k \sin 2\pi x \sin y$. Show that the deflection at any instant is $k \sin 2\pi x \sin \pi y \cos(\sqrt{5} \pi ct)$.

32. Show that the deflection of a rectangular membrane ($0 \leq x \leq a$; $0 \leq y \leq b$) with fixed boundary and starting from rest satisfying $u(x, y, 0) = xy(a - x)(b - y)$ is given by

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \cos ckt \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b},$$

where

$$A_{mn} = \frac{16a^2b^2}{m^3n^3\pi^6} (1 - \cos m\pi)(1 - \sin n\pi) \text{ and}$$

$$k^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right).$$

4 Fourier Series

In the early eighteenth century, the work of C. Maclaurin and B. Taylor led to the representation of functions like $\sin x$, $\cos x$, e^x , and $\arctan x$ as power series expansions. By the middle of the eighteenth century it became important to study the possibility of representation of the 'given function by infinite series other than the power series. Since many phenomena like vibration of string, the voltages and currents in electrical networks, electromagnetic signals, and movement of pendulum are periodic in nature, physicist, and mathematicians discussed the possibility of representing a periodic function as an infinite series involving sinusoidal ($\sin x$ and $\cos x$) functions. In his classic *Théorie analytique de la chaleur*, published in 1822, the French physicist Jean Baptiste Joseph Fourier announced in his work on *heat conduction* that an arbitrary periodic function could be expanded in a series of sinusoidal functions. Thus, the aim of the theory of Fourier series is to determine the conditions under which the periodic functions can be represented as linear combinations of sine and cosine functions. These combinations are called *Fourier series* and the coefficients that occur in the combinations are called *Fourier-coefficients*.

4.1 TRIGONOMETRIC SERIES

Let T denote the period of the periodic functions. Our aim is to approximate arbitrary periodic function as linear combination of sine and cosine functions. Therefore in that situation the sine and cosine functions must also have period T . Obviously, the functions $\sin \frac{2\pi t}{T}$, $\cos \frac{2\pi t}{T}$, $\sin \frac{4\pi t}{T}$, $\cos \frac{4\pi t}{T}$, and so on have period T . The constant function also has period T . Thus the functions $\sin \frac{2\pi nt}{T}$ and $\cos \frac{2\pi nt}{T}$, $n \in \mathbb{N}$ have period T . Put $\frac{2\pi}{T} = \omega_0$. Then the functions $\sin n\omega_0 t$ and $\cos n\omega_0 t$, $n \in \mathbb{N}$ have period T .

The constant $\omega_0 = \frac{2\pi}{T}$ is called the *fundamental frequency*. The functions $\sin \omega_0 t$ and $\cos \omega_0 t$ will complete exactly one cycle on an interval of length T whereas functions $\sin n\omega_0 t$ and $\cos n\omega_0 t$ with $n > 1$ will complete several cycles (Fig. 4.1). Thus, frequencies of $\sin n\omega_0 t$ and $\cos n\omega_0 t$ are integer multiples of ω_0 . The linear combinations, called *superpositions*, are again periodic with period 2π .

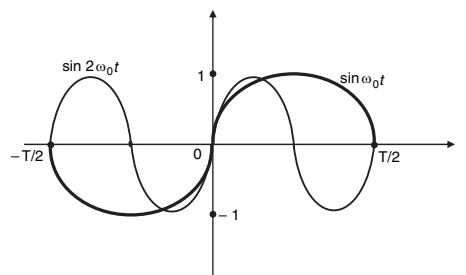


Figure 4.1 $\sin n\omega_0 t$ for $n = 1, 2$

Definition 4.1. An expression of the form

$$f(t) = a_0 + \sum_{k=1}^n (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t), \omega_0 = \frac{2\pi}{T},$$

where $a_0, a_1, a_2, \dots$ are constants, is called a *trigonometric polynomial* with period T .

If $|a_n| + |b_n| > 0$, then the number n is called the *order of the trigonometric polynomial*.

Definition 4.2. An expression of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t), \quad (1)$$

where $a_0, a_1, a_2, \dots$ are constants, is called a *trigonometric series*. The constants a_n, b_n are called *coefficients of the series*.

The free (first) term has been taken as $\frac{a_0}{2}$ so that it can be found directly from the formula for a_n , by taking $n = 0$, while finding Fourier coefficients.

The following trigonometric identities shall be required in the forthcoming discussion.

$$(a) \int_{-T/2}^{T/2} \cos n\omega_0 t \, dt = \left| \frac{\sin n\omega_0 t}{n\omega_0} \right|_{-T/2}^{T/2} = 0,$$

$$(b) \int_{-T/2}^{T/2} \sin n\omega_0 t \, dt = \left| \frac{-\cos n\omega_0 t}{n\omega_0} \right|_{-T/2}^{T/2} = 0,$$

$$(c) \int_{-T/2}^{T/2} \cos^2 n\omega_0 t \, dt = \frac{1}{2} \int_{-T/2}^{T/2} (1 + \cos 2n\omega_0 t) \, dt$$

$$= \frac{1}{2} \left[t + \frac{\sin 2n\omega_0 t}{2n\omega_0} \right]_{-T/2}^{T/2} = \frac{T}{2},$$

$$(d) \int_{-T/2}^{T/2} \sin^2 n\omega_0 t \, dt = \frac{1}{2} \int_{-T/2}^{T/2} (1 - \cos 2n\omega_0 t) \, dt$$

$$= \frac{1}{2} \left[t - \frac{\sin 2n\omega_0 t}{2n\omega_0} \right]_{-T/2}^{T/2} = \frac{T}{2},$$

$$(e) \int_{-T/2}^{T/2} \sin n\omega_0 t \cos m\omega_0 t \, dt$$

$$= \frac{1}{2} \int_{-T/2}^{T/2} [\sin(n+m)\omega_0 t + \sin(n-m)\omega_0 t] \, dt$$

$$= 0 \text{ using (b),}$$

(f) If $n, m \in \mathbb{N}$, $n \neq m$, then

$$\int_{-T/2}^{T/2} \cos n\omega_0 t \cos m\omega_0 t \, dt$$

$$= \frac{1}{2} \int_{-T/2}^{T/2} [\cos(n+m)\omega_0 t + \cos(n-m)\omega_0 t] \, dt$$

$$= 0 \text{ using (a),}$$

(g) If $n, m \in \mathbb{N}$, $n \neq m$, then

$$\int_{-T/2}^{T/2} \sin n\omega_0 t \sin m\omega_0 t \, dt$$

$$= \frac{1}{2} \int_{-T/2}^{T/2} [\cos(n-m)\omega_0 t - \cos(n+m)\omega_0 t] \, dt$$

$$= 0 \text{ using (a).}$$

4.2 FOURIER (OR EULER) FORMULAE

Let $f(t)$, defined on $[-\frac{T}{2}, \frac{T}{2}]$, be the sum of the trigonometric series (1). Thus

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t]. \quad (2)$$

Suppose that this trigonometric series converges uniformly in $[-\frac{T}{2}, \frac{T}{2}]$. Then term-by-term integration of the series is valid. Therefore, integration of (2) term-by-term yields

$$\int_{-T/2}^{T/2} f(t) \, dt = \int_{-T/2}^{T/2} \frac{a_0}{2} \, dt + \sum_{n=1}^{\infty} a_n \int_{-T/2}^{T/2} \cos n\omega_0 t \, dt$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-T/2}^{T/2} \sin n\omega_0 t \, dt$$

$$\text{and so} \quad = \frac{a_0}{2} T + 0 + 0 = \frac{a_0}{2} T,$$

$$a_0 = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \, dt \quad (3)$$

Now multiplying (2) by $\cos m\omega_0 t$ and integrating, we obtain

$$\int_{-T/2}^{T/2} f(t) \cos m\omega_0 t \, dt = \frac{a_0}{2} \int_{-T/2}^{T/2} \cos m\omega_0 t \, dt$$

$$+ \sum_{n=1}^{\infty} a_n \int_{-T/2}^{T/2} \cos m\omega_0 t \cos n\omega_0 t \, dt$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-T/2}^{T/2} \cos m\omega_0 t \sin n\omega_0 t \, dt$$

$$= 0 + \sum_{n=1}^{\infty} a_n \int_{-T/2}^{T/2} \cos m\omega_0 t \cos n\omega_0 t \, dt + 0$$

But

$$\int_{-T/2}^{T/2} \cos m\omega_0 t \cos n\omega_0 t \, dt = \begin{cases} 0 & \text{for } m \neq n \\ T/2 & \text{for } m = n. \end{cases}$$

Hence

$$\int_{-T/2}^{T/2} f(t) \cos n\omega_0 t \, dt = \frac{a_n}{2} T,$$

which yields

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos n\omega_0 t \, dt. \quad (4)$$

Similarly, multiplying (2) by $\sin m\omega_0 t$ and integrating, we get

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin n\omega_0 t \, dt. \quad (5)$$

Note that if we put $n = 0$ in (4), we obtain (3). That is why, we take $\frac{a_0}{2}$ in (1) instead of taking a constant a_0 .

In the above discussion, the interval of integration has been $(-\frac{T}{2}, \frac{T}{2})$, whose length is precisely *one period* T . However, to determine the coefficients a_n and b_n , we can integrate over any other interval of length T . Sometimes $(0, T)$ is also taken as interval of integration.

The formulae (4) and (5) are called *Fourier* or *Euler formulae*, the numbers a_n and b_n are called *Fourier coefficients* and the series $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$ is called *Fourier series* of the function f where a_n and b_n are Fourier coefficients determined by (4) and (5).

When the periodic function f is real, then a_n and b_n are real and the n th term $a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$, in the Fourier series, is called the n th *harmonic*. This term can also be written as a single cosine term in the following form:

$$a_n \cos n\omega_0 t + b_n \sin n\omega_0 t = \sqrt{a_n^2 + b_n^2} \cos(n\omega_0 t + \phi_n),$$

where

$$\tan \phi_n = \frac{b_n}{a_n} \text{ if } a_n \neq 0, \\ \phi_n = -\frac{\pi}{2} \text{ if } a_n = 0.$$

The factor $\sqrt{a_n^2 + b_n^2}$ is the *amplitude* of the n th harmonic and ϕ_n is the *initial phase*. The initial phase tells us how far the n th harmonic is shifted relative to $\cos n\omega_0 t$.

If a_n and b_n are Fourier coefficients for f , then we write

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t),$$

until and unless we know that the series converges to f . Thus we can replace ' $\sim$ ' by ' $=$ ' only if the Fourier series converges to f .

Deductions. (a) If f is even in $[-\frac{T}{2}, \frac{T}{2}]$, then $f(t) \cos n\omega_0 t$ is also even and so

$$a_n = \frac{4}{T} \int_0^{T/2} f(t) \cos n\omega_0 t \, dt.$$

Further, since product of an even function f with odd function $\sin n\omega_0 t$ is odd, we have

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin n\omega_0 t \, dt = 0.$$

Thus, if f is an even function, then its Fourier series will consist of cosine terms only.

(b) If f is an odd function in $[-\frac{T}{2}, \frac{T}{2}]$, then $f(t) \cos n\omega_0 t$ will be odd and $f(t) \sin n\omega_0 t$ will be even. Therefore, in this case,

$$a_n = 0 \text{ and } b_n = \frac{4}{T} \int_0^{T/2} f(t) \sin n\omega_0 t \, dt.$$

Thus, if f is an odd function, then its Fourier series will consist of sine terms only.

(c) As discussed above, to determine the Fourier coefficients a_n and b_n , we can, in general, integrate the integrand over any interval of length T (period). For example, if we take $T = 2l$ and the interval as $(-l, l)$, then $\omega_0 = \frac{2\pi}{T} = \frac{\pi}{l}$. Since $\sin \frac{\pi t}{l}$ and $\cos \frac{\pi t}{l}$ have period $2l$, the Fourier series valid in $(-l, l)$ takes the form

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{l} + b_n \sin \frac{n\pi t}{l} \right),$$

where

$$\begin{aligned}
 a_n &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos n\omega_0 t \, dt \\
 &= \frac{2}{2l} \int_{-l}^l f(t) \cos \frac{n\pi t}{l} \, dt \\
 &= \frac{1}{l} \int_{-l}^l f(t) \cos \frac{n\pi t}{l} \, dt \quad (6)
 \end{aligned}$$

and similarly

$$b_n = \frac{1}{l} \int_{-l}^l f(t) \sin \frac{n\pi t}{l} \, dt \quad (7)$$

In particular, if $T = 2\pi$ and interval of integration is $(-\pi, \pi)$, then

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt, \quad (8)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt. \quad (9)$$

Similarly, if we carry out integration over $(c, c + T)$, where T is the period of the function f , then we have

$$a_n = \frac{2}{T} \int_c^{c+T} f(t) \cos n\omega_0 t \, dt, \quad (10)$$

$$b_n = \frac{2}{T} \int_c^{c+T} f(t) \sin n\omega_0 t \, dt. \quad (11)$$

Taking $c = 0$, we get the interval of integration as $(0, T)$ and

$$a_n = \frac{2}{T} \int_0^T f(t) \cos n\omega_0 t \, dt, \quad (12)$$

$$b_n = \frac{2}{T} \int_0^T f(t) \sin n\omega_0 t \, dt. \quad (13)$$

4.3 PERIODIC EXTENSION OF A FUNCTION

Let f be a function defined on the interval $(-\frac{T}{2}, \frac{T}{2})$. By *periodic extension* of f we mean that f is defined by $f(t + kT) = f(t)$ for all $k \in \mathbb{Z}$. The extended function is then a periodic function of period T .

EXAMPLE 4.1

Consider the *sawtooth function* f defined on the interval $(-\frac{T}{2}, \frac{T}{2}) = (-\pi, \pi)$ by $f(t) = t$. The graph of f is shown in the Figure 4.2.

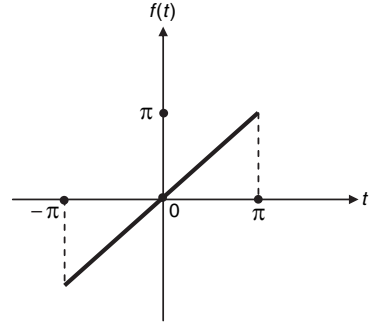


Figure 4.2 Graph of $f(t) = t$

The graph of the extended periodic function with period $T = 2\pi$ is then as shown in the Figure 4.3.

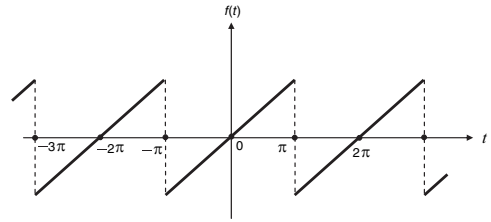


Figure 4.3 Extended Periodic Function $f(t) = t$ with Period 2π

It is a piecewise smooth function discontinuous at the points $t = (2k + 1)\pi$, $k = 0, \pm 1, \pm 2, \dots$

EXAMPLE 4.2

Consider the function f defined by $f(t) = t^2$ on $(-\pi, \pi)$. The graph of f is a parabola, shown in the Figure 4.4.

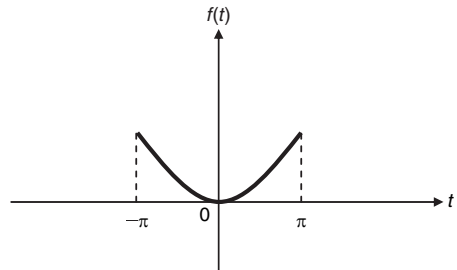


Figure 4.4 Graph of $f(t) = t^2$

The periodic extension of f is then a function of period 2π shown in the Figure 4.5.

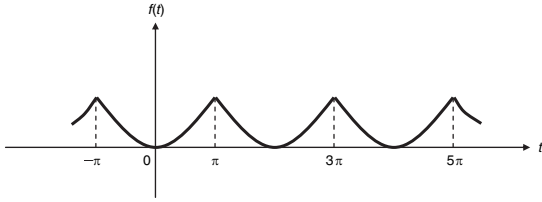


Figure 4.5 Periodically Extended Function $f(t) = t^2$

4.4 FOURIER COSINE AND SINE SERIES

We have seen that if f is an even function, then its Fourier series consists of cosine terms only, whereas for an odd function, the Fourier series consists of sine terms only. Sometimes one would like to obtain a Fourier series containing only cosine terms or sine terms for an *arbitrary function* on the interval $(0, \frac{T}{2})$. Such series are called *Fourier cosine series* and *Fourier sine series*, respectively.

To obtain Fourier cosine series for an arbitrary function f defined on $(0, \frac{T}{2})$, make the *even extension* of f from the interval $(0, \frac{T}{2})$ onto the interval $(-\frac{T}{2}, 0)$ by defining $f(-t) = f(t)$ for $-\frac{T}{2} < t < 0$ and, subsequently, extend the function periodically with period T . The function, thus, created is now an even function on the interval $(-\frac{T}{2}, \frac{T}{2})$ and so its Fourier series will consist of cosine terms only. This function is equal to the original function on the interval $(0, \frac{T}{2})$.

To obtain Fourier sine series, we first make *odd extension* of f from the interval $(0, \frac{T}{2})$ on the interval $(-\frac{T}{2}, 0)$ by defining $f(-t) = -f(t)$ for $-\frac{T}{2} < t < 0$ and, subsequently, extend the function periodically with period T . The function thus created is an odd function on the interval $(-\frac{T}{2}, \frac{T}{2})$ and so its Fourier series shall consist of only sine terms. This newly created function is equal to the original function on the interval $(0, \frac{T}{2})$.

The process of obtaining Fourier cosine series or Fourier sine series of an arbitrary function by making even or odd extension is called *forced series development*. The Fourier series so obtained is called *half-range series*.

EXAMPLE 4.3

Consider the function $f(t) = 1$ ($0 < t < \pi$) and suppose that we want to have odd extension of f to $(-\pi, 0)$. So we have to define $f(-t) = -f(t)$. Thus $f(-t) = -1$ in $(-\pi, 0)$. Thus, the graph of the extended function is as shown in Figure 4.6.

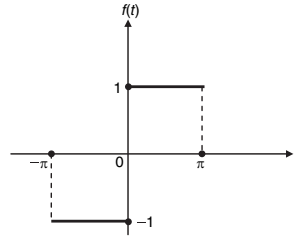


Figure 4.6 Odd Extension of f

The period of this extended function is 2π . We then extend this periodically to obtain the following graph (Fig. 4.7):

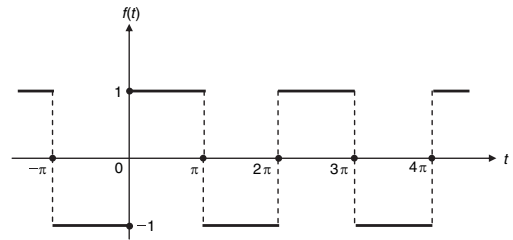


Figure 4.7 Periodic Extension of f

4.5 COMPLEX FOURIER SERIES

Let f be an integrable function on the interval $(-\frac{T}{2}, \frac{T}{2})$. Then the Fourier series of f is

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t), \quad (14)$$

where $\omega_0 = \frac{2\pi}{T}$ and a_n, b_n are determined by Euler's formulae

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos n\omega_0 t \, dt,$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin n\omega_0 t \, dt.$$

By Euler's formula, relating trigonometric and exponential functions, we have

$$\cos n\omega_0 t = \frac{e^{in\omega_0 t} + e^{-in\omega_0 t}}{2}$$

and

$$\sin n\omega_0 t = \frac{e^{in\omega_0 t} - e^{-in\omega_0 t}}{2i}.$$

Substituting these values into the expression (14), it follows that

$$\begin{aligned} & \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \frac{e^{in\omega_0 t} + e^{-in\omega_0 t}}{2} + b_n \frac{e^{in\omega_0 t} - e^{-in\omega_0 t}}{2i} \right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{1}{2}(a_n - ib_n)e^{in\omega_0 t} + \frac{1}{2}(a_n + ib_n)e^{-in\omega_0 t} \right] \\ &= c_0 + \sum_{n=1}^{\infty} (c_n e^{in\omega_0 t} + c_{-n} e^{-in\omega_0 t}) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}, \end{aligned}$$

where

$$c_0 = \frac{a_0}{2}, c_n = \frac{a_n - ib_n}{2}, c_{-n} = \frac{a_n + ib_n}{2} \text{ for } n \in \mathbb{N} \quad (15)$$

The form

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t} \quad (16)$$

is called *complex form of the Fourier series* of f . The coefficients c_n are *complex Fourier coefficients* of the function f . We note that

$$\begin{aligned} c_n &= \frac{1}{2}(a_n - ib_n) \\ &= \frac{1}{2} \left[\frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos n\omega_0 t dt - \frac{2i}{T} \int_{-T/2}^{T/2} f(t) \sin n\omega_0 t dt \right] \\ &= \frac{1}{T} \int_{-T/2}^{T/2} f(t) \left[\frac{e^{in\omega_0 t} + e^{-in\omega_0 t}}{2} - i \frac{e^{in\omega_0 t} - e^{-in\omega_0 t}}{2i} \right] dt \\ &= \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-in\omega_0 t} dt. \end{aligned} \quad (17)$$

Similarly,

$$c_{-n} = \frac{a_n + ib_n}{2} = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{in\omega_0 t} dt. \quad (18)$$

If f is real, then c_n and c_{-n} are complex conjugates. Thus, the *complex Fourier coefficients* c_n are defined by:

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-in\omega_0 t} dt, \quad n \in \mathbb{Z} \quad (19)$$

The term $e^{in\omega_0 t}$ in Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}$ is called *time-harmonic function*.

4.6 SPECTRUM OF PERIODIC FUNCTIONS

Let f be a periodic function defined for $t \in \mathbb{R}$. If t is time variable, then we say that the periodic function f is defined in the *time domain*. Further, each Fourier coefficient in the Fourier expansion of f is associated with a specific frequency $n\omega_0$. Also if the series converges to f , then the function f is completely determined by their Fourier coefficients. Therefore, we say that f is described by the Fourier coefficients in the *frequency domain*.

The signals are generally interpreted in terms of frequencies. For example, sound is expressed in terms of frequency as *pitch* whereas light is expressed in term of frequency as *colour*.

Definition 4.3. The sequence of Fourier coefficients c_n with $n \in \mathbb{Z}$, which describe a function in the frequency domain, is called the *spectrum of the function*. Since n assumes only integer values, the spectrum is called a *discrete spectrum* or *line spectrum*.

Definition 4.4. The sequence of absolute values of c_n , that is, $\{|c_n|\}$ is called *amplitude spectrum*, whereas, the sequence $\{\arg(c_n)\}$ is called *phase spectrum* of the function.

4.7 PROPERTIES OF FOURIER COEFFICIENTS

Following theorems describe the properties of Fourier coefficients:

Theorem 4.1. (Linearity). Let c_n and d_n be, respectively, the Fourier coefficients of f and g . Then the Fourier coefficients of $af + bg$, $a, b \in \mathbb{C}$ shall be $ac_n + bd_n$. (Thus Fourier coefficients of linear combinations of functions is equal to the same linear combinations of the Fourier coefficients of the individual functions).

Proof: Let $\hat{C}_n$ be the Fourier coefficients of $af + bg$. Then

$$\begin{aligned}\hat{C}_n &= \frac{1}{T} \int_{-T/2}^{T/2} [af(t) + bg(t)] e^{-in\omega_0 t} dt \\ &= \frac{a}{T} \int_{-T/2}^{T/2} f(t) e^{-in\omega_0 t} dt + \frac{b}{T} \int_{-T/2}^{T/2} g(t) e^{-in\omega_0 t} dt \\ &= ac_n + bd_n.\end{aligned}$$

Theorem 4.2. (Conjugation). If Fourier coefficients of f are c_n , then Fourier coefficients of $\overline{f(t)}$ are $\bar{c}_{-n}$.

Proof: Since $e^{in\omega_0 t} = e^{-in\omega_0 t}$, we have

$$\begin{aligned}&= \frac{1}{T} \int_{-T/2}^{T/2} \overline{f(t)} e^{-in\omega_0 t} dt \\ &= \frac{1}{T} \int_{-T/2}^{T/2} \overline{f(t) e^{in\omega_0 t}} dt \\ &= \frac{1}{T} \int_{-T/2}^{T/2} \overline{f(t) e^{-i(-n)\omega_0 t}} dt = \bar{c}_{-n}.\end{aligned}$$

Theorem 4.3. (Shift in Time). If c_n are Fourier coefficients of f , then the Fourier coefficient of $f(t - t_0)$ is $e^{-in\omega_0 t_0} c_n$.

Proof: We have

$$\begin{aligned}&\frac{1}{T} \int_{-T/2}^{T/2} f(t - t_0) e^{-in\omega_0 t} dt \\ &= e^{-in\omega_0 t_0} \cdot \frac{1}{T} \int_{-T/2}^{T/2} f(t - t_0) e^{-in\omega_0 (t - t_0)} dt \\ &= e^{-in\omega_0 t_0} \cdot \frac{1}{T} \int_{-T/2}^{T/2} f(u) e^{-in\omega_0 u} du, \quad t - t_0 = u \\ &= e^{-in\omega_0 t_0} c_n.\end{aligned}$$

Theorem 4.4. (Time reversal). If c_n are the Fourier coefficients of $f(t)$, then c_{-n} will be the Fourier coefficients of $f(-t)$.

Proof: Putting $-t = u$, we get

$$\begin{aligned}&\frac{1}{T} \int_{-T/2}^{T/2} f(-t) e^{-in\omega_0 t} dt \\ &= \frac{1}{T} \int_{T/2}^{-T/2} f(u) e^{-in\omega_0 (-u)} d(-u) \\ &= \frac{1}{T} \int_{-T/2}^{T/2} f(u) e^{-i(-n)\omega_0 u} du = c_{-n}.\end{aligned}$$

Theorem 4.5. (Bessel's inequality). Let c_n be the Fourier coefficients of piecewise continuous periodic function f with period T . Then

$$\sum_{n=-\infty}^{\infty} |c_n|^2 \leq \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt.$$

Proof: Let

$$S_n(t) = \sum_{k=-n}^n c_k e^{ik\omega_0 t} \quad (20)$$

be the partial sum of the Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}$. Let $-n \leq k \leq n$. Then

$$\begin{aligned}&\frac{1}{T} \int_{-T/2}^{T/2} [f(t) - S_n(t)] e^{-ik\omega_0 t} dt \\ &= \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-ik\omega_0 t} dt - \frac{1}{T} \int_{-T/2}^{T/2} S_n(t) e^{-ik\omega_0 t} dt \\ &= c_k - \frac{1}{T} \sum_{l=-n}^n c_l \int_{-T/2}^{T/2} e^{i(l-k)\omega_0 t} dt, \text{ using (20).}\end{aligned}$$

But

$$\int_{-T/2}^{T/2} e^{-i(l-k)\omega_0 t} dt = \begin{cases} 0 & \text{for } l \neq k \\ T & \text{for } l = k. \end{cases}$$

Hence

$$\begin{aligned} \frac{1}{T} \int_{-T/2}^{T/2} [f(t) - S_n(t)] e^{-ik\omega_0 t} dt \\ = c_k - \frac{1}{T} T c_k = c_k - c_k = 0 \end{aligned} \quad (21)$$

Using (21), it follows that

$$\begin{aligned} \frac{1}{T} \int_{-T/2}^{T/2} [f(t) - S_n(t)] \overline{S_n(t)} dt \\ = \sum_{k=-n}^n \bar{c}_k \int_{-T/2}^{T/2} [f(t) - S_n(t)] e^{-ik\omega_0 t} dt = 0. \end{aligned} \quad (22)$$

But (22) implies

$$\begin{aligned} \frac{1}{T} \int_{-T/2}^{T/2} [f(t) - S_n(t)] \overline{[f(t) - S_n(t)]} dt \\ = \int_{-T/2}^{T/2} [f(t) - S_n(t)] \overline{f(t)} dt \\ = \int_{-T/2}^{T/2} f(t) \overline{f(t)} dt - \int_{-T/2}^{T/2} S_n(t) \overline{f(t)} dt \\ = \int_{-T/2}^{T/2} |f(t)|^2 dt - \sum_{k=-n}^n c_k \int_{-T/2}^{T/2} e^{ik\omega_0 t} \overline{f(t)} dt \\ = \int_{-T/2}^{T/2} |f(t)|^2 dt - \sum_{k=-n}^n c_k \int_{-T/2}^{T/2} \overline{e^{-ik\omega_0 t} f(t)} dt \\ = \int_{-T/2}^{T/2} |f(t)|^2 dt - \sum_{k=-n}^n c_k T \bar{c}_k \quad (\text{by definition of } c_k) \\ = \int_{-T/2}^{T/2} |f(t)|^2 dt - T \sum_{k=-n}^n |c_k|^2. \end{aligned}$$

The integrand on the left-hand side is equal to $|f(t) - S_n(t)|^2 \geq 0$. Hence the left-hand side is also

greater than or equal to zero. Hence

$$T \sum_{k=-n}^n |c_k|^2 \leq \int_{-T/2}^{T/2} |f(t)|^2 dt. \quad (23)$$

Since (23) holds for any $n \in \mathbb{N}$, letting $n \rightarrow \infty$, we have

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \lim_{n \rightarrow \infty} \sum_{k=-n}^n |c_k|^2 \leq \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt,$$

which proves the theorem.

Remark 4.1. For real form of the Fourier series, we have $c_n = \frac{a_n - ib_n}{2}$ so that Bessel's inequality takes the form

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \leq \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt. \quad (24)$$

Theorem 4.6. (Riemann-Lebesgue Lemma). Let f be a piecewise continuous periodic function with Fourier coefficients c_n . Then $\lim_{n \rightarrow \infty} c_n = \lim_{n \rightarrow -\infty} c_n = 0$.

Proof: Since f is piecewise continuous, so is $|f(t)|^2$. Hence the integral $\int_{-T/2}^{T/2} |f(t)|^2 dt$ is finite. But, by Bessel's inequality, we have

$$\sum_{n=-\infty}^{\infty} |c_n|^2 \leq \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt.$$

Since right-hand side is finite, the series $\sum_{n=-\infty}^{\infty} |c_n|^2$ of positive terms is convergent. Hence $c_n \rightarrow 0$ as $n \rightarrow \pm\infty$.

Remark 4.2. Since $c_n = \frac{a_n - ib_n}{2}$, it follows from the Riemann-Lebesgue lemma that a_n, b_n tend to zero as $n \rightarrow \infty$.

Theorem 4.7. (Mean Convergence of Fourier Series in $L^2[-\pi, \pi]$). Let $f \in L^2[-\pi, \pi]$. Then the Fourier series of f converges in mean to f in $L^2[-\pi, \pi]$.

Proof: A function f is said to belong to the class $L^p[-\pi, \pi]$, $0 < p \leq \infty$, if $\int_{-\pi}^{\pi} |f(t)|^p dt < \infty$.

Further, a sequence of functions $f_n \in L^p[-\pi, \pi]$, $0 < p \leq \infty$ is said to *converge in norm* or *converge in mean with index p* to $f \in L^p[-\pi, \pi]$ if

$$\lim_{n \rightarrow \infty} \left[\int_{-\pi}^{\pi} |f_n(t) - f(t)|^p dt \right]^{\frac{1}{p}} = 0.$$

A necessary and sufficient condition for convergence of the sequence of functions $f_n \in L^p$ is that given $\varepsilon > 0$ there exists N such that

$$\|f_n - f_m\|_{L^p} < \varepsilon \quad \text{for } n, m \geq N,$$

$$\left[\int_{-\pi}^{\pi} |f_n - f_m|^p dt \right]^{\frac{1}{p}} < \varepsilon \quad \text{for } n, m \geq N.$$

Let $\{S_n(x)\}$ be the sequence of partial sums of the Fourier series $\sum_{k=0}^{\infty} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t)$ for $f \in L^2[-\pi, \pi]$. Then if $m < n$, we have

$$\begin{aligned} & \frac{1}{\pi} \int_{-\pi}^{\pi} |S_n(x) - S_m(x)|^2 dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \left| \sum_{k=m+1}^n a_k \cos k\omega_0 t + b_k \sin k\omega_0 t \right|^2 dt \\ &\leq \sum_{k=m+1}^n |a_k^2 + b_k^2|. \end{aligned}$$

But, by Bessel's inequality, $\sum_{k=1}^n |a_k^2 + b_k^2| < \infty$ for $f \in L^2[-\pi, \pi]$. Therefore there exists a number N such that $\sum_{k=m+1}^n |a_k^2 + b_k^2| < \varepsilon$ for $m > N$. Hence

$$\|S_n(x) - S_m(x)\|_{L^2} < \varepsilon \quad \text{for } n, m \geq N,$$

and so the Fourier series of f converges in mean to f .

Theorem 4.8. (Riesz-Fischer Theorem). If $\{a_k\}$ and $\{b_k\}$ are the sequences of real numbers such that

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) < \infty,$$

then there exists a function f such that $\int_{-\pi}^{\pi} |f(t)|^2 dt < \infty$ and whose Fourier coefficients are precisely a_k and b_k .

Proof: Let

$$S_n(t) = \frac{a_0}{2} + \sum_{k=1}^n (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t),$$

where $-\frac{T}{2} \leq t \leq \frac{T}{2}$. If $m < n$, then

$$\begin{aligned} \frac{2}{T} \int_{-T/2}^{T/2} (S_n - S_m)^2 dt &= \frac{2}{T} \int_{-T/2}^{T/2} \sum_{k=m+1}^n (a_k \cos k\omega_0 t \\ &\quad + b_k \sin k\omega_0 t)^2 dt \\ &= \sum_{k=m+1}^{\infty} (a_k^2 + b_k^2). \end{aligned} \quad (25)$$

But, by the given hypothesis, the series $\sum_{k=1}^{\infty} (a_k^2 + b_k^2)$ is convergent. Hence the tail $\sum_{k=m+1}^{\infty} (a_k^2 + b_k^2)$ tends to zero as $n \rightarrow \infty$. The left-hand side of (25) is nothing but $\frac{2}{T} \|S_n - S_m\|^2$. Therefore, $\|S_n - S_m\| \rightarrow 0$ as $m, n \rightarrow \infty$. Thus $\{S_n\}$ is a Cauchy sequence. But f is in $L^2[-\frac{T}{2}, \frac{T}{2}]$ space, which is complete. Hence this sequence of partial sum converges to some function f in $L^2[-\frac{T}{2}, \frac{T}{2}]$. Thus for any $p = 0, 1, 2, \dots$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{2}{T} \int_{-T/2}^{T/2} S_n(t) \cos p\omega_0 t dt \\ = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos p\omega_0 t dt. \end{aligned}$$

But if $n \geq p$, we have

$$\frac{2}{T} \int_{-T/2}^{T/2} S_n(t) \cos p\omega_0 t dt = a_p.$$

Hence

$$\frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos p\omega_0 t dt = a_p \quad (p = 0, 1, 2, \dots).$$

Thus a_p are the Fourier cosine coefficients of f . Similarly, it can be proved that b_p are the Fourier sine coefficients of f .

4.8 DIRICHLET'S KERNEL

Definition 4.5. The Dirichlet's kernel $D_n(t)$ is defined by

$$D_n(t) = \frac{1}{2} + \cos t + \dots + \cos nt. \quad (26)$$

This is a periodic function and plays an important role in the convergence of trigonometric series.

Multiplying both sides of (26) with $2 \sin \frac{t}{2}$, we get

$$2 \sin \frac{t}{2} D_n(t) = \sin \frac{t}{2} + 2 \cos t \sin \frac{t}{2} + 2 \cos 2t \sin \frac{t}{2} + \dots + 2 \cos nt \sin \frac{t}{2} \quad (27)$$

Applying the formula

$$2 \cos \alpha \sin \beta = \sin(\alpha + \beta) - \sin(\alpha - \beta),$$

the equation (27) reduces to

$$\begin{aligned} 2 \sin \frac{t}{2} D_n(t) &= \sin \frac{t}{2} + \left(\sin \frac{3}{2} t - \sin \frac{t}{2} \right) + \left(\sin \frac{5}{2} t - \sin \frac{3}{2} t \right) \\ &+ \dots + \left(\sin \left(n + \frac{1}{2} \right) t - \sin \left(n - \frac{1}{2} \right) t \right) \\ &= \sin \left(n + \frac{1}{2} \right) t. \end{aligned}$$

Hence

$$D_n(t) = \frac{\sin \left(n + \frac{1}{2} \right) t}{2 \sin \frac{t}{2}}. \quad (28)$$

We note that

- (a) Dirichlet's kernel is an *even periodic function*.
 (b) the expression (28) for Dirichlet's kernel implies that

$$D_n(t) \leq \frac{1}{2 \sin \frac{t}{2}}. \quad (29)$$

Simple differentiation shows that $\frac{\sin t}{t}$ decreases in the interval $(0, \frac{\pi}{2})$. Therefore,

$$\frac{\sin t}{t} \geq \frac{\sin \frac{\pi}{2}}{\frac{\pi}{2}} = \frac{2}{\pi},$$

which means

$$\frac{\sin t}{t} \geq \frac{2}{\pi} \text{ for } 0 \leq t \leq \frac{\pi}{2},$$

that is,

$$\frac{\sin \frac{t}{2}}{\frac{t}{2}} \geq \frac{2}{\pi} \text{ for } 0 \leq \frac{t}{2} \leq \frac{\pi}{2},$$

that is,

$$\sin \frac{t}{2} \geq \frac{t}{\pi} \text{ for } 0 \leq t \leq \pi.$$

Hence (29) yields

$$|D_n(t)| \leq \frac{\pi}{2t} \text{ for } 0 < |t| \leq \pi,$$

and so

$$|D_n(t)| = O\left(\frac{1}{t}\right). \quad (30)$$

Sometimes it will be required that if $\delta \leq |t| \leq \pi$, then

$$|D_n(t)| \leq \frac{\pi}{2\delta}. \quad (31)$$

(c) The expression (26) shows that

$$|D_n(t)| \leq \frac{1}{2} + \underbrace{1 + 1 + \dots + 1}_{n \text{ times}} = n + \frac{1}{2}. \quad (32)$$

(d) Integrating the expression (26) over the interval $[-\frac{T}{2}, \frac{T}{2}]$, we have

$$\int_{-T/2}^{T/2} D_n(t) dt = \frac{T}{2}. \quad (33)$$

(e) Since $D_n(t)$ is an even function, we have

$$L_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |D_n(t)| dt = \frac{2}{\pi} \int_0^{\pi} |D_n(t)| dt.$$

Expanding $\frac{\sin \left(n + \frac{1}{2} \right) t}{2 \sin \frac{t}{2}}$, we have $D_n(t) = \frac{\sin nt}{t} + O(1)$. Therefore,

$$L_n = \frac{2}{\pi} \int_0^{\pi} \left| \frac{\sin nt}{t} \right| dt + O(1).$$

Since $\int_0^{\pi} \left| \frac{\sin nt}{t} \right| dt \approx \frac{2}{\pi} \log n$, it follows that

$$L_n \approx \frac{4}{\pi^2} \log n.$$

Thus, we have shown that

$$L_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |D_n(t)| dt \approx \frac{4}{\pi^2} \log n. \quad (34)$$

The constant L_n in the expression (34) is called *Lebesgue constant*.

Remarks 4.3. (a) The expression

$$\bar{D}_n(t) = \sin t + \sin 2t + \dots + \sin nt$$

is called *conjugate Dirichlet's kernel*. It takes the form

$$\bar{D}_n(t) = \frac{\cos \frac{t}{2} - \cos \left(n + \frac{1}{2} \right) t}{2 \sin \frac{t}{2}}$$

and so

$$\bar{D}_n(t) \leq \frac{1}{\sin \frac{t}{2}}.$$

(b) The expressions

$$K_n(t) = \frac{1}{n+1} \sum_{p=0}^n D_p(x)$$

and

$$\bar{K}_n(t) = \frac{1}{n+1} \sum_{p=0}^n \bar{D}_p(x)$$

are called the *Fejér's kernel* and *conjugate Fejér's kernel*, respectively.

4.9 INTEGRAL EXPRESSION FOR PARTIAL SUMS OF A FOURIER SERIES

Let

$$S_n(t) = \frac{a_0}{2} + \sum_{k=1}^n (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t)$$

be the partial sum of the Fourier series. Replacing a_k and b_k by the defining integrals, we obtain

$$\begin{aligned} S_n(t) &= \frac{1}{T} \int_{-T/2}^{T/2} f(u) du \\ &\quad + \frac{2}{T} \sum_{k=1}^n \left[\left(\int_{-T/2}^{T/2} f(u) \cos k\omega_0 u du \right) \cos k\omega_0 t \right. \\ &\quad \left. + \left(\int_{-T/2}^{T/2} f(u) \sin k\omega_0 u du \right) \sin k\omega_0 t \right] \\ &= \frac{2}{T} \int_{-T/2}^{T/2} f(u) \left[\frac{1}{2} + \sum_{k=1}^n (\cos k\omega_0 u \cos k\omega_0 t \right. \\ &\quad \left. + \sin k\omega_0 u \sin k\omega_0 t) \right] du \\ &= \frac{2}{T} \int_{-T/2}^{T/2} f(u) \left[\frac{1}{2} + \sum_{k=1}^n \cos k\omega_0 (u-t) \right] du \\ &= \frac{2}{T} \int_{-T/2}^{T/2} f(u) D_n(u-t) du. \end{aligned}$$

Substituting $u - t = x$, we have

$$\begin{aligned} S_n(t) &= \frac{2}{T} \int_{(-T/2)-t}^{(T/2)-t} f(x+t) D_n(x) dx \\ &= \frac{2}{T} \int_{-T/2}^{T/2} f(x+t) D_n(x) dx, \end{aligned}$$

using the fact that the functions $f(x+t)$ and $D_n(x)$ are periodic in the variable x with period T and the length of the interval $[-\frac{T}{2}-t, \frac{T}{2}-t]$ is T and so the integral over $[-\frac{T}{2}-t, \frac{T}{2}-t]$ is same as the integral over $[-\frac{T}{2}, \frac{T}{2}]$.

Since $D_n(x)$ is even, that is, $D_n(-x) = D_n(x)$, the partial sum becomes

$$\begin{aligned} S_n(t) &= \frac{2}{T} \int_{-T/2}^0 f(x+t) D_n(x) dx + \frac{2}{T} \int_0^{T/2} f(x+t) D_n(x) dx \\ &= \frac{2}{T} \int_0^{T/2} f(t-x) D_n(-x) dx + \frac{2}{T} \int_0^{T/2} f(x+t) D_n(x) dx \\ &= \frac{2}{T} \int_0^{T/2} [f(x+t) + f(t-x)] D_n(x) dx \\ &= \frac{2}{T} \int_0^{T/2} [f(t+x) - f(t+) + f(t-x) - f(t-)] D_n(x) dx \\ &\quad + \frac{2}{T} \int_0^{T/2} [f(t+) + f(t-)] D_n(x) dx. \end{aligned} \quad (35)$$

Since $\int_0^{T/2} D_n(x) dx = \frac{T}{4}$ by (33), the second term in (35) is $\frac{f(t+) + f(t-)}{2}$. Thus, we get

$$\begin{aligned} S_n(t) &= \frac{f(t+) + f(t-)}{2} + \frac{2}{T} \int_0^{T/2} [f(t+x) - f(t+) \\ &\quad + f(t-x) - f(t-)] D_n(x) dx. \end{aligned}$$

4.10 FUNDAMENTAL THEOREM (CONVERGENCE THEOREM) OF FOURIER SERIES

The following theorem shows that the Fourier series of a piecewise smooth function converges to that function at each point of continuity.

Theorem 4.9. (Fundamental Theorem of Fourier Series). Let f be a piecewise smooth periodic function with period T defined on $\mathbb{R}$ with Fourier

coefficients a_n and b_n . Then for any $t \in \mathbb{R}$,

$$\begin{aligned} \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \\ = \frac{1}{2} [f(t+) + f(t-)]. \end{aligned}$$

[At the point of continuity $f(t+) = f(t-) = f(t)$ and so, in that case, the right-hand side becomes $f(t)$].

Proof: If $S_n(t)$ is the partial sum of the Fourier series, then we have established (above) that

$$\begin{aligned} S_n(t) = \frac{f(t+) + f(t-)}{2} \\ + \frac{2}{T} \int_{-T/2}^{T/2} [f(t+x) - f(t+) \\ + f(t-x) - f(t-)] D_n(x) dx, \end{aligned}$$

where $D_n(x)$ is the Dirichlet's kernel. To prove the theorem, it is sufficient to show that the term

$$\begin{aligned} I_n(t) = \frac{2}{T} \int_{-T/2}^{T/2} [f(t+x) - f(t+) + f(t-x) \\ - f(t-)] D_n(x) dx \end{aligned}$$

tends to zero as $n \rightarrow \infty$. To this end, we have

$$\begin{aligned} I_n(t) &= \frac{2}{T} \int_{-T/2}^{T/2} \frac{[f(t+x) - f(t+) + f(t-x) - f(t-)]}{x} \\ &\quad \times \frac{x \sin(n + \frac{1}{2})\omega_0 x}{2 \sin \omega_0 \frac{x}{2}} dx \\ &= \frac{2}{T} \int_{-T/2}^{T/2} Q(x) \sin\left(n + \frac{1}{2}\right) \omega_0 x dx, \quad (36) \end{aligned}$$

where

$$\begin{aligned} Q(x) &= \frac{f(t+x) - f(t+) + f(t-x) - f(t-)}{x} \\ &\quad \times \frac{x}{2 \sin \omega_0 \frac{x}{2}}. \end{aligned}$$

For $x = 0$, the denominator of $Q(x)$ equals 0 and so integral $I_n(t)$ is not defined for $x = 0$. But since f is piecewise smooth,

$$\lim_{x \rightarrow 0} \frac{f(t+x) - f(t+)}{x} = f'(t+)$$

(right-hand derivative)

and

$$\lim_{x \rightarrow 0} \frac{f(t-x) - f(t-)}{x} = f'(t-)$$

(left-hand derivative)

exist. Also $\lim_{x \rightarrow 0} \frac{x}{\sin \omega_0 \frac{x}{2}}$ exists. Hence, $Q(x)$ is piecewise continuous function. Further, both $Q(x)$ and $\sin(n + \frac{1}{2})\omega_0 x$ being odd, the integrand in (36) is an even function. Therefore, (36) can be written as

$$\begin{aligned} I_n(t) &= \frac{2}{T} \int_{-T/2}^{T/2} Q(x) \sin\left(n + \frac{1}{2}\right) \omega_0 x dx \\ &= \frac{1}{2} \cdot \frac{2}{T} \int_{-T/2}^{T/2} Q(x) \left[\sin n \omega_0 x \cos \omega_0 \frac{x}{2} \right. \\ &\quad \left. + \cos n \omega_0 x \sin \omega_0 \frac{x}{2} \right] dx \\ &= \frac{1}{2} \cdot \frac{2}{T} \int_{-T/2}^{T/2} Q(x) \cos \omega_0 \frac{x}{2} \sin n \omega_0 x dx \\ &\quad + \frac{1}{2} \cdot \frac{2}{T} \int_{-T/2}^{T/2} Q(x) \sin \omega_0 \frac{x}{2} \cos n \omega_0 x dx. \end{aligned}$$

But $\frac{2}{T} \int_{-T/2}^{T/2} Q(x) \cos \omega_0 \frac{x}{2} \sin n \omega_0 x dx$ is the Fourier coefficient b_n for the function $Q(x) \cos \omega_0 \frac{x}{2}$ whereas $\frac{2}{T} \int_{-T/2}^{T/2} Q(x) \sin \omega_0 \frac{x}{2} \cos n \omega_0 x dx$ is Fourier coefficient a_n for the function $Q(x) \sin \omega_0 \frac{x}{2}$. By Riemann-Lebesgue lemma, both of these coefficients tend to zero as $n \rightarrow \infty$. Hence $I_n(t) \rightarrow 0$ as $n \rightarrow \infty$. It follows, therefore, that the Fourier series converges to $\frac{1}{2}[f(t+) + f(t-)]$.

Remark 4.4. It follows from Theorem 4.9 that if two periodic piecewise smooth functions have the same Fourier series, that is, if their Fourier coefficients are equal, then these functions must be equal at all points of continuity. This assertion is known as *Uniqueness theorem*.

Remark 4.5. The assumption in the convergence theorem may be written as:

- (a) the function f is periodic and single-valued.
- (b) f is piecewise continuous.

(c) f has finite number of maxima and minima in a period.

These three conditions are called *Dirichlet's conditions*.

4.11 APPLICATIONS OF FUNDAMENTAL THEOREM OF FOURIER SERIES

As consequences of fundamental theorem of Fourier series, we have the following results:

Theorem 4.10. (Fourier Series of a Product of Functions). Let f and g be piecewise smooth periodic functions with Fourier coefficients c_n and d_n respectively. Then $h = fg$ has a convergent Fourier series with Fourier coefficients p_n given by

$$p_n = \sum_{k=-\infty}^{\infty} c_k d_{n-k}.$$

Proof: Since f and g are piecewise smooth periodic functions, so is $h = fg$. Therefore, by fundamental theorem of Fourier series, h has a convergent Fourier series. The Fourier coefficients p_n of h are given by

$$p_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) g(t) e^{-in\omega_0 t} dt.$$

Since f is piecewise smooth periodic function, by fundamental theorem of Fourier series, it can be replaced by its Fourier series at the points of continuity. But in integration, the values at the points of discontinuity are of no importance. Therefore,

$$p_n = \frac{1}{T} \int_{-T/2}^{T/2} \sum_{k=-\infty}^{\infty} c_k e^{ik\omega_0 t} g(t) e^{-in\omega_0 t} dt.$$

Changing the order of integration and summation, we have

$$p_n = \sum_{k=-\infty}^{\infty} c_k \frac{1}{T} \int_{-T/2}^{T/2} g(t) e^{-i(n-k)\omega_0 t} dt = \sum_{k=-\infty}^{\infty} c_k d_{n-k}.$$

Theorem 4.11. (Parseval's Identity). Let f and g be piecewise smooth periodic function with Fourier coefficients c_n and d_n , respectively. Then

$$\frac{1}{T} \int_{-T/2}^{T/2} f(t) \overline{g(t)} dt = \sum_{k=-\infty}^{\infty} c_k \bar{d}_k.$$

Proof: Since d_n is Fourier coefficient of $g(t)$, by Theorem 4.2, the Fourier coefficient of $\overline{g(t)}$ shall be $\bar{d}_{-n}$. Now if p_n be the Fourier coefficients of the product $f \bar{g}$, Theorem 4.10 implies that

$$p_n = \sum_{k=-\infty}^{\infty} c_k \bar{d}_{-(n-k)}.$$

In particular,

$$p_0 = \sum_{k=-\infty}^{\infty} c_k \bar{d}_k.$$

But, by definition,

$$p_0 = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \overline{g(t)} dt.$$

Hence

$$\frac{1}{T} \int_{-T/2}^{T/2} f(t) \overline{g(t)} dt = \sum_{k=-\infty}^{\infty} c_k \bar{d}_k.$$

Theorem 4.12. (Parseval's equality). Let f be a piecewise smooth periodic function with Fourier coefficient c_n . Then

$$\frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

Proof: Taking $f(t) = g(t)$ in Theorem 4.11, we have

$$\frac{1}{T} \int_{-T/2}^{T/2} f(t) \overline{f(t)} dt = \sum_{k=-\infty}^{\infty} c_k \bar{c}_k$$

and so

$$\frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2.$$

Definition 4.6. The integral $\frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt$ is called the *power* of periodic time continuous signal f . Thus, if f is piecewise smooth periodic function, then by Theorem 4.12, its power can be calculated using Fourier coefficients. In fact

$$P = \sum_{k=-\infty}^{\infty} |c_k|^2.$$

4.12 CONVOLUTION THEOREM FOR FOURIER SERIES

Definition 4.7. The *convolution product* of two piecewise smooth periodic functions f and g with period T is defined by

$$(f * g)(t) = \frac{1}{T} \int_{-T/2}^{T/2} f(u) g(t-u) du.$$

The convolution product is very useful in system analysis. We note that for $k \in \mathbb{Z}$,

$$\begin{aligned} (f * g)(t + kT) &= \frac{1}{T} \int_{-T/2}^{T/2} f(u) g(t + kT - u) du \\ &= \frac{1}{T} \int_{-T/2}^{T/2} f(u) g(t - u) du \\ &= (f * g)(t), \end{aligned}$$

since g being periodic, $g(t - u + kT) = g(t - u)$. It follows, therefore, that convolution product of periodic function is also *periodic* with the *same period*.

Theorem 4.13. (Convolution Theorem for Fourier Series). If f and g are piecewise smooth periodic functions with Fourier coefficients c_n and d_n , then $f * g$ has a convergent Fourier series with Fourier coefficients $c_n d_n$ [denoted by $(f * g)_n$].

Proof: Since f and g are piecewise smooth periodic function, $f * g$ is also piecewise smooth periodic function. Hence, by Fundamental theorem of Fourier series, it has a convergent Fourier series. Further

$$\begin{aligned} (f * g)_n &= \frac{1}{T} \int_{-T/2}^{T/2} (f * g)(t) e^{-in\omega_0 t} dt \\ &= \frac{1}{T^2} \int_{-T/2}^{T/2} \left(\int_{-T/2}^{T/2} f(u) g(t-u) du \right) e^{-in\omega_0 t} dt. \end{aligned}$$

Changing the order of integration, we get

$$\begin{aligned} (f * g)_n &= \frac{1}{T} \int_{-T/2}^{T/2} \left(\frac{1}{T} \int_{-T/2}^{T/2} g(t-u) e^{-in\omega_0(t-u)} dt \right) \\ &\quad \times f(u) e^{-in\omega_0 u} du \\ &= \frac{1}{T} d_n \int_{-T/2}^{T/2} f(u) e^{-in\omega_0 u} du = c_n d_n. \end{aligned}$$

4.13 INTEGRATION OF FOURIER SERIES

Sometimes, the Fourier series of a function is known but not the function itself. In such cases, the following problems arise:

(a) If Fourier series of the function f of period 2π is given, can we calculate $\int_a^b f(x) dx$ over arbitrary interval $[a, b]$?

(b) If Fourier series of the function f is known, can we find the Fourier series of the function $F(x) = \int_0^x f(t) dt$?

The following theorem provides the answer to the above-posed problems.

Theorem 4.14. Let

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

be the Fourier series of an absolutely integrable function of period 2π . Then $\int_a^b f(t) dt$ can be found by term-by-term integration of the Fourier series (irrespective of the convergence), that is,

$$\begin{aligned} \int_a^b f(t) dt &= \frac{a_0}{2} (b-a) \\ &\quad + \sum_{n=1}^{\infty} \frac{a_n (\sin nb - \sin na) - b_n (\cos nb - \cos na)}{n}. \end{aligned}$$

Moreover, the integral of f has the Fourier series expansion in $(-\pi, \pi)$ given by

$$\int_0^x f(t) dt = \sum \frac{b_n}{n} + \sum_{n=1}^{\infty} \frac{-b_n \cos nx + [a_n + (-1)^{n+1} a_0] \sin nx}{n}.$$

Proof: Let

$$F(x) = \int_0^x f(t) dt - \frac{a_0}{2} x.$$

Then F is continuous and has absolutely integrable derivative (except for a finite number of points).

Moreover,

$$\begin{aligned}
 F(x+2\pi) &= \int_0^{x+2\pi} f(t) dt - \frac{a_0}{2}(x+2\pi) \\
 &= \int_0^x f(t) dt + \int_x^{x+2\pi} f(t) dt - \frac{a_0}{2}x - \pi a_0 \\
 &= F(x) + \int_{-\pi}^{\pi} f(t) dt - \pi a_0 \\
 &= F(x) + \pi a_0 - \pi a_0, \text{ since } a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt \\
 &= F(x).
 \end{aligned}$$

Therefore, F is periodic with period 2π . Hence F can be expressed as a Fourier series

$$F(x) = \frac{A_0}{2} + \sum (A_n \cos nx + B_n \sin nx), \quad (37)$$

where

$$\begin{aligned}
 A_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \cos nx \, dx \\
 &= \frac{1}{\pi} \left[F(x) \frac{\sin nx}{n} \right]_{-\pi}^{\pi} - \frac{1}{n\pi} \int_{-\pi}^{\pi} F'(x) \sin nx \, dx \\
 &= 0 - \frac{1}{n\pi} \int_{-\pi}^{\pi} \left(f(x) - \frac{a_0}{2} \right) \sin nx \, dx \\
 &= -\frac{1}{n\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx + \frac{a_0}{2n\pi} \int_{-\pi}^{\pi} \sin nx \, dx \\
 &= -\frac{1}{n\pi} \cdot \pi b_n + 0 = -\frac{b_n}{n},
 \end{aligned}$$

and similarly

$$B_n = \frac{a_n}{n}.$$

Thus (37) reduces to

$$F(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{n}$$

and so

$$\begin{aligned}
 \int_0^x f(t) dt &= \frac{A_0}{2} + \frac{a_0}{2} x \\
 &+ \sum_{n=1}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{n} \quad (38)
 \end{aligned}$$

Putting $x = b$ and $x = a$ in (38) and subtracting, we get

$$\begin{aligned}
 \int_a^b f(t) dt &= \frac{a_0}{2} (b-a) \\
 &+ \sum_{n=1}^{\infty} \frac{a_n (\sin nb - \sin na) - b_n (\cos nb - \cos na)}{n}.
 \end{aligned}$$

It follows, therefore, that the *Fourier series (even divergent) can be integrated term-by-term in any interval*.

Now if we put $x = 0$ in (38), we get

$$\frac{A_0}{2} = \sum_{n=1}^{\infty} \frac{b_n}{n} \quad (39)$$

But (see Example 4.7)

$$\frac{x}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

Hence (38) reduces to

$$\begin{aligned}
 \int_0^x f(t) dt &= \sum_{n=1}^{\infty} \frac{b_n}{n} \\
 &+ \sum_{n=1}^{\infty} \frac{-b_n \cos nx + [a_n + (-1)^{n+1} a_0] \sin nx}{n}.
 \end{aligned}$$

Remark 4.6. The expression (39) shows that for any Fourier series, the series $\sum \frac{b_n}{n}$ converges. This fact helps us to differentiate the Fourier series of absolutely integrable functions from other trigonometric series. For example, the series

$$\sum_{n=2}^{\infty} \frac{\sin nx}{\log n},$$

converges everywhere but cannot be a Fourier series since the series

$$\sum \frac{b_n}{n} = \sum \frac{1}{n \log n}$$

is divergent.

4.14 DIFFERENTIATION OF FOURIER SERIES

Regarding differentiation of Fourier series, we have the following theorem:

Theorem 4.15. Let f be a continuous function of period 2π having an absolutely integrable derivative (except at certain points). Then the Fourier series of f' can be obtained from the Fourier series of the function f by term-by-term differentiation.

Proof: By Convergence theorem, the Fourier series of f converges to f . So let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \quad (40)$$

If a'_n and b'_n denote Fourier coefficients of f' , then

$$a'_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) dx = f(\pi) - f(-\pi) = 0,$$

$$\begin{aligned} a'_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \cos nx dx \\ &= \frac{1}{\pi} [\cos nx f(x)]_{-\pi}^{\pi} + \frac{n}{\pi} \int_{-\pi}^{\pi} \sin nx f(x) dx \\ &= 0 + \frac{n}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = nb_n, \end{aligned}$$

$$\begin{aligned} b'_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \sin nx dx \\ &= \frac{1}{\pi} [f(x) \sin nx]_{-\pi}^{\pi} - \frac{n}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \\ &= 0 - na_n = -na_n. \end{aligned}$$

Hence the Fourier series of f' is given by

$$f'(x) \sim \sum_{n=1}^{\infty} n(b_n \cos nx - a_n \sin nx),$$

which is nothing but the series obtained from (40) by term-by-term differentiation.

4.15 EXAMPLES OF EXPANSIONS OF FUNCTIONS IN FOURIER SERIES

EXAMPLE 4.4

Expand in Fourier series the function f defined by

$$f(x) = \begin{cases} 0 & \text{for } -\pi \leq x < 0 \\ 1 & \text{for } 0 \leq x \leq \pi. \end{cases}$$

Deduce that sum of the Gregory series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$ is $\frac{\pi}{4}$.

Solution. Taking periodic extension of the function, the graph of f is shown in Figure 4.8. The extended function is of period 2π . So, we have

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_0^{\pi} \cos nx dx = \frac{1}{\pi} \left[\frac{\sin nx}{n} \right]_0^{\pi} \\ &= \begin{cases} 0 & \text{for } n = 1, 2, \dots, \\ 1 & \text{for } n = 0 \end{cases} \end{aligned}$$

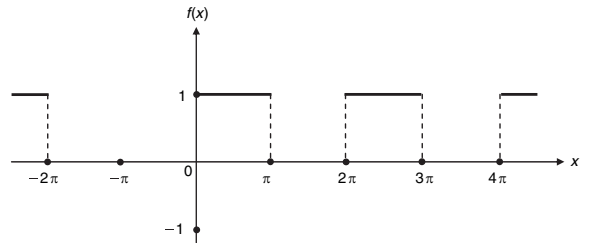


Figure 4.8 Graph of Periodically Extended f

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_0^{\pi} \sin nx dx = \frac{1 - \cos n\pi}{n\pi} \\ &= \begin{cases} \frac{2}{n\pi} & \text{for } n = 1, 3, 5, \dots \\ 0 & \text{for } n = 2, 4, 6, \dots \end{cases} \end{aligned}$$

Hence the Fourier series of f is given by

$$f \sim \frac{1}{2} + \frac{2}{\pi} \left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right].$$

We have used the symbol $\sim$ because the series does not converge to f . In fact, we note that $f(0) = 1$ by definition of f . But $x = 0$ in the series yields the sum as $\frac{1}{2}$.

At $x = \frac{\pi}{2}$, we have

$$1 = \frac{1}{2} + \frac{2}{\pi} \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots \right],$$

which yields

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots = \frac{\pi}{4}.$$

EXAMPLE 4.5

Determine Fourier series of the function f defined by

$$f(x) = \begin{cases} -\pi & \text{for } -\pi < x < 0 \\ x & \text{for } 0 < x < \pi. \end{cases}$$

Prove that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$.

Solution. Taking periodic extension of the given function, we have a function of period 2π . Then

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^0 f(x) dx + \frac{1}{\pi} \int_0^{\pi} f(x) dx \\ &= \frac{1}{\pi} \int_{-\pi}^0 (-\pi) dx + \frac{1}{\pi} \int_0^{\pi} x dx = \frac{1}{\pi} \left[-\pi^2 + \frac{\pi^2}{2} \right] = -\frac{\pi}{2}, \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \left[\int_{-\pi}^0 -\pi \cos nx dx + \int_0^{\pi} x \cos nx dx \right] \\ &= \frac{1}{\pi} \left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2} \right] = \begin{cases} -\frac{2}{\pi n^2} & \text{for odd } n \\ 0 & \text{for even } n, \end{cases} \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \\ &= \frac{1}{\pi} \left[\int_{-\pi}^0 -\pi \sin nx dx + \int_0^{\pi} x \sin nx dx \right] = \frac{1}{n} (1 - 2 \cos n\pi). \end{aligned}$$

Hence Fourier series expansion of f is

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \\ &= -\frac{\pi}{4} - \frac{2}{\pi} \left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right) \\ &\quad + 3 \sin x - \frac{\sin 2x}{2} + \frac{3 \sin 3x}{3} - \frac{\sin 4x}{4} + \dots \end{aligned}$$

Taking $x = 0$, we get

$$f(0) = -\frac{\pi}{4} - \frac{2}{\pi} \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right).$$

But, by Convergence theorem, we have

$$f(0) = \frac{1}{2} [f(0+) + f(0-)] = \frac{1}{2} (-\pi + 0) = -\frac{\pi}{2}.$$

Hence

$$-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi} \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right),$$

which yields

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}.$$

EXAMPLE 4.6

Expand $f(x) = x^2$, $-\pi < x < \pi$ in Fourier series and show that

- $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$
- $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$
- $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

Solution. The function is defined in the interval $(-\pi, \pi)$. The periodic extension of f is continuous and smooth (see Example 4.2). Since $f(x) = f(-x)$, the function is even. Hence, the Fourier coefficients $b_n = 0$ for $n = 1, 2, \dots$. To calculate a_n , we use integration by parts and get

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2\pi^2}{3}, \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx \\ &= \frac{2}{\pi} \left[x^2 \frac{\sin nx}{n} - 2x \left(\frac{-\cos nx}{n^2} \right) + 2 \left(\frac{-\sin nx}{n^3} \right) \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[2\pi \frac{\cos n\pi}{n^2} \right] = \frac{4}{n^2} (-1)^n, \text{ since } \cos n\pi = (-1)^n. \end{aligned}$$

Since f is continuous and smooth, the Fourier series of f converges to f and so the Fourier series is

$$f(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx. \quad (41)$$

Derivations. (a) Substituting $x = \pi$ in (41), we get

$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} (-1)^n = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which yields

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(b) Now putting $x = 0$ in (41), we get

$$0 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

Thus

$$4 \left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \dots \right] = -\frac{\pi^2}{3}, \quad (42)$$

that is,

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}. \quad (43)$$

Adding (42) and (43), we get

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

or

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

(c) Applying Parseval's equality, we get

$$\int_{-\pi}^{\pi} [f(x)]^2 dx = 2\pi a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2),$$

that is,

$$\int_{-\pi}^{\pi} (x^2)^2 dx = 2\pi \left(\frac{\pi^2}{3} \right)^2 + \pi \sum_{n=1}^{\infty} \frac{16}{n^4},$$

that is,

$$\frac{2}{5} \pi^5 = \frac{2}{9} \pi^5 + \pi \sum_{n=1}^{\infty} \frac{16}{n^4},$$

which yields

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

EXAMPLE 4.7

Obtain the Fourier series for the function $f(x) = 2x + 1$, $-\pi < x < \pi$. Hence deduce Fourier series for x and the line $y = mx + c$.

Solution. Using Fourier formulae, we have

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (2x + 1) dx = \frac{1}{\pi} [x^2 + x]_{-\pi}^{\pi} = 2,$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} (2x + 1) \cos nx \, dx \\ &= \frac{1}{\pi} \left[(2x + 1) \left(\frac{\sin nx}{n} \right) - 2 \left(-\frac{\cos nx}{n^2} \right) \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left[\frac{2 \cos n\pi}{n^2} - \frac{2 \cos n\pi}{n^2} \right] = 0, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} (2x + 1) \sin nx \, dx \\ &= \frac{1}{\pi} \left[(2x + 1) \left(-\frac{\cos nx}{n} \right) - 2 \left(-\frac{\sin nx}{n^2} \right) \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left[\frac{(2\pi + 1)(-\cos n\pi)}{n} + \frac{(-2\pi + 1)(\cos n\pi)}{n} \right] \\ &= \frac{1}{\pi} \left[-\frac{4\pi \cos n\pi}{n} \right] = -\frac{4}{n} (-1)^n. \end{aligned}$$

Since the function is continuous and smooth, by Fundamental theorem, Fourier series converges to f and we have

$$f(x) = 2x + 1 = 1 - 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx, \quad -\pi < x < \pi,$$

that is,

$$x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx,$$

which is the Fourier series for x in $[-\pi, \pi]$.

Comparing $mx + c$ with $2x + 1$, we get

$$mx + c = c - 2m \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx.$$

EXAMPLE 4.8

Find the Fourier series for the function f defined by $f(x) = x - x^2$, $-\pi < x < \pi$. Deduce that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$.

Solution. The periodic extension of f is of period 2π . Using Euler's formulae, we have

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} (x-x^2) dx = \frac{-2\pi^2}{3},$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} (x-x^2) \cos nx \, dx \\ &= \frac{1}{\pi} \left[(x-x^2) \left(\frac{\sin nx}{n} \right) - (1-2x) \left(-\frac{\cos nx}{n^2} \right) + (-2) \left(-\frac{\sin nx}{n^3} \right) \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left[\frac{(1-2\pi) \cos n\pi}{n^2} - \frac{(1+2\pi) \cos n\pi}{n^2} \right] = -\frac{4}{n^2} (-1)^n. \end{aligned}$$

Similarly, one can show that

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x-x^2) \sin nx \, dx = -\frac{2(-1)^n}{n}.$$

Hence the Fourier series is

$$\begin{aligned} f(x) &= -\frac{\pi^2}{3} + 4 \left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \dots \right] \\ &\quad + 2 \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right]. \end{aligned}$$

Putting $x = 0$, we get

$$0 = -\frac{\pi^2}{3} + 4 \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots \right)$$

and so

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}.$$

EXAMPLE 4.9

If a is a real number, find the Fourier series of the function f defined by

$$f(x) = e^{ax}, \quad -\pi < x < \pi$$

$$f(x+2\pi) = f(x), \quad x \in \mathbb{R}.$$

Deduce the value of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2}$.

Solution. Using Euler's formulae, we have

$$\begin{aligned} a_n + ib_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax+inx} \, dx = \frac{1}{\pi(a+in)} [e^{ax+inx}]_{-\pi}^{\pi} \\ &= \frac{1}{\pi(a+in)} e^{in\pi} [e^{a\pi} - e^{-a\pi}] \end{aligned}$$

$$\begin{aligned} &= \frac{2(-1)^n}{\pi(a+in)} \sinh a\pi \\ &= \frac{2(-1)^n(a-in)}{\pi(a^2+n^2)} \sinh a\pi. \end{aligned}$$

Equating real and imaginary parts, we have

$$a_n = \frac{2a(-1)^n \sinh a\pi}{\pi(a^2+n^2)}, \quad b_n = \frac{2n \sinh a\pi}{\pi(a^2+n^2)},$$

$$a_0 = \frac{2a \sinh a\pi}{\pi a^2} = \frac{2 \sinh a\pi}{\pi a}.$$

The series shall also converge to f due to piecewise continuity and smoothness. Hence

$$f(x) = \frac{\sinh a\pi}{\pi} \left[\frac{1}{a} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2} (a \cos nx - n \sin nx) \right].$$

Putting $x = 0$, we get

$$e^0 = 1 = \frac{\sinh a\pi}{\pi} \left[\frac{1}{a} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n a}{a^2+n^2} \right]$$

and so

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2} = \frac{1}{2a^2} (a\pi \operatorname{cosech} a\pi - 1).$$

EXAMPLE 4.10

Find the Fourier series of the function

$$f(x) = |x|, \quad -2 \leq x \leq 2$$

$$f(x) = f(x+4).$$

Solution. The period of the given function is 4. Therefore, the Fourier series shall be

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{2} + b_n \sin \frac{n\pi x}{2} \right),$$

where

$$a_0 = \frac{1}{2} \int_{-2}^2 |x| \, dx = 2,$$

$$\begin{aligned}
 a_n &= \frac{1}{2} \int_{-2}^2 |x| \cos \frac{n\pi x}{2} dx \\
 &= \int_0^2 x \cos \frac{n\pi x}{2} dx \quad (\text{integrand is even}) \\
 &= \left[\frac{2x}{n\pi} \sin \frac{n\pi x}{2} \right]_0^2 - \int_0^2 \frac{2}{n\pi} \sin \frac{n\pi x}{2} dx \\
 &= 0 + \frac{4}{n^2 \pi^2} [-1 + (-1)^n] = \begin{cases} 0 & \text{for even } n \\ -\frac{8}{n^2 \pi^2} & \text{for odd } n. \end{cases}
 \end{aligned}$$

Since the given function is even, $b_n = 0$ for $n = 1, 2, \dots$. Also, the function is continuous on $[0, \frac{1}{2}]$. Hence, by Convergence theorem,

$$f(x) = 1 - \frac{8}{\pi^2} \left[\cos x + \frac{1}{9} \cos 3x + \frac{1}{25} \cos 5x + \dots \right].$$

EXAMPLE 4.11

Expand $f(t) = 1 - t^2$, $-1 \leq t \leq 1$ in Fourier series.

Solution. Periodically extended function of the given function is of period 2. Therefore, its Fourier series shall be

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi t.$$

By Euler's formula, we have

$$\begin{aligned}
 a_0 &= \frac{1}{1} \int_{-1}^1 (1 - t^2) dt = \frac{4}{3}, \\
 a_n &= \frac{1}{1} \int_{-1}^1 (1 - t^2) \cos n\pi t dt \\
 &= \left[(1 - t^2) \frac{\sin n\pi t}{n\pi} \right]_{-1}^1 - \int_{-1}^1 (-2t) \frac{\sin n\pi t}{n\pi} dt \\
 &= 0 + \frac{2}{n\pi} \int_{-1}^1 t \sin n\pi t dt \\
 &= \frac{2}{n\pi} \left[\frac{t(-\cos n\pi t)}{n\pi} \right]_{-1}^1 - \frac{2}{n\pi} \int_{-1}^1 \frac{-\cos n\pi t}{n\pi} dt \\
 &= \frac{-4}{n^2 \pi^2} (-1)^n + \frac{2}{n^2 \pi^2} \left[\frac{\sin n\pi t}{n\pi} \right]_{-1}^1 = -\frac{4}{n^2 \pi^2} (-1)^n.
 \end{aligned}$$

Hence the Fourier series is

$$\begin{aligned}
 f(x) &\sim \frac{2}{3} + \sum_{n=1}^{\infty} \frac{-4}{n^2 \pi^2} (-1)^n \cos n\pi t \\
 &= \frac{2}{3} + \frac{4}{\pi^2} \left[\cos \pi t - \frac{\cos 2\pi t}{2^2} + \frac{\cos 3\pi t}{3^2} - \dots \right].
 \end{aligned}$$

EXAMPLE 4.12

Determine the Fourier series for sawtooth function f defined by

$$f(t) = t, \quad t \in (-\pi, \pi)$$

$$f(t) = f(t + 2\pi)$$

Solution. The periodic extension of the function is piecewise smooth and its graph is shown in Figure 4.9. This extended function is an odd function with period $T = 2\pi$. Therefore, the Fourier series will have only sine terms. Using Euler formulae, we have

$$a_0 = \frac{2}{T} \int_{-T/2}^{T/2} f(t) dt = \frac{1}{\pi} \int_{-\pi}^{\pi} t dt = 0,$$

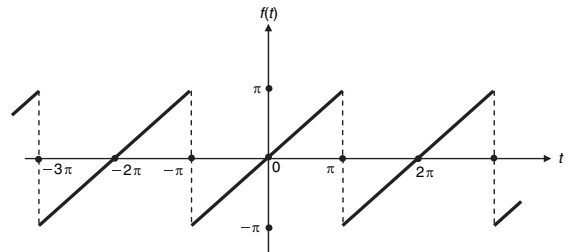


Figure 4.9

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} t \sin nt dt \\
 &= -\frac{1}{n\pi} [t \cos nt]_{-\pi}^{\pi} + \frac{1}{n\pi} \int_{-\pi}^{\pi} \cos nt dt \\
 &= -\frac{2}{n} \cos n\pi + 0 = -\frac{2}{n} (-1)^n, \quad n \in \mathbb{N}.
 \end{aligned}$$

Since the function is piecewise continuous and smooth, we have

$$\begin{aligned}
 f(t) &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin nt \\
 &= 2 \left(\sin t - \frac{\sin 2t}{2} + \frac{\sin 3t}{3} - \dots \right).
 \end{aligned}$$

EXAMPLE 4.13

Determine the Fourier series of the *square wave function* f defined by

$$f(x) = \begin{cases} -k & \text{for } -\pi < x < 0 \\ k & \text{for } 0 < x < \pi, \end{cases}$$

$$f(x) = f(x + 2\pi).$$

Deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} = \frac{\pi}{4}$.

Solution. The graph of periodically extended square wave function f is shown in the Figure 4.10. The function is of period $T = 2\pi$. Further,

$$\begin{aligned} f(-x) &= \begin{cases} -k & \text{for } -\pi < -x < 0, \text{ that is, } 0 < x < \pi \\ k & \text{for } 0 < -x < \pi, \text{ that is, } -\pi < x < 0 \end{cases} \\ &= - \begin{cases} k & \text{for } 0 < x < \pi \\ -k & \text{for } -\pi < x < 0 \end{cases} \\ &= -f(x). \end{aligned}$$

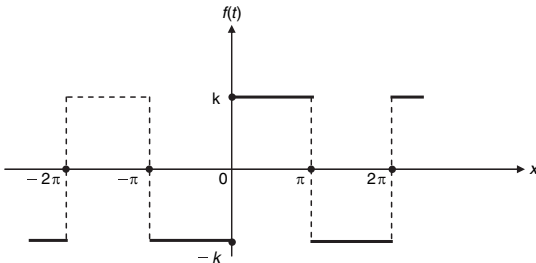


Figure 4.10

Thus, f is an odd function and so its Fourier series consists of sine terms only. We have

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} k \sin nx \, dx \\ &= \frac{2k}{\pi} \left[-\frac{\cos nx}{n} \right]_0^{\pi} = \begin{cases} 0 & \text{for even } n \\ \frac{4k}{n\pi} & \text{for odd } n. \end{cases} \end{aligned}$$

Hence, the function being piecewise continuous and smooth, the Fourier series for f is given by

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \frac{4k}{n\pi} \sin nx, \, n \text{ odd} \\ &= \frac{4k}{\pi} \left[\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right]. \end{aligned}$$

Taking $x = \frac{\pi}{2}$, we get

$$k = \frac{4k}{\pi} \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right]$$

and so

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}.$$

EXAMPLE 4.14

Determine the Fourier series for the *periodic triangle function* f with period T defined for $0 < a \leq \frac{T}{2}$ on $(-\frac{T}{2}, \frac{T}{2})$ by

$$f(x) = \begin{cases} 1 - \frac{|x|}{a} & \text{for } |x| \leq a \\ 0 & \text{for } a < |x| \leq T/2. \end{cases}$$

Solution. The graph of the periodically extended triangle function is shown in the Figure 4.11.

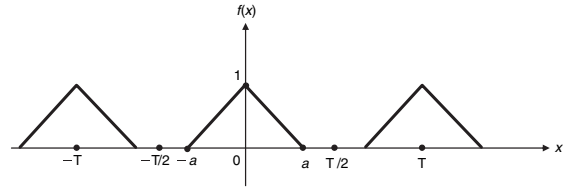


Figure 4.11

The function is periodic with period T and is also even. So the Fourier coefficients b_n are zero. For a_n , we have

$$\begin{aligned} a_n &= \frac{4}{T} \int_0^{T/2} f(x) \cos n\omega_0 x \, dx, \, \omega_0 = \frac{2\pi}{T} \\ &= \frac{4}{T} \int_0^a \left(1 - \frac{x}{a}\right) \cos n\omega_0 x \, dx \\ &= \frac{4}{Tn\omega_0} \left[\left\{ \left(1 - \frac{x}{a}\right) \sin n\omega_0 x \right\}_0^a + \frac{1}{a} \int_0^a \sin n\omega_0 x \, dx \right] \\ &= \frac{4}{aTn\omega_0} \int_0^a \sin n\omega_0 x \, dx = \frac{4}{aTn\omega_0} \left[-\frac{\cos n\omega_0 x}{n\omega_0} \right]_0^a \\ &= \frac{4}{aTn^2\omega_0^2} [1 - \cos n\omega_0 a] = \frac{8}{aTn^2\omega_0^2} \sin^2 \left(\frac{n\omega_0 a}{2} \right), \end{aligned}$$

and

$$a_0 = \frac{4}{T} \int_0^a \left(1 - \frac{x}{a}\right) dx = \frac{2a}{T}.$$

Hence

$$f(x) = \frac{a}{T} + \frac{8}{aT\omega_0^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin^2(n\omega_0 \frac{a}{2}) \cos nx.$$

EXAMPLE 4.15

Determine the Fourier series for *periodic block function* f with period $T > 0$ and $0 \leq a \leq T$ and defined by

$$f(x) = \begin{cases} 1 & \text{for } |x| \leq a/2 \leq T/2 \\ 0 & \text{for } a/2 < |x| \leq T/2. \end{cases}$$

Solution. The graph of periodically extended function with period T is shown in the Figure 4.12. This function is also even.

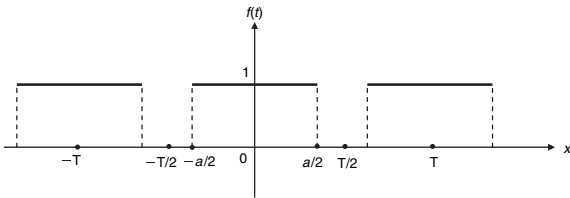


Figure 4.12

We have

$$\begin{aligned} a_n &= \frac{2}{T} \int_{-T/2}^{T/2} f(x) \cos n\omega_0 x \, dx, \quad \omega_0 = \frac{2\pi}{T} \\ &= \frac{4}{T} \int_0^{a/2} \cos n\omega_0 x \, dx = \frac{4}{T} \left[\frac{\sin n\omega_0 x}{n\omega_0} \right]_0^{a/2} \\ &= \frac{4}{Tn\omega_0} \sin(n\omega_0 \frac{a}{2}), \quad n \neq 0, \end{aligned}$$

whereas

$$a_0 = \frac{4}{T} \int_0^{a/2} dx = \frac{2a}{T}.$$

Since f is piecewise continuous and smooth, the Convergence theorem of Fourier series yields

$$f(x) = \frac{a}{T} + \frac{4}{T\omega_0} \sum_{n=1}^{\infty} \frac{1}{n} \sin(n\omega_0 \frac{a}{2}) \cos nx.$$

EXAMPLE 4.16

Expand $f(x) = x^2$, $0 < x < 2\pi$ in a Fourier series assuming that the function is of period 2π .

Solution. We are given that $T = 2\pi$. The graph of the periodically extended function is shown in the Figure 4.13.

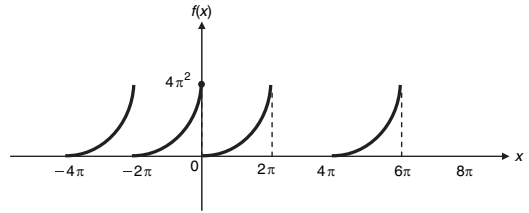


Figure 4.13

Using Fourier formulae, we have

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_0^{2\pi} x^2 \cos nx \, dx \\ &= \frac{1}{\pi} \left[x^2 \frac{\sin nx}{n} - 2x \left(-\frac{\cos nx}{n^2} \right) + 2 \left(-\frac{\sin nx}{n^3} \right) \right]_0^{2\pi} \\ &= \frac{4}{n^2}, \quad n \neq 0. \end{aligned}$$

For $n = 0$, we get

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} x^2 \, dx = \frac{8\pi^2}{3}.$$

Similarly,

$$b_n = \frac{1}{\pi} \int_0^{2\pi} x^2 \sin nx \, dx = -\frac{4\pi}{n}.$$

Thus the required Fourier series is

$$\frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \left(\frac{4}{n^2} \cos nx - \frac{4\pi}{n} \sin nx \right).$$

At $x = 0$ and $x = 2\pi$, the series converges to $2\pi^2$.

EXAMPLE 4.17

Develop Fourier series for $f(x) = x \sin x$, $0 < x < 2\pi$.

Solution. Using Euler's formulae, we have

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{2\pi} x \sin x \, dx \\ &= \frac{1}{\pi} [x(-\cos x)]_0^{2\pi} - \frac{1}{\pi} \int_0^{2\pi} (-\cos x) dx = -2, \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} x \sin x \cos nx \, dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} x [\sin(n+1)x - \sin(n-1)x] dx \\ &= \frac{1}{2\pi} \left[\int_0^{2\pi} x \sin(n+1)x \, dx - \int_0^{2\pi} x \sin(n-1)x \, dx \right] \\ &= \frac{1}{2\pi} \left\{ \left[\frac{x(-\cos(n+1)x)}{n+1} + \frac{\sin(n+1)x}{(n+1)^2} \right]_0^{2\pi} \right. \\ &\quad \left. - \left[\frac{x(-\cos(n-1)x)}{n-1} + \frac{\sin(n-1)x}{(n-1)^2} \right]_0^{2\pi} \right\} \\ &= \frac{1}{2\pi} \left[\frac{-2\pi}{n+1} + \frac{2\pi}{n-1} \right] = \frac{2}{n^2-1}, \quad n \neq 1, \end{aligned}$$

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} x \sin x \cos x \, dx = \frac{1}{2\pi} \int_0^{2\pi} x \sin 2x \, dx = -\frac{1}{2},$$

while

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_0^{2\pi} x \sin x \sin nx \, dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} x [\cos(n-1)x - \cos(n+1)x] dx \\ &= \frac{1}{2\pi} \left[\frac{1}{(n-1)^2} - \frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} + \frac{1}{(n+1)^2} \right] \\ &= 0, \quad n \neq 1 \end{aligned}$$

and

$$\begin{aligned} b_1 &= \frac{1}{\pi} \int_0^{2\pi} x \sin^2 x \, dx = \frac{1}{2\pi} \int_0^{2\pi} x(1 - \cos 2x) dx \\ &= \frac{1}{2\pi} [2\pi^2] = \pi. \end{aligned}$$

Hence the Fourier expansion of f is

$$\begin{aligned} \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \\ = -1 - \frac{1}{2} \cos x + \pi \sin x + 2 \sum_{n=2}^{\infty} \frac{\cos nx}{n^2-1}. \end{aligned}$$

EXAMPLE 4.18

Find the Fourier series of $f(x) = x$, $0 < x < 2\pi$.

Solution. The periodic extension of the given function f is shown in the Figure 4.14.

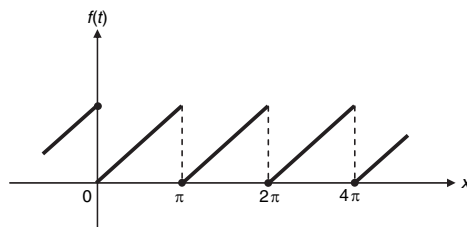


Figure 4.14

We have

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{2\pi} x \, dx = 2\pi, \\ a_n &= \frac{1}{\pi} \int_0^{2\pi} x \cos nx \, dx = 0 - \frac{1}{\pi n} \int_0^{2\pi} \sin nx \, dx = 0, \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} x \sin nx \, dx = -\frac{2}{n}. \end{aligned}$$

Therefore

$$x = \pi - 2 \left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots \right), \quad 0 < x < 2\pi.$$

EXAMPLE 4.19

Expand $f(x) = e^{-x}$, $0 < x < 2\pi$.

Solution. Using integration by parts, we have

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_0^{2\pi} e^{-x} \cos nx dx \\ &= \left(\frac{1 - e^{-2\pi}}{\pi} \right) \cdot \frac{1}{n^2 + 1}, \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_0^{2\pi} e^{-x} \sin nx dx \\ &= \frac{1 - e^{-2\pi}}{\pi} \left(\frac{n}{n^2 + 1} \right). \end{aligned}$$

Hence

$$\begin{aligned} f(x) &\sim \frac{1 - e^{-2\pi}}{\pi} \left[\frac{1}{2} + \left(\frac{1}{2} \cos x + \frac{1}{5} \cos 2x + \dots \right) \right. \\ &\quad \left. + \left(\frac{1}{2} \sin x + \frac{2}{5} \sin 2x + \dots \right) \right]. \end{aligned}$$

EXAMPLE 4.20

Find the Fourier series for $f(x) = \pi x$, $0 \leq x \leq 2$.

Solution. Comparing the interval with $(0, T)$, we have $T = 2$. The function is odd and, therefore, the Fourier series shall consist of only sine terms. We have

$$\begin{aligned} b_n &= \frac{2}{T} \int_0^T f(x) \sin n\pi x dx = \int_0^2 \pi x \sin n\pi x dx \\ &= \pi \left[x \frac{(-\cos n\pi x)}{n\pi} \right]_0^2 + \frac{1}{n} \int_0^2 (\cos n\pi x) dx = -\frac{2}{n}. \end{aligned}$$

Hence, the required Fourier series is $-2 \sum_{n=1}^{\infty} \frac{\sin n\pi x}{n}$.

EXAMPLE 4.21

Determine the Fourier series of the *half-wave rectified sinusoidal* defined by

$$\begin{aligned} f(t) &= \begin{cases} \sin t & \text{for } 0 < t < \pi \\ 0 & \text{for } \pi < t < 2\pi, \end{cases} \\ f(t) &= f(t + 2\pi). \end{aligned}$$

Deduce that

$$(a) \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots = \frac{1}{2}$$

$$(b) \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots = \frac{\pi-2}{4}.$$

Solution. The graph of the function is shown in the Figure 4.15.

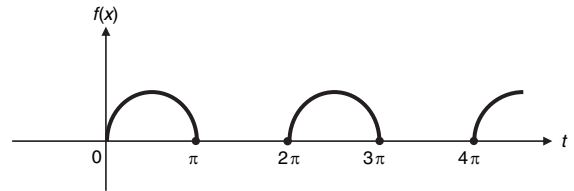


Figure 4.15

Using the Euler formulae, we have

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(t) dt = \frac{1}{\pi} \int_0^{\pi} \sin t dt = \frac{2}{\pi},$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos nt dt = \frac{1}{\pi} \int_0^{\pi} \sin t \cos nt dt$$

$$= \frac{1}{2\pi} \int_0^{\pi} [\sin(n+1)t - \sin(n-1)t] dt$$

$$= \frac{1}{2\pi} \left[-\frac{\cos(n+1)t}{n+1} + \frac{\cos(n-1)t}{n-1} \right]_0^{\pi}$$

$$= \begin{cases} -\frac{2}{\pi(n^2-1)} & \text{for even } n \\ 0 & \text{for odd } n. \end{cases}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin nt dt = \frac{1}{\pi} \int_0^{\pi} \sin t \sin nt dt$$

$$= \frac{1}{2\pi} \int_0^{\pi} [\cos(n-1)t - \cos(n+1)t] dt$$

$$= \frac{1}{2\pi} \left[\frac{\sin(n-1)t}{n-1} - \frac{\sin(n+1)t}{n+1} \right]_0^{\pi} = 0, n \neq 1$$

For $n = 1$, we get

$$b_1 = \frac{1}{\pi} \int_0^{\pi} \sin t \sin t dt = \frac{1}{\pi} \int_0^{\pi} \sin^2 t dt = \frac{1}{2}.$$

Hence the required Fourier series is

$$\begin{aligned} f(t) &\sim \frac{1}{\pi} - \frac{2}{3\pi} \cos 2t - \frac{2}{15\pi} \cos 4t \\ &\quad - \frac{2}{35\pi} \cos 6t + \dots + \frac{1}{2} \sin t \\ &= \frac{1}{\pi} \left[1 + \frac{\pi}{2} \sin t - \frac{2}{3} \cos 2t - \frac{2}{3.5} \cos 4t \right. \\ &\quad \left. - \frac{2}{3.5.7} \cos 6t - \dots \right] \end{aligned}$$

Deductions. (a) Putting $t = \pi$ in this Fourier series, we get $f(\pi) = 0$ and, therefore,

$$0 = \frac{1}{\pi} \left[1 - \frac{2}{1.3} - \frac{2}{3.5} - \frac{2}{5.7} - \dots \right]$$

or

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots = \frac{1}{2}.$$

(b) Putting $t = \frac{\pi}{2}$, we have $f\left(\frac{\pi}{2}\right) = \sin \frac{\pi}{2} = 1$. Therefore, the Fourier series reduces to

$$1 = \frac{1}{\pi} \left[1 + \frac{\pi}{2} + \frac{2}{1.3} - \frac{2}{3.5} + \frac{2}{5.7} - \dots \right],$$

which yields

$$\frac{\pi - 2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots$$

EXAMPLE 4.22

Expand $f(x) = |\sin x|$ in Fourier series.

Solution. The given function is defined for all x and is continuous, piecewise smooth, and even. Its graph is shown in the Figure 4.16

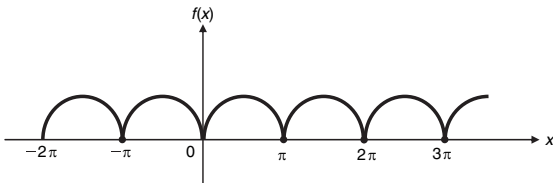


Figure 4.16

By Fundamental theorem of Fourier series, the function $f(x) = |\sin x|$ is everywhere equal to its convergent Fourier series. Since f is even, the series

shall consists of only cosine terms. We have

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} \sin x dx = \frac{4}{\pi}$$

and

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} \sin x \cos nx dx \\ &= \frac{1}{\pi} \int_0^{\pi} [\sin(n+1)x - \sin(n-1)x] dx \\ &= -\frac{1}{\pi} \left[\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1} \right]_0^{\pi} \\ &= -\frac{1}{\pi} \left[\frac{(-1)^{n+1} - 1}{n+1} - \frac{(-1)^{n+1} - 1}{n-1} \right] \\ &= \frac{-2}{\pi(n^2 - 1)} [(-1)^n + 1], \text{ for } n \neq 1, \end{aligned}$$

while for $n = 1$, we get

$$a_1 = \frac{2}{\pi} \int_0^{\pi} \sin x \cos x dx = \frac{1}{\pi} \int_0^{\pi} \sin 2x dx = 0.$$

Hence

$$\begin{aligned} f(x) &= |\sin x| \\ &= \frac{2}{\pi} - \frac{4}{\pi} \left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \dots \right). \end{aligned}$$

EXAMPLE 4.23

Determine *sine series* expansion of the function f defined by $f(x) = 1$, $0 < x < \pi$.

Solution. The graph of the given function is shown in the Figure 4.17. We wish to obtain a Fourier sine series for this function. Making the odd extension of f onto the interval $(-\pi, 0)$ produces a discontinuity at $x = 0$. We, thus, get an odd function on the interval $(-\pi, \pi)$. We then extend it periodically

with

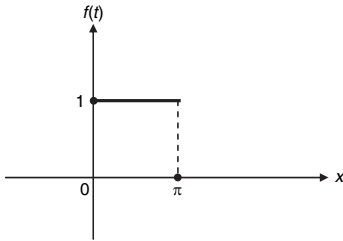


Figure 4.17

period 2π over the whole x -axis. The odd extension and subsequent periodic extensions are shown in the Figure 4.18.

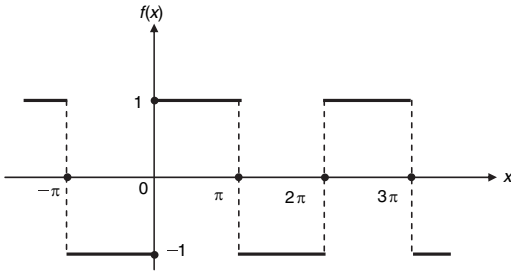


Figure 4.18

The Fundamental theorem of Fourier series is applicable. The Fourier series will converge to 1 for $0 < x < \pi$. Outside the interval $0 < x < \pi$, it converges to the function as shown in the Figure 4.18 with the sum of the series being equal to zero at the points $0, \pm\pi, \pm2\pi, \pm3\pi, \dots$. Since the function, thus, created is odd, $a_n = 0$ for $n = 0, 1, 2, \dots$. For b_n , we have

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt = \frac{1}{\pi} \int_{-\pi}^0 (-1) \sin nt \, dt + \frac{1}{\pi} \int_0^{\pi} \sin nt \, dt \\ &= \frac{2}{\pi} \int_0^{\pi} \sin nt \, dt = \frac{2}{\pi} \left[\frac{-\cos nt}{n} \right]_0^{\pi} = \frac{2}{\pi n} [1 - (-1)^n]. \end{aligned}$$

Hence

$$\begin{aligned} f(x) &= 1 = \sum_{n=1}^{\infty} \frac{2}{\pi n} [1 - (-1)^n] \sin nx \\ &= \frac{4}{\pi} \left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right] \text{ for } 0 < x < \pi. \end{aligned}$$

EXAMPLE 4.24

Find *cosine series* for the function f defined by

$$f(x) = \begin{cases} x & \text{for } 0 \leq x \leq L/2 \\ L - x & \text{for } L/2 \leq x \leq L. \end{cases}$$

Solution. The even extension of f in $(-L, L)$ and subsequent periodic extension are shown in the Figure 4.19.

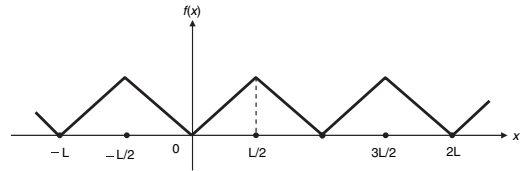


Figure 4.19

We have

$$\begin{aligned} a_0 &= \frac{2}{L} \int_0^L f(x) \, dx = \frac{2}{L} \left[\int_0^{L/2} x \, dx + \int_{L/2}^L (L - x) \, dx \right] \\ &= \frac{2}{L} \left[\frac{2L^2}{8} \right] = \frac{L}{2}, \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx \\ &= \frac{2}{L} \left[\int_0^{L/2} x \frac{\cos nx\pi}{L} \, dx + \int_{L/2}^L (L - x) \cos \frac{n\pi x}{L} \, dx \right] \\ &= \frac{L}{n\pi} \sin \frac{n\pi}{2} + \frac{2L}{n^2\pi^2} \cos \frac{n\pi}{2} - \frac{2L}{n^2\pi^2} \\ &\quad - \frac{L}{n\pi} \sin \frac{n\pi}{2} - \frac{2L}{n^2\pi^2} \cos n\pi + \frac{2L}{n^2\pi^2} \cos \frac{n\pi}{2} \\ &= \frac{4L}{n^2\pi^2} \cos \frac{n\pi}{2} - \frac{2L}{n^2\pi^2} \cos n\pi - \frac{2L}{n^2\pi^2} \\ &= \frac{4L}{n^2\pi^2} \cos \frac{n\pi}{2} - \frac{2L}{n^2\pi^2} [1 + \cos n\pi] \\ &= \frac{4L}{n^2\pi^2} \cos \frac{n\pi}{2} - \frac{2L}{n^2\pi^2} \left(2 \cos^2 \frac{n\pi}{2} \right) \\ &= \frac{4L}{n^2\pi^2} \cos \frac{n\pi}{2} \left[1 - \cos \frac{n\pi}{2} \right] \\ &= \frac{4L}{n^2\pi^2} \cos \frac{n\pi}{2} \cdot 2 \sin^2 \frac{n\pi}{4}. \end{aligned}$$

Thus

$$a_1 = 0, a_2 = -\frac{2L}{\pi^2}, a_3 = 0, a_4 = 0, a_5 = 0,$$

$$a_6 = \frac{2L}{9\pi^2}, a_7 = a_8 = a_9 = 0, a_{10} = -\frac{2L}{25\pi^2}, \dots$$

Therefore,

$$f(x) = \frac{L}{4} - \frac{2L}{\pi^2} \left[\cos \frac{2\pi x}{L} + \frac{1}{3^2} \cos \frac{6\pi x}{L} + \frac{1}{5^2} \cos \frac{10\pi x}{L} + \dots \right].$$

EXAMPLE 4.25

Expand $f(x) = \sin x$ ($0 < x < \pi$) in *cosine series*.

Solution. The graph of the given function is shown in the Figure 4.20.

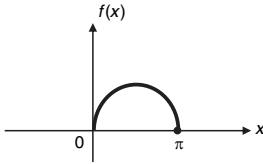


Figure 4.20

We extend $\sin x$ to an even function on the interval $(-\pi, \pi)$ and then extend it periodically with period 2π . The graph of the extended function then becomes as shown in Figure 4.21.

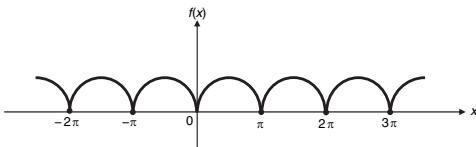


Figure 4.21

Since the function so created is even, $b_n = 0$. For this extended function, we have already calculated a_n , $n \neq 1$, a_1 and a_0 in Example 4.22. The Fourier series is, therefore, same as in Example 4.22.

EXAMPLE 4.26

Determine *half-range sine series* for the function f defined by $f(t) = t^2 + t$, $0 \leq t \leq \pi$.

Solution. Extending f to an odd function we get the graph of the extended odd function as shown in the Figure 4.22.

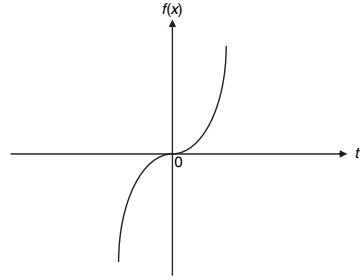


Figure 4.22

Since the extended function is odd, $a_n = 0$. For b_n , we have

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt = \frac{1}{\pi} \int_{-\pi}^0 f(t) \sin nt \, dt \\ &\quad + \frac{1}{\pi} \int_0^{\pi} f(t) \sin nt \, dt \\ &= \frac{2}{\pi} \int_0^{\pi} (t^2 + t) \sin nt \, dt \\ &= \frac{2}{\pi} \left[(t^2 + t) \left(-\frac{\cos nt}{n} \right) \right]_0^{\pi} \\ &\quad - \frac{2}{\pi} \int_0^{\pi} (2t + 1) \left(-\frac{\cos nt}{n} \right) dt \\ &= \frac{2}{\pi} \left[-\frac{(\pi^2 + \pi)}{n} (-1)^n + \frac{2}{n^3} ((-1)^n - 1) \right]. \end{aligned}$$

Thus, the required Fourier series is

$$\begin{aligned} f(t) &= \frac{2}{\pi} \sum_{n=1}^{\infty} \left[\frac{(\pi^2 + \pi)}{n} (-1)^{n+1} \right. \\ &\quad \left. + \frac{2}{n^3} ((-1)^n - 1) \right] \sin nt, t \in (0, \pi). \end{aligned}$$

EXAMPLE 4.27

Find the *half-range sine series* for the function f defined by

$$f(x) = \begin{cases} x & \text{for } 0 < x < \pi/2 \\ \pi - x & \text{for } \pi/2 < x < \pi. \end{cases}$$

Solution. Extending f as an odd function in the interval $(-\pi, \pi)$, we have $a_n = 0$ for $n = 0, 1, 2, \dots$

For b_n , we have

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \\ &= \frac{2}{\pi} \left[\int_0^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx \right] \\ &= \frac{2}{\pi} \left[x \left(-\frac{\cos nx}{n} \right) + \frac{\sin nx}{n^2} \right]_0^{\pi/2} \\ &\quad + \frac{2}{\pi} \left[(\pi - x) \left(-\frac{\cos nx}{n} \right) - \frac{\sin nx}{n} \right]_0^{\pi/2} = \frac{4}{\pi n^2} \sin \frac{n\pi}{2}. \end{aligned}$$

Therefore, the forced series development of f is

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \frac{4}{\pi n^2} \sin \frac{n\pi}{2} \sin nx \\ &= \frac{4}{\pi} \left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \dots \right]. \end{aligned}$$

EXAMPLE 4.28

Find the Fourier series of the following function:

$$f(x) = \begin{cases} x^2 & \text{for } 0 \leq x \leq \pi \\ -x^2 & \text{for } -\pi \leq x \leq 0. \end{cases}$$

Solution. The given function is an odd extension of the function $f(x) = x^2$, $0 \leq x \leq \pi$ to the interval $-\pi \leq x \leq \pi$. Since the extended function is odd, the Fourier series shall consist of only sine terms. We have

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} x^2 \sin x \, dx \\ &= \frac{2}{\pi} \left\{ \left[x^2 \left(-\frac{\cos nx}{n} \right) \right]_0^{\pi} - \int_0^{\pi} 2x \left(-\frac{\cos nx}{n} \right) dx \right\} \\ &= \frac{-2}{n\pi} [(-1)^n \pi^2] + \frac{4}{n\pi} \left[x \frac{\sin nx}{n} \right]_0^{\pi} - \frac{4}{n^2 \pi} \int_0^{\pi} \sin nx \, dx \\ &= \frac{-2}{n\pi} [(-1)^n \pi^2] - \frac{4}{n^2 \pi} \left[-\frac{\cos nx}{n} \right]_0^{\pi} \\ &= -\frac{2}{n\pi} [(-1)^n \pi^2] + \frac{4}{n^3 \pi} [(-1)^n - 1]. \end{aligned}$$

Hence, the Fourier sine series is given by

$$\begin{aligned} f(x) &\sim 2 \left(\pi - \frac{4}{\pi} \right) \sin x - \pi \sin 2x \\ &\quad + \frac{2}{3} \left(\pi - \frac{4}{9\pi} \right) \sin 3x - \frac{\pi}{2} \sin 4x + \dots \end{aligned}$$

EXAMPLE 4.29

Sketch the amplitude spectrum of the function f defined by

$$f(t) = t, \quad t \in (-\pi, \pi).$$

Solution. The Fourier coefficients of this saw tooth function, as derived in Example 4.12, are

$$a_n = 0, \quad b_n = -\frac{2}{n}(-1)^n.$$

Therefore

$$c_n = \frac{a_n - ib_n}{2} = \frac{(-1)^n i}{n}.$$

So the line spectrum is $\left\{ \frac{(-1)^n i}{n} \right\}$. The amplitude spectrum is then given by

$$|c_n| = \frac{1}{|n|}, \quad n \neq 0, \quad |c_0| = 0.$$

The sketch of the amplitude spectrum is thus as shown in the Figure 4.23.

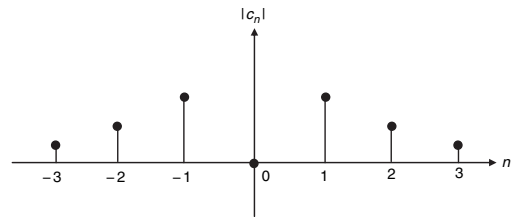


Figure 4.23

EXAMPLE 4.30

Find complex Fourier series for the function f defined by $f(x) = e^x$, $-\pi < x < \pi$

$$f(x + 2\pi) = f(x).$$

Solution. By definition of complex Fourier coefficients, we have

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-in\omega_0 t} dt$$

Here $T = 2\pi$. Therefore, $\omega_0 = \frac{2\pi}{T} = 1$. Thus

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x \cdot e^{-inx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x-inx} dx \\ &= \frac{1}{2\pi(1-in)} [e^{x-inx}]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi(1-in)} [e^{(1-in)\pi} - e^{-(1-in)\pi}] \\ &= \frac{(-1)^n}{2(1-in)} [e^{\pi} - e^{-\pi}], \quad \text{since } e^{-inx} = (-1)^n \\ &= \frac{2(-1)^n(1+in)}{2\pi(1+n^2)} \sinh \pi \\ &= \frac{(-1)^n(1+in)}{\pi(1+n^2)} \sinh \pi. \end{aligned}$$

Hence, the complex Fourier series is given by

$$f(x) \sim \sum_{n=-\infty}^{\infty} \frac{(-1)^n(1+in)}{\pi(1+n^2)} \sinh \pi e^{inx}.$$

EXAMPLE 4.31

Using shift property, derive the Fourier coefficients of $g(t) = t - \pi$, $0 < t < 2\pi$ from the Fourier coefficients of $f(t) = t$, $-\pi < t < \pi$, $f(t) = f(t + 2\pi)$.

Solution. In Example 4.12, we have seen that Fourier coefficients of $f(t) = t$, $\pi < t < \pi$ are $a_n = 0$ and $b_n = -\frac{2}{n}(-1)^n$.

Therefore,

$$c_n = \frac{a_n - ib_n}{2} = \frac{i}{n}(-1)^n.$$

The periodically extended graph of $g(t)$ is shown in the Figure 4.24.

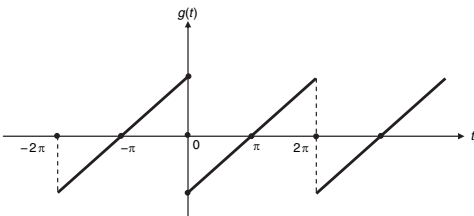


Figure 4.24

Clearly, $g(t) = f(t - \pi)$. Hence, by shift property, the coefficients of g are

$$\begin{aligned} d_n &= c_n e^{-in\omega_0\pi}, \quad \omega_0 = \frac{2\pi}{2\pi} = 1 \\ &= c_n e^{-i\pi n} = \frac{i}{n}(-1)^n \cdot e^{-i\pi n} = \frac{i}{n}(-1)^{2n} = \frac{i}{n}. \end{aligned}$$

Verification: For the function $g(t)$, we have $a_n = 0$ and

$$\begin{aligned} b_n &= \frac{2}{2\pi} \int_0^{2\pi} (t - \pi) \sin nt \, dt \\ &= \frac{-1}{n\pi} [(t - \pi) \cos nt]_0^{2\pi} \\ &\quad + \frac{1}{n\pi} \int_0^{2\pi} (1 - \pi) \cos nt \, dt = -\frac{2}{n}, \end{aligned}$$

and so

$$c_n = \frac{a_n - ib_n}{2} = \frac{0 + 2\frac{i}{n}}{2} = \frac{i}{n}.$$

EXAMPLE 4.32

Find complex Fourier coefficients for the function f defined by

$$f(t) = \begin{cases} 0 & \text{for } -\pi \leq t < 0 \\ 1 & \text{for } 0 \leq t \leq \pi. \end{cases}$$

Using time reversal property of Fourier coefficients, deduce the Fourier coefficients of $f(-t)$.

Solution. The Fourier coefficients of f (see Example 4.4) are $a_n = 0$ for $n = 1, 2, \dots$, and $b_n = \frac{1-(-1)^n}{n\pi}$. Therefore,

$$c_n = \frac{a_n - ib_n}{2} = \left[\frac{(-1)^n - 1}{2n\pi} \right] i.$$

Let d_n be the Fourier coefficients of $f(-t)$. Then, by time reversal property,

$$d_n = c_{-n} = \frac{a_n + ib_n}{2} = \left(\frac{1 - (-1)^n}{2n\pi} \right) i.$$

Verification: We have

$$f(-t) = \begin{cases} 0 & \text{for } -\pi < -t < 0 \\ 1 & \text{for } 0 \leq -t \leq \pi. \end{cases}$$

that is,

$$f(-t) = \begin{cases} 0 & \text{for } 0 < t < \pi \\ 1 & \text{for } -\pi < t < 0. \end{cases}$$

For this function $a_n = 0$, $n = 1, 2, \dots$ and

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dx = \frac{1}{\pi} \left[-\frac{\cos nt}{n} \right]_{-\pi}^0 \\ &= \frac{\cos n\pi - 1}{n\pi} = \frac{(-1)^n - 1}{n\pi}. \end{aligned}$$

So

$$d_n = \frac{a_n - ib_n}{2} = \left(\frac{1 - (-1)^n}{2n\pi} \right) i.$$

EXAMPLE 4.33

Given the series $x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$, show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Solution. The Parseval's equality states that

$$\int_{-\pi}^{\pi} [f(x)]^2 \, dx = 2\pi a_0^2 + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

For this example, the left-hand side of this equality is

$$\int_{-\pi}^{\pi} x^2 \, dx = \left[\frac{x^3}{3} \right]_{-\pi}^{\pi} = \frac{2}{3} \pi^3.$$

Since $a_n = 0$, $b_n = \frac{2(-1)^{n+1}}{n}$, the right-hand side of the equality is $\pi \sum_{n=1}^{\infty} \frac{4}{n^2}$. Hence, $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

EXAMPLE 4.34

Expand $f(x) = x$ ($0 < x < 2\pi$) in Fourier series and deduce that $\sum_{n=1}^{\infty} \frac{4(-1)^n}{4n^2} = \frac{3\pi^2}{16}$.

Solution. In Example 4.18, we have seen that the Fourier expansion of the given function is

$$x = \pi - 2 \sum_{n=1}^{\infty} \frac{1}{n} \sin nx.$$

Using integration of Fourier series with $t = \frac{\pi}{2}$, we get

$$\int_0^{\pi/2} x \, dx = \pi \left(\frac{\pi}{2} - 0 \right) - 2 \sum_{n=1}^{\infty} -\frac{1}{n} \left[\frac{1}{n} \cos \frac{nx}{2} \right]_0^{\pi/2},$$

that is,

$$\begin{aligned} \frac{\pi^2}{8} &= \frac{\pi^2}{2} + 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \left[\cos \frac{n\pi}{2} - \cos 0 \right] \\ &= \frac{\pi^2}{2} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)^2} - 2 \sum_{n=1}^{\infty} \frac{1}{n^2}. \end{aligned}$$

Therefore,

$$\frac{3\pi^2}{16} = \sum_{n=1}^{\infty} \frac{4 - (-1)^2}{4n^2}.$$

EXAMPLE 4.35

Expand $f(x) = x^2$ ($-\pi < x < \pi$) in Fourier series and deduce the value of the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^3}.$$

Solution. As per Example 4.6, the Fourier series expansion of $f(x) = x^2$ ($-\pi < x < \pi$) is

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx.$$

Using integration of Fourier series with $t = \pi/2$, we have

$$\int_0^{\pi/2} x^2 \, dx = \frac{\pi^2}{3} \left(\frac{\pi}{2} \right) + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin(n\pi/2),$$

or

$$\begin{aligned} \frac{\pi^3}{24} &= \frac{\pi^3}{6} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin(n\pi/2) \\ &= \frac{\pi^3}{6} + 4 \left[- \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^3} \right], \end{aligned}$$

which yields

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^3} = \frac{1}{4} \left[\frac{\pi^3}{6} - \frac{\pi^3}{24} \right] = \frac{\pi^3}{32}.$$

EXAMPLE 4.36

Expand $f(x) = x$ ($-\pi < x < \pi$) in Fourier series and deduce the Fourier series for $f(x) = x^2$ ($-\pi < x < \pi$).

Solution. As per Example 4.12, we have

$$x = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^n \sin nx.$$

Term-by-term integration yields

$$\frac{x^2}{2} = \sum_{n=1}^{\infty} \frac{2}{n^2} (-1)^n \cos nx + A(\text{constant}). \quad (44)$$

Integrating both sides with respect to x between the limits $-\pi$ and π , we have

$$\int_{-\pi}^{\pi} \frac{x^2}{2} dx = \sum_{n=1}^{\infty} \frac{2}{n^2} (-1)^n \int_{-\pi}^{\pi} \cos nx dx + \int_{-\pi}^{\pi} A dx,$$

which yields $A = \frac{\pi^2}{6}$. Hence (44) reduces to

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx,$$

which is the required Fourier series for $f(x) = x^2$ ($-\pi < x < \pi$).

EXAMPLE 4.37

Verify Riemann-Lebesgue lemma for the function $f(x) = 2x + 1$, $-\pi < x < \pi$.

Solution. The Fourier coefficients of this function (see Example 4.7) are

$$a_n = 0, n = 1, 2, \dots \text{ and } b_n = -\frac{4}{n}(-1)^n.$$

Clearly $b_n \rightarrow 0$ as $n \rightarrow \infty$. Thus Riemann-Lebesgue lemma is valid.

EXAMPLE 4.38

Let f and g be periodic function with period 2π defined on $(-\pi, \pi)$ by $f(x) = g(x) = x$. Find the Fourier series of fg over $(-\pi, \pi)$.

Solution. In Example 4.12, we have seen that the Fourier coefficients of f are $a_n = 0$ and $b_n = -\frac{2}{n}(-1)^n$.

Therefore, $c_n = d_n = \frac{a_n - ib_n}{2} = \frac{(-1)^n i}{n}$. Thus

$$c_n = d_n = \begin{cases} \frac{(-1)^n i}{n} & \text{for } n \neq 0 \\ 0 & \text{for } n = 0. \end{cases}$$

If f_n is complex Fourier coefficients of fg , then we know that

$$f_n = \sum_{k=-\infty}^{\infty} c_k d_{n-k}.$$

Therefore,

$$f_0 = \sum_{k=-\infty}^{\infty} c_k d_{-k} = \sum_{k=-\infty}^{\infty} c_k c_{-k}.$$

But $c_{-k} = \frac{a_{-k} + ib_{-k}}{2} = \frac{-(-1)^n i}{n}$. Hence $f_0 = 2 \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{3}$, since $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ (by Examples 4.6, 4.33).

Further,

$$\begin{aligned} f_n &= \sum_{\substack{k=-\infty \\ k \neq 0, k \neq n}}^{\infty} c_k d_{n-k} = \sum_{\substack{k=-\infty \\ k \neq 0, k \neq n}}^{\infty} c_k c_{n-k} \\ &= \sum_{\substack{k=-\infty \\ k \neq 0, k \neq n}}^{\infty} \frac{i}{k} (-1)^k \cdot \frac{i}{(n-k)} (-1)^{n-k} \\ &= -(-1)^n \sum_{\substack{k=-\infty \\ k \neq 0, k \neq n}}^{\infty} \frac{1}{k(n-k)} \\ &= -(-1)^n \frac{1}{n} \sum_{\substack{k=-\infty \\ k \neq 0, k \neq n}}^{\infty} \left(\frac{1}{k} - \frac{1}{n-k} \right) \\ &= \frac{-(-1)^n}{n} \left[\left(-\frac{1}{n} + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{1}{k} \right) \right. \\ &\quad \left. + \left(-\frac{1}{n} + \sum_{\substack{k=-\infty \\ k \neq n}}^{\infty} \frac{1}{n-k} \right) \right] \\ &= \frac{-(-1)^n}{n} \left[-\frac{2}{n} \right] = \frac{2}{n^2} (-1)^n. \end{aligned}$$

Hence

$$f_n = \begin{cases} \frac{2}{n^2} (-1)^n & \text{for } n \neq 0 \\ \frac{\pi^2}{3} & \text{for } n = 0. \end{cases}$$

Therefore,

$$x^2 = \frac{\pi^2}{3} + 2 \sum_{n=-\infty}^{\infty} \frac{1}{n^2} (-1)^n e^{inx}.$$

Verification: We have seen in Example 4.6 that Fourier coefficients of $f(x) = x^2$, $-\pi < x < \pi$ are $a_0 = \frac{2\pi^2}{3}$ and $a_n = \frac{4}{n^2} (-1)^n$. Then complex coefficients are

$$f_0 = \frac{\pi^2}{3} \text{ and } f_n = \frac{a_n - ib_n}{2} = \frac{2}{n^2} (-1)^n, n = 1, 2, \dots$$

4.16 METHOD TO FIND HARMONICS OF FOURIER SERIES OF A FUNCTION FROM TABULAR VALUES

The harmonics of a Fourier series of a function can even be found if the function is not defined explicitly but the tabular values of the function are given. This is possible using mean value of a function.

The mean value a function f over (a, b) is defined by

$$\frac{1}{b-a} \int_a^b f(x) dx.$$

By the definition of Fourier coefficients, we have

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{2\pi} f(x) dx = 2 \left[\frac{1}{2\pi} \int_0^{2\pi} f(x) dx \right] \\ &= 2 \times \text{mean value of } f \text{ over } (0, 2\pi), \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx = 2 \left[\frac{1}{2\pi} \int_0^{2\pi} f(x) \cos nx dx \right] \\ &= 2 \times \text{mean value of } f \cos nx \text{ over } (0, 2\pi), \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx = 2 \left[\frac{1}{2\pi} \int_0^{2\pi} f(x) \sin nx dx \right] \\ &= 2 \times \text{mean value of } f(x) \sin nx \text{ over } (0, 2\pi), \end{aligned}$$

Thus the Fourier coefficient can be determined by the mean values of $f(x)$, $f(x) \cos nx$ and $f(x) \sin nx$. Hence the n th harmonic $a_n \cos nx + b_n \sin nx$ can be determined using the tabular values of f , $f(x) \cos nx$ and $f(x) \sin nx$.

The following example illustrates the method discussed above.

EXAMPLE 4.39

Find the first two harmonics of the Fourier Series of $y = f(x)$ from the data:

x° :	0	30	60	90	120	150	180	210	240	270	300	330
y :	298	356	373	337	254	155	80	51	60	93	147	221

Solution. The *Mean value* of a function f over (a, b) is defined by $\frac{1}{b-a} \int_a^b f(x) dx$. The Fourier coefficients in terms of mean values can be expressed as

$$\begin{aligned} a_0 &= 2 \left(\frac{1}{2\pi} \right) \int_0^{2\pi} f(x) dx = 2 \\ &\quad [\text{mean value of } f(x) \text{ in } (0, 2\pi)], \end{aligned}$$

$$\begin{aligned} a_n &= 2 \left(\frac{1}{2\pi} \right) \int_0^{2\pi} f(x) \cos nx dx = 2 \\ &\quad [\text{mean value of } f(x) \cos nx \text{ in } (0, 2\pi)], \end{aligned}$$

$$\begin{aligned} b_n &= 2 \left(\frac{1}{2\pi} \right) \int_0^{2\pi} f(x) \sin nx dx = 2 \\ &\quad [\text{mean value of } f(x) \sin nx \text{ in } (0, 2\pi)]. \end{aligned}$$

We want to find the first two harmonics, that is, $a_1 \cos x + b_1 \sin x$ and $a_2 \cos 2x + b_2 \sin 2x$.

The values of $\sin x$, $\sin 2x$, $\cos x$, $\cos 2x$, $f(x) \sin x$, $f(x) \sin 2x$, $f(x) \cos x$ and $f(x) \cos 2x$ are given in the following table:

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
$f(x)$	298	356	373	337	254	155	80	51	60	93	147	221
$\sin x$	0	0.5	0.87	1	0.87	0.5	0	-0.5	-0.87	-1.0	-0.87	-0.5
$\sin 2x$	0	0.87	0.87	0	-0.87	-0.87	0	0.87	0.87	0	-0.87	-0.87
$\cos x$	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0	0.5	0.87
$\cos 2x$	1	0.5	0.5	-1	-0.5	-0.5	1	0.5	-0.5	-1	-0.5	0.5
$f(x) \sin x$	0	178	324.51	337	220.98	75	0	-25.5	-52.2	-93	-127.89	-110.5
$f(x) \sin 2x$	2x0	309.72	324.51	0	-220.98	-134.85	0	44.37	52.2	0	-127.89	-192.27
$f(x) \cos x$	298	309.72	186.5	0	-127	-134.85	-80	-44.37	-30	0	73.5	192.27
$f(x) \cos 2x$	298	178	186.5	-337	-127	-75	80	25.5	-30	-93	-73.5	110.5

We note that

$$\begin{aligned}\sum f(x) &= 2425, \quad \sum f(x) \cos x = 643.77, \\ \sum f(x) \sin x &= 726.40 \\ \sum f(x) \cos 2x &= 143.0 \text{ and} \\ \sum f(x) \sin 2x &= 54.81.\end{aligned}$$

Therefore

$$\begin{aligned}a_1 &= 2 \left[\frac{\sum f(x) \cos x}{12} \right] = \frac{643.77}{6} = 107.295 \\ a_2 &= 2 \left[\frac{\sum f(x) \cos 2x}{12} \right] = \frac{143.0}{6} = 23.83 \\ b_1 &= 2 \left[\frac{\sum f(x) \sin x}{12} \right] = \frac{726.4}{6} = 121.07 \\ b_2 &= 2 \left[\frac{\sum f(x) \sin 2x}{12} \right] = \frac{54.81}{6} = 9.135.\end{aligned}$$

Hence the first two harmonics are

$$a_1 \cos x + b_1 \sin x = 107.295 \cos x + 121.07 \sin x$$

and

$$a_2 \cos 2x + b_2 \sin 2x = 23.83 \cos 2x + 9.135 \sin 2x.$$

4.17 SIGNALS AND SYSTEMS

Definition 4.8. A *signal* is a function of one or more independent variable(s) which convey information. The independent variable may be *time*, *space*, etc. For example, in electrical network, the voltage $E(t)$ is a signal, which is defined as a function of time.

Definition 4.9. A *system* is a mapping F which assigns a unique output to an input.

Definition 4.10. Let y be a uniquely determined output corresponding to an input x under the system F , then y is called *response* of the system to the input x and we write $y = Fx$ or $x \rightarrow y$.

4.18 CLASSIFICATION OF SIGNALS

Definition 4.11. If the signal f , as a mapping, is real-valued, then it is called a *real signal*.

Definition 4.12. If the signal f , as a mapping, is complex-valued, then it is called *complex signal*. It is of the form $f = f_1 + if_2$, where f_1 is called real part of the complex signal and f_2 is called the imaginary part of f . If $f_1 = f_2 = 0$, then the signal is called the *null signal*.

Definition 4.13. A signal which is a function of time variable t , $t \in \mathbb{R}$, is called *continuous time signal*.

For example, in electrical networks and mechanical systems, the signals are functions of the time variable. Similarly, temperature of a room, and speech signals are continuous time signals.

A continuous time signal f is said to be *bounded* if there exists a positive constant K such that $|f(t)| \leq K$, $t \in \mathbb{R}$.

Definition 4.14. Signals which are defined at discrete time are called *discrete signals*. Thus discrete time signals can be considered as a function defined on $\mathbb{Z}$ or a part of $\mathbb{Z}$ (the set of integers).

For example, energy consumption in a state in the years 2001, 2002, ..., 2006 is a discrete time signal.

A discrete time signal $f[n]$ is called *bounded* if there exists a positive constant K such that $|f[n]| \leq K$, $n \in \mathbb{N}$.

Definition 4.15. A continuous time signal f is called *periodic* with period $T > 0$ if $f(t + T) = f(t)$, $t \in \mathbb{R}$.

For example, sinusoidal ($\sin t$, $\cos t$) are periodic signals. The sinusoidal are real signals, which in the continuous time case can be written as $f(t) = A \cos(\omega t + \phi_0)$, $t \in \mathbb{R}$, where A is the *amplitude*, ω is the *radial frequency*, and ϕ_0 the *initial phase* of the signal. The frequency ω equals $\frac{2\pi}{T}$, where T is *period*.

Definition 4.16. A discrete time signal $f[n]$ is called *periodic* with period $N \in \mathbb{N}$ if $f[n + N] = f[n]$, $n \in \mathbb{Z}$.

In discrete-time case, the sinusoidal signals have the form $f[n] = A \cos(\omega n + \phi_0)$, $n \in \mathbb{N}$, where A is amplitude, ω is frequency, ϕ_0 is initial phase, and period $N = 2\pi/\omega$.

Definition 4.17. If, both, dependent and independent variables of a signal are continuous in nature, then it is called an *analog signal*. These signals arise when a physical wave form is converted into an electrical signal.

For example, telephone speech signals and TV signals are analog signals.

Definition 4.18. If both dependent and independent variables of a signal are *discrete in nature*, then it is called a *digital signal*. These signals comprise of pulses occurring at discrete intervals of time.

For example, telegraph and teleprinter signals are digital signals.

Definition 4.19. A complex signal $f(t)$ is called a *time-harmonic continuous time signal* if $f(t) = ce^{i\omega t}$, $t \in \mathbb{R}$, where c is a complex variable.

A time-harmonic continuous time signal is *bounded* since

$$|ce^{i\omega t}| = |c| |e^{i\omega t}| = |c| \text{ for } t \in \mathbb{R}.$$

If we set $c = Ae^{i\phi_0}$, where $A = |c|$ and ϕ_0 is the argument, then

$$f(t) = Ae^{i\phi_0} \cdot e^{i\omega t} = Ae^{i(\omega t + \phi_0)}.$$

Thus, $f(t)$ can be represented in the complex plane by a point on the circle with origin as the centre and A as the radius. At $t = 0$, the argument is equal to ϕ_0 , the initial phase (Figure 4.25).

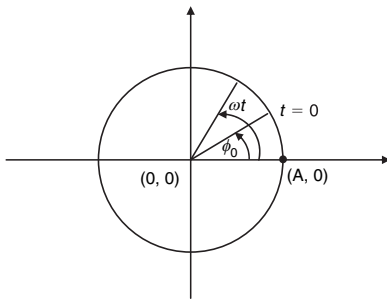


Figure 4.25

In the complex plane, the signal $f(t)$ corresponds to a circular movement with constant angular velocity $|\omega|$. Further, time-harmonic signal $f(t)$ is periodic with period $\frac{2\pi}{|\omega|}$. The real number ω is called *frequency* of the time harmonic signal, A the *amplitude* and ϕ_0 the initial phase. Further, by Euler's formula

$$\begin{aligned} f(t) &= A e^{i(\omega t + \phi_0)} \\ &= A[\cos(\omega t + \phi_0) + i \sin(\omega t + \phi_0)]. \end{aligned}$$

Thus *sinusoidal signal* is the real part of a *time-harmonic signal*. Also

$$\begin{aligned} A \cos(\omega t + \phi_0) &= A \left[\frac{e^{i(\omega t + \phi_0)} + e^{-i(\omega t + \phi_0)}}{2} \right] \\ &= \frac{ce^{i\omega t} + \bar{c}e^{-i\omega t}}{2}, \quad c = A e^{i\phi_0}. \end{aligned}$$

Definition 4.20. The *power* P of a continuous time signal $f(t)$ is defined by

$$P = \lim_{A \rightarrow \infty} \frac{1}{2A} \int_{-A}^A |f(t)|^2 dt.$$

Definition 4.21. The *power of a periodic continuous time signal* $f(t)$ with period T is defined by

$$P = \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt.$$

Definition 4.22. A continuous time signal whose power is *finite* is called a *power signal*.

For example, a periodic signal is a power signal. In particular, sinusoidal waves are power signals.

Definition 4.23. The *energy-content (total energy)* of a continuous-time signal is defined by

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt.$$

Definition 4.24. A continuous time signal with a *finite energy content* is called an *energy signal*.

For example, *rectangular pulse* is an energy signal.

Definition 4.25. The *power* P of a discrete-time signal $f[n]$, is defined by

$$P = \lim_{M \rightarrow \infty} \frac{1}{2M} \sum_{n=-M}^M |f[n]|^2.$$

Definition 4.26. If the power of a discrete-time signal is finite, then the signal is called a *discrete time power-signal*.

Definition 4.27. The *power of a periodic discrete-time signal* $f[n]$ with period N is defined by

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |f[n]|^2.$$

Definition 4.28. The energy content E of a discrete-time signal $f[n]$ is defined by

$$E = \sum_{n=-\infty}^{\infty} |f[n]|^2.$$

Definition 4.29. If the energy content of a discrete-time signal is finite, then the signal is called an *energy-signal*.

Definition 4.30. A continuous time signal $f(t)$ is called *causal* if $f(t) = 0$ for $t < 0$.

Definition 4.31. A discrete-time signal $f[n]$ is called *causal* if $f[n] = 0$ for $n < 0$.

It follows from the above definitions that *periodic signals (except the null sequence) are not causal*.

Definition 4.32. Let $f(t)$ be a signal. If there exists t_0 such that $f(t) = 0$ for $t < t_0$, then t_0 is called the *switch on time* of the signal $f(t)$.

4.19 CLASSIFICATION OF SYSTEMS

Definition 4.33. A system $F(t)$ is called linear if for two inputs x_1 and x_2 and arbitrary complex numbers a and b ,

$$F(ax_1 + bx_2) = aF(x_1) + bF(x_2).$$

Thus for continuous time system, we can write

$$ax_1(t) + bx_2(t) \rightarrow a(Fx_1)(t) + b(Fx_2)(t),$$

whereas for discrete time signal

$$ax_1[n] + bx_2[n] \rightarrow a(Fx_1)[n] + b(Fx_2)[n].$$

For example, system F defined by $F(t) = 3t$ is linear. In fact,

$$\begin{aligned} F(t_1 + t_2) &= 3(t_1 + t_2) = 3t_1 + 3t_2 = F(t_1) + F(t_2), \\ F(\alpha t) &= 3(\alpha t) = \alpha(3t) = \alpha F(t). \end{aligned}$$

Definition 4.34. A system for which power of the output equals the power of the input is called *all-pass system*.

Definition 4.35. A continuous time system is called *time-invariant* if for each input $u(t)$ and each $t_0 \in \mathbb{R}$,

$$u(t) \rightarrow y(t) \text{ implies } u(t - t_0) \rightarrow y(t - t_0).$$

Similarly, a discrete time system is called time invariant if for each input $u[n]$ and each $n_0 \in \mathbb{Z}$,

$$u[n] \rightarrow y[n] \text{ implies } u[n - n_0] \rightarrow y[n - n_0].$$

Definition 4.36. A system which is both linear and time-invariant is called a *linear time-invariant system* (or *LTI system*).

We now show that for linear time-invariant system $F(t)$, the response (whenever exists) to a time-harmonic signal is again a time-independent signal with the same frequency.

Theorem 4.16. Let F be a linear time invariant system, u a time-harmonic input with frequency ω for which response exist. Then the output y is also a time-harmonic signal with the same frequency ω .

Proof: Let $u(t)$ be the time-harmonic input with frequency ω and $y(t)$ the corresponding output. Thus, $u(t) = c e^{i\omega t}$, where c is a complex number and $\omega \in \mathbb{R}$. Since the system is time invariant, we have

$$F(u(t - t_0)) = y(t - t_0).$$

But

$$u(t - t_0) = c e^{i\omega(t-t_0)} = c e^{-i\omega t_0} \cdot e^{i\omega t} = e^{-i\omega t_0} u(t).$$

Since F is linear, we have

$$\begin{aligned} F(u(t - t_0)) &= F(e^{-i\omega t_0} u(t)) = e^{-i\omega t_0} F(u(t)) \\ &= e^{-i\omega t_0} y(t). \end{aligned}$$

Thus

$$F(u(t - t_0)) = y(t - t_0) = e^{-i\omega t_0} y(t).$$

Putting $t = 0$, we have

$$y(-t_0) = e^{-i\omega t_0} y(0).$$

Replacing $-t_0$ by t , we have

$$y(t) = y(0) e^{i\omega t} = c e^{i\omega t}, \text{ } c \text{ a complex constant,}$$

which shows that the response is again a time harmonic signal with frequency ω .

Remark 4.7. The complex constant c is a function of the frequency ω . This function is called *frequency response* (system function or transfer function) of the system. For continuous time systems, the transfer function is denoted by $H(\omega)$ and for discrete time systems by $H(e^{i\omega})$. Thus

$e^{j\omega t} \rightarrow H(\omega) e^{j\omega t}$ for continuous time system,

and

$e^{j\omega n} \rightarrow H(e^{j\omega}) e^{j\omega n}$ for a discrete time system.

Since $H(\omega)$ is complex, we can write it as

$$H(\omega) = |H(\omega)| e^{i\Phi(\omega)},$$

where $|H(\omega)|$ and $\Phi(\omega)$ are, respectively, the *modulus* and *argument* of $H(\omega)$. The function $|H(\omega)|$ is called the *amplitude response* and $\Phi(\omega)$ is called *phase response*.

Definition 4.37. A system F is said to be *stable* if the response of each bounded signal is again bounded. For example, let time-harmonic signal be the input signal, then response $ce^{j\omega t}$ to this input signal exists and is bounded.

Definition 4.38. A system F is called *real* if the response to every real input is again real.

Consider the sinusoidal input $u(t) = A \cos(\omega t + \phi)$. We can consider it as the real part of the time harmonic signal $ce^{j\omega t}$ with $c = Ae^{i\phi_0}$. The response to this harmonic signal is $c H(\omega) ce^{j\omega t}$. If the system is real, the response of sinusoidal input $u(t)$ is equal the real part of $c H(\omega) ce^{j\omega t}$. But $H(\omega) = |H(\omega)| e^{i\Phi(\omega)}$. Hence

$$\begin{aligned} y(t) &= \operatorname{Re}[A e^{i\phi_0} |H(\omega)| e^{i\Phi(\omega)} e^{j\omega t}] \\ &= A |H(\omega)| \cos(\omega t + \phi_0 + \Phi(\omega)). \end{aligned}$$

Definition 4.39. A continuous-time system F is called *causal* if for each two inputs $u(t)$ and $v(t)$ and for each $t_0 \in \mathbb{R}$,

$$u(t) = v(t) \Rightarrow (Fu)(t) = (Fv)(t) \text{ for } t < t_0.$$

Similarly, a discrete time system F is called *causal* if for each two inputs $u[n]$ and $v[n]$ and for each $n_0 \in \mathbb{Z}$,

$$u[n] = v[n] \Rightarrow (Fu)[n] = (Fv)[n] \text{ for } n < n_0.$$

Regarding causal systems we have the following theorem:

Theorem 4.17. A linear time-invariant system F is causal if and only if the response to each causal input is again causal.

4.20 RESPONSE OF A STABLE LINEAR TIME INVARIANT CONTINUOUS TIME SYSTEM (LTC SYSTEM) TO A PIECEWISE SMOOTH AND PERIODIC INPUT

We know that the response to the time-harmonic signal $e^{j\omega t}$ of frequency ω is equal to $H(\omega) e^{j\omega t}$, where $H(\omega)$ is the transfer function. Let the Fourier expansion of the periodic input $u(t)$ be

$$u(t) = \sum_{n=-\infty}^{\infty} u_n e^{in\omega_0 t},$$

where $\omega_0 = \frac{2\pi}{T}$ and u_n is the line spectrum of $u(t)$.

Since $u(t)$ is piecewise smooth, Convergence Theorem of Fourier series is applicable.

The following theorem gives the line spectrum y_n of the response $y(t)$.

Theorem 4.18. Let $y(t)$ be the response of a stable LTC system to a piecewise smooth and periodic input $u(t)$ with period T , fundamental frequency ω_0 and line spectrum u_n . Let $H(\omega)$ be the transfer function of the system. Then $y(t)$ is again periodic with period T and the line spectrum y_n of $y(t)$ is given by

$$y_n = H(n\omega_0) u_n, \quad n = 0, \pm 1, \pm 2, \dots$$

and so

$$y(t) = \sum_{n=-\infty}^{\infty} u_n H(n\omega_0) e^{in\omega_0 t}.$$

Proof: Since u_n is the line spectrum of the periodic input $u(t)$, we have

$$u(t) = \sum_{n=-\infty}^{\infty} u_n e^{in\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T}.$$

Since response to time-harmonic signal $e^{j\omega t}$ of frequency ω is $H(\omega) e^{j\omega t}$, the response of $e^{in\omega_0 t}$ is $H(n\omega_0) e^{in\omega_0 t}$. Therefore, by the linearity of the system, we have

$$u_n e^{in\omega_0 t} \rightarrow u_n H(n\omega_0) e^{in\omega_0 t}.$$

Therefore, the line spectrum y_n of $y(t)$ is

$$y_n = H(n\omega_0) u_n.$$

Now, by superposition rule, we have

$$y(t) = \sum_{n=-\infty}^{\infty} H(n\omega_0) u_n e^{in\omega_0 t},$$

where $y(t)$ is clearly periodic with period T .

4.21 APPLICATION TO DIFFERENTIAL EQUATIONS

Consider the following differential equation of order m :

$$\begin{aligned} a_m \frac{d^m y}{dt^m} + a_{m-1} \frac{d^{m-1} y}{dt^{m-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y \\ = b_n \frac{d^n u}{dt^n} + b_{n-1} \frac{d^{n-1} u}{dt^{n-1}} + \dots + b_1 \frac{du}{dt} + b_0 u, \end{aligned} \quad (45)$$

with $n \leq m$, where $a_0, a_1, \dots, a_m$ and $b_0, b_1, \dots, b_n$ are constants with $a_m \neq 0$ and $b_n \neq 0$. This equation describes the relation between an input $u(t)$ and the corresponding output $y(t)$. Let

$$P(s) = a_m s^m + a_{m-1} s^{m-1} + \dots + a_1 s + a_0$$

and

$$Q(s) = b_n s^n + b_{n-1} s^{n-1} + \dots + b_1 s + b_0.$$

The polynomial $P(s)$ is called the *characteristics polynomial* of the differential equation (45). The transfer function $H(\omega)$ can be found by the following theorem:

Theorem 4.19. Let the differential equation (45) describes an LTC-system and have $P(s)$ as the characteristic polynomial. If for all ω , $P(i\omega) \neq 0$, then

$$H(\omega) = \frac{Q(i\omega)}{P(i\omega)}.$$

Proof: To get frequency response, we substitute the input $u(t) = e^{i\omega t}$ in the given differential equation. Then the response $y(t)$ is of the form $H(\omega) e^{i\omega t}$. Since the derivative of $e^{i\omega t}$ is $i\omega e^{i\omega t}$, substitution into the differential equation yields

$$P(i\omega) H(\omega) e^{i\omega t} = Q(i\omega) e^{i\omega t}$$

and so

$$H(\omega) = \frac{Q(i\omega)}{P(i\omega)}.$$

EXAMPLE 4.40

Solve the differential equation

$$\frac{dy}{dt} + 3y = \cos 3t, y(0) = 0.$$

Solution. The characteristic equation is $s + 3 = 0$ so that $s = -3$.

Therefore, the *homogeneous solution* (eigen function) is $c_1 e^{-3t}$.

Further, taking $P(s) = s + 3$, $Q(s) = 1$, we have

$$H(\omega) = \frac{Q(i\omega)}{P(i\omega)} = \frac{1}{i\omega + 3}.$$

Since

$$u(t) = \cos 3t = \frac{e^{i3t} + e^{-i3t}}{2}$$

and

$$\begin{aligned} e^{i3t} &\rightarrow H(3) e^{3it} \\ e^{-i3t} &\rightarrow H(-3) e^{-3it}, \end{aligned}$$

therefore,

$$\begin{aligned} \cos 3t &\rightarrow \frac{H(3)e^{3it} + H(-3)e^{-3it}}{2} \\ &= \frac{e^{3it}}{2(3+3i)} + \frac{e^{-3it}}{2(3-3i)} \\ &= \frac{1}{6} \left[\frac{(\cos 3t + i \sin 3t)(1-i) + (\cos 3t - i \sin 3t)(1+i)}{(1+i)(1-i)} \right] \\ &= \frac{1}{6} [\cos 3t + \sin 3t]. \end{aligned}$$

Hence the complete solution is

$$y(t) = [\cos 3t + \sin 3t] + c_1 e^{-3t}.$$

Putting $t = 0$ and using the initial condition $y(0) = 0$, we have $0 = \frac{1}{6} + c_1$ and so $c_1 = -\frac{1}{6}$. Thus, the complete solution is

$$y(t) = \frac{1}{6} [\cos 3t + \sin 3t] - \frac{1}{6} e^{-3t}.$$

EXAMPLE 4.41

Solve $\frac{d^2 y}{dt^2} + 5 \frac{dy}{dt} + 6y = 2 \sin t$, $t \geq 0$ subject to the conditions $y'(0) = 0$, $y(0) = 0$.

Solution. The characteristic equation $s^2 + 5s + 6 = 0$ yields $s = -3, -6$. Thus the homogeneous solution is

$$c_1 e^{-3t} + c_2 e^{-2t}.$$

Now taking

$$P(s) = s^2 + 5s + 6 \text{ and } Q(s) = 2,$$

we have

$$H(\omega) = \frac{Q(i\omega)}{P(i\omega)} = \frac{2}{6 + 5i\omega - \omega^2}.$$

Now

$$u(t) = \frac{2(e^{it} - e^{-it})}{2i} = \frac{e^{it} - e^{-it}}{i}$$

and

$$e^{it} \rightarrow H(1) e^{it}$$

$$e^{-it} \rightarrow H(-1) e^{-it}.$$

Therefore,

$$\begin{aligned} \frac{e^{it} - e^{-it}}{i} &\rightarrow \frac{1}{i} [H(1) e^{it} - H(-1) e^{-it}] \\ &= \frac{1}{i} \left[\frac{2}{5+5i} e^{it} - \frac{2}{5-5i} e^{-it} \right] = \frac{2}{5i} \left[\frac{e^{it}}{1+i} - \frac{e^{-it}}{1-i} \right] \\ &= \frac{2}{5i} \left[\frac{(\cos t + i \sin t)(1-i) - (\cos t - i \sin t)(1+i)}{2} \right] \\ &= \frac{2}{5} [\sin t - \cos t]. \end{aligned}$$

Hence the complete solution is

$$y(t) = c_1 e^{-3t} + c_2 e^{-2t} + \frac{2}{5} (\sin t - \cos t).$$

Putting $t = 0$, we get

$$c_1 + c_2 = \frac{2}{5}.$$

Putting $t = 0$ in $y'(t)$, we get

$$3c_1 + 2c_2 = \frac{2}{5}.$$

Solving these equations, we get $c_1 = -\frac{2}{5}$, $c_2 = \frac{4}{5}$.
Hence

$$y(t) = \frac{2}{5} [2e^{-2t} - e^{-3t} + \sin t - \cos t].$$

EXAMPLE 4.42

Find the power of the output for the following electric network (Fig. 4.26).

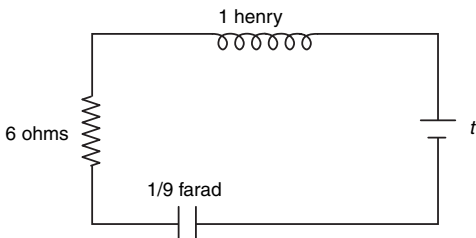


Figure 4.26

Solution. By Kirchhoff's law, the differential equation for the given circuit is

$$L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = E(t).$$

Here $L = 1$ henry, $R = 6$ ohms, $C = \frac{1}{9}$ farad, and $E(t) = t$. Hence the differential equation for the system is

$$\frac{d^2 Q}{dt^2} + 6 \frac{dQ}{dt} + 9Q = t, -\pi < t < \pi.$$

The input $u(t) = t$ is periodic with period 2π . The characteristic polynomial is $P(s) = s^2 + 6s + 9$, and $Q(s) = 1$.

Therefore, the transfer function is given by

$$H(\omega) = \frac{Q(i\omega)}{P(i\omega)} = \frac{1}{9 + 6i\omega - \omega^2}.$$

The line spectrum (see Example 4.29) of the input function is $\left\{ \frac{(-1)^n}{n} i \right\}$. The amplitude spectrum is

$$|c_n| = \frac{1}{|n|}, \quad n \neq 0, \quad |c_0| = 0.$$

Therefore, by Theorem 4.17, the amplitude spectrum $|y_n|$ of the output is equal to

$$\begin{aligned} |y_n| &= |H(n\omega_0)c_n| = \left| \frac{1}{-(n^2\omega_0^2 + 9)} c_n \right| \\ &= \frac{1}{|n| (n^2\omega_0^2 + 9)} \quad \text{for } n \neq 0 \end{aligned}$$

and

$$|y_n| = 0 \quad \text{for } n = 0.$$

Therefore, power of $y(t)$ is

$$\begin{aligned} P &= \frac{1}{2\pi} \int_0^{2\pi} |y(t)|^2 dt = \sum_{n=-\infty}^{\infty} |y_n|^2 \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{n^2 (n^2\omega_0^2 + 9)^2}. \end{aligned}$$

EXAMPLE 4.43

Show that the following LTC system representing an electric network consisting of L , C , R , and with $\alpha = \frac{1}{RC}$ is an all-pass system:

$$\frac{d^2 y}{dt^2} - 2\alpha \frac{dy}{dt} + \alpha^2 y = \frac{d^2 u}{dt^2} - \alpha^2 u,$$

with $u(t)$ a periodic input.

Solution. Let

$$P(s) = s^2 - 2\alpha + \alpha^2 \quad (\text{characteristic polynomial}),$$

$$Q(s) = s^2 - \alpha^2.$$

Then the transfer function (frequency response) is

$$H(\omega) = \frac{Q(i\omega)}{P(i\omega)} = \frac{(i\omega)^2 - \alpha^2}{(i\omega)^2 - 2i\omega + \alpha^2} = \frac{i\omega + \alpha}{i\omega - \alpha}.$$

Now $u(t)$ is a periodic input with period T . Let u_n be the line spectrum of $u(t)$. Then the line spectrum of the response is

$$y_n = H(n\omega_0) u_n = \frac{in\omega_0 + \alpha}{in\omega_0 - \alpha} u_n.$$

Therefore, the amplitude spectrum $|y_n|$ of the response (output) is

$$|y_n| = \left| \frac{in\omega_0 + \alpha}{in\omega_0 - \alpha} u_n \right| = \left| \frac{in\omega_0 + \alpha}{in\omega_0 - \alpha} \right| |u_n| = |u_n|.$$

It follows, therefore, that the amplitude spectrum is invariant under the given system. Using Parseval's identity for periodic function, we have

$$\begin{aligned} P &= \frac{1}{T} \int_0^T |y(t)|^2 dt = \sum_{n=-\infty}^{\infty} |y_n|^2 = \sum_{n=-\infty}^{\infty} |u_n|^2 \\ &= \frac{1}{T} \int_0^T |u(t)|^2 dt. \end{aligned}$$

Hence, the power of the output is equal to the power of the input and so the system is all-pass system.

4.22 APPLICATION TO PARTIAL DIFFERENTIAL EQUATIONS

In this section, we solve some partial differential equations of second order with the aid of Fourier series. For the sake of simplicity, we consider only one-dimensional partial differential equations.

EXAMPLE 4.44

Solve one-dimensional heat equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad x \in (0, L),$$

with initial condition $u(x, 0) = f(x)$ and the boundary conditions $u(0, t) = 0$, $u(L, t) = 0$, $t \geq 0$.

Solution. The boundary conditions show that the temperature at both ends of the rod is kept at zero. We shall use the method of separation of variables. So let,

$$u(x, t) = T(t)X(x), \quad (46)$$

be a trial solution of the given partial differential equation. Then the given heat equation reduces to

$$T'(t)X(x) = k X''(x)T(t).$$

Separating the variables, we get

$$\frac{T'(t)}{T(t)} = k \frac{X''(x)}{X(x)} \quad (47)$$

The left-hand side of (47) is a function of t only whereas the right-hand side is a function of x only. Since t and x are different variables, equality in (47) can occur only if the left-hand side and right-hand sides are both equal to a constant, say C . Thus, we have

$$\frac{T'(t)}{T(t)} = k \frac{X''(x)}{X(x)} = C \quad (\text{constant of separation}).$$

The boundary conditions imply that

$$u(0, t) = T(t)X(0) = 0 \quad (48)$$

and

$$u(L, t) = T(t)X(L) = 0 \quad (49)$$

Since we do not want trivial solution $T(t) = 0$, the relations (48) and (49) yield $X(0) = 0$ and $X(L) = 0$. If $C = 0$, then $X''(x) = 0$ and so $X(x) = ax + b$. Therefore, $X(0) = 0$ implies $b = 0$ and $X(L) = 0$ implies $aL = 0$. Thus $a = b = 0$ and as such $X(x)$ is a trivial solution. Hence $C = 0$ is discarded.

Let $C \neq 0$, then the characteristic equation of $X''(x) - \frac{C}{k}X(x) = 0$ is $s^2 - \frac{C}{k} = 0$ which yields two roots s_1 and s_2 such that $s_2 = -s_1$. The fundamental (general) solution is, therefore,

$$X(x) = \alpha e^{s_1 x} + \beta e^{-s_1 x}. \quad (50)$$

The boundary condition $X(0) = 0$ implies $\alpha + \beta = 0$ and so $\beta = -\alpha$.

The boundary condition $X(L) = 0$ implies $\alpha(e^{s_1 L} - e^{-s_1 L}) = 0$ (since $\beta = -\alpha$). Now α cannot be zero, because $\alpha = 0$ gives $\beta = 0$ and so (50) has trivial solution. Hence $e^{s_1 L} - e^{-s_1 L} = 0$, which yields $e^{2s_1 L} = 1$ and so $s_1 = \frac{in\pi}{L}$, where $n \neq 0$ is an integer. Thus $\frac{C}{k} = s_1^2$ yields $C = \frac{-kn^2\pi^2}{L^2}$. Hence (50) reduces to

$$X_n(x) = \alpha \left[e^{\frac{in\pi x}{L}} - e^{-\frac{in\pi x}{L}} \right] = 2\alpha \left[\sin \frac{n\pi x}{L} \right].$$

Also the equation $T'(t) - CT(t) = 0$ has characteristic equation as $s - C = 0$ and so $s = C$. So the fundamental solution is

$$T_n(t) = \alpha_0 e^{-Ct} = \alpha_0 e^{-\frac{kx^2 \pi^2}{L^2} t} = \alpha_0 e^{-\left(\frac{n\pi}{L}\right)^2 k t}$$

Hence, the fundamental solution of the heat equation is given by

$$\begin{aligned} u_n(x, t) &= T_n(t)X_n(x) \\ &= A_n e^{-\left(\frac{n\pi}{L}\right)^2 k t} \sin \frac{n\pi x}{L} \text{ for } n = 1, 2, \dots \end{aligned}$$

By principle of superposition (linear combinations), we have

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{L}\right)^2 k t} \sin \left(\frac{n\pi x}{L} \right).$$

Using initial condition $u(x, 0) = f(x)$, we have

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \left(\frac{n\pi x}{L} \right), \quad 0 \leq x \leq L. \quad (51)$$

Thus, the coefficients A_n are nothing but the Fourier sine coefficients of the function $f(x)$ with respect to the system $\sin \frac{n\pi x}{L}$, that is,

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \left(\frac{n\pi x}{L} \right) dx.$$

Substituting the value of A_n in (51) shall yield the solution of the given heat equation.

For example, if $u(x, 0) = x$, $0 < x < 2\pi$, then

$$b_n = \frac{2}{L} \int_0^L x \sin \frac{n\pi x}{L} dx = \frac{1}{\pi} \int_0^{2\pi} x \sin \frac{nx}{2} dx = -\frac{4}{n} (-1)^n.$$

and so in that case the solution of the heat equation becomes

$$\begin{aligned} u(x, t) &= \sum -\frac{4}{n} (-1)^n e^{-n^2 \pi^2 k t / L^2} \sin \left(\frac{n\pi x}{L} \right), L = 2\pi \\ &= \sum -\frac{4}{n} (-1)^n e^{-n^2 k t / 4} \sin \left(\frac{nx}{2} \right). \end{aligned}$$

EXAMPLE 4.45

Solve

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 2\pi,$$

with the condition

$$u(x, 0) = x^2, \quad u(0, t) = u(2\pi, t) = 0$$

Solution. From Example 4.43, we have

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} A_n e^{-n^2 \pi^2 k t / L^2} \sin \frac{n\pi x}{L}, \quad L = 2\pi \\ &= \sum_{n=1}^{\infty} A_n e^{-n^2 k t / L^2} \sin \frac{nx}{2}, \end{aligned}$$

where

$$\begin{aligned} A_n &= \frac{1}{\pi} \int_0^{2\pi} x^2 \sin \frac{nx}{2} dx = \frac{-2}{\pi n} \left[x^2 \frac{nx}{2} \right]_0^{2\pi} \\ &\quad + \frac{4}{\pi n} \int_0^{2\pi} x \cos \frac{nx}{2} dx \\ &= -\frac{8\pi}{n} (-1)^n + \frac{8}{\pi n^2} \left[x \sin \frac{nx}{2} \right]_0^{2\pi} \\ &\quad - \frac{8}{\pi n^2} \int_0^{2\pi} \sin \frac{nx}{2} dx \\ &= \frac{8}{n} (-1)^n \left[\frac{2}{\pi n^2} - \pi \right]. \end{aligned}$$

Hence

$$u(x, t) = \sum_{n=1}^{\infty} \frac{8}{n} (-1)^n \left[\frac{2}{\pi n^2} - \pi \right] e^{-n^2 k t / 4} \sin \frac{nx}{2}.$$

EXAMPLE 4.46

Solve the heat equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

with the boundary conditions $u(0, t) = u(L, t) = 0$, $t \geq 0$ and initial condition

$$u(x, 0) = \begin{cases} 0 & \text{for } 0 \leq x \leq L/2 \\ L - x & \text{for } L/2 \leq x \leq L. \end{cases}$$

Solution. The solution to this problem is

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi}{L}\right)^2 k t} \sin \left(\frac{n\pi x}{L} \right)$$

where

$$A_n = \frac{2}{L} \int_0^L u(x, 0) \sin \left(\frac{n\pi x}{L} \right) dx.$$

But (see Exercise 16) the coefficients A_n are given by

$$A_n = \frac{4L}{n^2\pi^2} \sin \frac{n\pi}{2}, n = 1, 2, \dots$$

$$= \frac{4L}{\pi^2} \left[\frac{1}{(2n+1)^2} \sin(2n+1) \frac{\pi}{2} \right], n = 0, 1, 2, \dots$$

Hence

$$u(x, t) = \sum_{n=0}^{\infty} \frac{4L}{\pi^2} \left(\frac{1}{2n+1} \right)^2$$

$$\times \sin((2n+1) \frac{\pi}{2}) e^{-(2n+1)^2 \pi^2 kt/L^2} \sin(2n+1) \frac{\pi x}{L}$$

$$= \frac{4L}{\pi^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} e^{-(2n+1)^2 \pi^2 kt/L^2} \sin(2n+1) \frac{\pi x}{L}.$$

EXAMPLE 4.47

Solve the heat conduction equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, t > 0$$

with the boundary condition $u_x(0, t) = 0$, $u_x(L, t) = 0$, $t \geq 0$ and the initial condition $u(x, 0) = f(x)$, $0 \leq x \leq L$.

Solution. The boundary conditions show that the ends of the rod are insulated. As in Example 4.43, we have

$$\frac{T'(t)}{T(t)} = k \frac{X''(x)}{X(x)} = C \quad (52)$$

Differentiating $u(x, t) = T(t) X(x)$ with respect to x , we have

$$u_x(x, t) = T(t) X'(x).$$

Therefore, the boundary value conditions yield

$$u_x(0, t) = T(t) X'(0) = 0$$

and

$$u_x(L, t) = T(t) X'(L) = 0,$$

and so

$$X'(0) = X'(L) = 0.$$

Now if $C = 0$, then (52) implies that $X''(x) = 0$ and so $X(x) = ax + b$. Therefore, $X'(x) = a$ and so $X'(0) = a = 0$ and so $X(x) = b$. Therefore, $C = 0$ is an eigenvalue with eigen function a constant.

If $C \neq 0$, then the characteristic equation of $X''(x) - \frac{C}{k} X(x) = 0$ is $s^2 - \frac{C}{k} = 0$ which yield,

two roots s_1 and s_2 with $s_2 = -s_1$. Therefore, the fundamental solution is

$$X(x) = \alpha e^{s_1 x} + \beta e^{-s_1 x}. \quad (53)$$

Differentiating (53), we have

$$X'(x) = \alpha s_1 e^{s_1 x} - \beta e^{-s_1 x}.$$

So the boundary condition $X'(0) = 0$ and $X'(L) = 0$ implies

$$\alpha - \beta = 0 \text{ yielding } \alpha = \beta$$

and

$$\alpha s_1 (e^{s_1 L} - e^{-s_1 L}) = 0.$$

If $\alpha = 0$, then $\beta = 0$, and so we will have a trivial solution. Therefore, $e^{s_1 L} - e^{-s_1 L} = 0$, which gives $e^{2s_1 L} = 1$ and so $s_1 = \frac{in\pi}{L}$. Then $\frac{C}{k} = s_1^2$ implies $C = -\frac{kn^2\pi^2}{L^2}$. Hence (53) becomes

$$X_n(x) = \alpha [e^{in\pi x/L} + e^{-in\pi x/L}] = 2\alpha \cos \frac{n\pi x}{L}.$$

Moreover (see Example 4.43), $T'(t) - CT(t) = 0$ has fundamental solution as

$$T_n(t) = \alpha_0 e^{-n^2\pi^2 kt/L^2}.$$

Thus, the fundamental solution of the heat equation becomes

$$u_n(x, t) = T_n(t) X_n(x) = \frac{A_0}{2} + A_n e^{-\frac{n^2\pi^2 kt}{L^2}} \cos \frac{n\pi x}{L}.$$

Now superposition of fundamental solutions yields

$$u(x, t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2 kt}{L^2}} \cos \frac{n\pi x}{L} \quad (54)$$

The initial condition $u(x, 0) = f(x)$, $0 \leq x \leq L$ gives

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}, \quad 0 \leq x \leq L.$$

Thus, the Fourier coefficients A_n of $f(x)$ with respect to the system $\cos \frac{n\pi x}{L}$ are given by

$$A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, \dots$$

Substituting the value of A_n in (54) will yield the solution of the given heat conduction equation.

EXAMPLE 4.48

Solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, t > 0,$$

where a is a constant related to tension in the vibrating string of length L having fixed ends. The boundary conditions and initial conditions are

$$u(0, t) = u(L, t) = 0, t \geq 0,$$

$$u(x, 0) = f(x), 0 \leq x \leq L,$$

$$u_t(x, 0) = 0, 0 \leq x \leq L.$$

Solution. Let

$$u(x, t) = T(t)X(x) \quad (55)$$

be a trial solution of the given partial differential equation. Then the wave equation takes the form

$$\text{or} \quad T''(t)X(x) = a^2 X''(x)T(t)$$

$$\frac{T''(t)}{T(t)} = a^2 \frac{X''(x)}{X(x)} = C \quad (\text{constant of separation}).$$

The boundary conditions $u(0, t) = u(L, t) = 0, t \geq 0$ imply $X(0) = 0$ and $X(L) = 0$. Further, the condition $u_t(x, 0) = 0$ gives $T'(0)X(x) = 0, 0 \leq x \leq L$. Therefore, $T'(0) = 0$.

The auxiliary equation for $X(x)$ is $s^2 - \frac{C}{a^2} = 0$ which yields $s_2 = -s_1$ as the two roots. Therefore, the fundamental solution is

$$X(x) = \alpha e^{s_1 x} + \beta e^{-s_1 x} \quad (56)$$

The boundary condition $X(0) = 0$ and $X(L) = 0$ gives $\alpha + \beta = 0$, that is, $\beta = -\alpha$ and

$$0 = \alpha e^{s_1 L} + \beta e^{-s_1 L} = \alpha(e^{s_1 L} - e^{-s_1 L})$$

Since $\alpha = 0$ implies $\beta = 0$ and the solution then becomes trivial, so $\alpha \neq 0$ and thus $e^{s_1 L} = e^{-s_1 L}$ and so $e^{2s_1 L} = 1$. Thus $s_1 = \frac{i n \pi}{L}$, $n \neq 0$ being an integer. Therefore, $s_1^2 = \frac{C}{a^2}$ yields $C = -\frac{a^2 n^2 \pi^2}{L^2}$. The fundamental solution (56) now takes the form

$$X_n(x) = \alpha[e^{i n \pi x / L} - e^{-i n \pi x / L}] = 2\alpha \sin \frac{n \pi x}{L}.$$

Further, characteristic equation for $T''(t) - CT(t) = 0$ is $s^2 - C = 0$, that is, $s^2 + \frac{a^2 n^2 \pi^2}{L^2} = 0$. Thus the fundamental solution is

$$T_n(t) = \alpha \cos\left(\frac{n \pi a t}{L}\right) + \beta \sin\left(\frac{n \pi a t}{L}\right).$$

We have

$$T'(t) = \frac{n \pi \alpha}{L} \left(-\alpha \sin \frac{n \pi a t}{L} + \beta \cos \frac{n \pi a t}{L} \right).$$

The condition $T'(t)$ yields $\beta = 0$ and so

$$T_n(t) = \alpha \cos\left(\frac{n \pi a t}{L}\right), n = 1, 2, 3, \dots$$

Hence the fundamental solution for the wave equation is

$$u_n(x, t) = T_n(t)X_n(x) = A_n \cos \frac{n \pi a t}{L} \sin \frac{n \pi x}{L},$$

$$n = 1, 2, \dots$$

Superposition of the fundamental solutions implies

$$u(x, t) = \sum_{n=1}^{\infty} A_n \cos \frac{n \pi a t}{L} \sin \frac{n \pi x}{L}. \quad (57)$$

Using initial condition $u(x, 0) = f(x)$, we have

$$f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n \pi x}{L}, \quad 0 \leq x \leq L.$$

The Fourier coefficients A_n of $f(x)$ with respect to the system $\sin \frac{n \pi x}{L}$ is given by

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n \pi x}{L} dx.$$

Putting the values of A_n in (57), we get the solution of the wave equation.

4.23 MISCELLANEOUS EXAMPLES

EXAMPLE 4.49

Expand $f(x) = \pi x$ from $x = -c$ to $x = c$ as a Fourier series.

Solution. The given function is an odd function of period $2c$. Therefore $a_n = 0$ and

$$\begin{aligned} b_n &= \frac{\pi}{c} \int_{-c}^c x \sin \frac{n \pi x}{c} dx \\ &= \frac{\pi}{c} \left\{ \left[x \left(-\frac{\cos \frac{n \pi x}{c}}{\frac{n \pi}{c}} \right) \right]_{-c}^c + \frac{c}{n \pi} \int_{-c}^c \cos \frac{n \pi x}{c} dx \right\} \\ &= \frac{\pi}{c} \left\{ \left[\frac{x c}{n \pi} \left(-\cos \frac{n \pi x}{c} \right) \right]_{-c}^c + \frac{c}{n \pi} \left[\frac{\sin \frac{n \pi x}{c}}{\frac{n \pi}{c}} \right]_{-c}^c \right\} \\ &= -\frac{1}{n} \left[x \cos \frac{n \pi x}{c} \right]_{-c}^c + \frac{c}{n^2 \pi} \left[\sin \frac{n \pi x}{c} \right]_{-c}^c \\ &= -\frac{1}{n} [c \cos n \pi + c \cos n \pi] + 0 \\ &= -\frac{2c}{n} \cos n \pi = -\frac{2c}{n} (-1)^n. \end{aligned}$$

Hence

$$f(x) = -2c \sum_{n=1}^{\infty} \frac{1}{n} (-1)^n \sin \frac{n\pi x}{c}.$$

EXAMPLE 4.50

Express $f(x) = x \sin x$, $0 < x < \pi$ as a Fourier cosine series. Hence, deduce the sum of the series

$$1 + \frac{2}{1.3} - \frac{2}{3.5} + \frac{2}{5.7} - \dots \infty = \frac{\pi}{2}.$$

Solution. Similar to Example 4.17 except the limits. We note that $x \sin x$ is an even function and so the Fourier coefficients $b_n = 0$. Further

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin x \, dx = \frac{2}{\pi} \int_0^{\pi} x \sin x \, dx \\ &= \frac{2}{\pi} [-x \cos x + \sin x]_0^{\pi} = \frac{2}{\pi} [-\pi \cos \pi] = 2. \end{aligned}$$

and

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin x \cos nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} x \sin x \cos nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} \frac{1}{2} x [\sin(n+1)x - \sin(n-1)x] \, dx \\ &= \frac{1}{\pi} \left\{ \left[\frac{x(-\cos(n+1)x)}{n+1} + \frac{\sin(n+1)x}{(n+1)^2} \right]_0^{\pi} \right. \\ &\quad \left. - \left[\frac{x(-\cos(n-1)x)}{n-1} + \frac{\sin(n-1)x}{(n-1)^2} \right]_0^{\pi} \right\} \\ &= \frac{1}{\pi} \left[\frac{-\pi \cos(n+1)\pi}{n+1} + \frac{\pi \cos(n-1)\pi}{n-1} \right], \quad n \neq 1 \\ &= \frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} = \frac{2(-1)^{n-1}}{n^2-1}, \quad n \neq 1. \end{aligned}$$

Also

$$\begin{aligned} a_1 &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin x \cos x \, dx = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos x \, dx \\ &= \frac{1}{\pi} \int_0^{\pi} x \sin 2x \, dx = \frac{1}{\pi} \left[x \left(\frac{-\cos 2x}{2} \right) + \frac{\sin 2x}{4} \right]_0^{\pi} \\ &= \frac{1}{\pi} \left[\frac{-\pi \cos 2\pi}{2} \right] = -\frac{1}{2}. \end{aligned}$$

Hence

$$\begin{aligned} x \sin x &= 1 - \frac{1}{2} \cos x + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}}{n^2-1} \cos nx \\ &= 1 - \frac{1}{2} \cos x \\ &\quad - 2 \left[\frac{\cos 2x}{1.3} - \frac{\cos 3x}{2.4} + \frac{\cos 4x}{3.5} - \dots \right] \end{aligned}$$

Putting $x = \frac{\pi}{2}$, we get

$$\frac{\pi}{2} = 1 + \frac{2}{1.3} - \frac{2}{3.5} + \frac{2}{5.7} - \dots$$

EXAMPLE 4.51

Find the Fourier series for $f(x) = \frac{(\pi-x)^2}{4}$ in the interval $(0, 2\pi)$ and hence deduce

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

Solution. We have $f(x) = \frac{(\pi-x)^2}{4}$ and the interval is $(0, 2\pi)$. Therefore the Fourier coefficients are

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{2\pi} f(x) \, dx = \frac{1}{\pi} \int_0^{2\pi} \frac{(\pi-x)^2}{4} \, dx \\ &= \frac{1}{4\pi} \left[\frac{(\pi-x)^3}{-3} \right]_0^{2\pi} = \frac{\pi^2}{6}, \\ a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \int_0^{2\pi} \frac{(\pi-x)^2}{4} \cos nx \, dx \\ &= \frac{1}{4\pi} \left[\left\{ (\pi-x)^2 \frac{\sin nx}{n} \right\}_0^{2\pi} + \int_0^{2\pi} 2(\pi-x) \frac{\sin nx}{n} \, dx \right] \\ &= \frac{-1}{2\pi n^2} (-2\pi) = \frac{1}{n^2}, \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_0^{2\pi} \frac{(\pi-x)^2}{4} \sin nx \, dx \\ &= \frac{1}{4\pi} \left[\left\{ (\pi-x)^2 \frac{\cos nx}{-n} \right\}_0^{2\pi} - \int_0^{2\pi} 2(\pi-x) \frac{\cos nx}{n} \, dx \right] \\ &= \frac{1}{2\pi n^2} \left[\frac{\cos nx}{n} \right]_0^{2\pi} = 0. \end{aligned}$$

Therefore the Fourier series is

$$\begin{aligned} f(x) &\sim \frac{a_0}{2} + \sum (\cos nx + \sin nx) \\ &= \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} \\ &= \frac{\pi^2}{12} + \frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \frac{\cos 4x}{4^2} + \dots \end{aligned}$$

Putting $n = 0$ in the above equation, we get

$$\frac{\pi^2}{6} = \frac{1}{12} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

EXAMPLE 4.52

(i) Examine whether the function $f(x) = x \sin x$ is even or odd and find its Fourier series in the interval $(-\pi, \pi)$.

(ii). Examine whether the function $f(x) = x \cos x$ is even or odd and find its Fourier series in $(-\pi, \pi)$.

Solution. (i). Since the functions x and $\sin x$ are both odd functions, their product $x \sin x$ is even. Hence the Fourier coefficients b_n are zero. Further,

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi} x \sin x \, dx \\ &= \frac{2}{\pi} \left[x(-\cos x) - \int (-\cos x) dx \right]_0^{\pi} \\ &= \frac{2}{\pi} [-x \cos x + \sin x]_0^{\pi} = 2, \\ a_n &= \frac{2}{\pi} \int_0^{\pi} x \sin x \cos nx \, dx \\ &= \frac{1}{\pi} \int_0^{\pi} x [\sin(n+1)x - \sin(n-1)x] dx \\ &= \frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1}, \quad n \neq 1 \end{aligned}$$

If n is odd and $n \neq 1$, then

$$a_n = \frac{1}{n-1} - \frac{1}{n+1} = \frac{2}{n^2-1}.$$

If n is even, then

$$a_n = \frac{-1}{n-1} + \frac{1}{n+1} = \frac{-2}{n^2-1}.$$

Also

$$\begin{aligned} a_1 &= \frac{2}{\pi} \int_0^{\pi} x \sin x \cos x \, dx = \frac{1}{\pi} \int_0^{\pi} x \sin 2x \, dx \\ &= \frac{1}{\pi} \left[-\frac{\pi \cos 2\pi}{2} \right] = -\frac{1}{2}. \end{aligned}$$

Hence the Fourier series for $x \sin x$ is

$$1 - \frac{1}{2} \cos x - 2 \left[\frac{\cos 2x}{2^2-1} - \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} - \dots \right].$$

(ii) We have $f(x) = x \cos x$ as a product of odd and even functions x and $\cos x$ respectively. But the product of odd and even function is odd. Hence f is odd. Since f is odd, $a_n = 0$. Further,

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} x \cos x \sin nx \, dx \\ &= \frac{1}{\pi} \int_0^{\pi} 2x \cos x \sin nx \, dx \\ &= \frac{1}{\pi} \int_0^{\pi} x [\sin(n+1)x + \sin(n-1)x] dx \\ &= \frac{1}{\pi} \left[x \left\{ -\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1} \right\} \right. \\ &\quad \left. - \left\{ -\frac{\sin(n+1)x}{(n+1)^2} - \frac{\sin(n-1)x}{(n-1)^2} \right\} \right]_0^{\pi} \\ &= \frac{1}{\pi} \left[\pi \left\{ -\frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1} \right\} \right], \quad n \neq 1 \\ &= - \left[\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} \right], \quad n \neq 1. \end{aligned}$$

If $n \neq 1$ and n is odd, then both $n-1$ and $n+1$ are even. Therefore

$$b_n = - \left[\frac{-1}{n+1} - \frac{1}{n-1} \right] = \frac{2n}{n^2-1}$$

If $n \neq 1$ and n is even, then both $n-1$ and $n+1$ are odd. Therefore

$$b_n = - \left[\frac{-1}{n+1} + \frac{1}{n-1} \right] = - \frac{2n}{n^2-1}.$$

When $n = 1$, we have

$$\begin{aligned} b_1 &= \frac{2}{\pi} \int_0^{\pi} x \cos x \sin x \, dx = \frac{1}{\pi} \int_0^{\pi} x \sin 2x \, dx \\ &= -\frac{1}{2}. \end{aligned}$$

Hence

$$f(x) = x \cos x = -\frac{1}{2} \sin x - \frac{4}{3} \sin 2x + \frac{3}{4} \sin 3x - \dots$$

EXAMPLE 4.53

Obtain a half-range cosine series for

$$f(x) = \begin{cases} kx, & 0 \leq x \leq \frac{l}{2} \\ k(l-x), & \frac{l}{2} \leq x \leq l. \end{cases}$$

Deduce the sum of the series

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Solution. As in Example 4.24, we have

$$a_0 = \frac{kL}{2} \quad \text{and} \quad a_n = \frac{8kL}{n^2\pi^2} \cos \frac{n\pi}{2} \sin^2 \frac{n\pi}{4}.$$

Therefore

$$\begin{aligned} f(x) &= \frac{kL}{4} \\ &\quad - \frac{2kL}{\pi^2} \left[\cos \frac{2\pi x}{L} + \frac{1}{3^2} \cos \frac{6\pi x}{L} \right. \\ &\quad \left. + \frac{1}{5^2} \cos \frac{10\pi x}{L} + \dots \right] \end{aligned}$$

Putting $x = 0$, we get

$$0 = \frac{kL}{4} - \frac{2kL}{\pi^2} \left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

or

$$\frac{-kL}{4} = -\frac{2kL}{\pi^2} \left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

or

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

EXAMPLE 4.54

Find the Fourier series of periodicity 2 for $f(x) =$

$$\begin{cases} x, & -1 < x \leq 0 \\ x+2, & 0 < x \leq 1. \end{cases} \quad \text{Hence, show}$$

that the sum of the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$.

Solution. The period of the function is 2. Therefore the Fourier series shall be

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n \omega_0 t + b_n \sin n \omega_0 t), \quad \omega_0 = \frac{2\pi}{T} = \pi.$$

or

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n \pi t + b_n \sin n \pi t),$$

where

$$a_0 = \frac{2}{T} \int_{-1}^1 f(x) \, dx, \quad a_n = \frac{2}{T} \int_{-1}^1 f(x) \cos n \pi x \, dx,$$

$$b_n = \frac{2}{T} \int_{-1}^1 f(x) \sin n \pi x \, dx,$$

With $T = 2$, we are given that

$$f(x) = \begin{cases} x, & -1 < x < 0 \\ x+2, & 0 < x \leq 1. \end{cases}$$

Therefore

$$\begin{aligned} a_0 &= \int_{-1}^1 f(x) \, dx = \int_{-1}^0 f(x) \, dx + \int_0^1 f(x) \, dx \\ &= \int_{-1}^0 x \, dx + \int_0^1 (x+2) \, dx = \left[\frac{x^2}{2} \right]_{-1}^0 + \left[\frac{x^2}{2} + 2x \right]_0^1 = 2, \end{aligned}$$

$$\begin{aligned} a_n &= \int_{-1}^1 f(x) \cos n \pi x \, dx \\ &= \int_{-1}^0 x \cos n \pi x \, dx + \int_0^1 (x+2) \cos n \pi x \, dx \\ &= \left[x \frac{\sin n \pi x}{n \pi} - \left(-\frac{\cos n \pi x}{n^2 \pi^2} \right) \right]_{-1}^0 \\ &\quad + \left[(x+2) \frac{\sin n \pi x}{n \pi} - \left(-\frac{\cos n \pi x}{n^2 \pi^2} \right) \right]_0^1 \\ &= \left[\frac{1}{n^2 \pi^2} - \frac{\cos n \pi}{n^2 \pi^2} \right] + \left[\frac{\cos n \pi}{n^2 \pi^2} - \frac{1}{n^2 \pi^2} \right] = 0, \end{aligned}$$

$$\begin{aligned} b_n &= \int_{-1}^1 f(x) \sin n \pi x \, dx = \int_{-1}^0 x \sin n \pi x \, dx \\ &\quad + \int_0^1 (x+2) \sin n \pi x \, dx \end{aligned}$$

$$\begin{aligned}
&= \left[x \left(\frac{-\cos n\pi x}{n\pi} \right) - \left(\frac{-\sin n\pi x}{n^2\pi^2} \right) \right]_{-1}^0 \\
&\quad + \left[(x+2) \left(\frac{-\cos n\pi x}{n\pi} \right) - \left(\frac{-\sin n\pi x}{n^2\pi^2} \right) \right]_0^1 \\
&= -\frac{\cos n\pi}{n\pi} + \left[-\frac{3\cos n\pi}{n\pi} + \frac{2}{n\pi} \right] \\
&= -4\frac{\cos n\pi}{n\pi} + \frac{2}{n\pi} = \frac{2}{n\pi} [1 - 2(-1)^n].
\end{aligned}$$

Hence

$$\begin{aligned}
f(x) &= 1 + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} [1 - 2(-1)^n] \sin n\pi x \\
&= 1 + \frac{2}{\pi} \left[3\sin \pi x + \frac{3}{3}\sin 3\pi x + \frac{3}{5}\sin 5\pi x + \dots \right]
\end{aligned}$$

Putting $x = \frac{1}{2}$, we get

$$\frac{1}{2} + 2 = 1 + \frac{2}{\pi} \left[3 - \frac{3}{3} + \frac{3}{5} - \frac{3}{7} + \dots \right]$$

or

$$-\frac{3}{2} = \frac{2}{\pi} \left[3 - \frac{3}{3} + \frac{3}{5} - \frac{3}{7} + \dots \right]$$

or

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

EXERCISES

1. Find the Fourier series to represent x^2 in the interval $(-l, l)$.

Hint: see Example 4.6

2. Find the Fourier series of the function

$$f(x) = \begin{cases} \sin \frac{x}{2} & \text{for } 0 \leq x \leq \pi \\ -\sin \frac{x}{2} & \text{for } \pi \leq x \leq 2\pi \end{cases}$$

$$f(x) = f(x + 2\pi)$$

$$\text{Ans. } -\frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n \sin(2nx)}{(2n+1)(2n-1)}$$

3. Derive Fourier series for e^{-ax} , $-\pi < x < \pi$ and deduce series for $\frac{\pi}{\sinh \pi}$.

Hint: Similar to Example 4.9

$$\begin{aligned}
\text{Ans. } \frac{2\sinh a\pi}{\pi} &\left[\left(\frac{1}{2a} - \frac{a\cos \alpha}{1^2+a^2} + \frac{a\cos 2\alpha}{2^2+a^2} - \dots \right) \right. \\
&\quad \left. + \left(\frac{\sin x}{1^2+a^2} - \frac{2\sin 2x}{2^2+a^2} + \dots \right) \right] \\
\frac{\pi}{\sinh \pi} &= 2 \left(\frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - \dots \right)
\end{aligned}$$

4. Show that for $-\pi < x < \pi$,

$$\cosh ax = \frac{2a^2}{\pi} \sinh a\pi \left[\frac{1}{2a^2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{a^2+n^2} \cos nx \right].$$

Hint: $\cosh ax = \frac{1}{2}(e^{ax} + e^{-ax})$, so add the series of Example 4.9 and Exercise 3 (given above).

5. An alternating current, after passing through a rectifier, has the form

$$i = \begin{cases} I_0 \sin x & \text{for } 0 \leq x \leq \pi \\ 0 & \text{for } \pi \leq x \leq 2\pi, \end{cases}$$

where I_0 is maximum current and the period is 2π . Express i as a Fourier series.

Hint: see Example 4.21.

$$\begin{aligned}
\text{Ans. } \frac{I_0}{\pi} &\left(1 + \frac{\pi}{2} \sin \theta - \frac{2}{1.3} \cos 2\theta \right. \\
&\quad \left. - \frac{2}{3.5} \cos 4\theta - \frac{2}{5.7} \cos 6\theta + \dots \right)
\end{aligned}$$

6. Determine Fourier series expansion of the function

$$f(x) = \begin{cases} 2 & \text{for } 0 < x < \frac{2\pi}{3} \\ 1 & \text{for } \frac{2\pi}{3} < x < \frac{4\pi}{3} \\ 0 & \text{for } \frac{4\pi}{3} < x < 2\pi. \end{cases}$$

$$\text{Ans. } 1 + \frac{3}{\pi} \left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \frac{\sin 5x}{5} + \dots \right)$$

7. Expand $f(t) = t^2$ in Fourier sine series on the interval $(0, 1)$.

$$\text{Ans. } \sum_{n=1}^{\infty} \frac{2}{n\pi} \left[\frac{2((-1)^n - 1)}{n^2\pi^2} - (-1)^n \right] \sin n\pi t$$

8. Find Fourier series expansion of the function f defined by

$$\begin{aligned}
f(t) &= \begin{cases} \pi^2 & \text{for } -\pi < t < 0 \\ (t-\pi)^2 & \text{for } 0 \leq t < \pi, \end{cases} \\
f(t) &= f(t + 2\pi).
\end{aligned}$$

Determine the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$$\text{Ans. } f(t) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \left[\frac{2}{n^2} \cos nt + \frac{(-1)^n}{n} \pi \sin nt \right] - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)t}{(2n-1)^2}$$

Putting $t = 0$, we get $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

9. Find half range cosine series for the function

$$f(x) = \begin{cases} x & \text{for } 0 < x < \frac{\pi}{2} \\ \pi - x & \text{for } \frac{\pi}{2} < x < \pi. \end{cases}$$

$$\text{Ans. } \frac{\pi}{4} - \frac{2}{\pi} \left(\cos 2x + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \dots \right)$$

10. Find the Fourier sine series for

$$f(x) = \begin{cases} \frac{1}{4} - x & \text{for } 0 < x < \frac{1}{2} \\ x - \frac{3}{4} & \text{for } \frac{1}{2} < x < 1. \end{cases}$$

Hint: Taking odd extension, the extended function is odd over $(-1, 1)$ and so find b_n using

$$b_n = \frac{1}{1} \int_0^1 f(x) \sin n\pi x \, dx.$$

$$\text{Ans. } \left(\frac{1}{\pi} - \frac{4}{\pi^2} \right) \sin \pi x + \left(\frac{1}{3\pi} + \frac{4}{3^2\pi^2} \right) \sin 3\pi x + \left(\frac{1}{5\pi} - \frac{4}{5^2\pi^2} \right) \sin 5\pi x + \dots$$

11. Find Fourier cosine series for $f(x) = (x-1)^2$, $0 < x < 1$.

$$\text{Ans. } \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi x$$

12. Verify Riemann-Lebesgue lemma for the function $f(x) = e^{ax}$, $-\pi < x < \pi$.

Hint: see Example 4.9, show that $a_n \rightarrow 0$, $b_n \rightarrow 0$ as $n \rightarrow \infty$.

13. Find half-range sine series for e^x , $0 < x < 1$.

$$\text{Ans. } 2\pi \sum_{n=1}^{\infty} \frac{n[1 - e^{(-1)^n}]}{(1+n^2\pi^2)} \sin n\pi x$$

14. Given the series

$$t(\pi - t) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)t}{(2n-1)^3}, \quad 0 \leq t \leq \pi,$$

$$\text{show that } \sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^6}{960}$$

Hint: Use Parseval's equality.

15. Find Fourier series for e^{-x} in $(0, \pi)$.

$$\text{Hint: } b_n = \frac{2}{\pi} \int_0^{\pi} e^{-x} \sin nx \, dx = \frac{2n}{\pi} \frac{[(-1)^n e^{-\pi} - 1]}{1+n^2} \text{ and}$$

$$\text{so the series is } \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{n[e^{-\pi}(-1)^n - 1]}{1+n^2} \sin nx$$

16. Find Fourier sine series for the function f defined by

$$f(x) = \begin{cases} x & \text{for } 0 \leq x \leq L/2 \\ L - x & \text{for } L/2 \leq x \leq L. \end{cases}$$

Hint: Extend f as an odd function in $(-L, L)$. Then

$$\begin{aligned} b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx = \frac{2}{L} \left[\int_0^{L/2} + \int_{L/2}^L \right] \\ &= -\frac{L}{n\pi} \cos \frac{n\pi}{2} + \frac{2L}{n^2\pi^2} \sin \frac{n\pi}{2} + \frac{L}{n\pi} \cos \frac{n\pi}{2} \\ &\quad + \frac{2L}{n^2\pi^2} \sin \frac{n\pi}{2} \\ &= \frac{4L}{n^2\pi^2} \sin \frac{n\pi}{2} \end{aligned}$$

Hence

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \frac{4L}{n^2\pi^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{L} \\ &= \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{L}. \end{aligned}$$

17. Solve the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < 2$, $t > 0$

$$u_x(0, t) = 0, u(2, t)$$

$$u(x, 0) = \begin{cases} 1 & \text{for } 0 < x < 1 \\ 2 - x & \text{for } 1 \leq x \leq 2. \end{cases}$$

$$\text{Hint: } u(x, t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 t/L^2} \cos \frac{n\pi x}{L}, \quad L = 2$$

$$= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 t/4} \cos \frac{n\pi x}{2},$$

$$A_0 = \int_0^2 f(x) \, dx = \int_0^1 dx$$

$$+ \int_1^2 (2-x) \, dx = \frac{3}{2},$$

$$A_n = \int_0^1 x \cos \frac{n\pi x}{2} \, dx + \int_1^2 (2-x) \cos \frac{n\pi x}{2} \, dx$$

$$= \frac{-4}{n^2\pi^2} \left[\cos \frac{n\pi}{2} - \cos n\pi \right].$$

Thus

$$u(x, t) = \frac{3}{4} - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left[\cos \frac{n\pi}{2} - \cos n\pi \right] e^{-n^2 \pi^2 t/4} \cos \frac{n\pi x}{2}.$$

18. Solve $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$, $0 < x < \pi$ subject to

$$u(x, 0) = e^{-x}, \quad 0 < x < \pi$$

$$u(0, t) = 0, u(\pi, t) = 0, t \geq 0.$$

Ans. $u(x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{n[(-1)^n e^{-\pi} - 1]}{1+n^2} e^{-n^2 kt} \sin nx$

5 Laplace Transform

The study of Laplace transform is essential for engineers and scientists because these transforms provide easy and powerful means of solving differential and integral equations. The Laplace transforms directly provides the solution of differential equations with given boundary values without finding the general solution first. A Laplace transform is an extension of the continuous-time Fourier transform motivated by the fact that this transform can be used to a wider class of signals than the Fourier transform can. In fact, Fourier transform does not converge for many signals whereas the Laplace transform does. Fourier transform is not applicable to initial-value problems whereas Laplace transform is applicable. Also some functions like sinusoidal functions and polynomials do not have Fourier transform in the usual sense without the introduction of generalized functions. Causal functions (which assume zero value for $t < 0$) are best handled by Laplace transforms.

5.1 DEFINITION AND EXAMPLES OF LAPLACE TRANSFORM

Definition 5.1. Let $f(t)$ be a function of t defined for $t > 0$. Then the *one-sided Laplace transform* (or merely *Laplace transform*) of $f(t)$, denoted by $L\{f(t)\}$ or $F(s)$, is defined by

$$L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt, \quad s \in \mathbb{R} \text{ or } \mathbb{C},$$

provided that the integral converges for some value of s .

The defining integral is called the *Laplace integral*.

Definition 5.2. The *two-sided Laplace transform* of a function $f(t)$ is defined by

$$L\{f(t)\} = \int_{-\infty}^{\infty} e^{-st} f(t) dt$$

for all values of s , real or complex for which the integral converges.

The defining integral in this case is called *two-sided Laplace integral*.

The symbol L , which transforms $f(t)$ into $F(s)$ is called *Laplace operator*. Thus the Laplace transform of a function exists only if the Laplace integral converges. The following theorem provides sufficient conditions for the existence of Laplace transform.

Theorem 5.1. Let $f(t)$ be piecewise continuous in every finite interval $0 \leq t \leq N$ and be of exponential order γ for $t > N$. Then Laplace transform of $f(t)$ exists for all $s > \gamma$.

Proof: Since $f(t)$ is piecewise continuous on every finite interval $[0, N]$ and e^{-st} is also piecewise continuous on $[0, N]$ for $N > 0$, it follows that $e^{-st} f(t)$ is integrable on $[0, N]$. For any positive number N , we have

$$\int_0^{\infty} e^{-st} f(t) dt = \int_0^N e^{-st} f(t) dt + \int_N^{\infty} e^{-st} f(t) dt.$$

By the above arguments, the first integral on the right exists. Further, since $f(t)$ is of exponential order γ for $t > N$, there exists constant M such that $|f(t)| \leq M e^{\gamma t}$ for $t \geq 0$ and so

$$\begin{aligned} \left| \int_N^{\infty} e^{-st} f(t) dt \right| &\leq \int_N^{\infty} |e^{-st} f(t)| dt \leq \int_0^{\infty} |e^{-st} f(t)| dt \\ &\leq \int_0^{\infty} e^{-st} M e^{\gamma t} dt = \frac{M}{s - \gamma}. \end{aligned}$$

Thus the Laplace transform $L\{f(t)\}$ exists for $s > \gamma$.

Remark 5.1. The conditions of the Theorem 5.1 are *sufficient but not necessary* for the existence of Laplace transform of a function. Thus Laplace transform may exist even if these conditions are not satisfied. For example, $f(t) = t^{-1/2}$ does not satisfy these conditions but its Laplace transform does exist (see Example 5.8).

EXAMPLE 5.1

Find Laplace transform of *unit step function* f defined by $f(t) = 1, t \geq 0$.

Solution. By definition of Laplace transform, we have

$$\begin{aligned} L\{f(t)\} &= \int_0^{\infty} e^{-st} dt = \lim_{T \rightarrow \infty} \int_0^T e^{-st} dt \\ &= \lim_{T \rightarrow \infty} \left[\frac{e^{-st}}{-s} \right]_0^T = \lim_{T \rightarrow \infty} \frac{1 - e^{-sT}}{s} = \frac{1}{s} \text{ if } s > 0. \end{aligned}$$

EXAMPLE 5.2

Find the Laplace transform of the *unit ramp function* f defined by $f(t) = t, t \geq 0$.

Solution. Using integration by parts, we get

$$\begin{aligned} L\{f(t)\} &= \int_0^{\infty} t e^{-st} dt = \lim_{T \rightarrow \infty} \int_0^T t e^{-st} dt \\ &= \lim_{T \rightarrow \infty} \left\{ \left[t \left(\frac{e^{-st}}{-s} \right) \right]_0^T - \left[\frac{e^{-st}}{s^2} \right]_0^T \right\} \\ &= \lim_{T \rightarrow \infty} \left(\frac{1}{s^2} - \frac{e^{-sT}}{s^2} - \frac{T e^{-sT}}{s} \right) = \frac{1}{s^2} \text{ if } s > 0. \end{aligned}$$

EXAMPLE 5.3

Find $L\{f(t)\}$, where $f(t) = [t], t > 0$.

Solution. We have

$$\begin{aligned} L\{f(t)\} &= \int_0^{\infty} e^{-st} f(t) dt \\ &= \int_0^1 e^{-st}(0) dt + \int_1^2 e^{-st} dt + \int_2^3 e^{-st} 2 dt + \dots \\ &= \left| \frac{e^{-st}}{-s} \right|_1^2 + 2 \left| \frac{e^{-st}}{-s} \right|_2^3 + 3 \left| \frac{e^{-st}}{-s} \right|_3^4 + \dots \end{aligned}$$

$$\begin{aligned} &= \frac{e^{-s}}{s} (1 - e^{-s}) + 2 \frac{e^{-2s}}{s} (1 - e^{-s}) \\ &\quad + 3 \frac{e^{-3s}}{s} (1 - e^{-s}) + \dots \\ &= \frac{e^{-s}}{s} (1 - e^{-s}) [1 + 2e^{-s} + 3e^{-2s} + \dots] \\ &= \frac{e^{-s}}{s} (1 - e^{-s}) \frac{1}{(1 - e^{-s})^2} \\ &= \frac{e^{-s}}{s(1 - e^{-s})} = \frac{1}{s(e^s - 1)}. \end{aligned}$$

EXAMPLE 5.4

Find Laplace transform of $f(t) = e^{at}, t \geq 0$.

Solution. By the definition of Laplace transform, we have

$$\begin{aligned} L\{f(t)\} &= \int_0^{\infty} e^{-st} e^{at} dt = \lim_{T \rightarrow \infty} \int_0^T e^{-(s-a)t} dt \\ &= \lim_{T \rightarrow \infty} \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^T = \lim_{T \rightarrow \infty} \frac{1 - e^{-(s-a)T}}{s-a} \\ &= \frac{1}{s-a}, \text{ if } s > a. \end{aligned}$$

The result of this example holds for complex numbers also.

EXAMPLE 5.5

Find Laplace transforms of $f(t) = \sin at$ and $g(t) = \cos at$.

Solution. Since

$$\int e^{at} \sin bt dt = \frac{e^{at}(a \sin bt - b \cos bt)}{a^2 + b^2},$$

and

$$\int e^{at} \cos bt dt = \frac{e^{at}(a \cos bt + b \sin bt)}{a^2 + b^2},$$

we have

$$\begin{aligned} L\{\sin at\} &= \int_0^{\infty} e^{-st} \sin at dt \\ &= \lim_{T \rightarrow \infty} \int_0^T e^{-st} \sin at dt \\ &= \lim_{T \rightarrow \infty} \left[\frac{e^{-st}(-s \sin at - a \cos at)}{s^2 + a^2} \right]_0^T \end{aligned}$$

$$\begin{aligned}
&= \lim_{T \rightarrow \infty} \left[\frac{a}{s^2 + a^2} - \frac{e^{-st}(s \sin aT + a \cos aT)}{s^2 + a^2} \right] \\
&= \frac{a}{s^2 + a^2} \text{ if } s > 0,
\end{aligned}$$

and

$$\begin{aligned}
L\{\cos at\} &= \int_0^{\infty} e^{-st} \cos at \, dt \\
&= \lim_{T \rightarrow \infty} \int_0^T e^{-st} \cos at \, dt \\
&= \lim_{T \rightarrow \infty} \left[\frac{e^{-st}(-s \cos at + a \sin at)}{s^2 + a^2} \right]_0^T \\
&= \lim_{T \rightarrow \infty} \left[\frac{s}{s^2 + a^2} - \frac{e^{-st}(s \cos aT - a \sin aT)}{s^2 + a^2} \right] \\
&= \frac{s}{s^2 + a^2} \text{ if } s > 0.
\end{aligned}$$

EXAMPLE 5.6

Find the Laplace transforms of $f(t) = \sinh at$ and $g(t) = \cosh at$.

Solution. Since $\sinh at = \frac{e^{at} - e^{-at}}{2}$, we have

$$\begin{aligned}
L\{\sinh at\} &= L\left\{\frac{e^{at} - e^{-at}}{2}\right\} = \int_0^{\infty} e^{-st} \left(\frac{e^{at} - e^{-at}}{2}\right) dt \\
&= \frac{1}{2} \int_0^{\infty} e^{-st} e^{at} dt - \frac{1}{2} \int_0^{\infty} e^{-st} e^{-at} dt \\
&= \frac{1}{2} L\{e^{at}\} - \frac{1}{2} L\{e^{-at}\} \\
&= \frac{1}{2} \left[\frac{1}{s-a} - \frac{1}{s+a} \right] = \frac{a}{s^2 - a^2}, \quad s > |a|.
\end{aligned}$$

Again, since $\cosh at = \frac{e^{at} + e^{-at}}{2}$, proceeding as above, we have

$$\begin{aligned}
L\{\cosh at\} &= \frac{1}{2} L\{e^{at}\} + \frac{1}{2} L\{e^{-at}\} \\
&= \frac{1}{2} \left[\frac{1}{s-a} + \frac{1}{s+a} \right] \\
&= \frac{s}{s^2 - a^2}, \quad s > |a|.
\end{aligned}$$

EXAMPLE 5.7

Find Laplace transform of $f(t) = t^n$, where n is a positive integer.

Solution. Putting $st = u$, we have

$$\begin{aligned}
L\{f(t)\} &= \int_0^{\infty} e^{-st} t^n \, dt = \int_0^{\infty} e^{-u} \left(\frac{u}{s}\right)^n \cdot \frac{du}{s} \\
&= \frac{1}{s^{n+1}} \int_0^{\infty} e^{-u} u^{(n+1)-1} \, du = \frac{\Gamma(n+1)}{s^{n+1}}
\end{aligned}$$

for $s > 0$ and $n+1 \geq 0$,

by the definition of gamma function. Since n is positive integer, $\Gamma(n+1) = n!$ and so

$$L\{t^n\} = \frac{n!}{s^{n+1}}.$$

Remark 5.2. Integrating the defining formula for Laplace transform of t^n by parts, we have

$$\begin{aligned}
L\{t^n\} &= \int_0^{\infty} t^n e^{-st} \, dt \\
&= \left[\frac{-t^n e^{-st}}{s} \right]_0^{\infty} + \frac{n}{s} \int_0^{\infty} t^{n-1} e^{-st} \, dt.
\end{aligned}$$

The integral on the right exists and the lower limit can be used in the first term if $n \geq 1$. Since $s > 0$, the exponent in the first term goes to zero as t tends to infinity. Thus, we obtain a general *recurrence formula*,

$$L\{t^n\} = \frac{n}{s} L\{t^{n-1}\}, \quad n \geq 1.$$

Hence, by induction, we get the sequence

$$L\{t^0\} = L\{1\} = \frac{1}{s} \quad (\text{by Example 5.1})$$

$$L\{t\} = \frac{1}{s} L\{t^0\} = \frac{1}{s^2}$$

$$L\{t^2\} = \frac{2}{s^3}$$

.....

.....

$$L\{t^n\} = \frac{n!}{s^{n+1}}.$$

EXAMPLE 5.8

Find the Laplace transform of $f(t) = t^{-1/2}$.

Solution. The condition $n + 1 > 0$ of Example 5.7 is satisfied and so

$$L\{t^{-1/2}\} = \frac{\Gamma(1/2)}{s^{1/2}} = \frac{\sqrt{\pi}}{s^{1/2}} = \sqrt{\frac{\pi}{s}}.$$

It may be mentioned here that the function $f(t) = t^{-1/2}$ does not satisfy the conditions of Theorem 5.1, even then the Laplace transform of this function exists. Thus the conditions of Theorem 5.1 are sufficient but not necessary for the existence of Laplace transform of a given function.

EXAMPLE 5.9

Find the Laplace transform of the function f defined by

$$f(t) = \begin{cases} t & \text{for } 0 \leq t \leq 3 \\ 0 & \text{for } t > 3. \end{cases}$$

Solution. The graph of the function f is shown in the Figure 5.1.

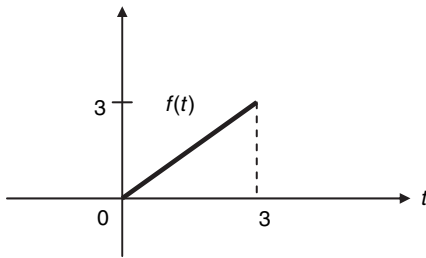


Figure 5.1

Using integration by parts, we have

$$\begin{aligned} L\{f(t)\} &= \int_0^{\infty} e^{-st} f(t) dt = \int_0^3 e^{-st} t dt \\ &= \left[t \cdot \frac{e^{-st}}{-s} \right]_0^3 - \int_0^3 \frac{e^{-st}}{-s} dt \\ &= -\frac{3}{s} e^{-3s} + \frac{1}{s} \left[\frac{e^{-st}}{-s} \right]_0^3 \\ &= -\frac{3}{s} e^{-3s} - \frac{1}{s^2} [e^{-3s} - 1] \\ &= \frac{1}{s^2} [1 - e^{-3s}] - \frac{3}{s} e^{-3s} \quad \text{for } s > 0. \end{aligned}$$

EXAMPLE 5.10

Find the Laplace transform of the function f defined by $f(t) = \sqrt{t^n}$, $n \geq 1$ and odd integer.

Solution. Integration by parts yields

$$\begin{aligned} L\{f(t)\} &= \int_0^{\infty} \sqrt{t^n} e^{-st} dt \\ &= - \left[\frac{\sqrt{t^n} e^{-st}}{s} \right]_0^{\infty} + \frac{n}{2s} \int_0^{\infty} \sqrt{t^{n-2}} e^{-st} dt. \end{aligned}$$

If $n \geq 1$, the lower limit can be used in the first term on the right and thus the integral exists. Thus

$$L\{\sqrt{t^n}\} = \frac{n}{2s} L\{\sqrt{t^{n-2}}\}, \quad n \geq 1 \text{ and odd.}$$

Thus we obtain a sequence of formulas given below:

$$L\left\{\frac{1}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{\sqrt{s}} \quad (\text{Example 5.8})$$

$$L\{\sqrt{t}\} = \frac{\sqrt{\pi}}{2\sqrt{s^3}}$$

$$L\{\sqrt{t^3}\} = \frac{3\sqrt{\pi}}{4\sqrt{s^5}}$$

.....

.....

$$L\{\sqrt{t^n}\} = \frac{(n+1)! \sqrt{\pi}}{2^{(n+1)} \left(\frac{n+1}{2}\right)! \sqrt{s^{(n+2)}}}.$$

EXAMPLE 5.11

Find the Laplace transform of $\text{erf}(\sqrt{z})$ and $\text{erf}(z)$.

Solution. Recall that the *error function* is defined by the integral

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt,$$

where the variable z may be real or complex.

The graph of $\text{erf}(z)$, where z is real is shown in the Figure 5.2.

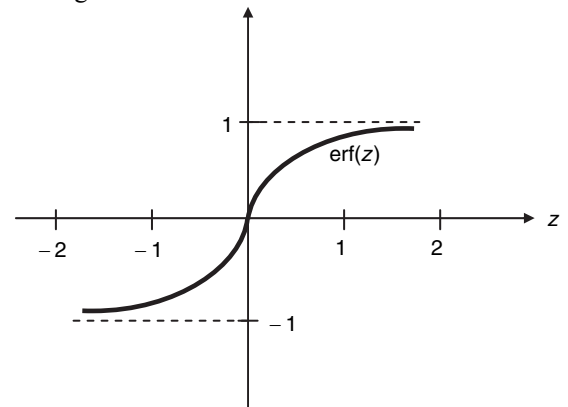


Figure 5.2 The Error Function

Let us find Laplace transform of $\operatorname{erf}(\sqrt{z})$. Using series expansion of e^{-t^2} , we have

$$\begin{aligned}
 & \mathcal{L}\{\operatorname{erf}(\sqrt{z})\} \\
 &= \mathcal{L}\left\{\frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} \left[1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \dots\right] dt\right\} \\
 &= \frac{2}{\sqrt{\pi}} \mathcal{L}\left\{t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5.2!} - \frac{t^{7/2}}{7.3!} + \dots\right\} \\
 &= \frac{2}{\sqrt{\pi}} \left\{ \frac{\Gamma(3/2)}{s^{3/2}} - \frac{\Gamma(5/2)}{3s^{5/2}} \right. \\
 &\quad \left. + \frac{\Gamma(7/2)}{5.2! s^{7/2}} - \frac{\Gamma(9/2)}{7.3! s^{9/2}} + \dots \right\} \\
 &= \frac{1}{s^{3/2}} - \frac{1}{2s^{5/2}} + \frac{1.3}{2.4s^{7/2}} - \frac{1.3.5}{2.4.6s^{9/2}} + \dots \\
 &= \frac{1}{s^{3/2}} \left[1 - \frac{1}{2s} + \frac{1.3}{2.4s^2} - \frac{1.3.5}{2.4.6s^3} + \dots \right] \\
 &= \frac{1}{s^{3/2}} \left(1 + \frac{1}{s} \right)^{-1/2} \\
 &= \frac{1}{s\sqrt{s+1}}, \quad s > -1.
 \end{aligned}$$

We now find the Laplace transform of $\operatorname{erf}(z)$. We have

$$\begin{aligned}
 \mathcal{L}\{\operatorname{erf}(z)\} &= \int_0^\infty e^{-st} \operatorname{erf}(z) dz \\
 &= \int_0^\infty e^{-st} \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2} dx dz
 \end{aligned}$$

Changing the order of integration (Figure 5.3), we get

$$\begin{aligned}
 \mathcal{L}\{\operatorname{erf}(z)\} &= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} \int_x^\infty e^{-st} dt dx = \frac{2}{s\sqrt{\pi}} \int_0^\infty e^{-(x^2+sx)} dx \\
 &= \frac{2}{s\sqrt{\pi}} e^{s^2/4} \int_0^\infty e^{-(x+\frac{s}{2})^2} dx,
 \end{aligned}$$

because $x^2 + sx = (x + \frac{s}{2})^2 - \frac{s^2}{4}$.

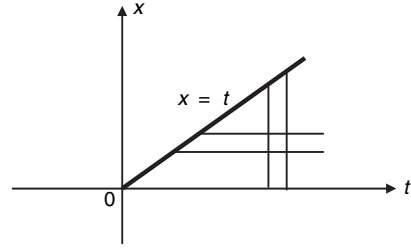


Figure 5.3

Taking $u = x + \frac{s}{2}$, we have

$$\mathcal{L}\{\operatorname{erf}(z)\} = \frac{2}{s\sqrt{\pi}} e^{s^2/4} \int_{s/2}^\infty e^{-u^2} du$$

and so

$$\mathcal{L}\{\operatorname{erf}(z)\} = \frac{2}{s\sqrt{\pi}} e^{s^2/4} \operatorname{erfc}\left(\frac{s}{2}\right), \quad s > 0.$$

EXAMPLE 5.12

Find the Laplace transform of $f(t) = \sin \sqrt{t}$.

Solution. The series expansion for $\sin \sin \sqrt{t}$ is

$$\begin{aligned}
 \sin \sqrt{t} &= t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \dots \\
 &= \sum_{n=0}^\infty \frac{(-1)^n t^{n+1/2}}{(2n+1)!}
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \mathcal{L}\{\sin \sqrt{t}\} &= \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!} \mathcal{L}\{t^{n+1/2}\} \\
 &= \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!} \frac{\Gamma(n+\frac{3}{2})}{s^{n+\frac{3}{2}}} \\
 &= \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!} \cdot \frac{(2n+2)!}{2^{2n+2}(n+1)!} \sqrt{\pi} \cdot \frac{1}{s^{n+\frac{3}{2}}} \\
 &= \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!} \cdot \frac{(2n+1)!}{2^{2n+1}n!} \cdot \frac{\sqrt{\pi}}{s^{n+\frac{3}{2}}} \\
 &= \frac{1}{2s} \sqrt{\frac{\pi}{s}} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \left(\frac{1}{4s}\right)^n \\
 &= \frac{1}{2s} \sqrt{\frac{\pi}{s}} e^{-\frac{1}{4s}}, \quad s > 0.
 \end{aligned}$$

EXAMPLE 5.13

Find the Laplace transform of the *pulse of unit height and duration T*.

Solution. The pulse of unit height and duration T is defined by

$$f(t) = \begin{cases} 1 & \text{for } 0 < t < T \\ 0 & \text{for } T < t. \end{cases}$$

Therefore,

$$L\{f(t)\} = \int_0^T e^{-st} dt = \frac{1 - e^{-sT}}{s}.$$

EXAMPLE 5.14

Find the Laplace transform of sinusoidal (sine) pulse.

Solution. Sinusoidal pulse is defined by

$$f(t) = \begin{cases} \sin at & \text{for } 0 < t < \pi/a \\ 0 & \text{for } \pi/a < t. \end{cases}$$

Therefore, by the definition of Laplace transform we have

$$L\{f(t)\} = \int_0^{\infty} \sin at e^{-st} dt = \frac{a(1 + e^{-s\pi/a})}{s^2 + a^2}.$$

The denominator of $L\{f(t)\}$ here is zero at $s = \pm ia$. But, since $e^{\pm ia} = -1$, the numerator also becomes zero. Thus $L\{f(t)\}$ have no poles and is an entire function.

EXAMPLE 5.15

Find the Laplace transform of *triangular pulse of duration T*.

Solution. The triangular pulse of duration T is defined by

$$f(t) = \begin{cases} \frac{2}{T}t & \text{for } 0 < t < T/2 \\ 2 - \frac{2}{T}t & \text{for } T/2 < t < T \\ 0 & \text{for } T < t. \end{cases}$$

By the definition of Laplace transform, we have

$$\begin{aligned} L\{f(t)\} &= \frac{2}{T} \int_0^{T/2} t e^{-st} dt + \int_{T/2}^T \left(2 - \frac{2}{T}t\right) e^{-st} dt \\ &= \frac{2}{T} \left(\frac{1 - 2e^{-sT/2} + e^{-sT}}{s^2} \right). \end{aligned}$$

EXAMPLE 5.16

Find the Laplace transform of a function defined by

$$f(t) = \begin{cases} \frac{t}{a} & \text{for } 0 < t < a \\ 1 & \text{for } t > a. \end{cases}$$

Solution. Integration by parts, we

$$\begin{aligned} L\{f(t)\} &= \int_0^a \frac{t}{a} e^{-st} dt + \int_a^{\infty} e^{-st} dt \\ &= \frac{-e^{-sa}}{s} - \frac{e^{-sa} - 1}{as^2} + \frac{e^{-sa}}{s} = \frac{1 - e^{-sa}}{as^2}. \end{aligned}$$

EXAMPLE 5.17

Find Laplace transform of f defined by

$$f(t) = \begin{cases} e^t & \text{for } 0 < t < 1 \\ 0 & \text{for } t > 1. \end{cases}$$

Solution. By definition of Laplace transform, we have

$$L\{f(t)\} = \int_0^1 e^t \cdot s^{-st} dt = \int_0^1 e^{(-s+1)t} dt = \frac{e^{1-s} - 1}{1-s}.$$

EXAMPLE 5.18

Find the Laplace transform of a function f defined by

$$f(t) = \begin{cases} \sin t & \text{for } 0 < t < \pi \\ 0 & \text{for } t > \pi. \end{cases}$$

Solution. The integration by parts yields

$$\begin{aligned} L\{f(t)\} &= \int_0^{\pi} e^{-st} \sin t dt = 0 + \frac{1}{s} \int_0^{\pi} e^{-st} \cos t dt \\ &= \frac{1}{s} \left[\frac{e^{-st}}{-s} \cos t \right]_0^{\pi} + \frac{1}{s} \int_0^{\pi} \frac{e^{-st}}{-s} \sin t dt \\ &= \frac{1}{s^2} [1 + e^{-s\pi}] - \frac{1}{s^2} \int_0^{\pi} e^{-st} \sin t dt. \end{aligned}$$

Thus,

$$\left(\frac{s^2 + 1}{s^2} \right) L\{f(t)\} = \frac{1}{s^2} [1 + e^{-s\pi}],$$

which yields

$$L\{f(t)\} = \frac{1 + e^{-s\pi}}{s^2 + 1}.$$

EXAMPLE 5.19

Find the Laplace transform of the function f_ε defined by

$$f_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon} & \text{for } 0 \leq t \leq \varepsilon \\ 0 & \text{for } t > \varepsilon, \end{cases}$$

where $\varepsilon > 0$. Deduce the Laplace transform of the *Dirac delta function*.

Solution. The graph of the function f_ε is shown in the Figure 5.4.

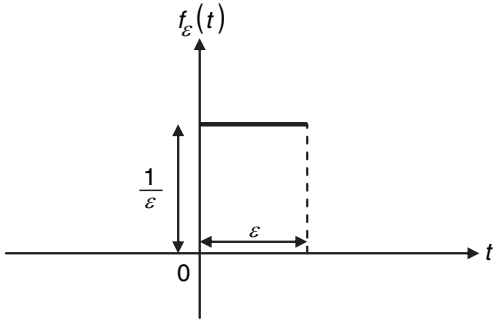


Figure 5.4

We observe that as $\varepsilon \rightarrow 0$, the height of the rectangle increases indefinitely and the width decreases in such a way that its area is always equals to 1.

The Laplace transform of f_ε is given by

$$\begin{aligned} L\{f_\varepsilon(t)\} &= \int_0^\infty e^{-st} f_\varepsilon(t) dt = \int_0^\varepsilon e^{-st} f_\varepsilon(t) dt \\ &+ \int_\varepsilon^\infty e^{-st} f_\varepsilon(t) dt \\ &= \frac{1}{\varepsilon} \int_0^\varepsilon e^{-st} dt = \frac{1 - e^{-s\varepsilon}}{s\varepsilon}. \end{aligned}$$

Further we note that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} L\{f_\varepsilon(t)\} &= \lim_{\varepsilon \rightarrow 0} \frac{1 - e^{-s\varepsilon}}{s\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1 - (1 - s\varepsilon + \frac{s^2\varepsilon^2}{2!} - \dots)}{s\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \left(1 - \frac{s\varepsilon}{2} + \dots\right) = 1. \end{aligned}$$

Also, we observe from the definition of f_ε that $\lim_{\varepsilon \rightarrow 0} f_\varepsilon(t)$ does not exist and so $L\{\lim_{\varepsilon \rightarrow 0} f_\varepsilon(t)\}$ is not

defined. Even then it is useful to define a function δ as $\delta(t) = \{\lim_{\varepsilon \rightarrow 0} f_\varepsilon(t)\}$ such that

$$L\{\delta(t)\} = \lim_{\varepsilon \rightarrow 0} \{f_\varepsilon(t)\} = 1.$$

The function $\delta(t)$ is called the *Dirac delta function* or *unit impulse function* having the properties

$$\delta(t) = 0, t \neq 0, \text{ and } \int_0^\infty \delta(t) dt = 1.$$

EXAMPLE 5.20

Find the Laplace transform of Heaviside's unit step function defined by

$$H(t-a) = \begin{cases} 1 & \text{for } t > a \\ 0 & \text{for } t < a. \end{cases}$$

Solution. The Heaviside's unit step function is also known as *delayed unit step function* and occurs in the electrical systems. It delays the output until $t = a$ and then assumes a constant value of one unit. Its graph is shown in the Figure 5.5.

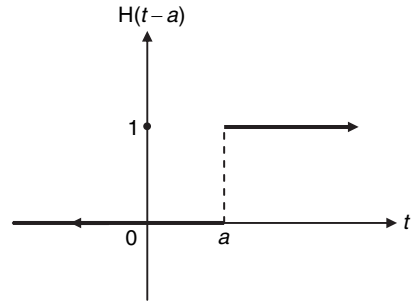


Figure 5.5

The Laplace transform of Heaviside's unit function is given by

$$\begin{aligned} L\{H(t-a)\} &= \int_0^\infty e^{-st} H(t-a) dt = \int_a^\infty e^{-st} dt \\ &= \lim_{T \rightarrow \infty} \int_a^T e^{-st} dt = \lim_{T \rightarrow \infty} \left[\frac{e^{-st}}{-s} \right]_a^T = \frac{e^{-sa}}{s}. \end{aligned}$$

EXAMPLE 5.21

Find the Laplace transform of *rectangle function* defined by

$$g(t) = \begin{cases} 1 & \text{for } a < t < b \\ 0 & \text{otherwise.} \end{cases}$$

Solution. The graph of this function is shown in the Figure 5.6. Clearly, this function can be expressed in terms of Heaviside's unit function as

$$g(t) = H(t - a) - H(t - b).$$

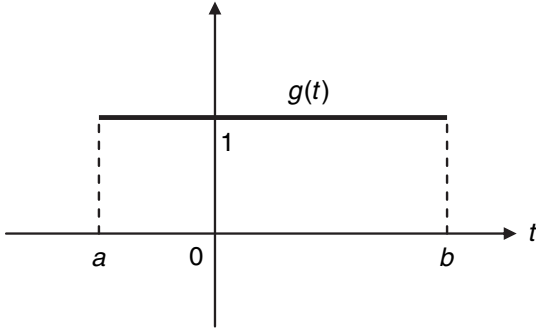


Figure 5.6

Further if $a = 0$, then g becomes pulse of unit height and duration b (Example 5.13). The Laplace transform of rectangle function g is given by

$$\begin{aligned} L\{g(t)\} &= L\{H(t - a) - H(t - b)\} \\ &= \frac{e^{-sa}}{s} - \frac{e^{-sb}}{s} \\ &= \frac{e^{-sa} - e^{-sb}}{s}. \end{aligned}$$

5.2 PROPERTIES OF LAPLACE TRANSFORMS

While studying the following properties of Laplace transforms, we assume that the Laplace transforms of the given functions exist.

Theorem 5.2. (Linearity of the Laplace Transform). If c_1 and c_2 are arbitrary constants (real or complex) and $f_1(t)$ and $f_2(t)$ are functions with Laplace transforms $F_1(s)$ and $F_2(s)$, respectively, then

$$\begin{aligned} L\{c_1 f_1(t) + c_2 f_2(t)\} &= c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\} \\ &= c_1 F_1(s) + c_2 F_2(s). \end{aligned}$$

Thus L is a linear operator.

Proof: Using the definition of Laplace transform and the linearity property of integral, we have

$$\begin{aligned} &L\{c_1 f_1(t) + c_2 f_2(t)\} \\ &= \int_0^{\infty} e^{-st} [c_1 f_1(t) + c_2 f_2(t)] dt \\ &= c_1 \int_0^{\infty} e^{-st} f_1(t) dt + c_2 \int_0^{\infty} e^{-st} f_2(t) dt \\ &= c_1 F_1(s) + c_2 F_2(s). \end{aligned}$$

EXAMPLE 5.22

Find the Laplace transform of $f(t) = \sin^2 3t$.

Solution. Since

$$\sin^2 3t = \frac{1 - \cos 6t}{2} = \frac{1}{2} - \frac{1}{2} \cos 6t,$$

we have

$$\begin{aligned} L\{\sin^2 3t\} &= L\left\{\frac{1}{2} - \frac{1}{2} \cos 6t\right\} = \frac{1}{2} L\{1\} - \frac{1}{2} L\{\cos 6t\} \\ &= \frac{1}{2} \left(\frac{1}{s}\right) - \frac{1}{2} \left(\frac{s}{s^2 + 6^2}\right), \quad s > 0 \\ &= \frac{1}{2} \left[\frac{1}{s} - \frac{s}{s^2 + 36}\right] = \frac{18}{s(s^2 + 36)}, \quad s > 0. \end{aligned}$$

EXAMPLE 5.23

Find the Laplace transform of $f(t) = e^{4t} + e^{2t} + t^3 + \sin^2 t$.

Solution. Since $\sin^2 t = \frac{1 - \cos 2t}{2}$, by Theorem 5.2, we have

$$\begin{aligned} L\{f(t)\} &= L\{e^{4t} + e^{2t} + t^3 + \sin^2 t\} \\ &= L\{e^{4t}\} + L\{e^{2t}\} + L\{t^3\} + \frac{1}{2} L\{1\} \\ &\quad - \frac{1}{2} L\{\cos 2t\} \\ &= \frac{1}{s - 4} + \frac{1}{s - 2} + \frac{6}{s^4} + \frac{1}{2s} - \frac{s}{2(s^2 + 4)} \\ &= \frac{1}{s - 4} + \frac{1}{s - 2} + \frac{6}{s^4} + \frac{2}{s(s^2 + 4)}, \quad s > 0. \end{aligned}$$

EXAMPLE 5.24

Find Laplace transform of $f(t) = \sin^3 2t$.

Solution. Since $\sin 3t = 3 \sin t - 4 \sin^3 t$, we have

$$\sin^3 t = \frac{3}{4} \sin t + \frac{1}{4} \sin 3t$$

and so

$$\sin^3 2t = \frac{3}{4} \sin 2t - \frac{1}{4} \sin 6t.$$

Hence, by linearity of L , we get

$$\begin{aligned} L\{f(t)\} &= \frac{3}{4} L\{\sin 2t\} - \frac{1}{4} L\{\sin 6t\} \\ &= \frac{3}{4} \left[\frac{2}{s^2 + 4} \right] - \frac{1}{4} \left[\frac{6}{s^2 + 36} \right], \quad s > 0 \\ &= \frac{3}{2} \left[\frac{1}{s^2 + 4} - \frac{1}{s^2 + 36} \right] \\ &= \frac{48}{(s^2 + 4)(s^2 + 36)}. \end{aligned}$$

EXAMPLE 5.25

Find the Laplace transform of $f(t) = \sin at \sin bt$.

Solution. We have

$$\begin{aligned} f(t) &= \frac{1}{2} (2 \sin at \sin bt) \\ &= \frac{1}{2} [\cos(at - bt) - \cos(at + bt)] \\ &= \frac{1}{2} \cos(a - b)t - \frac{1}{2} \cos(a + b)t. \end{aligned}$$

Therefore, using linearity, we have

$$\begin{aligned} L\{f(t)\} &= \frac{1}{2} L\{\cos(a - b)t\} - \frac{1}{2} L\{\cos(a + b)t\} \\ &= \frac{1}{2} \left[\frac{s}{s^2 + (a - b)^2} \right] - \frac{1}{2} \left[\frac{s}{s^2 + (a + b)^2} \right], \quad s > 0 \\ &= \frac{2abs}{(s^2 + (a - b)^2)(s^2 + (a + b)^2)}, \quad s > 0. \end{aligned}$$

EXAMPLE 5.26

Find the Laplace transform of $f(t) = \sin(\omega t + \phi)$, $t \geq 0$.

Solution. Since

$$\sin(\omega t + \phi) = \sin \omega t \cos \phi + \cos \omega t \sin \phi,$$

we have by linearity of the operator L ,

$$\begin{aligned} L\{f(t)\} &= \cos \phi L\{\sin \omega t\} + \sin \phi L\{\cos \omega t\} \\ &= \cos \phi \left(\frac{\omega}{s^2 + \omega^2} \right) + \sin \phi \left(\frac{s}{s^2 + \omega^2} \right), \quad s > 0 \\ &= \frac{1}{s^2 + \omega^2} (\omega \cos \phi + s \sin \phi), \quad s > 0. \end{aligned}$$

EXAMPLE 5.27

Determine Laplace transform of the *square wave function* f defined by

$$\begin{aligned} f(t) &= H(t) - 2H(t - a) + 2H(t - 2a) \\ &\quad - 2H(t - 3a) + \dots \end{aligned}$$

Solution. We note that

$$f(t) = H(t) - 2H(t - a) = 1 - 2(0) = 1, \quad 0 < t < a$$

$$f(t) = H(t) - 2H(t - a) + 2H(t - 2a)$$

$$= 1 - 2(1) + 2(0) = -1, \quad 0 < a < t < 2a$$

and so on. Thus the graph of the function is as shown in the Figure 5.7.

By linearity of Laplace operator, we have

$$\begin{aligned} L\{f(t)\} &= L\{H(t) - 2L\{H(t - a)\} \\ &\quad + 2L\{H(t - 2a)\} - 2L\{H(t - 3a)\} + \dots \\ &= \frac{1}{s} - 2 \frac{e^{-sa}}{s} + 2 \frac{e^{-2sa}}{s} - 2 \frac{e^{-3sa}}{s} + \dots \end{aligned}$$

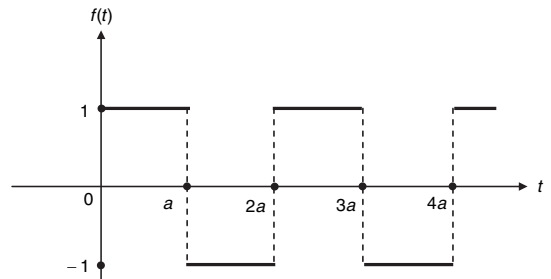


Figure 5.7

$$\begin{aligned}
&= \frac{1}{s} [1 - 2e^{-sa} \{1 - e^{-sa} + e^{-2sa} - \dots\}] \\
&= \frac{1}{s} \left[1 - 2e^{-sa} \left(\frac{1}{1 + e^{-sa}} \right) \right] \\
&= \frac{1}{s} \left(\frac{1 - e^{-sa}}{1 + e^{-sa}} \right) = \frac{1}{s} \left(\frac{e^{\frac{sa}{2}} - e^{-\frac{sa}{2}}}{e^{\frac{sa}{2}} + e^{-\frac{sa}{2}}} \right) \\
&= \frac{1}{s} \tanh\left(\frac{sa}{2}\right).
\end{aligned}$$

EXAMPLE 5.28

Find the Laplace transform of $f(t) = (\sin t - \cos t)^2$, $t \geq 0$.

Solution. Since

$$\begin{aligned}
(\sin t - \cos t)^2 &= \sin^2 t + \cos^2 t - 2 \sin t \cos t \\
&= 1 - \sin 2t,
\end{aligned}$$

we have

$$\begin{aligned}
L\{f(t)\} &= L\{1\} - L\{\sin 2t\} = \frac{1}{s} - \frac{2}{s^2 + 4} \\
&= \frac{s^2 - 2s + 4}{s(s^2 + 4)}, \quad s > 0.
\end{aligned}$$

EXAMPLE 5.29

Find Laplace transform of $f(t) = 2 + \sqrt{t} + \frac{1}{\sqrt{t}}$, $t > 0$.

Solution. By linearity of L, we have

$$\begin{aligned}
L\{f(t)\} &= 2L\{1\} + L\{\sqrt{t}\} + L\left\{\frac{1}{\sqrt{t}}\right\} \\
&= \frac{2}{s} + \frac{\Gamma(\frac{1}{2} + 1)}{s^{\frac{1}{2} + 1}} + \frac{\sqrt{\pi}}{s} \\
&= \frac{2}{s} + \frac{\sqrt{\pi}}{2s^{3/2}} + \sqrt{\frac{\pi}{s}}, \quad s > 0.
\end{aligned}$$

EXAMPLE 5.30

Find Laplace transform of $e^{at} \cos bt$ and $e^{at} \sin bt$, where a and b are real.

Solution. Let $f(t) = e^{(a+ib)t}$. Then (see Example 5.4)

$$\begin{aligned}
L\{f(t)\} &= \frac{1}{s - (a + ib)} = \frac{1}{s - a - ib} \\
&= \frac{1}{(s - a) - ib} \cdot \frac{(s - a) + ib}{(s - a) + ib} \\
&= \frac{(s - a) + ib}{(s - a)^2 + b^2} \quad (1)
\end{aligned}$$

Also

$$\begin{aligned}
e^{(a+ib)t} &= e^{at} [\cos bt + i \sin bt] \\
&= e^{at} \cos bt + i e^{at} \sin bt
\end{aligned}$$

Hence

$$\begin{aligned}
L\{f(t)\} &= L\{e^{at} \cos bt + i e^{at} \sin bt\} \\
&= L\{e^{at} \cos bt\} + i L\{e^{at} \sin bt\} \\
&\quad \text{(by linearity of L)} \quad (2)
\end{aligned}$$

Thus, by (1) and (2), we have

$$L\{e^{at} \cos bt\} + i L\{e^{at} \sin bt\} = \frac{(s - a) + ib}{(s - a)^2 + b^2}.$$

Comparing real and imaginary parts, we have

$$L\{e^{at} \cos bt\} = \frac{s - a}{(s - a)^2 + b^2}$$

and

$$L\{e^{at} \sin bt\} = \frac{b}{(s - a)^2 + b^2}.$$

EXAMPLE 5.31

Find the Laplace transform of $f(t) = \int_0^t \frac{\sin u}{u} du$.

Solution. Using series expansion of $\sin u$, we have

$$\begin{aligned}
\int_0^t \frac{\sin u}{u} du &= \int_0^t \frac{1}{u} \left[u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \dots \right] du \\
&= \int_0^t \left[1 - \frac{u^2}{3!} + \frac{u^4}{5!} - \frac{u^6}{7!} + \dots \right] du \\
&= t - \frac{t^3}{3.3!} + \frac{t^5}{5.5!} - \frac{t^7}{7.7!} + \dots
\end{aligned}$$

Therefore,

$$\begin{aligned}
L\{f(t)\} &= L\left\{t - \frac{t^3}{3.3!} + \frac{t^5}{5.5!} - \frac{t^7}{7.7!} + \dots\right\} \\
&= \frac{1}{s^2} - \frac{1}{3.3!} \cdot \frac{3!}{s^4} + \frac{1}{5.5!} \cdot \frac{5!}{s^6} - \frac{1}{7.7!} \cdot \frac{7!}{s^8} \\
&\quad + \dots \text{(by linearity of L)} \\
&= \frac{1}{s^2} - \frac{1}{3s^4} + \frac{1}{5s^6} - \frac{1}{7s^8} + \dots \\
&= \frac{1}{s} \left[\frac{1}{s} - \frac{(1/s)^3}{3} + \frac{(1/s)^5}{5} - \frac{(1/s)^7}{7} + \dots \right] \\
&= \frac{1}{s} \tan^{-1} \frac{1}{s}.
\end{aligned}$$

EXAMPLE 5.32

Find Laplace transform $f(t) = \cosh at - \cos at$.

Solution. By linearity of the Laplace operator, we have

$$\begin{aligned} L\{f(t)\} &= L\{\cosh at - \cos at\} \\ &= L\{\cosh at\} - L\{\cos at\} \\ &= \frac{s}{s^2 - a^2} - \frac{s}{s^2 + a^2}, \quad s > |a| \\ &= \frac{s^3 + a^2s - s^3 + a^2s}{s^4 - a^4} = \frac{2a^2s}{s^4 - a^4}. \end{aligned}$$

EXAMPLE 5.33

Find the Laplace transform of *Bessel's function of order zero*.

Solution. Recall that Bessel's function of order zero is defined by

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 \cdot 4^2} - \frac{t^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots$$

Therefore,

$$\begin{aligned} L\{J_0(t)\} &= L\left\{1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 \cdot 4^2} - \frac{t^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots\right\} \\ &= L\{1\} - \frac{1}{2^2} L\{t^2\} + \frac{1}{2^2 \cdot 4^2} L\{t^4\} \\ &\quad - \frac{1}{2^2 \cdot 4^2 \cdot 6^2} L\{t^6\} + \dots \\ &= \frac{1}{s} - \frac{1}{2^2} \frac{2!}{s^3} + \frac{1}{2^2 \cdot 4^2} \frac{4!}{s^5} - \frac{1}{2^2 \cdot 4^2 \cdot 6^2} \frac{6!}{s^7} + \dots \\ &= \frac{1}{s} \left[1 - \frac{1}{2} \left(\frac{1}{s^2} \right) + \frac{1.3}{2.4} \left(\frac{1}{s^4} \right) \right. \\ &\quad \left. - \frac{1.3.5}{2.4.6} \left(\frac{1}{s^6} \right) + \dots \right] \\ &= \frac{1}{s} \left[\left(1 + \frac{1}{s^2} \right)^{-1/2} \right] \text{ (using binomial theorem)} \\ &= \frac{1}{\sqrt{s^2 + 1}}. \end{aligned}$$

Theorem 5.3. [First Shifting (Translation) Property]. If $f(t)$ is a function of t for $t > 0$ and $L\{f(t)\} = F(s)$, then

$$L\{e^{at} f(t)\} = F(s - a).$$

Proof: We are given that

$$L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt.$$

By the definition of Laplace transform, we have

$$\begin{aligned} L\{e^{at} f(t)\} &= \int_0^{\infty} e^{-st} (e^{at} f(t)) dt = \int_0^{\infty} e^{-(s-a)t} f(t) dt \\ &= F(s - a). \end{aligned}$$

EXAMPLE 5.34

Find the Laplace transform of $g(t) = e^{-t} \sin^2 t$.

Solution. We have (see Example 5.23)

$$L\{\sin^2 t\} = F(s) = \frac{2}{s(s^2 + 4)}.$$

Therefore, using first shifting property, we get

$$\begin{aligned} L\{g(t)\} &= F(s - a) \\ &= \frac{2}{(s + 1)[(s + 1)^2 + 4]}, \quad \text{since } a = -1. \\ &= \frac{2}{(s + 1)(s^2 + 2s + 5)}. \end{aligned}$$

EXAMPLE 5.35

Find Laplace transform of $g(t) = t^3 e^{-3t}$.

Solution. Since

$$L\{t^3\} = F(s) = \frac{3!}{s^4} = \frac{6}{s^4},$$

we have by shifting property,

$$L\{g(t)\} = L\{e^{-3t} \cdot t^3\} = F(s - a) = \frac{6}{(s + 3)^4},$$

since $a = -3$.

EXAMPLE 5.36

Using first-shifting property, find Laplace transforms of $t \sin at$ and $t \cos at$.

Solution. Since $L\{t\} = \frac{1}{s^2}$, we have

$$\begin{aligned} L\{t e^{iat}\} &= L\{t \cos at\} + i L\{t \sin at\} \\ &= F(s - a) = \frac{1}{(s - ia)^2} = \frac{(s + ia)^2}{[(s - ia)(s + ia)]^2} \\ &= \frac{(s^2 - a^2) + i(2as)}{(s^2 + a^2)^2}. \end{aligned}$$

Equating real and imaginary parts, we have

$$L\{t \cos at\} = \frac{s^2 - a^2}{(s^2 + a^2)^2}$$

and

$$L\{t \sin at\} = \frac{2as}{(s^2 + a^2)^2}.$$

EXAMPLE 5.37

Find the Laplace transform of $f(t) = e^{at} \cosh bt$.

Solution. Since

$$L\{\cosh bt\} = \frac{s}{s^2 - b^2}, \quad s > |b|,$$

the shifting property yields

$$\begin{aligned} L\{e^{at} \cosh bt\} &= F(s - a) \\ &= \frac{s - a}{(s - a)^2 - b^2}, \quad s > |b| + a. \end{aligned}$$

EXAMPLE 5.38

Find Laplace transform of $f(t) = e^{-3t} (2 \cos 5t + 3 \sin 5t)$.

Solution. Since

$$\begin{aligned} L\{2 \cos 5t - 3 \sin 5t\} &= 2L\{\cos 5t\} - 3L\{\sin 5t\} \\ &= \frac{2s}{s^2 + 25} - \frac{3 \times 5}{s^2 + 25} = \frac{2s - 15}{s^2 + 25} = F(s), \end{aligned}$$

therefore, shifting property yields

$$\begin{aligned} L\{f(t)\} &= F(s - a) \text{ with } a = -3 \\ &= \frac{2(s + 3) - 15}{(s + 3)^2 + 25} = \frac{2s - 9}{s^2 + 6s + 34}. \end{aligned}$$

EXAMPLE 5.39

Find Laplace transform of $f(t) = \sinh 3t \cos^2 t$.

Solution. We know that $\cos^2 t = \frac{1 + \cos 2t}{2}$. Therefore,

$$\begin{aligned} L\{\cos^2 t\} &= \frac{1}{2}L\{1\} + \frac{1}{2}L\{\cos 2t\} \\ &= \frac{1}{2} \left[\frac{1}{s} + \frac{s}{s^2 + 4} \right] = \frac{s^2 + 2}{s(s^2 + 4)}, \quad s > 0. \end{aligned}$$

Therefore, by first shifting theorem, we have

$$\begin{aligned} L\{f(t)\} &= L\{\sinh 3t \cos^2 t\} = L\left\{\frac{e^{3t} - e^{-3t}}{2} \cos^2 t\right\} \\ &= \frac{1}{2}L\{e^{3t} \cos^2 t\} - \frac{1}{2}L\{e^{-3t} \cos^2 t\} \\ &\quad (\text{by linearity of } L) \\ &= \frac{1}{2} \left[\frac{(s - 3)^2 + 2}{(s - 3)[(s - 3)^2 + 4]} \right] \\ &\quad - \frac{1}{2} \left[\frac{(s + 3)^2 + 2}{(s + 3)[(s + 3)^2 + 4]} \right] \\ &= \frac{1}{2} \left[\frac{s^2 - 6s + 11}{(s - 3)(s^2 - 6s + 13)} \right. \\ &\quad \left. - \frac{s^2 + 6s + 11}{(s + 3)(s^2 + 6s + 13)} \right]. \end{aligned}$$

EXAMPLE 5.40

Find the Laplace transform of $\cosh at \sin bt$.

Solution. Let $F(s)$ be Laplace transform of $f(t)$, $t > 0$ and let

$$g(t) = f(t) \cosh at.$$

Then

$$\begin{aligned} L\{g(t)\} &= L[f(t) \cosh at] = L\left[\frac{e^{at} + e^{-at}}{2} f(t)\right] \\ &= \frac{1}{2}L(e^{at}f(t)) + \frac{1}{2}L(e^{-at}f(t)) \\ &= \frac{1}{2}[F(s - a) + F(s + a)] \\ &\quad (\text{use of first shifting theorem}). \end{aligned}$$

We take $f(t) = \sin bt$. Then $F(s) = \frac{b}{s^2 + b^2}$ and, therefore, using above result, we have

$$\begin{aligned} L\{(\cosh at) \sin bt\} &= \frac{1}{2} \left[\frac{b}{(s - a)^2 + b^2} + \frac{b}{(s + a)^2 + b^2} \right]. \end{aligned}$$

EXAMPLE 5.41

Find the Laplace transform of $f(t) = \cosh 4t \sin 6t$.

Solution. Taking $a = 4$, $b = 6$ in Example 5.40, we get

$$L\{\cosh 4t \sin 6t\} = \frac{6(s^2 + 52)}{s^4 + 40s^2 + 2704}.$$

Theorem 5.4. (Second Shifting Property). Let $F(s)$ be the Laplace transform of $f(t)$, $t > 0$ and let g be a function defined by

$$g(t) = \begin{cases} f(t-a) & \text{for } t > a \\ 0 & \text{for } t < a. \end{cases}$$

Then

$$L\{g(t)\} = e^{-as}F(s).$$

Proof: Using the substitution $t - a = u$, we have

$$\begin{aligned} L\{g(t)\} &= \int_0^{\infty} e^{-st} g(t) dt \\ &= \int_0^a e^{-st} g(t) dt + \int_a^{\infty} e^{-st} g(t) dt \\ &= 0 + \int_a^{\infty} e^{-st} f(t-a) dt \\ &= \int_0^{\infty} e^{-s(u+a)} f(u) du \\ &= e^{-as} \int_0^{\infty} e^{-su} f(u) du = e^{-as} F(s). \end{aligned}$$

EXAMPLE 5.42

Find the Laplace transform of the function f defined by

$$f(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right) & \text{for } t > \frac{2\pi}{3} \\ 0 & \text{for } t < \frac{2\pi}{3}. \end{cases}$$

Solution. We know that $L\{\cos t\} = \frac{s}{s^2+1}$, $s > 0$. Therefore, by second shifting property,

$$L\{f(t)\} = e^{-\frac{2\pi s}{3}} L\{\cos t\} = \frac{se^{-\frac{2\pi s}{3}}}{s^2+1}, \quad s > 0.$$

EXAMPLE 5.43

Find the Laplace transform of the *sine function switched on at time $t = 3$* .

Solution. The given function is defined by

$$f(t) = \begin{cases} \sin t & \text{for } t \geq 3 \\ 0 & \text{for } t < 3. \end{cases}$$

Using Heaviside's unit step function H , this function can be expressed as

$$f(t) = H(t-3) \sin t.$$

To use second-shift theorem, we first write $\sin t$ as $\sin t = \sin(t-3+3)$

$$= \sin(t-3) \cos 3 + \cos(t-3) \sin 3.$$

Then

$$\begin{aligned} L\{f(t)\} &= L\{H(t-3) \sin(t-3) \cos 3\} \\ &\quad + L\{H(t-3) \cos(t-3) \sin 3\} \\ &= \cos 3 e^{-3s} L\{\sin t\} + \sin 3 e^{-3s} L\{\cos t\} \\ &= \cos 3 e^{-3s} \frac{1}{s^2+1} + \sin 3 e^{-3s} \frac{s}{s^2+1} \\ &= \frac{e^{-3s}}{s^2+1} (\cos 3 + s \sin 3). \end{aligned}$$

EXAMPLE 5.44

Find the Laplace transform of the function f defined by

$$f(t) = \begin{cases} (t-1)^2 & \text{for } t \geq 1 \\ 0 & \text{for } 0 \leq t < 1. \end{cases}$$

Solution. This function is just the function $g(t) = t^2$ delayed by 1 unit of time and its graph is shown in Figure 5.8.

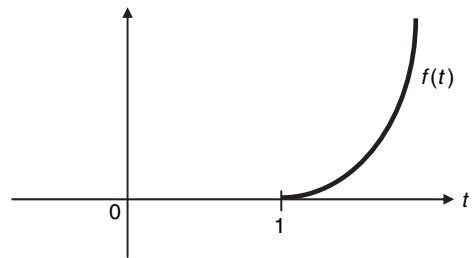


Figure 5.8

Therefore, by second shift property, we have

$$L\{f(t)\} = e^{-s} L\{t^2\} = \frac{2e^{-s}}{s^3}, \quad \operatorname{Re}(s) > 0.$$

EXAMPLE 5.45

Find Laplace transform of the function f defined by

$$f(t) = \begin{cases} (t-4)^5 & \text{for } t > 4 \\ 0 & \text{for } t < 4. \end{cases}$$

Solution. Using second shift property, we have

$$\mathcal{L}\{f(t)\} = e^{-4s}[\mathcal{L}\{t^5\}] = e^{-4s} \cdot \frac{5!}{s^6} = 120 \frac{e^{-4s}}{s^6}.$$

Theorem 5.5. (Change of Scale Property). If $F(s)$ is the Laplace transform of $f(t)$ for $t > 0$, then for any positive constant a ,

$$\mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right).$$

Proof: We are given that

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt.$$

Taking $u = at$, we have

$$\begin{aligned} \mathcal{L}\{f(at)\} &= \int_0^{\infty} e^{-st} f(at) dt = \int_0^{\infty} e^{-su/a} f(u) \frac{du}{a} \\ &= \frac{1}{a} \int_0^{\infty} e^{-su/a} f(u) du = \frac{1}{a} F\left(\frac{s}{a}\right). \end{aligned}$$

EXAMPLE 5.46

Find the Laplace of $f(t) = \cos 6t$.

Solution. Since $\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$, the change of scale property implies

$$\begin{aligned} \mathcal{L}\{\cos 6t\} &= \frac{1}{6} \left(\frac{s/6}{(s/6)^2 + 1} \right) \\ &= \frac{1}{6} \left(\frac{s}{6[(s^2/36) + 1]} \right) = \frac{s}{s^2 + 36}. \end{aligned}$$

EXAMPLE 5.47

Using change of scale property, find the Laplace transform $J_0(at)$.

Solution. Let $J_0(t)$ be Bessel's function of order zero. By Example 5.33, $\mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$. Therefore, by change of scale property,

$$\mathcal{L}\{J_0(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right) = \frac{1}{a} \cdot \frac{1}{\sqrt{(s/a)^2 + 1}} = \frac{1}{\sqrt{s^2 + a^2}}.$$

Theorem 5.6. (Laplace Transform of Derivatives).

Let f be a function such that

- (a) f is continuous for all t , $0 \leq t \leq N$
- (b) f is of exponential order γ for $t > N$
- (c) f' is sectionally continuous for $0 \leq t \leq N$.

Then the Laplace transform of f' exists and is given by

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0),$$

where $F(s)$ is the Laplace transform of f .

Proof: The existence of the Laplace transform is established by Theorem 5.1. Further, integrating by parts, we have

$$\begin{aligned} \mathcal{L}\{f'(t)\} &= \int_0^{\infty} e^{-st} f'(t) dt = \lim_{T \rightarrow \infty} \int_0^T e^{-st} f'(t) dt \\ &= \lim_{T \rightarrow \infty} \left\{ [e^{-st} f(t)]_0^T + s \int_0^T e^{-st} f(t) dt \right\} \\ &= \lim_{T \rightarrow \infty} \left\{ [e^{-sT} f(T) - f(0)] + s \int_0^T e^{-st} f(t) dt \right\} \\ &= s \int_0^{\infty} e^{-st} f(t) dt - f(0) \\ &= sF(s) - f(0), \end{aligned}$$

the last but one step being the consequence of the fact that f is of exponential order and so $\lim_{T \rightarrow \infty} e^{-sT} f(T) = 0$ for $s > \gamma$.

EXAMPLE 5.48

Find Laplace transform of $g(t) = \sin at \cos at$.

Solution. Let $f(t) = \sin^2 at$. Then

$$f'(t) = 2a \sin at \cos at.$$

Since

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0),$$

we have

$$\begin{aligned} &\mathcal{L}\{2a \sin at \cos at\} \\ &= s\mathcal{L}\{\sin^2 at\} - 0 = s\mathcal{L}\{\sin^2 at\} \\ &= \frac{2a^2}{s(s^2 + 4a^2)} \quad (\text{see Example 5.22}). \end{aligned}$$

Hence

$$\mathcal{L}\{\sin at \cos at\} = \frac{a}{(s^2 + 4a^2)}.$$

EXAMPLE 5.49

Using Laplace transform of $\cos bt$, find the Laplace transform of $\sin bt$.

Solution. We want to find $L\{\sin bt\}$ from $L\{\cos bt\}$. So, let $f(t) = \cos bt$. Then $f'(t) = -b \sin bt$ and so

$$\begin{aligned} L\{f'(t)\} &= sF(s) - f(0) = sL\{\cos bt\} - 1 \\ &= s \left(\frac{s}{s^2 + b^2} \right) - 1 = \frac{s^2}{s^2 + b^2} - 1 \\ &= \frac{-b^2}{s^2 + b^2}. \end{aligned}$$

Thus

$$L\{-b \sin bt\} = \frac{-b^2}{s^2 + b^2}.$$

Hence

$$L\{\sin bt\} = \frac{b}{s^2 + b^2}.$$

EXAMPLE 5.50

Find Laplace transform of *Bessel's function of order 1*.

Solution. Let $J_1(t)$ be Bessel's function of order 1. We know that

$$\frac{d}{dt}\{t^n J_n(t)\} = t^n J_{n-1}(t).$$

If $n = 0$, we have

$$J'_0(t) = J_{-1}(t) = -J_1(t).$$

Hence

$$\begin{aligned} L\{J_1(t)\} &= -L\{J'_0(t)\} = -[sL\{J_0(t)\} - J_0(0)] \\ &= -\left[\frac{s}{\sqrt{s^2 + 1}} - 1 \right] \\ &= 1 - \frac{s}{\sqrt{s^2 + 1}} \text{ (see Example 5.33)} \\ &= \frac{\sqrt{s^2 + 1} - s}{\sqrt{s^2 + 1}}. \end{aligned}$$

Theorem 5.7. If $L\{f(t)\} = F(s)$, then

$$L\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$

if $f(t)$ and $f'(t)$ are continuous for $0 \leq t \leq N$ and of exponential order for $t > N$ whereas $f''(t)$ is sectionally continuous for $0 \leq t \leq N$.

Proof: By Theorem 5.6, we have

$$L\{g'(t)\} = s G(s) - g(0).$$

Taking $g(t) = f'(t)$, we have

$$\begin{aligned} L\{f''(t)\} &= sL\{f'(t)\} - f'(0) \\ &= s[sF(s) - f(0)] - f'(0) \\ &= s^2 F(s) - sf(0) - f'(0). \end{aligned}$$

EXAMPLE 5.51

Using Theorem 5.7, find $L\{\sin at\}$, $t \geq 0$.

Solution. Let $f(t) = \sin at$. Then

$$f'(t) = a \cos at, \quad f''(t) = -a^2 \sin at.$$

By Theorem 5.7,

$$L\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$

and so

$$L\{-a^2 \sin at\} = s^2 L\{\sin at\} - a$$

which yields

$$(s^2 + a^2)L\{\sin at\} = a$$

and hence

$$L\{\sin at\} = \frac{a}{s^2 + a^2}, \quad s > 0.$$

EXAMPLE 5.52

Using Laplace transform of derivatives, find $L\{t \cos at\}$.

Solution. Let $f(t) = t \cos at$. Then

$$f'(t) = \cos at - at \sin at$$

$$f''(t) = -2a \sin at - a^2 t \cos at.$$

But

$$\begin{aligned} L\{f''(t)\} &= s^2 L\{f(t)\} - sf(0) - f'(0) \\ &= s^2 L\{f(t)\} - 1 \end{aligned}$$

and so

$$L\{-2a \sin at - a^2 t \cos at\} = s^2 L\{t \cos at\} - 1,$$

that is,

$$\begin{aligned} (s^2 + a^2)L\{t \cos at\} &= 2aL\{\sin at\} + 1 \\ &= -2a \left(\frac{a}{s^2 + a^2} \right) + 1 \\ &= \frac{s^2 - a^2}{s^2 + a^2}, \end{aligned}$$

and so

$$L\{t \cos at\} = \frac{s^2 - a^2}{(s^2 + a^2)^2}.$$

Theorem 5.7. can be generalized to higher order derivatives in the form of the following result:

Theorem 5.8. Let $L\{f(t)\} = F(s)$. Then

$$\begin{aligned} L\{f^{(n)}(t)\} &= s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) \\ &\quad - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0), \end{aligned}$$

if $f(t), f'(t), \dots, f^{(n-1)}(t)$ are continuous for $0 \leq t \leq N$ and of exponential order for $t > N$ whereas $f^{(n)}(t)$ is piecewise continuous for $0 \leq t \leq N$.

Proof: We shall prove our result using mathematical induction. By Theorems 5.6 and 5.7, we have

$$\begin{aligned} L\{f'(t)\} &= sF(s) - f(0), \\ L\{f''(t)\} &= s^2F(s) - sf(0) - f'(0). \end{aligned}$$

Thus the theorem is true for $f'(t)$ and $f''(t)$. Suppose that the result is true for $f^{(n)}(t)$. Then

$$L\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - \dots - f^{(n-1)}(0)$$

Then application of Theorem 5.6 yields

$$\begin{aligned} L\{f^{(n+1)}(t)\} &= s[s^n F(s) - s^{n-1}f(0) - \dots - f^{(n-1)}(0)] - f^{(n)}(0) \\ &= s^{n+1}F(s) - s^n f(0) - \dots - sf^{(n-1)}(0) + f^{(n)}(0), \end{aligned}$$

which shows that the result holds for $(n+1)^{\text{th}}$ derivative also. Hence by mathematical induction, the result holds.

EXAMPLE 5.53

Using Theorem 5.8, find $L\{t^n\}$.

Solution. We have $f(t) = t^n$. Therefore,

$$f'(t) = nt^{n-1}, f''(t) = n(n-1)t^{n-2}, \dots, f^{(n)}(t) = n!$$

Now use of Theorem 5.8 yields

$$\begin{aligned} L\{f^{(n)}(t)\} &= L\{n!\} = s^n L\{t^n\} \\ &\quad - s^{n-1}f(0) - \dots - f^{(n-1)}(0) \end{aligned}$$

But

$$f(0) = f'(0) = f''(0) = \dots = f^{(n-1)}(0) = 0.$$

Therefore

$$L\{n!\} = s^n L\{t^n\},$$

which gives

$$L\{t^n\} = \frac{L\{n!\}}{s^n} = \frac{n! L\{1\}}{s^n} = \frac{n!}{s^{n+1}}.$$

Theorem 5.9. (Multiplication by t^n). If $L\{f(t)\} = F(s)$, then

$$L\{tf(t)\} = -\frac{d}{ds}F(s),$$

and in general

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s).$$

Proof: By definition of Laplace transform,

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt.$$

Then, by Leibnitz-rule for differentiating under the integral sign, we have

$$\begin{aligned} \frac{dF}{ds} &= \frac{d}{ds} \int_0^{\infty} e^{-st} f(t) dt = \int_0^{\infty} \frac{d}{ds} (e^{-st} f(t)) dt \\ &= \int_0^{\infty} -t e^{-st} f(t) dt = - \int_0^{\infty} e^{-st} (tf(t)) dt \\ &= -L\{tf(t)\} \end{aligned}$$

and so

$$L\{tf(t)\} = -\frac{d}{ds}F(s).$$

Thus the theorem is true for $n = 1$. To obtain the general form, we use mathematical induction. So, assume that the result is true for $n = m$. Thus

$$L\{t^m f(t)\} = (-1)^m \frac{d^m}{ds^m} F(s) = (-1)^m F^{(m)}(s).$$

Therefore,

$$\frac{d}{ds} [L\{t^m f(t)\}] = (-1)^m F^{(m+1)}(s),$$

that is,

$$\frac{d}{ds} \int_0^{\infty} e^{-st} t^m f(t) dt = (-1)^m F^{(m+1)}(s),$$

which, on using Leibnitz rule, yields

$$-\int_0^{\infty} e^{-st} t^{m+1} f(t) dt = (-1)^m F^{(m+1)}(s),$$

and so

$$L\{t^{m+1} f(t)\} = (-1)^{m+1} F^{(m+1)}(s).$$

Hence, the result follows by mathematical induction.

EXAMPLE 5.54

Find Laplace transform of $f(t) = t \sin^2 t$.

Solution. Let $f(t) = t \sin^2 t$. We know that $L\{\sin^2 t\} = \frac{2}{s(s^2+4)}$. Therefore

$$L\{t \sin^2 t\} = -\frac{d}{ds} \left\{ \frac{2}{s(s^2+4)} \right\} = 2 \left[\frac{3s^2+4}{s^2(s^2+4)^2} \right].$$

EXAMPLE 5.55

Find Laplace transform of $f(t) = te^{-t} \cosh t$.

Solution. We know that

$$L\{\cosh t\} = \frac{s}{s^2-1}.$$

Therefore,

$$L\{t \cosh t\} = -\frac{d}{ds} \left(\frac{s}{s^2-1} \right) = \frac{s^2+1}{(s^2-1)^2}.$$

Then, by Theorem 5.3, we have

$$L\{e^{-t} t \cosh t\} = \frac{(s+1)^2+1}{((s+1)^2-1)^2} = \frac{s^2+2s+2}{(s^2+2s)^2}.$$

EXAMPLE 5.56

Using Theorem 5.9, evaluate $\int_0^{\infty} t e^{-2t} \sin t dt$ and $\int_0^{\infty} t e^{-2t} \cos t dt$.

Solution. We know that $L\{\sin t\} = \frac{1}{s^2+1}$.

Therefore, by Theorem 5.9, we have

$$L\{t \sin t\} = -\frac{d}{ds} \left(\frac{1}{s^2+1} \right) = \frac{2s}{(s^2+1)^2}.$$

But $\int_0^{\infty} e^{-2t} (t \sin t) dt$ is the Laplace transform of $t \sin t$ with $s = 2$. Hence

$$\int_0^{\infty} e^{-2t} (t \sin t) dt = L\{t \sin t\} = \left[\frac{2s}{(s^2+1)^2} \right]_{s=2} = \frac{4}{25}.$$

In a similar way, we can show that

$$\begin{aligned} \int_0^{\infty} t e^{-2t} \cos t dt &= L\{t \cos t\} \text{ with } s = 2 \\ &= \left[\frac{s^2-1}{(s^2+1)^2} \right]_{s=2} = \frac{3}{25}. \end{aligned}$$

EXAMPLE 5.57

Evaluate the integral $I = \int_0^{\infty} e^{-x^2} dx$.

Solution. Putting $t = x^2$, we get

$$\begin{aligned} I &= \frac{1}{2} \int_0^{\infty} e^{-t} t^{-1/2} dt = \frac{1}{2} L\{t^{-1/2}\} \text{ with } s = 1 \\ &= \frac{1}{2} \cdot \frac{\Gamma(-\frac{1}{2}+1)}{s^{1/2}} \text{ with } s = 1 \\ &= \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}. \end{aligned}$$

EXAMPLE 5.58

Find Laplace transform of $f(t) = e^{-2t} t \cos t$.

Solution. We know that $L\{\cos t\} = \frac{s}{s^2+1}$. Therefore, by Theorem 5.9, we have

$$L\{t \cos t\} = -\frac{d}{ds} \left(\frac{s}{s^2+1} \right) = \frac{s^2-1}{(s^2+1)^2}.$$

Now using first shifting property, we have

$$L\{e^{-2t} t \cos t\} = \frac{(s+2)^2-1}{((s+2)^2+1)^2} = \frac{s^2+4s+3}{(s^2+4s+5)^2}.$$

EXAMPLE 5.59

Find the Laplace transform of $f(t) = t^2 e^{-2t} \cos t$.

Solution. As in Example 5.58, $L\{t \cos t\} = \frac{s^2-1}{(s^2+1)^2}$. Therefore,

$$L\{t^2 \cos t\} = -\frac{d}{ds} \left(\frac{s^2-1}{(s^2+1)^2} \right).$$

Then using first-shifting property, we have

$$L\{t^2 e^{-2t} \cos t\} = 2 \left(\frac{s^3+10s^2+25s+22}{(s^2+4s+5)^3} \right).$$

EXAMPLE 5.60

Find Laplace transform of $f(t) = t^n e^{-at}$.

Solution. Since $L\{e^{-at}\} = \frac{1}{s+a}$, we have

$$L\{t^n e^{-at}\} = (-1)^n \frac{d^n}{ds^n} \left(\frac{1}{s+a} \right) = (-1)^n \frac{n!}{(s+a)^{n+1}}.$$

Theorem 5.10. (Division by t). If $L\{f(t)\} = F(s)$, then

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u) du,$$

provided $\lim_{t \rightarrow 0} \frac{f(t)}{t}$ exists.

Proof: Put $g(t) = \frac{f(t)}{t}$. So, $f(t) = t g(t)$ and

$$\begin{aligned} L\{f(t)\} &= L\{tg(t)\} \\ &= -\frac{d}{ds} L\{g(t)\}, \text{ by Theorem 5.9} \\ &= -\frac{dG}{ds}. \end{aligned}$$

Then integration yields

$$G(s) = -\int_s^\infty F(u) du = \int_s^\infty F(u) du,$$

that is,

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u) du.$$

Remark 5.3. By Theorem 5.10, we have

$$L\left\{\frac{f(t)}{t}\right\} = \int_0^\infty e^{-st} \frac{f(t)}{t} dt = \int_s^\infty F(u) du.$$

Letting $s \rightarrow 0+$ and assuming that the integral converges, it follows that

$$\int_0^\infty \frac{f(t)}{t} dt = \int_0^\infty F(u) du.$$

For example, if $f(t) = \sin t$, then $F(s) = \frac{1}{s^2+1}$ and so

$$\int_0^\infty \frac{\sin t}{t} dt = \int_0^\infty \frac{du}{u^2+1} = [\tan^{-1} u]_0^\infty = \frac{\pi}{2}.$$

EXAMPLE 5.61

Find the Laplace transform of $f(t) = \frac{\cos 2t - \cos 3t}{t}$.

Solution. By linearity of L , we have

$$\begin{aligned} L\{\cos 2t - \cos 3t\} &= L\{\cos 2t\} - L\{\cos 3t\} \\ &= \frac{s}{s^2+4} - \frac{s}{s^2+9}. \end{aligned}$$

Therefore, by Theorem 5.10, we get

$$\begin{aligned} L\left\{\frac{\cos 2t - \cos 3t}{t}\right\} &= \int_s^\infty \frac{u}{u^2+4} du - \int_s^\infty \frac{u}{u^2+9} du \\ &= \frac{1}{2} [\log(u^2+4)]_s^\infty - \frac{1}{2} [\log(u^2+9)]_s^\infty \\ &= \frac{1}{2} \left[\log \frac{u^2+4}{u^2+9} \right]_s^\infty \\ &= \frac{1}{2} \lim_{u \rightarrow \infty} \left(\log \frac{u^2+4}{u^2+9} \right) - \frac{1}{2} \log \frac{s^2+4}{s^2+9} \\ &= \frac{1}{2} \log \left(\lim_{u \rightarrow \infty} \frac{1+(4/u^2)}{1+(9/u^2)} \right) - \frac{1}{2} \log \frac{s^2+4}{s^2+9} \\ &= 0 + \frac{1}{2} \log \frac{s^2+9}{s^2+4} = \frac{1}{2} \log \frac{s^2+9}{s^2+4}. \end{aligned}$$

EXAMPLE 5.62

Find the Laplace transform of $f(t) = \frac{e^{-at} - e^{-bt}}{t}$.

Solution. We have

$$\begin{aligned} L\{e^{-at} - e^{-bt}\} &= L\{e^{-at}\} - L\{e^{-bt}\} \\ &= \frac{1}{s+a} - \frac{1}{s+b}. \end{aligned}$$

Therefore, proceeding as in Example 5.61, we have

$$\begin{aligned} L\{f(t)\} &= \int_s^\infty \left[\frac{1}{u+a} - \frac{1}{u+b} \right] du = \left[\log \frac{u+a}{u+b} \right]_s^\infty \\ &= \lim_{u \rightarrow \infty} \log \frac{u+a}{u+b} - \log \frac{s+a}{s+b} \\ &= 0 - \log \frac{s+a}{s+b} = \log \frac{s+b}{s+a}. \end{aligned}$$

EXAMPLE 5.63

Find the Laplace transform of $f(t) = \frac{1 - \cos 2t}{t}$.

Solution. We have

$$L\{1 - \cos 2t\} = L\{1\} - L\{\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}.$$

Therefore, by Theorem 5.10, we get

$$\begin{aligned} \mathcal{L}\left\{\frac{1 - \cos 2t}{t}\right\} &= \int_s^\infty \left[\frac{1}{u} - \frac{u}{u^2 + 4}\right] du \\ &= \left[\log u - \left(\frac{1}{2}\log(u^2 + 4)\right)\right]_s^\infty \\ &= \left[\frac{1}{2}\log u^2 - \frac{1}{2}\log(u^2 + 4)\right]_s^\infty \\ &= \frac{1}{2}\log\left(\frac{s^2 + 4}{s^2}\right). \end{aligned}$$

EXAMPLE 5.64

Using Remark 5.3, evaluate the integral

$$\int_0^\infty \frac{e^{-t} - e^{-3t}}{t} dt.$$

Solution. By Remark 5.3, we have

$$\begin{aligned} \int_0^\infty \frac{e^{-t} - e^{-3t}}{t} dt &= \int_0^\infty L\{e^{-t} - e^{-3t}\} ds = \int_0^\infty \left[\frac{1}{u+1} - \frac{1}{u+3}\right] du \\ &= [\log(u+1) - \log(u+3)]_0^\infty = \left[\log \frac{u+1}{u+3}\right]_0^\infty \\ &= \log \left[\lim_{u \rightarrow \infty} \frac{u+1}{u+3}\right] - \log \frac{1}{3} \\ &= \log \left[\lim_{u \rightarrow \infty} \frac{1 + (1/u)}{1 + (3/u)}\right] - \log \frac{1}{3} \\ &= \log 1 - \log \frac{1}{3} = \log 3. \end{aligned}$$

EXAMPLE 5.65

Find the Laplace transform of $f(t) = \frac{1-e^t}{t}$.

Solution. Since $\mathcal{L}\{1 - e^t\} = \mathcal{L}\{1\} - \mathcal{L}\{e^t\} = \frac{1}{s} - \frac{1}{s-1}$, we have

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \int_s^\infty \left(\frac{1}{u} - \frac{1}{u-1}\right) du \\ &= [\log u - \log(u-1)]_s^\infty \\ &= \left[\log \frac{u}{u-1}\right]_s^\infty = -\log \left[\frac{1}{1 - (1/s)}\right] \\ &= \log\left(\frac{s-1}{s}\right). \end{aligned}$$

EXAMPLE 5.66

Find Laplace transform of $f(t) = \frac{\sin at}{t}$.

Solution. We know that

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}.$$

Therefore, by Theorem 5.10,

$$\mathcal{L}\left\{\frac{\sin at}{t}\right\} = a \int_s^\infty \frac{du}{u^2 + a^2} = \tan^{-1}\left(\frac{a}{s}\right).$$

Theorem 5.11. (Laplace Transform of Integrals). If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\left\{\int_0^t f(u) du\right\} = \frac{F(s)}{s}.$$

Proof: The function $f(t)$ should be integrable in such a way that

$$g(t) = \int_0^t f(u) du$$

is of exponential order. Then $g(0) = 0$ and $g'(t) = f(t)$. Therefore,

$$\mathcal{L}\{g'(t)\} = s \mathcal{L}\{g(t)\} - g(0) = s \mathcal{L}\{g(t)\}$$

and so

$$\begin{aligned} \mathcal{L}\left\{\int_0^t f(u) du\right\} &= \mathcal{L}\{g(t)\} = \frac{\mathcal{L}\{g'(t)\}}{s} \\ &= \frac{\mathcal{L}\{f(t)\}}{s} = \frac{F(s)}{s}. \end{aligned}$$

EXAMPLE 5.67

Find Laplace transform of $\int_0^t \frac{\sin u}{u} du$.

Solution. From Example 5.66, we have

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left(\frac{1}{s}\right).$$

Therefore, by Theorem 5.11, we have

$$\mathcal{L}\left\{\int_0^t \frac{\sin u}{u} du\right\} = \frac{1}{s} \tan^{-1}\left(\frac{1}{s}\right).$$

(The function $\int_0^t \frac{\sin u}{u} du$ is called *sine integral function*, denoted by $\text{Si}(t)$, which is used in optics).

EXAMPLE 5.68

Find the Laplace transform of $\int_0^t \cos 2u \, du$.

Solution. We know that $L\{\cos 2t\} = \frac{s}{s^2+4}$ and so

$$L\left\{\int_0^t \cos 2u \, du\right\} = \frac{1}{s} \cdot \frac{s}{s^2+4} = \frac{1}{s^2+4}.$$

EXAMPLE 5.69

Find Laplace transform of *cosine integral function*

$$Ci(t) = \int_{-\infty}^t \frac{\cos u}{u} \, du, \quad t > 0$$

Solution. We know that $L\{\cos t\} = \frac{s}{s^2+1}$. Therefore, by Theorem 5.10

$$\begin{aligned} L\left\{\frac{\cos t}{t}\right\} &= \int_s^\infty \frac{u}{u^2+1} \, du = \frac{1}{2} \int_s^\infty \frac{2u}{u^2+1} \, du \\ &= \frac{1}{2} \log(1+s^2). \end{aligned}$$

Now Theorem 5.11 yields the Laplace transform of cosine integral function $Ci(t)$ as given below:

$$\begin{aligned} L\{Ci(t)\} &= L\left\{\int_{-\infty}^t \frac{\cos u}{u} \, du\right\} = L\left\{-\int_t^\infty \frac{\cos u}{u} \, du\right\} \\ &= -L\left\{\int_t^\infty \frac{\cos u}{u} \, du\right\} \\ &= -\left[\frac{1}{s} \left(\frac{1}{2} \log(1+s^2)\right)\right] \\ &= -\frac{1}{2s} \log(1+s^2). \end{aligned}$$

EXAMPLE 5.70

Find the Laplace transform of the *exponential integral* defined by

$$Ei(t) = \int_t^\infty \frac{e^{-u}}{u} \, du, \quad t > 0.$$

Solution. We know that $L\{e^{-u}\} = \frac{1}{s+1}$. Therefore, by Theorem 5.10, we have

$$L\left\{\frac{e^{-u}}{u}\right\} = \int_s^\infty \frac{1}{u+1} \, du = \log(1+s).$$

Now application of Theorem 5.11 yields

$$L\{Ei(t)\} = \frac{1}{s} \log(1+s).$$

EXAMPLE 5.71

Find Laplace transform of $\int_0^t e^t \frac{\sin t}{t} \, dt$.

Solution. We know that $L\{\sin t\} = \frac{1}{s^2+1}$. Therefore,

$$\begin{aligned} L\left\{\frac{\sin t}{t}\right\} &= \int_s^\infty \frac{1}{u^2+1} \, du = [\tan^{-1} u]_s^\infty \\ &= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s. \end{aligned}$$

Therefore, by first-shifting property, we have

$$L\left\{e^t \frac{\sin t}{t}\right\} = \cot^{-1}(s-1).$$

Hence

$$L\left\{\int_0^t e^t \frac{\sin t}{t} \, dt\right\} = \frac{1}{s} \cot^{-1}(s-1).$$

Theorem 5.12. (Laplace Transform of a Periodic Function). Let f be a periodic function with period T so that $f(t) = f(t+T)$. Then

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) \, dt.$$

Proof: We begin with the definition of Laplace transform and evaluate the integral using periodicity of $f(t)$. Thus we have

$$\begin{aligned} L\{f(t)\} &= \int_0^\infty e^{-st} f(t) \, dt \\ &= \int_0^T e^{-st} f(t) \, dt + \int_T^{2T} e^{-st} f(t) \, dt \\ &\quad + \int_{2T}^{3T} e^{-st} f(t) \, dt + \dots \\ &\quad + \int_{(n-1)T}^{nT} e^{-st} f(t) \, dt + \dots \end{aligned}$$

provided that the series on the right-hand side converges. This is true since the function f satisfies the condition for the existence of its Laplace transform. Consider the integral

$$\int_{(n-1)T}^{nT} e^{-st} f(t) dt$$

and substitute $u = t - (n-1)T$. Since f has period T , we have

$$\int_{(n-1)T}^{nT} e^{-st} f(t) dt = e^{-s(n-1)T} \int_0^T e^{-su} f(u) du,$$

$$n = 1, 2, \dots$$

Thus, we have

$$\begin{aligned} L\{f(t)\} &= (1 + e^{-sT} + e^{-2sT} + \dots) \int_0^T e^{-st} f(t) dt \\ &= \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt \quad (\text{summing the G.P.}), \end{aligned}$$

since $1 + r + r^2 + r^3 + \dots = \frac{1}{1-r}$, $|r| < 1$.

EXAMPLE 5.72

Find Laplace transform of the *half-wave rectified sinusoidal*

$$f(t) = \begin{cases} \sin t & \text{for } 2n\pi < t < (2n+1)\pi \\ 0 & \text{for } (2n+1)\pi < t < (2n+2)\pi, \end{cases}$$

for $n = 0, 1, 2, \dots$

Solution. The graph of the half-wave rectified sinusoidal function f is shown in the Figure 5.9

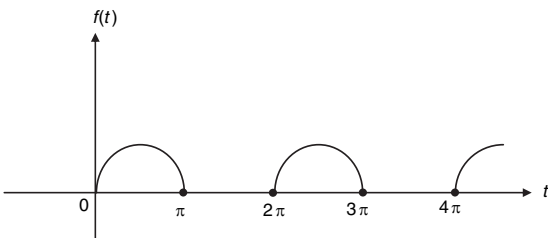


Figure 5.9

The function is of period 2π . So, Theorem 5.12 yields

$$\begin{aligned} L\{f(t)\} &= \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt \\ &= \frac{1}{1 - e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t) dt \\ &= \frac{1}{1 - e^{-2\pi s}} \int_0^{\pi} e^{-st} \sin t dt \\ &= \frac{1}{1 - e^{-2\pi s}} \left[\frac{e^{-st}(-s \sin t - \cos t)}{s^2 + 1} \right]_0^{\pi} \\ &= \frac{1}{1 - e^{-2\pi s}} \left(\frac{1 + e^{-\pi s}}{s^2 + 1} \right) \\ &= \frac{1}{(1 - e^{-\pi s})(s^2 + 1)}. \end{aligned}$$

EXAMPLE 5.73

Find the Laplace transform of *full rectified sine wave* defined by the expression

$$\begin{aligned} f(t) &= \begin{cases} \sin t & \text{for } 0 < t < \pi \\ -\sin t & \text{for } \pi < t < 2\pi \end{cases} \\ f(t) &= f(t + \pi). \end{aligned}$$

Solution. The graph of the rectified sine wave is shown in the Figure 5.10.

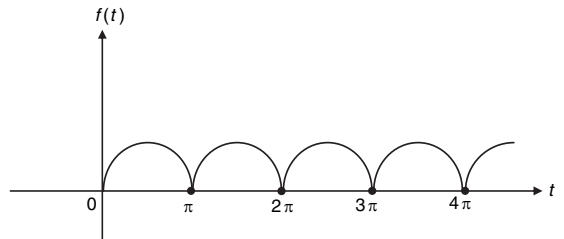


Figure 5.10

The function f has period π and, therefore, application of Theorem 5.12 yields

$$\begin{aligned} L\{f(t)\} &= \frac{1}{1 - e^{-\pi s}} \int_0^{\pi} e^{-st} \sin t \, dt \\ &= \frac{1}{1 - e^{-\pi s}} \left[\frac{1 + e^{-\pi s}}{s^2 + 1} \right] \\ &= \frac{1 + e^{-\pi s}}{(1 - e^{-\pi s})(s^2 + 1)}. \end{aligned}$$

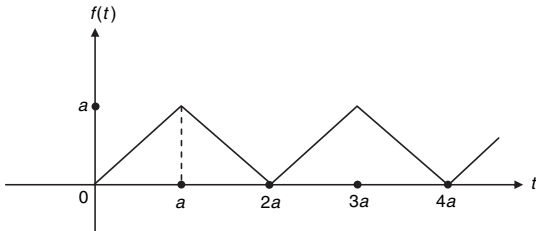
EXAMPLE 5.74

Find Laplace transform of the *triangular wave function* defined by

$$f(t) = \begin{cases} t & \text{for } 0 \leq t < a \\ 2a - t & \text{for } a \leq t < 2a, \end{cases}$$

$$f(2a + t) = f(t).$$

Solution. The graph of triangular wave function is shown in the Figure 5.11.

**Figure 5.11**

Using the formula for the Laplace transform of a periodic function, we have

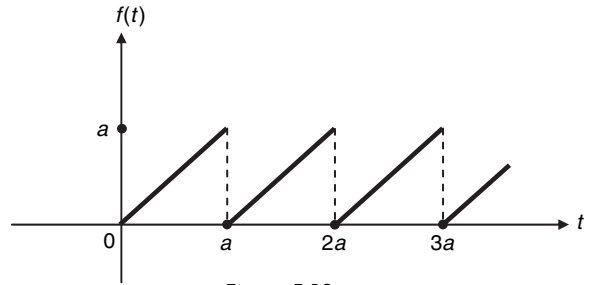
$$\begin{aligned} L\{f(t)\} &= \frac{1}{1 - e^{-2as}} \int_0^{2a} e^{-st} f(t) \, dt \\ &= \frac{1}{1 - e^{-2as}} \left[\int_0^a t e^{-st} \, dt + \int_a^{2a} (2a - t) e^{-st} \, dt \right]. \end{aligned}$$

Now integration by parts gives

$$\begin{aligned} L\{f(t)\} &= \frac{1}{1 - e^{-2as}} (e^{-as} - 1)^2 \cdot \frac{1}{s^2} \\ &= \frac{1}{s^2} \left(\frac{1 - e^{-as}}{1 + e^{-as}} \right) = \frac{1}{s^2} \tanh\left(\frac{as}{2}\right). \end{aligned}$$

EXAMPLE 5.75

Find the Laplace transform of the *saw tooth wave function*, whose graph is shown in the Figure 5.12.

**Figure 5.12**

Solution. It is a periodic function with period a . Therefore,

$$\begin{aligned} L\{f(t)\} &= \frac{1}{1 - e^{-as}} \int_0^a e^{-st} f(t) \, dt = \frac{1}{1 - e^{-as}} \int_0^a t e^{-st} \, dt \\ &= \frac{1}{1 - e^{-as}} \left\{ \left[\frac{te^{-st}}{-s} \right]_0^a - \int_0^a \frac{e^{-st}}{-s} \, dt \right\} \\ &= \frac{1}{1 - e^{-as}} \left\{ \left[\frac{te^{-st}}{-s} \right]_0^a - \left[\frac{e^{-st}}{s^2} \right]_0^a \right\} \\ &= \frac{1}{1 - e^{-as}} \left[t \frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^a \\ &= \frac{1}{1 - e^{-as}} \left[\frac{ae^{-sa}}{-s} - \frac{e^{-sa}}{s^2} + \frac{1}{s^2} \right] \\ &= \frac{1}{1 - e^{-as}} \left[-\frac{ae^{-sa}}{s} + \frac{1 - e^{-sa}}{s^2} \right] \\ &= \frac{1}{s^2} - \frac{ae^{-sa}}{s(1 - e^{-as})}. \end{aligned}$$

5.3 LIMITING THEOREMS

Theorem 5.13. (Initial Value). Let $f(t)$ and its derivative $f'(t)$ be piecewise continuous and of exponential order, then

$$\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s),$$

provided the limits exist.

Proof: We know that

$$L\{f'(t)\} = sF(s) - f(0). \quad (3)$$

Further, since $f'(t)$ obeys the criteria for the existence of Laplace transform, we have

$$\begin{aligned} \left| \int_0^{\infty} e^{-st} f'(t) dt \right| &\leq \int_0^{\infty} |e^{-st} f'(t)| dt \\ &\leq \int_0^{\infty} e^{-st} e^{Mt} dt \\ &= -\frac{1}{M-s} \rightarrow 0 \text{ as } s \rightarrow \infty. \end{aligned}$$

Hence (3) implies $sF(s) - f(0) \rightarrow 0$ as $s \rightarrow \infty$, and so

$$\lim_{s \rightarrow \infty} sF(s) = f(0) = \lim_{t \rightarrow 0} f(t).$$

Theorem 5.14. (Final Value). If the limits indicated exist, then

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s).$$

Proof: Using Laplace transform of the derivative, we have

$$L\{f'(t)\} = \int_0^{\infty} e^{-st} f'(t) dt = sF(s) - f(0) \quad (4)$$

Then

$$\begin{aligned} \lim_{s \rightarrow 0} \int_0^{\infty} e^{-st} f'(t) dt &= \lim_{s \rightarrow 0} \lim_{T \rightarrow \infty} \int_0^T e^{-st} f'(t) dt \\ &= \lim_{s \rightarrow 0} \lim_{T \rightarrow \infty} \{e^{-sT} f(T) - f(0)\} \\ &= \lim_{T \rightarrow \infty} f(T) - f(0) \\ &= \lim_{t \rightarrow \infty} f(t) - f(0). \end{aligned}$$

Hence relation (4) yields

$$\lim_{t \rightarrow \infty} f(t) - f(0) = \lim_{s \rightarrow 0} sF(s) - f(0)$$

and so

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s).$$

EXAMPLE 5.76

Verify the validity of limiting theorems by considering $f(t) = e^t$.

Solution. We observe that

$$L\{e^t\} = F(s) = \frac{1}{s-1}.$$

$$\lim_{t \rightarrow 0} f(t) = f(0) = 1,$$

$$\lim_{s \rightarrow \infty} sF(s) = \lim_{s \rightarrow \infty} \frac{s}{s-1} = 1.$$

Thus, initial value theorem is verified in this case.

On the other hand,

$$\lim_{t \rightarrow \infty} f(t) = \lim_{t \rightarrow \infty} e^t = \infty,$$

$$\lim_{s \rightarrow 0} sF(s) = \lim_{s \rightarrow 0} \frac{s}{s-1} = 0,$$

and so the final value theorem also holds for e^t .

EXAMPLE 5.77

Suppose Laplace transform of a function f is given by

$$F(s) = \frac{18}{s(s^2 + 36)}.$$

Find $f(0)$ and $f'(\infty)$.

Solution. By initial value theorem, we have

$$\begin{aligned} f(0) &= \lim_{s \rightarrow \infty} f(s) = \lim_{s \rightarrow \infty} sF(s) \\ &= \lim_{s \rightarrow \infty} \frac{18s}{s(s^2 + 36)} = 0, \end{aligned}$$

and

$$\begin{aligned} f'(\infty) &= \lim_{s \rightarrow 0} [s^2 F(s) - sf(0)] \\ &= \lim_{s \rightarrow 0} \left[\frac{18s^2}{s(s^2 + 36)} - 0 \right] = 0. \end{aligned}$$

EXAMPLE 5.78

Let $F(s) = \frac{1}{s-1}$. Can we find $f(\infty)$ using final value theorem?

Solution. In this case, the final value theorem does not apply because

$$\lim_{t \rightarrow \infty} f(t) = \lim_{t \rightarrow \infty} e^t \rightarrow \infty \text{ as } t \rightarrow \infty.$$

5.4 MISCELLANEOUS EXAMPLES

EXAMPLE 5.79

Find the Laplace Transforms of (i) $e^{4t} \sin 2t \cos t$, (ii) $\sinh t \cos^2 t$ and (iii) $e^{-t} \cos^2 t$.

Solution. (i). We have

$$\sin 2t \cos t = \frac{1}{2} [\sin 3t + \sin t].$$

Therefore

$$\begin{aligned} L\{\sin 2t \cos t\} &= \frac{1}{2} L\{\sin 3t\} + \frac{1}{2} L\{\sin t\} \\ &= \frac{1}{2} \left[\frac{3}{s^2 + 9} + \frac{1}{s^2 + 1} \right]. \end{aligned}$$

Now using shifting property, we get

$$\begin{aligned} L\{e^{4t} \sin 2t \cos t\} &= \frac{1}{2} \left[\frac{3}{(s-4)^2 + 9} + \frac{1}{(s-4)^2 + 1} \right] \\ &= \frac{1}{2} \left[\frac{3}{s^2 - 8s + 25} + \frac{1}{s^2 - 8s + 17} \right]. \end{aligned}$$

(ii). Since $\cos^2 t = \frac{1 + \cos 2t}{2}$ Therefore

$$\begin{aligned} L\{\cos^2 t\} &= \frac{1}{2} L\{1\} + \frac{1}{2} L\{\cos 2t\} \\ &= \frac{1}{2} \left[\frac{1}{s} + \frac{s}{s^2 + 4} \right] = \frac{s^2 + 2}{s(s^2 + 4)}, \quad s > 0 \end{aligned}$$

Therefore, by shifting property,

$$\begin{aligned} L\{\sinh t \cos^2 t\} &= L\left\{\frac{e^t - e^{-t}}{2} \cos^2 t\right\} \\ &= \frac{1}{2} L\{e^t \cos^2 t\} - \frac{1}{2} L\{e^{-t} \cos^2 t\} \\ &= \frac{1}{2} \left\{ \frac{(s-1)^2 + 2}{(s-1)[(s-1)^2 + 4]} \right\} - \frac{1}{2} \left\{ \frac{(s+1)^2 + 2}{(s+1)[(s+1)^2 + 4]} \right\} \\ &= \frac{1}{2} \left[\frac{s^2 - 2s + 3}{(s-1)(s^2 - 2s + 5)} - \frac{s^2 + 2s + 3}{(s+1)(s^2 + 2s + 5)} \right]. \end{aligned}$$

(iii). As in part (ii),

$$L\{\cos^2 t\} = \frac{s^2 + 2}{s(s^2 + 4)}, \quad s > 0$$

Therefore, by shifting property,

$$\begin{aligned} L\{e^{-t} \cos^2 t\} &= \frac{(s+1)^2 + 2}{(s+1)[(s+1)^2 + 4]} \\ &= \frac{s^2 + 2s + 3}{(s+1)(s^2 + 2s + 5)} \end{aligned}$$

EXAMPLE 5.80

Find the Laplace transforms of the following:

(i) $e^{-3t}(2 \cos 5t - 3 \sin 5t)$ and (ii) $\frac{1}{t}(\cos at - \cos bt)$

Solution. (i). Using linearity and shifting properties, we have

$$\begin{aligned} &L\{e^{-3t}(2 \cos 5t - 3 \sin 5t)\} \\ &= 2L\{e^{-3t} \cos 5t\} - 3L\{e^{-3t} \sin 5t\} \\ &= 2 \frac{s+3}{(s+3)^2 + 25} - 3 \frac{5}{(s+3)^2 + 25} \\ &= \frac{2s-9}{(s+3)^2 + 25} = \frac{2s-9}{s^2 + 6s + 34}. \end{aligned}$$

(ii). As in Example 5.64, we have

$$\begin{aligned} \int_0^\infty \frac{\cos at - \cos bt}{t} dt &= \int_0^\infty L\{\cos at - \cos bt\} ds \\ &= \int_0^\infty \left[\frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2} \right] ds \\ &= \left[\frac{1}{2} \log(s^2 + a^2) - \frac{1}{2} \log(s^2 + b^2) \right]_0^\infty \\ &= \frac{1}{2} \left[\log \frac{s^2 + a^2}{s^2 + b^2} \right]_0^\infty \\ &= \frac{1}{2} \left[\log \left\{ \lim_{s \rightarrow \infty} \frac{1 + \frac{a^2}{s^2}}{1 + \frac{b^2}{s^2}} \right\} \right] - \frac{1}{2} \log \frac{a^2}{b^2} \\ &= -\frac{1}{2} \log \frac{a^2}{b^2} = \frac{1}{2} \log \frac{b^2}{a^2} \end{aligned}$$

EXAMPLE 5.81

Find the Laplace transform of $\cosh at \sin at$.

Solution. Let

$$g(t) = \cosh at \sin at.$$

Then

$$\begin{aligned} L\{g(t)\} &= L\left[\frac{e^{at} + e^{-at}}{2} \sin at \right] \\ &= \frac{1}{2} L\{e^{at} \sin at + e^{-at} \sin at\} \\ &= \frac{1}{2} L\{e^{at} \sin at\} + \frac{1}{2} L\{e^{-at} \sin at\}. \end{aligned}$$

Now

$$L\{\sin at\} = \frac{a}{s^2 + a^2}.$$

Therefore, by shifting property,

$$L\{e^{at} \sin at\} = \frac{a}{(s-a)^2 + a^2},$$

$$L\{e^{-at} \sin at\} = \frac{a}{(s+a)^2 + a^2}.$$

Hence

$$L\{g(t)\} = \frac{1}{2} \left[\frac{a}{(s-a)^2 + a^2} + \frac{a}{(s+a)^2 + a^2} \right]$$

EXAMPLE 5.82

Find the Laplace transform of $f(t) = t^2 \sin^2 at$.

Solution. Let $f(t) = t^2 \sin^2 at$. We know that

$$L\{\sin^2 at\} = \frac{2a^2}{s[s^2 + (2a)^2]}.$$

Therefore

$$\begin{aligned} L\{t^2 \sin^2 at\} &= (-1)^2 \frac{d^2}{ds^2} \left[\frac{2a^2}{s(s^2 + 4a^2)} \right] \\ &= 2a^2 \frac{d^2}{ds^2} \left[\frac{1}{s(s^2 + 4a^2)} \right] \\ &= 8a^2 \left[\frac{3s^4 + 6a^2s^2 - 8a^4}{s^2(s^2 + 4a^2)^3} \right]. \end{aligned}$$

EXAMPLE 5.83

Evaluate:

$$L\{te^{-2t} \sin 4t\}$$

Solution. We know that

$$L\{\sin 4t\} = \frac{1}{s^2 + 16}.$$

Now, by shifting property,

$$L\{e^{-2t} \sin 4t\} = \frac{1}{(s+2)^2 + 16}.$$

Further,

$$\begin{aligned} L\{e^{-2t} \sin 4t\} &= -\frac{d}{ds} \left[\frac{1}{(s+2)^2 + 16} \right] \\ &= \frac{2s+4}{(s^2 + 4s + 20)^2} \end{aligned}$$

EXAMPLE 5.84

Express $f(t) = \begin{cases} 1 & \text{if } 0 < t \leq 1 \\ t & \text{if } 1 < t \leq 2 \\ t^2 & \text{if } t > 2 \end{cases}$ in terms of unit step function and hence find $L\{f(t)\}$.

Solution. In terms of Heavyside's unit step function, we have

$$f(t) = H(t) + tH(t-1) + t^2H(t-2).$$

Therefore

$$\begin{aligned} L\{f(t)\} &= L\{H(t)\} + L\{tH(t-1)\} \\ &\quad + L\{t^2H(t-2)\} \\ &= \frac{1}{s} - \frac{d}{ds} \left\{ \frac{e^{-s}}{s} \right\} + (-1)^2 \frac{d}{ds} \left\{ \frac{e^{-2s}}{s} \right\} \\ &= \frac{1}{s} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} - \frac{2e^{-2s}}{s} \\ &= \frac{1}{s} [1 - e^{-s} - 2e^{-2s}] - \frac{1}{s^2} [e^{-s} + e^{-2s}]. \end{aligned}$$

EXAMPLE 5.85

Find the Laplace Transform of $e^{3t} [t \cos 2t]$ and $\frac{1-e^{-2t}}{t}$.

Solution. We know that

$$F(s) = L\{t \cos 2t\} = \frac{s^2 - 4}{(s^2 + 4)^2}.$$

Therefore, by shifting property,

$$\begin{aligned} L\{e^{3t}(t \cos 2t)\} &= F(s-3) \\ &= \frac{(s-3)^2 - 4}{((s-3)^2 + 4)^2} = \frac{s^2 - 6s + 5}{(s^2 - 6s + 13)^2}. \end{aligned}$$

Also, As in Example 5.65,

$$L\{1 - e^{-2t}\} = L\{1\} - L\{e^{-2t}\} = \frac{1}{s} - \frac{1}{s+2}.$$

Therefore

$$\begin{aligned} L\{f(t)\} &= \int_s^\infty \left(\frac{1}{u} - \frac{1}{u+2} \right) du = [\log u - \log(u+2)]_s^\infty \\ &= \log \left[\frac{u}{u+2} \right]_s^\infty = -\log \left[\frac{1}{1+\frac{2}{s}} \right] = \log \left[\frac{s+2}{s} \right]. \end{aligned}$$

EXAMPLE 5.86

Evaluate:

$$L\left\{\frac{1 - \cos at}{t}\right\}$$

Solution. See Example 5.63. We have

$$L\{1 - \cos at\} = L\{1\} - L\{\cos at\} = \frac{1}{s} - \frac{s}{s^2 + a^2}.$$

Therefore

$$\begin{aligned} L\left\{\frac{1 - \cos at}{t}\right\} &= \int_s^\infty \left[\frac{1}{u} - \frac{u}{u^2 + a^2}\right] du \\ &= \left[\log u - \left(\frac{1}{2} \log(u^2 + a^2)\right)\right]_s^\infty \\ &= \left[\frac{1}{2} \log u^2 - \frac{1}{2} \log(u^2 + a^2)\right]_s^\infty \\ &= \frac{1}{2} \log \frac{s^2 + a^2}{s^2}. \end{aligned}$$

EXAMPLE 5.87Using Laplace transform evaluate $\int_0^\infty \frac{e^{-at} \sin^2 t}{t} dt$ **Solution.** We have

$$L\{\sin^2 t\} = L\left\{\frac{1}{2}(1 - \cos 2t)\right\} = \frac{1}{2} \left[\frac{1}{s} - \frac{s}{s^2 + 4}\right]$$

Therefore

$$L\{e^{-at} \sin^2 t\} = \frac{1}{2} \left[\frac{1}{s+a} - \frac{s+a}{(s+a)^2 + 4}\right].$$

Further,

$$\begin{aligned} \int_0^\infty \frac{e^{-at} \sin^2 t}{t} dt &= \int_0^\infty L\{e^{-at} \sin^2 t\} ds \\ &= \frac{1}{2} \left[\int_0^\infty \frac{ds}{s+a} - \int_0^\infty \frac{s+a}{(s+a)^2 + 4} ds \right] \\ &= \frac{1}{2} \left[\log \frac{s+a}{\sqrt{(s+a)^2 + 4}} \right]_0^\infty \\ &= \frac{1}{2} \left[0 - \frac{1}{2} \log \frac{a}{\sqrt{a^2 + 4}} \right] \\ &= \frac{1}{2} \log \frac{\sqrt{a^2 + 4}}{a}. \end{aligned}$$

EXERCISES

1. Find the Laplace transforms of

(a) $4t + 6e^{4t}$ **Ans.** $\left(\frac{4}{s^2} + \frac{6}{s-4}\right)$

(b) $e^{-4t} \sin 5t$ **Ans.** $\left(\frac{5}{s^2 + 8s + 41}\right)$

(c) $\frac{1}{t}(\sin at - at \cos at)$ **Ans.** $\tan^{-1}\left(\frac{a}{s}\right) - \frac{as}{s^2 + a^2}$

(d) $\frac{t}{2a} \sin at$ **Ans.** $\left(\frac{s}{(s^2 + a^2)^2}\right)$

(e) $\frac{t^{x-1} e^{at}}{\Gamma(x)}$ **Ans.** $\frac{1}{(s-a)^x}, x > 0$

(f) $f(t) = \begin{cases} t+1 & \text{for } 0 \leq t \leq 2 \\ 3 & \text{for } t > 2 \end{cases}$ **Ans.** $\frac{1}{s} + \frac{1}{s^2}(e^{-2s} - 1)$

(g) $f(t) = t H(t-a)$ **Ans.** $\frac{(1+as)e^{-as}}{s^2}$

(h) Null function defined by $\int_0^t n(t) dt = 0$ for all t . **Ans.** 0

2. Show that the Laplace transforms of the following functions do not exist:

(a) e^{t^2} **Hint:** Not of exponential order

(b) $e^{1/t}$ **Hint:** Not defined at $t = 0$

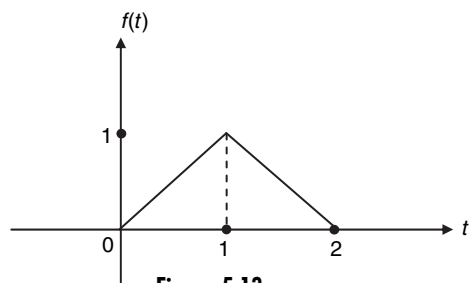
(c) $f(t) = \begin{cases} 1 & \text{for even } t \\ 0 & \text{for odd } t \end{cases}$

Hint: has infinite number of finite jumps and so condition of piecewise continuity is not satisfied

(d) t^{-n} , n is positive integer.

3. Find $L\left\{\frac{\sinh \omega t}{t}\right\}$ **Ans.** $\frac{1}{2} \log \frac{s+\omega}{s-\omega}, s > |\omega|$

4. Find the Laplace transform of a function whose graph is shown in the Figure 5.13.

**Figure 5.13****Hint:** The function is defined by

$$f(t) = \begin{cases} t & \text{for } 0 \leq t < 1 \\ 2-t & \text{for } 1 \leq t < 2. \end{cases}$$

Therefore, $L\{f(t)\} = \int_0^1 te^{-st} dt + \int_1^2 (2-t)e^{-st} dt + \int_2^\infty e^{-st} dt = \frac{1}{s^2}(1 - e^{-s})^2$.

5. Find the Laplace transform of step function f defined by $f(t) = n, n \leq t < n+1, n = 0, 1, 2, \dots$

Hint: The graph of step function is shown in the Figure 5.14.

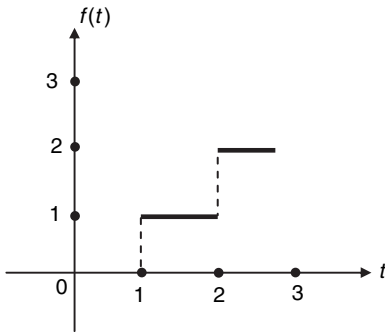


Figure 5.14

$$\begin{aligned} L\{f(t)\} &= \int_0^\infty e^{-st} f(t) dt \\ &= \int_1^2 e^{-st} dt + 2 \int_2^3 e^{-st} dt + 3 \int_3^4 e^{-st} dt + \dots \\ &= \left[\frac{e^{-st}}{-s} \right]_1^2 + 2 \left[\frac{e^{-st}}{-s} \right]_2^3 + 3 \left[\frac{e^{-st}}{-s} \right]_3^4 + \dots \\ &= \frac{e^{-s}}{s} (1 + e^{-s} + e^{-2s} + \dots) \\ &= \frac{e^{-s}}{s(e^s - 1)}. \end{aligned}$$

6. Find the Laplace transforms of

(a) $t e^{-2t} \sin 2t$ **Ans.** $\frac{4(s-2)}{(s^2-4s+8)^2}$

(b) $t^2 \cos at$ **Ans.** $\frac{2s^3-6a^2s}{(s^2+a^2)^3}$

(c) $t \sin 3t \cos 2t$.

Hint: $\sin 3t \cos 2t = \frac{1}{2} (2 \sin 3t \cos 2t)$

$$= \frac{1}{2} (\sin(3t+2t) + \sin(3t-2t)) = \frac{1}{2} \sin 5t + \frac{1}{2} \sin t.$$

Therefore,

$$L\{\sin 3t \cos 2t\} = \frac{3s^2 + 15}{(s^2 + 1)(s^2 + 25)}, \quad s > 0 \text{ and so}$$

$$\begin{aligned} L\{s \sin 3t \cos 2t\} &= -\frac{d}{ds} \frac{(3s^2 + 15)}{(s^2 + 1)(s^2 + 25)} \\ &= \frac{5s}{(s^2 + 25)^2} + \frac{s}{(s^2 + 1)^2}. \end{aligned}$$

7. Find Laplace transforms of

(a) $\frac{e^{-t} \sin t}{t}$ **Hint:** First find $L\{e^{-t} \sin t\}$ and then use

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s) ds. \quad \text{Ans. } \cot^{-1}(s+1).$$

(b) $\frac{e^{at} - \cos 6t}{t}$ **Ans.** $\log \frac{s^2-36}{s-a}$

8. Evaluate $I = \int_0^\infty \frac{e^{-t} \sin^2 t}{t} dt$.

Hint: $I = \int_0^\infty L\{f(t)\} du$, where

$$f(t) = e^{-t} \sin^2 t. \text{ But } L\{e^{-t} \sin^2 t\} = \frac{2}{(s+1)(s^2+2s+5)}.$$

Therefore,

$$I = \int_0^\infty \frac{2}{(s+1)(s^2+2s+5)} ds \quad \text{Ans. } \frac{1}{2} \log 5$$

9. Show that $L\{tf'(t)\} = -\{s \frac{d}{ds} F(s) + F(s)\}$.

10. Show that $L\{\frac{1}{t}(\sin at - at \cos at)\} = \tan^{-1}\left(\frac{a}{s}\right) - \frac{as}{s^2+a^2}$.

11. Show that $L\{t^2 f''(t)\} = s^2 \frac{d^2}{ds^2} F(s) + 4s \frac{d}{ds} F(s) + 2F(s)$.

12. Use initial value theorem to find $f(0)$ and $f'(0)$ for the function f for which $F(s) = \frac{s}{s^2-5s+12}$

Ans. $f(0) = 1, f'(0) = 5$

13. Find the Laplace transform of the square wave function with graph shown in the Figure 5.15.

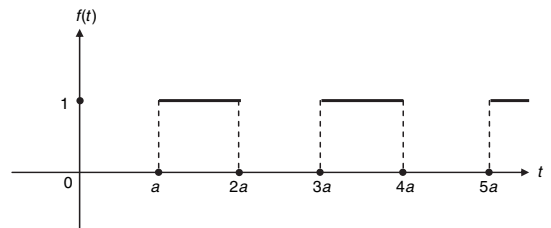


Figure 5.15

Hint: The function is of period $2a$. Therefore,

$$L\{f(t)\} = \frac{1}{1 - e^{-2as}} \int_a^{2a} e^{-st} dt = \frac{e^{-as}}{s(1 + e^{-as})}$$

14. Express the function $f(t)$ in Exercise 13 in terms of Heaviside's unit step function and then find its Laplace transform.

Hint: $f(t) = H(t - a) - H(t - 2a) + H(t + 3a) - H(t + 4a) + \dots$

15. Find the Laplace transform of the square wave function with graph given in the Figure 5.16.

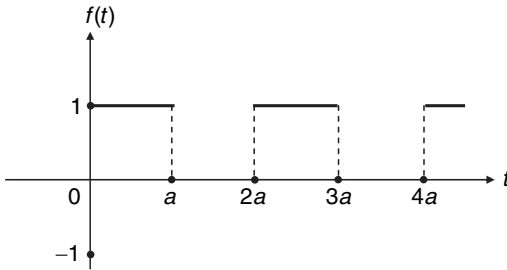


Figure 5.16

Hint: Here the function is of period $2a$. Therefore,

$$\begin{aligned} L\{f(t)\} &= \frac{1}{1 - e^{-2as}} \int_0^a e^{-st} dt \\ &= \frac{1}{1 - e^{-2as}} \left[\frac{e^{-st}}{-s} \right]_0^a = \frac{1}{s(1 + e^{-as})}. \end{aligned}$$

16. Solve Example 5.27 using periodicity

Hint: $L\{f(t)\} = \frac{1}{1 - e^{-2as}} \int_0^{2a} e^{-st} f(t) dt = \frac{1}{1 - e^{-2as}} \times \left[\int_0^a e^{-st} dt + \int_a^{2a} -e^{-st} dt \right]$ which on using

Examples 5.13 and 5.15 yields the transform as

$$\frac{1}{s} \tanh \frac{as}{2}.$$

17. Find Laplace transform of the half-wave rectified sine function f defined by

$$f(t) = \begin{cases} \sin \omega t & \text{for } \frac{2n\pi}{\omega} < t < \frac{(2n+1)\pi}{\omega} \\ 0 & \text{for } \frac{(2n+1)\pi}{\omega} < t < \frac{(2n+2)\pi}{\omega} \end{cases}$$

Hint: The function is periodic with period $\frac{2\pi}{\omega}$. Therefore,

$$\begin{aligned} L\{f(t)\} &= \frac{1}{1 - e^{-2\pi s/\omega}} \int_0^{2\pi/\omega} e^{-st} \sin \omega t dt \\ &= \frac{1}{1 - e^{-2\pi s/\omega}} \int_0^{\pi/\omega} e^{-st} \sin \omega t dt \\ &= \frac{\omega}{s^2 + \omega^2} (1 + e^{-\pi s/\omega}). \end{aligned}$$

18. Establish relation between Laplace transform and Fourier transform of a function.

Hint: Let ϕ be a function defined by

$$\phi(t) = \begin{cases} e^{-xt} f(t) & \text{for } t > 0 \\ 0 & \text{for } t < 0. \end{cases}$$

Then the Fourier transform of ϕ is

$$\begin{aligned} F(y) &= F\{\phi(t)\} = \int_0^{\infty} e^{-iyt} \cdot e^{-xt} f(t) dt \\ &= \int_0^{\infty} e^{-(x+iy)t} f(t) dt \\ &= \int_0^{\infty} e^{-st} f(t) dt = L\{f(t)\}. \end{aligned}$$

Thus Laplace transform of $f(t)$ is equal to the Fourier transform of $e^{-xt} f(t)$.

6

Inverse Laplace Transform

Like the operations of addition, multiplication, and differentiation, the Laplace transform has also its inverse. During the process of solving physical problems like differential equations, it is necessary to invoke the inverse transform of the Laplace transform. Thus given a Laplace transform $F(s)$ of a function f , we would like to know what f is. Hence, we are concerned with the solution of the integral equation,

$$\int_0^{\infty} e^{-st} f(t) dt = F(s).$$

6.1 DEFINITION AND EXAMPLES OF INVERSE LAPLACE TRANSFORM

Definition 6.1. Let f have Laplace transform $F(s)$, that is, $L\{f(t)\} = F(s)$, then $f(t)$ is called an *inverse Laplace transform* of $F(s)$ and we write

$$L^{-1}\{F(s)\} = f(t), t \geq 0.$$

The transformation L^{-1} is called *inverse Laplace operator* and it maps the Laplace transform of a function back to the original function.

We know that Laplace transform $F(s)$ of a function $f(t)$ is uniquely determined due to the properties of integrals. However, this is not true for the inverse transform. For example, if $f(t)$ and $g(t)$ are two functions that are identical except for a finite number of points, they have the same transform $F(s)$ since their integrals are identified. Therefore, either $f(t)$ or $g(t)$ is the inverse transform of $F(s)$. Thus inverse transform of a given function $F(s)$ is uniquely determined only upto an additive null function [a function $n(t)$ for which $\int_0^t n(u) du = 0$ for all t]. The following examples show that $L^{-1}\{F(s)\}$ can be more than one function.

(a) Let $f(t) = \sin \omega t, t \geq 0$. Then $L\{f(t)\} = \frac{\omega}{s^2 + \omega^2}$. Thus

$$L^{-1}\left\{\frac{\omega}{s^2 + \omega^2}\right\} = \sin \omega t.$$

Now let

$$g(t) = \begin{cases} \sin \omega t & \text{for } t > 0 \\ 1 & \text{for } t = 0. \end{cases}$$

Then

$$L\{g(t)\} = \frac{\omega}{s^2 + \omega^2},$$

and so

$$L^{-1}\left\{\frac{\omega}{s^2 + \omega^2}\right\} = g(t).$$

Hence there are two inverse transforms of $\frac{\omega}{s^2 + \omega^2}$.

(b) Let $f(t) = e^{-3t}$ and

$$g(t) = \begin{cases} 0 & \text{for } t = 1 \\ e^{-3t} & \text{otherwise.} \end{cases}$$

Then both $f(t)$ and $g(t)$ have same Laplace transform $\frac{1}{s+3}$. Thus $\frac{1}{s+3}$ has two inverse Laplace transforms $f(t)$ and $g(t)$.

But the following theorem shows that the Laplace transform is one-one mapping.

Theorem 6.1. (Lerch's Theorem). Distinct continuous functions on $[0, \infty)$ have distinct Laplace transforms.

Thus, if we restrict ourselves to continuous functions on $[0, \infty)$, then the inverse transform $L^{-1}\{F(s)\} = f(t)$ is uniquely defined. Since many of the functions, we generally deal with, are solutions to the differential equations and hence continuous, the assumption of the theorem is satisfied.

EXAMPLE 6.1

Find inverse Laplace transform of $\frac{a}{s^2+a^2}$, $\frac{s}{s^2+a^2}$, $\frac{1}{s-a}$, $\frac{e^{-sa}}{s}$, $\frac{s}{s^2-a^2}$ and $\frac{n!}{s^{n+1}}$.

Solution. We know that

$$L\{\sin at\} = \frac{a}{s^2 + a^2}, \quad L\{\cos at\} = \frac{s}{s^2 + a^2},$$

$$L\{e^{at}\} = \frac{1}{s-a}, \quad L\{h(t-a)\} = \frac{e^{-sa}}{s},$$

$$L\{\cosh at\} = \frac{s}{s^2 - a^2}, \quad \text{and}$$

$$L\{t^n\} = \frac{n!}{s^{n+1}}, \quad n \text{ being non-negative integer.}$$

Therefore,

$$L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at,$$

$$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at,$$

$$L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at},$$

$$L^{-1}\left\{\frac{e^{-sa}}{s}\right\} = H(t-a), \text{ Heavyside's unit step function.}$$

$$L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at,$$

and

$$L^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n, \quad n \text{ being non-negative integer,}$$

EXAMPLE 6.2

Find $L^{-1}\left(\frac{1}{\sqrt{s}}\right)$.

Solution. Since $L\left\{\frac{1}{t^{1/2}}\right\} = \frac{\Gamma(1/2)}{s^{1/2}} = \sqrt{\frac{\pi}{s}}$, it follows that

$$L^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi t}}.$$

6.2 PROPERTIES OF INVERSE LAPLACE TRANSFORM

The operational properties used in finding the Laplace transform of a function are also used in constructing the inverse transform. We, thus, have the following properties of inverse transform.

Theorem 6.2. (Linearity Property). If $F_1(s)$ and $F_2(s)$ are Laplace transforms of $f_1(t)$ and $f_2(t)$, respectively, and a_1 and a_2 are arbitrary constants, then

$$\begin{aligned} L^{-1}\{a_1 F_1(s) + a_2 F_2(s)\} \\ &= a_1 L^{-1}\{F_1(s)\} + a_2 L^{-1}\{F_2(s)\} \\ &= a_1 f_1(t) + a_2 f_2(t). \end{aligned}$$

Proof: Since

$$\begin{aligned} L\{a_1 f_1(t) + a_2 f_2(t)\} &= a_1 L\{f_1(t)\} + a_2 L\{f_2(t)\} \\ &= a_1 F_1(s) + a_2 F_2(s), \end{aligned}$$

we have

$$\begin{aligned} L^{-1}\{a_1 F_1(s) + a_2 F_2(s)\} \\ &= a_1 f_1(t) + a_2 f_2(t) \\ &= a_1 L^{-1}\{F_1(s)\} + a_2 L^{-1}\{F_2(s)\}. \end{aligned}$$

EXAMPLE 6.3

Find the inverse Laplace transform of

$$\frac{1}{2s} + \frac{4}{3(s-a)} + \frac{s}{s^2+16}.$$

Solution. By linearity of inverse Laplace transform, we have

$$\begin{aligned} L^{-1}\left\{\frac{1}{2s} + \frac{4}{3(s-a)} + \frac{s}{s^2+16}\right\} \\ &= \frac{1}{2}L^{-1}\left\{\frac{1}{s}\right\} + \frac{4}{3}L^{-1}\left\{\frac{1}{s-a}\right\} \\ &\quad + L^{-1}\left\{\frac{s}{s^2+16}\right\} = \frac{1}{2} + \frac{4}{3}e^{at} + \cos 4t. \end{aligned}$$

EXAMPLE 6.4

Find inverse Laplace transform of

$$\frac{5}{s-3} + \frac{s}{s^2+4} + \frac{3}{s-7}.$$

Solution. By linearity, we have

$$\begin{aligned} L^{-1}\left\{\frac{5}{s-3} + \frac{s}{s^2+4} + \frac{3}{s-7}\right\} \\ &= 5L^{-1}\left\{\frac{1}{s-3}\right\} + L^{-1}\left\{\frac{s}{s^2+4}\right\} \\ &\quad + 3L^{-1}\left\{\frac{1}{s-7}\right\} = 5e^{3t} + \cos 2t + 3e^{7t}. \end{aligned}$$

EXAMPLE 6.5

Find inverse Laplace transform of

$$-\frac{3}{2s} + \frac{4}{3(s-1)} + \frac{1}{6(s+2)}.$$

Solution. By linearity of inverse Laplace transform, we have

$$\begin{aligned} L^{-1} \left\{ -\frac{3}{2s} + \frac{4}{3(s-1)} + \frac{1}{6(s+2)} \right\} \\ = -\frac{3}{2} L^{-1} \left\{ \frac{1}{s} \right\} + \frac{4}{3} L^{-1} \left\{ \frac{1}{s-1} \right\} + \frac{1}{6} L^{-1} \left\{ \frac{1}{s+2} \right\} \\ = -\frac{3}{2} + \frac{4}{3} e^t + \frac{1}{6} e^{-2t}. \end{aligned}$$

EXAMPLE 6.6Find $L^{-1} \left\{ \frac{1}{\sqrt{s+a}} \right\}$.**Solution.** Since

$$\frac{1}{\sqrt{s+a}} = \frac{1}{\sqrt{s}} - \frac{a}{\sqrt{s}(\sqrt{s+a})} = \frac{1}{\sqrt{s}} - \frac{a(\sqrt{s}-a)}{\sqrt{s}(s-a^2)},$$

therefore,

$$\begin{aligned} L^{-1} \left\{ \frac{1}{\sqrt{s+a}} \right\} &= L^{-1} \left\{ \frac{1}{\sqrt{s}} \right\} - a L^{-1} \left\{ \frac{\sqrt{s}-a}{\sqrt{s}(s-a^2)} \right\} \\ &= L^{-1} \left\{ \frac{1}{\sqrt{s}} \right\} - a L^{-1} \left\{ \frac{1}{s-a^2} \right\} \\ &\quad + a^2 L^{-1} \left\{ \frac{1}{\sqrt{s}(s-a^2)} \right\} \\ &= \frac{1}{\sqrt{\pi t}} - a e^{a^2 t} + a e^{a^2 t} \operatorname{erf}(a\sqrt{t}) \\ &= \frac{1}{\sqrt{\pi t}} - a e^{a^2 t} (1 - \operatorname{erf}(a\sqrt{t})) \\ &= \frac{1}{\sqrt{\pi t}} - a e^{a^2 t} \operatorname{erfc}(a\sqrt{t}) \end{aligned}$$

Theorem 6.3. (First Shifting Property). If $F(s)$ is Laplace transform of $f(t)$, then

$$L^{-1} \{ F(s-a) \} = e^{at} f(t).$$

Proof: We know that

$$L \{ e^{at} f(t) \} = F(s-a).$$

Therefore,

$$L^{-1} \{ F(s-a) \} = e^{at} f(t).$$

EXAMPLE 6.7Find inverse Laplace transform of $\frac{s-5}{s^2+6s+13}$.**Solution.** Since

$$\frac{s-5}{s^2+6s+13} = \frac{(s+3)-8}{(s+3)^2+4},$$

we have

$$\begin{aligned} L^{-1} \left\{ \frac{s-5}{s^2+6s+13} \right\} &= L^{-1} \left\{ \frac{s+3}{(s+3)^2+4} \right\} \\ &\quad - 4 L^{-1} \left\{ \frac{2}{(s+3)^2+4} \right\} \\ &= e^{-3t} \cos 2t - 4e^{-3t} \sin 2t \\ &= e^{-3t} (\cos 2t - 4 \sin 2t). \end{aligned}$$

EXAMPLE 6.8Find inverse Laplace transform of $\frac{2s-3}{s^2+4s+13}$.**Solution.** Since

$$\frac{2s-3}{s^2+4s+13} = \frac{2s+4-7}{(s+2)^2+9} = \frac{2(s+2)-7}{(s+2)^2+9},$$

we have

$$\begin{aligned} L^{-1} \left\{ \frac{2s-3}{s^2+4s+13} \right\} &= L^{-1} \left\{ \frac{2(s+2)}{(s+2)^2+9} \right\} \\ &\quad - 7 L^{-1} \left\{ \frac{1}{(s+2)^2+9} \right\} \\ &= 2e^{-2t} \cos 3t - \frac{7}{3} e^{-2t} \sin 3t. \end{aligned}$$

EXAMPLE 6.9Find inverse Laplace transform of $\frac{s}{(s+1)^2}$.**Solution.** We note that

$$\frac{s}{(s+1)^2} = \frac{s+1-1}{(s+1)^2} = \frac{1}{s+1} - \frac{1}{(s+1)^2}.$$

Therefore,

$$\begin{aligned} L^{-1} \left\{ \frac{s}{(s+1)^2} \right\} &= L^{-1} \left\{ \frac{1}{s+1} \right\} - L^{-1} \left\{ \frac{1}{(s+1)^2} \right\} \\ &= e^{-t} \cdot 1 - e^{-t} \cdot t = e^{-t} (1-t). \end{aligned}$$

EXAMPLE 6.10

Find inverse Laplace transform of

$$\frac{1}{s^2 + 4s + 13} - \frac{s + 4}{s^2 + 8s + 97} + \frac{s + 2}{s^2 - 4s + 29}.$$

Solution. We have

$$\begin{aligned} F(s) &= \frac{1}{s^2 + 4s + 13} - \frac{s + 4}{s^2 + 8s + 97} + \frac{s + 2}{s^2 - 4s + 29} \\ &= \frac{1}{(s + 2)^2 + 9} \\ &\quad - \frac{s + 4}{(s + 4)^2 + 81} + \frac{s - 2 + 4}{(s - 2)^2 + 25}. \end{aligned}$$

Therefore,

$$\begin{aligned} L^{-1}\{F(s)\} &= \frac{1}{3}L^{-1}\left\{\frac{3}{(s + 2)^2 + 9}\right\} \\ &\quad - L^{-1}\left\{\frac{s + 4}{(s + 4)^2 + 81}\right\} \\ &\quad + L^{-1}\left\{\frac{s - 2}{(s - 2)^2 + 25}\right\} \\ &\quad + \frac{4}{5}L^{-1}\left\{\frac{5}{(s - 2)^2 + 25}\right\} \\ &= \frac{1}{3}e^{-2t}\sin 3t - e^{-4t}\cos 9t \\ &\quad + e^{2t}\cos 5t + \frac{4}{5}e^{2t}\sin 5t. \end{aligned}$$

EXAMPLE 6.11Find the inverse Laplace transform of $\frac{s+2}{s^2-4s+13}$.**Solution.** We have

$$\begin{aligned} \frac{s + 2}{s^2 - 4s + 13} &= \frac{(s - 2) + 4}{(s - 2)^2 + 9} \\ &= \frac{(s - 2)}{(s - 2)^2 + 9} + \frac{4}{(s - 2)^2 + 9} \\ &= \frac{s - 2}{(s - 2)^2 + 9} + \frac{4}{3} \cdot \frac{3}{(s - 2)^2 + 9} \end{aligned}$$

Therefore, by linearity of inverse Laplace transform and shifting property, we get

$$\begin{aligned} L^{-1}\left\{\frac{s + 2}{s^2 - 4s + 13}\right\} &= L^{-1}\left\{\frac{s - 2}{(s - 2)^2 + 9}\right\} \\ &\quad + \frac{4}{3}L^{-1}\left\{\frac{3}{(s - 2)^2 + 9}\right\} \\ &= e^{2t}\cos 3t + \frac{4}{3}e^{2t}\sin 3t \\ &= e^{2t}(\cos 3t + \frac{4}{3}\sin 3t). \end{aligned}$$

Theorem 6.4. (Second Shifting Property). If $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\{e^{-sa}F(s)\} = g(t)$, where

$$g(t) = \begin{cases} f(t - a) & \text{for } t > a \\ 0 & \text{for } t < a. \end{cases}$$

Proof: Since $L\{g(t)\} = e^{-sa}F(s)$, it follows that

$$L^{-1}\{e^{-sa}F(s)\} = g(t).$$

Second Proof. By definition of Laplace transform, we have

$$F(s) = \int_0^{\infty} e^{-st}f(t) dt.$$

Therefore,

$$\begin{aligned} e^{-sa}F(s) &= \int_0^{\infty} e^{-sa}e^{-st}f(t) dt = \int_0^{\infty} e^{-s(t+a)}f(t) dt \\ &= \int_a^{\infty} e^{-su}f(u - a) du, \quad t + a = u. \\ &= \int_0^a e^{-su}(0) du + \int_a^{\infty} e^{-su}f(u - a) du \\ &= L\{g(t)\}. \end{aligned}$$

Hence

$$L^{-1}\{e^{-sa}F(s)\} = g(t).$$

EXAMPLE 6.12Find inverse Laplace transform of $-\frac{e^{-\pi s/2}}{s^2 + 1}$.**Solution.** We have

$$\begin{aligned} -\frac{e^{-\pi s/2}}{s^2 + 1} &= e^{-\pi s/2}\left(-\frac{1}{s^2 + 1}\right) \\ &= e^{-\pi s/2}F(s), \text{ where } F(s) = -\frac{1}{s^2 + 1}. \end{aligned}$$

But

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{-\frac{1}{s^2+1}\right\} = -\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}.$$

Therefore, by second-shifting property,

$$\mathcal{L}^{-1}\left\{-\frac{e^{-\pi s/2}}{s^2+1}\right\} = g(t),$$

where

$$\begin{aligned} g(t) &= \begin{cases} -\sin(t - \pi/2) & \text{for } t > \pi/2 \\ 0 & \text{for } t < \pi/2 \end{cases} \\ &= -\sin\left(t - \frac{\pi}{2}\right) \left[H\left(t - \frac{\pi}{2}\right)\right] \\ &= \cos t \left[H\left(t - \frac{\pi}{2}\right)\right], \end{aligned}$$

where $H(t)$ denotes Heavyside's unit step function.

EXAMPLE 6.13

Find inverse Laplace transform of $\frac{\omega e^{-sa}}{s^2 + \omega^2}$.

Solution. We have

$$\mathcal{L}^{-1}\left\{\frac{\omega}{s^2 + \omega^2}\right\} = \sin \omega t.$$

Therefore, by second-shifting property,

$$\mathcal{L}^{-1}\left\{e^{-sa} \frac{\omega}{s^2 + \omega^2}\right\} = g(t),$$

where

$$g(t) = \begin{cases} \sin \omega(t - a) & \text{for } t > a \\ 0 & \text{for } t < a. \end{cases}$$

EXAMPLE 6.14

Find inverse transform of $\frac{2e^{-s}}{s^3}$, $\text{Re}(s) > 0$.

Solution. We have

$$\frac{2e^{-s}}{s^3} = 2e^{-s} \cdot \frac{1}{s^3}.$$

Since $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = t^2$, therefore, by second-shifting property,

$$\mathcal{L}^{-1}\left\{\frac{2e^{-s}}{s^3}\right\} = g(t),$$

where

$$g(t) = \begin{cases} 2(t-1)^2 & \text{for } t \geq 1 \\ 0 & \text{for } 0 \leq t < 1. \end{cases}$$

EXAMPLE 6.15

Find $\mathcal{L}^{-1}\left\{\frac{A}{s} - \frac{s}{s^2+1} e^{-sa}\right\}$, where A is a constant.

Solution. We have

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1 \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t.$$

Therefore, using linearity property and second-shifting property, we have

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{A}{s} - \frac{s}{s^2+1} e^{-sa}\right\} \\ = A\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{e^{-sa} \frac{s}{s^2+1}\right\} = A - g(t), \end{aligned}$$

where

$$\begin{aligned} g(t) &= \begin{cases} \cos(t-a) & \text{for } t > a \\ 0 & \text{for } t < a. \end{cases} \\ &= H(t-a) \cos(t-a). \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{A}{s} - \frac{s}{s^2+1} e^{-sa}\right\} \\ = A - H(t-a) \cos(t-a). \end{aligned}$$

EXAMPLE 6.16

Find the inverse Laplace transform of $\frac{e^{-7s}}{(s-3)^3}$.

Solution. Since

$$\mathcal{L}^{-1}\left(\frac{1}{(s-3)^3}\right) = \frac{1}{2}t^2 e^{3t},$$

by second-shifting property, we have

$$\mathcal{L}^{-1}\left\{e^{-7s} \frac{1}{(s-3)^3}\right\} = g(t),$$

where

$$\begin{aligned} g(t) &= \begin{cases} \frac{1}{2}(t-7)^2 e^{3(t-7)} & \text{for } t > 7 \\ 0 & \text{for } 0 \leq t \leq 7 \end{cases} \\ &= \frac{1}{2}H(t-7)(t-7)^2 e^{3(t-7)}. \end{aligned}$$

Theorem 6.5. (Change of Scale Property). If $\mathcal{L}^{-1}\{F(s)\} = f(t)$, then

$$\mathcal{L}^{-1}\{F(as)\} = \frac{1}{a}f\left(\frac{t}{a}\right).$$

Proof: By the definition of Laplace transform,

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt.$$

Therefore,

$$\begin{aligned} F(as) &= \int_0^{\infty} e^{-ast} f(t) dt \\ &= \frac{1}{a} \int_0^{\infty} e^{-su} f\left(\frac{u}{a}\right) du, \quad u = at \\ &= \frac{1}{a} L\left\{f\left(\frac{t}{a}\right)\right\}. \end{aligned}$$

Hence

$$L^{-1}\{F(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right).$$

Remark 6.1. It follows from Theorem 6.5 that if $L^{-1}\{F(s)\} = f(t)$, then $L^{-1}\left\{F\left(\frac{s}{a}\right)\right\} = af(at)$ for $a > 0$.

EXAMPLE 6.17

Find the inverse transform of $\frac{s}{(s/2)^2 + 4}$.

Solution. Since

$$L^{-1}\left\{\frac{s}{s^2 + 4}\right\} = \cos 2t,$$

we have

$$\begin{aligned} L^{-1}\left\{\frac{s}{(s/2)^2 + 4}\right\} &= 2L^{-1}\left\{\frac{s/2}{(s/2)^2 + 4}\right\} \\ &= 2.2 \cos(2(2t)) = 4 \cos 4t. \end{aligned}$$

Theorem 6.6. (Inverse Laplace Transform of Derivatives). If $L^{-1}\{F(s)\} = f(t)$, then

$$L^{-1}\{F^{(n)}(s)\} = (-1)^n t^n f(t).$$

Proof: Since

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s),$$

we have

$$L^{-1}\{F^{(n)}(s)\} = (-1)^n t^n f(t)$$

EXAMPLE 6.18

Find $L^{-1}\left\{\frac{1-s^2}{(s^2+1)^2}\right\}$.

Solution. We know that

$$L^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t.$$

Further

$$\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \frac{1-s^2}{(s^2+1)^2}.$$

Therefore, by Theorem 6.6, we have

$$L^{-1}\left\{\frac{1-s^2}{(s^2+1)^2}\right\} = -t \cos t.$$

EXAMPLE 6.19

Find $L^{-1}\left\{\log \frac{s+a}{s+b}\right\}$.

Solution. We note that

$$\begin{aligned} \frac{d}{ds} \log \frac{s+a}{s+b} &= \frac{d}{ds} [\log(s+a) - \log(s+b)] \\ &= \frac{1}{s+a} - \frac{1}{s+b}. \end{aligned}$$

Therefore, the use of Theorem 6.6 yields

$$L^{-1}\left\{\frac{1}{s+a} - \frac{1}{s+b}\right\} = -t f(t)$$

and so

$$L^{-1}\left\{\frac{1}{s+a}\right\} - L^{-1}\left\{\frac{1}{s+b}\right\} = -t f(t),$$

that is,

$$e^{-at} - e^{-bt} = -t f(t).$$

Hence

$$f(t) = \frac{1}{t} (e^{-bt} - e^{-at}).$$

EXAMPLE 6.20

Find $L^{-1}\left\{\log \frac{s^2+a^2}{s^2+b^2}\right\}$.

Solution. Since

$$\begin{aligned} \frac{d}{ds} \log \frac{s^2+a^2}{s^2+b^2} &= \frac{d}{ds} [\log(s^2+a^2) - \log(s^2+b^2)] \\ &= \frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}, \end{aligned}$$

Theorem 6.6 yields

$$L^{-1}\left\{\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right\} = -t f(t),$$

or

$$2 \cos at - 2 \cos bt = -t f(t),$$

or

$$f(t) = \frac{2}{t}(\cos bt - \cos at).$$

EXAMPLE 6.21

Find $L^{-1}\{\log \frac{1+s}{s}\}$.

Solution. Since

$$\begin{aligned} \frac{d}{ds} \log \frac{1+s}{s} &= \frac{d}{ds} [\log(1+s) - \log s] \\ &= \frac{1}{s+1} - \frac{1}{s}, \end{aligned}$$

Therefore,

$$L^{-1}\left\{\frac{1}{s+1} - \frac{1}{s}\right\} = -tf'(t)$$

or

$$e^{-t} - 1 = -tf'(t)$$

or

$$f(t) = \frac{1 - e^{-t}}{t}.$$

EXAMPLE 6.22

Find $L^{-1}\{\log(1 + \frac{1}{s^2})\}$.

Solution. Since

$$\begin{aligned} \frac{d}{ds} \left\{ \log \frac{s^2+1}{s^2} \right\} &= \frac{d}{ds} [\log(s^2+1) - \log s^2] \\ &= \frac{2s}{s^2+1} - \frac{2s}{s^2} = 2 \left[\frac{s}{s^2+1} - \frac{1}{s} \right], \end{aligned}$$

we have

$$L^{-1}\left\{2\left(\frac{s}{s^2+1} - \frac{1}{s}\right)\right\} = -tf'(t)$$

and so

$$2 \cos t - 2 = -tf'(t)$$

or

$$f(t) = \frac{2(1 - \cos t)}{t}.$$

EXAMPLE 6.23

Find $L^{-1}\{\tan^{-1} \frac{1}{s}\}$, $s > 0$,

Solution. Since

$$\begin{aligned} \frac{d}{ds} \left(\tan^{-1} \frac{1}{s} \right) &= \frac{1}{1 + (1/s)^2} \left(-\frac{1}{s^2} \right) \\ &= \frac{-1}{1 + (1/s^2)} \left(\frac{1}{s^2} \right) = -\frac{1}{s^2 + 1}, \end{aligned}$$

it follows that

$$L^{-1}\left\{-\frac{1}{s^2+1}\right\} = -tf'(t),$$

that is,

$$-1(\sin t) = -tf'(t).$$

Hence

$$f(t) = \frac{\sin t}{t}.$$

EXAMPLE 6.24

Find $L^{-1}\left\{\log \frac{s^2+1}{(s-1)^2}\right\}$.

Solution. Since

$$\begin{aligned} \frac{d}{ds} \left(\log \frac{s^2+1}{(s-1)^2} \right) &= \frac{d}{ds} [\log(s^2+1) - 2\log(s-1)] \\ &= \frac{2s}{s^2+1} - \frac{2}{s-1}, \end{aligned}$$

we have

$$L^{-1}\left\{\frac{2s}{s^2+1} - \frac{2}{s-1}\right\} = -tf'(t),$$

which yields

$$2[\cos t - e^t] = -tf'(t)$$

or

$$f(t) = \frac{2(e^t - \cos t)}{t}.$$

EXAMPLE 6.25

Find $L^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\}$.

Solution. We have

$$\frac{s+2}{(s^2+4s+5)^2} = \frac{s+2}{((s+2)^2+1)^2}$$

Therefore,

$$\begin{aligned} L^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\} &= L^{-1}\left\{\frac{s+2}{((s+2)^2+1)^2}\right\} \\ &= e^{-2t} \cdot L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}. \end{aligned}$$

We now find $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$. We note that

$$\frac{d}{ds} \left(\frac{1}{(s^2+1)} \right) = -\frac{2s}{(s^2+1)^2}.$$

Therefore, by Theorem 6.6

$$L^{-1}\left\{-\frac{2s}{(s^2+1)^2}\right\} = -t L^{-1}\left\{\frac{1}{s^2+1}\right\} = -t \sin t$$

and so

$$L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2} t \sin t.$$

Hence

$$L^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\} = \frac{1}{2} t e^{-2t} \sin t.$$

EXAMPLE 6.26

Find $L^{-1}\left\{\frac{s+3}{(s^2+6s+13)^2}\right\}$.

Solution. We have

$$\frac{s+3}{(s^2+6s+13)^2} = \frac{s+3}{((s+3)^2+4)^2}.$$

Therefore,

$$\begin{aligned} L^{-1}\left\{\frac{s+3}{(s^2+6s+13)^2}\right\} &= L^{-1}\left\{\frac{s+3}{((s+3)^2+4)^2}\right\} \\ &= e^{-3t} L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}. \end{aligned}$$

To find $L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}$, we note that

$$\frac{d}{ds}\left(\frac{1}{s^2+4}\right) = \frac{-2s}{(s^2+4)^2}.$$

Therefore,

$$\begin{aligned} L^{-1}\left\{-\frac{2s}{(s^2+4)^2}\right\} &= -t L^{-1}\left\{\frac{1}{s^2+4}\right\} \\ &= -\frac{t}{2} \sin 2t \end{aligned}$$

and so

$$L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = \frac{1}{4} t \sin 2t.$$

Consequently, we get

$$L^{-1}\left\{\frac{s+3}{(s^2+6s+13)^2}\right\} = \frac{1}{4} t e^{-3t} \sin 2t.$$

Theorem 6.7. (Inverse Laplace Transform of Integrals). If $L^{-1}\{F(s)\} = f(t)$, then

$$L^{-1}\left\{\int_s^\infty F(u) du\right\} = \frac{f(t)}{t}.$$

Proof: Since,

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u) du,$$

we have

$$L^{-1}\left\{\int_s^\infty F(u) du\right\} = \frac{f(t)}{t}.$$

EXAMPLE 6.27

Find

$$L^{-1}\left\{\int_s^\infty \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du\right\}.$$

Solution. By Theorem 6.7, we have

$$L^{-1}\left\{\int_s^\infty \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du\right\} = \frac{L^{-1}\left\{\frac{1}{u-1} - \frac{1}{u+1}\right\}}{t}.$$

But

$$L^{-1}\left\{\frac{1}{u-1} - \frac{1}{u+1}\right\} = e^t - e^{-t}.$$

Therefore,

$$L^{-1}\left\{\int_s^\infty \left(\frac{1}{u-1} - \frac{1}{u+1}\right) du\right\} = \frac{e^t - e^{-t}}{t}.$$

Theorem 6.8. (Multiplication by s^n). If $L^{-1}\{F(s)\} = f(t)$ and $f(0) = 0$, then

$$L^{-1}\{sF(s)\} = f'(t).$$

Proof: We know that

$$L\{f'(t)\} = s F(s) - f(0)$$

and so

$$L^{-1}\{sF(s) - f(0)\} = f'(t),$$

that is,

$$L^{-1}\{sF(s)\} = f'(t) + f(0)\delta(t),$$

Remark 6.2. If $f(0) \neq 0$, then

$$L^{-1}\{sF(s) - f(0)\} = f'(t)$$

and so

$$L^{-1}\{sF(s)\} = f'(t) + f(0)\delta(t),$$

where $\delta(t)$ is the Dirac delta function.

EXAMPLE 6.28

Find $L^{-1}\left\{\frac{s^2}{s^2+1}\right\}$.

Solution. We know that

$$L^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t = f(t)$$

and $f(0) = 1 \neq 0$ for $t = 0$. Therefore,

$$\begin{aligned} L^{-1}\left\{\frac{s^2}{s^2+1}\right\} &= L^{-1}\{sF(s)\} \\ &= f'(t) + f(0)\delta(t) = \delta(t) - \sin t. \end{aligned}$$

EXAMPLE 6.29

Find $L^{-1}\left\{\frac{s^3}{s^2+1}\right\}$.

Solution. From the Example 6.28,

$$L^{-1}\left\{\frac{s^2}{s^2+1}\right\} = \delta(t) - \sin t = f(t).$$

Since $f(0) = 0$, we have

$$\begin{aligned} L^{-1}\left\{\frac{s^3}{s^2+1}\right\} &= L^{-1}\{sF(s)\} = f'(t) \\ &= \frac{d}{dt}\{\delta(t) - \sin t\} = \delta'(t) - \cos t. \end{aligned}$$

Theorem 6.9. (Division by s). If $L^{-1}\{F(s)\} = f(t)$, then

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u) du.$$

Proof: We know that

$$L\left\{\int_0^t f(u) du\right\} = \frac{F(s)}{s}.$$

Therefore,

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u) du.$$

Remark 6.3. Consider

$$g(t) = \int_0^t \int_0^v f(u) du dv.$$

Then

$$g'(t) = \int_0^t f(u) du \quad \text{and} \quad g''(t) = f(t).$$

Also $g(0) = g'(0) = 0$. Thus

$$\begin{aligned} L\{g''(t)\} &= s^2 L\{g(t)\} - s g(0) - g'(0) \\ &= s^2 L\{g(t)\}, \end{aligned}$$

that is,

$$L\{f(t)\} = s^2 L\{g(t)\}$$

or

$$F(s) = s^2 L\{g(t)\}$$

or

$$L\{g(t)\} = \frac{F(s)}{s^2}$$

Hence

$$\begin{aligned} L^{-1}\left\{\frac{F(s)}{s^2}\right\} &= g(t) = \int_0^t \int_0^v f(u) du dv \\ &= \int_0^t \int_0^t f(t) dt^2. \end{aligned}$$

In general, we have

$$L^{-1}\left\{\frac{F(s)}{s^n}\right\} = \underbrace{\int_0^t \int_0^t \cdots \int_0^t}_{n \text{ times}} f(t) dt^n.$$

EXAMPLE 6.30

Find $L^{-1}\left\{\frac{1}{s(s+1)}\right\}$.

Solution. Since $L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$, we have

$$\begin{aligned} L^{-1}\left\{\frac{1}{s(s+1)}\right\} &= \int_0^t e^{-u} du = [-e^{-u}]_0^t \\ &= -[e^{-t} - 1] = 1 - e^{-t}. \end{aligned}$$

EXAMPLE 6.31

Find $L^{-1}\left\{\frac{1}{s^2(s+1)}\right\}$.

Solution. From Example 6.30, we have

$$L^{-1}\left\{\frac{1}{s(s+1)}\right\} = 1 - e^{-t}$$

Therefore, by Theorem 6.9, we have

$$\begin{aligned} L^{-1}\left\{\frac{1}{s^2(s+1)}\right\} &= \int_0^t (1 - e^{-u}) du = [u + e^{-u}]_0^t \\ &= t + e^{-t} - 1. \end{aligned}$$

EXAMPLE 6.32

Find $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$ and deduce $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$ from it.

Solution. Since

$$\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = -\frac{2as}{(s^2+a^2)^2},$$

we have

$$L^{-1}\left\{-\frac{2as}{(s^2+a^2)^2}\right\} = -tf'(t),$$

where $f(t) = L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$ and so

$$-2a L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = -t \sin at$$

Hence

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t \sin at}{2a}$$

Now Theorem 6.9 yields

$$\begin{aligned} L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} &= \frac{1}{2a} \int_0^t u \sin au \, du \\ &= \frac{1}{2a} \left\{ \left[u \frac{-\cos au}{a} \right]_0^t - \int_0^t -\frac{\cos au}{a} \, du \right\} \\ &= \frac{1}{2a} \left[-\frac{t \cos at}{a} + \frac{\sin at}{a^2} \right] \\ &= \frac{1}{2a^3} (\sin at - at \cos at). \end{aligned}$$

6.3 PARTIAL FRACTIONS METHOD TO FIND INVERSE LAPLACE TRANSFORM

While working with physical problems, we come across some functions that are not immediately recognizable as the Laplace transform of some elementary functions. In such cases, we decompose the given function into partial fractions and then write down the inverse of each fraction. This method is used when inverse transform of a rational function is required.

We can also use the concept of *simple poles* of the rational function $F(s)$ to find inverse Laplace transform. Let

$$F(s) = \frac{P(s)}{Q(s)} = \frac{P(s)}{(s-\alpha_1)(s-\alpha_2)\dots(s-\alpha_n)},$$

$$\alpha_i \neq \alpha_j,$$

where $P(s)$ is a polynomial of degree less than n . In terms of complex analysis, we call $\alpha_1, \alpha_2, \dots, \alpha_n$ the simple poles of $F(s)$. The partial fraction decomposition shall be

$$F(s) = \frac{A_1}{s-\alpha_1} + \frac{A_2}{s-\alpha_2} + \dots + \frac{A_n}{s-\alpha_n}.$$

Multiplying both sides of this equation by $s-\alpha_i$ and letting $s \rightarrow \alpha_i$, we have

$$A_i = \lim_{s \rightarrow \alpha_i} (s-\alpha_i)F(s).$$

Again, in terms of complex analysis, A_i is called the *residue of $F(s)$ at the poles α_i* .

Now, let $L\{f(t)\} = F(s)$. Then

$$\begin{aligned} f(t) = L^{-1}\{F(s)\} &= \sum_{i=1}^n L^{-1}\left(\frac{A_i}{s-\alpha_i}\right) = \sum_{i=1}^n A_i e^{\alpha_i t} \\ &= \sum_{i=1}^n \lim_{s \rightarrow \alpha_i} (s-\alpha_i)F(s) e^{\alpha_i t}. \end{aligned}$$

EXAMPLE 6.33

Find $L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}$.

Solution. Write

$$\frac{1}{s^3(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds+E}{s^2+1},$$

which yields

$$\begin{aligned} 1 &= As^2(s^2+1) + Bs(s^2+1) \\ &\quad + C(s^2+1) + (Ds+E)s^3. \end{aligned}$$

Putting $s=0$, we get $C=1$. Comparing coefficients of s on both sides, we get $B=0$. Comparing coefficients of s^2 on both sides, we get $A+C=0$ which yields $A=-C=-1$. Comparing coefficients of s^3 , we get $B+E=0$ and so $E=-B=0$. Comparing coefficients of s^4 , we have $A+D=0$ and so $D=-A=1$. Thus

$$\frac{1}{s^3(s^2+1)} = -\frac{1}{s} + \frac{1}{s^3} + \frac{s}{s^2+1}.$$

Hence, by Example 6.1,

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} &= \mathcal{L}^{-1}\left\{-\frac{1}{s} + \frac{1}{s^3} + \frac{s}{s^2+1}\right\} \\ &= -1 + \frac{1}{2}t^2 + \cos t \\ &= \frac{1}{2}(t^2 + 2\cos t - 2). \end{aligned}$$

EXAMPLE 6.34

Find $\mathcal{L}^{-1}\left\{\frac{2}{(s+1)(s^2+1)}\right\}$.

Solution. Write

$$\frac{2}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}.$$

Clearing fractions, we have

$$2 = A(s^2+1) + (Bs+C)(s+1).$$

Putting $s = -1$, we have $A = 1$. Comparing the coefficient of s^2 and that of s^0 , we get

$$0 = A + B, \text{ which yields } B = -1$$

and

$$2 = A + C, \text{ which yields } C = 1.$$

Hence

$$\begin{aligned} \frac{2}{(s+1)(s^2+1)} &= \frac{1}{s+1} + \frac{-s+1}{s^2+1} \\ &= \frac{1}{s+1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}. \end{aligned}$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{2}{(s+1)(s^2+1)}\right\} = e^{-t} - \cos t + \sin t.$$

EXAMPLE 6.35

Find $\mathcal{L}^{-1}\left\{\frac{s^2}{(s-1)(s+2)(s-3)}\right\}$.

Solution. The simple poles of the rational function $F(s)$ are $s = 1, -2$, and 3 . Therefore,

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F(s)\} = \lim_{s \rightarrow 1} (s-1) F(s) e^t \\ &\quad + \lim_{s \rightarrow -2} (s+2) F(s) e^{-2t} + \lim_{s \rightarrow 3} (s-3) F(s) e^{3t} \\ &= -\frac{1}{6}e^t + \frac{4}{(-3)(-5)}e^{-2t} + \frac{9}{(2)(5)}e^{3t} \\ &= -\frac{1}{6}e^t + \frac{4}{15}e^{-2t} + \frac{9}{10}e^{3t}. \end{aligned}$$

EXAMPLE 6.36

Find $\mathcal{L}^{-1}\left\{\frac{4s+5}{(s-1)^2(s+2)}\right\}$.

Solution. Write

$$\frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2},$$

which gives

$$4s+5 = A(s+2)(s-1) + B(s+2) + C(s-1)^2.$$

Putting $s = 1$, we get $9 = 3B$ and so $B = 3$. Equating the coefficients of s and s^2 , we get $A + B - 2C = 4$ and $A + C = 0$. These equations give $C = -\frac{1}{3}$ and $A = \frac{1}{3}$. Thus

$$\frac{4s+5}{(s-1)^2(s+2)} = \frac{1}{3(s-1)} + \frac{3}{(s-1)^2} - \frac{1}{3(s+2)}.$$

Hence

$$\mathcal{L}^{-1}\left\{\frac{4s+5}{(s-1)^2(s+2)}\right\} = \frac{1}{3}e^t + 3te^t - \frac{1}{3}e^{-2t}.$$

EXAMPLE 6.37

Find $\mathcal{L}^{-1}\left\{\frac{s^2}{s^3+6s^2+11s+6}\right\}$.

Solution. We have

$$F(s) = \frac{s^2}{s^3+6s^2+11s+6}.$$

Clearly $s = -1, -2$, and -3 are roots of the polynomial $s^3 + 6s^2 + 11s + 10$. Hence the simple poles of $F(s)$ are $-1, -2$, and -3 . Therefore,

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F(s)\} = \lim_{s \rightarrow -1} (s+1) F(s) e^{-t} \\ &\quad + \lim_{s \rightarrow -2} (s+2) F(s) e^{-2t} + \lim_{s \rightarrow -3} (s+3) F(s) e^{-3t} \\ &= \frac{1}{2}e^{-t} - 4e^{-2t} + \frac{9}{2}e^{-3t}. \end{aligned}$$

EXAMPLE 6.38

Find $\mathcal{L}^{-1}\left\{\frac{s^3}{s^4-a^4}\right\}$.

Solution. We have

$$\begin{aligned} \frac{s^3}{s^4-a^4} &= \frac{s^3}{(s-a)(s+a)(s^2+a^2)} \\ &= \frac{A}{s-a} + \frac{B}{s+a} + \frac{Cs+D}{s^2+a^2}, \end{aligned}$$

which yields

$$\begin{aligned}s^3 &= A(s+a)(s^2+a^2) + B(s-a)(s^2+a^2) \\ &\quad + (Cs+D)(s^2-a^2) \\ &= A(s^3+as^2+a^2s+a^3) + B(s^3-as^2+a^2s-a^3) \\ &\quad + C(s^3-sa^2) + D(s^2-a^2).\end{aligned}$$

Putting $s = a, -a$, we get $A = B = \frac{1}{4}$. Comparing coefficients of s , and s^2 , we get $C = \frac{1}{2}$, $D = 0$.

Thus
$$\frac{s^3}{s^4-a^4} = \frac{1}{4(s-a)} + \frac{1}{4(s+a)} + \frac{s}{2(s^2+a^2)}.$$

Hence
$$\begin{aligned}L^{-1}\left\{\frac{s^3}{s^4-a^4}\right\} &= \frac{1}{4}e^{at} + \frac{1}{4}e^{-at} + \frac{1}{2}\cos at \\ &= \frac{1}{2}\cos at + \frac{1}{2}\left(\frac{e^{at}+e^{-at}}{2}\right) \\ &= \frac{1}{2}[\cos at + \cosh at].\end{aligned}$$

EXAMPLE 6.39

Find $L^{-1}\left\{\frac{s^2}{(s^2-a^2)(s^2-b^2)(s^2-c^2)}\right\}$.

Solution. Let

$$\begin{aligned}F(s) &= \frac{s^2}{(s^2-a^2)(s^2-b^2)(s^2-c^2)} \\ &= \frac{1}{(s-a)(s+a)(s-b)(s+b)(s-c)(s+c)}.\end{aligned}$$

The simple poles of $F(s)$ are $a, -a, b, -b, c$, and $-c$.

Therefore,

$$\begin{aligned}L^{-1}\{F(s)\} &= \lim_{s \rightarrow a} (s-a)F(s)e^{at} + \lim_{s \rightarrow -a} (s+a)F(s)e^{-at} \\ &\quad + \lim_{s \rightarrow b} (s-b)F(s)e^{bt} + \lim_{s \rightarrow -b} (s+b)F(s)e^{-bt} \\ &\quad + \lim_{s \rightarrow c} (s-c)F(s)e^{ct} + \lim_{s \rightarrow -c} (s+c)F(s)e^{-ct} \\ &= \frac{a^2}{2a(a^2-b^2)(a^2-c^2)}e^{at} \\ &\quad - \frac{a^2}{2a(a^2-b^2)(a^2-c^2)}e^{-at} \\ &\quad + \frac{b^2}{2b(b^2-a^2)(b^2-c^2)}e^{bt} \\ &\quad - \frac{b^2}{2b(b^2-a^2)(b^2-c^2)}e^{-bt} \\ &\quad + \frac{c^2}{2c(c^2-a^2)(c^2-b^2)}e^{ct} \\ &\quad - \frac{c^2}{2c(c^2-a^2)(c^2-b^2)}e^{-ct}\end{aligned}$$

$$\begin{aligned}&= \frac{a(e^{at}-e^{-at})}{2(a^2-b^2)(a^2-c^2)} + \frac{b(e^{bt}-e^{-bt})}{2(b^2-a^2)(b^2-c^2)} \\ &\quad + \frac{c(e^{ct}-e^{-ct})}{2(c^2-a^2)(c^2-b^2)} = \frac{a \sinh at}{(a^2-b^2)(a^2-c^2)} \\ &\quad + \frac{b \sinh bt}{(b^2-a^2)(b^2-c^2)} + \frac{c \sinh ct}{(c^2-a^2)(c^2-b^2)}.\end{aligned}$$

EXAMPLE 6.40

Find $L^{-1}\left\{\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right\}$.

Solution. Let

$$F(s) = \frac{2s^2-6s+5}{s^3-6s^2+11s-6}.$$

We observe that $s = 1, s = 2$, and $s = 3$ are the roots of the equation $s^3 - 6s^2 + 11s - 6 = 0$. Thus

$$F(s) = \frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}.$$

Therefore,

$$\begin{aligned}L^{-1}\{F(s)\} &= \lim_{s \rightarrow 1} (s-1)F(s)e^t + \lim_{s \rightarrow 2} (s-2)F(s)e^{2t} \\ &\quad + \lim_{s \rightarrow 3} (s-3)F(s)e^{3t} = \frac{1}{2}e^t - e^{2t} + \frac{5}{2}e^{3t}.\end{aligned}$$

EXAMPLE 6.41

Find $L^{-1}\left\{\frac{1}{s^3-a^3}\right\}$.

Solution. We have

$$\begin{aligned}F(s) &= \frac{1}{s^3-a^3} = \frac{1}{(s-a)(s^2+as+a^2)} \\ &= \frac{A}{s-a} + \frac{Bs+C}{s^2+as+a^2}.\end{aligned}$$

Therefore,

$$1 = A(s^2+as+a^2) + (Bs+C)(s-a).$$

Putting $s = a$, we have $A = \frac{1}{3a^2}$. Comparing the coefficients of constant terms and of s^2 , we get respectively, $a^2A - aC = 1$ and $A + B = 0$ which in turn imply $B = -A = -\frac{1}{3a^2}$ and $C = -\frac{2}{3a}$. Thus

$$F(s) = \frac{1}{3a^2(s-a)} - \frac{1}{3a^2}\left(\frac{s+2a}{s^2+as+a^2}\right).$$

Hence

$$\begin{aligned} \mathcal{L}^{-1}\{F(s)\} &= \frac{1}{3a^2} \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} - \frac{1}{3a^2} \mathcal{L}^{-1}\left\{\frac{(s+\frac{a}{2})+\frac{3a}{2}}{(s+\frac{a}{2})^2+\frac{3a^2}{4}}\right\} \\ &= \frac{1}{3a^2} e^{at} - \frac{1}{3a^2} \left[\mathcal{L}^{-1}\left\{\frac{s+\frac{a}{2}}{(s+\frac{a}{2})^2+\frac{3a^2}{4}}\right\} \right. \\ &\quad \left. + \mathcal{L}^{-1}\left\{\frac{\frac{3a}{2}}{(s+\frac{a}{2})^2+\frac{3a^2}{4}}\right\} \right] \\ &= \frac{1}{3a^2} e^{at} - \frac{1}{3a^2} \left[e^{-\frac{at}{2}} \left(\cos \frac{\sqrt{3}a}{2} t + \sqrt{3} \sin \frac{\sqrt{3}a}{2} t \right) \right] \\ &= \frac{1}{3a^2} \left[e^{at} - e^{-\frac{at}{2}} \left(\cos \frac{\sqrt{3}a}{2} t + \sqrt{3} \sin \frac{\sqrt{3}a}{2} t \right) \right]. \end{aligned}$$

EXAMPLE 6.42

Find $\mathcal{L}^{-1}\left\{\frac{4s+5}{(s-1)^2(s+2)}\right\}$.

Solution. Write

$$F(s) = \frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2},$$

and so

$$4s+5 = A(s-1)(s+2) + B(s+2) + C(s-1)^2.$$

Taking $s = 1$, we get $B = 3$. Similarly, taking $s = -2$, we get $C = -\frac{1}{3}$. Now equating coefficients of s^2 , we have $A + C = 0$ and so $A = -C = \frac{1}{3}$. Hence

$$F(s) = \frac{1}{3(s-1)} + \frac{3}{(s-1)^2} - \frac{1}{3(s+2)}.$$

Hence

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{3} e^t + 3t e^t - \frac{1}{3} e^{-2t}.$$

EXAMPLE 6.43

Find $\mathcal{L}^{-1}\left\{\frac{2s+1}{(s+2)^2(s-1)^2}\right\}$.

Solution. Write

$$\begin{aligned} F(s) &= \frac{2s+1}{(s+2)^2(s-1)^2} \\ &= \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2} \end{aligned}$$

and so

$$\begin{aligned} 2s+1 &= A(s+2)(s-1)^2 + B(s-1)^2 \\ &\quad + C(s-1)(s+2)^2 + D(s+2)^2. \end{aligned}$$

Putting $s = 1$ yields $D = \frac{1}{3}$. Putting $s = -2$, we have $B = -\frac{1}{3}$. Comparing coefficients of s^3 , we have $A + C = 0$ and comparing coefficient of s^2 , we have $B + D - 3C = 0$. Thus $C = A = 0$ and so

$$F(s) = \frac{2s-1}{(s+2)^2(s-1)^2} = -\frac{1}{3(s+2)^2} + \frac{1}{3(s-1)^2}.$$

Hence

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{3} t e^{-2t} + \frac{1}{3} t e^t = \frac{1}{3} t (e^t - e^{-2t}).$$

6.4 HEAVISIDE'S EXPANSION THEOREM

Let $F(s) = \frac{P(s)}{Q(s)}$, where $P(s)$ and $Q(s)$ have no common factors and the degree of $Q(s)$ is greater than the degree of $P(s)$. If $Q(s)$ has distinct zeros, then the following theorem gives us the inverse Laplace transform of $F(s)$.

Theorem 6.10. (Heaviside's Expansion Formula). Let $P(s)$ and $Q(s)$ be polynomials in s , where degree of $Q(s)$ is greater than that of $P(s)$. If $Q(s)$ has n distinct zeros $a_1, a_2, \dots, a_n$, then

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{i=1}^n \frac{P(a_i)}{Q'(a_i)} e^{a_i t}.$$

Proof: Since $Q(s)$ has n distinct zeros, it factorizes into n linear factors and so by partial fractions, we have

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s-a_1} + \frac{A_2}{s-a_2} + \dots + \frac{A_i}{s-a_i} + \dots + \frac{A_n}{s-a_n}.$$

Multiplying throughout by $s - a_i$ and letting $s \rightarrow a_i$, we obtain

$$A_i = \lim_{s \rightarrow a_i} \frac{P(s)}{Q(s)} (s - a_i) = \lim_{s \rightarrow a_i} P(s) \left(\frac{s - a_i}{Q(s)} \right)$$

$$= \lim_{s \rightarrow a_i} P(s) \lim_{s \rightarrow a_i} \left(\frac{s - a_i}{Q(s)} \right)$$

$$= P(a_i) \lim_{s \rightarrow a_i} \frac{1}{Q'(s)} \quad (\text{using L'Hospital's rule})$$

$$= \frac{P(a_i)}{Q'(a_i)}.$$

Hence

$$\frac{P(s)}{Q(s)} = \sum_{i=1}^n \frac{P(a_i)}{Q'(a_i)} \frac{1}{(s - a_i)}$$

and so

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{P(s)}{Q(s)} \right\} &= \sum_{i=1}^n \frac{P(a_i)}{Q'(a_i)} \mathcal{L}^{-1} \left\{ \frac{1}{s - a_i} \right\} \\ &= \sum_{i=1}^n \frac{P(a_i)}{Q'(a_i)} e^{a_i t}. \end{aligned}$$

EXAMPLE 6.44

Find $\mathcal{L}^{-1} \left\{ \frac{s}{(s-3)(s^2+4)} \right\}$ using Heaviside's expansion formula.

Solution. Let

$$F(s) = \frac{P(s)}{Q(s)} = \frac{s}{(s-3)(s^2+4)}.$$

Thus

$$\begin{aligned} P(s) &= s, \quad Q(s) = (s-3)(s^2+4) \\ &= s^3 - 3s^2 + 4s - 12 \end{aligned}$$

and

$$Q'(s) = 3s^2 - 6s + 4.$$

Also roots of $Q(s) = 0$ are 3, $2i$, and $-2i$. Therefore, by Heaviside's expansion formula

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{P(s)}{Q(s)} \right\} &= \sum_{i=1}^3 \frac{P(a_i)}{Q'(a_i)} e^{a_i t} = \frac{P(3)}{Q'(3)} e^{3t} \\ &\quad + \frac{P(2i)}{Q'(2i)} e^{2it} + \frac{P(-2i)}{Q'(-2i)} e^{-2it} \\ &= \frac{3}{13} e^{3t} + \frac{2i}{-8-12i} e^{2it} - \frac{2i}{-8+12i} e^{-2it} \\ &= \frac{3}{13} e^{3t} - \frac{(3+2i)}{26} (\cos 2t + i \sin 2t) \\ &\quad + \frac{2i-3}{26} (\cos 2t - i \sin 2t) \\ &= \frac{3}{13} e^{3t} - \frac{3}{13} \cos 2t + \frac{2}{13} \sin 2t. \end{aligned}$$

EXAMPLE 6.45

Find $\mathcal{L}^{-1} \left\{ \frac{3s+7}{(s-3)(s+1)} \right\}$ using Heaviside's expansion formula.

Solution. We have

$$F(s) = \frac{P(s)}{Q(s)} = \frac{3s+7}{(s-3)(s+1)} = \frac{3s+7}{s^2-2s-3}.$$

Here $P(s) = 3s+7$, $Q(s) = s^2-2s-3$, $Q'(s) = 2s-2$, and zeros of $Q(s)$ are 3 and -1 . Hence, by Heaviside's expansion formula,

$$\begin{aligned} \mathcal{L}^{-1} \{F(s)\} &= \sum_{i=1}^2 \frac{P(a_i)}{Q'(a_i)} e^{a_i t} = \frac{P(3)}{Q'(3)} e^{3t} + \frac{P(-1)}{Q'(-1)} e^{-t} \\ &= \frac{16}{4} e^{3t} + \frac{4}{-4} e^{-t} = 4e^{3t} - e^{-t}. \end{aligned}$$

6.5 SERIES METHOD TO DETERMINE INVERSE LAPLACE TRANSFORM

Let $\mathcal{L}\{f(t)\} = F(s)$. In certain situation, we observed that Laplace transform of $f(t)$ can be obtained by expressing $f(t)$ in terms of power series and then by taking transform term-by-term. The same technique proves useful in finding inverse Laplace transforms. Thus, if

$$F(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3} + \dots,$$

then

$$f(t) = a_0 + a_1 + a_2 \frac{t^2}{2!} + \dots,$$

where the series on the right-hand side may be summable to a known function.

EXAMPLE 6.46

Find $\mathcal{L}^{-1} \left\{ \frac{e^{-1/s}}{s} \right\}$.

Solution. Expansion in series yields

$$\begin{aligned} \frac{1}{s} e^{-1/s} &= \frac{1}{s} \left[1 - \frac{1}{s} + \frac{1}{2! s^2} - \frac{1}{3! s^3} + \frac{1}{4! s^4} - \dots \right] \\ &= \frac{1}{s} - \frac{1}{s^2} + \frac{1}{2! s^3} - \frac{1}{3! s^4} + \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n! s^{n+1}}. \end{aligned}$$

Inversion of the series term-by-term yields

$$\begin{aligned} \mathcal{L}^{-1} \{F(s)\} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathcal{L}^{-1} \left\{ \frac{1}{s^{n+1}} \right\} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{(n!)^2} = J_0(2\sqrt{t}), \end{aligned}$$

where $J_0(t)$ is the Bessel function of order zero.

EXAMPLE 6.47

Find $\mathcal{L}^{-1} \left\{ \frac{1}{s} \sin \frac{1}{s} \right\}$.

Solution. We have

$$\begin{aligned} \frac{1}{s} \sin \frac{1}{s} &= \frac{1}{s} \left[\frac{1}{s} - \frac{(1/s)^3}{3!} - \frac{(1/s)^5}{5!} - \frac{(1/s)^7}{7!} + \dots \right] \\ &= \frac{1}{s^2} - \frac{1}{s^4 \cdot 3!} + \frac{1}{s^6 \cdot 5!} - \frac{1}{s^8 \cdot 7!} + \dots \end{aligned}$$

Therefore,

$$\begin{aligned} L^{-1}\left\{\frac{1}{s}\sin\frac{1}{s}\right\} &= L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{3!}L^{-1}\left\{\frac{1}{s^4}\right\} \\ &\quad + \frac{1}{5!}L^{-1}\left\{\frac{1}{s^6}\right\} - \frac{1}{7!}L^{-1}\left\{\frac{1}{s^8}\right\} + \dots \\ &= t - \frac{1}{(3!)^2}t^3 - \frac{1}{(5!)^2}t^5 - \frac{1}{(7!)^2}t^7 + \dots \end{aligned}$$

EXAMPLE 6.48

Find $L^{-1}\{\tan^{-1}\frac{1}{s}\}$.

Solution. We have

$$\begin{aligned} \tan^{-1}\left(\frac{1}{s}\right) &= \frac{1/s}{1} - \frac{(1/s)^3}{3} + \frac{(1/s)^5}{5} - \frac{(1/s)^7}{7} + \dots \\ &= \frac{1}{s} - \frac{1}{3s^3} + \frac{1}{5s^5} - \frac{1}{7s^7} + \dots \end{aligned}$$

Therefore, inversion of the series term-by-term yields

$$\begin{aligned} L^{-1}\left\{\tan^{-1}\left(\frac{1}{s}\right)\right\} &= 1 - \frac{1}{3} \cdot \frac{t^2}{2!} + \frac{1}{5} \cdot \frac{t^4}{4!} - \frac{1}{7} \cdot \frac{t^6}{6!} + \dots \\ &= 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \dots \\ &= \frac{1}{t} \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots \right) = \frac{\sin t}{t}. \end{aligned}$$

EXAMPLE 6.49

Find $L^{-1}\left\{\frac{1}{\sqrt{s+a}}\right\}$.

Solution. Let

$$\begin{aligned} F(s) &= \frac{1}{\sqrt{s+a}} = \frac{1}{\sqrt{s}} \left(1 + \frac{a}{s}\right)^{-1/2} \\ &= \frac{1}{\sqrt{s}} \left[1 - \frac{1}{2} \left(\frac{a}{s}\right) + \frac{\frac{1}{2} \cdot \frac{3}{2}}{2!} \left(\frac{a}{s}\right)^2 - \frac{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2}}{3!} \left(\frac{a}{s}\right)^3 + \dots \right] \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 1.3.5 \dots (2n-1)a^n}{2^n n! s^{n+(1/2)}} |s| > |a|. \end{aligned}$$

Hence

$$\begin{aligned} L^{-1}\{F(s)\} &= \sum_{n=0}^{\infty} \frac{(-1)^n 1.3.5 \dots (2n-1)a^n t^{n-(1/2)}}{2^n n! \Gamma(n+1/2)} \\ &= \frac{1}{\sqrt{t}} \sum_{n=0}^{\infty} \frac{(-1)^n 1.3.5 \dots (2n-1)a^n t^n}{2^n n! \Gamma(n+1/2)}. \end{aligned}$$

But, using $\Gamma(v+1) = v\Gamma(v)$, we have

$$\begin{aligned} \Gamma\left(n + \frac{1}{2}\right) &= \Gamma\left(\frac{1}{2}\right) \left(\frac{1.3.5 \dots (2n-1)}{2^n}\right) \\ &= \sqrt{\pi} \left(\frac{1.3.5 \dots (2n-1)}{2^n}\right). \end{aligned}$$

Hence

$$L^{-1}\{F(s)\} = \frac{1}{\sqrt{t}} \sum_{n=0}^{\infty} \frac{(-1)^n a^n t^n}{\sqrt{\pi} n!} = \frac{1}{\sqrt{\pi t}} e^{-at}.$$

6.6 CONVOLUTION THEOREM

The convolution of two functions plays an important role in a number of physical applications. It is generally expedient to resolve a Laplace transform into the product of two transforms when the inverse transform of both transforms are known.

Definition 6.2. The *convolution* (or *Faltung*) of two given functions $f(t)$ and $g(t)$, $t > 0$ is defined by the integral

$$(f * g)(t) = \int_0^t f(u)g(t-u) du,$$

which, of course, exists if f and g are piecewise continuous.

The integral $\int_0^t f(u)g(t-u) du$ represents a *superposition* of effects of magnitude $f(u)$ occurring at time $t = u$ for which $g(t-u)$ is the *influence function* or *response* of a system to a unit impulse defined at time $t = u$.

If we substitute $v = t - u$, then

$$(f * g)(t) = \int_0^t g(v)f(t-v)dv = (g * f)(t).$$

Hence, the convolution is *commutative*. Also, it follows from the definition that

$$\begin{aligned} a(f * g) &= af * g = f * ag, \text{ a constant} \\ f * (g + h) &= (f * g) + (f * h) \text{ (distributive property)}. \end{aligned}$$

Further, we note that

$$\begin{aligned} [f * (g * h)](t) &= \int_0^t f(u)(g * h)(t-u)du \\ &= \int_0^t f(u) \left(\int_0^{t-u} g(v)h(t-u-v)dv \right) du \end{aligned}$$

$$\begin{aligned}
&= \int_0^t f(u) \int_u^t g(z-u) h(t-z) dz du, \quad v = z - u \\
&= \int_0^t \left(\int_0^z f(u) g(z-u) du \right) h(t-z) dz \\
&= [(f * g) * h](t).
\end{aligned}$$

Hence

$$f * (g * h) = (f * g) * h,$$

and so *convolution is associative*.

EXAMPLE 6.50

Evaluate $\cos t * \sin t$.

Solution. By definition,

$$\begin{aligned}
\cos t * \sin t &= \int_0^t \cos u \sin(t-u) du \\
&= \int_0^t \frac{1}{2} [\sin t + \sin(t-2u)] du \\
&= \frac{1}{2} \sin t [u]_0^t + \frac{1}{4} [\cos(t-2u)]_0^t \\
&= \frac{1}{2} t \sin t + \frac{1}{4} [\cos(-t) - \cos t] = \frac{1}{2} t \sin t.
\end{aligned}$$

EXAMPLE 6.51

Evaluate $e^t * t$.

Solution. By definition of convolution,

$$\begin{aligned}
e^t * t &= \int_0^t e^u (t-u) du \\
&= [t e^u]_0^t - [(u e^u - e^u)]_0^t = e^t - t - 1.
\end{aligned}$$

Convolution is very significant in the sense that the Laplace transform of the convolution of two functions is the product of their respective Laplace transforms. We prove this assertion in the form of the following theorem.

Theorem 6.11. (Convolution Theorem). If f and g are piecewise continuous on $[0, \infty)$ and of exponential order γ , then

$$L\{(f * g)(t)\} = L\{f(t)\} \cdot L\{g(t)\} = F(s)G(s),$$

or equivalently,

$$L^{-1}\{F(s)G(s)\} = (f * g)(t).$$

Proof: Using the definition of the Laplace transform, we have

$$\begin{aligned}
L\{(f * g)(t)\} &= \int_0^\infty e^{-st} \int_0^t f(u) g(t-u) du dt \\
&= \int_0^\infty \int_0^t e^{-st} f(u) g(t-u) du dt.
\end{aligned}$$

The domain of this double integral is an infinite wedge bounded by $t = 0$, $t = \infty$, $u = 0$, and $u = t$ as displayed in Figure 6.1. Due to hypotheses on f and g , the double integral on the right hand side converges absolutely and so we can perform change of order of integration.

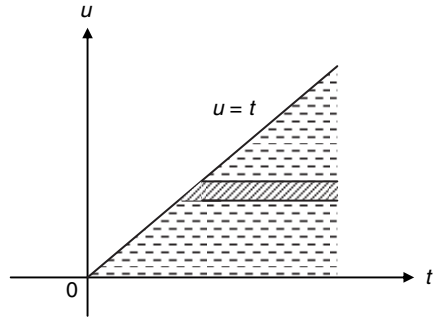


Figure 6.1 Region of Integration

We take a strip parallel to the axis of t . Then t varies from u to ∞ and u varies from 0 to ∞ . Hence

$$\begin{aligned}
L\{f(t) * g(t)\} &= \int_0^\infty \int_u^\infty e^{-st} f(u) g(t-u) dt du \\
&= \int_0^\infty \int_u^\infty e^{-su} \cdot e^{-s(t-u)} f(u) g(t-u) dt du \\
&= \int_0^\infty e^{-su} f(u) \int_u^\infty e^{-s(t-u)} g(t-u) dt du.
\end{aligned}$$

Implementing the change of variable $t-u = v$ in the inner integral, we get

$$\begin{aligned}
L\{f(t) * g(t)\} &= \int_0^\infty e^{-su} f(u) \int_0^\infty e^{-sv} g(v) dv du \\
&= \int_0^\infty e^{-su} f(u) G(s) du \\
&= G(s) \int_0^\infty e^{-su} f(u) du \\
&= G(s) F(s) = F(s) G(s).
\end{aligned}$$

Consequently,

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t).$$

EXAMPLE 6.52

Find $\mathcal{L}\{e^{at} * e^{bt}\}$.

Solution. By Convolution theorem,

$$\mathcal{L}\{e^{at} * e^{bt}\} = \mathcal{L}\{e^{at}\} \cdot \mathcal{L}\{e^{bt}\} = \frac{1}{(s-a)(s-b)}.$$

EXAMPLE 6.53

Find $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\}$.

Solution. By Convolution theorem

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\} = e^{at} * e^{bt}$$

$$\int_0^t e^{ut} e^{b(t-u)} du = \frac{e^{at} - e^{bt}}{a-b}, a \neq b.$$

EXAMPLE 6.54

Using Convolution theorem, find $\mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}$.

Solution. Let $F(s) = \frac{1}{s^3}$, $G(s) = \frac{1}{s^2+1}$ and so $f(t) = \frac{1}{2}t^2$ and $g(t) = \sin t$.

By Convolution theorem and integration by parts, we have

$$\begin{aligned} \mathcal{L}^{-1}\{F(s)G(s)\} \\ &= f * g = g * f = \frac{1}{2} \int_0^t \sin u (t-u)^2 du \\ &= \frac{1}{2} \left\{ [-(t-u)^2 \cos u]_0^t - \int_0^t 2(t-u) \cos u du \right\} \\ &= \frac{1}{2} \left[t^2 - 2 \left\{ [(t-u) \sin u]_0^t + \int_0^t \sin u du \right\} \right] \\ &= \frac{1}{2} [t^2 + 2 \cos t - 2]. \end{aligned}$$

EXAMPLE 6.55

Using Convolution theorem, find $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-1)}\right\}$.

Solution. Let $F(s) = \frac{1}{s^2}$ and $G(s) = \frac{1}{s-1}$. Then, $f(t) = t$ and $g(t) = e^t$. Therefore, by Convolution theorem,

$$\begin{aligned} \mathcal{L}^{-1}\{F(s)G(s)\} &= f * g = t * e^t \\ &= \int_0^t e^u (t-u) du = e^t - t - 1. \end{aligned}$$

EXAMPLE 6.56

Use Convolution theorem to find $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$. (see also Example 6.33).

Solution. Let

$$\frac{s}{(s^2+a^2)^2} = \frac{s}{s^2+a^2} \cdot \frac{1}{s^2+a^2} = F(s)G(s).$$

Then

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \cos at \quad \text{and}$$

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \frac{\sin at}{a}.$$

By Convolution theorem,

$$\begin{aligned} \mathcal{L}^{-1}\{F(s)G(s)\} &= f * g = \int_0^t \cos au \frac{\sin a(t-u)}{a} du \\ &= \frac{1}{2a} \int_0^t [\sin at + \sin a(t-2u)] du \\ &= \frac{1}{2a} \sin at [u]_0^t + \frac{1}{4a^2} [\cos a(t-2u)]_0^t \\ &= \frac{1}{2a} t \sin at + \frac{1}{4a^2} [\cos a(-t) - \cos at] \\ &= \frac{1}{2a} t \sin at. \end{aligned}$$

EXAMPLE 6.57

Using Convolution theorem, find $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\}$.

Solution. Let

$$\frac{1}{s^2(s^2+a^2)} = \frac{1}{s^2} \cdot \frac{1}{(s^2+a^2)} = F(s)G(s).$$

Then

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = t \quad \text{and}$$

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \frac{\sin at}{a}.$$

By Convolution theorem,

$$\begin{aligned}
 L^{-1}\{F(s)G(s)\} &= \frac{1}{a} \sin at * t = \frac{1}{a} \int_0^t (t-u) \sin au \, du \\
 &= \frac{1}{a} \left\{ -\left[\frac{1}{a} (t-u) \cos au \right]_0^t - \int_0^t \frac{\cos au}{a} \, du \right\} \\
 &= \frac{1}{a} \left\{ \frac{t}{a} - \left[\frac{\sin au}{a^2} \right]_0^t \right\} = \frac{1}{a} \left[\frac{t}{a} - \frac{\sin at}{a^2} \right] \\
 &= \frac{1}{a^2} \left[t - \frac{1}{a} \sin at \right].
 \end{aligned}$$

EXAMPLE 6.58

Using Convolution theorem, find $L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$.

Solution. Let

$$\frac{1}{s^2(s+1)^2} = \frac{1}{s^2} \cdot \frac{1}{(s+1)^2} = F(s)G(s).$$

Then

$$f(t) = L^{-1}\{F(s)\} = t \quad \text{and}$$

$$g(t) = L^{-1}\{G(s)\} = t e^{-t}$$

So, by Convolution theorem, we have

$$L^{-1}\{F(s)G(s)\} = g * f$$

$$\begin{aligned}
 &= \int_0^t (ue^{-u}) (t-u) \, du = \int_0^t (tu - u^2) e^{-u} \, du \\
 &= -[(tu - u^2)e^{-u}]_0^t - \int_0^t -(t-2u)e^{-u} \, du \\
 &= \int_0^t (t-2u) e^{-u} \, du = -[(t-2u)e^{-u}]_0^t - 2 \int_0^t e^{-u} \, du \\
 &= te^{-t} + t - 2 \left[\frac{e^{-u}}{-1} \right]_0^t = te^{-t} + t + 2e^{-t} - 2 \\
 &= (t+2)e^{-t} + t - 2.
 \end{aligned}$$

EXAMPLE 6.59

Using Convolution theorem, find $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$.

Solution. Let

$$\frac{1}{(s^2+a^2)^2} = \frac{1}{(s^2+a^2)} \cdot \frac{1}{(s^2+a^2)} = F(s)G(s).$$

Then

$$f(t) = g(t) = \frac{\sin at}{a}.$$

Therefore, by Convolution theorem,

$$\begin{aligned}
 L^{-1}\{F(s)G(s)\} &= f * g = \frac{1}{a^2} \int_0^t \sin au \sin a(t-u) \, du \\
 &= \frac{1}{a^2} \int_0^t \frac{1}{2} [\cos a(2u-t) - \cos at] \, du \\
 &= \frac{1}{2a^2} \left[\int_0^t \cos a(2u-t) \, du - \int_0^t \cos at \, du \right] \\
 &= \frac{1}{2a^2} \left\{ \left[\frac{\sin a(2u-t)}{2a} \right]_0^t - \cos at [u]_0^t \right\} \\
 &= \frac{1}{2a^2} \left[\frac{\sin at}{2a} - \frac{\sin a(-t)}{2a} - t \cos at \right] \\
 &= \frac{1}{2a^3} [\sin at - at \cos at].
 \end{aligned}$$

EXAMPLE 6.60

Use Convolution theorem to find $L^{-1}\left\{\frac{a^2}{s(s+a)^2}\right\}$.

Solution. Let

$$\frac{a^2}{s(s+a)^2} = \frac{1}{s} \cdot \frac{a^2}{(s+a)^2} = F(s)G(s).$$

Then

$$f(t) = L^{-1}\left\{\frac{1}{s}\right\} = 1.$$

To find $g(t) = L^{-1}\left\{\frac{a^2}{s(s+a)^2}\right\}$, we note that

$$\frac{d}{ds} \left(\frac{1}{s+a} \right) = \frac{-1}{(s+a)^2}.$$

Therefore,

$$L^{-1}\left\{\frac{-1}{(s+a)^2}\right\} = -t L^{-1}\left\{\frac{1}{s+a}\right\} = -t e^{-at}$$

and so

$$L^{-1}\left\{\frac{a^2}{(s+a)^2}\right\} = a^2 t e^{-at}.$$

Application of Convolution theorem yields

$$\begin{aligned}
 \mathcal{L}^{-1}\{F(s)G(s)\} &= \int_0^t a^2 u e^{-au} du = a^2 \int_0^t u e^{-au} du \\
 &= a^2 \left\{ \left[\frac{u e^{-au}}{-a} \right]_0^t - \int_0^t \frac{e^{-au}}{-a} du \right\} \\
 &= a^2 \left\{ \frac{te^{-at}}{-a} - \left[\frac{e^{-au}}{a^2} \right]_0^t \right\} \\
 &= -at e^{-at} - e^{-at} + 1 \\
 &= 1 - e^{-at}(at + 1).
 \end{aligned}$$

EXAMPLE 6.61

Find $\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s(s-1)}}\right\}$ using Convolution theorem.

Solution. Let

$$\frac{1}{\sqrt{s(s-1)}} = \frac{1}{\sqrt{s}} \cdot \frac{1}{s-1} = F(s)G(s).$$

Then

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{\sqrt{\pi t}} \quad \text{and}$$

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = e^t.$$

Hence Convolution theorem yields

$$\begin{aligned}
 \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s(s+1)}}\right\} &= \mathcal{L}^{-1}\{F(s)G(s)\} = f * g \\
 &= \int_0^t \frac{1}{\sqrt{\pi u}} e^{(t-u)} du = \frac{e^t}{\sqrt{\pi}} \int_0^t \frac{e^{-u}}{\sqrt{u}} du \\
 &= \frac{2e^t}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-v^2} dv, u = v^2 \\
 &= e^t \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-v^2} dv = e^t \operatorname{erf} \sqrt{t},
 \end{aligned}$$

where $\operatorname{erf} \sqrt{t}$ is the *error function*.

EXAMPLE 6.62

Using Convolution theorem, establish Euler's formula for Beta function:

$$\beta(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}.$$

Solution. We know that Beta function is defined by

$$\beta(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx.$$

Let

$$f(t) = t^{a-1}, g(t) = t^{b-1}, a, b > 0.$$

Then

$$(f * g)(t) = \int_0^t u^{a-1} (t-u)^{b-1} du.$$

Putting $u = vt$, we get

$$\begin{aligned}
 (f * g)(t) &= t^{a+b-1} \int_0^1 v^{a-1} (1-v)^{b-1} dv \\
 &= t^{a+b-1} \beta(a, b).
 \end{aligned}$$

But by Convolution theorem,

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\} \mathcal{L}\{g(t)\}.$$

Therefore,

$$\begin{aligned}
 \mathcal{L}\{t^{a+b-1} \beta(a, b)\} &= \mathcal{L}\{t^{a-1}\} \mathcal{L}\{t^{b-1}\} \\
 &= \frac{\Gamma(a)}{s^a} \cdot \frac{\Gamma(b)}{s^b} = \frac{\Gamma(a) \Gamma(b)}{s^{a+b}}
 \end{aligned}$$

and so

$$\begin{aligned}
 t^{a+b-1} \beta(a, b) &= \mathcal{L}^{-1}\left\{\frac{\Gamma(a) \Gamma(b)}{s^{a+b}}\right\} \\
 &= \Gamma(a) \Gamma(b) \mathcal{L}^{-1}\left\{\frac{1}{s^{a+b}}\right\} \\
 &= \Gamma(a) \Gamma(b) \frac{t^{a+b-1}}{\Gamma(a+b)},
 \end{aligned}$$

which implies

$$\beta(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)},$$

the required Euler's Formula for Beta function.

Remark 6.4. It also follows from Example 6.62 that

$$\mathcal{L}\left\{\int_0^t u^{a-1} (t-u)^{b-1} du\right\} = \frac{\Gamma(a) \Gamma(b)}{s^{a+b}}.$$

EXAMPLE 6.63

Using Convolution theorem, show that

$$J_0(t) = \frac{2}{\pi} \int_0^1 \frac{\cos tx}{\sqrt{1-x^2}} dx,$$

where J_0 represents Bessel's function of order zero.

Solution. We know that

$$L\{J_0(t)\} = \frac{1}{\sqrt{s^2 + 1}} = \frac{1}{\sqrt{s+i}} \cdot \frac{1}{\sqrt{s-i}}$$

and so

$$J_0(t) = L^{-1}\left\{\frac{1}{\sqrt{s+i}} \cdot \frac{1}{\sqrt{s-i}}\right\} = L^{-1}\{F(s)G(s)\},$$

where

$$F(s) = \frac{1}{\sqrt{s+i}}, G(s) = \frac{1}{\sqrt{s-i}}.$$

Since

$$L^{-1}\left\{\frac{1}{\sqrt{s+a}}\right\} = \frac{1}{\sqrt{\pi t}} e^{-at},$$

we have

$$f(t) = \frac{1}{\sqrt{\pi t}} e^{-it} \quad \text{and} \quad g(t) = \frac{1}{\sqrt{\pi t}} e^{it}.$$

Hence, by Convolution theorem

$$\begin{aligned} J_0(t) &= L^{-1}\{F(s)G(s)\} = f * g \\ &= \int_0^t \frac{1}{\sqrt{\pi u}} e^{-iu} \left(\frac{1}{\sqrt{\pi(t-u)^{1/2}}} e^{i(t-u)} \right) du \\ &= \frac{1}{\pi} \int_0^t \frac{e^{i(t-2u)}}{\sqrt{u(t-u)}} du. \end{aligned}$$

Putting $u = tv$, we get

$$J_0(t) = \frac{1}{\pi} \int_0^1 \frac{e^{it(1-2v)}}{\sqrt{v(1-v)}} dv.$$

Now putting $x = 1 - 2v$, we have

$$\begin{aligned} J_0(t) &= \frac{1}{\pi} \int_{-1}^1 \frac{e^{itx}}{\sqrt{1-x^2}} dx \\ &= \frac{1}{\pi} \int_{-1}^1 \frac{\cos tx}{\sqrt{1-x^2}} dx + \frac{i}{\pi} \int_{-1}^1 \frac{\sin tx}{\sqrt{1-x^2}} dx \\ &= \frac{2}{\pi} \int_0^1 \frac{\cos tx}{\sqrt{1-x^2}} dx, \end{aligned}$$

the second integral vanishes because $\sin x$ is odd.

6.7 COMPLEX INVERSION FORMULA

The complex inversion formula is a technique for computing directly the inverse of a Laplace transform. For this technique, we require the parameter s to be a complex variable. The contour integration is the main tool applied in this technique.

Let f be a continuous function possessing a Laplace transform and defined in $(-\infty, \infty)$ such that $f(t) = 0$ for $t < 0$. Let $s = x + iy$ be a complex variable. Then

$$\begin{aligned} L\{f(t)\} &= F(s) = \int_0^\infty e^{-st} f(t) dt \\ &= \int_{-\infty}^\infty e^{-(x+iy)t} f(t) dt \\ &= \int_{-\infty}^\infty e^{-iyt} (e^{-xt} f(t)) dt. \end{aligned}$$

The integral on the right is the Fourier transform of the function $e^{-xt} f(t)$. Thus the Laplace transform of $f(t)$ is the Fourier transform of the function $e^{-xt} f(t)$.

Suppose that f and its derivative f' are piecewise continuous on $(-\infty, \infty)$, that is, both f and f' are continuous in any finite interval except possibly for a finite number of jump discontinuities. Suppose, further, that f is absolutely integrable, that is, $\int_{-\infty}^\infty |f(t)| dt < \infty$. Then Fourier integral theorem asserts that

$$\frac{f(t+) + f(t-)}{2} = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(u) e^{is(t-u)} du ds,$$

where the integration with respect to s is in the Cauchy principal value sense.

If f were continuous on $(-\infty, \infty)$, then $\frac{f(t+) + f(t-)}{2}$ converts to $f(t)$ in the above expression. Further, in terms of Fourier transform of $f(t)$, the above expression becomes

$$\frac{f(t+) + f(t-)}{2} = \frac{1}{2\pi} \int_{-\infty}^\infty F(s) e^{-ist} ds,$$

where $F(s) = \int_{-\infty}^\infty e^{-isu} f(u) du$ is Fourier transform of $f(t)$.

Theorem 6.12. (Complex Inversion Formula). Let f be continuous on $[0, \infty)$, $f(t) = 0$ for $t < 0$, f be of exponential order γ and f' be piecewise continuous on $[0, \infty)$. If $L\{f(t)\} = F(s)$ and the real number γ is so chosen that all the singularities of $F(s)$ lie in the left of the line $\operatorname{Re}(s) > \gamma$, then

$$f(t) = L^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} F(s) ds, \quad t > 0.$$

Proof: By definition of Laplace transform,

$$F(s) = \int_0^{\infty} e^{-su} f(u) du.$$

We have then

$$\lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{\gamma-iT}^{\gamma+iT} e^{st} F(s) ds = \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{\gamma-iT}^{\gamma+iT} \int_0^{\infty} e^{st} e^{-su} f(u) du ds.$$

Let $s = \gamma + iy$ so that $ds = i dy$. Therefore,

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{\gamma-iT}^{\gamma+iT} e^{st} F(s) ds &= \lim_{T \rightarrow \infty} \frac{1}{2\pi} e^{\gamma t} \int_{-T}^T e^{iyt} dy \int_0^{\infty} e^{-iyu} e^{-\gamma u} f(u) du \\ &= \frac{1}{2\pi} e^{\gamma t} \int_{-\infty}^{\infty} \int_0^{\infty} e^{iy(t-u)} (e^{-\gamma u} f(u)) du dy. \end{aligned}$$

The given hypothesis and Theorem 6.1 imply that $L\{f(t)\}$ converges absolutely for $\operatorname{Re}(s) > \gamma$, that is,

$$\int_0^{\infty} |e^{-xt} f(t)| dt = \int_{-\infty}^{\infty} e^{-xt} |f(t)| dt < \infty, x > \gamma.$$

Thus $e^{-xt} f(t)$ is absolutely integrable and so Fourier integral theorem asserts that

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{-\gamma u} f(u)) e^{iy(t-u)} du dy \\ = e^{-\gamma t} f(t) \text{ for } t > 0. \end{aligned}$$

Hence

$$\lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{\gamma-iT}^{\gamma+iT} e^{st} F(s) ds = \frac{1}{2\pi} \cdot 2\pi e^{-\gamma t} f(t) e^{\gamma t}$$

and so

$$f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} F(s) ds, \quad t > 0.$$

Remark 6.5. The expression

$$f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} F(s) ds$$

is called *complex inversion formula*, *Bromwich integral formula*, *Fourier-Mellin inversion formula*, or *fundamental theorem of Laplace transform*. In practice, the integral in the above expression is evaluated by considering the contour integration along the contour Γ_R , ABCDEA, known as *Bromwich contour* shown in the Figure 6.2.

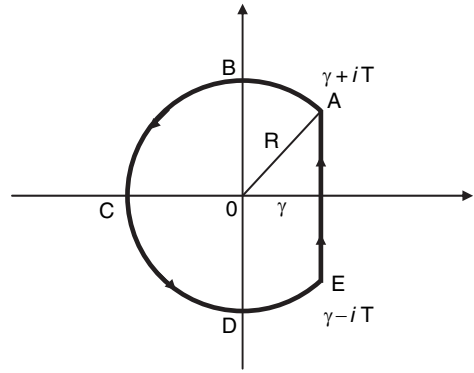


Figure 6.2 Bromwich Contour

The vertical line at γ is known as *Bromwich line*. Thus the contour Γ_R consists of arc C_R (ABCDE) of radius R and centre at the origin and the Bromwich line EA . Thus,

$$\begin{aligned} \frac{1}{2\pi i} \int_{\Gamma_R} e^{st} F(s) ds &= \frac{1}{2\pi i} \int_{C_R} e^{st} F(s) ds \\ &\quad + \frac{1}{2\pi i} \int_{EA} e^{st} F(s) ds. \end{aligned}$$

Since $F(s)$ is analytic for $\operatorname{Re}(s) = x > \gamma$, all singularities of $F(s)$ must lie to the left of the Bromwich line. Thus, by Cauchy residue theorem, we have

$$\frac{1}{2\pi i} \int_{\Gamma_R} e^{st} F(s) ds = \sum_{k=1}^n \operatorname{Res}(z_k),$$

where $\operatorname{Res}(z_k)$ is the residue of the function at the pole $s = z_k$. Since $e^{st} \neq 0$, multiplying $F(s)$ by e^{st} does not affect the status of the poles z_k of $F(s)$.

If we can show that

$$\lim_{R \rightarrow \infty} \int_{C_R} e^{st} F(s) ds = 0,$$

then letting $R \rightarrow \infty$, we get

$$f(t) = \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{\gamma-iT}^{\gamma+iT} e^{ts} F(s) ds = \sum_{k=1}^n \text{Res}(z_k),$$

and so inverse function f can be determined.

The following theorem shows that *Laplace transform is one-to-one*.

Theorem 6.13. Let $f(t)$ and $g(t)$ be two piecewise smooth functions of exponential order and let $F(s)$ and $G(s)$ be the Laplace transforms of $f(t)$ and $g(t)$ respectively. If $F(s) = G(s)$ in a half-plane $\text{Re}(s) > \gamma$, then $f(t) = g(t)$ at all points where f and g are continuous.

Proof: Suppose f and g are continuous at $t \in \mathbb{R}$. By complex inversion formula, we have

$$f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} F(s) ds,$$

$$g(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} G(s) ds.$$

Since $F(s) = G(s)$, it follows that $f(t) = g(t)$. Thus $L\{f(t)\} = L\{g(t)\}$ implies that $f(t) = g(t)$ and so Laplace operator is one-to-one.

Generally, we see that most of the Laplace transforms satisfy the growth restriction

$$|F(s)| \leq \frac{M}{|s|^p}$$

for all sufficiently large values of $|s|$ and some $p > 0$. Obviously, $F(s) \rightarrow 0$ as $|s| \rightarrow \infty$. Therefore, the following result (stated without proof) is helpful.

Theorem 6.14. Let for s on C_R , $F(s)$ satisfies the growth restriction

$$|F(s)| \leq \frac{M}{|s|^p} \text{ for } p > 0, \text{ all } R > R_0.$$

Then

$$\lim_{R \rightarrow \infty} \int_{C_R} e^{st} F(s) ds = 0, t > 0.$$

EXAMPLE 6.64

Show that $F(s) = \frac{s}{s^2 - a^2}$ satisfies growth restriction condition.

Solution. We have

$$F(s) = \frac{s}{s^2 - a^2},$$

and so

$$|F(s)| \leq \frac{|s|}{|s^2 - a^2|} \leq \frac{|s|}{|s|^2 - |a|^2}.$$

If $|s| \geq 2|a|$, then $|a|^2 \leq \frac{|s|^2}{4}$ and so $|s|^2 - |a|^2 \geq \frac{3}{4}|s|^2$ and we have

$$|F(s)| \leq \frac{4/3}{|s|}.$$

EXAMPLE 6.65

Find the Laplace transform of $f(t) = \cosh at$ and verify the inversion formula.

Solution. By Example 6.6, we have $F(s) = \frac{s}{s^2 - a^2}$. The function $F(s)$ is analytic except at poles $s = a$ and $s = -a$. The Bromwich contour is shown in the Figure 6.3.

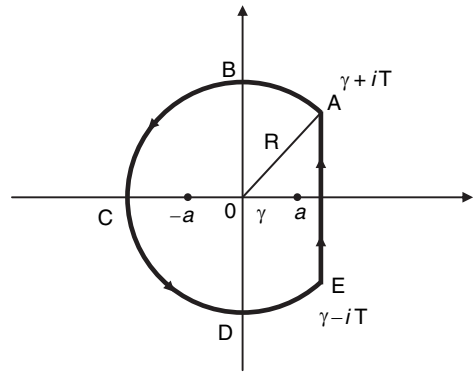


Figure 6.3

By inversion formula, we have

$$\begin{aligned} L^{-1} \left\{ \frac{s}{s^2 - a^2} \right\} &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} \left(\frac{s}{s^2 - a^2} \right) ds \\ &= \frac{1}{2\pi i} \int_{\Gamma_R} \frac{se^{st}}{s^2 - a^2} ds. \end{aligned}$$

Further, $F(s)$ satisfies growth restriction condition. Therefore, integral over contour C_R (arc ABCDE) tends to zero as $R \rightarrow \infty$. Now

$$\text{Res}(a) = \lim_{s \rightarrow a} (s-a) e^{st} F(s) = \lim_{s \rightarrow a} \frac{s e^{st}}{s+a} = \frac{a e^{at}}{2a},$$

$$\text{Res}(-a) = \lim_{s \rightarrow (-a)} (s+a) e^{st} F(s) = \lim_{s \rightarrow (-a)} \frac{s e^{st}}{s-a} = \frac{e^{-at}}{-2a}(-a).$$

Hence

$$L^{-1}\{F(s)\} = \frac{e^{at} + e^{-at}}{2} = \cosh at,$$

and so inversion formula is verified.

EXAMPLE 6.66

Find $L^{-1}\left\{\frac{\omega}{s^2 + \omega^2}\right\}$, $s > 0$ using inversion formula.

Solution. We have

$$F(s) = \frac{\omega}{s^2 + \omega^2}, s > 0$$

The function $F(s)$ has two simple poles at $s = \pm i\omega$.

By inversion formula

$$\begin{aligned} L^{-1}\{F(s)\} &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} \left(\frac{\omega}{s^2 + \omega^2} \right) ds \\ &= \frac{\omega}{2\pi i} \int_{\Gamma_R} \frac{e^{st}}{s^2 + \omega^2} ds, \end{aligned}$$

where Γ_R is the Bromwich contour shown in the Figure 6.4:

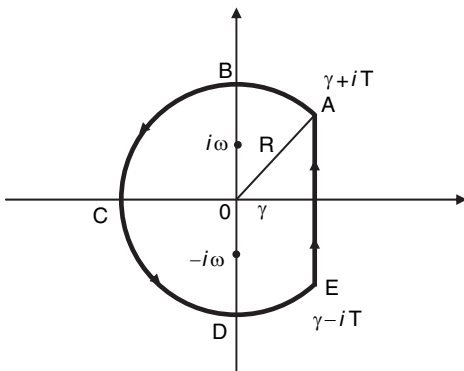


Figure 6.4

We have

$$\text{Res}(i\omega) = \lim_{s \rightarrow i\omega} (s-i\omega) e^{st} F(s) = \lim_{s \rightarrow i\omega} \frac{e^{st}}{s+i\omega} = \frac{e^{i\omega t}}{2i\omega},$$

$$\text{Res}(-i\omega) = \lim_{s \rightarrow -i\omega} (s+i\omega) e^{st} F(s) = \lim_{s \rightarrow -i\omega} \frac{e^{st}}{s-i\omega} = \frac{e^{-i\omega t}}{-2i\omega}.$$

Further, let $s = \gamma + R e^{i\theta}$, $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$. Then the integral over the contour C_R yields

$$\left| \int_{\pi/2}^{3\pi/2} \frac{e^{\gamma t} e^{t R(\cos \theta + i \sin \theta)} R e^{i \theta}}{(\gamma + R e^{i \theta})^2 + \omega^2} d\theta \right| \leq \frac{\pi e^{\gamma t} R}{R^2 - \gamma^2 - \omega^2} \rightarrow 0 \text{ as } R \rightarrow \infty.$$

Hence

$$\begin{aligned} L^{-1}\{F(s)\} &= \frac{\omega}{2\pi i} 2\pi i (\text{sum of residue at } \pm i\omega) \\ &= \frac{e^{i\omega t} - e^{-i\omega t}}{2i} = \sin \omega t. \end{aligned}$$

EXAMPLE 6.67

Find $L^{-1}\left\{\frac{1}{s(s^2 + a^2)}\right\}$ using inversion formula.

Solution. We have

$$F(s) = \frac{1}{s(s^2 + a^2)} = \frac{1}{s(s-ia)^2(s+ia)^2}.$$

Thus $F(s)$ has a simple pole at $s = 0$ and a pole of order 2 at $s = \pm ia$. Further, $F(s)$ satisfies growth restriction condition. Therefore, integral over the contour C_R goes to zero as $R \rightarrow \infty$. Further,

$$\text{Res}(0) = \lim_{s \rightarrow 0} s e^{st} F(s) = \lim_{s \rightarrow 0} \frac{e^{st}}{(s^2 + a^2)^2} = \frac{1}{a^4}$$

$$\begin{aligned} \text{Res}(ia) &= \lim_{s \rightarrow ia} \frac{d}{ds} (s-ia)^2 e^{ts} F(s) \\ &= \lim_{s \rightarrow ia} \frac{d}{ds} \left(\frac{e^{ts}}{s(s+ia)^2} \right) = \frac{it}{4a^3} e^{iat} - \frac{e^{iat}}{2a^4}, \end{aligned}$$

$$\begin{aligned} \text{Res}(-ia) &= \lim_{s \rightarrow -ia} \frac{d}{ds} (s+ia)^2 e^{ts} F(s) \\ &= \lim_{s \rightarrow -ia} \frac{d}{ds} \left(\frac{e^{ts}}{s(s-ia)^2} \right) = \frac{-it}{4a^3} e^{-iat} - \frac{e^{-iat}}{2a^4}. \end{aligned}$$

Hence

$$\begin{aligned} f(t) &= \frac{1}{a^4} + \frac{it}{4a^3} (e^{iat} - e^{-iat}) - \frac{1}{2a^4} (e^{iat} + e^{-iat}) \\ &= \frac{1}{a^4} \left(1 - \frac{a}{2} t \sin at - \cos at \right). \end{aligned}$$

EXAMPLE 6.68

Find $L^{-1}\left\{\frac{s}{(s+1)^3(s-1)^2}\right\}$ using inversion formula.

Solution. We have

$$F(s) = \frac{s}{(s+1)^3(s-1)^2}.$$

The function $F(s)$ has poles of multiplicity 2 at $s = 1$ and poles of multiplicity 3 at $s = -1$. Residue at $s = 1$ is given by

$$\begin{aligned}\text{Res}(1) &= \lim_{s \rightarrow 1} \frac{d}{ds} \left[\frac{(s-1)^2 s e^{st}}{(s+1)^3 (s-1)^2} \right] \\ &= \lim_{s \rightarrow 1} \frac{d}{ds} \left(\frac{s e^{st}}{(s+1)^3} \right) \\ &= \frac{1}{16} e^t (2t - 1),\end{aligned}$$

$$\begin{aligned}\text{Res}(-1) &= \lim_{s \rightarrow -1} \frac{1}{2!} \frac{d^2}{ds^2} \left[\frac{(s+1)^3 s e^{st}}{(s+1)^3 (s-1)^2} \right] \\ &= \lim_{s \rightarrow -1} \frac{1}{2} \frac{d^2}{ds^2} \left(\frac{s e^{st}}{(s-1)^2} \right) \\ &= \frac{1}{16} e^{-t} (1 - 2t^2).\end{aligned}$$

The value of the integral over the contour C_R tends to zero as $R \rightarrow \infty$. Hence

$$\begin{aligned}f(t) &= \text{sum of residues at the poles} \\ &= \frac{1}{16} e^{-t} (1 - 2t^2) + \frac{1}{16} e^t (2t - 1).\end{aligned}$$

EXAMPLE 6.69

Derive Heaviside's expansion formula using complex inversion formula.

Solution. Let $F(s) = \frac{P(s)}{Q(s)}$, where $P(s)$ and $Q(s)$ are polynomials having no common factors (roots) and degree of $Q(s)$ is greater than the degree of $P(s)$. Suppose $Q(s)$ has simple zeros at $z_1, z_2, \dots, z_m$. If degree of $P(s)$ and $Q(s)$ are n and m , respectively, then for $a_0 \neq 0, b_m \neq 0$,

$$\begin{aligned}\frac{P(s)}{Q(s)} &= \frac{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0}{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0} \\ &= \frac{a_n + \frac{a_{n+1}}{s} + \dots + \frac{a_0}{s^n}}{s^{m-n} \left(b_m + \frac{b_{m-1}}{s} + \dots + \frac{b_0}{s^m} \right)}.\end{aligned}$$

For sufficiently large $|s|$, we have

$$\left| a_n + \frac{a_{n-1}}{s} + \dots + \frac{a_0}{s^n} \right| \leq |a_n| + |a_{n-1}| + \dots + |a_0| = C_1 \text{ say,}$$

$$\left| b_m + \frac{a_{m-1}}{s} + \dots + \frac{b_0}{s^m} \right| \geq |b_m| - \frac{|b_{m-1}|}{|s|} - \dots - \frac{|b_0|}{|s|^m} \geq \frac{|b_m|}{2} = C_2 \text{ say,}$$

and so

$$|F(s)| = \left| \frac{P(s)}{Q(s)} \right| \leq \frac{C_1/C_2}{|s|^{m-n}}.$$

Thus, $F(s)$ satisfies growth restriction condition. Further,

$$\begin{aligned}\text{Res}(z_n) &= \lim_{s \rightarrow z_n} (s - z_n) e^{st} F(s) \\ &= \lim_{s \rightarrow z_n} \frac{e^{st} P(s)}{\frac{Q(s) - Q(z_n)}{s - z_n}}, \quad \text{since } Q(z_n) = 0 \\ &= \frac{e^{z_n t} P(z_n)}{Q'(z_n)}.\end{aligned}$$

Hence, by inversion formula, we have

$$f(z) = \sum_{n=1}^m \frac{P(z_n)}{Q'(z_n)} e^{z_n t},$$

which is the required Heaviside's expansion formula.

EXAMPLE 6.70

Find $L^{-1} \left(\frac{1}{s(1+e^{as})} \right)$, using complex inversion formula.

Solution. Let $F(s) = \frac{1}{s(1+e^{as})}$. Then $F(s)$ has a simple pole at $s = 0$. Further, $1 + e^{as} = 0$ yields

$$e^{as} = -1 = e^{(2n-1)\pi i}, \quad n = 0, \pm 1, \pm 2, \dots$$

and so $s_n = \left(\frac{2n-1}{a} \right) \pi i$, $n = 0, \pm 1, \pm 2, \dots$ are also poles of $F(s)$. Also, $\frac{d}{ds} (1 + e^{as})_{s=s_n} = -a \neq 0$. Therefore, s_n are simple poles. Now

$$\text{Res}(0) = \lim_{s \rightarrow 0} s e^{st} F(s) = \frac{1}{2},$$

$$\begin{aligned}\text{Res} \left[\left(\frac{2n-1}{a} \right) \pi i \right] &= \lim_{s \rightarrow s_n} (s - s_n) e^{st} F(s) \\ &= \lim_{s \rightarrow s_n} \frac{(s - s_n) e^{st}}{s(1 + e^{as})} \left(\frac{0}{0} \text{ form} \right) \\ &= \lim_{s \rightarrow s_n} \frac{e^{st} + t e^{st} (s - s_n)}{a s e^{as} + 1 + e^{as}}, \text{ by L'Hospital rule} \\ &= \frac{e^{tsn}}{a s_n e^{asn}} = -\frac{e^t \left(\frac{2n-1}{a} \right) \pi i}{(2n-1) \pi i}.\end{aligned}$$

Also, it can be shown that $F(s)$ satisfies growth restriction condition. Hence, by inversion formula,

at the points of continuity of f , we have

$f(t)$ = sum of residues at the poles

$$\begin{aligned} &= \frac{1}{2} - \sum_{n=-\infty}^{\infty} \frac{1}{(2n-1)\pi i} e^{t\left(\frac{2n-1}{a}\right)\pi i} \\ &= \frac{1}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin\left(\frac{(2n-1)}{a}\right) \pi t. \end{aligned}$$

EXAMPLE 6.71

Find $L^{-1}\{e^{-a\sqrt{s}}\}$, $a > 0$.

Solution. We know (Exercise 12) that

$$L^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\} = \operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right),$$

that is,

$$L\left\{\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)\right\} = e^{-\frac{a\sqrt{s}}{s}}.$$

Therefore,

$$\begin{aligned} L\left\{\frac{d}{dt}\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)\right\} &= s F(s) - f(0) \\ &= s \frac{e^{-a\sqrt{s}}}{s} \end{aligned}$$

because $\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right) \rightarrow 0$ as $t \rightarrow 0$.

Thus

$$L\left\{\frac{a}{2\sqrt{\pi t^3}} e^{-a^2/(4t)}\right\} = e^{-a\sqrt{s}}.$$

Hence

$$L^{-1}\{e^{-a\sqrt{s}}\} = \frac{a}{2\sqrt{\pi t^3}} e^{-a^2/(4t)}.$$

6.8 MISCELLANEOUS EXAMPLES

EXAMPLE 6.72

Find $L^{-1}\left[\frac{1}{(s-3)^2}\right]$.

Solution. Since

$$L\{e^{at}\} = \frac{1}{(s-a)^2}$$

by shifting property, we have

$$L^{-1}\left[\frac{1}{(s-3)^2}\right] = te^{3t}$$

EXAMPLE 6.73

Find the inverse Laplace transforms of:

(i) $\frac{3s+1}{s^2(s^2+4)}e^{-3s}$ and (ii) $\tan^{-1}\left(\frac{2}{s}\right)$.

Solution. (i). We first find partial fraction of $\frac{3s+1}{s^2(s^2+4)}$. We have

$$\frac{3s+1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$$

or

$$3s+1 = As(s^2+4) + B(s^2+4) + s(Cs+D).$$

Putting $s = 0$, we get

$$1 = 4B \text{ which yields } B = \frac{1}{4}.$$

Taking $s = -4$, we get

$$-11 = 16C - 4D \quad (1)$$

Comparing coefficients of s , we get

$$3 = 4A + D \quad (2)$$

Comparing coefficients of s^2 , we get

$$0 = B + C \text{ which yields } C = -B = -\frac{1}{4}.$$

Then (1) yields $D = \frac{16C+11}{4} = \frac{7}{4}$. Now (2) gives

$$A = \frac{3-D}{4} = \frac{3-\frac{7}{4}}{4} = \frac{5}{16}.$$

Hence

$$\frac{3(s+1)}{s^2(s^2+4)} = \frac{5}{16s} + \frac{1}{4s^2} - \frac{s}{4(s^2+4)} + \frac{7}{4(s^2+4)}.$$

Therefore

$$L^{-1}\left\{\frac{3s+1}{s^2(s^2+4)}\right\} = \frac{5}{16} + \frac{1}{4}t - \frac{1}{4}\cos 2t + \frac{7}{8}\sin 2t.$$

Using second shifting property, we have

$$\begin{aligned} L^{-1}\left\{\frac{3s+1}{s^2(s^2+4)}e^{-3s}\right\} &= \frac{5}{16} + \frac{1}{4}(t-3) - \frac{1}{4}\cos 2(t-3) \\ &\quad + \frac{7}{8}\sin 2(t-3) \quad \text{for } t > 3 \end{aligned}$$

(ii). We note that

$$\begin{aligned}
 \frac{d}{ds} \left(\tan^{-1} \frac{2}{s^2} \right) &= \frac{1}{1 + \frac{4}{s^4}} \left(\frac{-4}{s^3} \right) = \frac{-4s}{s^4 + 4} \\
 &= -\frac{4s}{(s^2 + 2)^2 - (2s)^2} \\
 &= -\frac{4s}{(s^2 + 2 + 2s)(s^2 + 2 - 2s)} \\
 &= -\frac{(s^2 + 2s + 2) - (s^2 - 2s + 2)}{(s^2 + 2s + 2)(s^2 - 2s + 2)} \\
 &= -\frac{1}{s^2 - 2s + 2} + \frac{1}{s^2 + 2s + 2} \\
 &= -\frac{1}{(s-1)^2 + 1} + \frac{1}{(s+1)^2 + 1}.
 \end{aligned}$$

Therefore

$$L^{-1} \left\{ -\frac{1}{(s-1)^2 + 1} + \frac{1}{(s+1)^2 + 1} \right\} = -t f(t),$$

that is,

$$-e^t \sin t + e^{-t} \sin t = -t f(t).$$

Hence

$$f(t) = \frac{\sin t (e^t - e^{-t})}{t} = \frac{2 \sin t \sinh t}{t}.$$

EXAMPLE 6.74

Find the inverse Laplace transform of $(5s+3)/(s-1)(s^2+2s+5)$.

Solution. Using partial fractions, we have

$$\begin{aligned}
 \frac{5s+3}{(s-1)(s^2+2s+5)} &= \frac{1}{s-1} + \frac{-s+2}{s^2+2s+5} \\
 &= \frac{1}{s-1} - \frac{(s+1)-3}{(s+1)^2+4} \\
 &= \frac{1}{s-1} - \frac{s+1}{(s+1)^2+4} + \frac{3}{(s+1)^2+4}
 \end{aligned}$$

Therefore

$$\begin{aligned}
 L^{-1} \left\{ \frac{5s+3}{(s-1)(s^2+2s+5)} \right\} \\
 = e^t - e^{-t} \cos 2t + 3e^{-t} \sin 2t.
 \end{aligned}$$

EXAMPLE 6.75

Find inverse Laplace transform of $(se^{-s/2} + \pi e^{-s})/(s^2 + \pi^2)$.

Solution. Using linearity and shifting properties, we have

$$\begin{aligned}
 L^{-1} \left\{ \frac{se^{-s/2} + \pi e^{-s}}{s^2 + \pi^2} \right\} &= L^{-1} \left\{ \frac{se^{-s/2}}{s^2 + \pi^2} \right\} + L^{-1} \left\{ \frac{\pi e^{-s}}{s^2 + \pi^2} \right\} \\
 &= \cos \pi \left(t - \frac{1}{2} \right) H \left(t - \frac{1}{2} \right) \\
 &\quad + \sin \pi (t-1) H(t-1) \\
 &= \sin \pi t \left[H \left(t - \frac{1}{2} \right) \right] \\
 &\quad - \sin \pi t [H(t-1)] \\
 &= \sin \pi t \left[H \left(t - \frac{1}{2} \right) - H(t-1) \right],
 \end{aligned}$$

where $H(t)$ denotes Heavyside's unit step function.

EXAMPLE 6.76

Fine the Inverse Laplace transform of $\frac{1}{s^2(s^2+a^2)}$.

Solution. We know that

$$L\{\sin at\} = \frac{a}{s^2 + a^2}.$$

Therefore

$$L^{-1} \left\{ \frac{1}{s^2 + a^2} \right\} = \frac{1}{a} \sin at.$$

Then

$$\begin{aligned}
 L^{-1} \left\{ \frac{a}{s(s^2 + a^2)} \right\} &= \frac{1}{a} \int_0^t \sin at \, dt = \frac{1}{a} \left(-\frac{\cos at}{a} \right)_0^t \\
 &= -\frac{1}{a^2} [\cos at - 1]
 \end{aligned}$$

and

$$\begin{aligned}
 L^{-1} \left\{ \frac{a}{s^2(s^2 + a^2)} \right\} &= -\frac{1}{a^2} \int_0^t (\cos at - 1) dt. \\
 &= -\frac{1}{a^2} \left[\frac{\sin at}{a} - t \right]_0^t \\
 &= -\frac{1}{a^3} \sin at + \frac{t}{a^2} \\
 &= \frac{1}{a^2} \left[t - \frac{1}{a} \sin at \right].
 \end{aligned}$$

Note: - This question can also be solved using convolution theorem (see Example 6.57)

EXAMPLE 6.77

Find $L^{-1}\left[\cot^{-1}\left(\frac{2}{s+1}\right)\right]$.

Solution. Since

$$\begin{aligned}\frac{d}{ds}\left[\cot^{-1}\left(\frac{2}{s+1}\right)\right] &= \frac{-1}{1+\left(\frac{2}{s+1}\right)^2}\left(-\frac{2}{(s+1)^2}\right) \\ &= \frac{2}{s^2+2s+5},\end{aligned}$$

we have

$$L^{-1}\left\{\frac{2}{s^2+2s+5}\right\} = -tf(t)$$

or

$$L^{-1}\left\{\frac{2}{(s+1)^2+4}\right\} = -tf(t)$$

or

$$e^{-t} \sin 2t = -tf(t).$$

Hence

$$f(t) = -\frac{1}{t} e^{-t} \sin 2t.$$

EXAMPLE 6.78

Find:

$$L^{-1}\left\{\log \frac{s^2+1}{s(s+1)}\right\}$$

Solution. Similar to Example 6.22. In fact,

$$\begin{aligned}\frac{d}{ds}\left\{\log \frac{s^2+1}{s(s+1)}\right\} &= \frac{d}{ds}[\log(s^2+1) - \log s \\ &\quad - \log(s+1)] \\ &= \frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}.\end{aligned}$$

Therefore

$$L^{-1}\left\{\frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}\right\} = -tf(t)$$

or

$$2 \cos t - 1 - e^{-t} = -tf(t).$$

Hence

$$f(t) = \frac{1}{t}[e^{-t} + 1 - 2 \cos t].$$

EXAMPLE 6.79

Find the inverse Laplace transform of $\frac{s^2}{(s^2+4)^2}$.

Solution. Let $F(s) = \frac{s}{s^2+4} = G(s)$. Then

$$f(t) = L^{-1}\{F(s)\} = \cos 2t$$

and

$$g(t) = L^{-1}\{G(s)\} = \cos 2t.$$

Therefore, by convolution Theorem

$$\begin{aligned}L^{-1}\{F(s)G(s)\} &= f * g = \int_0^t \cos 2u \cos 2(t-u) du \\ &= \frac{1}{2} \int_0^t [\cos 2t + \cos(4u-2t)] du \\ &= \frac{1}{2} \left[u \cos 2t + \frac{\sin(4u-2t)}{4} \right]_0^t \\ &= \frac{1}{2} \left[t \cos 2t + \frac{\sin 2t}{4} \right] \\ &= \frac{1}{4} [2t \cos 2t + \sin 2t].\end{aligned}$$

EXAMPLE 6.80

Using Convolution Theorem find the Laplace inverse of $(s+2)/(s^2+4s+13)^2$.

Solution. We have

$$\frac{s+2}{(s^2+4s+13)^2} = \frac{s+2}{((s+2)^2+3^2)^2}.$$

Therefore

$$L^{-1}\left\{\frac{s+2}{(s^2+4s+13)^2}\right\} = e^{-2t} L^{-1}\left\{\frac{s}{(s^2+3^2)^2}\right\}$$

Let

$$\frac{s}{(s+3^2)^2} = \frac{s}{s^2+3^2} \cdot \frac{1}{s^2+3^2} = F(s) g(s).$$

Then $f(t) = L^{-1}\{F(s)\} = \cos 3t$ and $g(t) = L^{-1}\{G(s)\} = \frac{\sin 3t}{3}$.

By convolution Theorem

$$\begin{aligned} L^{-1}\{F(s)G(s)\} &= f * g = \int_0^t \cos 3u \frac{\sin 3(t-u)}{3} du \\ &= \frac{1}{3} \cdot \frac{1}{2} \int_0^t [\sin 3t + \sin 3(t-2u)] du \\ &= \frac{1}{6} \sin 3t \Big|_0^t + \frac{1}{12} [\cos 3(t-2u)]_0^t \\ &= \frac{1}{6} t \sin 3t + \frac{1}{12} [\cos 3(-t) - \cos 3t] \\ &= \frac{1}{6} t \sin 3t. \end{aligned}$$

Hence

$$\begin{aligned} L^{-1}\left\{\frac{s+2}{(s^2+4s+13)^2}\right\} &= e^{-2t} \left[\frac{1}{6} t \sin 3t\right] \\ &= \frac{1}{6} t e^{-2t} \sin 3t. \end{aligned}$$

EXAMPLE 6.81

Apply convolution theorem to evaluate $L^{-1}\left(\frac{1}{s^2+a^2}\right)^2$.

Solution. Proceeding as in Example 6.56, we have

$$F(s) = G(s) = \frac{1}{s^2 + a^2}.$$

Then

$$\begin{aligned} f(t) &= L^{-1}\{F(s)\} = \frac{\sin at}{a} \\ g(t) &= L^{-1}\{G(s)\} = \frac{\sin at}{a}. \end{aligned}$$

Therefore, by convolution theorem,

$$\begin{aligned} L^{-1}\{F(s)G(s)\} &= f * g = \int_0^t \frac{\sin au}{a} \cdot \frac{\sin a(t-u)}{a} du \\ &= \frac{1}{a^2} \int_0^t \frac{1}{2} [\cos a(2u-t) - \cos at] du \\ &= \frac{1}{2a^2} \left[\int_0^t \cos a(2u-t) du - \int_0^t \cos at du \right] \\ &= \frac{1}{2a^2} \left\{ \left[\frac{\sin a(2u-t)}{2a} \right]_0^t - t \cos at \right\} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2a^2} \left[\frac{\sin at}{2a} + \frac{\sin at}{2a} - t \cos at \right] \\ &= \frac{1}{2a^2} \left[\frac{\sin at}{a} - t \cos at \right] \\ &= \frac{1}{2a^3} [\sin at - at \cos at]. \end{aligned}$$

EXAMPLE 6.82

Find the inverse Laplace transform of $\frac{s}{(s^2+1)(s^2+4)}$ using convolution theorem.

Solution. Let

$$\frac{s}{(s^2+1)(s^2+4)} = \frac{1}{(s^2+1)} \cdot \frac{s}{s^2+4} = F(s)G(s).$$

Then

$$\begin{aligned} f(t) &= L^{-1}\{F(s)\} = \sin t \\ g(t) &= L^{-1}\{G(s)\} = \cos 2t. \end{aligned}$$

By convolution Theorem,

$$L^{-1}\{F(s)G(s)\} = \sin t * \cos 2t$$

$$\begin{aligned} &= \int_0^t \sin u \cos 2(t-u) du \\ &= \frac{1}{2} \int_0^t [\sin(2t-u) - \sin(2t-3u)] du \\ &= \frac{1}{2} \left[\frac{-\cos(2t-u)}{-1} + \frac{\cos(2t-3u)}{-3} \right]_0^t \\ &= \frac{1}{2} \left[\cos t - \frac{1}{3} \cos t - \cos 2t + \frac{1}{3} \cos 2t \right] \\ &= \frac{1}{3} \cos t - \frac{1}{3} \cos 2t \\ &= \frac{1}{3} (\cos t - \cos 2t). \end{aligned}$$

EXAMPLE 6.83

Using Convolution Theorem, find inverse Laplace transform of (i) $\frac{s}{(s^2+a^2)^2}$ and (ii) $\frac{1}{s(s+1)(s+2)}$

Solution. (i). Let

$$F(s) = \frac{s}{s^2+a^2} \text{ and } G(s) = \frac{1}{(s^2+a^2)^2}$$

Then (see Example 6.59)

$$f(t) = \cos at \text{ and } g(t) = \frac{1}{2a^3} [\sin at - at \cos at]$$

By Convolution Theorem, we have

$$\begin{aligned} L^{-1}\{F(s)G(s)\} &= f * g = \int_0^t \frac{1}{2a^3} \cos au [\sin a(t-u) \\ &\quad - a(t-u) \cos a(t-u)] du \\ &= \frac{1}{2a^3} \int_0^t [\cos au \sin a(t-u) \\ &\quad - a \cos au (t-u) \cos a(t-u)] du \\ &= \frac{1}{2a^3} \left[\int_0^t \cos au \sin a(t-u) du - a \right. \\ &\quad \left. \int_0^t (t-u) \cos au \cos a(t-u) du \right] \\ &= \frac{1}{2a^3} \int_0^t \cos au \sin a(t-u) du \\ &\quad - \frac{1}{2a^2} \left[\int_0^t (t-u) \frac{1}{2} \{ \cos at + \cos a(t-2u) \} du \right] \\ &= \frac{1}{4a^3} t \sin at - \frac{t^2}{8a^2} \cos at - \frac{1}{8a^3} t \sin at \\ &= \frac{t}{8a^3} [\sin at - at \cos at]. \end{aligned}$$

(ii). Let

$$F(s) = \frac{1}{s(s+1)} \text{ and } G(s) = \frac{1}{s+2}.$$

Then

$$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} = 1 - e^{-t}$$

and

$$g(t) = L^{-1}\{G(s)\} = L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}.$$

Therefore, by Convolution theorem,

$$\begin{aligned} L^{-1}\{F(s)G(s)\} &= f * g = \int_0^t (1 - e^{-u}) e^{-2(t-u)} du \\ &= e^{-2t} \int_0^t (e^{2u} - e^u) du \\ &= e^{-2t} \left[\frac{2u}{2} - e^u \right]_0^t = e^{-2t} \left[\frac{e^{2t}}{2} - e^t + \frac{1}{2} \right] \\ &= \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t} \end{aligned}$$

EXERCISES

1. Find inverse Laplace transform of

(a) $\frac{2s+6}{(s^2+6s+10)^2}$ **Ans.** $te^{-3t} \sin t$

(b) $\frac{s}{(s^2+4)^2}$ **Ans.** $\frac{1}{4} t \sin 2t$

(c) $\log \frac{s+1}{s-1}$ **Ans.** $\frac{2 \sinh t}{t}$

(d) $\cot^{-1} \frac{s}{\pi}$ **Hint:** $\frac{d}{ds} \left\{ \cot^{-1} \frac{s}{\pi} \right\} = -\frac{\pi}{s^2 + \pi^2}$
implies $L^{-1} \left\{ \frac{-\pi}{s^2 + \pi^2} \right\} = -tf(t)$ implies
 $\pi \left[\frac{1}{\pi} \sin \pi t \right] = tf(t)$ **Ans.** $f(t) = \frac{\sin \pi t}{t}$

(e) $\frac{1}{s} \log \left(1 + \frac{1}{s^2} \right)$ **Hint:** Use Example 6.22 and Theorem 6.9

Ans. $2 \int_0^t \frac{1 - \cos u}{u} du$

(f) $\frac{2s-3}{s^2+4s+13}$ **Ans.** $2e^{-2t} \cos 3t - \frac{7}{3} e^{-2t} \sin 3t$

(g) $\frac{s}{s^4+4a^4}$ **Hint:** Use partial fraction method
Ans. $\frac{1}{2a^2} \sin at \sinh at$

(h) $\frac{a(s^2-2a^2)}{s^4+4a^4}$ **Ans.** $\cos at \sinh at$

(i) $\frac{s^2+6}{(s^2+1)(s^2+4)}$ **Ans.** $f(t) = \frac{1}{3} (5 \sin t - \sin 2t)$

(j) $\frac{s}{s^4+s^2+1}$ (partial fraction method)
Ans. $\frac{2}{\sqrt{3}} \sinh \frac{t}{2} \sin \frac{\sqrt{3}}{2} t$

(k) $\frac{2s^2+5s-4}{s^3+s^2-2s}$ **Hint:** Has simple poles, so use residue method

Ans. $2 + e^t - e^{-2t}$

(l) $\frac{2s^2-4}{(s+1)(s-2)(s-3)}$ **Ans.** $-\frac{1}{6} e^{-t} - \frac{4}{3} e^{2t} + \frac{7}{2} e^{3t}$

2. Use first shift property to find $L^{-1} \left\{ \frac{1}{\sqrt{s+a}} \right\}$.

Hint: $L^{-1} \left\{ \frac{1}{\sqrt{s+a}} \right\} = e^{-at} L^{-1} \left\{ \frac{1}{\sqrt{s}} \right\} = e^{-at} \cdot \frac{t^{-1/2}}{\Gamma(1/2)}$
Ans. $\frac{e^{-at}}{\sqrt{\pi t}}$

3. Solve Exercise 1(k) using Heaviside's expansion formula.

4. Use Heaviside's expansion formula to find
 $L^{-1} \left\{ \frac{27-12s}{(s+4)(s^2+9)} \right\}$
Ans. $3e^{-4t} - 3 \cos 3t$

5. Use series method to find
- $L^{-1}\{e^{-\sqrt{s}}\}$
- .

$$\text{Ans. } \frac{1}{2\sqrt{\pi}} t^{3/2} e^{-\frac{1}{4t}}$$

6. Show that

$$L^{-1}\left\{\frac{1}{s} \cos \frac{1}{s}\right\} = 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \frac{t^6}{(6!)^2} + \dots$$

7. Evaluate
- $\sin t * t^2$
- .
- Ans.**
- $t^2 + 2\cos t - 2$

8. Find
- $L\{\sin t * t^2\}$
- .
- Ans.**
- $\frac{2}{(s^2+1)s^3}$

9. Use Convolution theorem to find the inverse Laplace transforms of the following:

$$(a) \frac{1}{s(s-a)} \quad \text{Ans. } \frac{e^{at}-1}{a}.$$

$$(b) \frac{a^2}{(s^2+a^2)^2} \quad \text{Ans. } \frac{1}{2a}(\sin at - at \cos at)$$

$$(c) \frac{4}{s^3+s^2+s+1} \quad \text{Ans. } 2(e^{-t} - \cos t + \sin t)$$

$$(d) \frac{s+2}{(s^2+4s+5)^2} \quad \text{Ans. } \frac{1}{2}te^{-2t} \sin t$$

10. Verify complex inversion formula for
- $F(s) = \frac{1}{s(s-a)}$
- .

Hint: Simple poles at 0 and a , satisfies growth restriction condition, $\text{Res}(0) = -1/a$, $\text{Res}(a) = e^{at}/a$

$$\text{Ans. } f(t) = \frac{1}{a}(e^{at} - 1)$$

11. Using complex inversion formula, find the inverse Laplace transform of the following:

$$(a) \frac{s}{s^2+a^2} \quad \text{Ans. } \cos at$$

$$(b) \frac{1}{(s+1)(s-2)^2} \quad \text{Ans. } \frac{1}{9}e^{-t} + \frac{1}{3}te^{2t} - \frac{1}{9}e^{2t}$$

$$(c) \frac{1}{(s^2+1)^2} \quad \text{Ans. } \frac{1}{2}(\sin t - t \cos t)$$

12. Find
- $L^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\}$
- ,
- $a > 0$
- .
- Ans.**
- $1 - \text{erf}\left(\frac{a}{2\sqrt{t}}\right)$
- or
- $\text{erfc}\left(\frac{a}{2\sqrt{t}}\right)$
- .

7

Applications of Laplace Transform

Laplace transform is utilized as a tool for solving linear differential equations, integral equations, and partial differential equations. It is also used to evaluate the integrals. The aim of this chapter is to discuss these applications.

7.1 ORDINARY DIFFERENTIAL EQUATIONS

Recall that a *differential equation* is an equation where the unknown is in the form of a derivative. The *order* of an ordinary differential equation is the highest derivative attained by the unknown. Thus the equation

$$\frac{d^2y}{dt^2} + \alpha \frac{dy}{dt} + \beta y = f(t)$$

is of *second order*, whereas the equation.

$$\left(\frac{dy}{dx}\right)^3 + y = \sin x$$

is a *first order* differential equation.

Theorem 5.8, opens up the possibility of using Laplace transform as a tool for solving *ordinary differential equations*. Laplace transforms, being linear, are useful only for solving linear differential equations. Differential equations containing powers of the unknown or expression such as $\tan x$, e^x cannot be solved using Laplace transforms.

The results

$$L\{f'(t)\} = sF(s) - f(0)$$

and

$$L\{f''(t)\} = s^2F(s) - sf'(0) - f''(0)$$

will be used frequently for solving ordinary differential equations. To solve linear ordinary differential equation by the Laplace transform method, we first convert the equation in the unknown function $f(t)$ into an equation in $F(s)$ and find $F(s)$. The inversion of $F(s)$ then yields $f(t)$.

Since $f(0)$, $f'(0)$, and $f''(0)$ appear in Laplace transform of derivatives of f , the Laplace transform method is best suited to *initial value problems* (where auxiliary conditions are *all* imposed at $t=0$). The solution by Laplace method with initial conditions automatically built into it. We need not add particular integral to complementary function and then apply the auxiliary conditions.

(a) Ordinary Differential Equations with Constant Solution

In case of an ordinary differential equation with constant coefficients, the transformed equation for $F(s)$ turns out to be an algebraic one and, therefore, the Laplace transform method is powerful tool for solving this type of ordinary differential equations. If

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_0 y = f(t)$$

with $y(0) = y_0$, $y'(0) = y_1$, ..., $y^{(n-1)}(0) = y_{n-1}$, then $f(t)$ is called *input*, *excitation*, or *forcing function* and $y(t)$ is called the *output* or *response*. Further, the following results suggests that if $f(t)$ is continuous and of exponential order, then $y(t)$ is also continuous and of exponential order.

Theorem 7.1. If $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y = f(t)$ is n^{th} order linear non-homogeneous equation with constant coefficients and f is continuous on $[0, \infty)$ and of exponential order, then $y(t)$ is also continuous and of exponential order.

EXAMPLE 7.1

Find the general solution of the differential equation

$$y''(t) + k^2 y(t) = 0.$$

Solution. Assume that the value of the unknown function at $t = 0$ be denoted by the constant A , and

the value of its first derivative at $t = 0$ by the constant B. Thus

$$y(0) = A \text{ and } y'(0) = B.$$

Taking Laplace transform of both sides of the given differential equations, we have

$$L\{y''(t)\} + k^2 L\{y(t)\} = 0$$

But

$$\begin{aligned} L\{y''(t)\} &= s^2 Y(s) - sy(0) - y'(0) \\ &= s^2 Y(s) - As - B. \end{aligned}$$

Therefore,

$$s^2 Y(s) - As - B + k^2 Y(s) = 0.$$

The solution of this algebraic equation in $Y(s)$ is

$$Y(s) = A \frac{s}{s^2 + k^2} + \frac{B}{k} \cdot \frac{k}{s^2 + k^2}.$$

Taking inverse Laplace transform, we get

$$y(t) = A \cos kt + \frac{B}{k} \sin kt,$$

where A and B are constants since the initial conditions were not given.

EXAMPLE 7.2

Solve

$$\frac{d^2 x}{dt^2} + x = A \sin t, \quad x(0) = x_0, \quad x'(0) = v_0.$$

Show that the phenomenon of *resonance* occurs in this case.

Solution. Taking Laplace transform, we get

$$s^2 X(s) - sx(0) - x'(0) + X(s) = \frac{A}{s^2 + 1}$$

or

$$(s^2 + 1) X(s) = \frac{A}{s^2 + 1} + sx_0 + v_0$$

or

$$X(s) = \frac{A}{(s^2 + 1)^2} + \frac{s}{s^2 + 1} x_0 + \frac{v_0}{s^2 + 1}.$$

Taking inverse Laplace transform, we have

$$x(t) = \frac{A}{2} (\sin t - t \cos t) + x_0 \cos t + v_0 \sin t.$$

We note that $x(t) \rightarrow \infty$ as $t \rightarrow \infty$ due to the term $t \cos t$. This term is called a *secular term*. The presence of secular term causes resonance, because the solution becomes unbounded.

Remark 7.1. If we consider the equation $\frac{d^2 x}{dt^2} + k^2 x = A \sin t$, $k \neq 1$, then there will be no secular term in the solution and so the system will be *purely oscillatory*.

EXAMPLE 7.3

Solve the initial value problem

$$y'(t) + 3y(t) = 0, \quad y(0) = 1.$$

Solution. Taking Laplace transform, we get

$$L\{y'(t)\} + 3L\{y(t)\} = 0,$$

which yields

$$sY(s) - y(0) + 3Y(s) = 0.$$

Since $y(0) = 1$, we have

$$sY(s) - 3Y(s) = 1,$$

an algebraic equation whose solution is

$$Y(s) = \frac{1}{s + 3}.$$

Taking inverse Laplace transform leads to

$$y(t) = e^{-3t}.$$

EXAMPLE 7.4

Solve the initial value problem

$$\frac{d^2 y}{dt^2} - 2 \frac{dy}{dt} - 8y = 0, \quad y(0) = 3, \quad y'(0) = 6.$$

Solution. The given equation is

$$y''(t) - 2y'(t) - 8y = 0, \quad y(0) = 3, \quad y'(0) = 6.$$

Laplace transform leads to

$$L\{y''(t)\} - 2L\{y'(t)\} - 8L\{y\} = 0,$$

that is,

$$s^2 Y(s) - sy(0) - y'(0) - 2\{sY(s) - y(0)\} - 8Y(s) = 0$$

and so using initial conditions, we have

$$(s^2 - 2s - 8)Y(s) - 3s = 0.$$

Hence

$$\begin{aligned} Y(s) &= \frac{3s}{s^2 - 2s - 8} = 3 \left[\frac{s - 1 + 1}{(s - 1)^2 - 9} \right] \\ &= 3 \left[\frac{s - 1}{(s - 1)^2 - 9} + \frac{1}{(s - 1)^2 - 9} \right]. \end{aligned}$$

Taking inverse Laplace transform, we get

$$\begin{aligned} y(t) &= 3L^{-1} \left\{ \frac{s - 1}{(s - 1)^2 - 9} \right\} + 3L^{-1} \left\{ \frac{1}{(s - 1)^2 - 9} \right\} \\ &= 3e^t \cosh 3t + e^t \sinh 3t. \end{aligned}$$

EXAMPLE 7.5

Solve the initial value problem

$$y''' + y'' = e^t + t + 1, \quad y(0) = y'(0) = y''(0) = 0.$$

Solution. Taking Laplace transform of both sides of the given equation, we have

$$L\{y'''(t) + L\{y''(t)\} = L\{e^t\} + L\{t\} + L\{1\},$$

that is,

$$\begin{aligned} s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0) \\ + s^2 Y(s) - sy(0) - y'(0) &= \frac{1}{s - 1} + \frac{1}{s^2} + \frac{1}{s} \end{aligned}$$

Since $y(0) = y'(0) = y''(0) = 0$, we have

$$s^3 Y(s) + s^2 Y(s) = \frac{1}{s - 1} + \frac{1}{s^2} + \frac{1}{s},$$

and so

$$Y(s) = \frac{2s^2 - 1}{s^4(s - 1)(s + 1)}.$$

Using partial fraction decomposition, we have

$$Y(s) = -\frac{1}{s^2} + \frac{1}{s^4} - \frac{1}{2(s + 1)} - \frac{1}{2(s - 1)}.$$

Taking inverse transform yields

$$\begin{aligned} y(t) &= L^{-1} \left\{ -\frac{1}{s^2} + \frac{1}{s^4} - \frac{1}{2(s + 1)} + \frac{1}{2(s - 1)} \right\} \\ &= -t + \frac{1}{6}t^3 - \frac{1}{2}e^{-t} + \frac{1}{2}e^t. \end{aligned}$$

Verification: We have

$$y' = -1 + \frac{1}{2}t^2 + \frac{1}{2}e^{-t} + \frac{1}{2}e^t,$$

$$y'' = t - \frac{1}{2}e^{-t} + \frac{1}{2}e^t,$$

$$y''' = 1 + \frac{1}{2}e^{-t} + \frac{1}{2}e^t.$$

Adding y'' and y''' , we get

$$y'' + y''' = t + e^t + 1 \text{ (the given equation).}$$

EXAMPLE 7.6

Solve

$$\frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} - 3y = \sin t, \quad y(0) = y'(0) = 0.$$

Solution. Taking Laplace transform of both sides of the given equation, we take

$$L\{y'''(t)\} + 2L\{y'(t)\} - 3L\{y(t)\} = L\{\sin t\},$$

which yields

$$\begin{aligned} s^2 Y(s) - sy(0) - y'(0) + 2\{sY(s) - y(0)\} - 3Y(s) &= L\{\sin t\} \\ &= \frac{1}{s^2 + 1}. \end{aligned}$$

Using the given initial conditions, we have

$$s^2 Y(s) + 2sY(s) - 3Y(s) = \frac{1}{s^2 + 1}$$

and so

$$\begin{aligned} Y(s) &= \frac{1}{(s^2 + 1)(s^2 + 2s - 3)} = \frac{s - 1}{2(s^2 + 1)} - \frac{s + 1}{2(s^2 + 2s - 3)} \\ &= \frac{s}{2(s^2 + 1)} - \frac{1}{2(s^2 + 1)} - \frac{1}{2} \left[\frac{s + 1}{(s + 1)^2 - 4} \right]. \end{aligned}$$

Taking inverse Laplace transform, we have

$$y(t) = \frac{1}{2} \cos t - \frac{1}{2} \sin t - \frac{1}{2} e^{-t} \sinh 2t.$$

EXAMPLE 7.7

Solve

$$\frac{d^2 y}{dt^2} - 6 \frac{dy}{dt} + 9y = t^2 e^{3t}, \quad y(0) = 2, \quad y'(0) = 6.$$

Solution. Taking Laplace transform, we get

$$s^2 Y(s) - sy(0) - y'(0) - 6(sY(s) - y(0)) + 9Y(s) = \frac{2}{(s-3)^3}.$$

Using initial conditions, we have

$$s^2 Y(s) - 2s - 6 - 6sY(s) + 12 + 9Y(s) = \frac{2}{(s-3)^3}$$

or

$$(s^2 - 6s + 9)Y(s) = 2(s-3) + \frac{2}{(s-3)^3}$$

or

$$Y(s) = \frac{2}{s-3} + \frac{2}{(s-3)^5}.$$

Taking inverse Laplace transform yields

$$y(t) = 2e^{3t} + \frac{1}{12}t^4 e^{-3t}.$$

EXAMPLE 7.8

Solve

$$y'' - 3y' + 2y = t, \quad y(0) = 0 \text{ and } y'(0) = 0.$$

Solution. Taking Laplace transform yields

$$s^2 Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s^2}.$$

Making use of initial value conditions, we have

$$(s^2 - 3s + 2)Y(s) = \frac{1}{s^2}$$

and so

$$Y(s) = \frac{1}{s^2(s^2 - 3s + 2)} = \frac{1}{4(s-2)} - \frac{1}{(s-1)} + \frac{3}{4s} + \frac{1}{2s^2}.$$

Taking inverse Laplace transform, we get

$$y(t) = \frac{1}{4}e^{2t} - e^t + \frac{3}{4} + \frac{t}{2}.$$

EXAMPLE 7.9

Solve

$$y' + 2y = 1 - H(t-1), \quad y(0) = 2,$$

where $H(t)$ is Heaviside's unit step function.

Solution. Taking Laplace transform leads to

$$sY(s) + 2 + 2Y(s) = \frac{1}{s} - \frac{e^{-s}}{s}$$

or

$$(s+2)Y(s) = \frac{1}{s} - \frac{e^{-s}}{s} - 2$$

or

$$Y(s) = \frac{1}{s(s+2)} - \frac{e^{-s}}{s(s+2)} - \frac{2}{s+2}.$$

But, by partial fraction, we have

$$\frac{1}{s(s+2)} = \frac{1}{2s} - \frac{1}{2(s+2)}.$$

Therefore,

$$Y(s) = \frac{1}{2s} - \frac{1}{2(s+2)} - \left(\frac{e^{-s}}{2s} - \frac{e^{-s}}{2(s+2)} \right) - \frac{2}{s+2}.$$

Taking inverse transform, we get

$$y(t) = \frac{1}{2} - \frac{1}{2}e^{-2t} - \frac{1}{2}H(t-1) + \frac{1}{2}e^{-2(t-1)}H(t-1) + 2e^{-2t} = \begin{cases} \frac{1}{2} + \frac{3}{2}e^{-2t} & \text{for } 0 \leq t < 1 \\ \frac{3}{2}e^{-2t} + \frac{1}{2}e^{-2(t-1)} & \text{for } t \geq 1. \end{cases}$$

EXAMPLE 7.10

Solve

$$\frac{d^2 y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t} \sin t, \quad y(0) = 0, \quad y'(0) = 1.$$

Solution. Taking Laplace transform yields

$$s^2 Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + 5Y(s) = \frac{1}{(s+1)^2 + 1}.$$

Using initial conditions, we get

$$s^2 Y(s) + 2sY(s) + 5Y(s) = 1 + \frac{1}{(s+1)^2 + 1}$$

or

$$(s^2 + 2s + 5)Y(s) = 1 + \frac{1}{(s+1)^2 + 1}$$

or

$$Y(s) = \frac{1}{s^2 + 2s + 5} + \frac{1}{(s^2 + 2s + 5)((s+1)^2 + 1)} = \frac{s^2 + 2s + 3}{(s^2 + 2s + 5)(s^2 + 2s + 2)}.$$

Using partial fractions, we have

$$Y(s) = \frac{2}{3(s^2 + 2s + 5)} + \frac{1}{3(s^2 + 2s + 2)}$$

$$= \frac{2}{3((s+1)^2 + 1)}.$$

Taking inverse transform, we get

$$y(t) = \frac{2}{3} \cdot \frac{1}{2} e^{-t} \sin 2t + \frac{1}{3} e^{-t} \sin t$$

$$= \frac{1}{3} e^{-t} (\sin t + \sin 2t).$$

EXAMPLE 7.11

Solve the equation of motion

$$m \frac{d^2 x}{dt^2} + k \frac{dx}{dt} = mv_0 \delta(t), \quad x(0) = x'(0) = 0,$$

which represents the motion of a pellet of mass m fired into a viscous gas from a gun at time $t = 0$ with a muzzle velocity v_0 and where $\delta(t)$ is Dirac delta function, $x(t)$ is displacement at time $t \geq 0$ and $k > 0$ is a constant.

Solution. The condition $x'(0) = 0$ implies that the pellet is initially at rest for $t < 0$. Taking the Laplace transform of both sides, we have

$$m[s^2 X(s) - sx(0) - x'(0)] + k[sX(s) - x(0)] = 1 \cdot mv_0.$$

Using the given conditions, this expression reduces to
or

$$X(s) = \frac{mv_0}{ms^2 + ks} = \frac{v_0}{s(s + \frac{k}{m})}$$

Use of partial fractions yields

$$X(s) = \frac{v_0}{s(s + \frac{k}{m})} = \frac{A}{s} + \frac{B}{s + \frac{k}{m}}$$

or

$$v_0 = A \left(s + \frac{k}{m} \right) + Bs.$$

Comparing coefficients, we get

$$v_0 = A \frac{k}{m}, \quad \text{which yields } A = \frac{mv_0}{k}$$

and

$$0 = A + B, \quad \text{which gives } B = -\frac{mv_0}{k}.$$

Hence

$$X(s) = \frac{mv_0}{ks} - \frac{mv_0}{k(s + \frac{k}{m})}.$$

Then, application of inverse Laplace transform yields

$$x(t) = \frac{mv_0}{k} (1 - e^{-\frac{k}{m}t}).$$

The graph of $x(t)$ is shown in the Figure 7.1

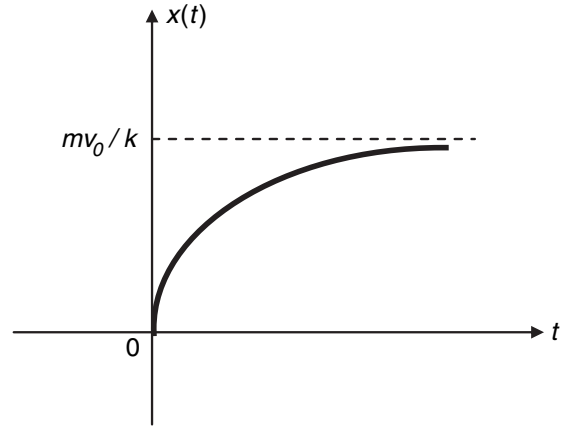


Figure 7.1

The velocity is given by

$$\frac{dx}{dt} = x'(t) = v_0 e^{-\frac{k}{m}t}.$$

We observe that $\lim_{t \rightarrow 0+} x'(t) = v_0$ and $\lim_{t \rightarrow 0-} x'(t) = 0$. This indicates instantaneous jump in velocity at $t=0$ from a rest state to the value v_0 . The graph of $x'(t)$ is shown in the Figure 7.2.

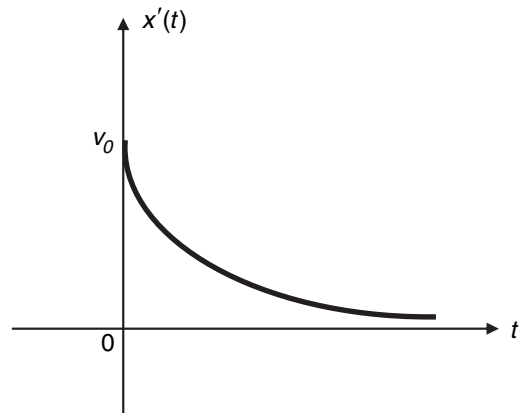


Figure 7.2

EXAMPLE 7.12

Solve boundary value problem

$$\frac{d^2 y}{dt^2} + 9y = \cos 2t, \quad y(0) = 1, \quad y\left(\frac{\pi}{2}\right) = -1.$$

Solution. Suppose that $y'(0) = A$. Then taking Laplace transform, we have

$$s^2 Y(s) - sy(0) - y'(0) + 9Y(s) = \frac{s}{s^2 + 4}$$

or

$$(s^2 + 9)Y(s) = s + A + \frac{s}{s^2 + 4},$$

and so

$$\begin{aligned} Y(s) &= \frac{s + A}{s^2 + 9} + \frac{s}{(s^2 + 9)(s^2 + 4)} \\ &= \frac{4s}{5(s^2 + 9)} + \frac{A}{s^2 + 9} + \frac{s}{5(s^2 + 4)} \\ &\quad \text{(partial fractions).} \end{aligned}$$

Taking inverse Laplace transform yields

$$y(t) = \frac{4}{5} \cos 3t + \frac{A}{3} \sin 3t + \frac{1}{5} \cos 2t.$$

Since $y\left(\frac{\pi}{2}\right) = 1$, putting $t = \frac{\pi}{2}$, we get $A = \frac{12}{5}$. Hence

$$y(t) = \frac{4}{5} \cos 3t + \frac{4}{5} \sin 3t + \frac{1}{5} \cos 2t.$$

EXAMPLE 7.13

Solve

$$\frac{d^2 x}{dt^2} + 6 \frac{dx}{dt} + 9x = \sin t \quad (t \geq 0)$$

subject to the conditions $x(0) = x'(0) = 0$.

Solution. Taking Laplace transform of both sides of the given equations yields

$$\begin{aligned} s^2 X(s) - sx(0) - x'(0) + 6(sX(s) - x(0)) \\ + 9X(s) &= \frac{1}{s^2 + 1}. \end{aligned}$$

Using the initial conditions, we have

$$(s^2 + 6s + 9)X(s) = \frac{1}{s^2 + 1}$$

or

$$\begin{aligned} X(s) &= \frac{1}{(s^2 + 1)(s + 3)^2} \\ &= \frac{A}{s + 3} + \frac{B}{(s + 3)^2} + \frac{Cs + D}{s^2 + 1}. \end{aligned}$$

Comparing coefficients of the powers of s , we get

$$\begin{aligned} A &= \frac{3}{50}, & B &= \frac{1}{10}, & C &= -\frac{3}{50}, \\ D &= \frac{2}{25}. \end{aligned}$$

Hence

$$\begin{aligned} X(s) &= \frac{3}{50(s + 3)} + \frac{1}{10(s + 3)^2} - \frac{3s}{50(s^2 + 1)} \\ &\quad + \frac{2}{25(s^2 + 1)}. \end{aligned}$$

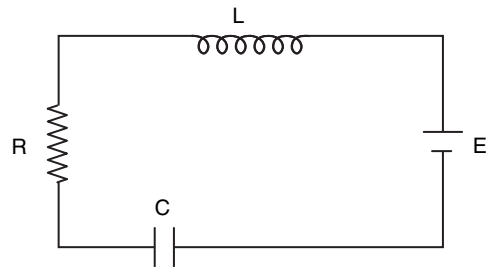
Application of inverse Laplace transform gives

$$\begin{aligned} x(t) &= \frac{3}{50} e^{-3t} + \frac{e^{-3t} t}{10} - \frac{3}{50} \cos t + \frac{2}{25} \sin t \\ &= \frac{e^{-3t}}{50} (5t + 3) - \frac{3}{50} \cos t + \frac{2}{25} \sin t. \end{aligned}$$

The term $\frac{e^{-3t}}{50}(5t + 3)$ is the particular solution, called the *transient response* since it dies away for large time, whereas the terms $-\frac{3}{50} \cos t + \frac{2}{25} \sin t$ is called the complementary function (sometimes called *steady state response* by engineers since it persists). However, there is nothing steady about it.

(b) Problems Related to Electrical Circuits

Consider the RCL circuit, shown in the Figure 7.3, consisting of *resistance*, *capacitor*, and *inductor* connected to a battery.

**Figure 7.3**

We know that resistance R is measured in *ohms*, capacitance C is measured in *farads*, and inductance is measured in *henrys*.

Let I denote the current flowing through the circuit and Q denote the charge. Then current I is related to Q by the relation $I = \frac{dQ}{dt}$. Also

(a) By Ohm's law, $\frac{V}{I} = R$ (resistance).

Therefore, the voltage drop V across a resistor R is RI .

(b) The voltage drop across the inductor L is $L \frac{dI}{dt}$.

(c) The voltage drop across a capacitor is $\frac{Q}{C}$.

Thus, if E is the voltage (potential difference) of the battery, then by Kirchhoff's law, we have

$$L \frac{dI}{dt} + RI + \frac{Q}{C} = E(t),$$

where L , C , and R are constants. In terms of current, this equation becomes

$$L \frac{dI}{dt} + RI + \frac{1}{C} \int_0^t I(u) du = E(t),$$

because $I = \frac{dQ}{dt}$ implies $Q = \int_0^t I(u) du$.

In terms of charge, this differential equation takes the form

$$L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = E(t),$$

which is a differential equation of second order with constant coefficients L , R , and $1/C$. The forcing function (input function) $E(t)$ is supplied by the battery (voltage source). The system described by the above differential equation is known as *harmonic oscillator*.

EXAMPLE 7.14

Given that $I = Q = 0$ at $t = 0$, find I in the LR circuit (Figure 7.4) for $t > 0$.

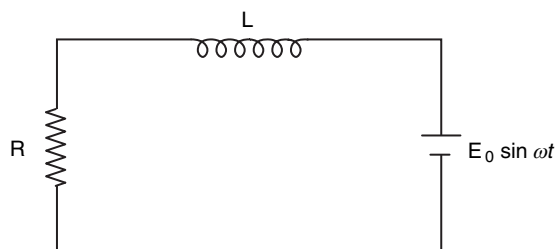


Figure 7.4

Solution. By Kirchhoff's law, the differential equation governing the given circuit is

$$L \frac{dI}{dt} + RI = E_0 \sin \omega t, \quad I(0) = 0,$$

where L , R , E_0 , and ω are constants. Taking Laplace transform of both sides, we have

$$L[sF(s) - I(0)] + RF(s) = \frac{E_0 \omega}{s^2 + \omega^2},$$

where $F(s)$ denotes the Laplace transform of I . Using the given initial condition, we have

$$(Ls + R)F(s) = \frac{E_0 \omega}{s^2 + \omega^2},$$

which yields

$$\begin{aligned} F(s) &= \frac{E_0 \omega}{(Ls + R)(s^2 + \omega^2)} = \frac{E_0 \frac{\omega}{L}}{(s + \frac{R}{L})(s^2 + \omega^2)} \\ &= \frac{A}{(s + \frac{R}{L})} + \frac{Bs + C}{s^2 + \omega^2}. \end{aligned}$$

Comparison of coefficients of different powers of s yields

$$A = \frac{E_0 L \omega}{L^2 \omega^2 + R^2}, \quad B = \frac{-E_0 L \omega}{L^2 \omega^2 + R^2}, \quad C = \frac{E_0 R \omega}{L^2 \omega^2 + R^2}.$$

Hence

$$\begin{aligned} F(s) &= \frac{E_0 L \omega}{(s + \frac{R}{L})(L^2 \omega^2 + R^2)} - \frac{s E_0 L \omega}{(s^2 + \omega^2)(L^2 \omega^2 + R^2)} \\ &\quad + \frac{E_0 R \omega}{(s^2 + \omega^2)(L^2 \omega^2 + R^2)}. \end{aligned}$$

Taking inverse Laplace transform yields

$$\begin{aligned} I(t) &= \frac{E_0 L \omega}{L^2 \omega^2 + R^2} e^{-\frac{R}{L}t} - \frac{E_0 L \omega}{L^2 \omega^2 + R^2} \cos \omega t \\ &\quad + \frac{E_0 R}{L^2 \omega^2 + R^2} \sin \omega t. \end{aligned}$$

EXAMPLE 7.15

Given that $I = Q = 0$ at $t = 0$, find charge Q and current I in the following circuit (Figure 7.5) for $t > 0$.

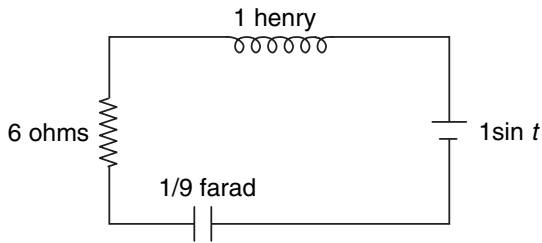


Figure 7.5

Solution. By Kirchhoff's law, the differential equation for the given circuit is

$$L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = E(t).$$

Here $L = 1$, $R = 6$, $C = \frac{1}{9}$, $E(t) = \sin t$. Thus we have

$$\frac{d^2 Q}{dt^2} + 6 \frac{dQ}{dt} + 9Q = \sin t \quad (t > 0),$$

subject to $Q(0) = 0$, $Q'(0) = I(0) = 0$. By Example 7.13, the solution of this equation is

$$Q(t) = \frac{e^{-3t}}{50} (5t + 3) - \frac{3}{50} \cos t + \frac{2}{25} \sin t.$$

Then

$$\begin{aligned} I(t) &= \frac{dQ}{dt} = \frac{5e^{-3t}}{50} - \frac{3}{50} (5t + 3)e^{-3t} + \frac{3}{50} \sin t + \frac{2}{25} \cos t \\ &= -\frac{e^{-3t}}{50} (15t + 4) + \frac{3}{50} \sin t + \frac{2}{25} \cos t. \end{aligned}$$

EXAMPLE 7.16

Solve

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = \delta(t) \quad (\text{Dirac delta function})$$

under conditions $q(0) = q'(0) = 0$.

Solution. Applying Laplace transform to both sides of the given equation, we find

$$\left(Ls^2 + Rs + \frac{1}{C} \right) Q(s) = 1$$

or

$$Q(s) = \frac{1}{Ls^2 + Rs + \frac{1}{C}} = \frac{1}{L(s^2 + \frac{R}{L}s + \frac{1}{LC})}.$$

Suppose the roots of $s^2 + \frac{R}{L}s + \frac{1}{LC}$ are s_1 and s_2 . Then

$$s_1 = \frac{-R + \sqrt{R^2 - (4L/C)}}{2L} \text{ and } s_2 = \frac{-R - \sqrt{R^2 - (4L/C)}}{2L}.$$

Let us suppose $R > 0$. Then three cases arise:

- (a) If $R^2 - \frac{4L}{C} < 0$, then s_1 and s_2 are complex and $s_1 = \bar{s}_2$.
- (b) If $R^2 - \frac{4L}{C} = 0$, then s_1 and s_2 are real and $s_1 = s_2$.
- (c) If $R^2 - \frac{4L}{C} > 0$, then s_1 and s_2 are real and $s_1 \neq s_2$.

Case (a). Using partial fractions, we have

$$Q(s) = \frac{1}{L(s-s_1)(s-s_2)} = \frac{1}{L(s_1-s_2)} \left[\frac{1}{s-s_1} - \frac{1}{s-s_2} \right].$$

Taking inverse Laplace transform yields

$$q(t) = \frac{1}{L(s_1-s_2)} [e^{s_1 t} - e^{s_2 t}].$$

If we put

$$\omega_0 = \frac{1}{2L} \sqrt{\frac{4L}{C} - R^2} \quad \text{and} \quad \sigma = -\frac{R}{L},$$

then $s_1 = \bar{s}_2 = \sigma + i\omega_0$ and so $s_1 - s_2 = 2i\omega_0$.

Therefore,

$$\begin{aligned} q(t) &= \frac{1}{2Li\omega_0} (e^{(\sigma+i\omega_0)t} - e^{(\sigma-i\omega_0)t}) \\ &= \frac{1}{L\omega_0} e^{\sigma t} \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i} \\ &= \frac{1}{L\omega_0} e^{\sigma t} \sin \omega_0 t, \quad \sigma < 0. \end{aligned}$$

Thus, the impulse response $q(t)$ is a *damped sinusoidal* with frequency ω_0 . That is why, this case is called *damped vibration* or *undercritical damping* (Figure 7.6).

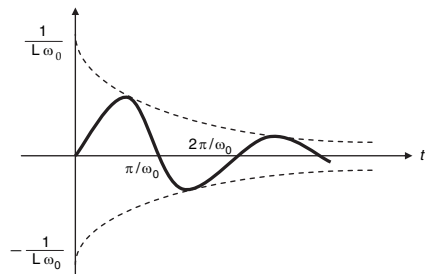


Figure 7.6

Case (b) In this case $s_1 = s_2 = -\frac{R}{2L}$ and so

$$Q(s) = \frac{1}{L(s - \sigma)^2}.$$

Taking inverse transform, we get

$$q(t) = \frac{te^{\sigma t}}{L}, \quad \sigma < 0.$$

This case is called *critical damping* (Figure 7.7)

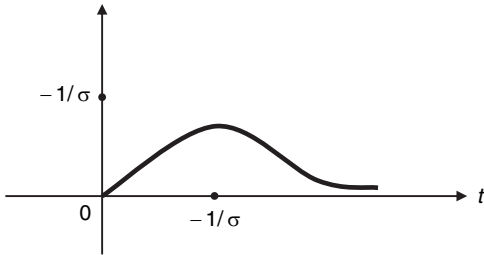


Figure 7.7

Case (c) As in case (a), we have

$$q(t) = \frac{1}{L(s_1 - s_2)} (e^{s_1 t} - e^{s_2 t}).$$

Since $L > 0$ and $C > 0$, we have $R > \sqrt{R^2 - \frac{4L}{C}}$ and so $s_2 < s_1 < 0$. Thus $q(t)$ is the sum of two exponentially damped functions. Put

$$\omega_0 = \frac{1}{2L} \sqrt{R^2 - \frac{4L}{C}} \quad \text{and} \quad \sigma = -\frac{R}{2L}.$$

Then, we have

$$s_1 = \sigma + \omega_0, \quad s_2 = \sigma - \omega_0.$$

Therefore,

$$s_1 - s_2 = 2\omega_0$$

and

$$\begin{aligned} q(t) &= \frac{1}{2L\omega_0} (e^{(\sigma+\omega_0)t} - e^{(\sigma-\omega_0)t}) = \frac{1}{L\omega_0} \sinh \omega_0 t \\ &= \frac{1}{2L\omega_0} e^{(\sigma+\omega_0)t} (1 - e^{-2\omega_0 t}), \quad \sigma < 0. \end{aligned}$$

Since $\sigma + \omega_0 < 0$, the impulse response $q(t)$ is damped hyperbolic sine. This case is called *over-damped* or *overcritical damping* (Figure 7.8).

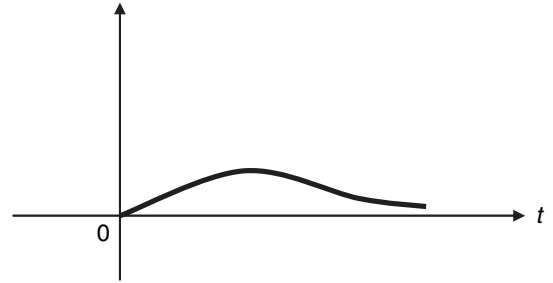


Figure 7.8

(c) Mechanical System (Mass-Spring System)

Let m be the mass suspended on a spring that is rigidly supported from one end (Figure 7.9). The rest position is denoted by $x = 0$, downward displacement by $x > 0$, and upward displacement is represented by $x < 0$. Let

- (i) $k > 0$ be the *spring constant* (or *stiffness*) and $a > 0$ be the *damping constant*.
- (ii) $a \frac{dx}{dt}$ be the damping force due to medium (air, etc.). Thus, damping force is proportional to the velocity.
- (iii) $f(t)$ represents all external impressed forces on m . It is also called *forcing* or *excitation*.

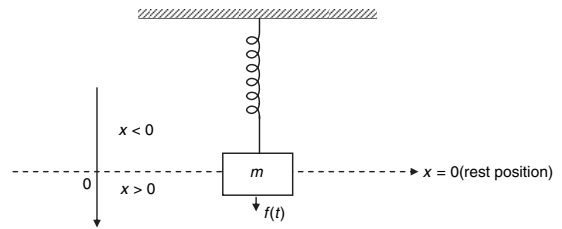


Figure 7.9

By Newton's second law of motion, the sum of forces acting on m equals $m \frac{d^2 x}{dt^2}$ and so

$$m \frac{d^2 x}{dt^2} = -kx - a \frac{dx}{dt} + f(t).$$

Thus the equation of motion is

$$m \frac{d^2 x}{dt^2} + a \frac{dx}{dt} + kx = f(t) \quad (1)$$

This is exactly the same differential equation which occurs in harmonic oscillator.

If $a = 0$, the motion is called *undamped* whereas if $a \neq 0$, the motion is called *damped*. Moreover, if $f(t) = 0$, that is, if there is no impressed forces, then the motion is called *forced*.

The equation (1) can be written as

$$\frac{d^2x}{dt^2} + \frac{a}{m} \frac{dx}{dt} + \frac{k}{m} = f(t)/m, \quad (2)$$

where $f(t)/m$ is now the external impressed force (or excitation force) per unit mass.

EXAMPLE 7.17

Solve the equation of motion

$$\frac{d^2x}{dt^2} + 2b \frac{dx}{dt} + \lambda^2 x = \delta(t), x(0) = x'(0) = 0$$

for $0 < b < \lambda$.

[Clearly this is equation (2) with $\frac{a}{m} = 2b$, $\frac{k}{m} = \lambda^2$]

Solution. We want to find the response of the given mechanical system to a unit impulse. Taking Laplace transform, we get

$$\{s^2 X(s) - sx(0) - x'(0)\} + 2b\{sX(s) - x(0)\} + \lambda^2 X(s) = 1.$$

Taking note of the given conditions, we have

$$\text{or} \quad (s^2 + 2bs + \lambda^2) X(s) = 1$$

$$X(s) = \frac{1}{s^2 + 2bs + \lambda^2} = \frac{1}{(s+b)^2 + (\lambda^2 - b^2)}.$$

Taking inverse Laplace transform yields

$$x(t) = e^{-bt} \left(\frac{1}{\sqrt{\lambda^2 - b^2}} \sin \sqrt{\lambda^2 - b^2} t \right),$$

which is clearly a case of damped oscillation (Figure 7.10).

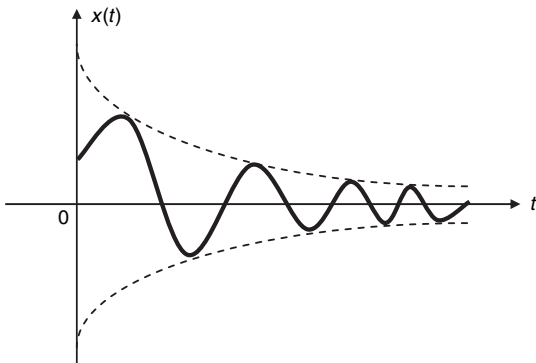


Figure 7.10

Also we note that

$$X(s) = \frac{1}{s^2 + 2bs + \lambda^2} L\{\delta(t)\}.$$

Thus we conclude that

$$\text{Response} = \text{Transfer function} \times \text{Input}.$$

(d) Ordinary Differential Equations with Polynomial (Variable) Coefficients

We know that

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s),$$

where $F(s) = L\{f(t)\}$. Thus for $n = 1$, we have

$$L\{tf(t)\} = -F'(s).$$

Hence, if $f'(t)$ satisfies the sufficient condition for the existence of Laplace transform, then

$$\begin{aligned} L\{tf'(t)\} &= -\frac{d}{ds} L\{f'(t)\} = -\frac{d}{ds} (sF(s) - f(0)) \\ &= -sF'(s) - F(s). \end{aligned}$$

Similarly for $f''(t)$,

$$\begin{aligned} L\{tf''(t)\} &= -\frac{d}{ds} L\{f''(t)\} = -\frac{d}{ds} \{s^2 F(s) - sf(0) - f'(0)\} \\ &= -s^2 F'(s) - 2sF(s) + f(0). \end{aligned}$$

The above-mentioned derivations are used to solve linear differential equations whose coefficients are first degree polynomials.

EXAMPLE 7.18

Solve

$$ty'' + y' + ty = 0, y(0) = 1, y'(0) = 0.$$

Solution. Taking Laplace transform, we have

$$L\{ty''\} + L\{y'\} + L\{ty\} = 0$$

or

$$-\frac{d}{ds} L\{y''(t)\} + \{sY(s) - y(0)\} - \frac{d}{ds} \{Y(s)\} = 0$$

or

$$\begin{aligned} -\frac{d}{ds} \{s^2 Y(s) - sy(0) - y'(0)\} + \{sY(s) - y(0)\} \\ - \frac{d}{ds} Y(s) = 0 \end{aligned}$$

which on using initial conditions yields

$$-\left[s^2 \frac{dY(s)}{ds} + 2sY(s)\right] + sY(s) - \frac{dY(s)}{ds} = 0$$

or

$$(s^2 + 1) \frac{dY(s)}{ds} + sY(s) = 0$$

or

$$\frac{dY(s)}{Y(s)} + \frac{s ds}{s^2 + 1} = 0.$$

Integrating, we have

$$\log Y(s) + \frac{1}{2} \log(s^2 + 1) = A \text{ (constant)}$$

and so

$$Y(s) = \frac{A}{\sqrt{s^2 + 1}}.$$

Taking inverse Laplace transform, we get

$$y(t) = A J_0(t),$$

where $J_0(t)$ is Bessel function of order zero.Putting $t = 0$ and using initial condition $y(0) = 1$, we have

$$1 = A J_0(0) = A.$$

Hence the required solution is

$$y(t) = J_0(t).$$

EXAMPLE 7.19

Solve

$$y'' + ty' - 2y = 4, \quad y(0) = -1, \quad y'(0) = 0.$$

Solution. Taking Laplace transform yields

$$L\{y''(t)\} + L\{ty'(t)\} - 2L\{y(t)\} = 4L\{1\}$$

or

$$s^2 Y(s) - sy(0) - y'(0) - \frac{d}{ds} L\{y'(t)\} - 2Y(s) = \frac{4}{s}$$

or

$$s^2 Y(s) - sy(0) - y'(0) - \frac{d}{ds} (sY(s) - y(0)) - 2Y(s) = \frac{4}{s}.$$

On using the initial values, we have

$$s^2 Y(s) + s - \left(s \frac{dY(s)}{ds} + Y(s) \right) - 2Y(s) = \frac{4}{s}$$

or

$$\frac{sdY(s)}{ds} - (s^2 - 3)Y(s) = -\frac{4}{s} + s$$

or

$$\frac{dY(s)}{ds} + \left(\frac{3}{s} - s \right) Y(s) = -\frac{4}{s^2} + 1.$$

The integrating factor is

$$e^{\int (\frac{3}{s} - s) ds} = s^3 e^{-\frac{s^2}{2}}.$$

Therefore,

$$\frac{d}{ds} [Y(s) \cdot s^3 e^{-\frac{s^2}{2}}] = -\frac{4}{s^2} s^3 e^{-\frac{s^2}{2}} + s^3 e^{-\frac{s^2}{2}},$$

and so integration yields

$$Y(s) s^3 e^{-\frac{s^2}{2}} = -4 \int s e^{-\frac{s^2}{2}} ds + \int s^3 e^{-\frac{s^2}{2}} ds.$$

Putting $u = -\frac{s^2}{2}$, we get

$$\begin{aligned} Y(s) s^3 e^{-\frac{s^2}{2}} &= 4 \int e^u du + 2 \int u e^u du \\ &= 4e^{-\frac{s^2}{2}} + 2 \left(\frac{-s^2}{2} e^{-\frac{s^2}{2}} - e^{-\frac{s^2}{2}} \right) + A \\ &= 2e^{-\frac{s^2}{2}} - s^2 e^{-\frac{s^2}{2}} + C. \end{aligned}$$

Thus,

$$Y(s) = \frac{2}{s^3} - \frac{1}{s} + \frac{C}{s^3} e^{\frac{s^2}{2}}.$$

Since $Y(s) \rightarrow 0$ as $s \rightarrow \infty$, we must have $C = 0$ and so

$$Y(s) = \frac{2}{s^3} - \frac{1}{s}.$$

Taking inverse Laplace transform, we get

$$y(t) = t^2 - 1.$$

EXAMPLE 7.20

Solve

$$ty'' + 2y' + ty = 0, \quad y(0) = 1, \quad y(\pi) = 0.$$

Solution. Let $y'(0) = A$ (constant). Taking Laplace transform of both sides, we obtain

$$-\frac{d}{ds} \{s^2 Y(s) - sy(0) - y'(0)\} + 2\{sY(s) - y(0)\} - \frac{d}{ds} \{Y(s)\} = 0$$

and so

$$\begin{aligned} -s^2 Y'(s) - 2sY(s) + y(0) + 2sY(s) - 2y(0) \\ -Y'(s) = 0. \end{aligned}$$

Using boundary conditions, we get

$$-(s^2 + 1)Y'(s) - 1 = 0$$

or

$$Y'(s) = \frac{-1}{s^2 + 1}.$$

Integration yields

$$Y(s) = -\tan^{-1} s + B \text{ (constant).}$$

Since $Y(s)$ tends to zero as $s \rightarrow \infty$, we must have $B = \pi/2$. Hence,

$$Y(s) = \frac{\pi}{2} - \tan^{-1} s = \tan^{-1} \left(\frac{1}{s} \right).$$

Taking inverse Laplace transform, we have (see Example 6.48).

$$y(t) = L^{-1} \left\{ \tan^{-1} \left(\frac{1}{s} \right) \right\} = \frac{\sin t}{t}.$$

This solution clearly satisfies $y(\pi) = 0$.

EXAMPLE 7.21

Solve

$$ty'' + y' + 2y = 0, \quad y(0) = 1.$$

Solution. Taking Laplace transform gives

$$-\frac{d}{ds}(s^2 Y(s) - sy(0) - y'(0)) + (sY(s) - y(0)) + 2Y(s) = 0$$

or

$$-s^2 Y'(s) - 2sY(s) + y(0) + sY(s) - y(0) + 2Y(s) = 0$$

or

$$-s^2 Y'(s) - sY(s) + 2Y(s) = 0$$

or

$$Y'(s) + \left(\frac{1}{s} - \frac{2}{s^2} \right) Y(s) = 0.$$

The integrating factor is

$$e^{\int \left(\frac{1}{s} - \frac{2}{s^2} \right) ds} = e^{\log s + \frac{2}{s}} = se^{\frac{2}{s}}.$$

Therefore,

$$\frac{d}{ds} \{ Y(s) se^{2/s} \} = 0.$$

Integrating, we have

$$Y(s) se^{2/s} = A \text{ (constant)}$$

or

$$Y(s) = \frac{A e^{-\frac{2}{s}}}{s}.$$

Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, taking $x = -\frac{2}{s}$, we have

$$Y(s) = A \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n! s^{n+1}}.$$

Taking inverse Laplace transform, we get

$$y(t) = A \sum_{n=0}^{\infty} \frac{(-1)^n 2^n t^n}{(n!)^2}.$$

The condition $y(0) = 1$ now yields $A = 1$. Hence

$$y(t) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n t^n}{(n!)^2} = J_0(2\sqrt{2}t),$$

where J_0 is Bessel's function of order zero.

EXAMPLE 7.22

Solve

$$ty'' - y' = -1, \quad y(0) = 0.$$

Solution. Taking Laplace transform of both sides of the given equation,

$$-\frac{d}{ds} \{ s^2 Y'(s) - sy(0) - y'(0) \} - \{ sY(s) - y(0) \} = -\frac{1}{s}$$

or

$$-s^2 Y'(s) - 2sY(s) + y(0) - sY(s) + y(0) = -\frac{1}{s}$$

or

$$-s^2 Y'(s) - 3sY(s) = -\frac{1}{s}$$

or

$$Y'(s) + \frac{3}{s} Y(s) = \frac{1}{s^3}.$$

The integrating factor is

$$e^{\int \frac{3}{s} ds} = e^{3 \log s} = s^3.$$

Therefore,

$$\frac{d}{ds} (Y(s)s^3) = \frac{1}{s^3} s^3 = 1.$$

Integrating

$$Y(s)s^3 = s + A(\text{constant})$$

and so

$$Y(s) = \frac{1}{s^2} + \frac{A}{s^3}.$$

Taking inverse Laplace transform, we get

$$y(t) = t + Bt^2,$$

where B is constant. Obviously, the solution satisfies $y(0) = 0$.

EXAMPLE 7.23

Solve

$$ty'' + (t+1)y' + 2y = e^{-t}, \quad y(0) = 0.$$

Solution. Taking Laplace transform of both sides gives

$$-\frac{d}{ds}\{s^2Y(s) - sy(0) - y'(0)\} - \frac{d}{ds}\{sY(s) - y(0)\} + \{sY(s) - y(0)\} + 2Y(s) = \frac{1}{s+1}$$

or

$$-s^2Y'(s) - 2sY(s) - \{sY'(s) + Y(s)\} + sY(s) - y(0) + 2Y(s) = \frac{1}{s+1}$$

or

$$-s^2Y'(s) - sY'(s) - 2sY(s) + sY(s) + Y(s) = \frac{1}{s+1}$$

or

$$-(s^2 + s)Y'(s) - (s-1)Y(s) = \frac{1}{s+1}$$

or

$$Y'(s) + \frac{s-1}{s^2+s} = \frac{-1}{s(s+1)^2}.$$

The integration factor is

$$e^{\int \frac{s-1}{s^2+s} ds} = e^{\int (-\frac{1}{s} + \frac{2}{s+1}) ds} = \frac{(s+1)^2}{s}.$$

Therefore,

$$\frac{d}{ds} \left(Y(s) \frac{(s+1)^2}{s} \right) = \frac{-1}{s(s+1)^2} \cdot \frac{(s+1)^2}{s} = -\frac{1}{s^2}.$$

Integrating, we get

$$Y(s) \frac{(s+1)^2}{s} = \int -\frac{1}{s^2} ds = \frac{1}{s} + C$$

and so

$$Y(s) = \frac{1}{(s+1)^2} + \frac{Cs}{(s+1)^2}.$$

By initial value theorem $y(0) = \lim_{s \rightarrow 0} sY(s) = 0$ and so $C = 0$. Hence

$$Y(s) = \frac{1}{(s+1)^2}.$$

Taking inverse Laplace transform, we get

$$y(t) = te^{-t}.$$

7.2 SIMULTANEOUS DIFFERENTIAL EQUATIONS

The Laplace transforms convert a pair of differential equations into simultaneous algebraic equations in parameters. After that we solve these equations for Laplace transforms of the variables and then apply inverse Laplace operators to get the required solution.

EXAMPLE 7.24

Solve the simultaneous differential equations

$$3x' + y' + 2x = 1, \quad x' + 4y' + 3y = 0$$

subject to the conditions $x(0) = 0, y(0) = 0$.

Solution. Taking Laplace transform, we get

$$3\{sX(s) - x(0)\} + \{sY(s) - y(0)\} + 2X(s) = \frac{1}{s}$$

and

$$sX(s) - x(0) + 4\{sY(s) - y(0)\} + 3Y(s) = 0.$$

Using the initial conditions, these equations reduce to

$$(3s+2)X(s) + sY(s) = \frac{1}{s} \quad (3)$$

and

$$sX(s) + (4s+3)Y(s) = 0. \quad (4)$$

Multiplying (3) and (4) by s and $(3s+2)$ respectively and then subtracting, we get

$$Y(s) = -\frac{1}{11s^2 + 17s + 6} = \frac{-1}{(11s+6)(s+1)},$$

and then using (4), we have

$$X(s) = \frac{4s+3}{s(11s+6)(s+1)}.$$

We deal with $X(s)$ first. Using partial fraction, we have

$$X(s) = \frac{1}{2s} - \frac{3}{10} \left(\frac{1}{s + (6/11)} \right) - \frac{1}{5(s+1)}.$$

Taking inverse transform, we have

$$\begin{aligned}x(t) &= \frac{1}{2} - \frac{3}{10}e^{-\frac{6}{11}t} - \frac{1}{5}e^{-t} \\&= \frac{1}{10}(5 - 3e^{-\frac{6}{11}t} - 2e^{-t}).\end{aligned}$$

Further, poles of $Y(s)$ are $-\frac{6}{11}$ and -1 . Hence

$$\begin{aligned}y(t) &= \lim_{s \rightarrow -\frac{6}{11}} \left(s + \frac{6}{11} \right) Y(s) e^{-\frac{6}{11}t} \\&+ \lim_{s \rightarrow -1} (s + 1) Y(s) e^{-t} = \frac{1}{5} (e^{-t} - e^{-\frac{6}{11}t}).\end{aligned}$$

EXAMPLE 7.25

Solve the simultaneous differential equations

$$\frac{dx}{dt} = 2x - 3y, \quad \frac{dy}{dt} = y - 2x$$

subject to the conditions $x(0) = 8, y(0) = 3$.

Solution. Taking Laplace transform and using the given conditions, we have

$$sX(s) = 2X(s) - 3Y(s) + 8$$

and

$$sY(s) = Y(s) - 2X(s) + 3.$$

Thus

$$(s - 2)X(s) + 3Y(s) = 8,$$

and

$$2X(s) + (s - 1)Y(s) = 3.$$

Solving these algebraic equations, we get

$$X(s) = \frac{8s - 17}{s^2 - 3s - 4} = \frac{8s - 17}{(s + 1)(s - 4)},$$

and

$$Y(s) = \frac{3s - 22}{s^2 - 3s - 4} = \frac{3s - 22}{(s + 1)(s - 4)}.$$

Using partial fractions, these yields

$$X(s) = \frac{5}{s + 1} + \frac{3}{s - 4}, \quad Y(s) = \frac{5}{s + 1} - \frac{2}{s - 4}.$$

Hence taking inverse Laplace transform, we get

$$x(t) = 5e^{-t} + 3e^{4t}, \quad y(t) = 5e^{-t} - 2e^{4t}.$$

EXAMPLE 7.26

Solve

$$\frac{dx}{dt} - y = e^t, \quad \frac{dy}{dt} + x = \sin t$$

subject to the conditions $x(0) = 1, y(0) = 0$.

Solution. Taking Laplace transform and using the given conditions, we have

$$sX(s) - Y(s) = \frac{1}{s - 1} + 1 = \frac{s}{s - 1},$$

$$sY(s) + X(s) = \frac{1}{s^2 + 1}.$$

Solving these equations, we get

$$X(s) = \frac{s^4 + s^2 + s - 1}{(s - 1)(s^2 + 1)^2},$$

$$Y(s) = \frac{-s^4 + s^3 - 2s^2}{s(s - 1)(s^2 + 1)^2} = \frac{-s^3 + s^2 - 2s}{(s - 1)(s^2 + 1)^2}.$$

Now

$$\begin{aligned}X(s) &= \frac{s^4 + s^2 + s - 1}{(s - 1)(s^2 + 1)^2} \\&= \frac{A}{s - 1} + \frac{Bs + C}{(s^2 + 1)} + \frac{Ds + E}{(s^2 + 1)^2}.\end{aligned}$$

Comparison of coefficients yields

$$A = B = C = \frac{1}{2} \quad \text{and} \quad E = 1.$$

Thus

$$X(s) = \frac{1}{2(s - 1)} + \frac{s}{2(s^2 + 1)} + \frac{1}{2(s^2 + 1)} + \frac{1}{(s^2 + 1)^2}.$$

Hence

$$\begin{aligned}x(t) &= \frac{1}{2}e^t + \frac{1}{2}\cos t + \frac{1}{2}\sin t + \frac{1}{2}(\sin t - t \cos t) \\&= \frac{1}{2}[e^t + \cos t + 2\sin t - t \cos t].\end{aligned}$$

Now consider $Y(s)$. We have

$$\begin{aligned}Y(s) &= \frac{-s^3 + s^2 - 2s}{(s - 1)(s^2 + 1)^2} \\&= \frac{A}{s - 1} + \frac{Bs + C}{(s^2 + 1)} + \frac{Ds + E}{(s^2 + 1)^2}.\end{aligned}$$

Comparing coefficients, we get

$$A = -\frac{1}{2}, B = \frac{1}{2}, C = -\frac{1}{2}, D = 2, E = 0,$$

and so

$$Y(s) = \frac{-1}{2(s - 1)} + \frac{s}{2(s^2 + 1)} - \frac{1}{2(s^2 + 1)} + \frac{2s}{(s^2 + 1)^2}.$$

Hence

$$y(t) = -\frac{1}{2}e^t + \frac{1}{2}\cos t - \frac{1}{2}\sin t + t \sin t.$$

EXAMPLE 7.27

The co-ordinates (x, y) of a particle moving along a plane curve at any time t are given by

$$\frac{dy}{dt} + 2x = \sin 2t, \quad \frac{dx}{dt} - 2y = \cos 2t, \quad t > 0.$$

If at $t = 0$, $x = 1$ and $y = 0$, show by using transforms, that the particle moves along the curve $4x^2 + 4xy + 5y^2 = 4$.

Solution. Using Laplace transform, we get

$$sY(s) - y(0) + 2X(s) = \frac{2}{s^2 + 4}$$

and

$$sX(s) - x(0) - 2Y(s) = \frac{s}{s^2 + 4}.$$

Using the given conditions, we have

$$sY(s) + 2X(s) = \frac{2}{s^2 + 4}$$

and

$$sX(s) - 2Y(s) = 1 + \frac{s}{s^2 + 4} = \frac{s^2 + s + 4}{s^2 + 4}.$$

Solving for $X(s)$ and $Y(s)$, and using partial fractions, we have

$$\begin{aligned} X(s) &= \frac{s^3 + s^2 + 4s + 4}{(s^2 + 4)^2} = \frac{s}{s^2 + 4} + \frac{1}{s^2 + 4} \\ &= \frac{s}{s^2 + 4} + \frac{1}{2} \frac{2}{s^2 + 4}, \\ Y(s) &= \frac{-2s^2 - 8}{(s^2 + 4)^2} = -\frac{2}{s^2 + 4}. \end{aligned}$$

Hence taking inverse transform, we get

$$\begin{aligned} x(t) &= \cos 2t + \frac{1}{2} \sin 2t, \\ y(t) &= -\sin 2t. \end{aligned}$$

We observe that

$$4x^2 + 4xy + 5y^2 = 4(\cos^2 2t + \sin^2 2t) = 4,$$

and hence the particle moves along the curve $4x^2 + 4xy + 5y^2 = 4$.

EXAMPLE 7.28

Solve the following system of equations:

$$x(t) - y''(t) + y(t) = e^{-t} - 1,$$

$$x'(t) + y'(t) - y(t) = -3e^{-t} + t,$$

subject to $x(0) = 0$, $y(0) = 1$, $y'(0) = -2$.

Solution. Taking Laplace transform yields

$$X(s) - \{s^2 Y(s) - sy(0) - y'(0)\} + Y(s) = \frac{1}{s+1} - \frac{1}{s}$$

and

$$\begin{aligned} sX(s) - x(0) + sY(s) - y(0) - Y(s) \\ = \frac{-3}{s+1} + \frac{1}{s^2}. \end{aligned}$$

Using the given conditions, we have

$$X(s) - s^2 Y(s) + s - 2 + Y(s) = \frac{-1}{s(s+1)}$$

and

$$sX(s) + sY(s) - 1 - Y(s) = \frac{-3s^2 + s + 1}{(s+1)s^2}$$

or

$$\begin{aligned} X(s) - (s^2 - 1)Y(s) &= 2 - s - \frac{1}{s(s+1)} \\ &= \frac{-s^3 + s^2 + 2s - 1}{s(s+1)} \end{aligned}$$

and

$$\begin{aligned} sX(s) + (s-1)Y(s) &= 1 - \frac{3s^2 - s - 1}{(s+1)s^2} \\ &= \frac{s^3 - 2s^2 + s + 1}{(s+1)s^2}. \end{aligned}$$

Solving for $X(s)$ and $Y(s)$, we have

$$X(s) = \frac{1}{s^2(s+1)} = \frac{1}{s^2} + \frac{1}{s+1} - \frac{1}{s},$$

$$Y(s) = \frac{s^2 - s - 1}{s^2(s+1)} = \frac{1}{s+1} - \frac{1}{s^2}.$$

Hence, taking inverse Laplace transform, we get

$$x(t) = t + e^{-t} - 1, y(t) = e^{-t} - t.$$

EXAMPLE 7.29

Given that $I(0) = 0$, find the current I in RL-network shown in the Figure 7.11.

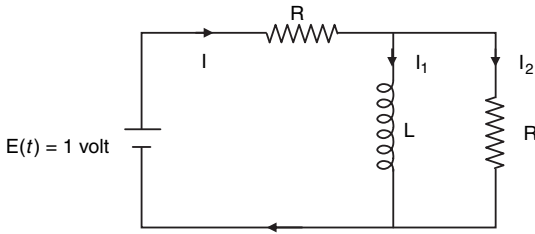


Figure 7.11

Solution. We note that $I = I_1 + I_2$ and so $RI = RI_1 + RI_2$, or equivalently, $RI_2 = RI - RI_1$. By Kirchhoff's law, we have

(a) In the closed loop containing R and L ,

$$RI + L \frac{dI_1}{dt} = E = 1 \quad (5)$$

(b) In the closed loop containing two resistances R ,

$$RI + RI_2 = E = 1$$

or

$$RI + RI - RI_1 = 1$$

or

$$2RI - RI_1 = 1. \quad (6)$$

We want to solve (5) and (6) under the conditions $I(0) = I_1(0) = 0$. Taking Laplace transform yields

$$RF(s) + L\{sG(s)I_1(0)\} = \frac{1}{s}$$

and

$$2R F(s) - RG(s) = \frac{1}{s}.$$

Using $I_1(0) = 0$, we have

$$RF(s) + LsG(s) = \frac{1}{s} \quad (7)$$

and

$$2RF(s) - RG(s) = \frac{1}{s}. \quad (8)$$

Multiplying (7) by R and (8) by Ls and adding, we get

$$(R^2 + 2RLs)F(s) = \frac{R}{s} + L = \frac{R + Ls}{s}$$

or

$$F(s) = \frac{R + Ls}{Rs(R + 2Ls)} = \frac{1}{R} \left(\frac{R + Ls}{s(R + 2Ls)} \right).$$

Using partial fractions, we get

$$F(s) = \frac{1}{R} \left(\frac{1}{s} - \frac{1}{2(s + (R/2L))} \right).$$

Taking inverse Laplace transform yields

$$I(t) = \frac{1}{R} \left[1 - \frac{1}{2} e^{-\frac{R}{2L}t} \right] = \frac{1}{2R} \left[2 - e^{-\frac{R}{2L}t} \right].$$

7.3 DIFFERENCE EQUATIONS

A relationship between the values of a function $y(t)$ and the values of the function at different arguments $y(t + h)$, h constant, is called a *difference equation*. For example,

$$y(n+2) - y(n+1) + y(n) = 2$$

and

$$y(n+2) - 2y(n) + y(n-1) = 1$$

are difference equations.

A relation between the terms of a sequence $\{x_n\}$ is also a difference equation. For example,

$$x_{n+1} + 2x_n = 8$$

is a difference equation.

Difference equations (also called *recurrence relations*) are closely related to differential equations and their theory is basically the same as that of differential equations.

Order of a difference equation is the difference between the largest and smallest arguments occurring in the difference equation divided by the unit of increment.

For example, the order of the difference equation $a_{n+2} - 3a_{n+1} + 2a_n = 5^n$ is $\frac{n+2-n}{1} = 2$.

Solution of a difference equation is an expression for y_n which satisfies the given difference equation.

The aim of this section is to solve difference equations using Laplace transform.

We first make the following *observations*:

(A) Let $f(t) = a^{[t]}$, where $[t]$ is the greatest integer less than or equal to t and $a > 0$. Then $f(t)$ is of exponential order and by definition,

$$\begin{aligned} L\{f(t)\} &= \int_0^{\infty} e^{-st} f(t) dt = \int_0^{\infty} e^{-st} a^{[t]} dt \\ &= \int_0^1 e^{-st} a^0 dt + \int_1^2 e^{-st} a^1 dt + \int_2^3 e^{-st} a^2 dt + \dots \end{aligned}$$

$$\begin{aligned}
&= \frac{1-e^{-s}}{s} + \frac{a(e^{-s}-e^{-2s})}{s} + \frac{a^2(e^{-2s}-e^{-3s})}{s} + \dots \\
&= \frac{1-e^{-s}}{s} [1 + ae^{-s} + a^2e^{-2s} + \dots] \\
&= \frac{1-e^{-s}}{s(1-ae^{-s})} \quad (\operatorname{Re}(s) > \max(0, \log a)).
\end{aligned}$$

(B) If $L^{-1}F\{s\} = f(t)$, then we know that

$$L^{-1}\{e^{-s}F(s)\} = \begin{cases} f(t-1) & \text{for } t > 1 \\ 0 & \text{for } t < 1. \end{cases}$$

Also, by observation (1) above, we have

$$L^{-1}\left\{\frac{1-e^{-s}}{s(1-ae^{-s})}\right\} = a^n \text{ for } n = 0, 1, 2, \dots, \\ n \leq t < n+1.$$

Therefore,

$$\begin{aligned}
L^{-1}\left\{\frac{(1-e^{-s})e^{-s}}{s(1-ae^{-s})}\right\} \\
&= f(t-1) \\
&= a^n \quad \text{for } n \leq t-1 < n+1, n = 0, 1, 2, \\
&= a^n \quad \text{for } n \leq t < n+1, n = 1, 2, 3, \dots
\end{aligned}$$

(C) If $f(t) = na^{n-1}$ for $n \leq t < n+1, n = 0, 1, 2, \dots$, then

$$\begin{aligned}
L\{f(t)\} &= \int_0^\infty e^{-st}f(t)dt \\
&= \int_1^2 e^{-st}dt + 2a \int_2^3 e^{-st}dt + 4a \int_3^4 e^{-st}dt + \dots \\
&= \frac{e^{-s}-e^{-2s}}{s} + 2a \left[\frac{e^{-2s}-e^{-3s}}{s} \right] + 4a^2 \left[\frac{e^{-3s}-e^{-4s}}{s} \right] + \dots \\
&= \frac{e^{-s}(1-e^{-s})}{s} [1 + 2ae^{-s} + 4a^2e^{-2s} + \dots] \\
&= \frac{e^{-s}(1-e^{-s})}{s} \cdot \frac{1}{(1-ae^{-s})^2} = \frac{e^{-s}(1-e^{-s})}{s(1-ae^{-s})^2}.
\end{aligned}$$

Hence

$$L^{-1}\left\{\frac{e^{-s}(1+e^{-s})}{s(1-ae^{-s})^2}\right\} = f(t) = na^{n-1}, \quad n = 0, 1, 2, \dots$$

EXAMPLE 7.30

Solve

$$a_{n+2} - 4a_{n+1} + 3a_n = 0, \quad a_0 = 0, a_1 = 1.$$

Solution. Let us define

$$y(t) = a_n, \quad n \leq t < n+1, \quad n = 0, 1, 2, \dots$$

Then the given difference equation reduces to

$$y(t+2) - 4y(t+1) + 3y(t) = 0.$$

Thus

$$L\{y(t+2)\} - 4L\{y(t+1)\} + 3L\{y(t)\} = 0 \quad (9)$$

Now

$$\begin{aligned}
L\{zy(t+2)\} &= \int_0^\infty e^{-st}y(t+2)dt \\
&= \int_2^\infty e^{-s(u-2)}y(u)du, \quad u=t+2 \\
&= e^{2s} \int_0^\infty e^{-su}y(u)du - e^{2s} \int_0^2 e^{-su}y(u)du. \\
&= e^{2s}L\{y(t)\} - e^{2s} \int_0^1 e^{-su}a_0du - e^{2s} \int_1^2 e^{-su}a_1du \\
&= e^{2s}L\{y(t)\} - e^{2s} \left(\frac{e^{-s}-e^{-2s}}{s} \right) \text{ since } a_0=0, \\
&\hspace{15em} a_1=1 \\
&= e^{2s}L\{y(t)\} - \frac{e^s}{s}(1-e^{-s}),
\end{aligned}$$

$$\begin{aligned}
L\{y(t+1)\} &= \int_0^\infty e^{-st}y(t+1)dt \\
&= \int_1^\infty e^{-s(u-1)}y(u)du, \quad u=t+1 \\
&= \int_0^\infty e^{-s(u-1)}y(u)du - \int_0^1 e^{-s(u-1)}y(u)du \\
&= e^s \int_0^\infty e^{-su}y(u)du - e^s \int_0^1 e^{-su}y(u)du \\
&= e^sL\{y(t)\} - e^s \int_0^1 e^{-su}a_0du \\
&= e^sL\{y(t)\} \text{ since } a_0=0.
\end{aligned}$$

Hence (9) becomes

$$\begin{aligned}
e^{2s}L\{y(t)\} - \frac{e^s}{s}(1-e^{-s}) - 4e^sL\{y(t)\} \\
+ 3L\{y(t)\} = 0,
\end{aligned}$$

which yields

$$\begin{aligned}
 L\{y(t)\} &= \frac{e^s(1-e^{-s})}{s(e^{2s}-4e^s+3)} \\
 &= \frac{e^s(1-e^{-s})}{2s} \left(\frac{1}{e^s-3} - \frac{1}{e^s-1} \right) \\
 &= \frac{1-e^{-s}}{2s} \left(\frac{1}{1-3e^{-s}} - \frac{1}{1-e^{-s}} \right) \\
 &= \frac{1-e^{-s}}{2s(1-3e^{-s})} - \frac{1-e^{-s}}{2s(1-e^{-s})} \\
 &= \frac{1}{2}L\{3^{[t]}\} - \frac{1}{2}L\{1\}, \text{ by observation (1).}
 \end{aligned}$$

Hence inversion yields

$$a_n = \frac{1}{2}[3^n - 1], \quad n = 0, 1, 2, \dots$$

EXAMPLE 7.31

Solve the difference equation

$$y(t+1) - y(t) = 1, \quad y(t) = 0, t < 1.$$

Solution. Taking Laplace transformation of both sides, we get

$$L\{y(t+1)\} - L\{y(t)\} = L\{1\}.$$

But, as in Example 7.30, we have

$$\begin{aligned}
 \text{and so} \quad L\{y(t+1)\} &= e^s L\{y(t)\}, \\
 e^s L\{y(t)\} - L\{y(t)\} &= \frac{1}{s}
 \end{aligned}$$

or

$$L\{y(t)\} = \frac{1}{s(e^s - 1)}.$$

Taking inverse Laplace transform, we have

$$\begin{aligned}
 y(t) &= L^{-1} \left\{ \frac{1}{s(e^s - 1)} \right\} \\
 &= [t], t > 0 \text{ (see Example 7.3)}
 \end{aligned}$$

EXAMPLE 7.32

Solve

$$a_{n+2} - 4a_{n+1} + 3a_n = 5^n, \quad a_0 = 0, \quad a_1 = 1$$

Solution. We define

$$y(t) = a_n, \quad n \leq t < n+1, \quad n = 0, 1, 2, \dots$$

Then the difference equation becomes

$$y(t+2) - 4y(t+1) + 3y(t) = 5^n. \quad (10)$$

By observation (B) and Example 7.30, we have

$$\begin{aligned}
 L\{5^n\} &= \frac{1-e^{-s}}{s(1-5e^{-s})}, \\
 L\{y(t+2)\} &= e^{2s}L\{y(t)\} - \frac{e^s}{s}(1-e^{-s}), \\
 L\{y(t+1)\} &= e^sL\{y(t)\}.
 \end{aligned}$$

Taking Laplace transform of both sides of (10), we have

$$L\{y(t+2)\} - 4L\{y(t+1)\} + 3L\{y(t)\} = L\{5^{[t]}\}$$

or

$$\begin{aligned}
 e^{2s}L\{y(t)\} - \frac{e^s}{s}(1-e^{-s}) - 4e^sL\{y(t)\} \\
 + 3L\{y(t)\} &= L\{5^{[t]}\}
 \end{aligned}$$

or

$$\{e^{2s} - 4e^s + 3\}L\{y(t)\} = \frac{e^s(1-e^{-s})}{s} + L\{5^{[t]}\}.$$

Hence

$$\begin{aligned}
 L\{y(t)\} &= \frac{e^s(1-e^{-s})}{s(e^{2s}-4e^s+3)} + \frac{L\{5^{[t]}\}}{e^{2s}-4e^s+3} \\
 &= \frac{1}{2}L\{3^{[t]}\} - \frac{1}{2}L\{1\} + \frac{L\{5^{[t]}\}}{e^{2s}-4e^s+3}.
 \end{aligned}$$

But

$$\begin{aligned}
 \frac{L\{5^{[t]}\}}{e^{2s}-4e^s+3} &= \frac{1-e^{-s}}{s(1-5e^{-s})} \cdot \frac{1}{e^{2s}-4e^s+3} \\
 &= \frac{e^s-1}{s(e^s-5)(e^s-3)(e^s-1)} \\
 &= \frac{e^s-1}{s} \left(\frac{1/8}{e^s-5} - \frac{1/4}{e^s-3} + \frac{1/8}{e^s-1} \right) \\
 &= \frac{1-e^{-s}}{s} \left(\frac{1/8}{1-5e^{-s}} - \frac{1/4}{1-3e^{-s}} + \frac{1/8}{1-e^{-s}} \right) \\
 &= \frac{1}{8}L\{1\} + \frac{1}{8}L\{5^{[t]}\} - \frac{1}{4}L\{3^{[t]}\}.
 \end{aligned}$$

Hence

$$L\{y(t)\} = \frac{-3}{8}L\{1\} + \frac{1}{4}L\{3^{[t]}\} + \frac{1}{8}L\{5^{[t]}\}$$

and so

$$a_n = \frac{-3}{8} + \frac{1}{4}3^n + \frac{1}{8}5^n.$$

EXAMPLE 7.33

Solve

$$a_{n+2} - 3a_{n+1} + 2a_n = 2^n, \quad a_0 = 0, \quad a_1 = 1.$$

Solution. We define

$$y(t) = a_n, \quad n \leq t < n+1.$$

Then the given equation reduces to

$$y(t+2) - 3y(t+1) + 2y(t) = 2^{[t]}.$$

Taking Laplace transform of both sides, we get

$$\begin{aligned} \mathcal{L}\{y(t+2)\} - 3\mathcal{L}\{y(t+1)\} + 2\mathcal{L}\{y(t)\} \\ = \mathcal{L}\{2^{[t]}\}. \end{aligned}$$

But, as in the previous examples,

$$\mathcal{L}\{y(t+2)\} = e^{2s}\mathcal{L}\{y(t)\} - \frac{e^s}{s}(1 - e^{-s}),$$

$$\mathcal{L}\{y(t+1)\} = e^s\mathcal{L}\{y(t)\}.$$

Therefore,

$$(e^{2s} - 3e^s + 2)\mathcal{L}\{y(t)\} = \frac{e^s}{s}(1 - e^{-s}) + \mathcal{L}\{2^{[t]}\},$$

which gives

$$\begin{aligned} \mathcal{L}\{y(t)\} &= \frac{e^s}{s(e^{2s} - 3e^s + 2)}(1 - e^{-s}) + \frac{\mathcal{L}\{2^{[t]}\}}{e^{2s} - 3e^s + 2} \\ &= \frac{e^s(1 - e^{-s})}{s} \left(\frac{1}{e^s - 2} - \frac{1}{e^s - 1} \right) + \frac{\mathcal{L}\{2^{[t]}\}}{e^{2s} - 3e^s + 2} \\ &= \frac{1 - e^{-s}}{s} \left(\frac{1}{1 - 2e^{-s}} - \frac{1}{1 - e^{-s}} \right) + \frac{\mathcal{L}\{2^{[t]}\}}{e^{2s} - 3e^s + 2} \\ &= \frac{1 - e^{-s}}{s(1 - e^{-s})} - \frac{1}{s} + \frac{\mathcal{L}\{2^{[t]}\}}{e^{2s} - 3e^s + 2} \\ &= \mathcal{L}\{2^{[t]}\} - \mathcal{L}\{1\} + \frac{\mathcal{L}\{2^{[t]}\}}{e^{2s} - 3e^s + 2}. \end{aligned}$$

But

$$\begin{aligned} \frac{\mathcal{L}\{2^{[t]}\}}{e^{2s} - 3e^s + 2} &= \frac{1 - e^{-s}}{s(1 - 2e^{-s})} \cdot \frac{1}{e^{2s} - 3e^s + 2} \\ &= \frac{e^s - 1}{s(e^s - 2)(e^s - 2)(e^s - 1)} \\ &= \frac{e^s - 1}{s} \left[\frac{-1}{e^s - 2} + \frac{1}{(e^s - 2)^2} + \frac{1}{e^s - 1} \right] \\ &= \frac{1 - e^{-s}}{s} \left[\frac{-1}{1 - 2e^{-s}} + \frac{e^{-s}}{(1 - 2e^{-s})^2} + \frac{1}{1 - e^{-s}} \right] \\ &= -\mathcal{L}\{2^{[t]}\} + \mathcal{L}\{n2^{n-1}\} + \mathcal{L}\{1\}. \end{aligned}$$

Therefore,

$$\mathcal{L}\{y(t)\} = \mathcal{L}\{n2^{n-1}\}.$$

Hence

$$a_n = n2^{n-1}, \quad n = 0, 1, 2, \dots$$

Verification. We note that

$$\begin{aligned} a_{n+1} &= (n+1)2^n, \quad a_{n+2} = (n+2)2^{n+1} \\ &= (n+2)2^{n+1}. \end{aligned}$$

Therefore,

$$\begin{aligned} a_{n+2} - 3a_{n+1} + 2a_n &= (n+2)2^{n+1} - (3n+3)2^n + 2n2^{n-1} \\ &= 2^n[2n+4-3n-3+n] = 2^n. \end{aligned}$$

EXAMPLE 7.34

Solve

$$y(t) - y(t - \pi) = \sin t, \quad y(t) = 0, \quad t \leq 0.$$

Solution. Taking Laplace transform, we have

$$\mathcal{L}\{y(t)\} - \mathcal{L}\{y(t - \pi)\} = \mathcal{L}\{\sin t\}. \quad (11)$$

But

$$\begin{aligned} \mathcal{L}\{y(t - \pi)\} &= \int_0^\infty e^{-st} y(t - \pi) dt \\ &= \int_{-\pi}^\infty e^{-s(u+\pi)} y(u) du, \quad u = t - \pi \\ &= e^{-s\pi} \int_0^\infty e^{-su} y(u) du, \quad y(u) = 0, \quad u < 0 \\ &= e^{-s\pi} \mathcal{L}\{y(t)\}. \end{aligned}$$

Hence (11) reduces to

$$\mathcal{L}\{y(t)\} - e^{-s\pi} \mathcal{L}\{y(t)\} = \frac{1}{s^2 + 1}$$

and so

$$\mathcal{L}\{y(t)\} = \frac{1}{(s^2 + 1)(1 - e^{-s\pi})}.$$

Taking inverse transform, we get (see Example 5.72)

$$\begin{aligned} y(t) &= \begin{cases} \sin t & \text{for } 0 < t < \pi \\ 0 & \text{for } \pi < t < 2\pi \end{cases} \\ &= \begin{cases} \sin t & \text{for } 2n\pi < t < (2n+1)\pi \\ 0 & \text{for } (2n+1)\pi < t < (2n+2)\pi \end{cases} \\ &\quad (\text{due to periodicity}), \text{ for } n = 0, 1, 2, \dots \end{aligned}$$

This is half-wave rectified sinusoidal function.

EXAMPLE 7.35

Find explicit formula (solution) for *Fibonacci sequence*:

$$a_{n+2} = a_{n+1} + a_n, \quad a_0 = 0, \quad a_1 = 1.$$

Solution. Define

$$y(t) = a_n, \quad n \leq t < n+1, \quad n = 0, 1, 2, \dots$$

Then the given difference equation reduces to

$$y(t+2) - y(t+1) - y(t) = 0.$$

Taking Laplace transform, we have

$$L\{y(t+2)\} - L\{y(t+1)\} - L\{y(t)\} = 0.$$

But

$$L\{y(t+2)\} = e^{2s}L\{y(t)\} - \frac{e^s(1 - e^{-s})}{s}$$

$$L\{y(t+1)\} = e^sL\{y(t)\}.$$

Therefore, we get

$$(e^{2s} - e^s - 1)L\{y(t)\} = \frac{e^s(1 - e^{-s})}{s}$$

or

$$\begin{aligned} L\{y(t)\} &= \frac{e^s(1 - e^{-s})}{s(e^{2s} - e^s - 1)} \\ &= \frac{e^s(1 - e^{-s})}{s} \left[\frac{\frac{1}{\sqrt{5}}}{e^s - \frac{1+\sqrt{5}}{2}} - \frac{\frac{1}{\sqrt{5}}}{e^s - \frac{1-\sqrt{5}}{2}} \right] \\ &= \frac{1 - e^{-s}}{s} \left[\frac{1}{\sqrt{5}} \left(\frac{1}{1 - \frac{1+\sqrt{5}}{2}e^{-s}} - \frac{1}{1 - \frac{1-\sqrt{5}}{2}e^{-s}} \right) \right] \\ &= \frac{1}{\sqrt{5}} \left[L\left\{ \left(\frac{1+\sqrt{5}}{2} \right)^{[t]} \right\} - L\left\{ \left(\frac{1-\sqrt{5}}{2} \right)^{[t]} \right\} \right]. \end{aligned}$$

Hence

$$a_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right], \quad n \geq 0.$$

EXAMPLE 7.36

Solve the difference equation

$$y(t) + y(t-1) = e^t, \quad y(t) = 0, \quad t \leq 0.$$

Solution. Taking Laplace transform of both sides of the given equation, we get

$$L\{y(t)\} + L\{y(t-1)\} = L\{e^t\}.$$

But

$$\begin{aligned} L\{y(t-1)\} &= \int_0^\infty e^{-st} y(t-1) dt \\ &= \int_{-1}^\infty e^{s(u+1)} y(u) du, \quad u = t-1 \\ &= e^{-s} \int_{-1}^0 e^{-su} y(u) du + e^{-s} \int_0^\infty e^{-su} y(u) du \\ &= e^{-s} \int_0^\infty e^{-su} y(u) du \quad \text{since } y(t) = 0 \text{ for } t \geq 0 \\ &= e^{-s} L\{y(t)\}. \end{aligned}$$

Therefore, we have

$$L\{y(t)\} + e^{-s} L\{y(t)\} = \frac{1}{s-1}$$

or

$$\begin{aligned} L\{y(t)\} &= \frac{1}{(s-1)(1+e^{-s})} \\ &= \frac{1}{(s-1)} [1 - e^{-s} + e^{-2s} - e^{-3s} + \dots] \\ &= \sum_{n=0}^\infty \frac{(-1)^n e^{-ns}}{s-1}. \end{aligned}$$

Hence

$$y(t) = \sum_{n=0}^{[t]} (-1)^n e^{t-n}.$$

EXAMPLE 7.37

Solve the *differential-difference equation*

$$y'(t) - y(t-1) = t, \quad y(t) = 0, \quad t \leq 0.$$

Solution. Taking Laplace transform of both sides, we have

$$L\{y'(t)\} - L\{y(t-1)\} = L\{t\}.$$

Now

$$L\{y'(t)\} = s L\{y(t)\} - y(0) = s L\{y(t)\}$$

and

$$L\{y(t-1)\} = e^{-s} L\{y(t)\}.$$

Therefore,

$$\{e^{-s} + s\} L\{y(s)\} = \frac{1}{s^2}$$

or

$$\begin{aligned} L\{y(t)\} &= \frac{1}{s^2(e^{-s} + s)} = \frac{1}{s^3(1 + \frac{e^{-s}}{s})} \\ &= \frac{1}{s^3} \left(1 - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s^2} - \frac{e^{-3s}}{s^3} + \dots \right) \\ &= \frac{1}{s^3} - \frac{e^{-s}}{s^4} + \frac{e^{-2s}}{s^5} - \frac{e^{-3s}}{s^6} + \dots \\ &= \sum_{n=0}^{\infty} \frac{e^{-ns}}{s^{n+3}}. \end{aligned}$$

But

$$L^{-1} \left\{ \frac{e^{-ns}}{s^{n+3}} \right\} = \begin{cases} \frac{(t-n)^{n+2}}{(n+2)!} & \text{for } t \geq n \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, if $[t]$ denotes the greatest integer less than or equal to t , then

$$y(t) = \sum_{n=0}^{[t]} \frac{(t-n)^{n+2}}{(n+2)!}.$$

EXAMPLE 7.38

Solve the differential-difference equation

$$y''(t) - y(t-1) = f(t), \quad y(t) = 0, \quad y'(t) = 0 \text{ for } t \leq 0,$$

$$f(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 2t & \text{for } t > 0. \end{cases}$$

Solution. Taking Laplace transform of both sides, we get

$$L\{y''(t)\} - L\{y(t-1)\} = L\{f(t)\}$$

or

$$s^2 L\{y(t)\} - sy(0) - y(0) - e^{-s} L\{y(t)\} = \frac{2}{s^2}$$

or

$$(s^2 - e^{-s}) L\{y(t)\} = \frac{2}{s^2}$$

or

$$\begin{aligned} L\{y(t)\} &= \frac{2}{s^2(s^2 - e^{-s})} = \frac{2}{s^4(1 - \frac{e^{-s}}{s^2})} \\ &= \frac{2}{s^4} \left(1 + \frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^4} + \frac{e^{-3s}}{s^6} + \dots \right) \\ &= 2 \left(\frac{1}{s^4} + \frac{e^{-s}}{s^6} + \frac{e^{-2s}}{s^8} + \frac{e^{-3s}}{s^{10}} + \dots \right) \\ &= 2 \sum_{n=0}^{\infty} \frac{e^{-ns}}{s^{2n+4}}. \end{aligned}$$

But

$$L^{-1} \left\{ \frac{e^{-ns}}{s^{n+4}} \right\} = \frac{(t-n)^{2n+3}}{(2n+3)!}.$$

Hence

$$y(t) = 2 \sum_{n=0}^{[t]} \frac{(t-n)^{2n+3}}{(2n+3)!}.$$

7.4 INTEGRAL EQUATIONS

Equations of the form

$$f(t) = g(t) + \int_a^b K(t, u) f(u) du$$

and

$$g(t) = \int_a^b K(t, u) f(u) du,$$

where the function $f(t)$ to be determined appears under the integral sign are called *integral equations*.

In an integral equation, $K(t, u)$ is called the *kernel*. If a and b are constants, the equation is called a *Fredholm integral equation*. If a is a constant and $b = t$, then the equation is called a *Volterra integral equation*.

If the kernel $K(t, u)$ is of the form $K(t-u)$, then the integral $\int_0^t K(t-u) f(u) du$ represents convolution. Thus, we have

$$f(t) = g(t) + \int_0^t K(t-u) f(u) du = g(t) + K(t) * f(t).$$

Such equations are called *convolution-type integral equations*. Taking Laplace transform of convolution-type integral equation, we have

$$\begin{aligned} L\{f(t)\} &= L\{g(t)\} + L\{K(t) * f(t)\} \\ &= L\{g(t)\} + L\{K(t)\} L\{f(t)\}, \end{aligned}$$

by using Convolution theorem. Hence

$$(1 - L\{K(t)\} L\{f(t)\}) = L\{g(t)\},$$

which implies

$$L\{f(t)\} = \frac{L\{g(t)\}}{1 - L\{K(t)\}}.$$

Taking inverse Laplace transform yields the solution $f(t)$.

EXAMPLE 7.39

Solve the integral equation

$$f(t) = e^{-t} + \int_0^t \sin(t-u)f(u) du.$$

Solution. Taking Laplace transform of both sides of the given equation, we get

$$L\{f(t)\} = L\{e^{-t}\} + L\{\sin t\} L\{f(t)\},$$

which yields

$$L\{f(t)\} = \frac{L\{e^{-t}\}}{1 - L\{\sin t\}} = \frac{s^2 + 1}{s^2(s+1)}.$$

Using partial fractions, we obtain

$$L\{f(t)\} = \frac{2}{s+1} + \frac{1}{s^2} - \frac{1}{s}.$$

Taking inverse Laplace transform yields

$$f(t) = 2e^{-t} + t - 1.$$

EXAMPLE 7.40

Solve the integral equation

$$f(t) = 1 + \int_0^t \sin(t-u)f(u) du.$$

Solution. We have

$$f(t) = 1 + \sin t * f(t).$$

Taking Laplace transform yields

$$L\{f(t)\} = L\{1\} + L\{f(t)\}L\{\sin t\}$$

or

$$\begin{aligned} L\{f(t)\} &= \frac{L\{1\}}{1 - L\{\sin t\}} = \frac{1}{s\left(1 - \frac{1}{s^2+1}\right)} \\ &= \frac{s^2+1}{s^3} = \frac{1}{s} + \frac{1}{s^3}. \end{aligned}$$

Taking inverse Laplace transform, we get

$$f(t) = 1 + \frac{t^2}{2}.$$

EXAMPLE 7.41

Solve

$$\int_0^t f(u)f(t-u) du = 16 \sin 4t.$$

Solution. The given equation in convolution form is

$$f(t) * f(t) = 16 \sin 4t.$$

Taking Laplace transform, we get

$$L\{f(t) * f(t)\} = 16 L\{\sin 4t\}$$

or

$$L\{f(t)\}L\{f(t)\} = 16 L\{\sin 4t\}$$

(using convolution theorem).

or

$$[L\{f(t)\}]^2 = \frac{16(4)}{s^2 + 16} = \frac{64}{s^2 + 16}.$$

or

$$L\{f(t)\} = \frac{\pm 8}{\sqrt{s^2 + 16}}.$$

Taking inverse Laplace transform yields

$$f(t) = \pm 8 J_0(4t),$$

where J_0 is Bessel's function of order zero.

EXAMPLE 7.42

Solve the integral equation

$$g(x) = f(x) - \int_0^t e^{t-u}f(u) du.$$

Solution. The given equation, in convolution form, is

$$g(t) = f(t) - e^t * f(t).$$

Taking Laplace transform of both sides, we get

$$L\{g(t)\} = L\{f(t)\} - L\{e^t\} L\{f(t)\}$$

or

$$\begin{aligned} L\{f(t)\} &= \frac{L\{g(t)\}}{1 - L\{e^t\}} = \frac{(s-1)L\{g(t)\}}{(s-2)} \\ &= L\{g(t)\} + \frac{L\{g(t)\}}{s-2} \\ &= L\{g(t)\} + L\{g(t)\} L\{e^{2t}\}. \end{aligned}$$

Taking inverse Laplace transform yields

$$\begin{aligned} f(t) &= g(t) + g(t) * e^{2t} \\ &= g(t) + \int_0^t g(u) e^{2(t-u)} du. \end{aligned}$$

Definition 7.1. The convolution-type integral equation of the form

$$\int_0^t \frac{f(u)}{(t-u)^n} du = g(t), \quad 0 < n < 1$$

is called *Abel's integral equation*.

We consider below examples of this type of integral equations.

EXAMPLE 7.43

Solve the integral equation

$$1 + 2t - t^2 = \int_0^t f(u) \frac{1}{\sqrt{t-u}} du.$$

Solution. The given equation is a special case of Abel's integral equation. The convolution form of this equation is

$$1 + 2t - t^2 = f(t) * \frac{1}{\sqrt{t}}.$$

Taking Laplace transform yields

$$L\{f(t)\} L\left\{\frac{1}{\sqrt{t}}\right\} = L\{1\} + 2L\{t\} - L\{t^2\}$$

or

$$L\{f(t)\} \sqrt{\frac{\pi}{s}} = \frac{1}{s} + \frac{2}{s^2} - \frac{2}{s^3}$$

or

$$L\{f(t)\} = \frac{1}{\sqrt{\pi}} \left[\frac{1}{s^{1/2}} + \frac{2}{s^{3/2}} - \frac{2}{s^{5/2}} \right].$$

Taking inverse transform, we get

$$\begin{aligned} f(t) &= \frac{1}{\sqrt{\pi}} \left[\frac{t^{-1/2}}{\Gamma(1/2)} + \frac{2t^{1/2}}{\Gamma(3/2)} - \frac{2t^{3/2}}{\Gamma(5/2)} \right] \\ &= \frac{1}{\sqrt{\pi}} \left[\frac{t^{-1/2}}{\sqrt{\pi}} + \frac{2t^{1/2}}{(1/2)\sqrt{\pi}} - \frac{2t^{3/2}}{(3/2)(1/2)\sqrt{\pi}} \right] \\ &= \frac{1}{\pi} [t^{-1/2} + 4t^{1/2} - 8t^{3/2}]. \end{aligned}$$

EXAMPLE 7.44

Solve the integral equation

$$\int_0^t f(u) \frac{1}{\sqrt{t-u}} du = 1 + t + t^2.$$

Solution. Proceeding as in Example 7.43 above, we have

$$L\{f(t)\} = \frac{1}{\sqrt{\pi}} \left[\frac{1}{s^{1/2}} + \frac{1}{s^{3/2}} + \frac{2}{s^{5/2}} \right],$$

which on inversion yields

$$f(t) = \frac{1}{\pi} [t^{-1/2} + 2t^{1/2} + \frac{8}{3} t^{3/2}].$$

EXAMPLE 7.45 (Tautochrone Curve)

A particle (bead) of mass m is to slide down a frictionless curve such that the duration T_0 of descent due to gravity is independent of the starting point. Find the shape of such curve (known as *Tautochrone curve*).

Solution. Velocity of the bead at the starting point is zero since it starts from rest at that point, say P with co-ordinates (x, y) . Let Q = (x, u) be some intermediate point during the motion. Let the origin O be the lowest point of the curve (Figure 7.12). Let the length of the arc OQ be s .

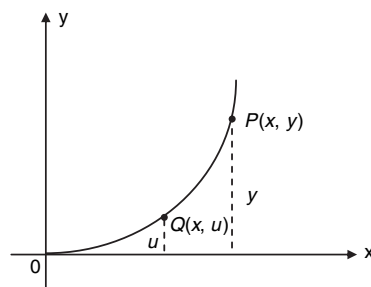


Figure 7.12

By law of conservation of energy, potential energy at P + kinetic energy at P = potential energy at Q + Kinetic energy at Q, that is,

$$mgy + 0 = mgu + \frac{1}{2} m \left(\frac{ds}{dt} \right)^2,$$

where $\frac{ds}{dt}$ is the instantaneous velocity of the particle at Q.

Thus

$$\left(\frac{ds}{dt} \right)^2 = 2g(y - u)$$

and so

$$\frac{ds}{dt} = -\sqrt{2g(y - u)},$$

negative sign since s decreases with time. The total time T_0 taken by the particle to go from P to Q is

$$T_0 = \int_0^{T_0} dt = \int_y^0 \frac{-ds}{\sqrt{2g(y-u)}} = \int_0^y \frac{ds}{\sqrt{2g(y-u)}}.$$

If $\frac{ds}{du} = f(u)$, then $ds = f(u) du$ and so

$$T_0 = \frac{1}{\sqrt{2g}} \int_0^y \frac{f(u)}{\sqrt{y-u}} du$$

The convolution form of this integral equation is

$$T_0 = \frac{1}{\sqrt{2g}} f(y) * \frac{1}{\sqrt{y}}.$$

Taking Laplace transform of both sides and using Convolution theorem, we have

$$L\{T_0\} = \frac{1}{\sqrt{2g}} L\{f(y)\} L\left\{\frac{1}{\sqrt{y}}\right\}$$

or

$$L\{f(y)\} = \frac{\sqrt{2g} T_0/s}{\sqrt{T/s}} = \frac{\sqrt{2g/\pi}}{s^{1/2}} T_0 = \frac{C_0}{s^{1/2}},$$

where C_0 is a constant. Inverse Laplace transform then yields

$$f(y) = \frac{C}{\sqrt{y}}.$$

Since $f(y) = \frac{dx}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$, we get

$$1 + \left(\frac{dx}{dy}\right)^2 = \frac{C^2}{y}$$

or

$$\left(\frac{dx}{dy}\right)^2 = \frac{C^2}{y} - 1 = \frac{C^2 - y}{y}$$

or

$$\frac{dx}{dy} = \sqrt{\frac{C^2 - y}{y}}$$

or

$$x = \int \sqrt{\frac{C^2 - y}{y}} dy.$$

Putting $y = C^2 \sin^2 \theta$, we get

$$x = \frac{C^2}{2} (\theta + \sin \theta), \quad y = \frac{C^2}{2} (1 - \cos \theta),$$

which are the parametric equations of a *cycloid*.

7.5 INTEGRO-DIFFERENTIAL EQUATIONS

An integral equation in which various derivatives of the unknown function $f(t)$ are also present is called an *integro-differential equation*. These types of equations can also be solved by the method of Laplace transform.

EXAMPLE 7.46

Solve the following integro-differential equation:

$$y'(t) = \int_0^t y(u) \cos(t-u) du, \quad y(0) = 1.$$

Solution. We write the given equation in convolution form as

$$y'(t) = y(t) * \cos t.$$

Taking Laplace transform and using Convolution theorem yields

$$L\{y'(t)\} = L\{y(t)\} L\{\cos t\}$$

or

$$sL\{y(t)\} - y(0) = L\{y(t)\} \frac{s}{s^2 + 1}$$

or

$$\left(s - \frac{s}{s^2 + 1}\right) L\{y(t)\} = 1, \quad \text{since } y(0) = 1$$

or

$$L\{y(t)\} = \frac{s^2 + 1}{s^3} = \frac{1}{s} + \frac{1}{s^3}.$$

Taking inverse Laplace transform, we get

$$y(t) = 1 + \frac{1}{2} t^2.$$

EXAMPLE 7.47

Solve

$$y'(t) + 5 \int_0^t y(u) \cos 2(t-u) dy = 10, \quad y(0) = 2.$$

Solution. Convolution form of the equation is

$$y'(t) + 5 \cos t * y(t) = 10.$$

Taking Laplace transform and using Convolution theorem, we have

$$sL\{y(t)\} - y(0) + \frac{5sL\{y(t)\}}{s^2 + 4} = \frac{10}{s}$$

or

$$\begin{aligned} \mathcal{L}\{y(t)\} &= \frac{2s^3 + 10s^2 + 8s + 40}{s^2(s^2 + 9)} \\ &= \frac{1}{9} \left\{ \frac{8}{s} + \frac{40}{s^2} + \frac{10s}{s^2 + 9} + \frac{50}{s^2 + 9} \right\}. \end{aligned}$$

Hence

$$y(t) = \frac{1}{9} \left(8 + 40t + 10 \cos t + \frac{50}{3} \sin 3t \right).$$

7.6 SOLUTION OF PARTIAL DIFFERENTIAL EQUATION

Consider the function $u = u(x, t)$, where $t \geq 0$ is a time variable. Suppose that $u(x, y)$, when regarded as a function of t , satisfies the sufficient conditions for the existence of its Laplace transform.

Denoting the Laplace transform of $u(x, t)$ with respect to t by $U(x, s)$, we see that

$$U(x, s) = \mathcal{L}\{u(x, t)\} = \int_0^\infty e^{-st} u(x, t) dt.$$

The variable x is the untransformed variable. For example,

$$\mathcal{L}\{e^{a(x-t)}\} = e^{ax} \mathcal{L}\{e^{-at}\} = e^{ax} \frac{1}{s+a}.$$

Theorem 7.2. Let $u(x, t)$ be defined for $t \geq 0$. Then

$$(a) \mathcal{L}\left\{\frac{\partial u}{\partial x}\right\} = \frac{d}{dx}(U(x, s))$$

$$(b) \mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} = sU(x, s) - u(x, 0)$$

$$(c) \mathcal{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{d^2}{dx^2}(U(x, s))$$

$$(d) \mathcal{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} = s^2 U(x, s) - su(x, 0) - \frac{\partial u}{\partial t}(x, 0).$$

Proof: (a) We have, by Leibnitz's rule for differentiating under the integration,

$$\begin{aligned} \mathcal{L}\left\{\frac{\partial u}{\partial x}\right\} &= \int_0^\infty e^{-st} \frac{\partial u}{\partial x} dt = \frac{d}{dx} \int_0^\infty e^{-st} u(x, t) dt \\ &= \frac{d}{dx}(U(x, s)). \end{aligned}$$

(b) Integrating by parts, we get

$$\begin{aligned} \mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} &= \int_0^\infty e^{-st} \frac{\partial u(x, t)}{\partial t} dt \\ &= \lim_{T \rightarrow 0} \int_0^T e^{-st} \frac{\partial u(x, t)}{\partial t} dt \\ &= \lim_{T \rightarrow 0} \left\{ \left[e^{-st} u(x, t) \right]_0^T + s \int_0^T e^{-st} u(x, t) dt \right\} \\ &= s \int_0^\infty e^{-st} u(x, t) dt - u(x, 0) \\ &= sU(x, s) - u(x, 0). \end{aligned}$$

(c) Taking $V = \frac{\partial u}{\partial x}$, we have by (a),

$$\begin{aligned} \mathcal{L}\left\{\frac{\partial^2 u}{\partial x^2}\right\} &= \mathcal{L}\left\{\frac{\partial V}{\partial x}\right\} = \frac{d}{dx}(V(x, s)) \\ &= \frac{d}{dx} \left(\frac{d}{dx}(U(x, s)) \right) = \frac{d^2}{dx^2}(U(x, s)). \end{aligned}$$

(d) Let $v = \frac{\partial u}{\partial t}$. Then

$$\begin{aligned} \mathcal{L}\left\{\frac{\partial^2 u}{\partial t^2}\right\} &= \mathcal{L}\left\{\frac{\partial v}{\partial t}\right\} = sV(x, s) - v(x, 0) \\ &= s[sU(x, s) - u(x, 0)] - \frac{\partial u}{\partial t}(x, 0) \\ &= s^2 U(x, s) - su(x, 0) - \frac{\partial u}{\partial t}(x, 0). \end{aligned}$$

Theorem 7.2 suggest that if we apply Laplace transform to both sides of the given partial differential equation, we shall get an ordinary differential equation in U as a function of single variable x . This ordinary differential equation is then solved by the usual methods.

EXAMPLE 7.48

Solve

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}, \quad u(x, 0) = x, \quad u(0, t) = t,$$

Solution. Taking Laplace transform, we get

$$\mathcal{L}\left\{\frac{\partial u}{\partial x}\right\} = \mathcal{L}\left\{\frac{\partial u}{\partial t}\right\}.$$

Using Theorem 7.2, we get

$$\frac{d}{dx}[U(x, s)] = sU(x, s) - u(x, 0) = sU(x, s) - x.$$

Thus, we have first order differential equation

$$\frac{d}{dx}[U(x, s)] - sU(x, s) = -x$$

The integrating factor is

$$e^{\int -s dx} = e^{-sx}.$$

Therefore,

$$\begin{aligned} U(x, s) e^{-sx} &= \int -xe^{-sx} dx \\ &= -\left[x \frac{e^{-sx}}{-s} - \int \frac{e^{-sx}}{-s} dx \right] + C \\ &= \frac{x e^{-sx}}{s} + \frac{e^{-sx}}{s^2} + C \\ &\quad (\text{constant of integration}). \end{aligned}$$

This yields

$$U(x, s) = \frac{x}{s} + \frac{1}{s^2} + C e^{sx}. \quad (12)$$

Now the boundary condition $u(0, t)$ is a function of t . Taking Laplace transform of this function, we have

$$U(0, s) = L\{u(0, t)\} = L\{t\} = \frac{1}{s^2}.$$

Then taking $x = 0$ in (12), we have

$$\frac{1}{s^2} = \frac{1}{s^2} + C$$

and so $C = 0$. Thus, we have

$$U(x, s) = \frac{x}{s} + \frac{1}{s^2}.$$

Taking inverse Laplace transform, we have

$$u(x, t) = x + t.$$

EXAMPLE 7.49

Solve the partial differential equation

$$\frac{\partial u}{\partial t} + x \frac{\partial u}{\partial x} = x, \quad x > 0, \quad t > 0$$

with the initial and boundary conditions $u(x, 0) = 0$, $x > 0$ and $u(0, t) = 0$ for $t > 0$.

Solution. Taking Laplace transform with respect to t , we get

$$L\left\{\frac{\partial u}{\partial t}\right\} + L\left\{x \frac{\partial u}{\partial x}\right\} = L\{x\}$$

which yields

$$sU(x, s) - u(x, 0) + x \frac{d}{dx}U(x, s) = \frac{x}{s}.$$

Since $u(x, 0) = 0$, this reduces to

$$\frac{d}{dx}U(x, s) + \frac{x}{s}U(x, s) = \frac{1}{s}. \quad (13)$$

The integrating factor is

$$e^{\int \frac{x}{s} dx} = e^{s \log x} = x^s.$$

Therefore solution of (13) is

$$U(x, s)x^s = \frac{1}{s} \int x^s dx + C = \frac{1}{s} \frac{x^{s+1}}{s+1} + C = \frac{x^{s+1}}{s(s+1)} + C$$

and so

$$U(x, s) = \frac{x}{s(s+1)} + C \quad (\text{constant of integration}). \quad (14)$$

Now since $U(0, t) = 0$, its Laplace transform is 0, that is, $U(0, s) = 0$. Therefore, (14) implies $C = 0$. Hence

$$U(x, s) = \frac{x}{s(s+1)} = x \left(\frac{1}{s} - \frac{1}{s+1} \right).$$

Taking inverse Laplace transform, we get the solution as

$$u(x, t) = x(1 - e^{-t}).$$

EXAMPLE 7.50

Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x > 0, \quad t > 0$$

under the conditions

$$u(x, 0) = 1, \quad u(0, t) = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} u(x, t) = 1.$$

Solution. The given equation is heat conduction equation in a solid, where $u(x, t)$ is the temperature at position x at any time t and diffusivity is 1. The boundary condition $u(0, t) = 0$ indicates that temperature at $x = 0$ is 0 and $\lim_{x \rightarrow \infty} u(x, t) = 1$ indicates that the temperature for large values of x is 1 whereas

$u(x, 0) = 1$ represents the initial temperature 1 in the semi-infinite medium ($x > 0$) (Figure 7.13).

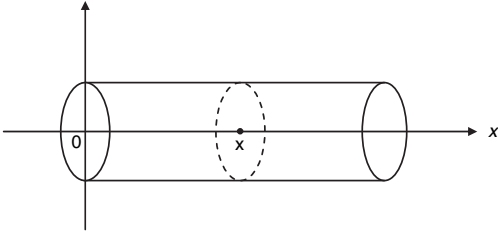


Figure 7.13

Taking Laplace transform, yields

$$sU(x, s) - u(x, 0) = \frac{d^2}{dx^2}U(x, s).$$

Since $u(x, 0) = 1$, we have

$$\frac{d^2}{dx^2}U(x, s) - sU(x, s) = -1.$$

The general solution of this equation is

$$\begin{aligned} U(x, s) &= \text{C.F.} + \text{P.I.} \\ &= [c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}] + \frac{1}{s}. \end{aligned} \quad (15)$$

The conditions $u(0, t) = 0$ yields

$$U(0, s) = L\{u(0, t)\} = 0, \quad (16)$$

whereas $\lim_{x \rightarrow \infty} u(x, t) = 1$ yields

$$\begin{aligned} \lim_{x \rightarrow \infty} U(x, s) &= \lim_{x \rightarrow \infty} L\{u(x, t)\} = L\{\lim_{x \rightarrow \infty} u(x, t)\} \\ &= L\{1\} = \frac{1}{s}. \end{aligned} \quad (17)$$

Now (15) and (17) imply $c_1 = 0$. Then (16) implies $c_2 = -\frac{1}{s}$. Hence

$$U(x, s) = \frac{1}{s} - \frac{e^{-\sqrt{s}x}}{s}.$$

Taking inverse Laplace transform, we get

$$\begin{aligned} u(x, t) &= 1 - L^{-1}\left\{\frac{e^{-\sqrt{s}x}}{s}\right\} \\ &= 1 - \left(1 - \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right)\right) = \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right). \end{aligned}$$

EXAMPLE 7.51

Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x > 0, t > 0$$

subject to the conditions

$$\begin{aligned} u(x, 0) &= 0, \quad x > 0, \\ u(0, t) &= t, \quad t > 0 \text{ and } \lim_{x \rightarrow \infty} u(x, t) = 0. \end{aligned}$$

Solution. Taking Laplace transform, we have

$$\frac{d^2}{dx^2}U(x, s) - sU(x, s) = 0.$$

The solution of this equation is

$$U(x, s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x} \quad (18)$$

Since $\lim_{x \rightarrow \infty} u(x, t) = 0$, we have

$$\begin{aligned} \lim_{x \rightarrow \infty} U(x, s) &= \lim_{x \rightarrow \infty} L\{u(x, t)\} \\ &= L\{\lim_{x \rightarrow \infty} u(x, t)\} = L\{0\} = 0 \text{ (finite)}. \end{aligned}$$

Therefore, $c_1 = 0$ and (18) reduces to

$$U(x, s) = c_2 e^{-\sqrt{s}x}. \quad (19)$$

Also, since $u(0, t) = t$, we have

$$U(0, s) = \frac{1}{s^2}.$$

Hence, (19) yields $c_2 = \frac{1}{s^2}$. Thus

$$U(x, s) = \frac{1}{s^2} e^{-\sqrt{s}x}.$$

Since $L^{-1}\{e^{-\sqrt{s}x}\} = \frac{x}{2\sqrt{\pi t^3}} e^{-\frac{x^2}{4t}}$, by Convolution theorem, we have

$$u(x, t) = \int_0^t (t-u) \frac{x}{2\sqrt{\pi u^3}} e^{-\frac{x^2}{4u}} du.$$

Putting $\lambda = \frac{x^2}{4u}$, we get

$$u(x, t) = \frac{2}{\sqrt{\pi}} \int_{x/2\sqrt{t}}^{\infty} e^{-\lambda^2} \left(t - \frac{x^2}{4\lambda^2}\right) d\lambda.$$

EXAMPLE 7.52

Solve

$$\frac{\partial u}{\partial t} = 2 \frac{\partial^2 u}{\partial x^2}$$

subject to the conditions $u(0, t) = 0$, $u(5, t) = 0$, $u(x, 0) = \sin \pi x$.

Solution. Taking Laplace transform and using $u(x, 0) = \sin \pi x$, we get

$$\frac{d^2}{dx^2} U(x, s) - \frac{s}{2} U(x, s) = -\frac{1}{2} \sin \pi x.$$

Complementary function for this equation is $c_1 e^{\sqrt{\frac{s}{2}}x} + c_2 e^{-\sqrt{\frac{s}{2}}x}$ and particular integral is $\frac{1}{2(\pi^2 + (s/2))} \sin \pi x$. Thus the complete solution is

$$U(x, s) = c_1 e^{\sqrt{\frac{s}{2}}x} + c_2 e^{-\sqrt{\frac{s}{2}}x} + \frac{1}{2(\pi^2 + (s/2))} \sin \pi x. \quad (20)$$

Since $u(0, t) = 0$, we have $U(0, s) = 0$ and since $u(5, t) = 0$, $U(5, s) = 0$. Therefore, (20) gives

$$c_1 + c_2 = 0 \quad \text{and} \quad c_1 e^{5\sqrt{s/2}} + c_2 e^{5(-\sqrt{s/2})} = 0.$$

These relations imply $c_1 = c_2 = 0$. Hence

$$U(x, s) = \frac{1}{2(\pi^2 + s/2)} \sin \pi x = \frac{1}{s + 2\pi^2} \sin \pi x.$$

Taking inverse Laplace transform, we get

$$U(x, t) = e^{-2\pi^2 t} \sin \pi x.$$

EXAMPLE 7.53

Solve one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}, \quad x > 0, \quad t > 0$$

subject to the condition $y(x, 0) = 0$, $x > 0$; $y_t(x, 0) = 0$, $x > 0$, $y(0, t) = \sin \omega t$ and $\lim_{x \rightarrow \infty} y(x, t) = 0$.

Solution. The displacement is only in the vertical direction and is given by $y(x, t)$ at position x and time t . For a vibrating string, the constant a equals $\sqrt{\frac{T}{\rho}}$, where T is tension in the string and ρ is mass per unit length of the vibrating string (Figure 7.14).

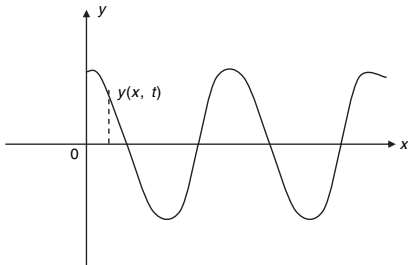


Figure 7.14

Taking Laplace transform, we get $s^2 Y(x, s) - sy(x, 0) - y_t(x, 0) - a^2 \frac{d^2}{dx^2} Y(x, s) = 0$ or

$$\frac{d^2}{ds^2} Y(x, s) - \frac{s^2}{a^2} = 0 \quad (21)$$

The general solution of (21) is

$$Y(x, s) = c_1 e^{\frac{s}{a}x} + c_2 e^{-\frac{s}{a}x} \quad (22)$$

The condition $\lim_{x \rightarrow \infty} y(x, t)$ implies $c_1 = 0$. Since $y(0, t) = \sin \omega t$, we have

$$Y(0, s) = \{y(0, t)\} = \frac{\omega}{s^2 + \omega^2}.$$

Therefore, (22) implies $c_2 = \frac{\omega}{s^2 + \omega^2}$ and so

$$Y(x, s) = \frac{\omega}{s^2 + \omega^2} e^{-\frac{s}{a}x}.$$

Taking inverse Laplace transform, we have

$$\begin{aligned} y(x, t) &= \begin{cases} \sin \omega \left(t - \frac{x}{a}\right) & \text{for } t > \frac{x}{a} \\ 0 & \text{for } t < \frac{x}{a} \end{cases} \\ &= \sin \omega \left(t - \frac{x}{a}\right) H\left(t - \frac{x}{a}\right). \end{aligned}$$

EXAMPLE 7.54

Solve

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}, \quad \text{for } 0 < x < 1, \quad t > 0$$

subject to $y(x, 0) = 0$, $0 < x < 1$; $y(0, t) = 0$, $t > 0$, $y(1, t) = 0$, $t > 0$ and $y_t(x, 0) = x$, $0 < x < 1$.

Solution. Taking Laplace transform and using $y(x, 0) = 0$ and $y_t(x, 0) = x$, we get

$$\frac{d^2 Y(x, s)}{dx^2} - s^2 Y(x, s) = x,$$

whose solution is given by

$$Y(x, s) = c_1 \cosh sx + c_2 \sinh sx - \frac{x}{s^2}.$$

Now $y(0, t) = 0$ implies that $Y(0, s) = 0$ and so $c_1 = 0$. Similarly, $y(1, t) = 0$ implies $Y(1, s) = 0$ and so $c_2 \sinh s - \frac{1}{s^2} = 0$. Thus $c_2 = \frac{1}{s^2 \sinh s}$. Hence

$$Y(x, s) = \frac{1}{s^2 \sinh s} \cdot \sinh sx - \frac{x}{s^2}.$$

This function has simple poles at $n\pi i$, $n = \pm 1, \pm 2, \dots$, and a pole of order 2 at $s = 0$. Now

$$\begin{aligned}\operatorname{Res}(n\pi i) &= \lim_{s \rightarrow n\pi i} (s - n\pi i) e^{ts} \cdot \frac{\sinh sx}{s^2 \sinh s} \\ &= \lim_{s \rightarrow n\pi i} \frac{(s - n\pi i)}{\sinh s} \lim_{s \rightarrow n\pi i} e^{ts} \frac{\sinh sx}{s^2} \\ &= \frac{1}{\cosh n\pi i} \cdot \frac{e^{n\pi i t} \sinh n\pi i x}{-n^2 \pi^2} \\ &= \frac{(-1)^{n+1}}{n^2 \pi^2} e^{n\pi i t} \sin n\pi x,\end{aligned}$$

$$\operatorname{Res}(0) = xt.$$

Hence, by Complex inversion formula,

$$\begin{aligned}y(x, t) &= xt + \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1}}{n^2 \pi^2} e^{n\pi i t} \sin n\pi x - xt \\ &= \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \sin n\pi x \sin n\pi t.\end{aligned}$$

7.7 EVALUATION OF INTEGRALS

Laplace transforms can be used to evaluate certain integrals. In some cases the given integral is a special case of a Laplace transform for a particular value of the transform variable s . To evaluate an integral containing a free parameter, we first take Laplace transform of the integrand with respect to the free parameter. The resulting integral is then easily evaluated. Then we apply inverse Laplace transform to get the value of the given integral. In some cases, Theorem 5.9, regarding Laplace transform is used to evaluate the given integral.

EXAMPLE 7.55

Evaluate the integral

$$I = \int_0^{\infty} e^{-t} \frac{\sin t}{t} dt,$$

and show that $\int_0^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}$.

Solution. We know that

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_0^{\infty} e^{-st} \frac{\sin t}{t} dt = \tan^{-1} \frac{1}{s}. \quad (23)$$

Setting $s = 1$, we get

$$\int_0^{\infty} e^{-t} \frac{\sin t}{t} dt = \tan^{-1} 1 = \frac{\pi}{4}.$$

Further, letting $s \rightarrow 0$ in (23), we get

$$\int_0^{\infty} \frac{\sin t}{t} dt = \tan^{-1} \infty = \frac{\pi}{2}.$$

EXAMPLE 7.56

Evaluate the integral

$$\int_0^{\infty} \frac{\sin tx}{x(1+x^2)} dx.$$

Solution. Let

$$f(t) = \int_0^{\infty} \frac{\sin tx}{x(1+x^2)} dx.$$

Taking Laplace transform with respect to t , we have

$$\begin{aligned}F(s) &= \int_0^{\infty} \frac{1}{x(1+x^2)} dx \int_0^{\infty} e^{-st} \sin tx dx \\ &= \int_0^{\infty} \frac{dx}{x(1+x^2)(s^2+x^2)} \\ &= \frac{1}{s^2-1} \int_0^{\infty} \left(\frac{1}{1+x^2} - \frac{1}{s^2+x^2} \right) dx \\ &= \frac{1}{s^2-1} \left(\frac{\pi}{2} - \frac{\pi}{2s} \right) = \frac{\pi}{2s^2-1} \left(\frac{s-1}{s} \right) \\ &= \frac{\pi}{2} \left(\frac{1}{s(s+1)} \right) = \frac{\pi}{2} \left(\frac{1}{s} - \frac{1}{s+1} \right).\end{aligned}$$

Taking inverse Laplace transforms, we get

$$f(t) = \frac{\pi}{2} (1 - e^{-t}).$$

EXAMPLE 7.57

Evaluate

$$\int_0^{\infty} \frac{\sin^2 tx}{x^2} dx.$$

Solution. We have

$$f(t) = \int_0^{\infty} \frac{\sin^2 tx}{x^2} dx = \int_0^{\infty} \frac{1 - \cos(2tx)}{2x^2} dx.$$

Taking Laplace transform with respect to t , we have

$$\begin{aligned} F(s) &= \frac{1}{2} \int_0^{\infty} \frac{1}{x^2} \left(\frac{1}{s} - \frac{s}{4x^2 + s^2} \right) dx = \frac{2}{s} \int_0^{\infty} \frac{dx}{4x^2 + s^2} \\ &= \frac{1}{s} \int_0^{\infty} \frac{dy}{y^2 + s^2} = \frac{1}{s^2} \left[\tan^{-1} \frac{y}{s} \right]_0^{\infty} = \pm \frac{\pi}{2s^2}. \end{aligned}$$

Thus, taking inverse Laplace transformation, we get

$$f(t) = \pm \frac{\pi t}{2} = \frac{\pi t}{2} \operatorname{sgn} t.$$

EXAMPLE 7.58

Evaluate

$$\int_{-\infty}^{\infty} \sin x^2 dx.$$

Solution. Let

$$f(t) = \int_0^{\infty} \sin t x^2 dx$$

Taking Laplace transform, we get

$$\begin{aligned} L\{f(t)\} &= \int_0^{\infty} e^{-st} dt \int_0^{\infty} \sin t x^2 dx \\ &= \int_0^{\infty} dx \int_0^{\infty} e^{-st} \sin t x^2 dt \\ &= \int_0^{\infty} L\{\sin t x^2\} dx = \int_0^{\infty} \frac{x^2}{s^2 + x^4} dx. \end{aligned}$$

Put $x^2 = s \tan \theta$, that is, $x = \sqrt{s} \sqrt{\tan \theta}$. Then

$$dx = \sqrt{s} \cdot \frac{1}{2} (\tan \theta)^{-1/2} \sec^2 \theta d\theta.$$

Therefore,

$$\begin{aligned} L\{f(t)\} &= \frac{1}{2} \int_0^{\pi/2} \frac{s^{3/2} \tan \theta (\tan \theta)^{-1/2} \sec^2 \theta d\theta}{s^2 (1 + \tan^2 \theta)} \\ &= \frac{1}{2\sqrt{s}} \int_0^{\pi/2} \sqrt{\tan \theta} d\theta = \frac{1}{2\sqrt{s}} \int_0^{\pi/2} \sin^{1/2} \theta \cos^{-1/2} \theta d\theta. \end{aligned}$$

But we know that

$$\int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta = \frac{1}{2} \beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{2\Gamma(m+n)}.$$

Taking $2m - 1 = \frac{1}{2}$ and $2n - 1 = -\frac{1}{2}$, we get $m = \frac{3}{4}$ and $n = \frac{1}{4}$. Hence using the relation $\Gamma(p) \Gamma(1-p) = \frac{\pi}{\sin p\pi}$, $0 < p < 1$, we have

$$L\{f(t)\} = \frac{1}{4\sqrt{s}} \frac{\Gamma(3/4) \Gamma(1/4)}{\Gamma(1)} = \frac{1}{4\sqrt{s}} \pi \sqrt{2} = \frac{\pi \sqrt{2}}{4\sqrt{s}}.$$

Taking inverse Laplace transform yields

$$f(t) = \frac{\pi \sqrt{2}}{4} \left(\frac{t^{-1/2}}{\sqrt{\pi}} \right) = \frac{\sqrt{2\pi}}{4} t^{-1/2}.$$

Putting $t = 1$, we get

$$\int_0^{\infty} \sin x^2 dx = \frac{1}{2} \sqrt{\frac{\pi}{2}}$$

and so

$$\int_{-\infty}^{\infty} \sin x^2 dx = \sqrt{\frac{\pi}{2}}.$$

EXAMPLE 7.59

Evaluate the integral

$$\int_0^{\infty} \frac{\cos tx}{x^2 + 1} dx, \quad t > 0.$$

Solution. Let

$$f(t) = \int_0^{\infty} \frac{\cos tx}{x^2 + 1} dx.$$

Taking Laplace transform with respect to t , we get

$$\begin{aligned} L\{f(t)\} &= L\left\{ \int_0^{\infty} \frac{\cos tx}{x^2 + 1} dx \right\} = \int_0^{\infty} L\left\{ \frac{\cos tx}{x^2 + 1} \right\} dx \\ &= \int_0^{\infty} \frac{s}{(x^2 + 1)(s^2 + x^2)} dx \\ &= s \int_0^{\infty} \frac{dx}{(x^2 + 1)(s^2 + x^2)} \\ &= \frac{s}{s^2 - 1} \int_0^{\infty} \left(\frac{1}{x^2 + 1} - \frac{1}{s^2 + x^2} \right) dx \\ &= \frac{s}{s^2 - 1} \left[\tan^{-1} x - \tan^{-1} \frac{x}{s} \right]_0^{\infty} \\ &= \frac{s}{s^2 - 1} \left(\frac{\pi}{2} - \frac{\pi}{2s} \right) = \frac{\pi/2}{s + 1}. \end{aligned}$$

Now, taking inverse Laplace transform, we have

$$f(t) = \frac{\pi}{2} e^{-t}, \quad t > 0.$$

EXAMPLE 7.60

Evaluate

$$\int_0^{\infty} t J_0(t) dt,$$

where J_0 is Bessel's function of order zero.

Solution. We know that

$$L\{J_0(t)\} = \frac{1}{\sqrt{s^2 + 1}}.$$

Therefore,

$$\begin{aligned} L\{tJ_0(t)\} &= -\frac{d}{ds} \{L\{J_0(t)\}\} = -\frac{d}{ds} \left\{ \frac{1}{\sqrt{s^2 + 1}} \right\} \\ &= -\frac{2s}{(s^2 + 1)^{3/2}}. \end{aligned}$$

But, by definition

$$L\{tJ_0(t)\} = \int_0^{\infty} e^{-st} t J_0(t) dt.$$

Hence

$$\int_0^{\infty} e^{-st} t J_0(t) dt = -\frac{2s}{(s^2 + 1)^{3/2}}.$$

Taking $s = 0$, we get

$$\int_0^{\infty} t J_0(t) dt = 0.$$

EXAMPLE 7.61

Evaluate

$$\int_0^{\infty} e^{-2t} \operatorname{erf} \sqrt{t} dt.$$

Solution. We have

$$L\{\operatorname{erf} \sqrt{t}\} = \int_0^{\infty} e^{-st} \operatorname{erf} \sqrt{t} dt = \frac{1}{s\sqrt{s+1}}.$$

Taking $s = 2$, we get

$$\int_0^{\infty} e^{-2t} \operatorname{erf} \sqrt{t} dt = \frac{1}{2\sqrt{3}}.$$

EXAMPLE 7.62

Evaluate

$$\int_0^t \operatorname{erf} \sqrt{u} \operatorname{erf} \sqrt{(t-u)} du.$$

Solution. Let

$$f(t) = \int_0^t \operatorname{erf} \sqrt{u} \operatorname{erf} \sqrt{t-u} du.$$

Then, by Convolution theorem, we have

$$\begin{aligned} F(t) &= L\{\operatorname{erf} \sqrt{t}\} L\{\operatorname{erf} \sqrt{t}\} \\ &= \frac{1}{s\sqrt{s+1}} \cdot \frac{1}{s\sqrt{s+1}} = \frac{1}{s^2(s+1)} \\ &= \frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1}. \end{aligned}$$

Taking inverse transform, we get

$$f(t) = -1 + t + e^{-t}.$$

7.8 MISCELLANEOUS EXAMPLES

EXAMPLE 7.63

Solve the differential equation, using Laplace transform $y'' + 4y' + 4y = e^{-t}$ given that $y(0) = 0$ and $y'(0) = 0$.

Solution. Taking Laplace transform, we have

$$L\{y''(t)\} + 4L\{y'(t)\} + 4L\{y(t)\} = L\{e^{-t}\}$$

or

$$s^2 Y(s) - sy(0) - y'(0) + 4\{sY(s) - y(0)\} + 4Y(s)$$

$$= \frac{1}{s+1}.$$

Using the initial conditions, we have

$$s^2 Y(s) + 4sY(s) + 4Y(s) = \frac{1}{s+1}$$

or

$$\begin{aligned} Y(s) &= \frac{1}{(s+1)(s^2 + 4s + 4)} = \frac{1}{(s+1)(s+2)^2} \\ &= \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}. \end{aligned}$$

Taking inverse Laplace transform, we get

$$y(t) = e^{-t} - e^{-2t} - te^{-2t}.$$

EXAMPLE 7.64

Using Laplace transform, solve

$$(D^2 + 5D - 6)y = x^2 e^{-x}, \quad y(0) = a, \quad y'(0) = b.$$

Solution. Applying Laplace transform to the given equation, we get

$$\begin{aligned} s^2 L\{y\} - sy(0) - y'(0) + 5[sL\{y\} - y(0)] - 6L\{y\} \\ = \frac{(-1)^4 2!}{(s+1)^3} = \frac{2}{(s+1)^3} \end{aligned}$$

or

$$\begin{aligned} s^2 L\{y\} - as - b + 5[sL\{y\} - a] - 6L\{y\} \\ = \frac{2}{(s+1)^3} \end{aligned}$$

or

$$L\{y\}[s^2 + 5s - 6] = \frac{2}{(s+1)^3} + as + b + 5a$$

or

$$\begin{aligned} L\{y\} &= \frac{as + 5a + b}{s^2 + 5s - 6} + \frac{2}{(s+1)^3(s^2 + 5s - 6)} \\ &= -\frac{1}{5} \frac{1}{(s+1)^3} - \frac{3}{50} \frac{1}{(s+1)^2} - \frac{19}{500} \frac{1}{(s+1)} \\ &\quad + \frac{1}{28} \frac{1}{(s-1)} + \frac{2}{875} \frac{1}{(s+6)} \quad (\text{by partial fractions}). \end{aligned}$$

Taking inverse Laplace transform, we have

$$\begin{aligned} y(x) &= \frac{a-b}{7} e^{-6x} + \frac{6a+b}{7} e^x - \frac{1}{50} x^2 e^{-x} \\ &\quad - \left(\frac{19}{500} e^{-x} + \frac{1}{28} e^x + \frac{2}{875} e^{-6x} \right). \end{aligned}$$

EXAMPLE 7.65

Solve the differential equation $\frac{d^2 x}{dt^2} + x = t \cos 2t$ under the conditions: $x(0) = x'(0) = 0$.

Solution. The given differential equation is

$$\frac{d^2 x}{dt^2} + x = t \cos 2t, \quad x(0) = x'(0) = 0.$$

Taking Laplace transform of both sides of the given equation, we have

$$L\{x''(t)\} + L\{x(t)\} = L\{t \cos 2t\}$$

or

$$s^2 X(s) - sx(0) - x'(0) + X(s) = \frac{s^2 - 4}{(s^2 + 4)^2}.$$

Using initial conditions $x(0) = x'(0) = 0$, we get

$$X(s)(s^2 + 1) = \frac{s^2 - 4}{(s^2 + 4)^2}$$

or

$$\begin{aligned} X(s) &= \frac{s^2 - 4}{(s^2 - 1)(s^2 + 4)^2} \\ &= \frac{5}{9(s^2 + 4)} + \frac{3}{3(s^2 + 4)^2} - \frac{5}{9(s^2 + 1)}. \end{aligned}$$

Taking inverse Laplace transform, we have

$$x(t) = \frac{5 \sin 2t}{9} + \frac{8}{3} [\sin 2t - 2t \cos 2t] - \frac{5}{9} \sin t.$$

EXAMPLE 7.66

Solve the differential equation

$$\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + y = e^{2x}, \quad y(0) = 2, \quad y'(0) = -1$$

by using Laplace transforms.

Solution. Taking Laplace transform, we have

$$\begin{aligned} s^2 Y(s) - sy(0) - y'(0) - 2\{sY(s) - y(0)\} + Y(s) \\ = \frac{1}{s-2}. \end{aligned}$$

Since $y(0) = 2$, $y'(0) = -1$, we have

$$s^2 Y(s) - 2s + 1 - 2sY(s) + 4 + Y(s) = \frac{1}{s-2}$$

or

$$s^2 Y(s) - 2sY(s) + Y(s) = 2s - 5 + \frac{1}{s-2}$$

or

$$(s^2 - 2s + 1)Y(s) = \frac{2s^2 - 7s + 11}{s-2}$$

or

$$\begin{aligned} Y(s) &= \frac{2s^2 - 9s + 11}{(s-2)(s-1)^2} \\ &= \frac{1}{s-2} + \frac{1}{s-1} - \frac{4}{(s-1)^2}. \end{aligned}$$

Taking inverse Laplace transform, we get

$$\begin{aligned} y(x) &= e^{2x} + e^x - 4 L^{-1} \left\{ \frac{1}{(s-1)^2} \right\} \\ &= e^{2x} + e^x - 4e^x(x) = e^x(1 - 4x) + e^{2x}. \end{aligned}$$

EXAMPLE 7.67

Solve the following differential equation using Laplace transforms:

$$\begin{aligned} \frac{d^3 y}{dt^3} - 3 \frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} - y &= t^2 e^t \text{ where} \\ y(0) = 1, \left(\frac{dy}{dt} \right)_{t=0} &= 0, \left(\frac{d^2 y}{dt^2} \right)_{t=0} = -2. \end{aligned}$$

Solution. The given differential equation is

$$y''' - 3y'' + 3y' - y = t^2 e^t$$

with conditions

$$y(0) = 1, y'(0) = 0, y''(0) = 2.$$

Taking Laplace transform, we get

$$\begin{aligned} s^3 Y(s) - s^2 y(0) - s y'(0) - y''(0) - 3[s^2 y(s) \\ - s y(0) - y'(0)] + 3[sY(s) - y(0)] - Y(s) \\ = \frac{2}{(s-1)^3}. \end{aligned}$$

Using the given initial conditions, we have

$$\begin{aligned} s^3 Y(s) - s^2 + 2 - 3[s^2 Y(s) - s] \\ + 3[sY(s) - 1] - Y(s) = \frac{2}{(s-1)^3} \end{aligned}$$

or

$$[s^3 - 3s^2 + 3s - 1]Y(s) - s^2 + 3s - 1 = \frac{2}{(s-1)^3}$$

or

$$\begin{aligned} (s-1)^3 Y(s) &= \frac{2}{(s-1)^3} + s^2 - 3s + 1 \\ &= \frac{2}{(s-1)^3} + (s-1)^2 - s \end{aligned}$$

Thus

$$\begin{aligned} Y(s) &= \frac{2}{(s-1)^6} + \frac{(s-1)^2}{(s-1)^3} - \frac{s}{(s-1)^3} \\ &= \frac{2}{(s-1)^6} + \frac{1}{s-1} - \left[\frac{(s-1)+1}{(s-1)^3} \right] \\ &= \frac{2}{(s-1)^6} + \frac{1}{s-1} - \frac{1}{(s-1)^3} - \frac{1}{(s-1)^3} \end{aligned}$$

Taking inverse Laplace transform, we get

$$y = \frac{t^5 e^t}{60} + e^t - t e^t - \frac{t^2}{2} e^t.$$

EXAMPLE 7.68

Solve the following simultaneous differential equations:

- (i) $3 \frac{dx}{dt} - y = 2t$, $\frac{dx}{dt} + \frac{dy}{dt} - y = 0$ with the condition $x(0) = y(0) = 0$.
 (ii) $\frac{dx}{dt} + 4 \frac{dy}{dt} - y = 0$, $\frac{dx}{dt} + 2y = e^{-t}$ with the condition $x(0) = y(0) = 0$.

Solution. (i). The given simultaneous differential equations are

$$\begin{aligned} 3x' - y &= 2t \\ x' + y' - y &= 0 \end{aligned}$$

with $x(0) = y(0) = 0$. Taking Laplace transforms, we get

$$3\{sX(s) - x(0)\} - Y(s) = \frac{2}{s^2}$$

and

$$sX(s) - x(0) + sY(s) - y(0) - Y(s) = 0.$$

Using the initial conditions, the above equations reduce to

$$3sX(s) - Y(s) = \frac{2}{s^2}$$

and

$$sX(s) + sY(s) - Y(s) = 0$$

or

$$3sX(s) - Y(s) = \frac{2}{s^2} \quad (24)$$

and

$$sX(s) + (s-1)Y(s) = 0 \quad (25)$$

Multiplying the equation (25) by 3 and then subtracting (24) from it, we get

$$(3s-2)Y(s) = -\frac{2}{s^2}$$

and so

$$Y(s) = -\frac{2}{s^2(3s-2)} = \frac{1}{s^2} + \frac{3}{2s} - \frac{3}{2(s-\frac{2}{3})} \quad (26)$$

(by partial fractions)

Taking inverse Laplace transform, we have

$$y = t + \frac{3}{2} - \frac{3}{2}e^{\frac{2t}{3}}. \quad (27)$$

Substituting the value of $Y(s)$ from (26) in (24), we get

$$3sX(s) = \frac{2}{s^2} + \frac{1}{s^2} + \frac{3}{2s} - \frac{3}{2(s-\frac{2}{3})}$$

so that

$$\begin{aligned} X(s) &= \frac{1}{s^3} + \frac{1}{2s^2} - \frac{1}{2s(s-\frac{2}{3})} \\ &= \frac{1}{s^3} + \frac{1}{2s^2} - \frac{3}{4} \left[\frac{1}{s-\frac{2}{3}} - \frac{1}{s} \right] \end{aligned}$$

Taking inverse Laplace transform, we have

$$x = \frac{t^2}{2} + \frac{t}{2} + \frac{3}{4} - \frac{3}{4}e^{\frac{2t}{3}} \quad (28)$$

Thus (27) and (28) provides the solution to the given system.

(ii). Taking Laplace transform, we have

$$sX(s) - x(0) + 4(sY(s) - y(0)) - Y(s) = 0$$

and

$$sX(s) - x(0) + 2Y(s) = \frac{1}{s+1}.$$

Using the given initial conditions, we get

$$sX(s) + (4s-1)Y(s) = 0 \quad (29)$$

and

$$sX(s) + 2Y(s) = 0 \quad (30)$$

Subtracting (30) from (29), we get

$$(4s-3)Y(s) = -\frac{1}{s+1}$$

or

$$Y(s) = \frac{1}{(s+1)(4s-3)} = \frac{1}{7} \left[\frac{1}{s+1} - \frac{1}{s-\frac{3}{4}} \right]. \quad (31)$$

Taking inverse Laplace transform, we get

$$y = \frac{1}{7} [e^{-t} - e^{\frac{3t}{4}}].$$

Putting the value of $Y(s)$ from (31) in (30), we get

$$sX(s) = \frac{5}{7(s+1)} + \frac{2}{7(s-\frac{3}{4})}$$

or

$$\begin{aligned} X(s) &= \frac{5}{7s(s+1)} + \frac{2}{7s(s-\frac{3}{4})} \\ &= \frac{5}{7} \left(\frac{1}{s} - \frac{1}{s+1} \right) + \frac{8}{21} \left[\frac{1}{s-\frac{3}{4}} - \frac{1}{s} \right]. \end{aligned}$$

Taking inverse transform, we get

$$x = \frac{1}{3} - \frac{5}{7}e^{-t} + \frac{8}{21}e^{\frac{3t}{4}}.$$

EXAMPLE 7.69

Solve the simultaneous differential equations using Laplace transforms:

$$\begin{aligned} x'(t) + y'(t) + x(t) &= -e^{-t}, \\ x'(t) + 2y'(t) + 2x(t) + 2y(t) &= 0 \\ \text{where } x(0) &= -1, y(0) = 1. \end{aligned}$$

Solution. We want to solve

$$\begin{aligned} x'(t) + y'(t) + x(t) &= -e^{-t}, \\ x'(t) + 2y'(t) + 2x(t) + 2y(t) &= 0, \end{aligned}$$

subject to the conditions $x(0) = -1$, $y(0) = 1$. Taking Laplace transform of the given equations, we have

$$sX(s) - x(0) + sY(s) - y(0) + X(s) = -\frac{1}{s+1}$$

and

$$sX(s) - x(0) + 2[sY(s) - y(0)] + 2X(s) + 2Y(s) = 0.$$

Using the given initial conditions, we get

$$sX(s) + X(s) + sY(s) = -\frac{1}{s+1}$$

and

$$sX(s) + 1 + 2sY(s) - 2 + 2X(s) + 2Y(s) = 0$$

that is,

$$(s+1)X(s) + sY(s) = -\frac{1}{s+1} \quad (32)$$

and

$$(s+2)X(s) + 2(s+1)Y(s) = 1 \quad (33)$$

Multiplying (32) by $2(s+1)$ and (33) by s and then subtracting, we get

$$[2(s+1)^2 - s(s+2)]X(s) = -2 - s$$

or

$$(s^2 + 2s + 2)X(s) = -(s+2)$$

or

$$\begin{aligned} X(s) &= -\frac{s+2}{s^2+2s+2} = -\frac{s+2}{(s+1)^2+1} \\ &= -\left[\frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1} \right]. \end{aligned} \quad (34)$$

Taking inverse Laplace transform, we get

$$x(t) = -e^{-t} \cos t - e^{-t} \sin t.$$

Further, putting the value of $X(s)$ in (33), we get

$$\begin{aligned} 2(s+1)Y(s) &= 1 + \frac{(s+2)(s+2)}{s^2+2s+2} \\ &= \frac{s^2+2s+2+s^2+4s+4}{s^2+2s+2} \\ &= \frac{2s^2+6s+6}{s^2+2s+2} \end{aligned}$$

or

$$\begin{aligned} Y(s) &= \frac{s^2+3s+3}{(s+1)(s^2+2s+2)} \\ &= \frac{1}{s+1} + \frac{1}{s^2+2s+2} \\ &= \frac{1}{s+1} + \frac{1}{(s+1)^2+1}. \end{aligned}$$

Taking inverse Laplace transform, we have

$$y(t) = e^{-t} - e^{-t} \sin t.$$

EXAMPLE 7.70

Use Laplace transform method to solve the simultaneous equations: $D^2x - Dy = \cos t$; $Dx + D^2y = -\sin t$, given that $x=0, Dx=0, y=0, Dy=1$, when $t=0$.

Solution. Taking Laplace transforms of the given equations, we have

$$s^2X(s) - sx(0) - x'(0) - sy(s) - y(0) = \frac{s}{s^2+1}$$

and

$$\begin{aligned} sX(s) - x(0) + s^2y(s) - sy(0) - y'(0) \\ = -\frac{1}{s^2+1}. \end{aligned}$$

Using the given condition, the above equations transform to

$$s^2X(s) - sY(s) = \frac{s}{s^2+1} \quad (35)$$

and

$$sX(s) + s^2Y(s) = 1 - \frac{1}{s^2+1} = \frac{s^2}{s^2+1}. \quad (36)$$

Multiplying (35) by s and adding to (36), we get

$$s^3X(s) + sX(s) = \frac{s^2}{s^2+1} + \frac{s^2}{s^2+1} = \frac{2s^2}{s^2+1}.$$

Thus

$$X(s) = \frac{2s^2}{(s^2+1)(s^3+s)} = \frac{2s}{(s^2+1)^2}.$$

Taking inverse Laplace transform, we have

$$x(t) = t \sin t.$$

Putting the value of $X(s)$ in (35), we get

$$\frac{2s^3}{(s^2+1)^2} - sY(s) = \frac{s}{s^2+1}$$

or

$$Y(s) = \frac{s^2-1}{(s^2+1)^2} = \frac{1}{s^2+1} - \frac{2}{(s^2+1)^2}.$$

Taking inverse Laplace transform, we get

$$y(t) = \sin t - (\sin t - t \cos t) = t \cos t.$$

Hence the required solution is

$$x(t) = t \sin t \quad \text{and} \quad y(t) = t \cos t.$$

EXAMPLE 7.71

Using Laplace transform, solve the integral equation

$$y(t) = 1 - e^{-t} + \int_0^t y(t-u) \sin u \, du.$$

Solution. Taking Laplace transform of both sides of the given integral equation, we have

$$\begin{aligned} L\{y\} &= L\{1\} - L\{e^{-t}\} + L\{y\} \cdot L\{\sin u\} \\ &= \frac{1}{s} - \frac{1}{s+1} + L\{y\} \cdot \frac{1}{s^2+1}. \end{aligned}$$

This relation yields

$$L\{y\} \left[1 - \frac{1}{s^2+1} \right] = \frac{1}{s} - \frac{1}{s+1}$$

or

$$\begin{aligned} L\{y\} &= \frac{s^2+1}{s^2} \left(\frac{1}{s} - \frac{1}{s+1} \right) = \frac{s^2+1}{s^3(s+1)} \\ &= \frac{2}{s} - \frac{2}{s+1} - \frac{1}{s^2} + \frac{1}{s^3} \end{aligned}$$

(by partial fractions).

Taking inverse Laplace transform, we get

$$y(t) = 2 - 2e^{-t} - t + \frac{1}{2}t^2$$

EXERCISES

1. Solve the following initial value problems:

(a) $y'(t) + 3y(t) = 0$, $x(1) = 1$.

Ans. $y(t) = e^{3(1-t)}$

(b) $\frac{d^2y}{dt^2} + y = 1$, $y(0) = y'(0) = 0$.

Ans. $1 - \cos t$

(c) $y'' + y = e^{-t}$, $y(0) = A$, $y'(0) = B$.

Ans. $y(t) = \frac{1}{2}e^{-t} + \left(A - \frac{1}{2}\right) \cos t + \left(B + \frac{1}{2}\right) \sin t$

(d) $\frac{d^2y}{dt^2} + y = 0$, $y(0) = 1$, $y'(0) = 0$.

Ans. $y(t) = \cos t$.

(e) $\frac{d^2y}{dt^2} + a^2y = f(t)$, $y(0) = 1$, $y'(0) = -2$.

Hint: $Y(s) = \frac{s-2}{s^2+a^2} + \frac{F(s)}{s^2+a^2}$. But by Convolution theorem

$$L^{-1}\left\{\frac{F(s)}{s^2+a^2}\right\} = f(t) * \frac{\sin at}{a} \text{ and so}$$

$$y(t) = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} - L^{-1}\left\{\frac{2}{s^2+a^2}\right\} + f(t) * \frac{\sin at}{a}$$

Ans. $\cos at - \frac{2 \sin at}{a} + \frac{1}{a} \int_0^t f(u) \sin a(t-u) \, du$

(f) $\frac{d^2y}{dt^2} + y = 3 \sin 2t$, $y(0) = 3$, $y'(0) = 1$.

Ans. $-\sin 2t + 3 \cos t + 3 \sin t$

(g) $\frac{d^2x}{dt^2} + 5 \frac{dx}{dt} + 6x = 2e^{-t}$ ($t \geq 0$), $x(0) = 1$ and $x'(0) = 0$

Ans. $e^{-t} + e^{-2t} - e^{-3t}$, $t \geq 0$

(h) $\frac{d^2x}{dt^2} + 6 \frac{dx}{dt} + 9x = 0$, $x(0) = x'(0) = 0$.

Ans. $x(t) = 0$

(i) $\frac{d^2x}{dt^2} + 2b \frac{dx}{dt} + \lambda^2 x = 0$, $x(0) = x'(0) = 0$.

Ans. $x(t) = e^{-bt} (c_1 \sin \sqrt{\lambda^2 - b^2} t + c_2 \cos \sqrt{\lambda^2 - b^2} t)$

2. Solve $y' - 2ty = 0$, $y(0) = 1$ and show that its solution does not have Laplace transform.

Ans. $y(t) = e^{t^2}$ (not of exponential order)

3. Solve $ty'' + y = 0$, $y(0) = 0$.

Hint: Proceed as in Example 7.20

Ans. $C \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{(n+1)! n!}$

4. Given that $I = Q = 0$ at $t = 0$, find current I in the LC circuit given for $t > 0$ in Figure 7.15.

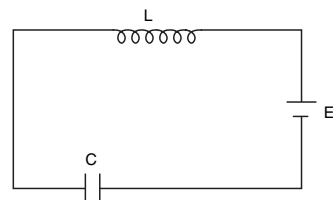


Figure 7.15

Hint: The differential equation governing the circuit is $L \frac{dI}{dt} + \frac{1}{C} \int I(u) \, du = E$. Application of Laplace transform yields

$$\mathcal{L}\{F(s) + \frac{F(s)}{Cs} = \frac{E}{s}, \text{ that is, } F(s) = \frac{EC}{LCs^2 + 1} = \frac{E}{L(s^2 + \frac{1}{LC})}$$

$$\text{Ans. } I(t) = E\sqrt{\frac{C}{L}} \sin \frac{1}{\sqrt{LC}} t.$$

5. Given that $I = Q = 0$ at $t = 0$, find charge and current in the circuit shown in Figure 7.16.

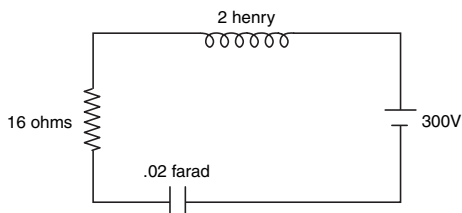


Figure 7.16

Hint: The governing equation is

$$\frac{d^2 Q}{dt^2} + 8 \frac{dQ}{dt} + 25Q = 150, F(s) = \frac{150}{s(s^2 + 8s + 25)} \text{ and so inversion gives } Q(t) = 6 - 6e^{-4t} \cos 3t - 8e^{-4t} \sin 3t. \text{ Then } I(t) = 50 e^{-4t} \sin 3t.$$

6. Solve the following systems of differential equations:

(a) $\frac{dx}{dy} + x - y = 1 + \sin t, \frac{dy}{dt} - \frac{dx}{dt} + y = t - \sin t$, with $x(0) = 0, y(0) = 1$

$$\text{Ans. } x(t) = t + \sin t, y(t) = t + \cos t$$

(b) $\frac{dy}{dt} = -z, \frac{dz}{dt} = y$ with $y(0) = 1, z(0) = 0$.

$$\text{Ans. } y(t) = \cos t, z(t) = \sin t$$

7. Solve $y'' + 4y = 4 \cos 2t, y(0) = y'(0) = 0$. Does resonance occur in this case? **Hint:** $Y(s) = \frac{4s}{(s^2 + 4)^2}$ and so $y(t) = 4 \left[\frac{t}{4} \sin 2t \right] = t \sin 2t$.

Note that $y(t) \rightarrow \infty$ as $t \rightarrow \infty$. Hence, there shall be resonance.

8. Solve the following difference equations:

(a) $3y(t) - 4y(t-1) + y(t-2) = t, y(t) = 0$ for $t < 0$.

$$\text{Ans. } y(t) = \frac{t}{3} + \frac{1}{2} \sum_{n=1}^{[t]} \left(1 - \frac{1}{3^n} \right) (t - n)$$

(b) $a_{n+2} - 2a_{n+1} + a_n = 0, a_0 = 0, a_1 = 1$.

$$\text{Ans. } a_n = n$$

(c) $a_n = a_{n-1} + 2a_{n-2}, a_0 = 1, a_1 = 8$.

$$\text{Ans. } a_n = 3(2^n) - 2(-1)^n, n \geq 0$$

(d) $a_n = 2a_{n-1} - a_{n-2}, a_1 = 1.5, a_2 = 3$

$$\text{Ans. } 1.5n$$

(e) $y(t) - y(t-1) = t^2$

$$\text{Ans. } y(t) = 2 \sum_{n=0}^{[t]} \frac{(t-n)^{n+3}}{(n+3)!}$$

(f) $y''(t) - y(t-1) = \delta(t), y(t) = y'(t) = 0, t < 0$.

Hint: $s^2 \mathcal{L}\{y(t)\} - e^{-s} \mathcal{L}\{y(t)\} = \mathcal{L}\{\delta(t)\}$ and so $\mathcal{L}\{y(t)\} = \frac{1}{s^2(1 - e^{-s})}$. But

$$\mathcal{L}^{-1}\left\{\frac{e^{-ns}}{s^{2n+2}}\right\} = \begin{cases} \frac{(t-n)^{2n+1}}{(2n+1)!} & \text{for } t \geq n \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{Hence } y(t) = \sum_{n=0}^{[t]} \frac{(t-n)^{2n+1}}{(2n+1)!}$$

9. Solve the integral equations:

(a) $f(t) = 1 + \int_0^t \cos(t-u)f(u) du$

$$\text{Ans. } f(t) = 1 + \frac{2}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2} t\right) e^{t/2}$$

(b) $y(t) = \sin t + 2 \int_0^t y(u) \cos(t-u) du$

$$\text{Ans. } y(t) = te^t$$

(c) $y(t) = t + \frac{1}{6} \int_0^t y(u) (t-u)^3 du$

$$\text{Ans. } y(t) = \frac{1}{2}(\sinh t + \sin t)$$

(d) $f(t) = \int_0^t \sin u(t-u) du$

$$\text{Ans. } 0$$

(e) $y'(t) + 3y(t) + 2 \int_0^t y(u) du = t, y(0) = 1$.

Hint: $\mathcal{L}\{y(t)\} = \frac{s^2 + 1}{s(s^2 + 3s + 2)}$. Use partial fractions and then use inversion to give $y(t) = \frac{1}{2} - 2e^{-t} + \frac{5}{2}e^{-2t}$

10. Solve the partial differential equation $x \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = x, x > 0, t > 0$ subject to the conditions $u(x, 0) = 0$ for $x > 0$ and $u(0, t) = 0$ for $t > 0$.

Hint: Using Laplace transform, $\frac{d}{dx} U(x, s) + x s U(x, s) = \frac{x}{s}$, integrating factor is $e^{\frac{1}{2}sx^2}$ and so $U(x, s) = \frac{1}{s^2} \left[1 - e^{-\frac{1}{2}sx^2} \right]$. Inversion yields $u(x, t) = \begin{cases} t & \text{for } t < x^2/2 \\ x^2/2 & \text{for } 2t > x^2 \end{cases}$.

11. Find the bounded solution of $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $x > 0$, $t > 0$ for $u(0, t) = 1$, $u(x, 0) = 0$.

Hint: Application of Laplace transform yields $U(x, s) = c_1 e^{\sqrt{s}x} + c_2 e^{-\sqrt{s}x}$ for bounded $u(x, t)$, $U(x, s)$ must be bounded and so $c_1 = 0$. Further $U(0, s) = L\{u(0, t)\} = L\{1\} = 1/s$. Therefore, $1/s = c_2$ and so $U(x, s) = \text{erfc}(x/2\sqrt{t})$.

12. Solve $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 v}{\partial x^2}$, $x > 0$, $t > 0$ for $u(x, 0) = 0$, $y(x, 0) = 0$, $x > 0$, $y(0, t) = f(t)$ with $f(0) = 0$ and $\lim_{x \rightarrow \infty} y(x, t) = 0$.

Hint: see Example 7.54

Ans. $y(x, t) = f\left(t - \frac{x}{a}\right) H\left(t - \frac{x}{a}\right)$

13. Solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, $x > 0$, $t > 0$ for $u(0, t) = 10 \sin 2t$, $u(x, 0) = 0$, $u_x(x, 0) = 0$ and $\lim_{x \rightarrow \infty} u(x, t) = 0$

Ans. $u(x, t) = \begin{cases} 10 \sin 2(t - x) & \text{for } t > x \\ 0 & \text{for } t < x. \end{cases}$

14. Evaluate the integrals:

(a) $\int_0^{\infty} J_0(t) dt$

Ans. 1

(b) $\int_0^{\infty} \frac{e^{-t} - e^{-2t}}{t} dt$

Hint: $L\{e^{-t} - e^{-2t}\} = \frac{1}{s+1} - \frac{1}{s+2}$ and so

$$L\left\{\frac{e^{-t} - e^{-2t}}{t}\right\} = \int_s^{\infty} \left(\frac{1}{t+1} - \frac{1}{t+2}\right) dt = \log\left(\frac{s+2}{s+1}\right),$$

that is,

$$\int_0^{\infty} e^{-st} \left(\frac{e^{-t} - e^{-2t}}{t}\right) dt = \log\left(\frac{s+2}{s+1}\right). \text{ Taking } s = 0,$$

we get the value of the given integral equal to $\log 2$

(c) $\int_0^{\infty} \cos x^2 dx$ (Proceed as in Example 7.59)

Ans. $\frac{1}{2} \sqrt{\frac{\pi}{2}}$

(d) $\int_0^{\infty} \frac{x \sin at}{x^2 + a^2} dx$; $a, t > 0$.

Ans. $f(t) = \frac{\pi}{2} e^{-t}$

Appendix: Model Question Papers

Model Paper I

1. Attempt any four parts of the following:

$$5 \times 4 = 20$$

- (a) Show that the particular solution of

$$(x^2 + 1) \frac{dy}{dx} + (y^2 + 1) = 0, y(0) = 1 \text{ is}$$

$$y = (1 - x) / (1 + x).$$

- (b) Show that the following differential equation is exact and solve:

$$e^y dx + (xe^y + 2y)dy = 0$$

- (c) Solve the following differential equation

$$(D^2 + 4)y = \sin 3x + \cos 2x \text{ where } D = \frac{d}{dx}.$$

- (d) Solve the following simultaneous differential equation

$$\frac{dx}{dt} = 3x + 2y$$

$$\frac{dy}{dt} = 5x + 3y$$

- (e) Solve by method of variation of parameters

$$\frac{d^2y}{dx^2} + a^2y = \sec ax$$

- (f) A resistance (R) of 5Ω and an inductance (L) of 0.1 H are connected in series with a battery of 12 V . Find the current in the circuit as a function of time using the following differential equation.

$$Ri + L \frac{di}{dt} = E$$

2. Attempt any four parts of the following:

$$5 \times 4 = 20$$

- (a) Using Laplace transform, evaluate

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt$$

- (b) If $L\{F(t)\} = f(s)$ then prove that

$$L\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$$

- (c) Using convolution theorem, find the inverse Laplace transform of the following

$$\frac{1}{s(s+1)(s+2)}$$

- (d) Solve the following simultaneous differential equations by Laplace transform

$$3 \frac{dx}{dt} - y = 2t$$

$$\frac{dx}{dt} + \frac{dy}{dt} - y = 0$$

With the condition $x(0) = y(0) = 0$

- (e) Solve the following differential equation using Laplace transform

$$\frac{d^3y}{dt^3} - 3 \frac{d^2y}{dt^2} + 3 \frac{dy}{dt} - y = t^2 e^t$$

where $y(0) = 1, \left(\frac{dy}{dt}\right)_{t=0} = 0,$

$$\left(\frac{d^2y}{dt^2}\right)_{t=0} = -2r$$

- (f) Find the Laplace transform of the triangular wave function of period $2e$ given by

$$f(t) = \begin{cases} t, & 0 < t < e \\ 2e - t, & e < t < 2e \end{cases}$$

3. Attempt any **two** parts of the following:

$$10 \times 2 = 20$$

- (a) Solve the following differential equation in series

$$x^2 \frac{d^2y}{dx^2} + x(x-1) \frac{dy}{dx} + (1-x)y = 0$$

(b) Show that

$$\begin{aligned} \text{(i)} \quad & \int_{-1}^{+1} f(x) P_n(x) dx \\ &= \frac{(-1)^n}{2^n n!} \int_{-1}^{+1} (x^2 - 1)^n f^{(n)}(x) dx \end{aligned}$$

where $P_n(x)$ is the Legendre polynomial of degree n and $f^{(n)}(x)$ is the n^{th} derivative of $f(x)$.

(ii) $P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$ where $(\cdot)$ denotes the differentiation.

(c) Show that

$$\text{(i)} \quad J_3(x) + 3J'_0(x) + 4J'''_0(x) = 0$$

$$\text{(ii)} \quad J_{1/2}(x) = \sqrt{2/(\pi x)} \cdot \sin x$$

where $J_n(x)$ is the Bessel's polynomial of degree n and dashes denote the differentiations.

4. Attempt any **two** parts of the following:

$$10 \times 2 = 20$$

(a) Find the Fourier series for $f(x) = \frac{(\pi-x)^2}{4}$ in the interval $(0, 2\pi)$ and hence deduce

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

(b) Examine whether the function $f(x) = x \cos x$ is even or odd. Hence expand it in the terms of Fourier series in the region $-\pi < x < \pi$.

(c) Solve the following partial differential equations:

$$\text{(i)} \quad x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$$

$$\text{(ii)} \quad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \cos mx \cdot \cos ny \quad \text{where } p = \frac{\partial z}{\partial x} \text{ and } q = \frac{\partial z}{\partial y}.$$

5. Attempt any two parts of the following:

$$10 \times 2 = 20$$

(a) Solve the following equation by the method of separation of variables

$$\frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$$

given that $u = 0$, when $t = 0$ and $\frac{\partial u}{\partial t} = 0$ when $x = 0$.

(b) Solve the following Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in a rectangle in the xy -plane with $u(x, 0) = 0$, $u(x, b) = 0$, $u(0, y) = 0$ and $u(a, y) = f(y)$ parallel to y -axis.

(c) In a telephone wire of length l , a steady voltage distribution of 20 volts at the source end and 12 volts at the terminal end is maintained. At time $t = 0$, the terminal end is grounded. Determine the voltage and current, assuming inductance (L) and conductance to ground (G) are negligible.

SOLUTIONS

1. (a) The given equation is

$$(x^2 + 1) \frac{dy}{dx} + (y^2 + 1) = 0, \quad (y(0) = 1).$$

Therefore

$$\frac{dy}{dx} = -\frac{y^2 + 1}{x^2 + 1}$$

or

$$\frac{dy}{y^2 + 1} = -\frac{dx}{x^2 + 1} \quad (\text{separation of variables})$$

Integrating, we get

$$\tan^{-1} u = -\tan^{-1} x + \tan^{-1} c$$

or

$$\tan^{-1} y + \tan^{-1} x = \tan^{-1} c$$

or

$$\tan^{-1} \frac{y+x}{1-xy} = \tan^{-1} c$$

or

$$\frac{y+x}{1-xy} = c$$

$$y+x = c(1-xy)$$

or

$$y(1+cx) = c-x$$

or

$$y = \frac{c - x}{1 + cx}.$$

Taking $y(0) = 1$, we get

$$1 = \frac{c}{1} \quad \text{or} \quad c = 1.$$

Therefore

$$y = \frac{1 - x}{1 + x}.$$

(b) We have

$$e^y dx + (xe^y + 2y)dy = 0$$

Comparing with $Mdx + Ndy = 0$, we get

$$M = e^y \quad \text{and} \quad N = xe^y + 2y.$$

Therefore

$$\frac{\partial M}{\partial y} = e^y \quad \text{and} \quad \frac{\partial N}{\partial x} = e^y.$$

Hence $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and so the equation is exact. It's solution is

$$\int_{y \text{ constant}} e^y dx + \int 2y dy = c$$

or

$$x e^y + y^2 = c.$$

(c) The given differential equation is

$$(D^2 + 4)y = \sin 3x + \cos 2x.$$

Its auxiliary equation is

$$m^2 + 4 = 0,$$

which yields $m = \pm 2i$. Therefore

$$\text{C.F.} = c_1 : \cos 2x + c_2 \sin 2x.$$

Further,

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 + 4} (\sin 3x) + \frac{1}{D^2 + 4} (\cos 2x) \\ &= \frac{1}{-(3^2 + 4)} (\sin 3x) + \frac{x}{2D} (\cos 2x) \\ &= -\frac{1}{5} \sin 3x + \frac{x}{2} \left(\frac{\sin 2x}{2} \right) \\ &= -\frac{1}{5} \sin 3x + \frac{x}{4} \sin 2x. \end{aligned}$$

Hence the complete solution of the given differential equation is

$$y = \text{C.F.} + \text{P.I.}$$

$$= c_1 \cos 2x + c_2 \sin 2x + \frac{x}{4} \sin 2x - \frac{1}{5} \sin 3x.$$

(d) The given equations are

$$\frac{dx}{dt} = 3x + 2y \quad \text{and} \quad \frac{dy}{dt} = 5x + 3y.$$

In symbolic form, we have

$$(D - 3)x - 2y = 0 \quad \text{and} \quad (1)$$

$$-5x + (D - 3)y = 0$$

or

$$(D - 3)^2 x - 2(D - 3)y = 0 \quad \text{and} \quad (2)$$

$$-10x + 2(D - 3)y = 0 \quad (3)$$

Adding (2) and (3), we get

$$[(D - 3)^2 x - 10]x = 0$$

or

$$(D^2 - 6D - 1)x = 0. \quad (4)$$

The auxiliary equation of (4) is

$$m^2 - 6m - 1 = 0,$$

which gives $m = 3 \pm \sqrt{10}$. Therefore

$$x = e^{3t} [c_1 \cosh \sqrt{10}t + c_2 \sinh \sqrt{10}t]. \quad (5)$$

Form equation (1), we have (using (5))

$$\begin{aligned} 2y &= \frac{dx}{dt} - 3x \\ &= e^{3t} \sqrt{10} [c_1 \sinh \sqrt{10}t + c_2 \cosh \sqrt{10}t] \\ &\quad + 3 e^{3t} \sqrt{10} [c_1 \cosh \sqrt{10}t + c_2 \sinh \sqrt{10}t] \\ &\quad - 3 e^{3t} \sqrt{10} [c_1 \cosh \sqrt{10}t + c_2 \sinh \sqrt{10}t] \\ &= e^{3t} [c_1 \sinh \sqrt{10}t + c_2 \cosh \sqrt{10}t]. \end{aligned}$$

Hence

$$y = \frac{1}{2} [e^{3t} \sqrt{10} [c_1 \sinh \sqrt{10}t + c_2 \cosh \sqrt{10}t].$$

(e) Please see Example 15.106

(Here roots of A. E are $\pm a i$)

(f) The equation of the given L R circuit is

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}. \quad (1)$$

We are given that $E = 12V$, $L = 0.1H$ and $R = 5 \text{ Ohms}$. Therefore (1) reduces to

$$\frac{di}{dt} + 50i = 120 \quad (2)$$

Thus is Leibnitz's linear equation with integrating factor $e^{\int 50t dt}$. Therefore the solution of (2) is

$$\begin{aligned} i e^{50t} &= 120 \int e^{50t} dt = 120(50e^{50t}) + c \\ &= 6000 e^{50t} + c. \end{aligned}$$

Hence

$$i = 6000 + c e^{-50t}.$$

2. (a) Proceeding as in Example 5.64, we have

$$\begin{aligned} \int_0^\infty \frac{e^{-at} - e^{-bt}}{t} dt &= \int_0^\infty L\{e^{-at} - e^{-bt}\} du \\ &= \int_0^\infty \left[\frac{1}{u+a} - \frac{1}{u+b} \right] du \\ &= [\log(u+a) - \log(u+b)]_0^\infty \\ &= \left[\log \frac{u+a}{u+b} \right]_0^\infty \\ &= \log \left[\lim_{u \rightarrow \infty} \frac{1 + \frac{a}{u}}{1 + \frac{b}{u}} \right] - \log \frac{a}{b} \\ &= \log 1 - \log \frac{a}{b} = -\log \frac{a}{b} \\ &= \log b - \log a. \end{aligned}$$

(b) Please see Theorem 5.5.

(c) Let

$$F(s) = \frac{1}{s(s+1)} \quad \text{and} \quad G(s) = \frac{1}{s+2}.$$

Then

$$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} = 1 - e^{-t}$$

and

$$g(t) = L^{-1}\{G(s)\} = L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}.$$

Therefore, by Convolution theorem,

$$\begin{aligned} L^{-1}\{F(s)G(s)\} &= f * g = \int_0^t (1 - e^{-u})e^{-2(t-u)} du \\ &= e^{-2t} \int_0^t (e^{2u} - e^u) du \\ &= e^{-2t} \left[\frac{e^{2u}}{2} - e^u \right]_0^t \\ &= e^{-2t} \left[\frac{e^{2t}}{2} - e^t + \frac{1}{2} \right] \\ &= \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t}. \end{aligned}$$

(d) The given simultaneous differential equations are

$$\begin{aligned} 3x' - y &= 2t \\ x' + y' - y &= 0 \end{aligned}$$

with $x(0) = y(0) = 0$. Taking Laplace transforms, we get

$$3\{sX(s) - x(0)\} - Y(s) = \frac{2}{s^2}$$

and

$$sX(s) - x(0) + sY(s) - y(0) - Y(s) = 0.$$

Using the initial conditions, the above equations reduce to

$$3sX(s) - Y(s) = \frac{2}{s^2}$$

and

$$sX(s) + sY(s) - Y(s) = 0$$

or

$$3sX(s) - Y(s) = \frac{2}{s^2} \quad (1)$$

and

$$sX(s) + (s-1)Y(s) = 0 \quad (2)$$

Multiplying the equation (2) by 3 and then subtracting (1) from it, we get

$$(3s - 2)Y(s) = -\frac{2}{s^2}$$

and so

$$Y(s) = -\frac{2}{s^2(3s - 2)} = \frac{1}{s^2} + \frac{3}{2s} - \frac{3}{2(s - \frac{2}{3})} \quad (3)$$

(by partial fractions)

Taking inverse Laplace transform, we have

$$y = t + \frac{3}{2} - \frac{3}{2}e^{\frac{2t}{3}}. \quad (4)$$

Substituting the value of $Y(s)$ from (3) in (1), we get

$$3sX(s) = \frac{2}{s^2} + \frac{1}{s^2} + \frac{3}{2s} - \frac{3}{2(s - \frac{2}{3})}$$

so that

$$\begin{aligned} X(s) &= \frac{1}{s^3} + \frac{1}{2s^2} - \frac{1}{2s(s - \frac{2}{3})} \\ &= \frac{1}{s^3} + \frac{1}{2s^2} - \frac{3}{4} \left[\frac{1}{s - \frac{2}{3}} - \frac{1}{s} \right] \end{aligned}$$

Taking inverse Laplace transform, we have

$$x = \frac{t^2}{2} + \frac{t}{2} + \frac{3}{4} - \frac{3}{4}e^{\frac{2t}{3}} \quad (5)$$

Thus (4) and (5) provides the solution to the given system.

(e) The given differential equation is

$$y''' - 3y'' + 3y' - y = t^2 e^t \quad (1)$$

with conditions

$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 2.$$

Taking Laplace transform, we get

$$\begin{aligned} s^3 Y(s) - s^2 y(0) - s y'(0) - y''(0) - 3[s^2 Y(s) \\ - s y(0) - y'(0)] + 3[s Y(s) - y(0)] - Y(s) \\ = \frac{2}{(s - 1)^3}. \end{aligned}$$

Using the given initial conditions, we have

$$\begin{aligned} s^3 Y(s) - s^2 + 2 - 3[s^2 Y(s) - s] + 3[s Y(s) - 1] - Y(s) \\ = \frac{2}{(s - 1)^3} \end{aligned}$$

or

$$[s^3 - 3s^2 + 3s - 1]Y(s) - s^2 + 3s - 1 = \frac{2}{(s - 1)^3}$$

or

$$\begin{aligned} (s - 1)^3 Y(s) &= \frac{2}{(s - 1)^3} + s^2 - 3s + 1 \\ &= \frac{2}{(s - 1)^3} + (s - 1)^2 - s \end{aligned}$$

Thus

$$\begin{aligned} Y(s) &= \frac{2}{(s - 1)^6} + \frac{(s - 1)^2}{(s - 1)^3} - \frac{s}{(s - 1)^3} \\ &= \frac{2}{(s - 1)^6} + \frac{1}{s - 1} - \left[\frac{(s - 1) + 1}{(s - 1)^3} \right] \\ &= \frac{2}{(s - 1)^6} + \frac{1}{s - 1} - \frac{1}{(s - 1)^3} - \frac{1}{(s - 1)^3} \end{aligned}$$

Taking inverse Laplace transform, we get

$$y = \frac{t^5 e^t}{60} + e^t - t e^t - \frac{t^2}{2} e^t.$$

(f) Example 5.74

3. (a) The given differential equation is

$$x^2 \frac{d^2 y}{dx^2} + x(x - 1) \frac{dy}{dx} + (1 - x)y = 0$$

or

$$\frac{d^2 y}{dx^2} + \frac{x - 1}{x} \frac{dy}{dx} + \frac{1 - x}{x^2} y = 0.$$

Comparing this equation with

$$\frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = 0,$$

we observe that

$$P(x) = \frac{x - 1}{x} \quad \text{and} \quad Q(x) = \frac{1 - x}{x^2}$$

Clearly, both $P(x)$ and $Q(x)$ are not analytic at $x = 0$ and hence $x = 0$ is a singular point. Further, $xP(x) = x - 1$ and $x^2 Q(x) = 1 - x$ are analytic at $x = 0$. Hence $x = 0$ is a regular singular point. So, let

$$y = \sum_{n=0}^{\infty} a_n x^{n+m}, \quad a_0 \neq 0$$

Differentiating twice in succession, we have

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} (n+m) a_n x^{n+m-1},$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m-2}.$$

Substituting the values of y , $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} (n+m)(n+m-1) a_n x^{n+m} \\ & + \sum_{n=0}^{\infty} (n+m) a_n x^{n+m+1} - \sum_{n=0}^{\infty} (n+m) a_n x^{n+m} \\ & + \sum_{n=0}^{\infty} a_n x^{n+m} + \sum_{n=0}^{\infty} (n+m-1) a_n x^{n+m+1} = 0 \end{aligned}$$

or

$$\begin{aligned} & \sum_{n=0}^{\infty} [(n+m)(n+m-1) - (n+m) + 1] a_n x^{n+m} \\ & + \sum_{n=1}^{\infty} (n+m-2) a_{n-1} x^{n+m} = 0 \end{aligned}$$

or

$$\begin{aligned} & \sum_{n=0}^{\infty} (n+m-1)^2 a_n x^{n+m} \\ & + \sum_{n=1}^{\infty} (n+m-2) a_{n-1} x^{n+m} = 0 \end{aligned}$$

or

$$\begin{aligned} & a_0(m-1)^2 x^m + \sum_{n=1}^{\infty} [(n+m-1)^2 a_n \\ & + (n+m-2) a_{n-1}] x^{n+m} = 0. \end{aligned}$$

Therefore the indicial equation is

$$(m-1)^2 = 0, \text{ which yields } m = 1, 1.$$

We note that the roots are equal. Equating to zero the other coefficients, we get

$$(n+m-1)^2 a_n = -(n+m-2) a_{n-1} \text{ for } n \geq 1. \quad (1)$$

For $m = 1$, we have from (1)

$$a_1 = 0 \quad a_2 = 0 \text{ and so on.}$$

Thus

$$y_1 = (a_0 x^m)_{m=1} = a_0 x.$$

Since the roots are equal,

$$y_2 = \left(\frac{\partial y}{\partial m} \right)_{m=1} (a_0 x^m \log x)_{m=1} = a_0 x \log x.$$

Hence the complete solution is

$$y = y_1 + y_2 = a_0 x + a_0 x \log x = Ax + Bx \log x.$$

(b) (i) By Rodrigue's formula, we have

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

Therefore integrating by parts, we have

$$\begin{aligned} \int_{-1}^1 f(x) P_n(x) dx &= \frac{1}{2^n n!} \int_{-1}^1 f(x) \frac{d^n}{dx^n} (x^2 - 1)^n dx \\ &= \frac{1}{2^n n!} \left[\left\{ f(x) \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n \right\}_{-1}^1 \right. \\ &\quad \left. - \int_{-1}^1 f'(x) \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n dx \right] \\ &= \frac{(-1)}{2^n n!} \int_{-1}^1 f'(x) \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^n dx \\ &= \frac{(-1)^2}{2^n n!} \int_{-1}^1 f(x) \frac{d^{n-1}}{dx^{n-1}} (x^2 - 1)^2 dx \end{aligned}$$

(using integration by parts second time)

$$\begin{aligned} & \dots\dots\dots \\ & \dots\dots\dots \\ & = \frac{(-1)^n}{2^n n!} \int_{-1}^1 f^{(n)}(x) (x^2 - 1)^n dx \end{aligned}$$

(using integration by parts n^{th} time)

(ii) Example 2.170 (iii)

(c) (i) We know that

$$\frac{d}{dx} [x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$$

(Please see Example 2.151 (ii))

Putting $n = 0$, we have

$$J_0'(x) = J_1(x).$$

Therefore

$$J_0''(x) = -J_1'(x) = -\frac{1}{2}[J_0 - J_2]$$

(Please see Example 2.151 (iv))

and

$$\begin{aligned} J_0'''(x) &= -\frac{1}{2}[J_0'(x) - J_2'(x)] \\ &= -\frac{1}{2}J_0'(x) + \frac{1}{2}\left[\frac{1}{2}\{J_1(x) - J_0(x)\}\right] \\ &= -\frac{1}{2}J_0'(x) + \frac{1}{4}J_1(x) - \frac{1}{4}J_3(x) \\ &= -\frac{1}{2}J_0'(x) + \frac{1}{4}[-J_0'(x)] - \frac{1}{4}J_3(x) \\ &= -\frac{3}{4}J_0'(x) - \frac{1}{4}J_3(x). \end{aligned}$$

Hence

$$4J_0'''(x) + 3J_0'(x) + J_3(x) = 0.$$

(ii) Example 2.153 (i)

4. (a) We have $f(x) = \frac{(\pi-x)^2}{4}$ and the interval is $(0, 2\pi)$. Therefore the Fourier coefficients are

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{2\pi} f(x) dx = \frac{1}{\pi} \int_0^{2\pi} \frac{(\pi-x)^2}{4} dx \\ &= \frac{1}{4\pi} \left[\frac{(\pi-x)^3}{-3} \right]_0^{2\pi} = \frac{\pi^2}{6}, \\ a_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_0^{2\pi} \frac{(\pi-x)^2}{4} \cos nx dx \\ &= \frac{1}{4\pi} \left[\left\{ (\pi-x)^2 \frac{\sin nx}{n} \right\}_0^{2\pi} + \int_0^{2\pi} 2(\pi-x) \frac{\sin nx}{n} dx \right] \\ &= \frac{-1}{2\pi n^2} (-2\pi) = \frac{1}{n^2}, \\ b_n &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_0^{2\pi} \frac{(\pi-x)^2}{4} \sin nx dx \\ &= \frac{1}{4\pi} \left[\left\{ (\pi-x)^2 \frac{\cos nx}{-n} \right\}_0^{2\pi} - \int_0^{2\pi} 2(\pi-x) \frac{\cos nx}{n} dx \right] \\ &= \frac{1}{2\pi n^2} \left[\frac{\cos nx}{n} \right]_0^{2\pi} = 0. \end{aligned}$$

Therefore the Fourier series is

$$\begin{aligned} f(x) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (\cos nx + \sin nx) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} \\ &= \frac{\pi^2}{12} + \frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \frac{\cos 4x}{4^2} + \dots \end{aligned}$$

Putting $n = 0$ in the above equation, we get

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

(b) We have $f(x) = x \cos x$ as a product of odd and even functions x and $\cos x$ respectively. But the product of odd and even function is odd. Hence f is odd. Since f is odd, $a_n = 0$. Further,

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} x \cos x \sin nx dx \\ &= \frac{1}{\pi} \int_0^{\pi} 2x \cos x \sin nx dx \\ &= \frac{1}{\pi} \int_0^{\pi} x [\sin(n+1)x + \sin(n-1)x] dx \\ &= \frac{1}{\pi} \left[x \left\{ -\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1} \right\} \right. \\ &\quad \left. - \left\{ -\frac{\sin(n+1)x}{(n+1)^2} - \frac{\sin(n-1)x}{(n-1)^2} \right\} \right]_0^{\pi} \\ &= \frac{1}{\pi} \left[\pi \left\{ -\frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1} \right\} \right], \quad n \neq 1 \\ &= - \left[\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} \right], \quad n \neq 1. \end{aligned}$$

If $n \neq 1$ and n is odd, then both $n-1$ and $n+1$ are even. Therefore

$$b_n = - \left[\frac{-1}{n+1} - \frac{1}{n-1} \right] = \frac{2n}{n^2-1}$$

If $n \neq 1$ and n is even, then both $n-1$ and $n+1$ are odd. Therefore

$$b_n = - \left[\frac{-1}{n+1} + \frac{1}{n-1} \right] = - \frac{2n}{n^2-1}.$$

When $n = 1$, we have

$$\begin{aligned} b_1 &= \frac{2}{\pi} \int_0^{\pi} x \cos x \sin x \, dx \\ &= \frac{1}{\pi} \int_0^{\pi} x \sin 2x \, dx = -\frac{1}{2}. \end{aligned}$$

Hence

$$f(x) = x \cos x = -\frac{1}{2} \sin x - \frac{4}{3} \sin 2x + \frac{3}{4} \sin 3x - \dots$$

(c) (i) Example 3.25

(ii) The symbolic form of the equation is

$$(D^2 + D^2)z = \cos mx \cos ny.$$

The auxiliary equation is

$$m^2 + 1 = 0, \text{ which yields } m = \pm i.$$

Therefore

$$\text{C.F.} = f_1(y + ix) + f_2(y - ix).$$

Further,

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 + D'^2} \cos mx \cos ny \\ &= \frac{1}{D^2 + D'^2} \left[\frac{1}{2} \{ \cos(mx + ny) + \cos(mx - ny) \} \right] \\ &= \frac{1}{2} \left[\frac{1}{D^2 + D'^2} \cos(mx + ny) \right. \\ &\quad \left. + \frac{1}{D^2 + D'^2} \cos(mx - ny) \right] \\ &= \frac{1}{2} \left[\frac{1}{m^2 + n^2} \iint \cos u \, du \, dv \right. \\ &\quad \left. + \frac{1}{m^2 + n^2} \iint \cos v \, dv \, dv \right], \end{aligned}$$

where $mx + ny = u$ and $mx - ny = v$

$$\begin{aligned} &= \frac{1}{2} \left[\frac{1}{m^2 + n^2} (-\cos u) \right. \\ &\quad \left. + \frac{1}{m^2 + n^2} (-\cos v) \right] \\ &= -\frac{1}{2(m^2 + n^2)} [\cos u + \cos v] \\ &= \frac{-1}{2(m^2 + n^2)} [\cos(mx + ny) \\ &\quad + \cos(mx - ny)] \\ &= \frac{-1}{m^2 + n^2} \cos mx \cos ny. \end{aligned}$$

Hence the solution is

$$\begin{aligned} z &= \text{C.F.} + \text{P.I.} \\ &= f_1(y + ix) + f_2(y - ix) \\ &\quad - \frac{1}{m^2 + n^2} \cos mx \cos ny. \end{aligned}$$

5. (a) Let

$$u = X(x)T(t)$$

be a trial solution of the given differential equation. Then

$$\begin{aligned} \frac{\partial u}{\partial t} &= X(x)T'(t), \\ \frac{\partial^2 u}{\partial x \partial t} &= T'(t)X'(x) \end{aligned}$$

Therefore the given equation transforms to

$$T'(t)X'(x) = e^{-t} \cos x.$$

The separation of variable yields

$$e^t T'(t) = \frac{\cos x}{X'(x)} = a \text{ (constant)}$$

Then

$$e^t T'(t) = a$$

or

$$dT = ae^{-t} dt \quad (1)$$

Integration of (1) yields

$$T = -ae^{-t} + c_1 \quad (2)$$

Further,

$$X'(x) = \frac{1}{a} \cos x \, dx \quad (3)$$

Integrating (3), we get

$$X = \frac{1}{a} \sin x + c_2 \quad (4)$$

From (2) and (4), we get

$$u(x, t) = \left(\frac{1}{a} \sin x + c_2 \right) (-ae^{-t} + c_1) \quad (5)$$

Since u is zero when $t = 0$, we have

$$0 = \left(\frac{1}{a} \sin x + c_2 \right) (-a + c_1)$$

This is possible if $(-a + c_1) = 0$ or if $c_1 = a$.

Further, the solution (5) implies

$$\frac{\partial u}{\partial t} = \left(\frac{1}{a} \sin x + c_2 \right) (ae^{-t}). \quad (6)$$

Using the condition that $\frac{\partial u}{\partial t} = 0$ when, $x = 0$, we get from (6) that

$$0 = c_2 a e^{-t}, \text{ which implies } c_2 = 0.$$

Substituting the values c_1 and c_2 in the solution (5), we have

$$u(x, t) = \frac{1}{a} \sin x [-ae^{-t} + a] = \sin x (1 - e^{-t})$$

(b) By Article 3.9(c), three possible solutions are

$$(i) \quad u(x, y) = (C_1 e^{px} + C_2 e^{-px}) (c_3 \cos py + c_4 \sin py)$$

$$(ii) \quad u(x, y) = (c_5 x + c_6) (c_7 y + c_8)$$

$$(iii) \quad u(x, y) = (c_9 \cos px + c_{10} \sin px) \times (c_{11} e^{py} + c_{12} e^{-py})$$

We check first the validity of (ii). We have

$$u(x, 0) = 0 \Rightarrow (c_5 x + c_6) c_8 = 0 \Rightarrow c_8 = 0 \quad (1)$$

and using (i),

$$u(x, b) = 0 \Rightarrow (C_5 x + c_6) c_7 b = 0 \Rightarrow c_7 = 0.$$

Thus $u(x, y) = 0$ in this case, which is impossible. Hence (ii) is discarded.

Now we check (iii)

$$\begin{aligned} u(x, 0) &= 0 \Rightarrow (c_9 \cos px + c_{10} \sin px) (c_{11} + c_{12}) \\ &= 0 \Rightarrow c_{11} + c_{12} = 0 \Rightarrow c_{12} = -c_{11}, \end{aligned}$$

and

$$\begin{aligned} u(x, b) &= (c_9 \cos px + c_{10} \sin px) (c_{11} e^{pb} + c_{12} e^{-pb}) = 0 \\ &\Rightarrow c_{11} e^{pb} + c_{12} e^{-pb} = 0 \\ &\Rightarrow c_{11} (e^{pb} - e^{-pb}) = 0 \\ &\Rightarrow c_{11} = 0 \end{aligned}$$

Thus $c_{11} = c_{12} = 0$ and so $u(x, y) = 0$, which is impossible. Hence (iii) is discarded. Thus solution (i) is suitable. Since $u(x, y) = 0$, we have

$$(c_1 e^{px} + c_2 e^{-px}) c_3 = 0 \text{ and so } c_3 = 0.$$

Therefore

$$u(x, y) = (c_1 e^{px} + c_2 e^{-px}) c_4 \sin py \quad (2)$$

Now $u(x, b) = 0$ implies

$$0 = (c_1 e^{px} + c_2 e^{-px}) c_4 \sin pb$$

Therefore $\sin bp = 0$ or $p = \frac{n\pi}{b}$. Hence (2) reduces to

$$u(x, y) = c_4 \sin \frac{n\pi y}{b} [c_1 e^{\frac{n\pi x}{b}} + c_2 e^{-\frac{n\pi x}{b}}].$$

Since $u(0, y) = 0$, we have

$$0 = c_4 \sin \frac{n\pi y}{b} [c_1 + c_2]$$

and so $c_1 + c_2 = 0$, which gives $c_2 = -c_1$

Therefore (3) reduces to

$$\begin{aligned} u(x, y) &= c_1 c_4 \sin \frac{n\pi y}{b} [e^{\frac{n\pi x}{b}} - e^{-\frac{n\pi x}{b}}] \\ &= 2c_1 c_4 \sin \frac{n\pi y}{b} \sinh \frac{n\pi x}{b} \\ &= b_n \sin \frac{n\pi y}{b} \sinh \frac{n\pi x}{b}. \end{aligned}$$

Hence the general solution is

$$u(x, y) = \Sigma b_n \sin \frac{n\pi y}{b} \sinh \frac{n\pi x}{b} \quad (3)$$

Further,

$$u(a, y) = f(y) = \Sigma b_n \sin \frac{n\pi y}{b} \sinh \frac{n\pi a}{b}.$$

Therefore

$$b_n = \frac{2}{b \sinh \frac{n\pi a}{b}} \int_0^b f(y) \sin \frac{n\pi y}{b} dy \quad (4)$$

Hence (3) is the solution, where b_n is given by (4).

(c) See Example 3.94

(There is change in data)

Ans.

$$v(x, t) = \frac{20(l-x)}{l} + \frac{24}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} e^{-\frac{n^2 \pi^2 t}{RCl^2}},$$

$$i(x, t) = \frac{20}{lR} + \frac{24}{lR} \Sigma (-1)^n \cos \frac{n\pi x}{l} e^{-\frac{n^2 \pi^2 t}{RCl^2}}.$$

Model Paper II

1. Attempt any **four** parts of the following:

$$5 \times 4 = 20$$

- (a) Solve

$$(y + \sqrt{x^2 + y^2})dx - xdy = 0, y(1) = 0.$$

- (b) Solve the following differential equation:

$$(\cos x - x \cos y)dy - (\sin y + y \sin x)dx = 0$$

- (c) Find the solution of following differential equation:

$$(D^2 - 4D - 5)y = e^{2x} + 3 \cos(4x + 3) \text{ where } D = \frac{d}{dx}.$$

- (d) Solve the following simultaneous differential equations:

$$\frac{dx}{dt} + 5x - 2y = t$$

$$\frac{dy}{dt} + 2x + y = 0$$

Also show that $x = y = 0$ when $t = 0$.

- (e) Solve by method of variation of parameters

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{-x} \log x.$$

- (f) An inductance (L) of 2.0 H and a resistance (R) of 20 Ω are connected in series with an e.m.f. E volt. If the current (i) is zero, when $t = 0$, find the current (i) at the end of 0.01 second if $E = 100$ V, using the following differential equation:

$$L \frac{di}{dt} + iR = E.$$

2. Attempt any **four** parts of the following:

$$5 \times 4 = 20$$

- (a) Using Laplace transform, evaluate

$$\int_0^\infty e^{-t} \frac{\sin^2 t}{t} dt.$$

- (b) State second shifting theorem for Laplace transform and hence find the Laplace

transform of the following function:

$$F(t) = \begin{cases} e^{t-a}, & t > a \\ 0, & t < a \end{cases}$$

- (c) Using convolution theorem, find the inverse Laplace transform of the following

$$\frac{s}{(s^2 + a^2)^3}.$$

- (d) Solve the following simultaneous differential equations by Laplace transform

$$\frac{dx}{dt} + 4\frac{dy}{dt} - y = 0$$

$$\frac{dx}{dt} + 2y = e^{-t}$$

With conditions $x(0) = y(0) = 0$.

- (e) Solve the following differential equation using Laplace transform

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = e^{-x} \sin x \text{ where } y(0), \left(\frac{dy}{dx}\right)_{x=0} = 1.$$

- (f) Using unit step function, find the Laplace transform of:

$$(i) (t-1)^2 \cdot u(t-1)$$

$$(ii) \sin t \cdot u(t-\pi).$$

3. Attempt any **two** parts of the following:

$$10 \times 2 = 20$$

- (a) Solve the following differential equation in series:

$$2x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + (x-5)y = 0$$

- (b) Show that

$$(i) \int_{-1}^{+1} [P_n(x)]^2 dx = \frac{2}{2n+1}$$

$$(ii) P_{2n}(0) = \frac{(-1)^n (2n)!}{2^{2n} (n!)^2}$$

where $P_n(x)$ is the Legendre polynomial of degree n .

- (c) Show that

$$(i) J_2'(x) = \left(1 - \frac{4}{x^2}\right) J_1(x) + \frac{2}{x} J_0(x)$$

$$(ii) J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + \dots) = 1$$

where $J_n(x)$ is the Bessel's polynomial of degree n and dash denote the differentiation.

4. Attempt any **two** parts of the following:

$$10 \times 2 = 20$$

(a) Obtain a Fourier series to represent $f(x) = x^2$ in the interval $(0, 2\pi)$ and hence deduce that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$.

(b) Examine whether the function $f(x) = x \sin x$ is even or odd. Hence expand it in the form of Fourier series in the interval $(-\pi, \pi)$.

(c) Solve the following partial differential equations:

$$(1) x(y^2 + z)p + y(x^2 + z)q = z(x^2 - y^2)$$

$$(2) \frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial y} = \sin x \cdot \cos 2y \text{ where } p = \frac{\partial z}{\partial x} \text{ and } q = \frac{\partial z}{\partial y}$$

5. Attempt any **two** parts of the following:

$$10 \times 2 = 20$$

(a) Solve the following equation by the method of separation of variables:

$$\frac{\partial u}{\partial x} = 4 \frac{\partial u}{\partial y} \text{ where } u(0, y) = 8 e^{-3y}.$$

(b) Solve the following Laplace equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in a rectangle with $u(0, y) = 0$, $u(a, y) = 0$, $u(x, b) = 0$ and $u(x, 0) = f(x)$ along x -axis.

(c) Assuming the resistance of wire (R) and conductance to ground (C_r) are negligible, find the voltage $v(x, t)$ and current $i(x, t)$ in a transmission line of length l , t seconds after the ends are suddenly grounded. The initial conditions are $v(x, 0) = v_0 \sin\left(\frac{\pi x}{l}\right)$ and $i(x, 0) = i_0$.

SOLUTIONS

1. (a) Example 2.17.

(b) We have

$$[\cos x - x \cos y]dy - [\sin y + y \sin x]dx = 0$$

Comparing with $Mdx + Ndy = 0$, we have

$$M = -\sin y - y \sin x \text{ and } N = \cos x - x \cos y.$$

Therefore

$$\frac{\partial M}{\partial y} = -\cos y - \sin x \text{ and } \frac{\partial N}{\partial x} = -\sin x - \cos y.$$

Hence $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and so the given differential equation is exact. Its solution is

$$\int_{y \text{ constant}} (-\sin y - y \sin x)dx + \int 0dy = c$$

or

$$-[x \sin y - y \cos x] = 0$$

or

$$y \cos x - x \sin y = c$$

(c) The A.E. is

$$m^2 - 4m - 5 = 0,$$

which yields $m = \frac{4 \pm \sqrt{16+20}}{2} = 5, -1$

Therefore

$$\text{C.F.} = c_1 e^{-x} + c_2 e^{5x}$$

Further,

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - 4D - 5} [e^{2x} + 3 \cos(4x + 3)] \\ &= \frac{1}{D^2 - 4D - 5} e^{2x} + \frac{3}{D^2 - 4D - 5} [\cos(4x + 3)] \\ &= \frac{1}{4 - 8 - 5} e^{2x} + \frac{3}{D^2 - 4D - 5} [\cos(4x + 3)] \\ &= -\frac{1}{9} e^{2x} + \frac{3}{-16 - 4D - 5} \cos(4x + 3) \\ &= -\frac{1}{9} e^{2x} + \frac{3}{-21 - 4D} \cos(4x + 3) \\ &= -\frac{1}{9} e^{2x} + \frac{3(21 - 4D)}{-(21 + 4D)(21 - 4D)} \cos(4x + 3) \\ &= -\frac{1}{9} e^{2x} - 3 \left[\frac{21 - 4D}{444 - 16D^2} \cos(4x + 3) \right] \\ &= -\frac{1}{9} e^{2x} - 3 \left[\frac{21 - 4D}{441 - 16(-16)} \cos(4x + 3) \right] \\ &= -\frac{1}{9} e^{2x} - 3 \left[\frac{21 - 4D}{697} \cos(4x + 3) \right] \end{aligned}$$

$$= -\frac{1}{9}e^{2x} - \frac{3}{697}[21\cos(4x+3) - 4D[\cos(4x+3)]]$$

$$= -\frac{1}{9}e^{2x} - \frac{3}{697}[21\cos(4x+3) + 16\sin(4x+3)]$$

(d) The symbolic forms of the equations are

$$(D+5)x - 2y = t \quad (1)$$

$$2x + (D+1)y = 0 \quad (2)$$

Operating (1) by $(D+1)$ and multiplying (2) by 2 and then adding, we get

$$(D^2 + 6D + 9)x = 1 + t.$$

The auxiliary equation is

$$m^2 + 6D + 9 = 0$$

and so $m = -3, -3$ and hence

$$\text{C.F.} = (c_1 + c_2t)e^{-3t}$$

Further,

$$\text{P.I.} = \frac{1}{(D+3)^2}(1+t) = \frac{1}{9}\left(1 + \frac{D}{3}\right)^{-2}(1+t)$$

$$= \frac{1}{9}\left[1 - \frac{2D}{3} + \dots\right](1+t) = \frac{1}{9}\left(t + \frac{1}{3}\right).$$

Hence

$$x = \text{C.F.} + \text{P.I.}$$

$$= (c_1 + c_2t)e^{-3t} + \frac{1}{9}\left(t + \frac{1}{3}\right). \quad (3)$$

Therefore

$$\frac{dx}{dt} = -3(c_1 + c_2t)e^{-3t} + c_2e^{-3t} + \frac{1}{9}.$$

Putting in the first (given) equation, we get

$$y = (c_1 + c_2t)e^{-3t} + \frac{1}{2}c_2e^{-3t} - \frac{2}{9}t + \frac{4}{27}. \quad (4)$$

Hence the solution is given by (3) and (4).

(e) The auxiliary equation for the given differential equation is

$$m^2 + 2m + 1 = 0,$$

Which yields $m = -1, -1$. Thus

$$\text{C.F.} = (c_1 + c_2x)e^{-x}.$$

To find P.I., let

$$y_1 = e^{-x} \quad \text{and} \quad y_2 = xe^{-x}.$$

Then

$$W = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & -xe^{-x} + e^{-x} \end{vmatrix} = e^{-2x}.$$

Therefore

$$\text{P.I.} = -y_1 \int \frac{y_2 F(x)}{W} dx + y_2 \int \frac{y_1 F(x)}{W} dx$$

$$= -e^{-x} \int \frac{xe^{-x} \cdot e^{-x} \log x}{e^{-2x}} dx$$

$$+ xe^{-x} \int \frac{e^{-x} \cdot e^{-x} \log x}{e^{-2x}} dx$$

$$= -e^{-x} \int x \log x dx + x e^{-x} \int \log x dx$$

$$= -e^{-x} \left[\frac{x^2}{2} \log x - \frac{x^2}{4} \right] + x e^{-x} [x \log x - x].$$

(f) The equation governing the circuit is

$$L \frac{di}{dt} + \frac{R}{L} = E$$

or

$$\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L}.$$

Here $R = 20$ Ohms, $L = 2H$ and c.m.c = E. Therefore

$$\frac{di}{dt} + 10 i = \frac{E}{2}$$

This is Leibnitz's linear equation with integrating factor $e^{\int \frac{R}{L} dt} = e^{\int 10 dt} = e^{10t}$. Therefore the solution is

$$i e^{10t} = \int \frac{E}{2} e^{10t} dt$$

$$= \frac{E}{20} e^{10t} + c \quad (1)$$

But $i = 0$ when $t = 0$. Therefore

$$0 = \frac{E}{20} + c \quad \text{or} \quad c = -\frac{E}{20}.$$

Hence (1) gives

$$i = \frac{E}{20} - \frac{E}{20} e^{-10t} = \frac{E}{20} [1 - e^{-10t}].$$

2. (a) Example 5.87

(b) Theorem 5.4 (for statement)

We have

$$F(t) = \begin{cases} e^{t-\alpha}, & t > \alpha \\ 0, & t < \alpha \end{cases}$$

By second shifting property, we have

$$L\{F(t)\} = e^{-s\alpha} \cdot L\{e^t\} = \frac{1}{s-1} e^{-s\alpha}$$

(c) Let

$$F(s) = \frac{s}{s^2 + a^2} \quad \text{and} \quad G(s) = \frac{1}{(s^2 + a^2)^2}$$

Then (see Example 6.59)

$$f(t) = \cos at \quad \text{and} \quad g(t) = \frac{1}{2a^3} [\sin at - at \cos at]$$

By Convolution Theorem, we have

$$\begin{aligned} L^{-1}\{F(s)G(s)\} &= f * g \\ &= \int_0^t \frac{1}{2a^3} \cos au [\sin a(t-u) \\ &\quad - a(t-u) \cos a(t-u)] du \\ &= \frac{1}{2a^3} \int_0^t [\cos au \sin a(t-u) \\ &\quad - a \cos au (t-u) \cos a(t-u)] du \\ &= \frac{1}{2a^3} \left[\int_0^t \cos au \sin a(t-u) du \right. \\ &\quad \left. - a \int_0^t (t-u) \cos au \cos a(t-u) du \right] \\ &= \frac{1}{2a^3} \int_0^t \cos au \sin a(t-u) du \\ &\quad - \frac{1}{2a^2} \left[\int_0^t (t-u) \frac{1}{2} \{ \cos at \right. \\ &\quad \left. + \cos a(t-2u) \} du \right] \\ &= \frac{1}{4a^3} t \sin at - \frac{t^2}{8a^2} \cos at \\ &\quad - \frac{1}{8a^3} t \sin at \\ &= \frac{t}{8a^3} [\sin at - at \cos at]. \end{aligned}$$

(d) Taking Laplace transform, we have

$$sX(s) - x(0) + 4(sY(s) - y(0) - Y(s)) = 0$$

and

$$sX(s) - x(0) + 2Y(s) = \frac{1}{s+1}.$$

Using the given initial conditions, we get

$$sX(s) + (4s-1)Y(s) = 0 \quad (1)$$

and

$$sX(s) + 2Y(s) = 0 \quad (2)$$

Subtracting (2) from 1, we get

$$(4s-3)Y(s) = -\frac{1}{s+1}$$

or

$$\begin{aligned} Y(s) &= \frac{1}{(s+1)(4s-3)} \\ &= \frac{1}{7} \left[\frac{1}{s+1} - \frac{1}{s-\frac{3}{4}} \right]. \end{aligned} \quad (3)$$

Taking inverse Laplace transform, we get

$$y = \frac{1}{7} [e^{-t} - e^{\frac{3t}{4}}].$$

Putting the value of $Y(s)$ from (3) in (2), we get

$$sX(s) = \frac{5}{7(s+1)} + \frac{2}{7(s-\frac{3}{4})}$$

or

$$\begin{aligned} X(s) &= \frac{5}{7s(s+1)} + \frac{2}{7s(s-\frac{3}{4})} \\ &= \frac{5}{7} \left(\frac{1}{s} - \frac{1}{s+1} \right) + \frac{8}{21} \left[\frac{1}{s-\frac{3}{4}} - \frac{1}{s} \right]. \end{aligned}$$

Taking inverse transform, we get

$$x = \frac{1}{3} - \frac{5}{7} e^{-t} + \frac{8}{21} e^{\frac{3t}{4}}.$$

(e) Example 7.10.

(f) (i) We have

$$\begin{aligned} L\{(t-1)^2 u(t-1)\} &= e^{-s} L\{t^2\} = e^{-s} \left(\frac{2}{s^3} \right) \\ &= \frac{2e^{-s}}{s^3}. \end{aligned}$$

(ii) We note that

$$\begin{aligned}\sin t u(t - \pi) &= \sin\{(t - \pi) + \pi\}u(t + \pi) \\ &= -\sin(t - \pi)u(t - \pi).\end{aligned}$$

Therefore

$$L\{\sin t \cdot u(t - \pi)\} = L\{-\sin t\}e^{-\pi s} = -\frac{e^{-\pi s}}{s^2 + 1}.$$

3. (a) Comparing the given equation with

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0,$$

we get

$$P(x) = -\frac{1}{2x}, \quad Q(x) = \frac{x-5}{2x^2},$$

At $x = 0$, both $P(x)$ and $Q(x)$ are not analytic. Thus $x = 0$ is a singular point. Further

$$xP(x) = -\frac{1}{2} \quad \text{and} \quad x^2Q(x) = \frac{x-5}{2},$$

which are analytic at $x = 0$. Hence $x = 0$ is a regular singular point. So, let

$$y = \sum_{n=0}^{\infty} a_n x^{n+m}, \quad a_n \neq 0.$$

Differentiating twice, we get

$$\begin{aligned}\frac{dy}{dx} &= \sum_{n=0}^{\infty} (n+m)x^{n+m-1} \\ \frac{d^2y}{dx^2} &= \sum_{n=0}^{\infty} (n+m)(n+m-1)x^{n+m-2}.\end{aligned}$$

Substituting the values of y , $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation, we have

$$\begin{aligned}&\sum_{n=0}^{\infty} 2(n+m)(n+m-1)a_n x^{n+m} \\ &- \sum_{n=0}^{\infty} (n+m)a_n x^{n+m} \\ &+ \sum_{n=0}^{\infty} a_n x^{n+m+1} - \sum_{n=0}^{\infty} 5a_n x^{n+m} = 0\end{aligned}$$

or

$$\begin{aligned}&\sum_{n=0}^{\infty} [2(n+m)(n+m-1) - (n+m) - 5]a_n x^{n+m} \\ &+ \sum_{n=0}^{\infty} a_n x^{n+m+1} = 0\end{aligned}$$

or

$$\begin{aligned}&\sum_{n=0}^{\infty} [2n^2 + 2m^2 + 4mn - 3n - 3m - 5]a_n x^{n+m} \\ &+ \sum_{n=0}^{\infty} a_n x^{n+m+1} = 0\end{aligned}$$

or

$$\begin{aligned}&\sum_{n=0}^{\infty} [2n^2 + 2m^2 + 4mn - 3n - 3m - 5]a_n x^{n+m} \\ &+ \sum_{n=0}^{\infty} a_n x^{n+m+1} = 0\end{aligned}$$

or

$$\begin{aligned}&\sum_{n=0}^{\infty} [2n^2 + 2m^2 + 4mn - 3n - 3m - 5]a_n x^{n+m} \\ &+ \sum_{n=1}^{\infty} a_{n-1} x^{n+m} = 0\end{aligned}$$

or

$$\begin{aligned}&(2m^2 - 3m - 5)a_0 x^m + \sum_{n=1}^{\infty} [(2n^2 + 2m^2 \\ &+ 4mn - 3n - 3m - 5)a_n + a_{n-1}]x^{m+n} = 0.\end{aligned}$$

Therefore the indicial equation is

$$2m^2 - 3m - 5 = 0, \text{ which yields } m = \frac{5}{2}, -1.$$

Equating to zero the other coefficients, we have

$$\begin{aligned}&(2n^2 + 2m^2 + 4mn - 3n - 3m - 5)a_n \\ &= -a_{n-1} \text{ for } n \geq 1.\end{aligned} \quad (1)$$

For $m = \frac{5}{2}$, we have from (1),

$$(2n^2 + 7n)a_n = -a_{n-1}, \quad a \geq 1$$

or

$$a_n = -\frac{a_{n-1}}{2n^2 + 7n}, \quad n \geq 1.$$

Thus, putting $n = 1, 2, 3, \dots$ we get

$$a_1 = -\frac{a_0}{9}$$

$$a_2 = \frac{a_0}{198}$$

$$a_3 = -\frac{a_0}{7722}$$

and so on. Therefore

$$y_1 = a_0 x^{\frac{5}{2}} \left[1 - \frac{x}{9} + \frac{x^2}{198} - \frac{x^3}{7722} + \dots \right].$$

For $m = -1$, the relation (1) yields

$$(2n^2 - 7n)a_n = -a_{n-1}, \quad n \geq 1$$

or

$$a_n = \frac{-a_{n-1}}{2n^2 - 7n}, \quad n \geq 1.$$

Therefore

$$a_1 = -\frac{a_0}{-5} = \frac{a_0}{5}$$

$$a_2 = -\frac{a_1}{-6} = \frac{a_0}{30}$$

$$a_3 = -\frac{a_2}{-3} = \frac{a_0}{90},$$

and so on. Hence

$$y_2 = a_0 x^{-1} \left[1 + \frac{x}{5} + \frac{x^2}{30} + \frac{x^3}{90} + \dots \right]$$

Therefore the general solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= A x^{\frac{5}{2}} \left[1 + \frac{x}{9} + \frac{x^2}{198} - \frac{x^3}{7722} + \dots \right] \\ &\quad + B x^{-1} \left[1 + \frac{x}{5} + \frac{x^2}{30} + \frac{x^3}{90} + \dots \right]. \end{aligned}$$

(b) (i) Example 2.168

(ii) Since $(1 - 2xt + t^2)^{-\frac{1}{2}}$ is generating function of the Legendre's polynomial, we have

$$(1 - 2x + t^2)^{-\frac{1}{2}} = \sum t^n P_n(x)$$

or

$$\begin{aligned} \sum t^n P_n(0) &= 1 - \frac{1}{2}t^2 + \frac{1.3}{2.4}t^4 \\ &\quad - \dots (-1)^r \frac{1.3.5 \dots (2r-1)}{2.4.6 \dots (2r)} t^{2r} + \dots \end{aligned}$$

Equating coefficients of t^{2n} on both sides, we get

$$\begin{aligned} P_{2n}(0) &= (-1)^n \frac{1.3.5 \dots (2n-1)}{2.4.6 \dots (2n)} \\ &= (-1)^n \frac{1.2.3.4 \dots (2n-1) 2^n}{[2.4.6 \dots (2n)]^2} \\ &= (-1)^n \frac{(2n)!}{[2^n (1.2 \dots n)]^2} = (-1)^n \frac{(2n)!}{2^{2n} (n!)^2}. \end{aligned}$$

(c) (i) We know that

$$J'_n(x) = \frac{1}{2} [J_{n-1}(x) - J_{n+1}(x)].$$

Therefore

$$J'_2(x) = \frac{1}{2} [J_1(x) - J_3(x)]. \quad (1)$$

Also

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x).$$

Therefore

$$\begin{aligned} J_3(x) &= \frac{4}{x} J_2(x) - J_1(x) \\ &= \frac{4}{x} \left[\frac{2}{x} J_1(x) - J_0(x) \right] - J_1(x) \\ &= \left(\frac{8}{x^2} - 1 \right) J_1(x) - \frac{4}{x} J_0(x). \end{aligned}$$

Putting this value in (1), we get

$$J'_2(x) = \left(1 - \frac{4}{x^2} \right) J_1(x) + \frac{2}{x} J_0(x).$$

(ii) From Example 2.160, the Jacobi's series are

$$J_0 + 2J_2 \cos 2\phi + 2J_4 \cos 4\phi + \dots = \cos(x \sin \phi)$$

and

$$\begin{aligned} 2J_1 \sin \phi + 2J_3 \sin 3\phi + 2J_5 \sin 5\phi + \dots \\ = \sin(x \sin \phi). \end{aligned}$$

Squaring these equations and integrating with respect to ϕ in the interval $(0, \pi)$, we get

$$\begin{aligned} \pi J_0^2 + 2\pi J_2^2 + 2\pi J_4^2 + \dots \\ = \int_0^\pi \cos^2(x \sin \phi) d\phi \end{aligned} \quad (1)$$

and

$$\begin{aligned} & 2\pi J_1^2 + 2\pi J_3^2 + 2\pi J_5^2 + \dots \\ &= \int_0^\pi \sin^2(x \sin \phi) d\phi \end{aligned} \quad (2)$$

Adding (1) and (2) we get

$$\pi J_0^2 + 2\pi J_1^2 + 2\pi J_2^2 + \dots = \int_0^\pi d\phi = \pi.$$

Hence

$$J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \dots = 1.$$

4. (a) Please see Example 4.6 and Example 4.16

(b) Since the functions x and $\sin x$ are both odd functions, their product $x \sin x$ is even. Hence the Fourier coefficients b_n are zero. Further,

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^\pi x \sin x \, dx \\ &= \frac{2}{\pi} \left[x(-\cos x) - \int (-\cos x) dx \right]_0^\pi \\ &= \frac{2}{\pi} [-x \cos x + \sin x]_0^\pi = 2, \\ a_n &= \frac{2}{\pi} \int_0^\pi x \sin x \cos nx \, dx \\ &= \frac{1}{\pi} \int_0^\pi x [\sin(n+1)x - \sin(n-1)x] dx \\ &= \frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1}, \quad n \neq 1 \end{aligned}$$

If n is odd and $n \neq 1$, then

$$a_n = \frac{1}{n-1} - \frac{1}{n+1} = \frac{2}{n^2-1}.$$

If n is even, then

$$a_n = \frac{-1}{n-1} + \frac{1}{n+1} = \left(\frac{-2}{n^2-1} \right)$$

Also

$$\begin{aligned} a_1 &= \frac{2}{\pi} \int_0^\pi x \sin x \cos x \, dx \\ &= \frac{1}{\pi} \int_0^\pi x \sin 2x \, dx \\ &= \frac{1}{\pi} \left[-\frac{\pi \cos 2\pi}{2} \right] = -\frac{1}{2}. \end{aligned}$$

Hence the Fourier series for $x \sin x$ is

$$1 - \frac{1}{2} \cos x - 2 \left[\frac{\cos 2x}{2^2-1} - \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} - \dots \right].$$

(c) (i) Example 3.25.

(ii) The symbolic form of the given equation is

$$(D^2 - 2DD')z = \sin x \cos 2y.$$

The auxiliary equation is

$$m^2 - 2m = 0, \text{ which yields } m = 0, 2.$$

Therefore

$$\text{C.F.} = f_1(y) + f_2(y+2x),$$

where f_1 and f_2 are arbitrary functions. Further,

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - 2DD'} \cdot \sin x \cos 2y \\ &= \frac{1}{D^2 - 2DD'} \cdot \frac{1}{2} [\sin(x+2y) + \sin(x-2y)] \\ &= \frac{1}{2} \left[\frac{1}{D^2 - 2DD'} \sin(x+2y) \right. \\ &\quad \left. + \frac{1}{D^2 - 2DD'} \sin(x-2y) \right] \\ &= \frac{1}{2} \left[\frac{1}{1^2 - 2(1)(2)} \iint \sin u \, du \, dv \right. \\ &\quad \left. + \frac{1}{1^2 - 2(1)(2)} \iint \sin v \, dv \, du \right], \end{aligned}$$

where $x+2y = u$, $x-2y = v$

$$\begin{aligned} &= \frac{1}{2} \left[\frac{1}{3} \sin u - \frac{1}{5} \sin v \right] \\ &= \frac{1}{6} \sin(x+2y) - \frac{1}{10} \sin(x-2y). \end{aligned}$$

Hence the complete solution of the given partial differential equation is

$$\begin{aligned} z &= \text{C.F.} + \text{P.I.} \\ &= f_1(y) + f_2(y+x) + \frac{1}{6} \sin(x+2y) \\ &\quad - \frac{1}{10} \sin(x-2y). \end{aligned}$$

5. (a) The given partial differential equation is

$$\frac{\partial u}{\partial x} = 4 \frac{\partial u}{\partial y}, \quad u(0, y) = 8e^{-3y} \quad (1)$$

Let

$$u = X(x)Y(y)$$

be a trial solution of the given partial differential equation. Then (1) implies

$$X'(x)Y(y) = 4X(x)Y'(y)$$

Separating the variables, we get

$$\frac{X'(x)}{X(x)} = 4 \frac{Y'(y)}{Y(y)} = a \text{ (constant)}$$

Thus we get two ordinary differential equations

$$\frac{X'(x)}{X(x)} = a \quad (2)$$

and

$$4 \frac{Y'(y)}{Y(y)} = a \quad (3)$$

Integrating (2), we get

$$\log X = ax + \log c_1$$

or

$$X = c_1 e^{ax} \quad (4)$$

Integrating (3), we have

$$\log Y = \frac{a}{4}y + \log c_2$$

or

$$Y = c_2 e^{\frac{ay}{4}} \quad (5)$$

Therefore

$$\begin{aligned} u &= X(x)Y(y) = c_1 c_2 e^{ax} e^{\frac{ay}{4}} \\ &= c_1 c_2 e^{ax + \frac{ay}{4}} = c_1 c_2 e^{\frac{4ax + ay}{4}} \end{aligned}$$

But $u(0, y) = 8e^{-3y}$. Therefore

$$8e^{-3y} = c_1 c_2 e^{\frac{ay}{4}},$$

which implies $c_1 c_2 = 8$ and $\frac{a}{4} = -3$. Thus $a = -12$.

Hence the solution is

$$u(x, y) = 8e^{\frac{-48x - 12y}{4}} = 8e^{-12x - 3y}.$$

(b) Please see Example 3.90. The only difference is that $u(x, 0) = f(x)$ in the present question. Thus

$$f(x) = \sum A_n \sin \frac{n\pi x}{a}$$

so that

$$A_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx. \quad (1)$$

Hence the solution is

$$u(x, y) = \sum A_n \sin \frac{n\pi x}{a} \cdot \frac{\sinh \frac{n\pi}{a}(b-y)}{\sinh \frac{n\pi b}{a}},$$

where A_n is given by (1).

(c) Example 3.96

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