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Engineering Mathematics III

For UPTU

Semester III

Babu Ram

Engineering Mathematics-III
for UPTU

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Engineering Mathematics-III for UPTU

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Delhi • Chennai • Chandigarh

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ISBN 9788131733370
eISBN 9789332506541

Head Office: A-8(A), Sector 62, Knowledge Boulevard, 7th Floor, NOIDA 201 309, India
Registered Office: 11 Local Shopping Centre, Panchsheel Park, New Delhi 110 017, India

In the Memory of

MY PARENTS

Smt. Manohari Devi and Sri. Makhan Lal

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Contents

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Preface

All branches of Engineering, Technology and Science require mathematics as a tool for the description of their contents. Therefore thorough knowledge of various topics in mathematics is essential to pursue study in Engineering, Technology and Science. The aim of this book is to provide the students with sound mathematics skills and their applications. Although the book is designed primarily for use by engineering students, it is also suitable for students pursuing bachelor degrees with mathematics as one of the subject and also for those who prepare for various competitive examinations. The material has been arranged to ensure the suitability of the book for class use and for individual self study. Accordingly, the contents of the book have been divided into eight chapters covering the complete syllabus prescribed for B.Tech. Semester-III of U.P. Technical University, Lucknow. A sufficient number of examples, figures, tables, and exercises have been provided to enable students to develop problem-solving skills. The language used is simple and amicable. Suggestions and feedback on this book are welcome.

ACKNOWLEDGEMENTS

I am extremely grateful to the reviewers for their valuable comments. My family members provided moral support during the preparation of this book. My son, Aman Kumar, Software Engineer, Adobe India Ltd., offered wise comments on some of the contents of the book. I am thankful to Sushma S. Pradeep for excellently typing the manuscript. Special thanks are due to Thomas Mathew Rajesh, Anita Yadav, and Vamanan Namboodiri of Pearson for their constructive support.

BABU RAM

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1 Functions of Complex Variables

In this chapter, we deal with functions of complex variables which are useful in evaluating a large number of new definite integrals, the theory of differential equations, the study of electric fields, thermodynamics, and fluid mechanics.

1.1 BASIC CONCEPTS

Definition 1.1. A *complex number z* is an ordered pair (x, y) of real numbers x and y.
If $z = (x, y)$ and $w = (u, v)$ are two complex numbers, then their addition and multiplication are defined as

$$z + w = (x, y) + (u, v) = (x + u, y + v)$$
$$zw = (x, y)\,(u, v) = (xu - yv, xv + yu).$$

With these operations of addition and multiplication, the complex numbers satisfy the same arithmetic properties as do the real numbers.

If we write the real number x as $(x, 0)$ and denote $i = (0, 1)$ (called imaginary number), then

$$\begin{aligned} z = (x, y) &= (x, 0) + (0, y) \\ &= (x, 0) + (y, 0)\,(0, 1) \\ &= x + iy. \end{aligned}$$

Thus, a complex number z can be expressed as $z = x + iy$, where x is called the *real part* of z and y is called the *imaginary part* of z. Thus

$$\mathrm{Re}(z) = x,\ \mathrm{Im}(z) = y.$$

Further,

$$i^2 = (0,1)\,(0,1) = (-1,0) = -1$$

and so $i = \sqrt{-1}$.

The set of complex numbers is denoted by $\mathbb{C}$. Since a real number x can be written as $x = (x, 0) = x + i0$, the set $\mathbb{C}$ is an extension of $\mathbb{R}$. Further, since the complex number $z = x + iy$ is an ordered pair (x, y), we can represent such numbers by points in xy plane, called the *complex plane* or *Argand diagram* (Fig. 1.1).

The *modulus* (or *absolute value*) of z is

$$|z| = r = \sqrt{x^2 + y^2}$$

and

$$|zw| = |z|\,|w|.$$

Further,

$$|z + w| \le |z| + |w| \quad \text{(triangle inequality)}.$$

Since $x = r\cos\theta$, $y = r\sin\theta$, we have

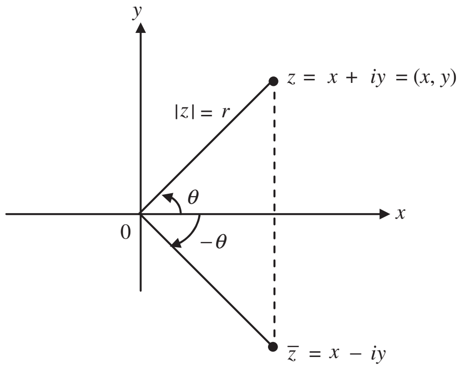


Figure 1.1 (Argand Diagram)

$$z = x + iy = r\cos\theta + i\, r\sin\theta$$
$$= r(\cos\theta + i\sin\theta),$$

which is called as the *polar form* of the complex number z. The angle θ is called the *amplitude* or *argument* of the complex number z and we have

$$\tan\theta = \frac{y}{x}.$$

Let $z = r\,(\cos\,\theta + i\,\sin\,\theta)$ and $w = R(\cos\,\phi + i\,\sin\,\phi)$ be two complex numbers. Then

$$zw = rR[(\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi)$$
$$= rR[\cos(\theta + \phi) + i\sin(\theta + \phi)].$$

Hence, the *arguments are additive under multiplication*, that is,

$$\arg(zw) = \arg z + \arg w.$$

Similarly, we can show that

$$\arg\left(\frac{z}{w}\right) = \arg z - \arg w$$

and

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}.$$

Definition 1.2. The *conjugate* of a complex number z is defined by

$$\bar{z} = x - iy.$$

We note that

$$z\bar{z} = (x + iy)\,(x - iy) = x^2 + y^2 = |z|^2,$$
$$\overline{z + w} = \bar{z} + \bar{w},$$
$$\overline{zw} = \bar{z}\,\bar{w}.$$

Consider complex numbers z with $|z| = 1$. All these numbers have distance 1 to the origin (0, 0) and so they form a circle with radius 1 and centre at the origin. This circle is called the *unit circle*.

Definition 1.3. For each $y \in \mathbb{R}$, the complex number e^{iy} is defined as

$$e^{iy} = \cos y + i \sin y,$$

which gives

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad 0 \le \theta < 2\pi,$$

known as *Euler's formula*. We note that

$$|e^{i\theta}| = \cos^2\theta + \sin^2\theta = 1,$$
$$\arg(e^{i\theta}) = \theta, \quad e^{i\pi} = 1, \quad \overline{e^{i\theta}} = e^{-i\theta},$$
$$e^{2\pi ik} = 1, k \in \mathbb{Z},\ e^{i(\theta+2k\pi)} = e^{i\theta},\ k \in \mathbb{Z}.$$

Since $e^{-i\theta} = \cos\theta - i\sin\theta$, we have

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2},\ \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

For $z = x + iy$, we define e^z by

$$e^z = e^{x+iy} = e^x\ e^{i\,y} = e^x(\cos y + i\sin y)$$

and so

$$\operatorname{Re}\left(e^z\right) = e^x \cos y,\ \operatorname{Im}\left(e^z\right) = e^z \sin y,$$
$$\arg\left(e^z\right) = \operatorname{Im} z,\ \left|e^z\right| = e^x = e^{\operatorname{Re}(z)}.$$

Definition 1.4. The complex number $z = r(\cos\theta + i\sin\theta)$, with $r = |z|$ can be written as $z = r\,e^{i\theta} = |z|\,e^{i\theta}$, which is called *exponential form* of the complex number z.

For any non-zero complex number z, we define

$$z^0 = 1 \quad z^{n+1} = z^n \cdot z \text{ for } n \ge 0$$

and

$$z^{-n} = \left(z^{-1}\right)^n \ if\ z \ne 0, n > 0.$$

Theorem 1.1. For any complex number $z = re^{i\theta}$ and n = 0, ±1, ±2,…, we have

$$z^n = r^n e^{in\theta}.$$

Proof: For $n = 0$, the result is trivial since $z^0 = 1$. For $n = 1,2, \ldots$ it can be proved by mathematical induction. For $n = -1, -2, \ldots$ let $n = -m$, where $m = 1, 2, \ldots$

Then

$$z^n = z^{-m} = \left(z^{-1}\right)^m = \left(\frac{1}{r}e^{-i\theta}\right)^m = \left(\frac{1}{r}\right)^m e^{-im\theta}$$
$$= r^{-m}e^{i(-m\theta)} = r^n e^{in\theta}.$$

Substituting $r = 1$, we have

$$z^n = \left(e^{i\theta}\right)^n = e^{in\theta}$$
$$\Rightarrow \left(\cos\theta + i\sin\theta\right)^n = \cos n\theta + i\sin n\theta, n \in I,$$

which is known as De-Moivre's theorem for integral index.

The De-Moivre's theorem also holds for rational index. To show it let $n = \frac{p}{q}$ be a rational number. Then

$$\begin{aligned}\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^q &= \cos q\frac{\theta}{q} + i\sin q.\frac{\theta}{q}\\ &= \cos\theta + i\sin\theta\end{aligned}$$

and so taking qth root of both sides, we note that $\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$ is one of the values of $(\cos\theta + i\sin\theta)^{\frac{1}{q}}$. Therefore, $\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^p$ is one of the values of $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$, that is, $\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$ is one of the values of $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$. Hence, $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$. This proves De-Moivre's theorem for rational index $\frac{p}{q}$. However, in general, the restriction $-\pi < \theta \le \pi$ is necessary. For example, if $\theta = -\pi$, $n = \frac{1}{2}$, then the result is not valid.

If n is a positive integer, then De-Moivre's formula

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

implies

$$\cos n\theta = \sum_{k=0,even}^{n} (-1)^{\frac{k}{2}}\binom{n}{k}\cos^{n-k}\theta\sin^k\theta$$

and

$$\sin n\theta = \sum_{k=1,odd}^{n} (-1)^{\frac{k-1}{2}}\binom{n}{k}\cos^{n-k}\theta\sin^k\theta.$$

Thus, expansion of $\cos n\theta$ and $\sin n\theta$ can be obtained using the above formulas. For example,

$$\begin{aligned}\cos 7\theta &= \sum_{k=0,\,even}^{7} (-1)^{\frac{k}{2}}\binom{7}{k}\cos^{7-k}\theta\sin^k\theta\\
&= \cos^7\theta - \binom{7}{2}\cos^5\theta\sin^2\theta + \binom{7}{4}\cos^3\theta\sin^4\theta - \binom{7}{6}\cos\theta\sin^6\theta\\
&= \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta\\
&= \cos^7\theta - 21\cos^5\theta\left(1-\cos^2\theta\right) + 35\cos^3\theta\left(1-\cos^2\theta\right)^2 - 7\cos\theta\left(1-\cos^2\theta\right)^3\\
&= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta\left(1-2\cos^2\theta+\cos^4\theta\right)\\
&\quad -7\cos\theta\left(1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta\right)\\
&= 64\cos^7\theta - 112\cos^5\theta + 56\cos^2\theta - 7\cos\theta.\end{aligned}$$

Similarly,

$$\sin 7\theta = \sum_{k=1,\, odd}^{7} (-1)^{\frac{k-1}{2}} \binom{7}{k} \cos^{7-k}\theta \sin^{k}\theta$$
$$= 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$$
$$= 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta.$$

Substituting $z = e^{i\theta}$, we have

$$z^n = \cos n\theta + i\sin n\theta \text{ and } z^{-n} = \cos\theta - i\sin n\theta.$$

Therefore,

$$z^n + \frac{1}{z^n} = 2\cos n\theta \text{ and } z^n - \frac{1}{z^n} = 2i\sin n\theta.$$

Thus,

$$z + \frac{1}{z} = 2\cos\theta \text{ and } z - \frac{1}{z} = 2i\sin\theta.$$

These expressions are useful in finding the expansion of $\cos^n\theta$ and $\sin^n\theta$. For example,

$$\left(z + \frac{1}{z}\right)^7 = \left(z^7 + z^{-7}\right) + 7\left(z^5 + z^{-5}\right) + 21\left(z^3 + z^{-3}\right) + 35\left(z + z^{-1}\right)$$

or

$$(2\cos\theta)^7 = 2\cos 7\theta + 14\cos 5\theta + 42\cos 3\theta + 70\cos\theta$$

or

$$\cos^7\theta = 2^{-7}[2\cos 7\theta + 14\cos 5\theta + 42\cos 3\theta + 70\cos\theta]$$
$$= 2^{-6}[\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta].$$

Similarly, the expansion of $\left(z - \frac{1}{z}\right)^n$ gives $\sin^n\theta$. For example,

$$\left(z - \frac{1}{z}\right)^5 = \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$

or

$$(2i\sin\theta)^5 = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$$

or

$$\sin^5\theta = 2^{-4}[\sin 5\theta - 5\sin 3\theta + 10\sin\theta].$$

The following theorem is helpful to determine the nth root of a non-zero complex number.

Theorem 1.2. For $z_0 \neq 0$, there exist n values of z satisfying the equation $z^n = z_0$.

Proof: We have $z^n = z_0$, that is, $z = z_0^{\frac{1}{n}}$. Let $z_0 = r_0 e^{i\theta_0}$, $-\pi < \theta_0 \leq \pi$, and $z = re^{i\theta}$. Then

$$z^n = z_0 \Rightarrow \left(re^{i\theta}\right)^n = r_0 e^{i\theta_0}$$
$$\Rightarrow r^n e^{in\theta} = r_0 e^{i\theta_0}$$
$$\Rightarrow r^n = r_0 \text{ and } n\theta = \theta_0 + 2k\pi, k \in I.$$

Therefore,

$$r = (r_0)^{\frac{1}{n}}, \quad \theta = \frac{\theta_0}{n} + \frac{2k\pi}{n},$$

where $(r_0)^{\frac{1}{n}}$ denotes the positive nth root of r_0. Hence, all values z given by

$$z = (r_0)^{\frac{1}{n}} e^{i\left(\frac{\theta_0 + 2k\pi}{n}\right)}, k = 0, 1, 2, \ldots, n-1$$

satisfy the given equation $z^n = z_0$. These n values of z are called nth roots of z_0. The root corresponding to $k = 0$, that is, $c = (r_0)^{\frac{1}{n}} e^{i\frac{\theta_0}{n}}$ is called the *principal root*. In terms of the principal root, the nth roots of z_0 are

$$c, cw_n, cw_n^2, \ldots, cw_n^{n-1} \quad where \quad w_n = e^{\frac{i2\pi}{n}}.$$

For the values of k other than 0, 1,…, n – 1, the roots start repeating.

For example, to find the fifth roots of unity, we put $z_0 = 1$ so that $z_0 = 1(\cos 0 + i \sin 0)$. Thus, $c = (r_0)^{\frac{1}{5}} e^0 = 1$ and $w_n = e^{\frac{i2\pi}{5}}$. Hence, the fifth roots are

$$1, e^{\frac{i2\pi}{5}}, e^{\frac{i4\pi}{5}}, e^{\frac{i6\pi}{5}}, e^{\frac{i8\pi}{5}}$$

or

$$\cos 0 + i\sin 0, \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}, \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5},$$
$$\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}, \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}.$$

As another example, we find fourth roots of the complex number $-8 - 8\sqrt{3}\, i$. We have

$$r^2 = 8^2 + \left(8\sqrt{3}\right)^2 = 256 \text{ so } r = 16.$$

Therefore, we can write $z_0 = 16e^{-\frac{i2\pi}{3}}$. Then $c = (16)^{\frac{1}{4}} e^{\frac{-i2\pi}{12}} = 2e^{-\frac{i\pi}{6}}$ and $w_n = e^{\frac{i2\pi}{4}}$. Hence the roots are

$$2e^{\frac{-i\pi}{6}}, 2e^{-\frac{i\pi}{2}}.e^{\frac{i\pi}{2}}, 2e^{\frac{-i\pi}{6}}.e^{i\pi}, 2e^{-\frac{i\pi}{6}}.e^{\frac{3i\pi}{4}}$$

or

$$2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right], \quad 2\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$$
$$2\left[\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right], \quad 2\left[\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right].$$

De-Moivre's theorem can also be used to solve equations. For example, consider the equation $z^4 - z^3 + z^2 - z + 1 = 0$. Multiplying both sides by $(z + 1)$, we get $z^5 + 1 = 0$. Therefore,

$$\begin{aligned} z^5 &= -1 = (\cos\pi + i\sin\pi) \\ &= \cos(2n+1)\pi + i\sin(2n+1)\pi, \\ n &= 0, 1, 2,\ldots \end{aligned}$$

Therefore, the roots of the equation are given by

$$\begin{aligned} &[\cos(2n+1)\pi + i\sin(2n+1)\pi]^{\frac{1}{5}} \\ &= \left[\cos(2n+1)\frac{\pi}{5} + i\sin(2n+1)\frac{\pi}{5}\right]. \end{aligned}$$

Hence, the roots are

$$\begin{aligned} &\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}, \quad \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5} \\ &\cos\pi - i\sin\pi = -1, \quad \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5} \\ &\cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}. \end{aligned}$$

But the root –1 corresponds to the factor $(z + 1)$. Therefore, the required roots are

$$\begin{aligned} &\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}, \ \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5} \\ &\cos\frac{\pi}{5} - i\sin\frac{\pi}{5} \text{ and } \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}. \end{aligned}$$

Logarithms of Complex Numbers

Let z and w be complex numbers. *If* $w = e^z$, then z is called a logarithm of w to the base e. Thus $\log_e w = z$. If $w = e^z$, then

$$e^{z+2n\pi i} = e^z . e^{2n\pi i} = e^z = w.$$

Therefore,

$$\log_e w = z + 2n\pi i, \quad n = 0, \pm 1, \pm 2, \ldots$$

Thus if z is logarithm of w, then $z + 2n\pi\, i$ is also logarithm of w. Hence, the *logarithm of a complex number has infinite values and so is a many-valued function.* The value $z + 2n\pi\, i$ is called the *general value* of $\log_e w$ and is denoted by $\text{Log}_e\, w$. Thus

$$\text{Log}_e w = z + 2n\pi i = 2n\pi i + \log_e w.$$

Substituting $n = 0$ in the general value, we get the *principal value* of z, that is, $\log_e w$.

Real and Imaginary Parts of Log (*x* + *iy*)

Let $x + iy = r(\cos\theta + i\sin\theta)$ so that $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\dfrac{y}{x}$. Then

$$\begin{aligned}
\mathrm{Log}(x+iy) &= \log(x+iy) + 2n\pi i \\
&= \log[r(\cos\theta + i\sin\theta)] + 2n\pi i \\
&= \log\left(re^{i\theta}\right) + 2n\pi i \\
&= \log r + \log e^{i\theta} + 2n\pi i \\
&= \log r + 2n\pi i + i\theta \\
&= \log\left(x^2+y^2\right)^{\frac{1}{2}} + 2n\pi i + i\theta \\
&= \frac{1}{2}\log\left(x^2+y^2\right) + 2n\pi i + i\tan^{-1}\frac{y}{x}.
\end{aligned}$$

Hence

$$\mathrm{Re}[\mathrm{Log}(x+iy)] = \frac{1}{2}\log\ (x^2+y^2)$$

and

$$\mathrm{Im}[\mathrm{Log}(x+iy)] = 2n\pi + \tan^{-1}\frac{y}{x}.$$

EXAMPLE 1.1

Separate the following into real and imaginary parts:
(i) Log (1 + *i*) (ii) Log (4 + 3*i*).

Solution. (i) We have $x + iy = 1 + i$ so that $r^2 = x^2 + y^2 = 1 + 1 = 2$.
Therefore

$$\mathrm{Re}[\mathrm{Log}(1+i)] = \frac{1}{2}\log\left(x^2+y^2\right) = \frac{1}{2}\log 2$$

and

$$\begin{aligned}
\mathrm{Im}[\mathrm{Log}(1+i)] &= 2n\pi + \tan^{-1}\frac{y}{x} = 2n\pi + \tan^{-1}\frac{1}{1} \\
&= 2n\pi + \frac{\pi}{4} = (8n-1)\frac{\pi}{4}.
\end{aligned}$$

(ii) We have $x + iy = 4 + 3i$ so that $r^2 = x^2 + y^2 = 25$.
Therefore

$$\mathrm{Re}[\log(4+3i)] = \frac{1}{2}\log 5^2\ \log 5$$

and

$$\mathrm{Im}[\log(4+3i)] = 2n\pi + \tan^{-1}\frac{3}{4}.$$

EXAMPLE 1.2

Find the general value of
(i) log (–3) (ii) log (– i).

Solution. (i) Since

$$-3 = 3(-1) = 3(\cos\pi + i\sin\pi) = 3e^{i\pi},$$

therefore

$$\begin{aligned}\text{Log}(-3) &= \text{Log}\left(3e^{i\pi}\right) = 2n\pi i + \log\left(3e^{i\pi}\right)\\ &= 2n\pi i + \log 3 + i\pi = \log 3 + i(2n+1)\pi.\end{aligned}$$

(ii) Since $-i = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) = e^{-\frac{i\pi}{2}}$, therefore

$$\begin{aligned}\text{Log}(-i) &= 2n\pi i + \log\left(e^{-\frac{i\pi}{2}}\right) = 2n\pi i - \frac{i\pi}{2}\\ &= (4n-1)\frac{\pi i}{2}.\end{aligned}$$

EXAMPLE 1.3

Show that

$$\tan\left(i\log\frac{a-ib}{a+ib}\right) = \frac{2ab}{a^2-b^2}.$$

Solution. Let $a + ib = r(\cos\theta + i\sin\theta)$. Therefore $a = r\cos\theta$, $b = r\sin\theta$ and $\tan\theta = \frac{b}{a}$. Then

$$\begin{aligned}\tan\left(i\log\frac{a-ib}{a+ib}\right) &= \tan\left[i\log\frac{r(\cos\theta - i\sin\theta)}{r(\cos\theta + i\sin\theta)}\right]\\ &= \tan\left[i\log\frac{e^{-i\theta}}{e^{i\theta}}\right]\\ &= \tan\left[i\log e^{-2i\theta}\right]\\ &= \tan\left[i(-2i\theta)\log e\right]\\ &= \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}.\\ &= \frac{\frac{2b}{a}}{1-\frac{b^2}{a^2}} = \frac{2ab}{a^2-b^2}.\end{aligned}$$

EXAMPLE 1.4

Show that

$$\cos\left[i\log\frac{a+ib}{a-ib}\right] = \frac{a^2-b^2}{a^2+b^2}.$$

Solution. Setting $a = r\cos\theta$, $b = r\sin\theta$, so that $\tan\theta = \frac{b}{a}$, we have

$$\log\left(\frac{a+ib}{a-ib}\right) = \log\frac{r(\cos\theta + i\sin\theta)}{r(\cos\theta - i\sin\theta)} = \log\frac{re^{i\theta}}{re^{-i\theta}}$$

$$= \log e^{2i\theta} = 2i\theta = 2i\tan^{-1}\frac{b}{a}.$$

Therefore

$$\cos\left[i\log\frac{a+ib}{a-ib}\right] = \cos[i(2i\theta)] = \cos 2\theta$$

$$= \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\dfrac{b^2}{a^2}}{1+\dfrac{b^2}{a^2}} = \frac{a^2-b^2}{a^2+b^2}.$$

Hyperbolic Functions

Let z be real or complex. Then

(i) $\frac{e^z - e^{-z}}{2}$ is called the *hyperbolic sine* of z and is denoted as sinh z

(ii) $\frac{e^z + e^{-z}}{2}$ is called the *hyperbolic cosine* of z and is denoted by $\cosh z$

The other hyperbolic functions are defined in terms of hyperbolic sine and cosine as follows:

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

$$\coth z = \frac{\cosh z}{\sinh z} = \frac{e^z + e^{-z}}{e^z - e^{-z}}$$

$$\text{sech } z = \frac{1}{\cosh z} = \frac{2}{e^z + e^{-z}}$$

$$\text{cosech } z = \frac{1}{\sinh z} = \frac{2}{e^z - e^{-z}}.$$

If follows from the above definitions that

(i) $\sinh 0 = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0$

(ii) $\cosh 0 = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2} = 1$

(iii) $\cosh z + \sinh z = \frac{e^z + e^{-z}}{2} + \frac{e^z - e^{-z}}{2} = e^z$

(iv) $\cosh z - \sinh z = \frac{e^z + e^{-z}}{2} - \frac{e^z - e^{-z}}{2} = e^{-z}.$

Relations Between Hyperbolic and Circular Functions

(i) By definition

$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}.$$

Substituting $\theta = iz$, we have

$$\sinh(iz) = \frac{e^{iz} - e^{-iz}}{2} = i\frac{e^{iz} - e^{-iz}}{2i} = i\sin z.$$

Similarly, by definition of circular function $\sin\theta$, we have

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Substituting $\theta = iz$, we get

$$\sin(iz) = \frac{e^{-z} - e^{z}}{2i} = -\frac{(e^{z} - e^{-z})}{2i}$$

$$= \frac{i^2 (e^{z} - e^{-z})}{2i} = i\frac{e^{z} - e^{-z}}{2} = i\sinh z$$

(ii) By definition

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}.$$

Substituting $\theta = iz$, we get

$$\cosh(iz) = \frac{e^{iz} + e^{-iz}}{2} = \cos z.$$

Similarly, by definition

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Substituting $\theta = iz$, we get

$$\cos(iz) = \frac{e^{-z} + e^{z}}{2} = \cosh z.$$

(iii) We note that

$$\tan(iz) = \frac{\sin(iz)}{\cos(iz)} = \frac{i\sinh z}{\cosh z} = i\tanh z$$

and

$$\tanh(iz) = \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = i\frac{\frac{e^{iz} - e^{iz}}{2i}}{\frac{e^{iz} + e^{-iz}}{2}} = i\frac{\sin z}{\cos z} = i\tan z.$$

(iv) We have

$$\cot(iz) = \frac{\cos(iz)}{\sin(iz)} = \frac{\cosh z}{i\sinh z} = \frac{i\cosh z}{i^2\sinh z} = -i\coth z$$

(v) We have

$$\sec(iz) = \frac{1}{\cos(iz)} = \frac{1}{\cosh z} = \text{sech } z$$

(vi) Lastly

$$\text{cosec}(iz) = \frac{1}{\sin(iz)} = \frac{1}{i\sinh z} = \frac{i}{i^2 \sinh z} = -i\,\text{cosech } z.$$

Periodicity of Hyperbolic Function

We note that

(i) $\sinh(z + 2n\pi\, i) = \sinh z$. Therefore, $\sinh z$ is a periodic function with period $2\pi\, i$
(ii) $\cosh(z + 2n\pi\, i) = \cosh z$ and so $\cosh z$ is also periodic with period $2\pi\, i$.
(iii) $\tanh(z + n\pi\, i) = \tanh z$ and so is periodic with period $\pi\, i$.

Further cosech z, sech z and coth z are reciprocals of sinh z, cosh z and tanh z, respectively, and are, therefore, periodic with period $2\pi\, i$, $2\pi\, i$, and $\pi\, i$, respectively.

EXAMPLE 1.5

Show that

(i) $\cosh^2 z - \sinh^2 z = 1$
(ii) $\text{sech}^2 z + \tanh^2 z = 1$
(iii) $\coth^2 z - \text{cosech}^2 z = 1$.

Solution. (i) Since $\cos^2\theta + \sin^2\theta = 1$, substituting $\theta = iz$, we get

$$\cos^2(iz) + \sin^2(iz) = 1$$

or

$$(\cosh z)^2 + (i\sinh z)^2 = 1$$

or

$$\cosh^2 z + i^2 \sinh^2 z = 1$$

or

$$\cosh^2 z - \sinh^2 z = 1.$$

(ii) Dividing both sides of the above expression by $\cosh^2 z$, we get

$$1 - \tanh^2 z = \text{sech}^2 z$$

or

$$\text{sech}^2 z + \tanh^2 z = 1.$$

(iii) Dividing both sides of (i) by $\sinh^2 z$, we get

$$\frac{\cosh^2 z}{\sinh^2 z} - 1 = \text{cosech}^2 z$$

or

$$\coth^2 z - \text{cosech}^2 z = 1.$$

EXAMPLE 1.6

Show that

$$\sinh 2z = 2\sinh z \cosh z = \frac{2\tanh z}{1-\tanh^2 z}.$$

Solution. Substitute $\theta = iz$ in trigonometric relation $\sin 2\theta = 2 \sin \theta \cos \theta$ to get

$$\sin(2iz) = 2\sin(iz)\cos(iz)$$

or

$$i\sinh 2z = 2i\sinh z \cosh z$$

or

$$\sinh 2z = 2\sinh z \cosh z.$$

Also, we know that

$$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}.$$

Substituting $\theta = iz$, we get

$$\sin(2iz) = \frac{2\tan iz}{1+\tan^2 iz} = \frac{2i\tanh z}{1+(i\tanh z)^2}$$

or

$$i\sinh 2z = \frac{2i\tanh z}{1+i^2\tanh^2 z} = \frac{2i\tanh z}{1-\tanh^2 z}$$

or

$$\sinh 2z = \frac{2\tanh z}{1-\tanh^2 z}.$$

EXAMPLE 1.7

Show that

$$\tanh 3z = \frac{3\tanh z + \tanh^3 z}{1+3\tanh^2 z}.$$

Solution. Substituting $\theta = iz$ in the trigonometric relation $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$, we get

$$\tan(3iz) = \frac{3\tan(iz) - \tan^3(iz)}{1-3\tan^2(iz)}$$

or

$$i\tanh 3z = \frac{3i\tanh z - (i\tanh z)^3}{1-3(i\tanh z)^2}$$

or

$$i\tanh 3z = \frac{3i\tanh z + i\tanh^3 z}{1+3\tanh^2 z}$$

or

$$\tanh 3z = \frac{3\tanh z + \tanh^2 z}{1+3\tanh^2 z}.$$

Remark 1.1. Proceeding as in the above example, the following formulae of hyperbolic function can also be derived.

(i) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

(ii) $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

(iii) $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

(iv) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1$

$$= 1 + 2\sinh^2 x = \frac{1+\tanh^2 x}{1-\tanh^2 x}$$

(v) $\tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}$

(vi) $\sinh 3x = 3 \sinh x + 4 \sinh^3 x$

(vii) $\cosh 3x = 4 \cosh^2 x - 3 \cosh x$

(viii) $\sinh(A+B) + \sinh(A-B) = 2\sinh A \cosh B$

(ix) $\sinh(A+B) - \sinh(A-B) = 2\cosh A \sinh B$

(x) $\cosh(A+B) + \cosh(A-B) = 2\cosh A \cosh B$

(xi) $\cosh(A+B) - \cosh(A-B) = 2\sinh A \sinh B$

(xii) $\sinh C + \sinh D = 2\sinh\dfrac{C+D}{2}\cosh\dfrac{C-D}{2}$

(xiii) $\sinh C - \sinh D = 2\cosh\dfrac{C+D}{2}\sinh\dfrac{C-D}{2}$

(xiv) $\cosh C + \cosh D = 2\cosh\dfrac{C+D}{2}\cosh\dfrac{C-D}{2}$

(xv) $\cosh C - \cosh D = 2\sinh\dfrac{C+D}{2}\sinh\dfrac{C-D}{2}.$

EXAMPLE 1.8

Separate the following into real and imaginary parts.
(i) $\tan(x + iy)$ (ii) $\sec(x + iy)$ (iii) $\tan^{-1}(x + iy)$.

Solution. (i) We have

$$\tan(x+iy) = \frac{\sin(x+iy)}{\cos(x+iy)} = \frac{2\sin(x+iy)}{2\cos(x+iy)}.\frac{\cos(x-iy)}{\cos(x-iy)}$$

$$= \frac{\sin 2x + \sin 2iy}{\cos 2x + \cos 2iy} = \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y}$$

$$= \frac{\sin 2x}{\cos 2x + \cosh 2y} + i\frac{\sinh 2y}{\cos 2x + \cosh 2y}.$$

Hence

$$\operatorname{Re}(\tan(x+iy)) = \frac{\sin 2x}{\cos 2x + \cosh 2y}$$

$$\operatorname{Im}(\tan(x+iy)) = \frac{\sinh 2y}{\cos 2x + \cosh 2y}.$$

(ii)

$$\sec(x+iy) = \frac{1}{\cos(x+iy)} = \frac{1}{2\cos(x+iy)} \cdot \frac{2\cos(x-iy)}{\cos(x-iy)}$$
$$= \frac{2(\cos x \cos iy + \sin x \sin iy)}{\cos 2x + \cos 2iy}$$
$$= \frac{2(\cos x \cosh y + i \sin x \sinh y)}{\cos 2x + \cosh 2y}$$
$$= \frac{2\cos x \cosh y}{\cos 2x + \cosh 2y} + i\frac{2\sin x \sinh y}{\cos 2x + \cosh 2y}.$$

Therefore,

$$\text{Re}[\sec(x+iy)] = \frac{2\cos x \cosh y}{\cos 2x + \cosh 2y}$$
$$\text{Im}[\sec(x+iy)] = \frac{2\sin x \sinh y}{\cos 2x + \cosh 2y}.$$

(iii) Suppose $\alpha + i\beta = \tan^{-1}(x+iy)$. Then $\alpha - i\beta = \tan^{-1}(x-iy)$. Addition of these two expressions yields

$$2\alpha = \tan^{-1}(x+iy) + \tan^{-1}(x-iy)$$
$$= \tan^{-1}\frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)}.$$

Therefore, $\alpha = \dfrac{1}{2}\tan^{-1}\dfrac{2x}{1-x^2-y^2}$.

Similarly, subtracting $\alpha - i\beta$ from $\alpha + i\beta$, we get

$$2i\beta = \tan^{-1}(x+iy) - \tan^{-1}(x-iy)$$
$$= \tan^{-1}\frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)}$$
$$= \tan^{-1} i\frac{2y}{1+x^2+y^2} = i\tanh^{-1}\frac{2y}{1+x^2+y^2}.$$

Hence

$$\beta = \frac{1}{2}\tanh^{-1}\frac{2x}{1+x^2+y^2}.$$

EXAMPLE 1.9

Separate the following into real and imaginary parts. (i) $\sinh(x+iy)$ (ii) $\coth(x+iy)$

Solution. (i) Since $\sin i\,\theta = i \sinh \theta$, we have

$$\sinh(x+iy) = \frac{1}{i}\sin i(x+iy) = \frac{1}{i}\sin\left(ix+i^2y\right)$$
$$= \frac{i}{i^2}\sin(ix-y)$$

$$
\begin{aligned}
&= -i\left(\sin ix\cos y - \cos ix \sin y\right)\\
&= -i\left(i\sinh x\cos y - \cosh x\sin y\right)\\
&= \sinh x\cos y + i\cosh x\sin y.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\text{Re}[\sinh(x+iy)] &= \sinh x\cos y\\
\text{Im}[\sinh(x+iy)] &= \cosh x\sin y.
\end{aligned}
$$

(ii) coth (x+iy)

$$
\begin{aligned}
&= \frac{\cosh(x+iy)}{\sinh(x+iy)} = \frac{\cos i(x+iy)}{\frac{1}{i}\sin i(x+iy)}\\
&= i\frac{\cos(ix-y)}{\sin(ix-y)} = i\frac{2\sin(ix+y)\cos(ix-y)}{2\sin(ix+y)\sin(ix-y)}\\
&= i\frac{\sin 2ix + \sin 2y}{\cos 2y - \cos 2ix} = i\frac{i\sinh 2x + \sin 2y}{\cos 2y - \cosh 2x}\\
&= \frac{-\sinh 2x}{\cos 2y - \cosh 2x} + i\frac{\sin 2y}{\cos 2y - \cosh 2x}\\
&= \frac{\sinh 2x}{\cosh 2x - \cos 2y} - i\frac{\sin 2y}{\cosh 2x - \cos 2y}.
\end{aligned}
$$

EXAMPLE 1.10

If $\sin(A+iB) = x+iy$, show that

(i) $x^2\operatorname{cosec}^2 A - y^2\sec^2 A = 1$

(ii) $\dfrac{x^2}{\cosh^2 B} + \dfrac{y^2}{\sinh^2 B} = 1.$

Solution. We have

$$
\begin{aligned}
x+iy = \sin(A+iB) &= \sin A\cos iB + \cos A\sin iB\\
&= \sin A\cosh B + i\cos A\sinh B.
\end{aligned}
$$

Therefore, real and imaginary parts are

$$x = \sin A\cosh B \text{ and } y = \cos A\sinh B$$

(i) Form above, we have

$$\frac{x}{\sin A} = \cosh B \quad \text{and} \quad \frac{y}{\cos A} = \sinh B.$$

Squaring and subtracting, we get

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = \cosh^2 B - \sinh^2 B = 1$$

or

$$x^2\operatorname{cosec}^2 A - y^2\sec^2 A = 1.$$

(ii) Again, from (i), we have

$$\frac{x}{\cosh B} = \sin A \text{ and } \frac{y}{\sinh B} = \cos A.$$

Squaring and adding we get

$$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1.$$

EXAMPLE 1.11

Show that

$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx,$$

where n is a positive integer.

Solution. We have

$$(\cosh x + \sin nhx)^n = \left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right)^n$$

$$= \left(e^x\right)^n = e^{xn} = \cosh nx + \sinh nx.$$

EXAMPLE 1.12

If $x + iy = \cosh(u + iv)$, show that

(i) $x^2 \sec^2 v - y^2 \operatorname{cosec}^2 v = 1$

(ii) $\dfrac{x^2}{\cosh^2 u} + \dfrac{y^2}{\sinh^2 u} = 1$

Solution. We are given that

$$x + iy = \cosh(u + iv) = \cos i(u + iv) = \cos(iu - v)$$

$$= \cos iu \cos v - \sin iu \sin v$$

$$= \cosh u \cos v + i \sinh u \sin v.$$

Equating the real and imaginary parts, we get

$$x = \cosh u \cos v \text{ and } y = \sinh u \sin v$$

(i) From above, we have

$$\frac{x}{\cos v} = \cosh u \text{ and } \frac{y}{\sin v} = \sinh u.$$

Squaring and subtracting, we get

$$\frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = \cosh^2 u - \sinh^2 u = 1$$

or

$$x^2 \sec^2 v - y^2 \operatorname{cosec}^2 v = 1$$

(ii) From above, we also have

$$\frac{x}{\cosh u} = \cos v \text{ and } \frac{y}{\sinh u} = \sin v.$$

Squaring and adding, we get

$$\frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = \sin^2 v + \cos^2 v = 1.$$

Definition 1.5. Let z_0 be a point in the complex plane and let ε be any positive number. Then the set of all points z such that $|z - z_0| < \varepsilon$ is called *ε-neighbourhood* of z_0.

A neighbourhood of a point z_0 from which z_0 is omitted is called a *deleted neighbourhood of* z_0. Thus $0 < |z - z_0| < \varepsilon$ is a deleted neighbourhood of z_0.

Definition 1.6. A point z_0 is called a *limit point*, *cluster point*, or *point of accumulation* of a point set S if every deleted neighbourhood of z_0 contains points of *S*.

We observe that if z_0 is a limit point of the point set *S*, then since ε is any positive number, *S* contains an infinite number of points. Hence, *a finite set has no limit point*.

Definition 1.7. The union of a set S and the set of its limit points is called the *closure* of S.

Definition 1.8. A set S is said to be *closed* if it contains all of its limit points.

Definition 1.9. A point z_0 is called an *interior point* of a point set S if there exists a neighbourhood of z_0 lying wholly in S.

Definition 1.10. A set S is said to be *open* if every point of S is an interior point.

Thus, a set S is open if for every $z \in$ S, there exists a neighbourhood lying wholly in S.

Definition 1.11. An open set is said to be *connected* if any two points of the set can be joined by a polynomial arc (path) lying entirely in the set.

Definition 1.12. An open connected set is called a *domain* or *open region*.

Definition 1.13. The closure of an open region or domain is called *closed region*.

Definition 1.14. If to a domain we add some, all, or none of its limit points, then the set obtained is called the region.

Definition 1.15. A function $w = f(z)$, which assign a complex number w to each complex variable z is called a *complex-valued function of a complex variable z*.

If only one value of w corresponds to each value of z, we say that $w = f(z)$ is a *single-valued function* of z or that $f(z)$ is *single valued*.

If more than one value of w corresponds to a value of z, then $f(z)$ is called *multiple-valued* or *many-valued* function of z.

EXAMPLE 1.13

The function $f(z) = z^2$ is single-valued function whereas the function $f(z) = z^{1/2} = r^{1/2} e^{\frac{\theta + 2k\pi}{2}}$, $k = 0, 1, \ldots, n - 1$ is multiple-valued having *n branches* (one for each value of *k*).

Consider

$$f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2xiy.$$

This shows that a complex-valued function can be expressed as

$$f(z) = \phi(x, y) + i\psi(x, y),$$

where $\phi(x, y)$, $\psi(x, y)$ are real functions of the real variables x and y. The function ϕ is called *real part* and ψ is called *imaginary part* of $f(z)$.

Definition 1.16. The function $f(z)$ is said to have the *limit* l as z approaches z_0 if given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - l| < \varepsilon \quad \textit{whenever } 0 < |z - z_0| < \delta.$$

We then write $\lim_{z \to z_0} f(z) = l$, provided that the limit is *independent of the direction of approach* of z *to* z_0.

Definition 1.17. The function $f(z)$ is said to be *continuous* at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$, provided that the limit is *independent of the direction of approach* of z to z_0.

For example, let $f(z) = z^2$ for all z. Then, we note that $\lim_{z \to i} f(z) = f(i) = -1$. Hence f is continuous at $z = i$.

Definition 1.18. The single-valued function $f(z)$ defined on a domain (open connected set) D is said to be *differentiable* at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists and is *independent of the direction of approach* of z to z_0.

If this limit exists, then the same is called *derivative* of $f(z)$ at z_0 and is denoted by $f'(z_0)$.

1.2 ANALYTIC FUNCTIONS

Definition 1.19. If $f(z)$ is differentiable at all points of some neighbourhood $|z - z_0| < r$ of z_0, then $f(z)$ is said to be *analytic* (or *holomorphic*) *at* z_0.

If $f(z)$ is analytic at each point of a domain D, then $f(z)$ is called *analytic in that domain*.

EXAMPLE 1.14

Consider

$$f(z) = \frac{1+z}{1-z}.$$

We note that

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{\dfrac{1 + (z + \Delta z)}{1 - (z + \Delta z)} - \dfrac{1+z}{1-z}}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{2}{(1 - z - \Delta z)(1 - z)} = \frac{2}{(1-z)^2}, \end{aligned}$$

independent of the direction of approach of Δz to 0, provided that $z \neq 1$. Thus $f(z)$ is analytic for all finite value of z except $z = 1$, where the derivative does not exist.

On the other hand, the function $f(z) = |z|^2$ is not analytic at any point since its derivative exists only at the point $z = 0$ and not throughout any neighbourhood.

Definition 1.20. A function which is analytic everywhere in the finite plane (that is everywhere except at ∞) is called an *entire function* or *integral function*.

For example, e^z, $\sin z$, and $\cos z$ are entire functions.

Definition 1.21. The point at which the function $f(z)$ is not analytic is called *singular point* of $f(z)$. We notice that $z = 1$ is the singular point of $f(z)$ in Example 1.14.

Definition 1.22. The point z_0 is called an *isolated singularity* or *isolated singular point* of $f(z)$ if we can find $\delta > 0$ such that the circle $|z - z_0| = \delta$ encloses no singular point other than z_0.

If no such δ can be found, then z_0 is called *non-isolated singularity*.

Definition 1.23. The point z_0 is called a *pole of order n* of $f(z)$ if there exists a positive integer n such that $\lim_{z \to z_0} (z - z_0)^n f(z) = A \neq 0$.

If $n = 1$, then z_0 is called a *simple pole*.

EXAMPLE 1.15

(i) $f(z) = \dfrac{1}{(z-1)(z-3)}$ has simple poles at $z = 1$ and $z = 3$.

(ii) $f(z) = \dfrac{1}{(z-2)^3}$ has a pole of order 3 at $z = 2$.

Regarding analyticity of a function $f(z)$, we have the following results.

Theorem 1.3. A necessary condition that $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D is that in D, the functions u, and v satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Proof: Let $f(z)$ be analytic in the domain D. Therefore, the limit

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

must exist independent of the manner in which Δz approaches zero. Since $\Delta z = \Delta x + i\Delta y$,

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{[u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - [u(x, y) + iv(x, y)]}{\Delta x + i\Delta y} \tag{1.1}$$

must exist independent of the manner in which Δx and Δy approach zero.

Two cases arise:

(i) If $\Delta y = 0$, $\Delta x \to 0$, then (1.1) becomes

$$\lim_{\Delta x \to 0} \frac{[u(x + \Delta x, y) + iv(x + \Delta x, y) - [u(x, y) + iv(x, y)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + \lim_{\Delta x \to 0} \frac{i[v(x + \Delta x, y) - v(x, y)]}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \tag{1.2}$$

provided the partial derivatives exist.

(ii) If $\Delta x = 0$ and $\Delta y \to 0$, then (1.1) becomes

$$\lim_{\Delta y \to 0}\left[\frac{u(x, y+\Delta y - u(x, y)}{i\Delta y} + \frac{v(x, y+\Delta y) - v(x, y)}{\Delta y}\right]$$
$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \tag{1.3}$$

For $f(z)$ to be analytic, these two limits should be identical. Hence a necessary condition for $f(z)$ to be analytic is

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

and so

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{1.4}$$

The equations given in (1.4) are called *Cauchy-Riemann Equations*.

Remark 1.2. The Cauchy-Riemann equations are not sufficient conditions for analyticity of a function. For example, we shall see that the function $f(z) = \sqrt{|x\,y|}$ is not analytic at the origin although Cauchy-Riemann equations are satisfied.

The following theorem provides us with sufficient conditions for a function to be analytic.

Theorem 1.4. If $f(z) = u(x, y) + iv(x, y)$ is defined in a domain D and the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous and satisfy Cauchy-Riemann equations, then $f(z)$ is analytic in D.

Proof: Since $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are continuous, we have

$$\begin{aligned}\Delta u &= u(x+\Delta x, y+\Delta y) - u(x, y)\\ &= [u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y)] + [u(x, y+\Delta y) - u(x, y)]\\ &= \left(\frac{\partial u}{\partial x} + \varepsilon_1\right)\Delta x + \left(\frac{\partial u}{\partial y} + \eta_1\right)\Delta y\\ &= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \varepsilon_1\Delta x + \eta_1\Delta y,\end{aligned}$$

where $\varepsilon_1 \to 0$ and $\eta_1 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$, respectively.

Similarly, the continuity of $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ implies

$$\Delta v = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \varepsilon_2\Delta x + \eta_2\Delta y,$$

where $\varepsilon_2 \to 0$ and $\eta_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$, respectively. Hence

$$\begin{aligned}\Delta f(z) &= \Delta w = \Delta u + i\Delta u\\ &= \left(\frac{\partial u}{\partial x} + \frac{i\partial u}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial u}{\partial y}\right)\Delta y + \varepsilon\Delta x + \eta\Delta y,\end{aligned}$$

where $\varepsilon = \varepsilon_1 + i\varepsilon_2 \to 0$ and $\eta = \eta_1 + i\eta_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$. But, by Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Therefore,

$$\begin{aligned}\Delta w &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\Delta y + \varepsilon\Delta x + \eta\Delta y \\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\Delta x + i\Delta y) + \varepsilon\Delta x + \eta\Delta y.\end{aligned}$$

Dividing by $\Delta z = \Delta x + i\Delta y$ and taking the limit as $\Delta z \to 0$, we get

$$\begin{aligned}\frac{dw}{dz} = f'(z) &= \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.\end{aligned}$$

Thus, the derivative exists and is unique. Hence $f(z)$ is analytic in D.

Remark 1.3. From above, we note that

$$\begin{aligned}f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad \text{using Cauchy – Riemann equations}\end{aligned}$$

and

$$\begin{aligned}f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= \frac{\partial v}{\partial x} + i\frac{\partial v}{\partial y} \quad \text{using Cauchy – Riemann equations.}\end{aligned}$$

EXAMPLE 1.16

Show that the function $f(z) = \overline{z}$ is not analytic at any point.

Solution. We have

$$\begin{aligned}f'(z) &= \lim_{\Delta z \to 0} \frac{\overline{z + \Delta z} - \overline{z}}{\Delta z} \\ &= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overline{x + iy + \Delta x + i\Delta y} - \overline{x + iy}}{\Delta x + i\Delta y} \\ &= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{x - iy + \Delta x - i\Delta y - (x - iy)}{\Delta x + i\Delta y} \\ &= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}.\end{aligned}$$

If we take $\Delta x = 0$, then the above limit is -1 and if we take $\Delta y = 0$, then this limit is 1. Since the limit depends on the manner in which $\Delta z \to 0$, the derivative does not exist and so $f(z)$ is not analytic.

Second Method: We have

$$f(z) = u + iv = \overline{z} = x - iy,$$

and so

$$u = x, v = -y,$$

$$\frac{\partial u}{\partial x} = 1, \frac{\partial v}{\partial y} = -1.$$

Thus Cauchy-Riemann equations are not satisfied. Hence the function is not analytic.

Theorem 1.5. If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then u and v are harmonic, that is, they satisfy

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Thus, for an analytic function $f(z)$, u, and v satisfy Laplace-equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

Proof: Since $f(z)$ is analytic in D, Cauchy-Riemann equations are satisfied and so

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{1.5}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{1.6}$$

Assuming that u and v have continuous second order partial derivatives, we differentiate both sides of (1.5) and (1.6) with respect to x and y, respectively, and get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \tag{1.7}$$

and

$$\frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2}. \tag{1.8}$$

The equations (1.7) and (1.8) imply

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$$

and so

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Hence u is harmonic.

Similarly, differentiating (1.5) and (1.6) w.r.t. y and x respectively, we get

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Hence v is harmonic.

Definition 1.24. If $f(z) = u + iv$ is analytic and u and v both satisfy Laplace's equation, then u and v are called *conjugate harmonic functions* or simply *conjugate functions*.

EXAMPLE 1.17

Show that

$$u = e^{-x}(x\sin y - y\cos y)$$

is harmonic.

Solution. We are given that

$$u = e^{-x}(x\sin y - y\cos y).$$

Therefore,

$$\frac{\partial u}{\partial x} = e^{-x}\sin y - x\,e^{-x}\sin y + y\,e^{-x}\cos y,$$

$$\frac{\partial^2 u}{\partial x^2} = -2\,e^{-x}\sin y + x\,e^{-x}\sin y - y\,e^{-x}\cos y,$$

$$\frac{\partial u}{\partial y} = x\,e^{-x}\cos y + y\,e^{-x}\sin y - e^{-x}\cos y,$$

$$\frac{\partial^2 u}{\partial y^2} = -xe^{-x}\sin y + 2\,e^{-x}\sin y + y\,e^{-x}\cos y.$$

Thus, we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and so u is harmonic.

EXAMPLE 1.18

If

$$u = e^{x}(x\cos y - y\sin y),$$

find v such that $f(z) = u + iv$ is analytic.

Solution. We want $f(z)$ to be analytic. So, by Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Thus

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{x}(x\cos y - y\sin y) + e^{x}\cos y$$

$$= x\,e^{x}\cos y - e^{x}y\sin y + e^{x}\cos y,$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = x\,e^{x}\sin y + e^{x}\sin y + e^{x}y\cos y.$$

Now

$$\text{dv} = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= (x\,e^{x}\sin y + e^{x}\sin y + e^{x}y\cos y)dx$$

$$+ (x\,e^{x}\cos y - e^{x}y\sin y + e^{x}\cos y)dy$$

Therefore,

$$v = \int_{y\text{ constant}} [xe^x \sin y + e^x \sin y + e^x y \cos y]\, dx$$

$$+ \int (xe^x \cos y - e^x y \sin y + e^x \cos y)\, dy$$

$$= e^x (x \sin y + \sin y + y \cos y) - e^x \sin y + C$$

$$= e^x (x \sin y + y \cos y) + \text{C (constant)}.$$

Hence

$$f(z) = u + iv = e^x [x \cos y - y \sin y + ix \sin y + iy \cos y] + Ci$$

$$= e^x (x + iy)(\cos y + i \sin y) + Ci$$

$$= (x + iy) e^{x+iy} + Ci$$

$$= z\, e^z + Ci.$$

EXAMPLE 1.19

If $u_1(x, y) = \dfrac{\partial u}{\partial y}$ and $u_2(x, y) = \dfrac{\partial u}{\partial x}$, show that

$$f'(z) = u_1(z, 0) - i\, u_2(z, 0).$$

Solution. By Remark 1.3 we have

$$f'(z) = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}$$

$$= u_1(x, y) - i u_2(x, y)$$

Substituting $y = 0$, we get

$$f'(x) = u_1(x, 0) - i u_2(x, 0)$$

Replacing x by z, we have

$$f'(z) = u_1(z, 0) - i u_2(z, 0) \tag{1.9}$$

Remark 1.4. (i) If $\dfrac{\partial v}{\partial y} = v_1(x, y)$ and $\dfrac{\partial v}{\partial x} = v_2(x, y)$, then as in Example 1.19, we have

$$f'(z) = v_1(z, 0) + i v_2(z, 0) \tag{1.10}$$

(ii) Integrating (1.9) and (1.10), we get $f(z)$. This method of constructing an analytic function is called *Milne-Thomson's method*.

EXAMPLE 1.20

If $u = e^{-x} (x \sin y - y \cos y)$, determine the analytic function $u + iv$.

Solution. We have

$$u_1(x, y) = \frac{\partial u}{\partial x} = e^{-x} \sin y - x\, e^{-x} \sin y + y\, e^{-x} \cos y$$

$$u_2(x, y) = \frac{\partial u}{\partial y} = x\, e^{-x} \cos y + y\, e^{-x} \sin y - e^{-x} \cos y$$

so that, by Example 1.19, we get

$$f'(z) = u_1(z,0) - iu_2(z,0)$$
$$= 0 - i(z\,e^{-z} - e^{-z}) = i\,e^{-z} - i\,z\,e^{-z}.$$

Integrating, we get

$$f(z) = \int ie^{-z}dz + ize^{-z} - \int ie^{-z}dz + Ci$$
$$= i\,z\,e^{-z} + Ci.$$

Also, on separating real and imaginary parts, we get

$$v = e^{-x}(y\sin y + x\cos y) + C$$

EXAMPLE 1.21

Find the analytic function of which the imaginary part is $v = 3x^2y - y^3$.

Solution. We are given that

$$v = 3x^2y - y^3.$$

Therefore,

$$\frac{\partial v}{\partial x} = 6xy, \frac{\partial v}{\partial y} = 3x^2 - 3y^2.$$

Thus

$$v_1(x,y) = \frac{\partial v}{\partial y} = 3x^2 - 3y^2,$$
$$v_2(x,y) = \frac{\partial v}{\partial x} = 6xy.$$

Therefore,

$$f'(z) = v_1(z,0) + i\,v_2(z,0) = 3z^2.$$

Hence

$$f(z) = \int 3z^2dz = 3\frac{z^2}{3} + C$$
$$= z^3 + C = (x+iy)^3 + C$$
$$= x^3 - 3xy^2 + 3ix^2y - iy^3 + C$$

Comparing real and imaginary parts, we have

$$u = x^3 - 3x^2 + C$$
$$v = 3x^2y - y^3.$$

EXAMPLE 1.22

Show that the function $u = \frac{1}{2}\log(x^2 + y^2)$ is harmonic and find its harmonic conjugate and the analytic function.

Solution. We have

$$u = \frac{1}{2}\log(x^2 + y^2).$$

Therefore,

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}, \quad \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2},$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2 - x^2}{x^2 + y^2}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{x^2 - y^2}{x^2 + y^2}.$$

Thus

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

and so u is harmonic. Further,

$$u_1(x, y) = \frac{x}{x^2 + y^2}, \quad u_2(x, y) = \frac{y}{x^2 + y^2}.$$

Therefore,

$$\begin{aligned} f'(z) &= u_1(z, 0) - i u_2(z, 0) \\ &= \frac{1}{z} - i\,0 = \frac{1}{z}. \end{aligned}$$

Hence the integration yields

$$\begin{aligned} f(z) &= \int \frac{1}{z} dz = \log z + C \\ &= \log(r\, e^{i\theta}) + C \\ &= \log r + i\theta + C \\ &= \log\,(x^2 + y^2)^{1/2} + i \tan^{-1}\frac{y}{x} + C \\ &= \frac{1}{2}\log\,(x^2 + y^2) + i \tan^{-1}\frac{y}{x} + C. \end{aligned}$$

Comparing real and imaginary parts, we get

$$u = \frac{1}{2}\log\,(x^2 + y^2) \text{ and } v = \tan^{-1}\frac{y}{x} + C$$

EXAMPLE 1.23

Find analytic function whose real part is

$$u = \frac{\sin 2x}{\cosh 2y - \cos 2x}.$$

Solution. We have

$$u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

So

$$\frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x)2\cos 2x - \sin 2x(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2}, \text{ and}$$

$$\frac{\partial u}{\partial y} = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}.$$

Therefore,

$$f'(z) = u_1(z,0) - iu_2(z,0) = \frac{2\cos 2z - 2}{(1-\cos 2z)^2} + i(0)$$

$$= -\frac{2}{1-\cos 2z} = \frac{-2}{2\sin^2 z} = -\text{cosec}^2 z.$$

Integrating w.r.t. z, we get

$$f(z) = \int -\text{cosec}^2 z \, dz = \cot z + Ci.$$

EXAMPLE 1.24

Find regular (analytic) function whose imaginary part is

$$v = \frac{x-y}{x^2+y^2}.$$

Solution. We are given that

$$v = \frac{x-y}{x^2+y^2}.$$

Therefore,

$$\frac{\partial v}{\partial x} = \frac{(x^2+y^2) - (x-y)2x}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2+2xy}{(x^2+y^2)^2},$$

$$\frac{\partial v}{\partial y} = \frac{(x^2+y^2)(-1) - (x-y)\,(2y)}{(x^2+y^2)^2}$$

$$= \frac{-x^2-y^2-2xy+2y^2}{(x^2+y^2)^2}.$$

Then

$$f'(z) = v_1(z,0) + iv_2(z,0)$$

$$= -\frac{z^2}{z^4} + \frac{i(-z^2)}{z^4}$$

$$= \frac{-z^2(1+i)}{z^4} = \frac{-(1+i)}{z^2}.$$

Hence, integration of $f'(z)$ yields

$$f(z) = \frac{1+i}{z} + C.$$

EXAMPLE 1.25

Find the regular function where imaginary part is

$$v = e^x \sin y.$$

Solution. We have

$$v_1(x, y) = \frac{\partial v}{\partial y} = e^x \cos y, \quad v_2(x, y) = \frac{\partial v}{\partial x} = e^x \sin y.$$

Therefore,

$$\begin{aligned} f'(z) &= v_1(z, 0) + iv_2(z, 0) \\ &= e^z + 0. \end{aligned}$$

Hence

$$f(z) = \int e^z dz = e^z + C.$$

EXAMPLE 1.26

In a two-dimensional fluid flow, the stream function ψ is given by $\psi = \tan^{-1} \frac{y}{x}$. Find the velocity potential.

Solution. The two-dimensional flow is represented by the function

$$f(z) = \phi + i\psi,$$

where ϕ is velocity potential and ψ is the stream potential. Thus, the imaginary part of the function is given as

$$\psi = \tan^{-1} \frac{y}{x}.$$

So

$$\begin{aligned} \frac{\partial \psi}{\partial x} &= \frac{1}{1 + \frac{y^2}{x^2}} \frac{d}{dx}\left(\frac{y}{x}\right) = -\frac{y}{x^2 + y^2} \\ \frac{\partial \psi}{\partial y} &= \frac{1}{1 + \frac{y^2}{x^2}} \frac{d}{dy}\left(\frac{y}{x}\right) = -\frac{x}{x^2 + y^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} f'(z) &= \psi_1(z, 0) + i\, \psi_2(z, 0) \\ &= \frac{z}{z^2} + i0 = \frac{1}{z}. \end{aligned}$$

Integrating, we get

$$\begin{aligned} f(z) &= \log z + C \\ &= \log(r\, e^{i\theta}) + C \\ &= \log r + i\theta. \end{aligned}$$

Hence, real part
$$= \phi = \log r$$
$$= \log (x^2 + y^2)^{1/2}$$
$$= \frac{1}{2}\log (x^2 + y^2).$$

EXAMPLE 1.27

If the potential function is log ($x^2 + y^2$), find the flux function and the complex potential function.

Solution. The complex potential function is given by
$$f(z) = \phi + i\psi,$$
where ϕ is potential function and ψ is flux function. We are given that
$$\phi = \log (x^2 + y^2).$$
To find $f(z)$ and ψ, we proceed as in Example 1.26 and get
$$f(z) = 2\log z + C,$$
$$\psi = 2\tan^{-1}\frac{y}{x}.$$

EXAMPLE 1.28

If $u - v = (x - y)(x^2 + 4xy + y^2)$ and $f(z) = u + iv$ is analytic function if $z = x + iy$, find $f(z)$ in terms of z.

Solution. We have
$$u + iv = f(z) \tag{1.11}$$
and so
$$iu - v = if(z) \tag{1.12}$$
Adding (1.11) and (1.12), we get
$$(u - v) + i(u + v) = (1 + i)f(z) = F(z)$$
$$= U + iV, \text{ say.}$$
Then $F(z) = U + iV$ is analytic function. We have
$$U = u - v = (x - y)(x^2 + 4xy + y^2).$$
Therefore,
$$\frac{\partial U}{\partial x} = 3x^2 + 6xy - 3y^2 = \phi_1(x, y),$$
$$\frac{\partial U}{\partial y} = 3x^2 - 6xy - 3y^2 = \phi_2(x, y).$$
Therefore, by Milne's method
$$F(z) = \int (\phi_1(z,0) - i\phi_2(z,0)]\, dz$$
$$= \int (3z^2 - i3z^2)\, dz$$
$$= (1 - i)z^3 + C.$$
Thus,
$$(1 + i)f(z) = (1 - i)z^3 + C.$$

Hence

$$f(z) = \frac{1-i}{1+i} z^3 + C$$
$$= -iz^3 + C.$$

EXAMPLE 1.29

If $f(z) = u + iv$ is an analytic function of $z = x + iy$, show that the family of curves $u(x, y) = C_1$ and $v(x, y) = C_2$ form an orthogonal system.

Solution. Recall that two family of curves form an orthogonal system if they intersect at right angles at each of their points of intersection. Differentiating $u(x, y) = C_1$, we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0$$

or

$$\frac{dy}{dx} = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = m_1, \text{ say.}$$

Similarly, differentiating $v(x, y) = C_2$, we get

$$\frac{dy}{dx} = -\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = m_2, \text{ say.}$$

Using Cauchy-Riemann equations, we have

$$m_1 m_2 = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \cdot \frac{-\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = \frac{-\frac{\partial v}{\partial y}}{\frac{\partial u}{\partial y}} \cdot \frac{\frac{\partial u}{\partial y}}{\frac{\partial v}{\partial y}} = -1.$$

Hence, the two curves $u(x, y) = C_1$ and $u(x, y) = C_2$ are orthogonal.

Remark 1.5. If $f(z) = u + iv$ is an analytic function, then Example 1.29 implies that u = constant and v = constant intersect at right angle in the z-plane.

EXAMPLE 1.30

Obtain polar form of Cauchy-Riemann equations.

Solution. Since $x = r \cos \theta$, $y = r \sin \theta$, we have

$$x^2 + y^2 = r^2 \text{ and } \theta = \tan^{-1} \frac{y}{x}.$$

Therefore,

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta,$$
$$\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta,$$

$$\frac{\partial\theta}{\partial x}=\frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right),$$

$$=-\frac{y}{x^2+y^2}=-\frac{\sin\theta}{r},$$

$$\frac{\partial\theta}{\partial y}=\frac{x}{x^2+y^2}=\frac{\cos\theta}{r}.$$

Now

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial\theta}\cdot\frac{\partial\theta}{\partial x}=\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial\theta}\frac{\sin\theta}{r},$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\sin\theta+\frac{\partial u}{\partial\theta}\cdot\frac{\cos\theta}{r},$$

$$\frac{\partial v}{\partial x}=\frac{\partial v}{\partial r}\cos\theta-\frac{\partial v}{\partial\theta}\frac{\sin\theta}{r},$$

$$\frac{\partial v}{\partial y}=\frac{\partial v}{\partial r}\sin\theta+\frac{\partial v}{\partial\theta}\cdot\frac{\cos\theta}{r}.$$

But, by Cauchy-Riemann equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

Hence

$$\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}=\frac{\partial v}{\partial r}\sin\theta+\frac{\partial v}{\partial\theta}\frac{\cos\theta}{r}, \tag{1.13}$$

and

$$\frac{\partial u}{\partial r}\sin\theta+\frac{\partial u}{\partial\theta}\frac{\cos\theta}{r}=-\frac{\partial v}{\partial r}\cos\theta+\frac{\partial v}{\partial\theta}\frac{\sin\theta}{r}. \tag{1.14}$$

Multiplying (1.13) by cos θ and (1.14) by sin θ and adding, we get

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial\theta} \tag{1.15}$$

Now multiplying (1.13) by – sin θ and (1.14) by cos θ and adding, we get

$$\frac{1}{r}\frac{\partial u}{\partial\theta}=-\frac{\partial v}{\partial r}. \tag{1.16}$$

The equations (1.15) and (1.16) are called *Cauchy-Riemann equations in polar form*.

EXAMPLE 1.31

Deduce from Example 1.30 that

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0.$$

Solution. The polar form of Cauchy-Riemann equations is

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial\theta},\quad \frac{1}{r}\frac{\partial u}{\partial\theta}=-\frac{\partial v}{\partial r},$$

that is,

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \tag{1.17}$$

and

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \tag{1.18}$$

Differentiating (1.17) with respect to r, we get

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta\,\partial r} \tag{1.19}$$

Differentiating (1.18) with respect to θ, we have

$$\frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial r\,\partial \theta} \tag{1.20}$$

Using (1.17), (1.19), and (1.20), we have

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

EXAMPLE 1.32

Find the analytic function $f(z) = u + iv$ if $u = a(1+\cos\theta)$.

Solution. By polar form of Cauchy-Riemann equations, we have

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \text{ and} \tag{1.21}$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}. \tag{1.22}$$

From (1.22), we have

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{1}{r}(-a\sin\theta) = \frac{a\sin\theta}{r}.$$

Integrating w.r.t r, we get

$$v = a\sin\theta\log r + \phi(\theta).$$

Hence, $f(z) = u + iv = a\,(1 + \cos\theta + i\sin\theta\log r) + \phi(\theta)$.

EXAMPLE 1.33

Show that the function $e^x(\cos y + i\sin y)$ is holomorphic and find its derivative.

Solution. Let

$$\begin{aligned} f(z) = u + iv &= e^x(\cos y + i\sin y) \\ &= e^x\cos y + i\,e^x\sin y. \end{aligned}$$

Thus

$$u = e^x\cos y, \quad v = e^x\sin y,$$

and so

$$\frac{\partial u}{\partial x} = e^x \cos y, \quad \frac{\partial u}{\partial y} = -e^x \sin y,$$
$$\frac{\partial v}{\partial x} = e^x \sin y, \quad \frac{\partial v}{\partial y} = e^x \cos y.$$

We note that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

and, hence, Cauchy-Riemann equations are satisfied. Since, partial derivative are continuous and Cauchy-Riemann equations are satisfied, it follows that $f(z)$ is analytic. Further,

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= e^x \cos y + ie^x \sin y \\ &= e^x(\cos y + i \sin y) = e^{x+iy} = e^z. \end{aligned}$$

We note that $f'(z) = f(z)$.

EXAMPLE 1.34

Show that the function

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} (z \neq 0),\ f(0) = 0$$

is continuous and satisfies Cauchy-Riemann equations at the origin, yet $f'(0)$ does not exist.

Solution. We observe that

$$\begin{aligned} \lim_{z\to 0} f(z) &= \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} \\ &= \lim_{y\to 0} \frac{-y^3(1-i)}{y^2} = \lim_{y\to 0}[-y(1-i)] = 0, \end{aligned}$$

$$\begin{aligned} \lim_{z\to 0} f(z) &= \lim_{\substack{y\to 0\\ x\to 0}} \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} \\ &= \lim_{x\to 0} \frac{x^3(1+i)}{x^2} = \lim_{x\to 0} x(1+i)] = 0. \end{aligned}$$

Also $f(0) = 0$. Now let both x and y tend to zero along the path $y = mx$. Then,

$$\begin{aligned} \lim_{z\to 0} f(z) &= \lim_{\substack{y\to mx\\ x\to 0}} \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} \\ &= \lim_{x\to 0} \frac{x^3(1+i) - m^3x^3(1-i)}{x^2 + m^2x^2} \\ &= \lim_{x\to 0} \frac{x[1+i-m^3(1-i)]}{1+m^2} = 0. \end{aligned}$$

Thus $\lim_{z\to 0} f(z) = f(0)$, whatever may be the path of z tending to zero. Hence f is continuous at the origin.

Now let

$$f(z) = u + iv,$$

where

$$u = \frac{x^3 - y^3}{x^2 + y^2}, \quad v = \frac{x^3 + y^3}{x^2 + y^2}.$$

Then

$$u(0,0) = 0, \; v(0,0) = 0.$$

Now, at the origin (0,0), we have

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0} \frac{x}{x} = 1,$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y \to 0} \frac{-y}{y} = -1,$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x \to 0} \frac{x}{x} = 1,$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y \to 0} \frac{y}{y} = 1.$$

Hence at the origin

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \, and \, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

and so the Cauchy-Riemann equations are satisfied.
But

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$$

$$= \lim_{z \to 0} \frac{x^3 - y^3 + i(x^3 + y^3)}{(x^2 + y^2)\,(x + iy)}.$$

If $z \to 0$ along $y = mx$, then

$$f'(0) = \lim_{x \to 0} \frac{x^3 - m^3x^3 + i(x^3 + m^3x^3)}{(x^2 + m^2x^2)\,(x + imx)}$$

$$= \frac{1 - m^3 + i(1 + m^3)}{(1 + m^2)\,(1 + im)}$$

and so that limit is not unique since it depends on m. Hence $f'(0)$ does not exist.

EXAMPLE 1.35

Show that function $f(z) = \sqrt{|xy|}$ is not regular at the origin, although the Cauchy-Riemann equations are satisfied at the origin.

Solution. We have

$$f(z) = u + iv = \sqrt{|xy|}.$$

Therefore,

$$u(x,y) = \sqrt{|xy|} \; and \; v(x,y) = 0.$$

Then, at the origin

$$\frac{\partial u}{\partial x} = \lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0}\frac{0-0}{x} = 0,$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0}\frac{0-0}{y} = 0,$$

$$\frac{\partial v}{\partial x} = 0 \quad \text{and} \quad \frac{\partial v}{\partial y} = 0.$$

Hence, Cauchy-Riemann equations are satisfied at the origin. But

$$f'(0) = \lim_{z\to 0}\frac{f(z)-f(0)}{z}$$
$$= \lim_{z\to 0}\frac{\sqrt{|xy|}}{(x+iy)}.$$

If $z \to 0$ along $y = mx$, then

$$f'(0) = \lim_{z\to 0}\frac{\sqrt{|mx^2|}}{x+imx} = \lim_{z\to 0}\frac{\sqrt{|m|}}{1+im}.$$

The limit is not unique since it depends on m. Hence $f'(0)$ does not exist.

EXAMPLE 1.36

Show that the function

$$f(z) = e^{-z^{-4}}\ (z \neq 0),\ f(0) = 0$$

is not analytic at the origin, although Cauchy-Riemann equations are satisfied at that point.

Solution. We have

$$f(z) = e^{-z^{-4}} = e^{-\frac{1}{(x+iy)^4}} = e^{-\frac{(x-iy)^4}{(x^2+y^2)^4}}$$
$$= e^{-\frac{1}{r^8}(x^4+y^4-6x^2y^2)}.e^{4ixy(x^2-y^2)/r^8}$$
$$= e^{-\frac{1}{r^8}(x^4+y^4-6x^2y^2)}\left[\cos\frac{4xy(x^2-y^2)}{r^8} + i\sin\frac{4xy(x^2-y^2)}{r^8}\right].$$

Thus

$$u(x,y) = e^{-\frac{1}{r^8}(x^4+y^4-6x^2y^2)}\cos\frac{4xy(x^2-y^2)}{r^8},$$

$$v(x,y) = e^{-\frac{1}{r^8}(x^4+y^4-6x^2y^2)}\sin\frac{4xy(x^2-y^2)}{r^8}.$$

Hence, at the origin,

$$\frac{\partial u}{\partial x} = \lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0}\frac{e^{-x^{-4}}}{x}$$
$$= \lim_{x\to 0}\frac{1}{x\left[1+\frac{1}{x^4}+\frac{1}{2x^8}+\ldots\right]} = \frac{1}{\infty} = 0.$$

Similarly,

$$\frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0.$$

Hence, Cauchy-Riemann equations are satisfied at the origin. But, taking $z = re^{i\pi/4}$, we have

$$f'(0) = \lim_{z\to 0} \frac{e^{-z^{-4}} - 0}{z} = \lim_{r\to 0} \frac{e^{r^{-4}}}{re^{i\pi/4}} = \infty.$$

Hence $f(z)$ is not analytic at $z = (0, 0)$.

EXAMPLE 1.37

Show that an analytic function with constant modulus is constant.

Solution. Let $f(z)$ be analytic with constant modulus.

Thus

$$|f(z)| = |u + iv| = C \,(\text{constant})$$

and so

$$u^2 + v^2 = C^2.$$

Then, we have

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \; and \; 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0.$$

Using Cauchy-Riemann equation, the above relations reduce to

$$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0 \tag{1.23}$$

and

$$u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x} = 0. \tag{1.24}$$

Multipling (1.23) by u, (1.24) by v and adding, we get

$$(u^2 + v^2)\frac{\partial u}{\partial x} = 0.$$

Thus $\frac{\partial u}{\partial x} = 0$ [if $f(z) \neq 0$]. Similarly, $\frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 0.$ Since all the four partial derivatives are zero, the functions u and v are constant and consequently $u + iv$ is constant.

EXAMPLE 1.38

If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and ψ any function of x and y with differential coefficient of first and second orders, then

$$\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 = \left\{\left(\frac{\partial \psi}{\partial u}\right)^2 + \left(\frac{\partial \psi}{\partial v}\right)^2\right\} |f'(z)|^2$$

and

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial v^2}\right) |f'(z)|^2.$$

Solution. We have

$$\frac{\partial\psi}{\partial x}=\frac{\partial\psi}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial\psi}{\partial v}\cdot\frac{\partial v}{\partial x}$$

and

$$\begin{aligned}\frac{\partial\psi}{\partial y}&=\frac{\partial\psi}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial\psi}{\partial v}\cdot\frac{\partial v}{\partial y}\\&=-\frac{\partial\psi}{\partial u}\cdot\frac{\partial v}{\partial x}+\frac{\partial\psi}{\partial v}\cdot\frac{\partial u}{\partial x}\text{ by Cauchy-Riemann equations.}\end{aligned}$$

Therefore,

$$\begin{aligned}&\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\\&=\left[\left(\frac{\partial\psi}{\partial u}\right)^2+\left(\frac{\partial\psi}{\partial v}\right)^2\right]\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]\\&=\left[\left(\frac{\partial\psi}{\partial u}\right)^2+\left(\frac{\partial\psi}{\partial v}\right)^2\right]|f'(z)|^2,\\&\text{since } f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}.\end{aligned}$$

Now let us prove the second result. We have

$$f(z)=w=u+iv,\text{and } \overline{w}=u-iv$$

and so

$$u=\frac{1}{2}(w+\overline{w}),v=\frac{1}{2i}(w-\overline{w}).$$

Therefore

$$\frac{\partial}{\partial\omega}=\frac{\partial}{\partial u}\frac{\partial u}{\partial\omega}+\frac{\partial}{\partial v}\cdot\frac{\partial v}{\partial\omega}=\frac{1}{2}\left(\frac{\partial}{\partial u}-i\frac{\partial}{\partial v}\right),$$

$$\frac{\partial}{\partial\overline{w}}=\frac{\partial}{\partial u}\cdot\frac{\partial u}{\partial\overline{w}}+\frac{\partial}{\partial v}\cdot\frac{\partial v}{\partial\overline{w}}=\frac{1}{2}\left(\frac{\partial}{\partial u}+i\frac{\partial}{\partial v}\right).$$

Thus

$$\frac{\partial}{\partial w}\cdot\frac{\partial}{\partial\overline{w}}=\frac{1}{4}\left(\frac{\partial}{\partial u}-i\frac{\partial}{\partial v}\right)\left(\frac{\partial}{\partial u}+i\frac{\partial}{\partial v}\right),$$

that is,

$$4\frac{\partial}{\partial w}\cdot\frac{\partial}{\partial\overline{w}}=\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial u^2}.$$

Hence

$$\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2}=4\frac{\partial^2\psi}{\partial w\,\partial\overline{w}}\tag{1.25}$$

But

$$4\frac{\partial^2}{\partial w\,\partial \overline{w}} = 4\left(\frac{\partial}{\partial z}\cdot\frac{\partial z}{\partial w}\right)\left(\frac{\partial}{\partial \overline{z}}\cdot\frac{\partial \overline{z}}{\partial \overline{w}}\right)$$

$$= 4\left(\frac{1}{f'(z)}\cdot\frac{\partial}{\partial z}\right)\left(\frac{1}{f'(z)}\cdot\frac{\partial}{\partial \overline{z}}\right)$$

$$= 4\left(\frac{1}{f'(z)\,f'(\overline{z})}\right).\frac{\partial^2}{\partial z\,\partial \overline{z}}$$

$$= \frac{4}{|\,f'(z)|^2}\cdot\frac{\partial}{\partial z}\cdot\frac{\partial}{\partial \overline{z}}$$

$$= \frac{4}{|\,f'(z)|^2}\left[\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right).\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\right]$$

$$= \frac{1}{|\,f'(z)|^2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right).$$

Hence (1.25) yields

$$\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2} = \frac{1}{|\,f'(z)|^2}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)$$

and so

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2} = \left(\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2}\right)|\,f'(z)|^2.$$

EXAMPLE 1.39

If $f(z)$ is a regular function of z, show that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\,f(z)|^2 = 4\,|\,f'(z)|^2\,.$$

Solution. Since $z = x + iy$, we have

$$x = \frac{1}{2}(z+\overline{z}) \quad \text{and} \quad y = -\frac{i}{2}(z-\overline{z}).$$

Therefore,

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial z}+\frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right),$$

$$\frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial \overline{z}}+\frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right).$$

Thus

$$\frac{\partial}{\partial z}\cdot\frac{\partial}{\partial \overline{z}} = \frac{1}{4}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$$

$$= \frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right).$$

Hence

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}[f(z)f(\bar{z})]$$

$$= 4\frac{\partial}{\partial z}[f(z)f'(\bar{z})]$$

$$= 4[f'(z)f'(\bar{z}) = 4\,|f'(z)|^2.$$

EXAMPLE 1.40

If $f(z)$ is a regular function of z such that $f'(z) \neq 0$, show that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0.$$

Solution. We have

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Therefore

$$\frac{\partial^2}{\partial z\,\partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right),$$

that is,

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\,\partial \bar{z}} \tag{1.26}$$

But

$$\log|f'(z)| = \frac{1}{2}\log|f'(z)|^2 = \frac{1}{2}\log[f'(z)f'(\bar{z})]$$

$$= \frac{1}{2}\log f'(z) + \frac{1}{2}\log f'(\bar{z}).$$

Therefore (1.26) yields

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f'(z)|$$

$$= 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left[\frac{1}{2}\log f'(z) + \frac{1}{2}\log f'(\bar{z})\right] = 0.$$

1.3 INTEGRATION OF COMPLEX-VALUED FUNCTIONS

The theory of Riemann-integrals can be extended to complex-valued functions. Integrals of complex-valued functions are calculated over certain types of curves in the complex plane. The following definitions are required for the complex integration.

Definition 1.25. A *continuous curve* or *arc* C in the complex plane joining the points $z(\alpha)$ and $z(\beta)$ are defined by the parametric representation

$$z(t) = x(t) + iy(t),\ \alpha \le t \le \beta,$$

where $x(t)$ and $y(t)$ are continuous real functions. The point $z(\alpha)$ is the *initial point* and $z(\beta)$ is the *terminal point* (Fig. 1.2).

If $z(\alpha) = z(\beta)$, $\alpha \neq \beta$, then the endpoints coincide and the curve is called *closed curve*. A closed curve which does not intersect itself anywhere is called a *simple closed curve* (Fig. 1.3). The curve is traversed counterclockwise.

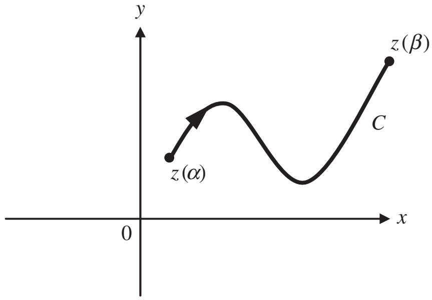


Figure 1.2 *Curve C*

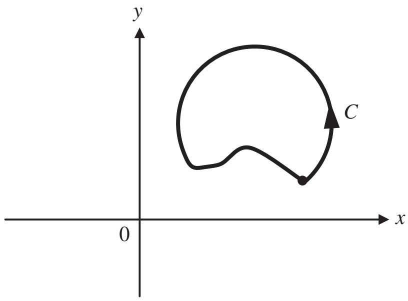


Figure 1.3 *Simple Closed Curve*

Definition 1.26. A continuous curve C: $z(t) = x(t) + iy(t)$, $\alpha \le t \le \beta$ is called *smooth curve* or *smooth arc* if $z'(t)$ is continuous in $[\alpha, \beta]$ and $z'(t) \neq 0$ in (α, β).

Definition 1.27. A piecewise smooth curve C is called a *contour*.

Thus, a curve C: $z(t) = x(t) + iy(t)$, $\alpha \le t \le \beta$ is a contour if there is a partition $\alpha = t_0 < t_1 < \ldots < t_n = \beta$ such that $z(t)$ is smooth on each subinterval $[t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$.

Definition 1.28. A region in which every closed curve can be contracted to a point without passing out of the region is called a *simply connected region*.

A region which is not simply connected is called *multiply connected*.

Figure 1.4 illustrates the simply-connected and multiply-connected regions.

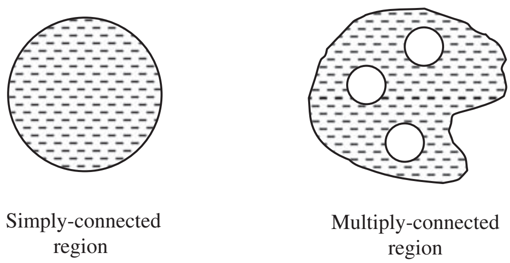


Figure 1.4

Thus, simply-connected region does not have any hole in it.

Definition 1.29. The *Riemann-integral* of $f(z)$ over a contour C is defined as

$$\int_C f(z)\,dz = \int_\alpha^\beta f(z(t))\,z'(t)\,dt.$$

The integral on the right-hand side exists because the integrand is piecewise continuous.

We note that the following properties hold for the integral.

(i) $-\int_C f(z)\,dz = \int_{-C} f(z)\,dz$

(ii) if $C_1, C_2, \ldots, C_n$ are disjoint contours, then

$$\int_{C_1+C_2+\ldots+C_n} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \ldots + \int_{C_n} f(z)dz$$

(iii) if $f(z)$ is continuous on contour C, then

$$\left|\int_C f(z)\,dz\right| = \left|\int_\alpha^\beta f(z(t))\,z'(t)\,dt\right| \le \int_\alpha^\beta |f(z(t))||z'(t)|\,dt$$
$$= \int_C |f(z)||dz|,$$

where

$$\int_C |dz| = \int_\alpha^\beta |z'(t)|\,dt = \int_\alpha^\beta \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$$
$$= L_c,\ \text{length of the curve } C.$$

Therefore, if $|f(z)| \le M$ on C, then

$$\left|\int_C f(z)\,dz\right| \le \int_C |f(z)||dz| \le M\,L_c.$$

EXAMPLE 1.41

Evaluate $\int_C \bar{z}\,dz$ from $z = 0$ to $z = 4 + 2i$ along the curve C given by $z = t^2 + it$.

Solution. We have

$$\int_0^{4+2i} \bar{z}\,dz = \int_C \overline{(t^2 + it)}\,dz$$
$$= \int_C \overline{(t^2 + it)}\,(2t + i)\,dt$$

The point $z = 0$ and $z = 4 + 2i$ correspond to $t = 0$ and $t = 2$, respectively. Hence the given integral is equal to

$$\int_0^2 (t^2 - it)(2t + i)\,dt = \int_0^2 (2t^3 - it^2 + t)\,dt = 10 - \frac{8i}{3}.$$

EXAMPLE 1.42

Evaluate $\int_0^{2+i} (\bar{z})^2\, dz$ along the line $y = \dfrac{x}{2}$.

Solution. Along the given line, we have $x = 2y$ and so $z = x + iy = 2y + iy = (2 + i)y$, $\bar{z} = (2 - i)y$, and $dz = (2 + i)dy$. Thus

$$\int_0^{2+i} (\bar{z})^2\, dz = \int_0^1 (2-i)^2\, y^2.(2+i)\, dy = 5(2-i)\left[\frac{y^3}{3}\right]_0^1 = \frac{5}{3}(2-i).$$

EXAMPLE 1.43

Evaluate $\int_0^{1+i} (x - y + ix^2)dz$ along the straight line from $z = 0$ to $z = 1 + i$.

Solution. As shown in Figure 1.5, the straight line from $z = 0$ to $z = 1 + i$ is OA. On this line, we have $y = x$ and so $z = x + iy$. Thus

$$dz = dx + i\ dy = dx + i\ dx = (1+i)dx.$$

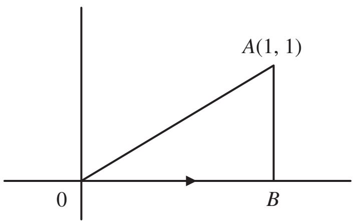


Figure 1.5

Hence

$$\int_{OA} (x - y + ix^2)\, dz = \int_0^1 (ix^2)\,(1+i)\, dx = (i-1)\left[\frac{x^3}{3}\right]_0^1 = \frac{i-1}{3}.$$

EXAMPLE 1.44

Evaluate $\int_0^{1+i} (x^2 + iy)\, dz$ along the path $y = x^2$.

Solution. We have $z = x + iy = x + ix^2$ and so $dz = dx + 2ixdx = (2ix + 1)dx$. Hence

$$\int_0^{1+i} (x^2 + iy)\, dz = \int_0^1 (x^2 + ix^2)(2ix+1)\, dx = \left[(2i-2)\frac{x^4}{4} + (1+i)\frac{x^3}{3}\right]_0^1 = \frac{5i-1}{6}.$$

EXAMPLE 1.45

Show that $\int_C \dfrac{dz}{z} = -\pi i$ or πi according as C is the semi-circular arc of $|z| = 1$ above or below the x-axis.

Solution. Taking $z = re^{i\theta}$, we have (Fig. 1.6) $dz = ir\, e^{i\theta}\, d\theta$. Therefore, for the semi-circular arc above the x-axis, we have

$$I_1 = \int_{C_1} \frac{dz}{z} = \int_\pi^0 \frac{1}{re^{i\theta}}\, i\, r\, e^{i\theta} d\theta = i\int_\pi^0 d\theta = -\pi i.$$

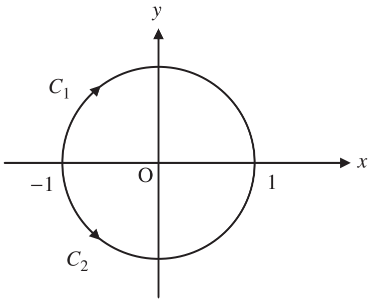


Figure 1.6

For the lower semi-circular arc, we have

$$I_2 = \int_{C_2} \frac{dz}{z} = i \int_{\pi}^{2\pi} d\theta = \pi i.$$

EXAMPLE 1.46

Evaluate $\int_{(0,3)}^{(2,4)} [(2y + x^2)dx + (3x - y)\, dy]$ along the parabola $x = 2t$, $y = t^2 + 3$.

Solution. The points (0, 3) and (2, 4) on the parabola correspond to $t = 0$ and $t = 1$, respectively. Thus, the given integral becomes

$$\int_0^1 [2(t^3 + 3) + 4t^2]\, 2\, dt + (6t - t^3 - 3)\, 2t\, dt = \int_0^1 (24t^2 - 2t^3 - 6t + 12)\, dt = \frac{33}{2}.$$

EXAMPLE 1.47

Evaluate $\int_C (z - z^2)\, dz$, where C is the upper half of the unit circle $|z| = 1$.

Solution. The contour is $|z| = 1$. So let $z = e^{i\theta}$. Then $dz = i\, e^{i\theta}\, d\theta$. As shown in the Figure 1.7, the limits of integration become 0 to π. Hence

$$\int_C (z - z^2)\, dz = \int_0^{\pi} (e^{i\theta} - e^{2i\theta}) i\, e^{i\theta}\, d\theta$$

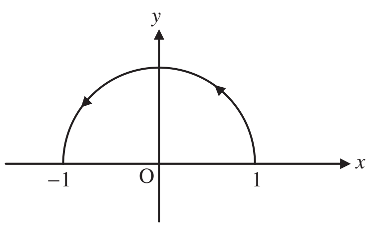


Figure 1.7

$$= i\int_0^{\pi}(e^{2i\theta} - e^{3i\theta})\, d\theta = i\left[\frac{e^{2i\theta}}{2i} - \frac{e^{3i\theta}}{3i}\right]_0^{\pi}$$

$$= \frac{1}{6}[3e^{2\pi i} - 2e^{3\pi i} - 3 + 2] = \frac{2}{3}.$$

EXAMPLE 1.48

Show that $\int_C (z-a)^n dz = 0,$ where n is any integer not equal to –1 and C is the circle $|z - a| = r$ with radius r and centre at a.

Solution. Substituting $z - a = r\, e^{i\theta}$, we have $dz = ir\, e^{i\theta}$ and so the given integral reduces to

$$\int_0^{2\pi} r^n\ e^{ni\theta} .ir\ e^{i\theta} d\theta$$

$$= i\ r^{n+1}\int_0^{2\pi} e^{(n+1)i\theta} d\theta = i\ r^{n+1}\left[\frac{e^{(n+1)i\theta}}{i(n+1)}\right]_0^{2\pi}$$

$$= \frac{r^{n+1}}{n+1}[e^{2(n+1)\pi i} - 1] = 0, \quad n \neq -1.$$

Theorem 1.6. If $f(z)$ is continuous on a contour C of length L and $|f(z)| \leq M$, then

$$\left|\int_C f(z)\, dz\right| \leq ML.$$

Proof: Since

$$\left|\sum_{s=1}^{n} f(\xi_r)\,(z_r - z_{r-1})\right| \leq \sum_{r=1}^{n} |\,f(\xi_r)\,|\,|\,z_r - z_{r-1}\,|,$$

taking limit as $n \to \infty$, we get

$$\left|\int_C f(z)\, dz\right| \leq \int_C |\,f(z)\,||\,dz\,|$$

$$\leq \mathrm{M}\int_C |\,dz\,|$$

$$\leq \mathrm{ML}, \text{ since } \int_C |\,dz\,| = \mathrm{L}.$$

Theorem 1.7. (Cauchy's Integral Theorem). If $f(z)$ is an analytic function and if $f'(z)$ is continuous at each point within and on a closed contour C, then

$$\int_C f(z)\, dz = 0.$$

Proof: Since $z = x + iy$, we can write

$$\int_C f(z)\,dz = \int_C (u+iv)\,(dx+idy)$$
$$= \int_C [(u\,dx - v\,dy) + i(v\,dx + udy)]$$
$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

Since $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ and $f'(z)$ in continuous, it follows that u_x, u_y, v_x, and v_y are all continuous in the region D enclosed by the curve C. Hence, by Green's Theorem, we have

$$\int_C f(z)\,dz$$
$$= \int\int_D -\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) dx\,dy + i\int\int_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$
$$= \int\int_D \left(\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\right) dx\,dy + i\int\int_D \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\,dy = 0,$$

the last but one step being the consequence of Cauchy-Riemann equations.

Theorem 1.7 was further generalized by Goursat in the form of the following theorem:

Theorem 1.8. (Cauchy-Goursat). Let $f(z)$ be analytic in a region R. Then for any closed contour C in R,

$$\int_C f(z)\,dz = 0.$$

(For proof, see E.C.Titchmarsh, *Theory of Functions*, Oxford University, Press).

Theorem 1.9. The function $F(z)$ defined by $F(z) = \int_a^z f(\xi)\,d\xi$, where z and a both are in domain D is an analytic function of z such that $F'(z) = f(z)$.

Proof: We have

$$F(z) = \int_a^z f(\xi)d\xi.$$

Therefore,

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{\int_a^z f(\xi)d\xi - \int_a^{z_0} f(\xi)d\xi}{z - z_0} - f(z_0)$$
$$= \frac{\int_{z_0}^z f(\xi)d\xi}{z - z_0} - f(z_0).$$

But,

$$f(z_0) = \frac{1}{z - z_0}\int_{z_0}^z f(z_0)d\xi.$$

Hence

$$\frac{F(z)-F(z_0)}{z-z_0}-f(z_0)=\frac{\left[\int_{z_0}^{z} f(\xi)-f(z_0)\right]}{z-z_0}d\xi.$$

Since f is continuous, we have

$$|f(\xi)-f(z_0)|<\varepsilon \text{ for } |\xi-z_0|<\delta.$$

Therefore

$$\left|\frac{F(z)-F(z_0)}{z-z_0}-f(z_0)\right|<\frac{\varepsilon}{z-z_0}\int_{z_0}^{z} d\xi=\varepsilon.$$

Thus, $F'(z_0)=f(z_0)$ and so $F(z)$ is differentiable and has $f(z_0)$ as its derivative. Hence F(z) is analytic.

Theorem 1.10. (Cauchy's Integral Formula). If $f(z)$ is analytic within and on any closed contour C and if a is a point within the contour C, then

$$f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz.$$

Proof: Let $z=a$ be any point within the contour C. Describe a small circle γ about $z=a$, whose radius is r and which lies entirely within C. Consider the function

$$\phi(z)=\frac{f(z)}{z-a}.$$

This function is analytic at all points in the ring-shaped region between C and γ but it has a simple pole at $z=a$. Now, we take a cross cut by joining any point of C to any point of γ. Thus, we obtain a closed contour Γ as shown in Figure 1.8.

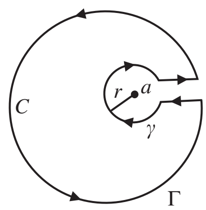


Figure 1.8

Hence, by Cauchy-Goursat theorem, we have

$$\int_\Gamma \phi(z)\,dz=0,$$

which yields

$$\int_C \phi(z)\,dz-\int_\gamma \phi(z)\,dz=0.$$

Thus,

$$\int_C \frac{f(z)}{z-a}\,dz=\int_\gamma \frac{f(z)}{z-a}\,dz=\int_\gamma \frac{f(z)-f(a)+f(a)}{z-a}\,dz$$

$$=\int_\gamma \frac{f(a)}{z-a}\,dz+\int_\gamma \frac{f(z)-f(a)}{z-a}\,dz \qquad (1.27)$$

Since $f(z)$ is continuous at $z = a$, to each $\varepsilon > 0$ there exists a positive δ such that

$$|f(z) - f(a)| < \varepsilon \text{ whenever } |z - a| < \delta.$$

Moreover, by substituting $z - a = r e^{i\theta}$, we get

$$\int_\gamma \frac{f(a)}{z-a}\,dz = f(a)\int_\gamma \frac{dz}{z-a} = f(a)\int_0^{2\pi} \frac{ir\,e^{i\theta}}{re^{i\theta}}\,d\theta$$

$$= i\,f(a)\int_0^{2\pi} d\theta = 2\pi i\,f(a).$$

Hence (1.27) yields

$$\int_C \frac{f(z)}{z-a}\,dz - 2\pi i\,f(a) = \int_\gamma \frac{f(z)-f(a)}{z-a}\,dz$$

and so

$$\left|\int_C \frac{f(z)}{z-a}\,dz - 2\pi i\,f(a)\right| \le \int_\gamma \left|\frac{f(z)-f(a)}{z-a}\right| dz$$

$$< \varepsilon\int_0^{2\pi} d\theta,\ z - a = r\,e^{i\theta}$$

$$< 2\pi\varepsilon.$$

The left-hand side is independent of ε, and so vanishes. Consequently,

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\,f(a),$$

and, therefore,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz.$$

Theorem 1.11. (Cauchy's Formula for Derivative of Analytic Function). If $f(z)$ is an analytic function in a region D, then its derivative at any point $z = a$ is represented by

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz,$$

where C is any closed contour in D surrounding the point $z = a$.

Proof: Suppose that 2δ is the shortest distance from the point a to the contour C. Thus $|z - a| \ge 2\delta$ for every point z on C. If $|h| \le \delta$, the point $a + h$ also lies within C, at a distance not less than δ from C (Fig. 1.9). Therefore, by Cauchy's integral formula, we have

$$f(a) = \frac{1}{2\pi i}\int \frac{f(z)}{z-a}\,dz, \text{ and so}$$

$$f(a+h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a-h}\,dz.$$

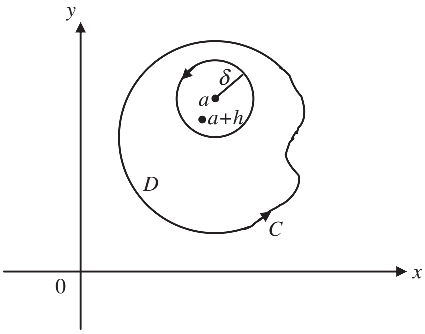


Figure 1.9

Thus

$$\begin{aligned}\frac{f(a+h)-f(a)}{h} &= \frac{1}{2\pi i\, h}\int_C \left(\frac{f(z)}{z-a-h}-\frac{f(z)}{z-a}\right) dz \\ &= \frac{1}{2\pi i}\int \frac{f(z)\, dz}{(z-a)\,(z-a-h)} \\ &= \frac{1}{2\pi i}\int \frac{z-a-h+h}{(z-a)^2(z-a-h)} f(z)dz \\ &= \frac{1}{2\pi i}\int \frac{z-a-h}{(z-a)^2(z-a-h)} f(z)dz + \frac{1}{2\pi i}\int_C \frac{hf(z)}{(z-a)^2(z-a-h)}\, dz \\ &= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\, dz + \frac{1}{2\pi i}\int_C \frac{hf(z)}{(z-a)^2(z-a-h)}\, dz\end{aligned}$$

and so

$$\frac{f(a+h)-f(a)}{h} - \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\, dz = \frac{1}{2\pi i}\int_C \frac{hf(z)}{(z-a)^2(z-a-h)}\, dz.$$

Now

$$|\, z-a-h\,| \geq |\, z-a\,| - |\, h\,| \geq 2\delta - \delta = \delta.$$

Since $f(z)$ is analytic on C, it is continuous and so is bounded. Thus there exists a constant $M > 0$ such that $|f(z)| \leq M$. Therefore,

$$\begin{aligned}&\left|\frac{f(a+h)-f(a)}{h} - \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\, dz\right| \\ &= \left|\frac{1}{2\pi i}\int_C \frac{hf(z)}{(z-a)^2(z-a-h)}\, dz\right| \leq \frac{|h|}{2\pi}\int_a \frac{|f(z)|}{|z-a|^2|z-a-h|}|\, dz\,| \leq \frac{M|h|}{2\pi}\int_C \frac{|\, dz\,|}{4\delta^2(\delta)} \\ &= \frac{|h|}{2\pi} \cdot \frac{ML}{4\delta^3} \text{ since} \int_C |\, dz\,| = \text{L(length of C)}.\end{aligned}$$

Letting $h \to 0$, we get

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz.$$

Hence

$$f'(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz.$$

Theorem 1.12. If $f(z)$ is analytic in a domain D, then $f(z)$ has, at any point $z = a$ *of* D, derivatives of all orders given by

$$f^{(n)}(a)=\frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz,$$

where C is any closed contour in D surrounding the point $z = a$.

Proof: By Cauchy's integral formulae, we have

$$f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz,$$

$$f'(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz.$$

Thus the result is true for $n = 0$ and $n = 1$. We use mathematical induction on n. Suppose that the result is true for $n = m$. Thus

$$f^{(m)}(a)=\frac{m!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{m+1}}\,dz.$$

Then

$$f^{(m+1)}(a)=\lim_{h\to 0}\frac{f^{(m)}(a+h)-f^{(m)}(a)}{h}$$

$$=\lim_{h\to 0}\frac{m!}{2\pi ih}\left[\int_C \frac{f(z)dz}{(z-a-h)^{m+1}}-\int_C \frac{f(z)dz}{(z-a)^{m+1}}\right]$$

$$=\lim_{h\to 0}\frac{m!}{2\pi ih}\int_C\left[\frac{1}{(z-a)^{m+1}}\left\{\left(1-\frac{h}{z-a}\right)^{-m+1}-1\right\}\right]f(z)\,dz$$

$$=\lim_{h\to 0}\frac{m!}{2\pi ih}\int_C\left[\frac{1}{(z-a)^{m+1}}\left\{(m+1)\frac{h}{z-a}+O(h^2)\right\}\right]f(z)\,dz$$

$$=\frac{(m+1)!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{m+1}}\,dz,$$

which shows that the theorem is also true for $n = m + 1$. Hence it is true for all values of n and we have

$$f^{(n)}(a)=\frac{n!}{2\pi i}\int \frac{f(z)\,dz}{(z-a)^{n+1}}.$$

Remark 1.6. Since each of $f'(a), f'(a) \ldots, f^{(n)}(a)$ have unique differential coefficient, it follows that *derivatives of an analytic function are also analytic functions*.

The following theorem is a sort of *converse of Cauchy's theorem*.

Theorem 1.13. (Morera's Theorem). If $f(z)$ is continuous in a region D and if the integral $\int f(z)dz$ taken round any closed contour in D is zero, then $f(z)$ is analytic inside D.

Proof: Let z_0 be any fixed and z any variable point of the domain D and let C_1, C_2 be any two continuous rectifiable curves in D joining z_0 to z (Fig. 1.10).

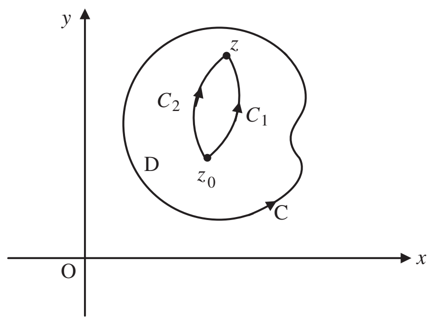


Figure 1.10

Then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0.$$

Thus the value of the integral is independent of the path. So, let

$$F(z) = \int_{z_0}^{z} f(\xi)\,d\xi.$$

Since $f(z) = \int_z^{z+h} \frac{f(z)}{h}\,d\xi$, we have

$$\frac{F(z+h)-F(z)}{h} - f(z)$$

$$= \frac{1}{h}\left[\int_{z_0}^{z+h} f(\xi)\,d\xi - \int_{z_0}^{z} f(\xi)d\xi\right] - \int_z^{z+h} \frac{f(z)}{h}\,d\xi$$

$$= \frac{1}{h}\left[\int_z^{z+h} [f(\xi)-f(z)]\,d\xi\right].$$

Since $f(z)$ is continuous, to every $\varepsilon > 0$ there exists $\eta > 0$ such that

$$|f(\xi)-f(z)| < \varepsilon \quad \text{whenever } |z-\xi| < \eta.$$

Thus

$$\left|\frac{F(z+h)-F(z)}{h}-f(z)\right| \le \frac{1}{h}\left|\int_{z}^{z+h}|f(\xi)-f(z)||d\xi| \le \frac{1}{|h|}\varepsilon|h| = \varepsilon.$$

Hence

$$F'(z) = f(z).$$

Since $F(z)$ is analytic, its derivative is also analytic. Therefore, $F'(z)$ is analytic and consequently $f(z)$ is analytic.

Theorem 1.14. (Cauchy's Inequality). If $f(z)$ is analytic within a circle $|z-a| = R$ *and if* $|f(z)| \le M$ *on C, then*

$$|f^{(n)}(a)| \le \frac{M.n!}{R^n}.$$

Proof: We know that

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz.$$

Therefore,

$$|f^{(n)}(a)| = \left|\frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz\right| \le \frac{n!}{2\pi}\int \frac{|f(z)|}{|(z-a)^{n+1}|}|dz|$$

$$= \frac{n!}{2\pi}\cdot\frac{M}{R^{n+1}}\int_C |dz|$$

$$= \frac{n!}{2\pi}\cdot\frac{M}{R^{n+1}}2\pi R$$

$$= \frac{Mn!}{R^n}.$$

Theorem 1.15. (Liouville's Theorem). A bounded entire function is constant.

Proof: Let $f(z)$ be bounded entire function. Then there exists a positive constant M such that $|f(z)| \le$ M. Let a be any point of the z-plane and C be the circumference of the circle $|z-a| = R$. Then, by Cauchy's integral formula, we have

$$|f'(a)| = \left|\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz\right| \le \frac{1}{2\pi}\int \frac{|f(z)|}{|(z-a)^2|}|dz| \le \frac{M}{2\pi R^2}\int_C |dz|$$

$$= \frac{M}{2\pi R^2}.2\pi R$$

$$= \frac{M}{R}.$$

Since $f(z)$ is an entire function, R may be taken arbitrarily large and, therefore, M/R tends to zero as $R \to \infty$. Hence, $|f(z)| \le M$ leads us to $|f'(a)| = 0$. Since a is arbitrary, we have $f'(a) = 0$ for all points in the z-plane. Hence $f(z)$ is constant.

Second Proof: By Cauchy inequality, we have

$$| f^{(n)}(a) | \le \frac{n!M}{R^n}.$$

Thus, for $n = 1$, we get

$$| f'(a) | < \frac{M}{R} \to 0 \text{ as } R \to \infty.$$

Therefore, $|f(z)| \le$ M implies $|f'(a)| = 0$. Since a is arbitrary, $f'(a) = 0$ for all points in the z-plane. Hence $f(z)$ is constant.

Remark 1.7. Since cos z and sin z are entire functions of complex variable z, it follows from Liouville's Theorem that *cos z and sin z are not bounded for complex z.*

Theorem 1.16. (Poisson's Integral Formula). Let $f(z)$ be analytic in the region $|z| \le R$ and let $u(r, \theta)$ be the real part of $f(r\,e^{i\theta})$, $z = re^{i\theta}$. Then for $0 < r < R$,

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)u(R,\phi)}{R^2 - 2Rr\cos(\theta - \phi) + r^2}\, d\phi,$$

where ϕ is the value of θ on the circle $|z| = R$.

Proof: Let C be the circle $|z| = R$. Suppose $z = re^{i\theta}$ is a point within the domain $|z| < R$ and let $\xi = R\,e^{i\phi}$ be a point on the circle $|z| = R$ (Fig. 1.11). Then, Cauchy's integral formula yields

$$f(z) = u + iv = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{(\xi - z)}\, d\xi \tag{1.28}$$

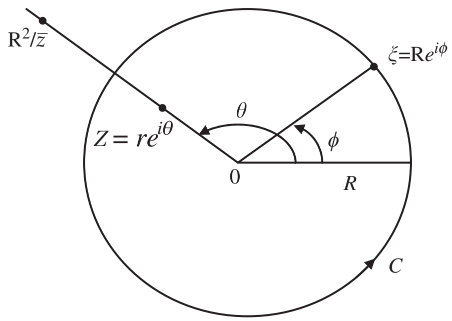


Figure 1.11

Now the point z being interior, the point $R^2/\bar{z}$ is the inverse point of z with respect to $|z| = R$ and, hence, lies outside the circle. Therefore, $\dfrac{f(\xi)}{\xi - \dfrac{R^2}{\bar{z}}}$ is analytic within C. Hence, by Cauchy'sGoursat theorem, we have

$$0 = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi - \left(\dfrac{R^2}{\overline{z}}\right) d\xi}. \tag{1.29}$$

Subtracting (1.29) from (1.28), we have

$$f(z) = \frac{1}{2\pi i}\int \frac{f(\xi)\left(\dfrac{R^2}{\overline{z}} - z\right)}{(\xi - z)\left(\dfrac{R^2}{\overline{z}} - \xi\right)} d\xi$$

$$= \frac{1}{2\pi i}\int \frac{R^2 - z\,\overline{z}}{(\xi - z)\,(R^2 - \xi\,\overline{z})}\; f(\xi)\, d\xi.$$

Substituting $\xi = Re^{i\phi}$, $z = re^{i\theta}$, we get

$$f(re^{i\theta}) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(Re^{i\phi})\,(R^2 - r^2)\,Re^{i\phi}}{(Re^{i\phi} - re^{i\theta})\,(R^2 - Re^{i\theta}.re^{-i\theta})}\, id\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{f(Re^{i\phi})(R^2 - r^2)}{(Re^{i\phi} - re^{i\theta})\,(-re^{-i\theta} + Re^{-i\phi})}\, d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{f(Re^{i\phi})\,(R^2 - r^2)}{R^2 + r^2 - 2Rr\cos(\theta - \phi)} d\phi.$$

Thus

$$u(r,\theta) + iv(r,\theta)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\,[u(R,\phi) + iv(R,\phi)]}{R^2 + r^2 - 2Rr\cos(\theta - \phi)} d\phi$$

Equating real and imaginary parts, we get

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\,u(R,\phi)}{R^2 + r^2 - 2Rr\cos(\theta - \phi)} d\phi$$

and

$$v(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\,v(R,\phi)}{R^2 + r^2 - 2Rr\cos(\theta - \phi)} d\phi.$$

EXAMPLE 1.49

If C is any simple closed curve, evaluate $\int_C \frac{dz}{z-a}$ if (*a*) a is outside C and (*b*) a is inside C.

Solution. Let

$$f(z) = \frac{1}{z-a}.$$

(i) If $z = a$ is outside C, then $f(z)$ is analytic everywhere inside C. Hence, by Cauchy's integral theorem $\int_C \frac{dz}{z-a} = 0.$

(ii) If $z = a$ is inside C, let Γ be the circle of radius r with centre at a so that Γ is inside C. Then

$$\int_C \frac{dz}{z-a} = \int_\Gamma \frac{dz}{z-a}.$$

Substituting $z - a = r\,e^{i\theta}$, we get $dz = ir\,e^{i\theta}\,d\theta$ and so

$$\int_C \frac{dz}{z-a} = \int_\Gamma \frac{dz}{z-a} = \int_0^{2\pi} \frac{ir\,e^{i\theta}}{re^{i\theta}}\,d\theta = i\int_0^{2\pi} d\theta = 2\pi i.$$

EXAMPLE 1.50

Evaluate $\int_C \frac{e^z}{z-2}\,dz$, where C is the circle

(i) $|z| = 3$ and (ii) $|z| = 1$.

Solution. (i) Let $f(z) = e^z$. Then $f(z)$ is analytic and $z = 2$ lies inside the circle $|z| = 3$. Therefore, by Cauchy's integral formula

$$f(2) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-2}\,dz = \frac{1}{2\pi i}\int_C \frac{e^z}{z-2}\,dz.$$

Thus

$$\int_C \frac{e^z}{z-2}\,dz = 2\pi i\; f(2) = 2\pi i\, e^2.$$

(ii) The point $z = 2$ lies outside the circle $|z| = 1$. Also the function $\frac{e^z}{z-2}$ is analytic within and on $|z| = 1$. Hence, by Cauchy's integral theorem

$$\int_C \frac{e^z}{z-2}\,dz = 0.$$

EXAMPLE 1.51

Evaluate

$$\int_{|z|=1/2} \frac{e^z}{z^2+1}\,dz.$$

Solution. The function $\frac{e^z}{z^2+1} = \frac{e^z}{(z+i)(z-i)}$ is analytic at all points except $z = \pm i$. Also the points $\pm i$ lie outside $|z| = 1/2$. Hence, by Cauchy-Goursat theorem, the given integral is equal to zero.

EXAMPLE 1.52

Using Cauchy's integral formula and Cauchy-Goursat theorem, evaluate the integral

$$\int_C \frac{z^2 - z + 1}{z - 1}\, dz,$$

where C is the circle

(i) $|z| = 1$ and (ii) $= |z| = \frac{1}{2}$.

Solution. (i) Let $f(z) = z^2 - z + 1$. Then f is analytic as the circle $|z| = 1$ and $z = 1$ lies on C. Therefore, by Cauchy's integral formula,

$$f(1) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - 1}\, dz.$$

But, $f(1) = 1$. Hence

$$\int_0 \frac{f(z)}{z - 1}\, dz = 2\pi i,$$

that is,

$$\int_C \frac{z^2 - z + 1}{z - 1} dz = 2\pi i.$$

(ii) The function $f(z) = z^2 - z + 1$ is analytic everywhere within $|z| = \frac{1}{2}$. Since $z = 1$ lies outside $|z| = \frac{1}{2}$. $\frac{z^2 - z + 1}{z - 1}$ is also analytic within $|z| = \frac{1}{2}$. Hence, by Cauchy-Goursat integral theorem

$$\int_{|z| = \frac{1}{2}} \frac{z^2 - z + 1}{z - 1}\, dz = 0.$$

EXAMPLE 1.53

Evaluate

$$\int_{|z|=3} \frac{\sin \pi z^2 + \cos \pi z^2}{(z - 1)(z - 2)}\, dz.$$

Solution. Since

$$\frac{1}{(z - 1)(z - 2)} = \frac{1}{z - 2} - \frac{1}{z - 1},$$

the given integral can be written as

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z - 1)(z - 2)}\, dz = \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z - 2}\, dz - \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z - 1}\, dz.$$

The points $z = 2$ and $z = 1$ lie within the circle $|z| = 3$ and the function $f(z) = \sin \pi z^2 + \cos \pi z^2$ is analytic within and on $|z| = 3$. Hence, by Cauchy's integral formula,

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\, dz = 2\pi i\ f(2)$$

$$= 2\pi i [\sin \pi + \cos \pi]$$

$$= -2\pi i,$$

and

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\, dz = 2\pi i\ f(1)$$

$$= 2\pi i [\sin \pi + \cos \pi]$$

$$= -2\pi i,$$

Hence

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz = 2\pi i - (-2\pi i) = 4\pi i.$$

EXAMPLE 1.54

Using Cauchy integral formula, evaluate the integral $\int_C \frac{e^{2z}}{(z-1)(z-2)}\, dz$, where C is the circle $|z| = 3$.

Solution. Let $f(z) = e^{2z}$. Then f is analytic within the circle $|z| = 3$. Also $z = 1$, $z = 2$ lie within $|z| = 3$. Hence, by Cauchy's integral formula, we have

$$\int_{|z|=3} \frac{e^{2z}}{(z-1)(z-2)}\, dz = \int_{|z|=3} \frac{e^{2z}}{z-2}\, dz - \int_{|z|=3} \frac{e^{2z}}{z-1}\, dz$$

$$= 2\pi i\ f(2) - 2\pi i\ f(1)$$

$$= 2\pi i (e^4 - e^2).$$

EXAMPLE 1.55

Evaluate the integral $\int_C \frac{ze^z}{(z-a)^3}\, dz$, where the point a lies within the closed contour C.

Solution. Let $f(z) = ze^z$. Then f is analytic (rather entire). The point a lies within C. Therefore,

$$f''(a) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-a)^3}\, dz$$

But,

$$f'(a) = ze^z + e^z, \quad f''(z) = ze^z + 2e^z.$$

So

$$f''(a) = (a+2)e^a.$$

Hence

$$\int_C \frac{ze^z}{(z-a)^3}\, dz = \pi i (a+2) e^a.$$

EXAMPLE 1.56

Evaluate
$$\int_{|z|=3} \frac{e^{2z}}{(z+1)^4}\,dz$$

Solution. The function $f(z) = e^{2z}$ is entire. The point $z = -1$ lies within the circle $|z| = 3$. Therefore, by Cauchy's integral formula, we have

$$f'''(-1) = \frac{3!}{2\pi i} \int_{|z|=3} \frac{e^{2z}}{(z+1)^4}\,dz.$$

But

$$f'(a) = 2e^{2z}, f''(z) = 4e^{2z}, \quad f'''(z) = 8e^{2z}$$

and so $f'''(-1) = 8\,e^{-2}$. Hence

$$\int_{|z|=3} \frac{e^{2z}}{(z+1)^4}\,dz = \frac{8\pi i\, e^{-2}}{3}.$$

EXAMPLE 1.57

Evaluate

$$I = \int_{|z+1-i|=2} \frac{z+4}{z^2+2z+5}\,dz$$

Solution. We have

$$I = \int_{|z+1-i|=2} \frac{z+4}{z^2+2z+5}\,dz$$

$$= \int_{|z+1-i|=2} \frac{z+4}{(z+1+2i)\,(z+1-2i)}\,dz$$

$$= \int_{|z+1-i|=2} \frac{f(z)}{(z+1-2i)}\,dz,$$

where $f(z) = \dfrac{z+4}{(z+1+2i)}$. The point $-1-2i$ lies outside the contour $|z+1-i| = 2$, whereas the point $-1+2i$ lies within the contour (infact taking $z = -1+2i$ in $|z+1-i|$, the value is less than 2). Hence, by Cauchy's integral formula, we have

$$I = 2\pi i\; f(-1+2i)$$

$$= 2\pi i\left(\frac{-1+2i+4}{-1+2i+1+2i}\right) = \frac{\pi}{2}(3+2i).$$

EXAMPLE 1.58

Evaluate, $\displaystyle\int_C \frac{dz}{(z-a)^n}$, $n = 2, 3$, where C is closed curve containing a.

Solution. Here $f(z) = 1$ so that $f'(z) = f''(z) = f'''(z) = 0$. By Cauchy's integral formula

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{dz}{(z-a)^{n+1}}.$$

Therefore,

$$\int_C \frac{dz}{(z-a)^2} = 2\pi i\ f'(a) = 0$$

$$\int_C \frac{dz}{(z-a)^3} = \frac{2\pi i}{2} f''(a) = 0.$$

EXAMPLE 1.59

Evaluate
$$\int_{|z|=\frac{1}{2}} \frac{3z^2+7z+1}{z+1}\,dz.$$

Solution. Let $f(z) = 3z^2 + 7z + 1$. Then $f(z)$ is analytic within $|z| = \frac{1}{2}$. The point $z = -1$ lies outside the curve $|z| = \frac{1}{2}$. The function $\frac{3z^2+7z+1}{z+1}$ is analytic within and on $|z| = \frac{1}{2}$. Hence, by Cauchy-Goursat theorem

$$\int_{|z|=\frac{1}{2}} \frac{3z^2+7z+1}{z+1}\,dz = 0.$$

EXAMPLE 1.60

Evaluate $\mathrm{I} = \int_{|z-1|=3} \frac{e^z}{(z+1)^2(z-2)}\,dz$

Solution. By partial fractions

$$\frac{1}{(z+1)^2(z-2)} = \frac{1}{9(z-2)} - \frac{1}{9(z+1)} - \frac{1}{3(z+1)^2}.$$

Hence

$$I = \frac{1}{9}\int_{|z-1|=3} \frac{e^z}{z-2}\,dz - \frac{1}{9}\int_{|z-1|=3} \frac{e^z}{z+1}\,dz - \frac{1}{3}\int_{|z-1|=3} \frac{e^z}{(z+1)^2}\,dz.$$

The function $f(z) = e^z$ is an entire function and the points $z = -1$ and $z = 2$ lies in $|z - 1| = 3$. Also $f'(z) = e^z$ and $f'(-1) = e^{-1}$. Hence, by Cauchy's integral formula,

$$\begin{aligned}\mathrm{I} &= \frac{1}{9}\cdot 2\pi i\ f(2) - \frac{1}{9}2\pi\ i\ f(-1) - \frac{1}{3}\cdot 2\pi i\ f'(-1)\\ &= \frac{2\pi i}{9}(e^2 - e^{-1} - 3e^{-1}) = \frac{2\pi i}{9}\left(e^2 - \frac{4}{e}\right).\end{aligned}$$

1.4 POWER SERIES REPRESENTATION OF AN ANALYTIC FUNCTION

Definition 1.30. A series of the form $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, where a_n and z_0 are fixed complex numbers and z is a complex variable, is called a *power series* in $(z - z_0)$.

The radius of the power series is given by

$$R = \frac{1}{\overline{\lim_{n\to\infty}}(|a_n|)^{1/n}}$$

or by

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|,$$

provided that the limit exists. If $R = 0$, the series converges only for $z = z_0$. It converges absolutely if $|z - z_0| < R$ and uniformly if $|z - z_0| \le R_0 < R$. The series diverges for $|z - z_0| > R$.

The circle $|z - z_0| = R$, $0 < R < \infty$, is called the *circle of convergence*.

Theorem 1.17. A power series represents an analytic function inside its circle of convergence.

Proof: Suppose the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges for $|z| < R$. Then if $\rho < R$, $a_n \rho^n$ is bounded and so $|a_n \rho^n| \le K$ for $K > 0$. Let $g(z) = \sum_{n=1}^{\infty} na_n z^{n-1}$. Then if $|z| < \rho$ and $|z| + |h| < \rho$, we have

$$\frac{f(z+h)-f(z)}{h} - g(z) = \sum_{n=0}^{\infty} a_n\left[\frac{(z+h)^n - z^n}{h} - n z^{n-1}\right].$$

But

$$\left|\frac{(z+h)^n - z^n}{h} - nz^{n-1}\right| \le \frac{(|z|+|h|)^n - |z|^n}{|h|} - n|z|^{n-1}.$$

Hence

$$\left|\frac{f(z+h)-f(z)}{h} - g(z)\right| \le \sum_{n=0}^{\infty}\frac{1}{\rho^n}\left[\frac{(|z|+|h|)^n - |z|^n}{|h|} - n|z|^{n-1}\right]$$

$$= K\left[\frac{1}{|h|}\left(\frac{\rho}{\rho-|z|-|h|} - \frac{\rho}{\rho-|z|}\right) - \frac{\rho}{(\rho-|z|)^2}\right]$$

$$= \frac{K\rho|h|}{(\rho-|z|-|h|)(\rho-|z|)^2} \to 0 \text{ as } h \to 0.$$

Hence $f(z)$ has the derivative $g(z)$ and so is analytic within the circle of convergence with radius R.

The converse of Theorem 1.17 is the following theorem due to Taylor.

Theorem 1.18. (Taylor). If $f(z)$ is analytic inside a disk $|z - z_0| < R$ with centre at z_0, then for all z in the disk

$$f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n,$$

where $f^{(n)}(z_0)$ represents nth derivative of $f(z)$ at z_0.

The coefficients $\dfrac{f^{(n)}(z_0)}{n!}$ are called *Taylor's coefficients*. The infinite series is convergent if $|z - z_0| < \delta$, where δ is the distance from z_0 to the nearest point of C. If $\delta_1 < \delta$, then the series converges uniformly in the region $|z - z_0| \le \delta_1$.

Proof: Choose $\delta_2 = \dfrac{\delta + \delta_1}{2}$ so that $0 < \delta_1 < \delta_2 < \delta$ (Fig. 1.12). Then, by the given hypothesis, $f(z)$ is analytic within and on a circle Γ defined by $|z - z_0| = \delta_2$.

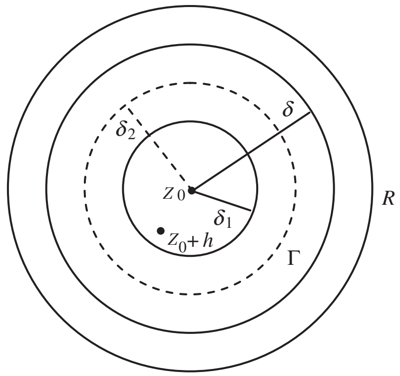


Figure 1.12

Let $z_0 + h$ be any point of the region defined by $|z - z_0| \le \delta_1$. Since $z_0 + h$ lies within the circle Γ, the Cauchy's integral formula yields

$$
\begin{aligned}
f(z_0 + h) &= \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - z_0 - h}\,dz \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z - z_0)\left(1 - \dfrac{h}{z - z_0}\right)}\,dz \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - z_0}\left[1 + \frac{h}{z - z_0} + \frac{h^2}{(z - z_0)^2} + \ldots + \frac{h^n}{(z - z_0)^n}\right. \\
&\quad \left. + \frac{h^{n+1}}{(z - z_0)^n (z - z_0 - h)}\right] dz \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - z_0}\,dz + \frac{h}{2\pi i}\int_\Gamma \frac{f(z)}{(z - z_0)^2}\,dz + \ldots + \frac{h^n}{2\pi i}\int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\,dz \\
&\quad + \frac{h^{n+1}}{2\pi i}\int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}(z - z_0 - h)}\,dz \\
&= f(z_0) + h\,f'(z_0) + \frac{h^2}{2!}f''(z_0) + \ldots + \frac{h^n}{n!}f^{(n)}(z_0) + \Delta_n
\end{aligned}
$$

where

$$\Delta_n = \frac{h^{n+1}}{2\pi i}\int_\Gamma \frac{f(z)\,dz}{(z - z_0)^{n+1}\,(z - z_0 - h)}.$$

Hence

$$f(z_0+h)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n+\Delta_n.$$

But $f(z)$ is bounded by virtue of its continuity. Thus, there exists a positive M such that $|f(z)|\le M$ on Γ. Further,

$$|z-z_0-h|\ge|z-z_0|-|h|>\delta_2-\delta_1$$

and since $|z-z_0|\le\delta_1$, we have

$$|h|=|z_0+h-z_0|\le\delta_1.$$

Since $\delta_1<\delta_2$, we get

$$\begin{aligned}|\Delta_n|&\le\frac{|h|^{n+1}}{|2\pi i|}\int_\Gamma\frac{|f(z)|}{|z-z_0|^{n+1}|z-z_0-h|}|dz|\\&\le\frac{M\,\delta_1^{n+1}}{2\pi\delta_2^{n+1}(\delta_2-\delta_1)}2\pi\delta_2\\&\le M\left(\frac{\delta_1}{\delta_2}\right)^n.\frac{\delta_1}{(\delta_2-\delta_1)}\to 0\ as\ n\to\infty.\end{aligned}$$

Hence

$$f(z_0+h)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n.$$

Substituting $z_0+h=z$, we get

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n. \tag{1.30}$$

Remark 1.8. (i) Theorem 1.18 was actually *invented by Cauchy* when he was in exile.
(ii) Substituting $z_0=0$, the Taylor's series reduces to

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n,$$

which is known as *Maclaurin's series*.
(iii) Using Cauchy's integral formula,

$$f^{(n)}(z_0)=\frac{n!}{2\pi i}\int_C\frac{f(z)}{(z-z_0)^{n+1}}dz,$$

the Taylor's series (1.30) reduces to

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n, \tag{1.31}$$

where

$$a_n=\frac{f^{(n)}(z_0)}{n!}=\frac{1}{2\pi i}\int\frac{f(z)}{(z-z_0)^{n+1}}dz.$$

Theorem 1.19. On the circumference of the circle of convergence of a power series, there must be at least one singular point of the function, represented by the series.

Proof: Suppose on the contrary that there is no singularity on the circumference $|z-z_0| = R$ of the circle of convergence of the power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n. \tag{1.32}$$

Then $f(z)$ is analytic in the disk $|z - z_0| < R + \varepsilon$, $\varepsilon > 0$. But, this implies that the series (1.32) converges in the disk $|z - z_0| < R + \varepsilon$. This contradicts the assumption that $|z - z_0| = R$ is the circle of convergence. Hence, there is at least one singular point of the function $f(z)$ on the circle of convergence of the power series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$.

EXAMPLE 1.61

If the function $f(z)$ is regular for $|z| < R$ and has the Taylor's expansion $\sum_{n=0}^{\infty} a_n z^n$, show that for $r < R$

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}.$$

Hence, show that if $|f(z)| \le M$, $|z| < R$, then

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \le M^2. \tag{1.33}$$

[The relation (1.33) is called *Parseval's inequality*].

Solution. Since $f(z)$ is regular in the region $|z| = r < R$, it has the Taylor's series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}, \; z = r\,e^{i\theta}.$$

If $\overline{a}_n$ enotes the conjugate of a_n, we have

$$f(\overline{z}) = \sum_{p=0}^{\infty} \overline{a}_p \, r^p e^{-i\,p\theta}.$$

Hence

$$|f(z)|^2 = f(z) f(\overline{z}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta} \sum_{p=0}^{\infty} \overline{a}_p \, r^p e^{-ip\theta}.$$

The two series for $f(z)$ and $f(\overline{z})$ are absolutely convergent and, hence, their product is uniformly convergent for the range $0 \le \theta \le 2\pi$. Thus, term-by-term integration is justified. On integration, all the terms for which $n \ne p$ vanish, for

$$\int_0^{2\pi} e^{il\theta} \, d\theta = 0, \quad l \ne 0$$

Hence, we have

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z)|^2\, d\theta = \frac{1}{2\pi}\sum_{n=0}^{\infty} a_n \bar{a}_n\, r^{2n} \int_0^{2\pi} d\theta$$

$$= \frac{1}{2\pi}\sum_{n=0}^{\infty} |a_n|^2\, r^{2n}\, 2\pi$$

$$= \sum_{n=0}^{\infty} |a_n|^2\, r^{2n}. \qquad (1.34)$$

If $|f(z)| \le M$, then (1.34) gives

$$\sum_{n=0}^{\infty} |a_n|^2\, r^{2n} = \frac{1}{2\pi}\int_0^{2\pi} |f(z)|^2\, d\theta$$

$$\le \frac{1}{2\pi}\int_0^{2\pi} M^2\, d\theta = M^2.$$

EXAMPLE 1.62

Show that a function which has no singularities in the finite part of the plane or at ∞ is a constant.

Solution. Since the function $f(z)$ has no singularities in the finite part of the plane, it can be expanded in the Taylor's series in any circle $|z| = K$ (arbitrarily large). Thus

$$f(z) = \sum_{r=0}^{\infty} A_r z^r$$

and so

$$f\left(\frac{1}{z}\right) = \sum_{r=0}^{\infty} A_r\, z^{-r}. \qquad (1.35)$$

Further, if $f(z)$ has no singularity at $z = \infty$, $f\left(\frac{1}{z}\right)$ has no singularity at $z = 0$. Since, $f(z)$ has no singularity in finite plane, $f\left(\frac{1}{z}\right)$ also has none in the finite plane. Hence, by Taylor's theorem,

$$f\left(\frac{1}{z}\right) = \sum_{r=0}^{\infty} B_r\, z^r \qquad (1.36)$$

From (1.35) and (1.36), we have

$$\sum_{r=0}^{\infty} B_r\, z^r = \sum_{r=0}^{\infty} A_r\, z^{-r}.$$

But this is not possible unless $B_r = A_r = 0$ for all values of r except zero in which case $A_0 = B_0$ and then $f(z) = A_0 = B_0 =$ constant.

EXAMPLE 1.63

If a function is analytic for all finite value of z as $|z| \to \infty$, and $|f(z)| = A|z|^k$, then show that $f(z)$ is a polynomial of degree less than or equal to k.

Solution. Let $f(z)$ be analytic in the finite part of z-plane and let $\lim_{|z|\to\infty} |f(z)| = A|z|^k$. We assume that $f(z)$ is analytic inside a circle $|z| = R$, where R is large but finite. Then $f(z)$ has Taylor's series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{1.37}$$

where

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^{n+1}}\, dz.$$

Hence

$$\begin{aligned}
|a_n| &\le \frac{1}{2\pi}\int \frac{|f(z)|}{|z|^{n+1}}|dz| \\
&= \frac{M}{2\pi\, R^{n+1}}\int_C |dz|, \quad M = \max|f(z)| \text{ on } C. \\
&= \frac{M}{2\pi R^{n+1}}.2\pi R \\
&= \frac{M}{R^n} = \frac{A|z|^k}{R^n} = \frac{AR^k}{R^n} \\
&= \frac{A}{R^{n-k}} \to 0 \; as \; R \to \infty \text{ for } n > k.
\end{aligned}$$

Thus, $a_n = 0$ for all $n > k$. Hence (1.37) implies that $f(z)$ is a polynomial of degree $\le k$.

EXAMPLE 1.64

Expand sin z in Taylor series about $z = \frac{\pi}{4}$.

Solution. We have $f(z) = \sin z$. So,

$$f'(z) = \cos z, f''(z) = -\sin z,$$
$$f'''(z) = -\cos z, f^{(4)}(z) = \sin z, \ldots$$

and

$$f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2},$$
$$f''\left(\frac{\pi}{4}\right) = \frac{-\sqrt{2}}{2}, f'''\left(\frac{\pi}{4}\right) = \frac{-\sqrt{2}}{2},$$
$$f^{(4)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \ldots$$

Hence,

$$\begin{aligned}
f(z) &= f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(z - \frac{\pi}{4}\right) + \frac{1}{2!}f''\left(\frac{\pi}{4}\right)\left(z - \frac{\pi}{4}\right)^2 + \frac{1}{3!}f'''\left(\frac{\pi}{4}\right)\left(z - \frac{\pi}{4}\right)^3 + \ldots \\
&= \frac{\sqrt{2}}{2}\left[1 + [z - (\pi/4)] - \frac{[z - (\pi/4)]^2}{2!} - \frac{[z - (\pi/4)]^3}{3!} + \ldots\right],
\end{aligned}$$

which is the required expansion.

Now suppose that $f(z)$ is not analytic in a disk but only in an *annular region* (*ring-shaped region*) bounded by two concentric circles C_1 and C_2 and is also analytic on C_1 and C_2. The function $f(z)$ can then be expressed in terms of two series by the following theorem known as *Laurent theorem.*

Theorem 1.20. (Laurent). Let $f(z)$ be analytic in the annular region bounded by two concentric circles C_1 and C_2 with centre z_0 and radii R_1 and R_2, respectively, with $0 < R_1 < R_2$. If z is any point of the annulus, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(\xi)d\xi}{(\xi - z_0)^{n+1}}, \quad n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(\xi)d\xi}{(\xi - z_0)^{1-n}}, \quad n = 1, 2, 3, \ldots$$

and integration over C_1 and C_2 is taken in anti-clockwise direction.

Proof: Since $f(z)$ is analytic on the circles and within the annular region between the two circles, the Cauchy integral formula yields

$$f(z) = \frac{1}{2\pi i}\left[\int_{C_2} \frac{f(\xi)}{\xi - z} d\xi - \int_{C_1} \frac{f(\xi)}{\xi - z} d\xi\right], \tag{1.38}$$

where z is any point in the region D (Fig. 1.13).

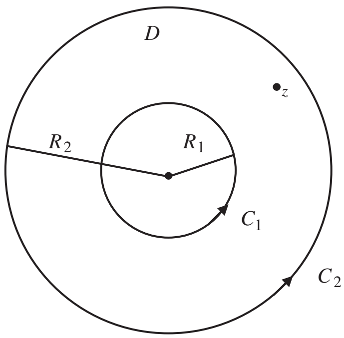


Figure 1.13

Now

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0 - (z - z_0)}$$

$$= \frac{1}{(\xi - z_0)\left(1 - \dfrac{z - z_0}{\xi - z_0}\right)} = \frac{1}{\xi - z_0}\left(1 - \frac{z - z_0}{\xi - z_0}\right)^{-1}$$

$$= \frac{1}{\xi - z_0}\left[1 + \frac{z - z_0}{\xi - z_0} + \left(\frac{z - z_0}{\xi - z_0}\right)^2 + \ldots + \left(\frac{z - z_0}{\xi - z_0}\right)^n + \ldots\right]$$

$$= \frac{1}{\xi - z_0}\sum_{n=0}^{\infty}\left(\frac{z - z_0}{\xi - z_0}\right)^n = \sum_{n=0}^{\infty}\frac{(z - z_0)^n}{(\xi - z_0)^{n+1}}.$$

Therefore,

$$\frac{1}{2\pi i}\int_{C_2}\frac{f(\xi)}{\xi - z}\,d\xi = \frac{1}{2\pi i}\int_{C_2} f(\xi)\sum_{n=0}^{\infty}\frac{(z - z_0)^n}{(\xi - z_0)^{n+1}}\,d\xi$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{C_2}\frac{f(\xi)\,(z - z_0)^n}{(\xi - z_0)^{n+1}}\,d\xi$$

$$= \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{1}{2\pi i}\int_{C_2}\frac{f(\xi)}{(\xi - z_0)^{n+1}}\,d\xi, \quad n = 0, 1, 2, \ldots.$$

Further,

$$-\frac{1}{\xi - z} = \frac{1}{z - z_0 - (\xi - z_0)} = \frac{1}{z - z_0}\left(1 - \frac{\xi - z_0}{z - z_0}\right)^{-1}$$

$$= \frac{1}{z - z_0}\left[1 + \frac{\xi - z_0}{z - z_0} + \left(\frac{\xi - z_0}{z - z_0}\right)^2 + \ldots\right]$$

$$= \frac{1}{z - z_0}\sum_{n=0}^{\infty}\left(\frac{\xi - z_0}{z - z_0}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{(\xi - z_0)^n}{(z - z_0)^{n+1}} = \sum_{n=1}^{\infty}\frac{(\xi - z_0)^{n-1}}{(z - z_0)^n}.$$

Therefore,

$$-\frac{1}{2\pi i}\int_{C_1}\frac{f(\xi)}{\xi - z}\,d\xi = \frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_{C_1} f(\xi)\,\frac{(\xi - z_0)^{n-1}}{(z - z_0)^n}\,d\xi$$

$$= \sum_{n=1}^{\infty}\frac{b_n}{(z - z_0)^n},$$

where

$$b_n = \frac{1}{2\pi i}\int_{C_1}\frac{f(\xi)}{(\xi - z_0)^{1-n}}\,d\xi.$$

Hence (1.38) becomes

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}, \tag{1.39}$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi,$$

$$b_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(\xi)}{(\xi - z_0)^{1-n}}\, d\xi.$$

Remark 1.9. Laurent's theorem is a generalization of Taylor's theorem. In fact, if $f(z)$ were analytic within and on C_2, then all the b_n are zero by Cauchy's theorem since the integrands are analytic within and on C_1. Also, then

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \quad n = 0, 1, 2, \ldots$$

EXAMPLE 1.65

Expand $\dfrac{z-1}{z^2}$ in a Taylor series in powers of $z - 1$ and determine the region of convergence.

Solution. We have

$$f(z) = \frac{z-1}{z^2} = \frac{1}{z} - \frac{1}{z^2},$$

$$f'(z) = -\frac{1}{z^2} + \frac{2}{z^3},$$

$$f''(z) = \frac{2z}{z^3} - \frac{3.2}{z^4},$$

$$\ldots\ldots\ldots\ldots$$

$$f^{(n)}(z) = \frac{(-1)^n n!}{z^{n+1}} + \frac{(-1)^{n+1}\,(n+1)!}{z^{n+2}}.$$

Hence

$$f(1) = 0, \quad \frac{f^{(n)}(1)}{n!} = (-1)^{n+1} n.$$

Hence

$$f(z) = f(1) + \sum_{n=1}^{\infty} \frac{f^{(n)}(1)}{n!} (z-1)^n$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} n (z-1)^n.$$

The region of convergence is $|\, z - 1 \,| < 1$.

EXAMPLE 1.66

Determine the two Laurent series expansions in power of z of the function

$$f(z) = \frac{1}{z(1+z^2)}.$$

Solution. The function ceases to be regular at $z = 0$ and $z = \pm i$.

When $0 < |z| < 1$, then

$$\begin{aligned} f(z) &= \frac{1}{z(1+z^2)} = \frac{1}{z}(1+z^2)^{-1} \\ &= \frac{1}{z}[1 - z^2 + z^4 - z^6 + \ldots] \\ &= \frac{1}{z} - z + z^3 - z^5 + \ldots \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n z^{2n-1}. \end{aligned}$$

When $|z| > 1$, then $|\frac{1}{z}| < 1$ and so

$$\begin{aligned} f(z) &= \frac{1}{z^3\left(1+\frac{1}{z^2}\right)} = \frac{1}{z^3}\left(1+\frac{1}{z^2}\right)^{-1} \\ &= \frac{1}{z^3}\left[1 - \frac{1}{z^2} + \frac{1}{z^4} - \ldots\right] \\ &= \frac{1}{z^3} - \frac{1}{z^5} + \frac{1}{z^7} - \ldots \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{z^{2n+1}}. \end{aligned}$$

EXAMPLE 1.67

Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in Taylor's/Laurent's series valid for the region

(i) $|z| < 1$, (ii) $1 < |z| < 3$, (*iii*) $|z| > 3$, (iv) $0 < |z+1| < 2$.

Solution. We have

$$f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}.$$

(i) When $|z| < 1$, we have

$$\begin{aligned} f(z) &= \frac{1}{2}(z+1)^{-1} - \frac{1}{2}(z+3)^{-1} \\ &= \frac{1}{2}(1+z)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1} \end{aligned}$$

$$= \frac{1}{2}[1 - z + z^2 - z^3 + \ldots] - \frac{1}{6}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \ldots\right]$$

$$= \frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 + \ldots$$

This is a Taylor's series valid for $|z| < 1$.

(ii) When $1 < |z| < 3$, we have for $|z| > 1$,

$$\frac{1}{2(z+1)} = \frac{1}{2z\left(1+\frac{1}{z}\right)} = \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \ldots\right)$$

$$= \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \ldots$$

and for $|z| < 3$,

$$\frac{1}{2(z+3)} = \frac{1}{6\left(1+\frac{z}{3}\right)} = \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \frac{z^3}{162} + \ldots$$

Hence the Laurent series for $f(z)$ for the annulus $1 < |z| < 3$ is

$$f(z) = \ldots - \frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \frac{z^3}{162} + \ldots$$

(iii) For $|z| > 3$, we have

$$f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right) - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= \frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \ldots$$

(iv) When $0 < |z+1| < 2$, we substitute $z + 1 = u$, then we have $0 < |u| < 2$ and

$$f(z) = \frac{1}{u(u+2)} = \frac{1}{2u\left(1+\frac{u}{2}\right)} = \frac{1}{2u}\left(1+\frac{u}{2}\right)^{-1}$$

$$= \frac{1}{2u} - \frac{1}{4} - \frac{u}{8} - \frac{u^2}{16} + \ldots$$

$$= \frac{1}{2(z+1)} - \frac{1}{4} + \frac{z+1}{8} - \frac{(z+1)^2}{16} + \ldots.$$

EXAMPLE 1.68

Obtain Taylor's/Laurent's series expansion for $f(z) = \dfrac{(z-2)(z+2)}{(z+1)(z+4)}$ which are valid

(i) When $|z| < 1$
(ii) When $1 < |z| < 4$
(iii) When $|z| > 4$.

Solution. (i) We have

$$f(z) = \frac{(z-2)(z+2)}{(z+1)(z+4)} = 1 - \frac{1}{z+1} - \frac{4}{z+4}.$$

When $|z| < 1$,

$$\begin{aligned}
f(z) &= 1-(1+z)^{-1} - \frac{4}{4}\left(1-\frac{z}{4}\right)^{-1} \\
&= 1-[1-z+z^2-\ldots]-\left[1-\frac{z}{4}+\left(\frac{z}{4}\right)^2-\ldots\right] \\
&= -1+(z-z^2+\ldots)+\left[\frac{z}{4}-\left(\frac{z}{4}\right)^2+\ldots\right] \\
&= -1+\sum_{n=1}^{\infty}(-1)^{n+1}(1+4^{-n})z^n,
\end{aligned}$$

which is a Maclaurine's series.

(ii) When $1 < |z| < 4$, we have $\dfrac{1}{|z|} < 1$ and $\dfrac{|z|}{4} < 1$. Thus

$$\begin{aligned}
f(z) &= 1-\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{4}{4}\left(1+\frac{z}{4}\right)^{-1} \\
&= \left[-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\ldots\right]-\left[-\frac{z}{4}+\frac{z^2}{4^2}-\frac{z^3}{4^3}+\ldots\right] \\
&= \sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{z^n}-\left(\frac{z}{4}\right)^n\right],
\end{aligned}$$

which is a Laurent series.

(iii) When $|z| > 4$, we have $\dfrac{4}{|z|} < 1$. Hence

$$\begin{aligned}
f(z) &= 1-\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{4}{z}\left(1+\frac{4}{z}\right)^{-1} \\
&= 1-\frac{1}{z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\ldots\right]-\frac{4}{z}\left[1-\frac{4}{z}+\frac{4^2}{z^2}-\ldots\right] \\
&= 1+\sum_{n=1}^{\infty}(-1)^n(1+4^n)\frac{1}{z^n},
\end{aligned}$$

which is again a Laurent's series.

EXAMPLE 1.69

Find series expansion of $f(z) = \dfrac{1}{(z-1)(z-2)}$ in the regions

(i) $1 < |z| < 2$, (ii) $|z| > 2$, (iii) $0 < |z-1| < 1$.

Solution. (i) We have

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}.$$

Now $|z| > 1$ implies $\dfrac{1}{|z|} < 1$ and $|z| < 2$ implies $\left|\dfrac{z}{2}\right| < 1$. Hence

$$\begin{aligned}
f(z) &= \frac{-1}{2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)} \\
&= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} \\
&= -\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\ldots\right) - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\ldots\right) \\
&= \ldots - \frac{1}{z^4} - \frac{1}{z^3} - \frac{1}{z^2} - \frac{1}{z} - \frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \ldots,
\end{aligned}$$

which is a Laurent's series

(ii) When $|z| > 2$, then $\left|\dfrac{2}{z}\right| < 1$ and so

$$\begin{aligned}
f(z) &= \frac{1}{z\left(1-\frac{2}{z}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)} \\
&= \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} \\
&= \frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}-\ldots\right) - \frac{1}{z}(1+z+z^2+\ldots) \\
&= \ldots + \frac{4}{z^3} + \frac{2}{z^2} - 1 - z - z^2 - \ldots
\end{aligned}$$

(iii) When $0 < |z-1| < 1$, we substitute $z - 1 = u$ and get $0 < |u| < 1$. Then

$$\begin{aligned}
f(z) &= \frac{1}{u-1} - \frac{1}{u} = -\frac{1}{1-u} - \frac{1}{u} \\
&= -(1-u)^{-1} - \frac{1}{u} = -(1+u+u^2+\ldots) - \frac{1}{u}
\end{aligned}$$

$$= -\frac{1}{u} - 1 - u - u^2 - \dots$$

$$= -\frac{1}{z-1} - 1 - (z-1) - (z-1)^2 - \dots,$$

which is a Laurent's series.

1.5 ZEROS AND POLES

Let $f(z)$ be analytic in a domain D. Then it can be expanded in Taylor's series about any point z_0 in D as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \ \dots$$

where

$$a_n = \frac{1}{2\pi i} \int \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

If $a_0 = a_1 = a_2 = \dots = a_{m-1} = 0$ but $a_m \neq 0$, then the first term in the above expansion is $a_m(z - z_0)^m$ and we say that $f(z)$ has a *zero of order m* at $z = z_0$.

If $f(z)$ satisfies the conditions of the Laurent's theorem, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

where

$$a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(\xi) d\xi}{(\xi - z_0)^{n+1}}, \quad n = 0, 1, 2, \dots$$

$$b_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(\xi) d\xi}{(\xi - z_0)^{1-n}}, \quad n = 1, 2, 3, \dots$$

The term $\sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$ is called the *principal part* of the function $f(z)$ at $z = z_0$.

Now there are the following three possibilities:

(i) If the principal part has only a *finite number of terms* given by

$$\frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \dots + \frac{b_n}{(z - a)^n}, b_n \neq 0,$$

then the point $z = z_0$ is called *a pole of order n*. If $n = 1$, then z_0 is called a *simple pole*.

(ii) If the principal part in Laurent expansion of $f(z)$ contains an *infinite number of terms*, then $z = z_0$ is called as *isolated essential singularity*.

(iii) If the principal part in Laurent expansion of $f(z)$ *does not contain any term*, that is, all b_n are zeros, then

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0^2) + \dots + a_n(z - z_0)^2 + \dots$$

and $z = z_0$ is called a *removable singularity*. Setting $f(z_0) = a_0$ makes $f(z)$ analytic at z_0.

From the Laurent expansion, it follows that a function $f(z)$ has a pole of order m at z_0 if and only if

$$f(z) = \frac{g(z)}{(z - z_0)^m},$$

where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$.

EXAMPLE 1.70

Find Laurent expansion of $\dfrac{z}{(z+1)(z+2)}$ about $z = -2$ and name the singularity.

Solution. We have

$$f(z) = \frac{z}{(z+1)(z+2)}.$$

Substitute $z + 2 = u$. Then

$$\begin{aligned}\frac{z}{(z+1)(z+2)} &= \frac{u-2}{(u-1)u} = \frac{2-u}{u(1-u)}\\ &= \frac{2-u}{u}(1+u+u^2+u^3+\ldots)\\ &= \frac{2}{u}+1+u+u^2+u^3+\ldots\\ &= \frac{2}{z+2}+1+(z+2)+(z+2)^2+\ldots\end{aligned}$$

Thus, the Laurent expansion about $z = -2$ has only one term in the principal part. Hence $z = -2$ is a *simple pole*.

EXAMPLE 1.71

Find Laurent's expansion of $f(z) = \dfrac{e^{2z}}{(z-1)^3}$ about $z = 1$ and name the singularity.

Solution. We have

$$f(z) = \frac{e^{2z}}{(z-1)^3}.$$

Substituting $z - 1 = u$, we get

$$\begin{aligned}\frac{e^{2z}}{(z-1)^3} &= \frac{e^{2(u+1)}}{u^3} = \frac{e^2}{u^3}.e^{2u}\\ &= \frac{e^2}{u^3}\left[1+2u+\frac{(2u)^2}{2!}+\frac{(2u)^3}{3!}+\ldots\right]\\ &= \frac{e^2}{u^3}+\frac{2e^2}{u^2}+\frac{2e^2}{u}+\frac{4e^2}{3}+\frac{2e^2u}{3}+\ldots\\ &= \frac{e^2}{(z-1)^3}+\frac{2e^2}{(z-1)^2}+\frac{2e^2}{z-1}+\frac{4e^2}{3}+\frac{2e^2(z-1)}{3}+\ldots.\end{aligned}$$

Thus, we obtain Laurent's series whose principal part consists of three terms. Hence, $f(z)$ has a pole of order 3 at $z = 1$. The function is analytic everywhere except for the pole of order 3 at $z = 1$. Hence, the series converges for all $z \neq 1$.

EXAMPLE 1.72

Find the Taylor and Laurent's series which represent the function $\dfrac{z^2-1}{z^2+5z+6}$ in the region

(i) $|z|<2$ (ii) $2<|z|<3$ (iii) $|z|>3$.

Solution. We have

$$f(z)=\frac{z^2-1}{(z+3)(z+2)}$$
$$=1+\frac{3}{z+2}-\frac{8}{z+3}.$$

(i) When $|z|<2$, we have $\left|\dfrac{z}{2}\right|<1$ and

$$f(z)=1+\frac{3}{2\left(1+\frac{z}{2}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}$$
$$=1+\frac{3}{2}\left(1+\frac{z}{2}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$
$$=1+\frac{3}{2}\left(1-\frac{z}{2}+\frac{z^2}{4}-\ldots\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}-\ldots\right)$$
$$=\left(1+\frac{3}{2}-\frac{8}{3}\right)-z\left(\frac{3}{4}-\frac{8}{9}\right)+z^2\left(\frac{3}{8}-\frac{8}{27}\right)-\ldots$$
$$=-\frac{1}{6}+\sum_{n=1}^{\infty}(-1)^n\left(\frac{3}{2^{n+1}}-\frac{8}{3^{n+1}}\right)z^n.$$

(ii) When $2<|z|<3$, we have $\left|\dfrac{2}{z}\right|<1$ and $\left|\dfrac{z}{3}\right|<1$. Hence

$$f(z)=1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}$$
$$=1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$
$$=1+\frac{3}{z}\left(1-\frac{2}{z}+\frac{4}{z^2}-\ldots\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}-\ldots\right)$$
$$=-\frac{5}{3}+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{8}{3^{n+1}}z^n+\sum_{n=1}^{\infty}\frac{3(-2)^{n+1}}{z^n},$$

which is a Laurent's series

(iii) When $|z| > 3$, we have $\left|\frac{3}{z}\right| < 1$. Hence

$$f(z) = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \frac{4}{z^2} - \ldots\right) - \frac{8}{z}\left(1 - \frac{3}{z} + \frac{9}{z^2} - \ldots\right)$$

$$= 1 + \sum_{n=1}^{\infty}\left[(-1)^n \left\{\frac{8(3)^{n-1} - 3(2)^{n-1}}{z^n}\right\}\right],$$

which is a Laurent's series.

EXAMPLE 1.73

Find the singularities of $f(z) = \dfrac{z}{(z^2+4)^2}$ and indicate the character of the singularities.

Solution. We have

$$f(z) = \frac{z}{(z^2+4)^2} = \frac{z}{[(z+2i)(z-2i)]^2}$$

$$= \frac{z}{(z+2i)^2 (z-2i)^2}.$$

Since $\lim\limits_{z\to 2i} (z-2i)^2 f(z) = \lim\limits_{z\to 2i} \dfrac{z}{(z+2i)^2} = \dfrac{1}{8i} \neq 0$, it follows that $z = 2i$ is a pole of order 2. Similarly, $z = -2i$ is a pole of order 2. Further, we can find δ such that no other singularity other than $z = 2i$ lies inside the circle $|z - 2i| = \delta$ (for example, we may take $\delta = 1$). Hence $z = 2i$ is an isolated singularity. Similarly, $z = -2i$ is also an isolated singularity.

EXAMPLE 1.74

Find the nature and the location of the singularities of $f(z) = \dfrac{1}{z(e^z - 1)}\cdot$ Show that if $0 < |z| < 2\pi$, the function can be expanded in Laurent's series.

Solution. We have

$$f(z) = \frac{1}{z(e^z - 1)}.$$

The function ceases to be regular at $z = 0$ and $e^z - 1 = 0$, that is, $e^z = 1$ or for $e^z = e^{\pm 2n\pi i}$ or for $z = \pm 2n\pi i$, $n = 0, \pm 1, \pm 2, \ldots$ Thus, $z = 0$ is a *double pole* (pole of order 2). The other singularities are simple poles. Hence, the function can be expanded in Laurent's series in the annulus $0 < |z| < 2\pi$. We note that

$$f(z) = \frac{1}{z\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots - 1\right)}$$

$$= \frac{1}{z^2\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \ldots\right)}$$

$$= \frac{1}{z^2}\left[1+\frac{z}{2!}+\frac{z^2}{3!}+\ldots\right]^{-1}$$

$$= \frac{1}{z^2}\left[1-\left(\frac{z}{2!}+\frac{z^2}{3!}+\ldots\right)+\left(\frac{z}{2!}+\frac{z^2}{3!}+\ldots\right)^2+\ldots\right]$$

$$= \frac{1}{z^2}\left[1-\frac{z}{2!}+\left(\frac{1}{4}-\frac{1}{6}\right)z^2+z^3\left(\frac{1}{24}+\frac{1}{6}-\frac{1}{8}\right)+\ldots\right]$$

$$= \frac{1}{z^2}-\frac{1}{2z}+\frac{1}{12}-\frac{1}{120}z+\ldots$$

EXAMPLE 1.75

Show that $\frac{e^z}{z^3}$ has a pole of order 3 at $z = 0$.

Solution. We have

$$\frac{e^z}{z^3} = \frac{1}{z^3}\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots\right)$$

$$= \frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{2!z}+\frac{1}{3!}+\frac{1}{4!}z+\ldots$$

Thus, the principal part of the Laurent expansion consists of three terms and so $\frac{e^z}{z^3}$ has a pole of order 3 at $z = 0$.

EXAMPLE 1.76

Show that $z \sin \frac{1}{z}$ has essential singularity at $z = 0$.

Solution. We have

$$z\sin\frac{1}{z} = z\left\{\frac{1}{z}-\frac{1}{3!z^3}+\frac{1}{5!z^5}-\ldots\right\}$$

$$= 1-\frac{1}{3!z^2}+\frac{1}{5!z^4}+\ldots$$

Since the series *does not terminate*, $z = 0$ is an *essential singularity*.

Definition 1.31. A function $f(z)$ is said to be *meromorphic* if it is analytic in the finite part of the plane except at a finite number of poles.

1.6 RESIDUES AND CAUCHY'S RESIDUE THEOREM

Definition 1.32. Let the Laurent series expansion of a function $f(z)$ at isolated singularity z_0 be

$$f(z) = \sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}+\sum_{n=0}^{\infty}a_n(z-z_0)^n \tag{1.40}$$

The coefficient b_1, in the principal part of the expansion, given by

$$b_1 = \frac{1}{2\pi i}\int_C f(\xi)\,d\xi \tag{1.41}$$

for the contour C: $|z - z_0| = r < R$ is called *residue* of $f(z)$ at z_0 and is denoted by Res(z_0).

The residue of $f(z)$ at $z = \infty$ is defined by

$$-b_1 = \frac{-1}{2\pi i}\int_C f(\xi)\, d\xi.$$

It is the coefficient of $\frac{1}{z}$ with its sign changed in the expansion of $f(z)$ in the neighbourhood of $z = \infty$.

If $f(z)$ has a pole of order m at z_0, then the Laurent expansion of $f(z)$ is

$$f(z) = \frac{b_m}{(z-z_0)^m} + \frac{b_{m-1}}{(z-z_0)^{m-1}} + \ldots + \frac{b_1}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0^2) + \ldots$$

Multiplying both sides by $(z - z_0)^m$, we have

$$(z-z_0)^m f(z) = b_m + b_{m-1}(z-z_0) + \ldots + b_1(z-z_0)^{m-1} + a_0(z-z_0)^m + a_1(z-z_0)^{m+1} + \ldots,$$

which is Taylor's series of the analytic function $(z - z_0)^m f(z)$. Differentiating both sides $m - 1$ times with respect to z, we have

$$\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)] = b_1(m-1)! + m(m-1)\ldots 2a_0(z-z_0) + \ldots$$

Letting $z \to z_0$, we get

$$\lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)] = b_1(m-1)!$$

and so

$$b_1 = \text{Res}(z_0) = \frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]. \tag{1.42}$$

If z_0 is a simple pole, that is, $m = 1$, then

$$\text{Res}(z_0) = b_1 = \lim_{z\to z_0}(z-z_0)f(z). \tag{1.43}$$

If

$$f(z) = \frac{p(z)}{q(z)},$$

where $p(z)$ and $q(z)$ are analytic at $z = z_0$, $p(z_0) \neq 0$ and $q(z)$ has a simple zero at z_0, that is, $f(z)$ has a simple pole at z_0. Then $q(z) = (z - z_0)\, g(z)$, $g(z_0) \neq 0$ and $q(z_0) = 0$. Hence (1.43) reduces to

$$\text{Res}(z_0) = \lim_{z\to z_0}(z-z_0)\, f(z) = \lim_{z\to z_0}(z-z_0)\frac{p(z)}{q(z)} = \lim_{z\to z_0}\frac{p(z)}{\dfrac{q(z)-q(z_0)}{z-z_0}} = \frac{p(z_0)}{q'(z_0)} \tag{1.44}$$

Thus, the residues at poles can be calculated using the formulas (1.42), (1.43), and (1.44).

EXAMPLE 1.77

Find residues of

(i) $f(z) = \dfrac{z^2 - 2z}{(z+1)^2\,(z^2+4)}$

(ii) $f(z) = \dfrac{e^{z^2}}{(z-i)^3}$

at all its poles.

Solution. (i) The function $f(z)$ has a pole of order 2 at $z = -1$ and simple poles at $z = \pm 2i$. Therefore,

$$\begin{aligned}\text{Res}(-1) &= \lim_{z\to -1} \frac{d}{dz}\left[(z+1)^2 \frac{z^2-2z}{(z+1)^2(z^2+4)}\right]\\ &= \lim_{z\to -1} \frac{(z^2+4)\,(2z-2)-(z^2-2z)(2z)}{(z^2+4)^2}\\ &= -\frac{14}{25}\end{aligned}$$

$$\begin{aligned}\text{Res}(2i) &= \lim_{z\to 2i} (z-2i)\frac{z^2-2z}{(z+1)^2\,(z^2+4)}\\ &= \lim_{z\to 2i} \frac{z^2-2z}{(z+1)^2(z+2i)} = \frac{7+i}{25}\end{aligned}$$

$$\begin{aligned}\text{Res}(-2i) &= \lim_{z\to -2i} (z+2i)\frac{z^2-2z}{(z+1)^2\,(z-2i)\,(z+2i)}\\ &= \frac{7-i}{25}\end{aligned}$$

(ii) $f(z)$ has a pole of order 3 at $z = i$. Hence

$$\begin{aligned}\text{Res}(i) &= \frac{1}{2!}\lim_{z\to i}\frac{d^2}{dz^2}[(z-i)^3\, f(z)]\\ &= \frac{1}{2}\lim_{z\to i}\frac{d^2}{dz^2}(e^{z^2})\\ &= \lim_{z\to i}[2z^2 e^{z^2} + e^{z^2}]\\ &= -\frac{1}{e}.\end{aligned}$$

EXAMPLE 1.78

Find the residue of $f(z) = \cot z$ at its poles.

Solution. We have

$$f(z) = \cot z = \frac{\cos z}{\sin z}.$$

The poles of $f(z)$ are given by $\sin z = 0$. Thus $z = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$ are the simple poles. Using formula (1.44), the residue at $z = n\pi$ is given by

$$\text{Res}(n\pi) = \lim_{z\to n\pi} \frac{\cos z}{\frac{d}{dz}(\sin z)} = \lim_{z\to n\pi} \frac{\cos z}{\cos z} = 1.$$

EXAMPLE 1.79

Find the residue at each pole of $f(z) = \dfrac{ze^{iz}}{z^2 + a^2}$.

Solution. We have

$$f(z) = \frac{ze^{iz}}{(z+ai)\,(z-ai)}$$

Therefore, $f(z)$ has simple poles at $z = \pm\, ai$. Now

$$\begin{aligned}\text{Res}(ai) &= \lim_{z\to ai} (z - ai) f(z)\\ &= \lim_{z\to ai} (z - ai)\frac{ze^{iz}}{(z+ai)\,(z-ai)}\\ &= \lim_{z\to ai} \frac{ze^{iz}}{z+ai} = \frac{e^{-a}}{2},\\ \text{Res}(-ai) &= \lim_{z\to -ai} (z+ai)\, f(z)\\ &= \lim_{z\to -ai} \frac{ze^{iz}}{z-ai} = \frac{e^{a}}{2}.\end{aligned}$$

EXAMPLE 1.80

Find the residue of $f(z) = \dfrac{1-e^{2z}}{z^4}$ at its poles.

Solution. The function $f(z)$ has a pole of order 4 at $z = 0$. Therefore,

$$\begin{aligned}\text{Res}(0) &= \frac{1}{3!}\lim_{z\to 0}\frac{d^3}{dz^3}[(z-0)^4\; f(z)]\\ &= \frac{1}{6}\lim_{z\to 0}\frac{d^3}{dz^3}[1-e^{2z}]\\ &= \frac{1}{6}\lim_{z\to 0}\frac{d^3}{dz^3}\left[1-\left\{1+2z+4z^2+8z^3+\frac{16z^4}{4!}+\ldots\right\}\right]\\ &= \frac{1}{6}\lim_{z\to 0}\frac{d^3}{dz^3}\left[-2z-2z^2-\frac{8}{6}z^3-\frac{16}{24}z^4-\ldots\right]\\ &= \frac{1}{6}\lim_{z\to 0}\left[-8-\frac{294}{24}z-\ldots\right] = -\frac{4}{3}.\end{aligned}$$

EXAMPLE 1.81

Find the residues of

$$f(z) = \frac{z^3}{(z-1)\,(z-2)\,(z-3)}.$$

at $z = 1, 2$, and 3 and ∞ and show that their sum is zero.

Solution. The function

$$f(z) = \frac{z^3}{(z-1)(z-2)(z-3)}$$

has simple poles at $z = 1, 2$, and 3. Now

$$\text{Res}(1) = \lim_{z\to 1}(z-1)\, f(z) = \lim_{z\to 1}\frac{z^3}{(z-2)(z-3)} = \frac{1}{2}$$

$$\text{Res}(2) = \lim_{z\to 2}(z-2)\, f(z) = \lim_{z\to 2}\frac{z^3}{(z-1)(z-3)} = -8$$

$$\text{Res}(3) = \lim_{z\to 3}(z-3)\, f(z) = \lim_{z\to 3}\frac{z^3}{(z-1)(z-2)} = \frac{27}{2}\cdot$$

To find residue at ∞, we expand $f(z)$ in the neighbourhood of $z = \infty$ as follows:

$$\begin{aligned} f(z) &= \frac{z^3}{z^3\left(1-\frac{1}{z}\right)\left(1-\frac{2}{z}\right)\left(1-\frac{3}{z}\right)} \\ &= \left(1-\frac{1}{z}\right)^{-1}\left(1-\frac{2}{z}\right)^{-1}\left(1-\frac{3}{z}\right)^{-1} \\ &= 1+\frac{6}{z}+ \textit{higher powers of } \frac{1}{z}. \end{aligned}$$

Now residue at ∞ is coefficient of $\frac{1}{z}$ with sign changed. Thus Res(∞) = –6. Hence, the sum of the residues equals $\frac{1}{2}-8+\frac{27}{2}-6 = 0.$

To compute the values of integrals in our study, we shall require the following theorem.

Theorem 1.21. (Cauchy's Residue Theorem). If $f(z)$ is analytic within and on a closed contour C except at finitely many poles lying in C, then

$$\int_C f(z)\, dz = 2\pi i\, \Sigma\, R,$$

where ΣR denotes the sum of residues of $f(z)$ at the poles within C.

Proof: Let $z_1, z_2, \ldots, z_n$ be the n poles lying in C. Let $C_1, C_2, \ldots, C_n$ be the circles with centre $z_1, z_2, \ldots, z_n$ and radius ρ such that all these circles lie entirely within C and do not overlap (Fig. 1.14)

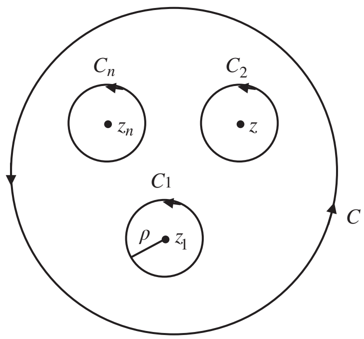


Figure 1.14

Then $f(z)$ is analytic in the region between C and the circles. Hence, by Cauchy-Goursat theorem

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \ldots + \int_{C_n} f(z)\,dz. \tag{1.45}$$

If $f(z)$ has a pole of order m_1 at $z = z_1$, then

$$f(z) = \phi_1(z) + \sum_{r=1}^{m_1} \frac{b_r}{(z-z_1)^r},$$

where $\phi_1(z)$ is regular within and on C_1. Therefore,

$$\int_{C_1} f(z)\,dz = \int_{C_1} \phi_1(z)\,dz + \int_{C_1} \frac{b_1}{(z-z_1)}\,dz + \int_{C_1} \frac{b_2}{(z-z_2)^2}\,dz + \ldots + \int_{C_1} \frac{bm_1}{(z-z_1)^r}\,dz. \tag{1.46}$$

Since $f(z)$ is analytic within and on C_1, by Cauchy-Goursat theorem

$$\int_{C_1} \phi_1(z)\,dz = 0.$$

Moreover, substituting $z - z_1 = \rho\, e^{i\theta}$, we have

$$\int_{C_1} \frac{bm_1}{(z-z_1)^r}\,dz = \int_0^{2\pi} \frac{bm_1 .\rho i e^{i\theta}}{e^{m_1}\, e^{m_1 i\theta}}\,d\theta$$

$$= \frac{ibm_1}{e^{m_1-1}} \int_0^{2\pi} e^{-(m_1-1)i\theta}\,d\theta = 0 \quad \text{for } m_1 \neq 1$$

and

$$\int_{C_1} \frac{b_1}{(z-z_1)}\,dz = \int_0^{2\pi} \frac{b_1 \rho i e^{i\theta}\,d\theta}{\rho e^{i\theta}} = 2\pi i\, b_1.$$

Hence (1.46) reduces to

$$\int_{C_1} f(z)\,dz = 2\pi i R_1,$$

where R_1 is the residue of $f(z)$ at $z = z_1$. Similarly,

$$\int_{C_2} f(z)\,dz = 2\pi i\, R_2,$$

....

$$\int_{C_n} f(z)\,dz = 2\pi i\, R_n,$$

where R_i is the residue of $f(z)$ at $z = z_i$. Hence (1.45) becomes

$$\int_C f(z)\,dz = 2\pi i(R_1 + R_2 + \ldots + R_n) = 2\pi i\, \Sigma R,$$

where $\Sigma R = R_1 + R_2 + \ldots + R_n$.

Remark 1.10. In the assumptions of Cauchy's integral formula, $f(z)$ is assumed to be analytic within and on a closed curve. Therefore, $\frac{f(z)}{z-a}$ has a simple pole at $z = a$. Then

$$\text{Res}(a) = \lim_{z\to a} (z-a)\frac{f(z)}{(z-a)} = f(a).$$

Hence, by Cauchy's Residue theorem, we have

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\ f(a),$$

that is,

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz.$$

It follows, therefore, that Cauchy's integral formula is a particular case of Cauchy's Residue theorem.

EXAMPLE 1.82

Evaluate

$$\int_C \frac{dz}{z^2(z+1)(z-1)}, \quad C:|z|=3.$$

Solution. The integrand has simple poles at $z = 1$ and $z = -1$ and double poles at $z = 0$ lying in C. Therefore,

$$\begin{aligned}
\text{Res}(1) &= \lim_{z\to 1}(z-1)f(z)\\
&= \lim_{z\to 1}\frac{1}{z^2(z+1)} = \frac{1}{2},\\
\text{Res}(-1) &= \lim_{z\to -1}(z+1)f(z)\\
&= \lim_{z\to -1}\frac{1}{z^2(z-1)} = -\frac{1}{2},\\
\text{Res}(0) &= \lim_{z\to 0}\frac{d}{dz}[(z-0)^2 f(z)]\\
&= \lim_{z\to 0}\frac{d}{dz}\left[\frac{1}{(z+2)(z-1)}\right]\\
&= \lim_{z\to 0}\frac{-2z-1}{(z^2+z-2)^2} = \frac{-1}{4}.
\end{aligned}$$

Hence, by Cauchy-Residue theorem,

$$\int_C \frac{dz}{z^2(z+1)(z-1)} = 2\pi i\left[\frac{1}{2}-\frac{1}{2}-\frac{1}{4}\right] = -\frac{\pi i}{2}.$$

EXAMPLE 1.83

Evaluate the integral

$$\int_{|z|=1} \frac{4z^2-4z+1}{(z-2)(4+z^2)}\,dz.$$

Solution. Let

$$f(z) = \frac{4z^2-4z+1}{(z-2)(4+z^2)}.$$

The poles of $f(z)$ are $z = 2$, $z = \pm 2i$. We note that none of these poles lie in the curve $|z| = 1$. Thus the function is analytic within and on $|z| = 1$. Hence, by Cauchy-Goursat theorem,

$$\int_{|z|=1} \frac{4z^2 - 4z + 1}{(z-2)(4+z^2)}\, dz = 0.$$

EXAMPLE 1.84

Evaluate $\int_C \frac{dz}{(z^2+4)^2}$, where C is the curve $|z - i| = 2$.

Solution. Let

$$f(z) = \frac{1}{(z^2+4)^2} = \frac{1}{(z+2i)^2\,(z-2i)^2}.$$

Thus $f(z)$ has two double poles at $z = 2i$ and $z = -2i$, out of which only $z = 2i$ lies within $|z - i| = 2$. Now

$$\text{Res}(2i) = \lim_{z\to 2i} \frac{d}{dz}[(z-2i)^2 . f(z)]$$

$$= \lim_{z\to 2i} \frac{d}{dz}\left[\frac{1}{(z+2i)^2}\right]$$

$$= \lim_{z\to 2i} \left[\frac{-2z-4i}{(z+2i)^4}\right] = \frac{-i}{32}.$$

Hence, by Cauchy's Residue theorem, we have

$$\int_C \frac{dz}{(z^2+4)^2} = 2\pi i\left(-\frac{i}{32}\right) = \frac{\pi}{16}.$$

EXAMPLE 1.85

Evaluate

$$\int_C \frac{1-\cos 2(z-3)}{(z-3)^3}\, dz,$$

where C is the curve $|z - 3| = 1$.

Solution. Expanding $\cos 2(z - 3)$, we have

$$f(z) = \frac{1-\cos 2(z-3)}{(z-3)^3}$$

$$= \frac{1}{(z-3)^3}\left[1 - 1 + \frac{4(z-3)^2}{2!} - \frac{16(z-3)^4}{4!} + \dots\right]$$

$$= \frac{2}{z-3} - \frac{16}{4!}(z-3) + \dots.$$

Thus $f(z)$ has a simple pole at $z = 3$. The Laurent's series is in the power of $z - 3$. The coefficient of $\frac{1}{z-3}$ is 2. Hence, the residue of $f(z)$ at $z = 3$ is 2 and so by Cauchy Residue theorem

$$\int_C f(z)dz = 2\pi i(2) = 4\pi i.$$

EXAMPLE 1.86

Evaluate

$$\int_{|z|=3} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)}\, dz.$$

Solution. The integrand has simple pole at $z = 2$ and a pole of order 2 at z = 1. But these poles lie within $|z| = 3$. Now

$$\text{Res}(2) = \lim_{z\to 2}(z-2)\, f(z) = \lim_{z\to 2} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2}$$
$$= \sin 4\pi + \cos 4\pi = 1,$$
$$\text{Res}(1) = \lim_{z\to 1} \frac{d}{dz}[(z-1)^2 f(z)] = \lim_{z\to 1} \frac{d}{dz}\left[\frac{i \sin \pi^2 + \cos \pi z^2}{z-2}\right]$$
$$= \lim_{z\to 1}\left[\frac{(z-2)\,(2\pi z \cos \pi z^2 - 2\pi z \sin \pi z^2) - (\sin \pi z^2 + \cos \pi z^2)}{(z-2)^2}\right]$$
$$= 2\pi + 1.$$

Hence, by Cauchy's Residue theorem, we have

$$\int_{|z|=3} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2\,(z-2)} dz = 2\pi i(2\pi + 2) = 4\pi i(\pi + 1).$$

EXAMPLE 1.87

Evaluate $I = \int_{|z|=3} \frac{z \sec z}{(1-z)^2}\, dz$

Solution. The integrand has a double pole at $z = 1$, which lies within the contour $|z| = 3$. Now

$$\text{Res}(1) = \lim_{z\to 1} \frac{d}{dz}\left[(z-1)^2 \frac{z \sec z}{(1-z)^2} dz\right]$$
$$= \lim_{z\to 1} \frac{d}{dz}[z \sec z]$$
$$= \lim_{z\to 1}[z \sec z \tan z + \sec z] = \sec 1[1 + \tan 1].$$

Hence, by Cauchy's Residue theorem

$$I = 2\pi i[\sec 1(1 + \tan 1)].$$

EXAMPLE 1.88

Evaluate $\int_{|z-1|=2} \frac{dz}{z^2 \sinh z}.$

Solution. Since

$$\sin hz = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \dots,$$

$$f(z) = \frac{1}{z^2 \sinh z} = \frac{1}{z^2\left[z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \dots\right]}$$

$$= \frac{1}{z^3}\left[1 + \left(\frac{z^2}{3!} + \frac{z^4}{5!} + \dots\right)\right]^{-1}$$

$$= \frac{1}{z^3}\left[1 - \left(\frac{z^2}{3!} + \frac{z^4}{5!} + \dots\right) + \left(\frac{z^2}{3!} + \frac{z^4}{5!} + \dots\right)^2 - \dots\right]$$

$$= \frac{1}{z^3}\left[1 - \frac{z^2}{6} - \frac{z^4}{120} + \frac{z^4}{36} + \dots\right]$$

$$= \frac{1}{z^3}\left[1 - \frac{z^2}{6} - \left(\frac{1}{36} - \frac{1}{120}\right)z^4 + \dots\right]$$

$$= \frac{1}{z^3} - \frac{1}{6z} + \frac{7}{360}z^4 - \dots$$

The coefficient of $\frac{1}{z}$ in this Laurent series in the powers of z is $-\frac{1}{6}$. Hence, residue at the pole $z = 0$ is

$$\text{Res}(0) = -\frac{1}{6}.$$

Hence, by Cauchy's Residue theorem,

$$I = 2\pi i\left(-\frac{1}{6}\right) = -\frac{\pi i}{3}.$$

1.7 EVALUATION OF REAL DEFINITE INTEGRALS

We shall now discuss the application of Cauchy's Residue theorem to evaluate real definite integrals.

(A) Integration Around the Unit Circle

We consider the integrals of the type

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta, \tag{1.47}$$

where the integrand is a rational function of sin θ and cos θ. Substitutet $z = e^{i\theta}$. Then, $dz = i\, e^{i\theta}\, d\theta = iz\, d\theta$ and

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

Thus (1.47) converts into the integral

$$\int_C \phi(z)dz, \tag{1.48}$$

where $\phi(z)$ is a rational function of z and C is the unit circle $|z| = 1$. The integral (1.48) can be solved using Cauchy's Residue theorem.

EXAMPLE 1.89

Show that

$$I = \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}, \quad a > b > 0.$$

Solution. Substituting $z = e^{i\theta}$, we get $d\theta = \dfrac{dz}{i\,z}$ and so

$$I = \frac{1}{i} \int_{|z|=1} \frac{dz}{z\left[a + \frac{b}{2}\left(z + \frac{1}{z}\right)\right]}$$

$$= \frac{2}{i} \int_{|z|=1} \frac{dz}{bz^2 + 2az + b}.$$

The poles of the integrand are given by

$$z = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b} = \frac{-a \pm \sqrt{a^2 - b^2}}{b}.$$

Thus the poles are

$$\alpha = \frac{-a + \sqrt{a^2 - b^2}}{b} \quad \text{and} \quad \beta = \frac{-a - \sqrt{a^2 - b^2}}{b}$$

Since $a > b > 0$, $|\beta| > 1$. But $|\alpha\beta| = 1$ (product of roots) so that $|\alpha| < 1$. Hence, $z = \alpha$ is the only simple pole lying within $|z| = 1$. Further

$$\text{Res}(\alpha) = \lim_{z \to \alpha} (z - \alpha).\frac{2}{bi(z - \alpha)(z - \beta)}$$

$$= \frac{2}{bi(\alpha - \beta)} = \frac{1}{i\sqrt{a^2 - b^2}}.$$

Hence

$$I = 2\pi i\left(\frac{1}{i\sqrt{a^2 - b^2}}\right) = \frac{2\pi}{\sqrt{a^2 - b^2}}.$$

EXAMPLE 1.90

Use calculus of residues to show that

$$\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta}\, d\theta = \frac{\pi}{6}.$$

Solution. We have

$$\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta}\, d\theta = \text{real part of} \int_0^{2\pi} \frac{e^{2i\theta}}{5 + 4\cos\theta}\, d\theta.$$

Now substituting $z = e^{i\theta}$, we get

$$\int_0^{2\pi} \frac{e^{2i\theta}}{5+4\cos\theta}\, d\theta$$

$$= \int_{|z|=1} \frac{z^2}{5+2\left(z+\frac{1}{z}\right)} \cdot \frac{dz}{iz} = \frac{1}{i}\int_{|z|=1} \frac{z^2}{2z^2+5z+2}\, dz$$

$$= \frac{1}{i}\int_{|z|=1} \frac{z^2}{(2z+1)(z+2)}\, dz = \frac{1}{2i}\int_{|z|=1} \frac{z^2}{\left(z+\frac{1}{2}\right)(z+2)}\, dz.$$

The integrand has simple poles at $z = -\frac{1}{2}$ and $z = -2$ of which only $z = -\frac{1}{2}$ lies inside $|z| = 1$. Now

$$\text{Res}(-\tfrac{1}{2}) = \lim_{z\to -\frac{1}{2}} \left(z+\frac{1}{2}\right) f(z) = \lim_{z\to -\frac{1}{2}} \frac{z^2}{2i(z+2)}$$

$$= \frac{1}{12i}.$$

Hence

$$\int_0^{2\pi} \frac{e^{2i\theta}}{5+4\cos\theta} = 2\pi i . \frac{1}{12i} = \frac{\pi}{6}.$$

Equating real and imaginary parts, we have

$$\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta = \frac{\pi}{6} \quad \text{and} \quad \int_0^{2\pi} \frac{\sin 2\theta}{5+4\cos\theta}\, d\theta = 0.$$

EXAMPLE 1.91

Show that

$$\int_0^{2\pi} e^{\cos\theta} \cos(n\theta - \sin\theta)\, d\theta = \frac{2\pi}{n!}.$$

Solution. The given integral is the real part of the integral

$$\int_0^{2\pi} e^{\cos\theta} e^{-(n\theta-\sin\theta)i}\, d\theta = \int_0^{2\pi} e^{\cos\theta + i\sin\theta} . e^{-in\theta}\, d\theta$$

$$= \int_0^{2\pi} e^{e^{i\theta}} . e^{-in\theta}\, d\theta = \frac{1}{i}\int_{|z|=1} \frac{e^z}{z^{n+1}}\, dz, \quad z = e^{i\theta}.$$

The integrand has a pole of order $n + 1$ at $z = 0$ which lies in $|z| = 1$. Then

$$\text{Res}(0) = \lim_{z\to 0} \frac{1}{n!} \frac{d^n}{dz^n} \left\{ z^{n+1} . \frac{e^z}{z^{n+1}} \right\}$$

$$= \lim_{z\to 0} \frac{1}{n!} \frac{d^n}{dz^n} \{e^z\} = \frac{1}{n!}.$$

Hence

$$I = 2\pi i \frac{1}{i} \cdot \frac{1}{n!} = \frac{2\pi}{n!}.$$

Equating real and imaginary parts, we get

$$\int_0^{2\pi} e^{\cos\theta} \cos(n\theta - \sin\theta) d\theta = \frac{2\pi}{n!}$$

and

$$\int_0^{2\pi} e^{\cos\theta} \sin(n\theta - \sin\theta)\, d\theta = 0.$$

EXAMPLE 1.92

Show that

$$\int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta}\, d\theta = \frac{2\pi}{b^2}\{a - \sqrt{a^2 - b^2}\}, 0 < b < a.$$

Solution. Let

$$I = \int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta}\, d\theta.$$

Substitute $z = e^{i\theta}$ so that $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$, and $dz = iz\, d\theta$. So

$$I = \frac{1}{i} \int_{|z|=1} \frac{\left[\frac{1}{2i}\left(z - \frac{1}{z}\right)\right]^2}{a + \frac{b}{2}\left(z + \frac{1}{z}\right)} \cdot \frac{dz}{z}$$

$$= -\frac{1}{2i} \int_{|z|=1} \frac{(z^2 - 1)^2}{z^2(2az + bz^2 + b)}\, dz$$

$$= -\frac{1}{2ib} \int_{|z|=1} \frac{(z^2 - 1)^2}{z^2\left(z^2 + \frac{2a}{b}z + 1\right)}\, dz.$$

The integrand has a double pole at $z = 0$ and simple poles at $z = \alpha$ and $z = \beta$, where

$$\alpha = \frac{-a + \sqrt{a^2 - b^2}}{b}, \quad \beta = \frac{-a - \sqrt{a^2 - b^2}}{b}.$$

Since $a > b > 0$, $|\beta| > 1$. But $|\alpha\beta| = 1$ so that $|\alpha| < 1$. Thus, the pole inside $|z| = 1$ is a double pole at $z = 0$ and a simple pole at $z = \alpha$. Now

$$\text{Res}(0) = \text{coefficient of } \frac{1}{z} \text{ in} - \frac{(z^2 - 1)^2}{2ibz^2\left(z^2 + \frac{2a}{b}z + 1\right)}$$

$$= \text{coefficient of } \frac{1}{z} \text{ in} - \frac{1}{2ibz^2}(z^4+1-2z^2)\left(z^2+\frac{2az}{b}+1\right)^{-1}$$

$$= \text{coefficient of } \frac{1}{z} \text{ in} - \frac{1}{2ibz^2}(1-2z^2+z^4)\left(1-\frac{2a}{b}z-z^2-\ldots\right)$$

$$= \frac{a}{i\,b^2} = -\frac{ai}{b^2},$$

$$\text{Res}(\alpha) = \lim_{z\to\alpha}(z-\alpha)\left[-\frac{1}{2ib}\frac{(z^2-1)^2}{z^2(z-2)\,(z-\beta)}\right]$$

$$= -\frac{(\alpha^2-1)^2}{2ib\alpha^2(\alpha-\beta)} = -\frac{1}{2ib}\frac{\left(\alpha-\frac{1}{\alpha}\right)^2}{(\alpha-\beta)}$$

$$= -\frac{1}{2ib}\frac{(\alpha-\beta)^2}{\alpha-\beta} \text{ since } \frac{1}{\alpha}=\beta$$

$$= -\frac{1}{2ib}(\alpha-\beta) = \frac{i}{b^2}\sqrt{a^2-b^2}.$$

Hence

$$I = 2\pi i\left[\frac{i}{b^2}\sqrt{a^2-b^2} - \frac{ai}{b^2}\right]$$

$$= \frac{2\pi}{b^2}[a-\sqrt{a^2-b^2}]$$

EXAMPLE 1.93
Evaluate

$$\int_0^{2\pi}\frac{\cos 3\theta}{5-4\cos\theta}\,d\theta$$

Solution. Let

$$I = \int_0^{2\pi}\frac{e^{i3\theta}}{5-4\cos\theta}\,d\theta$$

Putting $z = e^{i\theta}$, we get

$$I = \frac{1}{i}\int_{|z|=1}\frac{z^3}{5-2\left(z+\frac{1}{z}\right)}\frac{dz}{z} = \frac{1}{i}\int_{|z|=1}\frac{z^3}{5z-2z^2-2}\,dz$$

$$= -\frac{1}{i}\int_{|z|=1}\frac{z^3}{2\left(z^2-\frac{5}{2}z+1\right)}\,dz.$$

The poles of the integrand are given by $2z^2 - 5z + 2 = 0$ and so are $z = 2$ and $z = \frac{1}{2}$. Out of these poles, $z = \frac{1}{2}$ lies within $|z| = 1$. Then

$$\text{Res}\left(\frac{1}{2}\right) = -\frac{1}{i}\lim_{z\to 1/2}\left(z-\frac{1}{2}\right)\frac{z^3}{2\left(z-\frac{1}{2}\right)(z-2)}$$

$$= -\frac{1}{i}\lim_{z\to 1/2}\frac{z^3}{2(z-2)} = \frac{1}{24\,i}.$$

Hence

$$I = \frac{2\pi i}{24\,i} = \frac{\pi}{12}.$$

EXAMPLE 1.94

Evaluate

$$\int_0^{2\pi}\frac{d\theta}{(5-3\cos\theta)^2}.$$

Solution. Substituting $z = e^{i\theta}$, we have $dz = iz\,d\theta$ and so

$$I = \int_0^{2\pi}\frac{d\theta}{(5-3\cos\theta)^2} = \frac{1}{i}\int_{|z|=1}\frac{1}{\left[5-\frac{3}{2}\left(z+\frac{1}{z}\right)\right]^2}\frac{dz}{z}$$

$$= \frac{4}{i}\int_{|z|=1}\frac{z^2}{[10z-3z^2-3]^2}\,dz$$

$$= -\frac{4}{9\,i}\int_{|z|=1}\frac{z}{\left[z^2-\frac{10}{3}z+1\right]^2}\,dz.$$

The double poles of the integrand are given by $z^2 - \frac{10}{3} + 1 = 0$ and so the double poles are at $z = 3$ and $z = \frac{1}{3}$. The double pole at $z = \frac{1}{3}$ lies in $|z| = 1$. Now

$$\text{Res}\left(\frac{1}{3}\right) = -\frac{4}{9i}\lim_{z\to\frac{1}{3}}\frac{d}{dz}\left[\left(z-\frac{1}{3}\right)^2\frac{z}{[z-(1/3)]^2(z-3)^2}\right]$$

$$= -\frac{4}{9i}\lim_{z\to 1/3}\frac{d}{dz}\left[\frac{z}{(z-3)^2}\right]$$

$$= -\frac{4}{9i}\lim_{z\to 1/3}\left[\frac{(z-3)^2 - z[2(z-3)]}{(z-3)^4}\right]$$

$$= -\frac{4}{9i}\lim_{z\to\frac{1}{3}}\left[\frac{(z-3)-2z}{(z-3)^3}\right] = -\frac{4}{9i}\lim_{z\to 1/3}\left[\frac{-z-3}{(z-3)^3}\right]$$

$$= -\frac{4}{9i}\left[\frac{-10/3}{(-8/3)^3}\right] = -\frac{40}{512\,i}.$$

Hence

$$I = 2\pi i\left(\frac{40}{512i}\right) = -\frac{5\pi}{32}.$$

EXAMPLE 1.95

Evaluate

$$\int_0^{\pi}\frac{a\,d\theta}{a^2+\sin^2\theta}, a > 0.$$

Solution. Let

$$I = \int_0^{\pi}\frac{a\,d\theta}{a^2+\sin^2\theta} = \int_0^{\pi}\frac{2a\,d\theta}{2a^2+2\sin^2\theta}$$

$$= \int_0^{\pi}\frac{2a\,d\theta}{2a^2+(1-\cos 2\theta)} = \int_0^{2\pi}\frac{a d\phi}{2a^2+1-\cos\phi}, \quad 2\theta = \phi.$$

Substituting $z = e^{i}\phi$, we get

$$I = \frac{1}{i}\int_{|z|=1}\frac{adz}{z\left[2a^2+1-\frac{1}{2}\left(z+\frac{1}{z}\right)\right]}$$

$$= \frac{2a}{i}\int_{|z|=1}\frac{dz}{2z(2a^2+1)-z^2-1}$$

$$= 2ai\int_{|z|=1}\frac{dz}{z^2-2z(2a^2+1)-1}.$$

The poles α and β of the integrand are $z = 2a^2+1\pm 2a\sqrt{a^2+1}$. We note that $|\alpha| = |2a^2+1+2\sqrt{a^2+1}| > 1$. Since $|\alpha\beta| = 1$, we have $|\beta| = |2a^2+1-2\sqrt{a^2+1}| < 1$. Hence the pole β lies in $|z| = 1$.

$$\text{Res}(\beta) = 2ai\lim_{z\to\beta}(z-\beta)\,f(z)$$

$$= 2ai\lim_{z\to\beta}(z-\beta)\frac{1}{(z-\alpha)(z-\beta)}$$

$$= 2ai\lim_{z\to\beta}\frac{1}{(z-\alpha)} = 2ai\left[\frac{1}{(\beta-\alpha)}\right]$$

$$= 2ai\left(\frac{1}{-4a\sqrt{a^2+1}}\right) = -\frac{i}{2\sqrt{a^2+1}}.$$

Hence, by Cauchy's residue theorem,

$$I = 2\pi i\left(-\frac{i}{2\sqrt{a^2+1}}\right) = \frac{\pi}{\sqrt{a^2+1}}.$$

EXAMPLE 1.96

Evaluate

$$\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta + p^2} \quad (0 < p < 1).$$

Solution. We have

$$I = \int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2}.$$

Substitute $z = e^{i\theta}$ so that $dz = iz\, d\theta$. Thus

$$I = \frac{1}{i}\int_{|z|=1} \frac{1}{1-\frac{2p}{2i}\left(z-\frac{1}{z}\right)+p^2}\cdot\frac{dz}{z}$$

$$= \frac{1}{i}\int_{|z|=1} \frac{1}{z\left(i-pz+\frac{p}{z}+ip^2\right)}\,dz$$

$$= \frac{1}{i}\int_{|z|=1} \frac{i\,dz}{-pz^2+p+z(p^2+1)\,i}$$

$$= -\int_{|z|=1} \frac{dz}{pz^2-z(p^2+1)i-p}$$

$$= -\int \frac{dz}{(pz-i)\,(z-pi)}.$$

The poles of the integrand are given by $z = pi$ and $z = \frac{i}{p}$. Out of these simple poles, the pole at $z = pi$ lies in $|z| = 1$.

$$\text{Res}(pi) = \lim_{z\to pi}(z-pi)\frac{1}{(pz-i)\,(z-pi)}$$

$$= \lim_{z\to pi}\frac{1}{pz-i} = \frac{1}{p^2 i - i} = \frac{1}{i(p^2-1)}.$$

Hence, by Cauchy Residue theorems, we have

$$I = -2\pi i\left(\frac{1}{i(p^2-1)}\right) = \frac{2\pi}{1-p^2}.$$

(B) Definite Integral of the Type $\int_{-\infty}^{\infty} F(x)\,dx$

We know that if $|F(z)| \le M$ on a contour C and if L is the length of the curve C, then

$$\left|\int_C F(z)\,dz\right| \le ML. \tag{1.49}$$

Now, suppose, that $|F(z)| \le M/R^k$ for $z = R\,e^{i\theta}$, $k > 1$ and constant M. Then (1.49) implies

$$\left|\int_\Gamma F(z)\,dz\right| \le \frac{M}{R^k}(\pi R) = \frac{\pi M}{R^{k-1}},$$

where Γ is the semi-circular arc of radius R and length πR as shown in Figure 1.15.

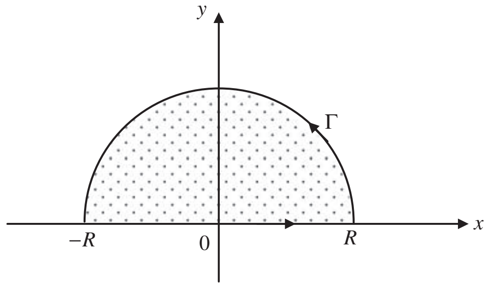


Figure 1.15

Then

$$\lim_{R\to\infty}\left|\int_\Gamma F(z)\,dz\right| = 0$$

and so

$$\lim_{R\to\infty}\int_\Gamma F(z)\,dz = 0.$$

We have thus proved the following result.

Theorem 1.22. If $|F(z)| \le M/R^k$ for $z = R\,e^{i\theta}$, $k > 1$ and constant M, then $\lim_{R\to\infty}\int_\Gamma F(z)\,dz = 0$, where Γ is the circular arc of radius R shown in Figure 1.15.

Equally important results are the following theorems:

Theorem 1.23. If C is an arc $\theta_1 \le \theta \le \theta_2$ of the circle $|z| = R$ and if $\lim_{R\to\infty} zF(z) = A$, then

$$\lim_{R\to\infty}\int_C F(z)\,dz = i(\theta_2 - \theta_1)A.$$

Proof: For sufficiently large value of R, we have

$$|zF(z) - A| < \varepsilon, \varepsilon > 0$$

or equivalently

$$z\,F(z) = A + \eta \text{ where } |\eta| < \varepsilon.$$

Therefore, substituting $z = R\,e^{i\theta}$

$$\int_C F(z)\,dz = \int_C \frac{A+\eta}{z}\,dz = \int_{\theta_1}^{\theta_2} \frac{(A+\eta)\,i\,R\,e^{i\theta}}{R\,e^{i\theta}}\,d\theta,$$

$$= Ai(\theta_2 - \theta_1) + \eta i(\theta_2 - \theta_1).$$

Thus

$$\left|\int F(z)dz - Ai(\theta_2 - \theta_1)\right| = |\eta i(\theta_2 - \theta_1)| = |\eta||i(\theta_2 - \theta_1)|$$

$$= |\eta|(\theta_2 - \theta_1) < (\theta_2 - \theta_1)\varepsilon.$$

Hence

$$\lim_{R\to\infty}\int_C F(z)\,dz = Ai(\theta_2-\theta_1).$$

Remark 1.11. (i) If $\lim\limits_{R\to\infty}$ F(z) = 0, then Theorem 1.23 implies that $\lim\limits_{R\to\infty}\int_C F(z)\,dz = 0.$

(ii) The Theorem 1.23 shall be applied to integrals of the form $\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}dx,$ where P(x) and Q(x) are polynomials such that

(i) The polynomial Q(x) has no real root

(ii) The degree of P(x) is at least two less than that of the degree of Q(x).

Theorem 1.24. (Jordan's Lemma). If $f(z)\to 0$ as $z\to\infty$ and $f(z)$ is meromorphic in the upper half-plane, then

$$\lim_{R\to\infty}\int_\Gamma e^{imz}f(z)\,dz = 0,\quad m>0,$$

where Γ denotes the semi-circle $|z| = R$, Im(z) > 0.

Proof: We shall use Jordan's inequality

$$\frac{2\theta}{\pi}\le\sin\theta\le\theta,\quad 0\le\theta\le\frac{\pi}{2}$$

to prove our theorem. We assume that $f(z)$ has no singularities on Γ for sufficiently large value of R. Since $f(z)\to 0$ as $R\to\infty$, there exists $\varepsilon > 0$ such that $|f(z)| < \varepsilon$ when $|z| = R \le R_0$, $R_0 > 0$. Let Γ be any semi-circle with radius $R \ge R_0$. Substituting $z = \mathrm{R}\,e^{i\theta}$, we get

$$\int_\Gamma e^{imz}f(z)\,dz = \int_0^\infty e^{im\,\mathrm{R}e^{i\theta}}f(R\,e^{i\theta})R\,i\,e^{i\theta}d\theta$$

$$= \int_0^\pi e^{imR(\cos\theta+i\sin\theta)}f(R\,e^{i\theta})\,Rie^{i\theta}d\theta$$

$$= \int_0^\pi e^{i\,mR\cos\theta}.e^{-m\,R\sin\theta}f(R\,e^{i\theta})\,Rie^{i\theta}d\theta.$$

Thus, using Jordan's inequality, we have

$$\left|\int_\Gamma e^{imz}f(z)\,dz\right|$$

$$\le\int_0^\pi|e^{i\,mz\cos\theta}||e^{-mR\sin\theta}||f(\mathrm{R}e^{i\theta}||Ri||e^{i\theta}|d\theta$$

$$\le\int_0^\pi e^{-mR\sin\theta}\varepsilon R\,d\theta\quad\text{using}\,|f(R\,e^{i\theta})|<\varepsilon.$$

$$=2\varepsilon R\int_0^{\pi/2}e^{-m\,R\sin\theta}d\theta = 2\varepsilon R\int_0^{\pi/2}e^{-2mR\,\theta/\pi}$$

$$=2\varepsilon R\frac{1-e^{-mR}}{2mR/\pi}=\frac{\varepsilon\pi}{m}(1-e^{-mR})<\frac{\varepsilon\pi}{m}.$$

Hence

$$\lim_{R\to\infty}\int_{\Gamma} e^{imz} f(z)\, dz = 0.$$

Remark 1.12. Jordan's lemma should be used to evaluate integrals of the form

$$\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\sin mx\, dx \quad \text{or} \quad \int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\cos mx\, dx, m>0,$$

where P(x) and Q(x) are polynomials such that

(i) degree of Q(x) exceeds the degree of P(x)

(ii) the polynomial Q(x) has no real roots.

We shall make use of Theorem 1.22, 1.23, and 1.24 in evaluating the definite integrals of the form $\int_{-\infty}^{\infty} F(x)\, dx$.

EXAMPLE 1.97

Using contour integration, show that

$$\int_0^{\infty}\frac{dx}{(1+x^2)^2} = \frac{\pi}{4}.$$

Solution. Consider the integral

$$\int_C \frac{dz}{(1+z^2)^2},$$

where C is the contour consisting of a large semi-circle C of radius R together with the part of real axis from −R to R traversed in the counter-clockwise sense (Fig. 1.15).

The double poles of the integrand are $z = \pm\, i$, out of which the double pole $z = i$ lies within the contour C. Now

$$\begin{aligned}\text{Res }(i) &= \lim_{z\to i}\frac{d}{dx}\left[(z-i)^2\frac{1}{(z+i)^2(z-i)^2}\right]\\ &= \lim_{z\to i}\frac{d}{dx}\left[\frac{1}{(z+i)^2}\right] = \lim_{z\to i}\left[\frac{2(z+i)}{(z+i)^4}\right]\\ &= \lim_{z\to i}\frac{-2}{(z+i)^3} = \frac{-2}{(2i)^3} = \frac{1}{4i}.\end{aligned}$$

Hence, by Cauchy's Residue theorem

$$\int_C \frac{dz}{(1+z^2)^2} = 2\pi i\left(\frac{1}{4i}\right) = \frac{\pi}{2},$$

that is,

$$\int_{\Gamma}\frac{dz}{(1+z^2)^2} + \int_{-R}^{R}\frac{dx}{(1+x^2)^2} = \frac{\pi}{2}. \tag{1.50}$$

But, substituting $z = \text{R}\, e^{i\theta}$, we have

$$\frac{1}{(1+z^2)^2} = \frac{1}{(1+R^2\, e^{2i\theta})^2} \le \frac{1}{(R^2-1)^2} \le \frac{1}{R^4} \to 0 \text{ as } R \to \infty.$$

Hence, letting $R \to \infty$ in (1.50), we get

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2}$$

and so

$$\int_{0}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{4}.$$

EXAMPLE 1.98

Evaluate $\int_0^\infty \frac{dx}{x^6+1}.$

Solution. Consider $\int_C \frac{dz}{z^6+1}$, where C is the closed contour consisting of the line from $-R$ to R and the semi-circle Γ traversed in the positive sense. The simple poles of $\frac{1}{z^6+1}$ are

$$z = e^{\pi i/6}\ e^{3\pi i/6}, e^{5\pi i/6}, e^{7\pi i/6}, e^{9\pi i/6}, e^{11\pi i/6}.$$

But only three simple poles $e^{\pi i/6}, e^{3\pi i/6}$, and $e^{5\pi i/6}$ lie within C. Now

$$\text{Res}(e^{\pi i/6}) = \lim_{z\to e^{\pi i/6}} \frac{1}{\frac{d}{dz}(z^6+1)} = \lim_{z\to e^{\pi i/6}} \frac{1}{6z^5}$$

$$= \frac{1}{6} e^{-5\pi i/6}.$$

$$\text{Res}(e^{3\pi i/6}) = \lim_{z\to e^{3\pi i/6}} \frac{1}{6z^5} = \frac{1}{6} e^{-5\pi i/2},$$

$$\text{Res}(e^{5\pi i/6}) = \lim_{z\to e^{5\pi i/6}} \frac{1}{6z^5} = \frac{1}{6} e^{-25\pi i/6}.$$

Thus

$$\int_C \frac{1}{z^6+1}\, dz = 2\pi i \Sigma R = 2\pi/3,$$

that is,

$$\int_\Gamma \frac{1}{z^6+1}\, dz + \int_{-R}^{R} \frac{1}{x^6+1}\, dx = \frac{2\pi}{3}.$$

But, $\lim_{z\to\infty} z\text{F}(z) = \lim_{z\to\infty} \frac{z}{z^6+1} = 0.$ Therefore by Theorem 1.23,

$$\int_\Gamma \frac{1}{z^6+1}\, dz \to 0 \text{ as } R \to \infty.$$

Hence, letting $R \to \infty$, we get

$$\int_{-\infty}^{\infty} \frac{1}{x^6+1}\, dx = \frac{2\pi}{3}$$

and so

$$\int_{0}^{\infty} \frac{1}{x^6+1}\, dx = \frac{\pi}{3}.$$

EXAMPLE 1.99

Show that

$$\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9} = \frac{5\pi}{2}.$$

Solution. Consider the integral

$$\int_C \frac{z^2-z+2}{z^4+10z^2+9}\, dz = \int_\Gamma \frac{z^2-z+2}{z^4+10z^2+9}\, dz + \int_{-R}^{R} \frac{z^2-z+2}{z^4+10z^2+9}\, dz.$$

The poles of the integrand are given by $z^4 + 10z^2 + 9 = 0$ which yields the simple poles at $z = \pm\, 3i, \pm i$. Out of these poles only $3i$ and i lie within semi-circle with radius R. Now

$$\begin{aligned}\text{Res}(3i) &= \lim_{z\to 3i} (z-3i)\frac{1}{(z-3i)\,(z+3i)\,(z-i)\,(z+i)}\\ &= \frac{7+3i}{48i}\\ \text{Res}(i) &= \lim_{z\to i}(z-i)\,\frac{1}{(z-i)\,(z+i)\,(z-3i)\,(z+3i)}\\ &= \frac{1-i}{16i}.\end{aligned}$$

Thus

$$\begin{aligned}&\int_\Gamma \frac{z^2-z+2}{z^4+10z^2+9}\, dz + \int_{-R}^{R} \frac{x^2-x+2}{x^4+10x^2+9}\, dx\\ &= 2\pi\text{i}\left[\frac{7+3i}{48i} + \frac{1-i}{16i}\right] = \frac{5\pi}{12}.\end{aligned}$$

Further, $z\,\text{F}(z) \to 0$ as $z \to \infty$. Therefore, by Theorem 1.23, we have

$$\int_\Gamma \frac{z^2-z+2}{z^4+10z^2+9}\, dz \to 0 \text{ as } R \to \infty.$$

Hence, letting $R \to \infty$, we get

$$\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\, dx = \frac{5\pi}{12}.$$

EXAMPLE 1.100

Evaluate $\int_0^\infty \frac{\cos ax}{x^2+1}\,dx.$

Solution. Consider the integral

$$\int_C \frac{e^{iaz}}{z^2+1}\,dz = \int_\Gamma \frac{e^{iaz}}{z^2+1}\,dz + \int_{-R}^{R} \frac{e^{iaz}}{z^2+1}\,dz.$$

The poles of the integrand are $z = \pm i$ of which $z = i$ lies in C. Hence

$$\int_\Gamma \frac{e^{iaz}}{z^2+1}\,dz + \int_{-R}^{R} \frac{\cos ax}{x^2+1}\,dz = 2\pi i.\,(\text{residue at } i)$$

But

$$\text{Res(i)} = \lim_{z\to i}(z-i)\frac{e^{iaz}}{(z-i)\,(z+i)} = \frac{e^{-a}}{2\,i}.$$

Since $f(z) \to 0$ as $z \to \infty$, by Jordan's lemma, the integral $\int_\Gamma \frac{e^{iaz}}{z^2+1}\,dz \to 0$ as $R \to \infty$. Hence, in the limit as $R \to \infty$, we get

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx = 2\pi i\frac{e^{-a}}{2i} = \pi\, e^{-a}$$

and

$$\int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx = \frac{\pi\, e^{-a}}{2}.$$

EXAMPLE 1.101

Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)\,(x^2+b^2)}\,dx, \quad a, b > 0.$

Solution. Consider

$$\int_C \frac{z^2}{(z^2+a^2)\,(z^2+b^2)}\,dz,$$

where C is the contour consisting of the line from $-R$ to R and semi-circle Γ of radius R traversed in the positive sense. Then

$$\int_C \frac{z^2}{(z^2+a^2)\,(z^2+b^2)}\,dz = \int_{-R}^{R} \frac{x^2}{(x^2+a^2)\,(x^2+b^2)}\,dx + \int_\Gamma \frac{z^2}{(z^2+a^2)\,(z^2+b^2)}\,dz.$$

But since $z\,f(z) \to \infty$ as $z \to \infty$, the second integral on the right tends to zero. Thus

$$\int_C \frac{z^2}{(z^2+a^2)\,(z^2+b^2)}\,dz = \int_{-R}^{R} \frac{x^2}{(x^2+a^2)\,(x^2+b^2)}\,dx.$$

But the poles of the integrand of the integral on the left are $z = \pm\, ai$ and $z = \pm\, bi$ out of which $z = ai$ and $z = bi$ lie within C. Now

$$\text{Res}(ai) = \lim_{z \to ai} (z - ai) \frac{z^2}{(z - ai)(z + ai)(z^2 + b^2)}$$

$$= \frac{a}{2i(a^2 - b^2)},$$

$$\text{Res}(bi) = \lim_{z \to bi} (z - bi) \frac{z^2}{(z^2 + a^2)(z + bi)(z - bi)}$$

$$= \frac{-b}{2i(a^2 - b^2)}.$$

Hence

$$\int_C \frac{z^2}{(z^2 + a^2)(z^2 + b^2)} dz = 2\pi i \left[\frac{a}{2i(a^2 - b^2)} - \frac{b}{2i(a^2 - b^2)} \right]$$

$$= \frac{\pi(a - b)}{a^2 - b^2} = \frac{\pi}{a + b}.$$

Hence, as $R \to \infty$,

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} dx = \frac{\pi}{a + b}.$$

EXAMPLE 1.102

Evaluate $\int_0^\infty \frac{x \sin x}{x^2 + a^2} dx, \ a > 0.$

Solution. Consider

$$\int_C \frac{z\, e^{iz}}{z^2 + a^2} dz,$$

where C is contour consisting of line from $-R$ to R and semi-circle with radius R traversed in positive sense. Then

$$\int_C \frac{z\, e^{iz}}{z^2 + a^2} dz = \int_{-R}^{R} \frac{x \sin x}{x^2 + a^2} dx + \int_\Gamma \frac{z\, e^{iz}}{z^2 + a^2} dz.$$

Since $\lim_{z \to \infty} \frac{z}{z^2 + a^2} = 0$, we have, by Jordan's lemma,

$$\int_\Gamma f(z)\, dz = 0.$$

The integrand has simple poles at $z = \pm ai$ of which $z = ai$ lies within C. Further

$$\text{Res}(ai) = \lim_{z \to ai} (z - ai) \frac{z\, e^{iz}}{(z - ai)(z + ai)}$$

$$= \lim_{z \to ai} \frac{z\, e^{iz}}{z + ai} = \frac{e^{-a}}{2}.$$

Hence

$$\int_C \frac{z\,e^{iz}}{z^2+a^2}\,dz = 2\pi i\left(\frac{e^{-a}}{2}\right) = \pi i\,e^{-a}.$$

Equating imaginary parts, we have

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2} = \pi e^{-a}$$

and so

$$\int_{0}^{\infty} \frac{x\sin x}{x^2+a^2}\,dx = \frac{\pi}{2}\,e^{-a}.$$

EXAMPLE 1.103

Use calculus of residue to show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx$$

$$= \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right), \quad a > b > 0.$$

Solution. The integrand is of the form $\dfrac{P(x)}{Q(x)}$. So let us consider

$$\int_C f(z)\,dz = \int_C \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,dz,$$

where C is semi-circle Γ with radius R and the line from $-R$ to R. We have

$$\int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz = 2\pi i\,\Sigma\mathrm{R}.$$

But, by Jordan's lemma

$$\int_\Gamma f(z)\,dz \to 0 \text{ as } R \to \infty.$$

Further, the poles of $f(z)$ are $z = \pm ai, \pm bi$, out of which $z = ai$ and $z = bi$ lie in the upper half-plane. Now

$$\mathrm{Res}(ai) = \lim_{z\to ai}(z-ai)\frac{e^{iz}}{(z+ai)(z^2+b^2)(z-ai)}$$

$$= \frac{e^{-a}}{2ai(b^2-a^2)},$$

$$\mathrm{Res}(bi) = \lim_{z\to bi}(z-bi)\frac{e^{iz}}{(z+bi)(z^2+a^2)(z-bi)}$$

$$= \frac{e^{-b}}{2bi(a^2-b^2)}.$$

Hence

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx$$

$$= 2\pi i\left[\frac{e^{-a}}{2ai(b^2-a^2)} + \frac{e^{-b}}{2bi(a^2-b^2)}\right]$$

$$= \frac{\pi}{a^2-b^2}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right].$$

EXAMPLE 1.104

Show that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5}\,dx = -\frac{\pi \sin 2}{e}.$$

Solution. Consider

$$\int_C f(z)\,dz = \int \frac{e^{iz}\,dz}{z^2+4z+5},$$

where C is the contour as in the above examples. Then

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz = 2\pi i \Sigma \text{R}.$$

Since $\frac{1}{z^2+4z+5} \to 0$ as $z \to \infty$, by Jordan's lemma, $\int_\Gamma f(z)\,dz \to 0$ as $R \to \infty$. Further, the poles of $f(z)$ are $-2 \pm i$. The pole $z = -2 + i$ lie in the upper half-plane. Then

$$\text{Res}(2+i) = \lim_{z \to -2+i} (z+2-i)\frac{e^{iz}}{z^2+4z+5}$$

$$= \frac{e^{-(1+2i)}}{2i}.$$

Hence

$$\int_C f(z)\,dz = \pi\, e^{-(1+2i)} = \frac{\pi}{e}(\cos 2 - i \sin 2).$$

Equating the imaginary parts, we get

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5}\,dx = -\frac{\pi}{e}\sin 2.$$

EXAMPLE 1.105

Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2-2x+2)}\,dx \text{ and } \int_{-\infty}^{\infty} \frac{1-\cos x}{(x^2-2x+2)}\,dx.$$

Solution. Consider

$$\int_C f(z)\, dz = \int_C \frac{1-e^{iz}}{z(z^2-2z+2)}\, dz,$$

where C is the contour consisting of a large semi-circle Γ of radius R in the upper half-plane and the real axis from $-R$ to R. We have

$$\int_C f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_\Gamma f(z)\, dz = 2\pi i \Sigma R.$$

We observe that

$$\left|\int_\Gamma f(z)\, dz\right| \le \int_0^\pi \left|\frac{1-e^{iRe^{i\theta}}}{Re^{i\theta}(R^2e^{2i\theta}-2Re^{i\theta}+2)} R\, ie^{i\theta} d\theta\right|$$

$$\le \int_0^\pi \frac{1-e^{-R\sin\theta}}{R^2-2R+2}\, d\theta \to 0 \; as \; R \to \infty,$$

since $\sin\theta$ is positive

Hence, when $R \to \infty$, we have

$$\int_C f(z)\, dz = \int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \Sigma R.$$

The function $f(z)$ has simple poles at $z = 1 \pm i$ of which $z = 1 + i$ lies in the upper half-plane. However, $z = 0$ is not a pole because expanding $1-e^{iz}$ we see that z is a common factor of numerator and denominator. Let $\alpha = 1 + i$ and $\beta = 1 - i$. Then

$$\text{Res}(\alpha) = \lim_{z\to\alpha} (z-\alpha)\frac{1-e^{iz}}{z(z-\beta)(z-\alpha)}$$

$$= \lim_{z\to\alpha} \frac{1-e^{iz}}{z(z-\beta)} = \frac{1-e^{i\alpha}}{\alpha(\alpha-\beta)}$$

$$= \frac{1-e^{i-1}}{(1+i)(2i)} = \frac{(1-i)(1-e^{i-1})}{4i}$$

$$= \frac{1-i}{4i}\left[1-\frac{1}{e}e^{i}\right]$$

$$= \frac{1-i}{4i}\left[1-\frac{1}{e}(\cos 1 + i\sin 1)\right].$$

Thus

$$\int_{-\infty}^{\infty} \frac{1-e^{ix}}{x(x^2-2x+2)}\, dx$$

$$= 2\pi i \left[\frac{1-i}{4i}\left\{1-\frac{1}{e}(\cos 1 + i\sin 1)\right\}\right]$$

$$= \frac{\pi}{2e}(1-i)\,[e-\cos 1 - i\sin 1].$$

Equating real and imaginary parts, we get

$$\int_{-\infty}^{\infty} \frac{1-\cos x}{x(x^2-2x+2)}\,dx = \frac{\pi}{2e}[\text{e-}\cos 1-\sin 1]$$

and

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2-2x+2)}\,dx = \frac{\pi}{2e}[\text{e-}\cos 1+\sin 1].$$

EXAMPLE 1.106

Show that

$$\int_{0}^{\infty} \frac{\log(1+x^2)}{1+x^2}\,dx = \pi \log 2.$$

Solution. Consider

$$\int_C f(z)\,dz = \int_C \frac{\log(z+i)}{1+z^2}\,dz,$$

where C is the contour as in the above examples. We have

$$\int_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz = 2\pi i \Sigma \text{R}.$$

Substituting $z = Re^{i\theta}$, we can show that $\int_\Gamma f(z)\,dz \to 0$ as $R \to \infty$. Hence when $R \to \infty$, we get

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \Sigma \text{R}.$$

But $f(z)$ has simple pole at $z = +i$ and a logarithmic singularity at $z = -i$, out of which $z = i$ lies inside C. Now

$$\begin{aligned}\text{Res}(i) &= \lim_{z\to i}(z-i)\frac{\log(z+i)}{(z-i)(z+i)}\\ &= \lim_{z\to i}\frac{\log(z+i)}{z+i} = \frac{\log 2i}{2i} = \frac{1}{2i}\left[\log 2 + i\frac{\pi}{2}\right].\end{aligned}$$

Thus

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\log(x+i)}{x^2+1}\,dx &= 2\pi i\left[\frac{1}{2i}\left(\log 2 + i\frac{\pi}{2}\right)\right]\\ &= \pi\left[\log 2 + \frac{i\pi}{2}\right].\end{aligned}$$

Comparing real parts

$$\int_{-\infty}^{\infty} \frac{\frac{1}{2}\log(1+x^2)}{1+x^2}\,dx = \pi \log 2$$

and so

$$\int_0^\infty \frac{\log(1+x^2)}{1+x^2}\,dx = \pi \log 2.$$

(C) Poles on the Real Axis

When the integrand has a simple pole on a real axis, we delete it from the region by indenting the contour. Indenting is done by drawing a small semi-circle having the pole as the centre. The procedure followed is called "indenting at a point."

EXAMPLE 1.107

Show that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2} \quad \text{and} \quad \int_0^\infty \frac{\cos x}{x}\,dx = 0.$$

Solution. Consider the integral

$$\int_C f(z)\,dz = \int_C \frac{e^{iz}}{z}\,dz,$$

where C is the contour (shown in Fig. 1.16) consisting of

(i) real axis from ρ to R, where ρ is small and R is large
(ii) the upper half of the circle $|z| = R$
(iii) the real axis from $-R$ to $-\rho$
(iv) the upper half of the circle $|z| = \rho$.

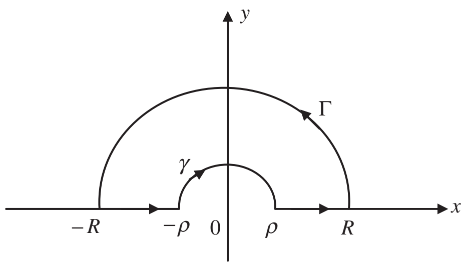


Figure 1.16

Since there is no singularity inside C, the Cauchy-Goursat theorem implies

$$\int_C f(z)\,dz = \int_\rho^R f(x)\,dx + \int_\Gamma f(z)\,dz + \int_{-R}^{\rho} f(x)\,dx + \int_\gamma f(z)\,dz = 0.$$

By Jordan's lemma, we have

$$\lim_{R\to\infty} \int_\Gamma f(z)\,dz = 0.$$

Further, since, $\lim\limits_{x\to\infty} z \to 0\, zf(z) = 1$, we have

$$\lim_{\rho\to 0} \int_\gamma f(z)\,dz = i(0-\pi).1 = -\pi i.$$

Hence as $\rho \to 0$ and $R \to \infty$, we get

$$\int_0^\infty f(x)\,dx + \int_{-\infty}^0 f(x)\,dx - \pi i = 0$$

and so

$$\int_{-\infty}^\infty f(x)\,dx = \pi i,$$

that is,

$$\int_{-\infty}^\infty \frac{e^{ix}}{x}\,dx = \pi \mathrm{i}.$$

Equating real and imaginary parts, we get

$$\int_{-\infty}^\infty \frac{\cos x}{x}\,dx = 0 \quad \text{and} \quad \int_{-\infty}^\infty \frac{\sin x}{x}\,dx = \pi.$$

Hence
$$\int_0^\infty \frac{\cos x}{x}\,dx = 0 \quad \text{and} \quad \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

EXAMPLE 1.108

Evaluate

$$\int_0^\infty \frac{\sin x}{x(x^2+a^2)}\,dx \quad \text{and} \quad \int_0^\infty \frac{\cos x}{x(x^2+a^2)}\,dx, \quad a > 0$$

Solution. Consider the integral

$$\int_C f(z)\,dz = \int_C \frac{e^{iz}}{z(z^2+a^2)}\,dz,$$

where C is the contour as shown in Figure 1.16.

Now $f(z)$ has simple poles at $z = 0, \pm ai$. Out of these, $z = 0$ lie on x-axis and $z = ai$ lies in the upper half-plane. Residue at $z = ai$ is

$$\text{Res}(ai) = \lim_{z\to ai} (z - ai)\frac{e^{iz}}{z(z-ai)(z+ai)}$$

$$= \lim_{z\to ai} \frac{e^{iz}}{z(z+ai)} = \frac{e^{-a}}{-2a^2}.$$

Hence

$$\int_C f(z)\,dz = \int_\rho^R f(x)dx + \int_\Gamma f(z)\,dz + \int_{-R}^{-\rho} f(x)dx + \int_\gamma f(z)\,dz$$

$$= 2\pi i\left[\frac{e^{-a}}{-2a^2}\right] = -\frac{\pi i}{a^2}\,e^{-a}.$$

Now

$$\int_\Gamma f(z)\,dz \to 0 \text{ as } \mathrm{R} \to \infty.$$

Also substituting $z = \rho e^{i\theta}$, we note that

$$\int_{\gamma} f(z)\, dz = -\frac{\pi i}{a^2} \text{ as } \rho \to 0.$$

Hence as $\rho \to 0$ and $R \to \infty$, we have

$$\int_0^{\infty} f(x)\, dx + \int_{-\infty}^{0} f(x)\, dx - \frac{\pi i}{a^2} = -\frac{\pi i}{a^2} e^{-a}$$

and so

$$\int_{-\infty}^{\infty} f(x)\, dx = \frac{\pi i}{a^2}(1 - e^{-a}),$$

that is,

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2 + a^2)} dx = \frac{\pi i}{a^2}(1 - e^{-a}).$$

Equating real and imaginary parts, we get

$$\int_{-\infty}^{\infty} \frac{\cos x}{x(x^2 + a^2)} dx = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\sin x}{x(x^2 + a^2)} dx = \frac{\pi}{a^2}(1 - e^{-a}).$$

Thus

$$\int_0^{\infty} \frac{\cos x}{x(x^2 + a^2)} dx = 0 \quad \text{and} \quad \int_0^{\infty} \frac{\sin x}{x(x^2 + a^2)} dx = \frac{\pi}{2a^2}(1 - e^{-a}).$$

EXAMPLE 1.109

Show that

$$\int_0^{\infty} \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin px}, 0 \le p \le 1.$$

Solution. Consider the integral $\int_C \frac{z^{p-1}}{1+z} dz$, where C is the contour shown in Figure 1.17 and where AB and GH are actually coincident with the x-axis. Here $z = 0$ is a branch point and the real axis is the branch line. The integrand has a simple pole at $z = -1 = e^{\pi i}$ ying on the x-axis and inside C. Now

$$\begin{aligned} \text{Res}(e^{\pi i}) &= \lim_{z \to -1} (z+1) \frac{z^{p-1}}{z+1} \\ &= \lim_{z \to -1} z^{p-1} = (e^{\pi i})^{p-1} = e^{(p-1)\pi i}. \end{aligned}$$

Hence

$$\int_C \frac{z^{p-1}}{1+z}\, dz = 2\pi i e^{(p-1)\pi i}.$$

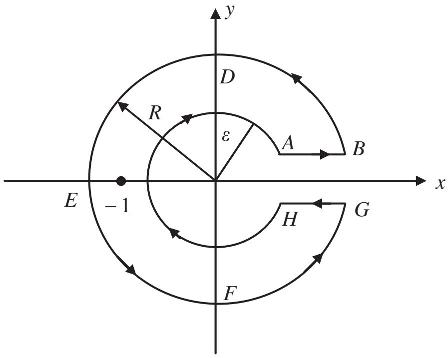


Figure 1.17

Thus

$$\int_\varepsilon^R \frac{x^{p-1}}{1+x}\, dx + \int_0^{2\pi} \frac{(Re^{i\theta})^{p-1} i\, R\, e^{i\theta}\, d\theta}{1+R\, e^{i\theta}} + \int_R^\varepsilon \frac{(x\, e^{2\pi i})^{p-1}}{1+xe^{2\pi i}}\, dx + \int_{2\pi}^0 \frac{(\varepsilon e^{i\theta})^{p-1} i\varepsilon e^{i\theta}}{1+\varepsilon\, e^{i\theta}}\, d\theta$$
$$= 2\pi i e^{(p-1)}\pi i.$$

Now taking the limit as $R \to \infty$ and $\varepsilon \to 0$, the second and fourth integral approaches zero. Therefore,

$$\int_0^\infty \frac{x^{p-1}}{1+x} dx + \int_\infty^0 \frac{xe^{2\pi i(p-1)} x^{p-1}}{1+x}\, dx = 2\pi i e^{(p-1)\pi i},$$

that is,

$$[1-e^{2\pi i(p-1)}] \int_0^\infty \frac{x^{p-1}}{1+x} dx = 2\pi i e^{(p-1)\pi i},$$

which yields

$$\int_0^\infty \frac{x^{p-1}}{1+x}\, dx = \frac{2\pi i}{e^{p\pi i} - e^{-p\pi i}} = \frac{\pi}{\sin p\pi}.$$

EXAMPLE 1.110

Using calculus of residue, evaluate

$$\int_0^\infty \sin x^2\, dx \quad \text{and} \quad \int_0^\infty \cos x^2\, dx.$$

Solution. Consider the integral $\int_C e^{iz^2}\, dz$, where C is the contour as shown in Figure 1.18. Here, AP is the arc of a circle with centre at the origin O and radius R.

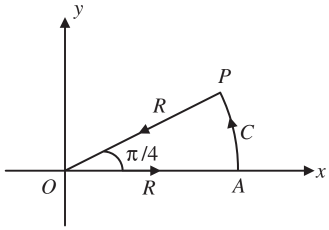


Figure 1.18

The function e^{iz^2} has no singularities within and on C. Hence by Cauchy-Goursat theorem

$$\int_C e^{iz^2} dz = 0.$$

Thus

$$\int_{OA} e^{iz^2} dz + \int_{AP} e^{iz^2} dz + \int_{PO} e^{iz^2} dz = 0,$$

that is,

$$\int_0^R e^{ix^2 dx} + \int_0^{\pi/4} e^{iR^2 e^{2i\theta}} iR\, e^{i\theta} d\theta + \int_R^0 e^{ir^2} e^{\pi i/2} e^{\pi i/4} dr = 0,$$

or

$$\int_0^R (\cos x^2 + i \sin x^2)\, dx = e^{\pi i/4} \int_0^R e^{-r^2} dr - \int_0^{\pi} e^{iR^2} e^{2i\theta} i\, \mathrm{Re}^{i\theta} d\theta$$

As $R \to \infty$

$$e^{\pi i/4} \int_0^{\infty} e^{-r^2} dr = \frac{\sqrt{\pi}}{2} e^{\pi i/4},$$

and

$$\left| \int_0^{\pi/4} e^{iR^2} e^{2i\theta} i\, R\, e^{i\theta} d\theta \right| \le \int_0^{\pi/4} e^{-R^2 \sin 2\theta} R\, d\theta$$

$$= \frac{R}{2} \int_0^{\pi/2} e^{-R^2 \sin\phi} d\phi, \quad \phi = 2\theta$$

$$\le \frac{R}{2} \int_0^{\pi/2} e^{-2R^2\phi/\pi}\, d\phi, \quad 0 \le \phi \le \frac{\pi}{2}$$

$$= \frac{\pi}{4R}(1 - e^{-R^2}) \to 0 \quad \text{as} \quad \mathrm{R} \to \infty.$$

Hence

$$\int_0^{\infty} (\cos x^2 + i \sin x^2)\, dx = \frac{\sqrt{\pi}}{2} e^{\pi i/4} = \frac{1}{2}\sqrt{\frac{\pi}{2}} + \frac{i}{2}\sqrt{\frac{\pi}{2}}.$$

Equating real and imaginary parts, we get

$$\int_0^{\infty} \cos x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} \quad \text{and} \quad \int_0^{\infty} \sin x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

1.8 CONFORMAL MAPPING

We know that a real-valued function $y = f(x)$ of a real variable x determines a curve in the xy-plane if x and y are interpreted as rectangular co-ordinates. But in case of analytic function $w = f(z)$ of a complex variable z, no such simple geometric interpretation is possible. In fact in this case, both z and w are complex numbers and, therefore, geometric representation of the function requires four real co-ordinates. But our geometry fails in a space of more than three dimensions. Thus, no geometric interpretation is possible of $w = f(z)$.

Suppose that we regard the points z and w as points in two different planes—the z-plane and the w-plane. Then we can interpret the functional relationship $w = f(z)$ as a mapping of points in the z-plane onto the points in the w-plane. Thus $f(z)$ if is regular on some set S in z-plane, there exists a set of points S' in the w-plane. The set S' is called the image of the set S under the function $w = f(x)$.

Let $f(z)$ be regular and single-valued in a domain D. If $z = x + iy$ and $w = u(x, y) + iv(x, y)$, the image of the continuous arc $x = x(t)$, $y = y(t)$, $(t_1 \le t \le t_2)$ is the arc $u = u(x(t), y(t))$, $v = v\,(x(t), y(t))$ under the mapping $w = f(z)$. Further, u and v are continuous in t if $x(t)$ and $y(t)$ are continuous. Therefore, $w = f(z)$ maps a continuous arc into a continuous arc.

Let the two curves C_1 and C_2 in the z-plane intersect at the point $P(x_0, y_0)$ at an angle and let C_1 and C_2 be mapped under $w = f(z)$ into the curves Γ_1 and Γ_2, respectively, in the w-plane. If Γ_1 and Γ_2 intersect at (u_0, v_0) at the same angle α such that the sense of angle is same in both cases, then $w = f(z)$ is called *conformal mapping. Thus, a mapping which preserves both the magnitude and the sense of the angles is called conformal.*

But, if a mapping preserves only the magnitude of angles but not necessarily the sense, then it is called *isogonal mapping.*

Theorem 1.25. The mapping $w = f(z)$ is conformal at every point z of a domain where $f(z)$ is analytic and $f'(z) \ne 0$.

Proof: Consider a smooth arc $z = z(t)$, which terminates at a point $z_0 = z(t_0)$ at which $f(z)$ is analytic. Let $w_0 = f(z_0)$ and $w = w(t) = f(z(t))$. Then

$$w - w_0 = \frac{f(z) - f(z_0)}{z - z_0}(z - z_0)$$

and so

$$\arg(w - w_0) = \arg\left[\frac{f(x) - f(z_0)}{z - z_0}\right] + \arg(z - z_0) \tag{1.51}$$

where $\arg(z - z_0)$ is the angle between the positive axis and the vector pointing from z_0 to z. If $z \to z_0$ along the smooth arc $z(t)$, then $\lim\limits_{z \to z_0} \arg(z - z_0)$ is the angle θ between the positive axis and the tangent to the arc at z_0. Similarly, $\arg(w - w_0)$ tends to the angle φ between the positive axis and the tangent to $w(t)$ at w_0. Hence, taking limit as $z \to z_0$, (1.51) reduces to

$$\phi = \arg f'(z) + \theta, \text{ provided } f'(z) \ne 0.$$

Thus, the difference $\phi - \theta$ depends only on the point z_0 and not on the smooth arc $z = f(t)$ for which the angle θ and ϕ were computed. If $z_1(t)$ is another smooth arc terminating at z_0 and if the corresponding tangential directions are given by the angles θ_1 and ϕ_1, then

$$\phi_1 - \theta_1 = \arg f'(z), \text{ provided } f'(z) \ne 0.$$

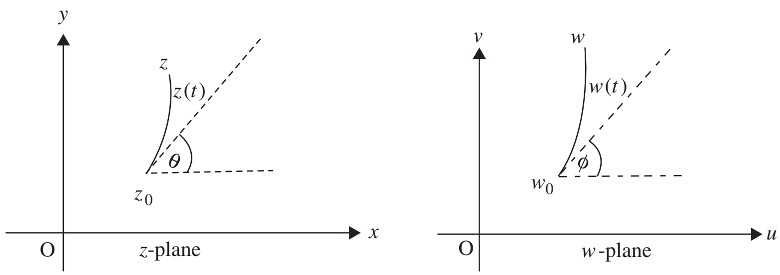


Figure 1.19

Hence

$$\phi_1 - \theta_1 = \phi - \theta$$

or

$$\phi_1 - \phi = \theta_1 - \theta, \tag{1.52}$$

where $\theta_1 - \theta$ is the angle between the arcs $z_1(t)$ and $z(t)$ and $\phi_1 - \phi$ is the angle between the images of these arcs. The expression (1.52) shows that the angle between the arcs is not changed by the mapping $w = f(z)$, provided $f' \neq 0$ at the point of intersection. Also (1.52) shows that the sense of angles is also preserved. Hence the mapping is conformal.

Bilinear (Mobius or Fractional) Transformation

Consider the transformation

$$w = \frac{az+b}{cz+d},\ ad - bc \neq 0, \tag{1.53}$$

where a, b, and c are complex constants. This can be written as

$$cwz + dw - az - b = 0, \tag{1.54}$$

which is linear in both w and z. Therefore, the mapping (1.53) is called *Bilinear* or *Mobius Transformation*. Also (1.53) can be written as

$$w = \frac{a}{c} + \frac{bc - ad}{c(cz+d)},\ ad - bc \neq 0. \tag{1.55}$$

The condition $ad - bc \neq 0$, called the *determinant of the transformation*, prevents (1.53) from degenerating into a constant.

A transformation $w = f(z)$ is said to be *univalent* if $z_1 \neq z_2$ implies $f(z_1) \neq f(z_2)$.

EXAMPLE 1.111

Show that the linear transformation $w = \dfrac{az+b}{cz+d}$ is a univalent transformation.

Solution. We have

$$w(z_1) = \frac{az_1 + b}{cz_1 + d},\ w(z_2) = \frac{az_2 + b}{cz_2 + d}.$$

Therefore,

$$w(z_1) - w(z_2) = \frac{(z_1 - z_2)(ad - bc)}{(cz_2 + d)(cz_1 + d)}.$$

Since $ad - bc \neq 0$, we note that $z_1 \neq z_2$ implies $w(z_1) \neq w(z_2)$. Hence $w = \dfrac{az+b}{cz+d}$ is univalent.

Particular cases of w $= \dfrac{a\,z+b}{c\,z+d}$**,** $ad - bc \neq 0$

(i) Substituting $c = 0$, $d = 1$, we get the transformation

$$w = az + b. \tag{1.56}$$

To find the effect of this transformation on a point in the z-plane, let us assume that $b = 0$. Thus $w = az$. Introducing polar co-ordinates we have $z = re^{i\theta}$. If $a = |\,a\,|\, e^{i\alpha}$, then

$$w = r|a|e^{i(\theta+\alpha)}$$

and so

$$|w| = r|a| \quad \text{and} \quad \arg w = \theta + \alpha.$$

Thus, under the mapping $w = az$, all distances from the origin are multiplied by the same factor $|\,a\,|$ and the argument of all numbers z are increased by the same amount α. Hence the transformation $z \to az$ results in a magnification or contraction according as $|\,a\,| > 1$ or $|\,a\,| < 1$ and rotation of any geometric figure in the z-plane. In particular, the mapping $z \to az$ maps a circle into a circle. The addition of b to $w = az$ amounts only to a translation. If b is real, all points are translated horizontally by the same amount and if b is complex, then we will also have vertical translation. *Hence $w = az + b$ will always transform a circle into circle.*

(ii) Substituting $a = d = 0$, $b = c$ in $w = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$, we get

$$w = \frac{1}{z}, \tag{1.57}$$

which is the translation, called *inversion*. Setting $z = re^{i\theta}$, (1.57) reduces to

$$w = \frac{e^{-i\theta}}{r}. \tag{1.58}$$

Thus

$$|w| = \frac{1}{r} = \frac{1}{|z|} \text{ or } |w||z| = 1 \text{ and } \arg w = -\theta.$$

This means that the points of the w-plane corresponding to z has a modulus which is the reciprocal of the modulus of z. *Thus the mapping $w = \dfrac{1}{z}$ transforms points in the interior of the unit circle into the points in its exterior and vice- versa. The circumference of the unit circle is transformed into itself. But since* arg $w = -$ arg *z, the circumference $|\,w\,| = 1$ is described in the negative sense if $|\,z\,| = 1$ is described in the positive sense.*

We note that $z = 0$ is mapped by $w = \dfrac{1}{z}$ to ∞ in the w-plane and $w = 0$ is mapped to ∞ in the z-plane. If we apply the mapping twice, we get the identity mapping. For any point z_0 in the z-plane, $\dfrac{1}{z_0}$ is *called the inverse of z_0 with respect to the circle $|\,z\,| = 1$*. That is why, the mapping $w = \dfrac{1}{z}$ is called inversion. The fixed points of the mapping are given by $z = \dfrac{1}{z}$, that is, by $z^2 = 1$. Hence $\pm\, 1$ are the fixed points of the inversion.

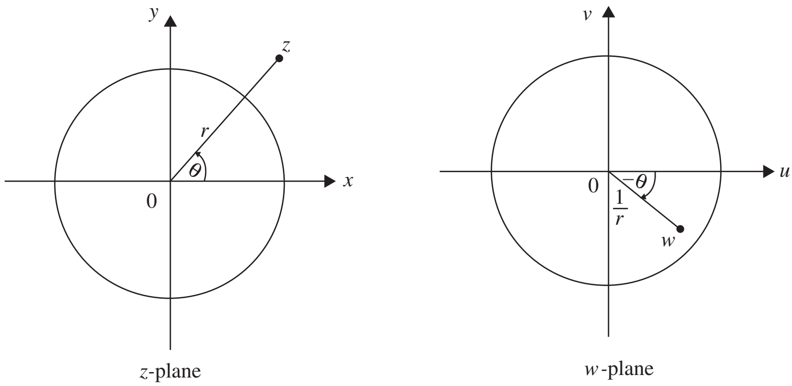


Figure 1.20

The mapping $w = \frac{1}{z}$ *transforms circles into circles*. To prove it, let the equation of circle in *xy*-plane be

$$x^2 + y^2 + Ax + By + C = 0,$$

where A, B, and C are real constants. Changing to polar co-ordinates, we have

$$r^2 + r(A\cos\theta + B\sin\theta) + C = 0. \tag{1.59}$$

If ρ, ϕ are polar coordinates in w-plane, then $w = \frac{1}{z} = \frac{e^{-i\theta}}{r}$ implies that $\rho = \frac{1}{r}$ and $\phi = -\theta$. Therefore, under the transformation $w = \frac{1}{z}$, the circle's equation (1.59) transforms to

$$\frac{1}{\rho^2} + \frac{1}{\rho}(A\cos\phi - B\sin\phi) + C = 0. \tag{1.60}$$

If $C \neq 0$, then

$$\rho^2 + \rho\left(\frac{A}{C}\cos\phi - \frac{B}{C}\sin\phi\right) + \frac{1}{C} = 0,$$

which is again the *equation of a circle in polar coordinates*.

If $C = 0$, then (1.60) reduces to

$$A\rho\cos\theta - B\rho\sin\theta + 1 = 0$$

If $w = u + iv$, we have

$$Au - Bv + 1 = 0.$$

Thus, the image of a circle $x^2 + y^2 + Ax + By = 0$ *passing through the origin, is a straight line. If we regard a straight line as a special case of a circle (namely degenerate circle) passing through the point at infinity, then it follows that the transformation* $w = \frac{1}{z}$ *transforms circles into circles.*

We now turn to the bilinear transformation $w = \frac{az + b}{cz + d}$, $ad - bc \neq 0$. We have

$$w = \frac{a}{c} + \frac{bc - ad}{c(cz + d)} = \frac{a}{c} + \frac{bc - ad}{cz_1}$$

$$= \frac{a}{c} + \frac{bc - ad}{c} z_2,$$

where

$$z_1 = cz + d \text{ and } z_2 = \frac{1}{z_1}.$$

Thus, the bilinear transformation $w = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$ splits into three successive transformations

$$z_1 = cz + d \tag{1.61}$$

$$z_2 = \frac{1}{z_1} \tag{1.62}$$

$$w = \frac{a}{c} + \frac{bc - cd}{c} z_2. \tag{1.63}$$

The transformations (1.61) and (1.63) are of the form $w = az + b$, whereas (1.62) is inversion. Hence, by the above discussion it follows that "The linear transformation $w = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$ maps circles in the z-plane onto circles in the w-plane. The point $z = -\dfrac{d}{c}$ is transformed by $w = \dfrac{az+b}{cz+d}$ into the point $w = \infty$, accordingly circles passing through the point $z = -\dfrac{d}{c}$ will be transformed into straight lines."

EXAMPLE 1.112

Find the condition where the transformation $w = \dfrac{az+b}{cz+d}$ transforms the unit circle in the w-plane into a straight line.

Solution. The given transformation is $w = \dfrac{az+b}{cz+d}$. Therefore,

$$|w| = 1 \Rightarrow w\bar{w} = 1 \Rightarrow \left(\frac{az+b}{cz+d}\right)\left(\frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\right) = 1$$

$$\Rightarrow \left(a\bar{a} - c\bar{c}\right) z\bar{z} + \left(a\bar{b} - c\bar{d}\right) z + \left(\bar{a}b - \bar{c}d\right)\bar{z} + b\bar{b} - d\bar{d} = 0.$$

In order that this equation represents a straight line, the coefficient of $z\bar{z}$ must vanish, that is,

$$a\bar{a} - c\bar{c} = 0 \text{ or } a\bar{a} = c\bar{c} \text{ or } |a| = |c|,$$

which is the required condition. If a and c are reals, then the condition becomes $a = c$.

EXAMPLE 1.113

Investigate the mapping $w = z^2$.

Solution. The given mapping is $w = z^2$. The derivative $\dfrac{dw}{dz} = 2z$ vanishes at the origin. Hence the mapping is not conformal at the origin. Taking $z = x + iy$ and $w = u + iv$, we have

$$u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy.$$

Separating the real and imaginary parts, we get

$$u = x^2 - y^2 \text{ and } v = 2xy.$$

Therefore, the straight lines $u = a$ and $v = b$ in the w-plane correspond to the *rectangular hyperbolas*

$$x^2 - y^2 = a \text{ and } 2xy = b.$$

These hyperbolas cut at right angles except in the case $a = 0$, $b = 0$, when they intersect at the angle $\frac{\pi}{4}$.

Now, let $x = a$ be a straight line in the z-plane parallel to the y-axis. Then

$$u = a^2 - y^2 \text{ and } v = 2ay.$$

Elimination of y yields

$$v^2 = 4a^2\left(a^2 - u\right),$$

which is a parabola in w-plane having its vertex at $u = a^2$ on the positive real axis in the w-plane. This parabola open towards the negative side of the u-axis. The line $y = b$ corresponds to the curve

$$u = x^2 - b^2, \quad v = 2bx.$$

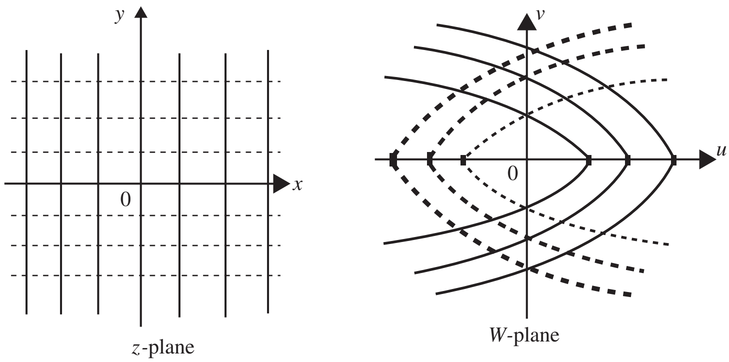


Figure 1.21

Elimination of x yields

$$v^2 = 4b^2\left(u + b^2\right),$$

which is again a parabola, but pointing in the opposite direction and having vertex at $u = -b^2$, $v = 0$, focus on the origin and opening towards the positive side of u-axis.

Hence the straight lines x = constant and y = constant correspond to the system of co-focal parabolas.

EXAMPLE 1.114

If a and c are reals, show that the transformation $w = z^2$ transforms the circle $|z - a| = c$ in the z-plane to a limacon in the w-plane.

Solution. We have $z - a = ce^{i\theta}$ so that

$$\begin{aligned} w - a^2 + c^2 &= \left(a + ce^{i\theta}\right)^2 - a^2 + c^2 = a^2 + c^2e^{2i\theta} \\ &\quad + 2ace^{i\theta} - a^2 + c^2 \\ &= ce^{i\theta}\left(ce^{i\theta} + 2a\right) + c^2 = ce^{i\theta}\left(ce^{i\theta} + 2a + ce^{-i\theta}\right) \\ &= ce^{i\theta}[2a + 2c\cos\theta] = 2ce^{i\theta}\left(a + c\cos\theta\right) \\ &= 2ce^{i\theta}\left(a + c\cos\theta\right). \end{aligned}$$

Substituting $w - a^2 + c^2 = R\,e^{i\theta}$, we get

$$Re^{i\phi} = 2ce^{i\theta}\left(a + c\cos\theta\right)$$

or

$$R = 2c\left(a + c\cos\phi\right), \quad \phi = \theta.$$

Therefore, polar equation of the curve in the w-plane is

$$R = 2c\left(a + c\cos\phi\right) = 2ac + 2c^2\cos\phi.$$

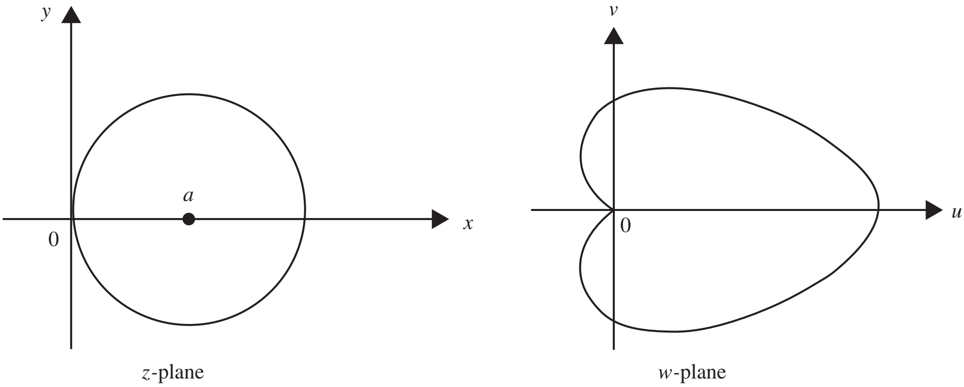


Figure 1.22

If we take $a = c$, that is, if the circle in z-plane touches the axis of y at (0, 0) and its centre is at $x = a$ and radius a, then the limacon degenerates into a cardiod $R = 2a^2(1 + cos\ \phi)$.

EXAMPLE 1.115

Find the image in the w-plane of the circle $|z - 3| = 2$ in the z-plane under the inverse mapping $w = \dfrac{1}{z}$.

Solution. The image in the w-plane of the given circle $|z - 3| = 2$ in the z-plane under the inverse mapping $w = \dfrac{1}{z}$ is given by

$$\left|\frac{1}{w} - 3\right| = 2$$

or

$$\left|\frac{1}{u + iv} - 3\right| = 2$$

or

$$\left|\frac{u - iv}{u^2 + v^2} - 3\right| = 2$$

or

$$\left|\frac{u - iv}{u^2 + v^2} - 3\right|^2 = 4$$

or

$$\left[\left(\frac{u}{u^2 + v^2} - 3\right) - \frac{iv}{u^2 + v^2}\right]\left[\left(\frac{u}{u^2 + v^2} - 3\right) + \frac{iv}{u^2 + v^2}\right] = 4$$

or

$$\left(\frac{u}{u^2+v^2}-3\right)^2+\frac{v^2}{\left(u^2+v^2\right)^2}=4$$

or

$$\frac{u^2+v^2}{\left(u^2+v^2\right)^2}-\frac{6u}{u^2+v^2}+5=0$$

or

$$1-6u+5\left(u^2+v^2\right)=0$$

or

$$\left(u-\frac{3}{5}\right)^2+v^2=\frac{4}{25}=\left(\frac{2}{5}\right)^2.$$

If follows that image of $|z-3|=2$ is a circle with centre $\left(\frac{3}{5},0\right)$ and radius $\frac{2}{5}$.

On the other hand, $w=\frac{1}{z}$ implies

$$u+iv=\frac{1}{x+iy}=\frac{x-iy}{x^2+y^2}=\frac{x}{x^2+y^2}-\frac{iy}{x^2+y^2}.$$

Equating real and imaginary parts, we get

$$u=\frac{x}{x^2+y^2},\quad v=\frac{-y}{x^2+y^2}.$$

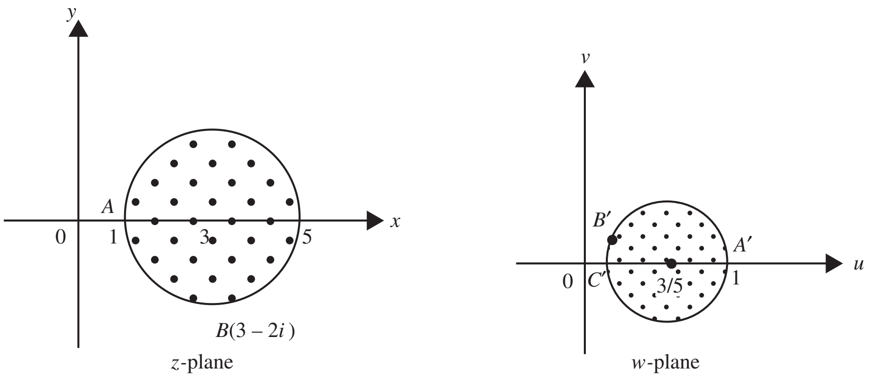


Figure 1.23

The centre (3, 0) of the circle in z-plane is mapped into $(u,v)=\left(\frac{1}{3},0\right)$ in the w-plane which is inside the mapped circle. Therefore, under $w=\frac{1}{z}$, the region under the circle $|z-3|=2$ is mapped onto the region inside the circle in the w-plane.

We note that the point $A(1+i0)$ is mapped into (1, 0), $B(3-2i)$ into $B'\left(\frac{3}{13},\frac{2}{13}\right)$, and $C(5+i0)$ is mapped into the point $C'\left(\frac{1}{5},0\right)$. Thus as the point z traverse the circle in the z-plane in an anticlockwise direction, the corresponding point w in the w-plane will also traverse the mapped circle is an anticlockwise direction.

EXAMPLE 1.116

Discuss the transformation $w = z + \frac{1}{z}$.

Solution. At $z = 0$, w becomes infinite. Further $\frac{dw}{dz} = 1 - \frac{1}{z^2}$ vanishes at $z = \pm 1$. Thus, $z = \pm 1$ are the critical points and the function $w = z + \frac{1}{z}$ is not conformal at 0, 1, and −1. Substituting $z = re^{i\theta}$ and $w = u + iv$, we have

$$u + iv = re^{i\theta} + \frac{1}{re^{i\theta}}$$

$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta.$$

Therefore,

$$u = \left(r + \frac{1}{r}\right)\cos\theta \tag{1.64}$$

$$v = \left(r - \frac{1}{r}\right)\sin\theta. \tag{1.65}$$

If $r = 1$, that is, if the radius of the circle in z-plane is unity, then we get $u = 2\cos\theta$, $v = 0$. Therefore, as θ varies from 0 to 2π in describing the unit circle in the z-plane, the domain described in the w-plane is the segment of the real axis between the points 2 and –2 twice, that is, the ellipse of minor axis 0 and major axis equal to 1.

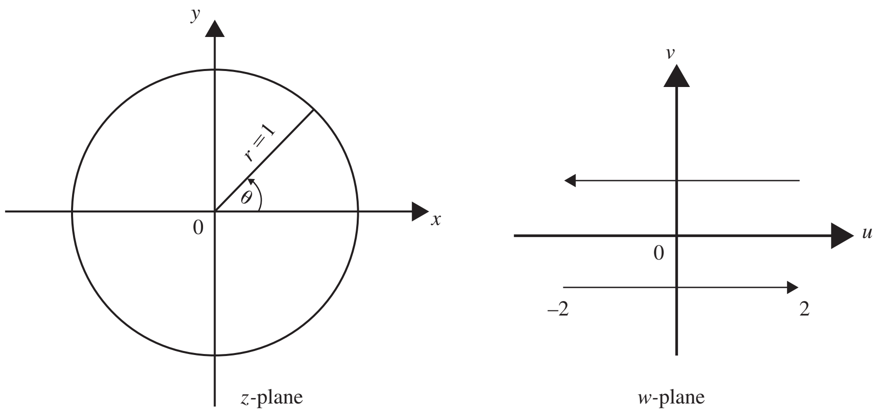


Figure 1.24

Moreover, (1.64) and (1.65) yield

$$\frac{u}{r + \frac{1}{r}} = \cos\theta, \quad \frac{v}{r - \frac{1}{r}} = \sin\theta.$$

Squaring and adding, we get

$$\frac{u^2}{\left(r+\frac{1}{r}\right)^2}+\frac{v^2}{\left(r-\frac{1}{r}\right)^2}=1,$$

which is an ellipse in the w-plane and it corresponds to each of the two circles $|z|=r$ and $|z|=\frac{1}{r}$, since the equation of the ellipse does not change on changing r to $\frac{1}{r}$. Thus the major and minor axis of the ellipse in w-plane are $r+\frac{1}{r}$ and $r-\frac{1}{r}$. As $r \to 0$ or $r \to \infty$, both semi-axis tends to infinity. Thus, the inside and the outside of the unit circle in the z-plane both correspond to the whole w-plane, cut along the real axis from –1 to 1.

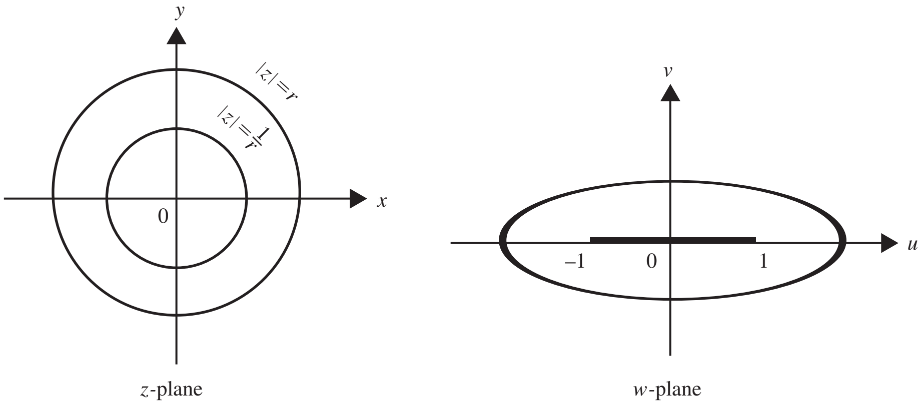


Figure 1.25

The fixed points of the given transformation are given by $z = z+\frac{1}{z}$. Therefore, $z=\infty$ in the fixed point.

EXAMPLE 1.117

Examine the exponential transformation $w = e^z$.

Solution. Substituting $z = x + iy$ and $w = u + iv$, the exponential transformation $w = e^z$ yields

$$u+iv = e^{x+iy} = e^x(\cos y + i\sin y).$$

Equating real and imaginary parts, we have

$$u = e^x \cos y \quad \text{and} \quad v = e^x \sin y$$

or

$$\cos y = \frac{u}{e^x} \text{ and } \sin y = \frac{v}{e^x}.$$

Squaring and adding, we get

$$u^2 + v^2 = e^{2x} \tag{1.66}$$

Also

$$\frac{v}{u} = \tan y. \tag{1.67}$$

Let $x = a$ be a line parallel to the imaginary axis. Then (1.66) yields

$$u^2 + v^2 = e^{2a}.$$

Thus, the line parallel to y-axis is transformed into circles with centre at the origin.

On the other hand, let $y = b$ be a line parallel to the x-axis. Then (1.67) yields

$$v = u \tan b.$$

Thus, the lines parallel to the x-axis are mapped by the transformation into rays emanating from the origin.

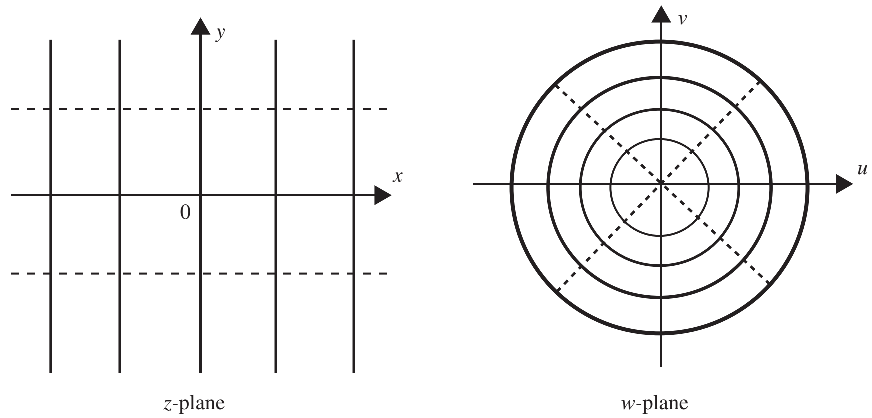


Figure 1.26

If $x = 0$, then we have $u^2 + v^2 = 1$. Hence the imaginary axis is mapped into unit circle $u^2 + v^2 = 1$ in the w-plane.

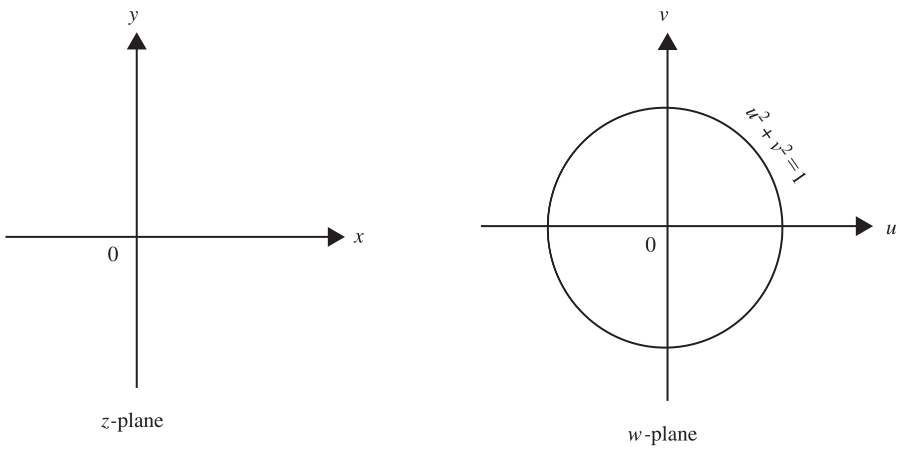


Figure 1.27

Moreover,

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y$$
$$= e^x(\cos y + i \sin y) = e^x e^{iy} = e^{x+iy} = e^z \neq 0.$$

Therefore, the mapping $w = e^z$ is conformal everywhere in the complex plane.

EXAMPLE 1.118

Discuss logarithmic transformation $w = \log z$.

Solution. Substituting $z = re^{i\theta}$ and $w = u + iv$, we have

$$u + iv = \log\left(re^{i\theta}\right) = \log r + i\theta.$$

Therefore,

$$u = \log r \text{ and } v = \theta.$$

Hence, the circles defined by r = constant in the z-planes are mapped onto straight lines parallel to the v-axis and the straight lines defined by θ = constant are mapped onto straight lines parallel to the u-axis.

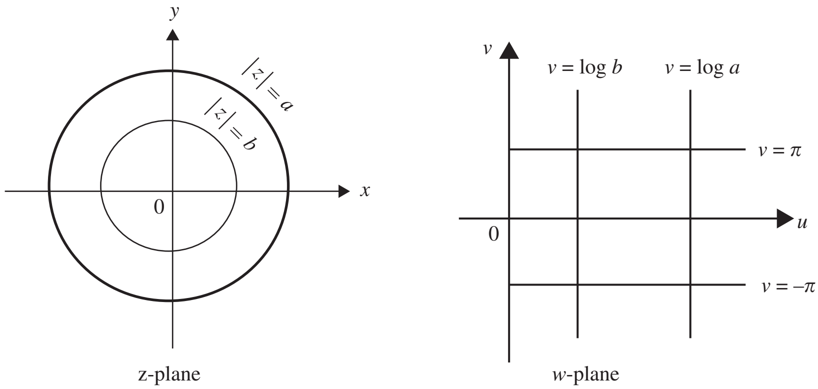


Figure 1.28

Since the derivative $\dfrac{dw}{dz} = \dfrac{1}{z}$ is infinite at the origin, the mapping is not conformal at the origin.

1.9 MISCELLANEOUS EXAMPLES

EXAMPLE 1.119

Separate log (6 + 8i) into real and imaginary parts.

Solution. We have $x + iy = 6 + 8i$ so that $r^2 = x^2 + y^2 = 100$. Therefore

$$\begin{aligned} \operatorname{Re}\left[\log(6+8i)\right] &= \frac{1}{2}\log(x^2 + y^2) \\ &= \frac{1}{2}\log 10^2 = \log 10 \end{aligned}$$

and

$$\operatorname{Im}\left[\log(6+8i)\right] = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{4}{3}.$$

EXAMPLE 1.120

If $\sin(\theta + ia)) = \cos a + i\sin a$, show that $\cos^4\theta = \sin^2 a$.

Solution. We have

$$\sin(\theta + i\alpha) = \sin\theta\cosh\alpha + i\cos\theta\sinh\alpha.$$
$$= \cos\alpha + i\sin\alpha.$$

Equating real and imaginary parts ,we have

$$\sin\theta\cosh\alpha = \cos\alpha \text{ and } \cos\theta\sinh\alpha = \sin\alpha \tag{1.68}$$

The relations in (1.68) yield

$$\cosh\alpha = \frac{\cos\alpha}{\sin\theta} \text{ and } \sinh\alpha = \frac{\sin\alpha}{\cos\theta}.$$

Squaring and subtracting we get

$$\cosh^2\alpha - \sinh^2\alpha = \frac{\cos^2\alpha}{\sin^2\theta} - \frac{\sin^2\alpha}{\cos^2\theta}$$

or

$$1 = \frac{\cos^2\alpha}{\sin^2\theta} - \frac{\sin^2\alpha}{\cos^2\theta}$$

or

$$\sin^2\theta\cos^2\theta = \cos^2\alpha\cos^2\theta - \sin^2\alpha\sin^2\theta.$$

or

$$(1-\cos^2\theta)\cos^2\theta = (1-\sin^2\alpha)\cos^2\theta - \sin^2\alpha(1-\cos^2\theta)$$

or

$$\cos^4\theta = \sin^2\alpha.$$

EXAMPLE 1.121

Show that the function

$$f(z) = \begin{cases} \dfrac{\mathrm{Im}(z)}{|z|}, & z \neq 0 \\ 0, & z \neq 0 \end{cases}$$

is not continuous at $z = 0$.

Solution. We have

$$f(z) = \begin{cases} \dfrac{y}{\sqrt{x^2+y^2}}, & z = x+iy \neq 0 \\ 0, & z = 0. \end{cases}$$

Therefore

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{y}{\sqrt{x^2+y^2}}.$$

If $z \to 0$ along $y = mx$, then

$$\lim_{z\to 0} f(z) = \lim_{x\to 0} \frac{mx}{\sqrt{x^2+m^2x^2}} = \lim_{x\to 0} \frac{mx}{x\sqrt{1+m^2}}$$

$$= \frac{m}{\sqrt{1+m^2}} \neq 0 \text{ for arbitary value of } m.$$

But $f(0) = 0$. Hence $\lim_{z \to 0} f(z) \neq f(0)$. Hence f is not continuous at the origin.

EXAMPLE 1.122

Show that an analytic function with constant real part is constant.

Solution. Here $u = C \Rightarrow \frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$. Then, by *CR* equations, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0$.

Thus $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$. Thus, both u and v are constant . Hence $f = u + iv$ is constant.

EXAMPLE 1.123

Show that $V = e^{2x}(y\cos 2y + x\sin 2y)$ is harmonic and find the corresponding analytic function $f(z) = u + iv$

Solution. Here

$$v = e^{2x}(y\cos 2y + x\sin 2y).$$

Therefore

$$\frac{\partial v}{\partial x} = 2e^{2x}(y\cos 2y + x\sin 2y) + e^{2x}\sin 2y$$

$$\frac{\partial v}{\partial y} = e^{2x}(\cos 2y - 2y\sin 2y + 2x\cos 2y)$$

$$\frac{\partial^2 v}{\partial x^2} = 4e^{2x}(\sin 2y + y\cos 2y + x\sin 2y)$$

$$\frac{\partial^2 v}{\partial y^2} = -4e^{2x}\sin 2y - 4ye^{2x}\cos 2y - 4xe^{2x}\sin 2y$$

We note that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$. Hence v is harmonic. Further,

$$v_1(x, y) = \frac{\partial v}{\partial y} = e^{2x}[\cos 2y - 2y\sin 2y + 2x\cos 2y]$$

$$v_2(x, y) = \frac{\partial v}{\partial x} = e^{2x}[2y\cos 2y - 2x\sin 2y + \sin 2y].$$

Therefore

$$f'(z) = v_1(z, 0) + i\, v_2(z, 0) = e^{2z}(1 + 2z)$$

and so

$$f(z) = \int e^{2z}dz + 2\int ze^{2z}dz = ze^{2z} + C.$$

EXAMPLE 1.124

Determine the analytic function whose real part is $e^{2x}(x\cos 2y - y\sin 2y)$.

Solution. Let $f(z) = u + iv$ be the required analytic function. We are given that

$$u = e^{2x}(x\cos 2y - y\sin 2y).$$

Then, by Cauchy – Riemann equations,

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=e^{2x}(2x\cos 2y - 2y\sin 2y+\cos 2y), \tag{1.69}$$

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=-e^{2x}\ (2x\sin 2y+\sin 2y+2y\cos 2y). \tag{1.70}$$

Integrating (1.69) with respect to y, treating x as constant, we have

$$v=\int_{x\ \text{constant}} e^{2x}(2x\cos 2y-2y\sin 2y+\cos 2y)dy$$

$$=e^{2x}\left[\frac{2x\sin 2y}{2}-\left\{2y\left(\frac{-\cos 2y}{2}\right)-2\int\frac{-\cos 2y}{2}dy\right\}+\frac{\sin 2y}{2}\right]+\phi(x)$$

$$=e^{2x}(x\sin 2y+y\cos 2y)+\phi(x),$$

where ϕ is a function of x. Now

$$\frac{\partial v}{\partial x}=2e^{2x}(x\sin 2y+y\cos 2y)+e^{2x}(\sin 2y)+\phi'(x)$$

$$=e^{2x}(2x\sin 2y+2y\cos 2y+\sin 2y)+\phi'(x) \tag{1.71}$$

From (1.70) and (1.71), we have $\varphi'(x)=0$ and so ϕ is constant. Hence

$$v=e^{2x}(x\sin 2y+y\cos 2y)+c.$$

Then

$$f=u+iv=e^{2x}\left[(x+iy)\cos 2y+i(x+iy)\sin 2y\right]+ic$$

$$=ze^{2x}.e^{2iy}+ic$$

$$=ze^{2(x+iy)}+ic$$

$$=ze^{2z}+ic.$$

EXAMPLE 1.125

Determine analytic function, whose real part is $\cos x\cosh y$.

Solution. We have $u=\cos x\cosh y$ .Therefore

$$\frac{\partial u}{\partial x}=-\sin x\cosh y \text{ and } \frac{\partial u}{\partial x}=\cos x\sinh\ y.$$

Then, by Milne's Method,

$$f'(z)=u_1(z,0)-iu_2(z,0)$$

$$=-\sin z\ \cosh 0-i(0)=-\sin z.$$

Integrating with respect to z, we have

$$f(z)=-\int\sin z\,dz+Ci=\cos z+Ci.$$

EXAMPLE 1.126

Find the analytic function whose imaginary part is

$$v = x^3 y - xy^3 + xy + x + y.$$

Solution. We have

$$\frac{\partial v}{\partial x} = 3x^2 y - y^3 + y + 1$$

$$\frac{\partial v}{\partial y} = x^3 - 3xy^2 + x + 1.$$

Thus

$$v_1(x, y) = \frac{\partial v}{\partial y} = x^3 - 3xy^2 + x + 1$$

$$v_2(x, y) = \frac{\partial v}{\partial x} = 3x^2 y - y^3 + y + 1.$$

Therefore

$$f'(z) = v_1(z, 0) + i v_2(z, 0) = z^3 + z + 1 + i.$$

Integrating, we get

$$f(z) = \int (z^3 + z + 1 + i)dz$$

$$= \frac{z^4}{4} + \frac{z^2}{2} + (1 + i)z + C$$

EXAMPLE 1.127

Prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) | \operatorname{Re} f(z) |^2 = 2 | f'(z) |^2$, where $w = f(z)$ is analytic.

Solution. Let $f(z) = u + iv$, then $\operatorname{Re} f(z) = u$ and

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = u_x + i v_x,$$

$$| f'(z) | = \sqrt{u_x^2 + v_x^2}.$$

Further,

$$\frac{\partial}{\partial x}(u^2) = 2u \frac{\partial u}{\partial x} = 2u\, u_x,$$

$$\frac{\partial^2}{\partial x^2}(u^2) = 2[u_x^2 + 2u_{xx}], \tag{1.72}$$

$$\frac{\partial}{\partial y}(u^2) = 2u \frac{\partial u}{\partial y} = 2u\, u_y,$$

$$\frac{\partial^2}{\partial y^2}(u^2) = 2[u_y^2 + 2u_{yy}]. \tag{1.73}$$

Adding (1.72) and (1.73), we get

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u^2 = 2\left[u_x^2+u_y^2+\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\right]$$

$$= 2[u_x^2+u_y^2] \text{ since } \nabla^2 u = 0 \text{ for analytic function}$$

$$= 2|f'(z)|^2.$$

EXAMPLE 1.128

Using C-R equations, show that $f(z) = z^3$ is analytic in the entire z-plane.

Solution. We have

$$f(z) = z^3 = (x+iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3) = u + iv, say.$$

Then

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^3 - 3xy^2) = 3x^2 - 3y^2$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^3 - 3xy^2) = -6xy$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(3x^2y - y^3) = 6xy$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(3x^2y - y^3) = 3x^2 - 3y^2.$$

We note that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Hance Cauchy-Riemann equations are satisfied. Further, all first order derivatives in the present case are continuous in the entire z-plane. Hence sufficient conditions for analyticity are satisfied. Hence the given function is analytic in the entire z-plane.

EXAMPLE 1.129

Evaluate $\int_c dz/(z-3)^2$ where C is the circle $|z| = 1$.

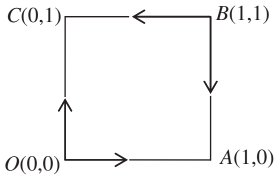
Solution. We have $\int_{|z|=1} \frac{dz}{(z-3)^2}$. The integrand is analytic except at $z = 3$. But $z = 3$ lies outside $|z| = 1$.

Hence, by Cauchy – Gaursat theorem, the given integral is equal to zero.

EXAMPLE 1.130

Show that $\int_C (z+1)dz = 0$, where C is the boundary of the square whose vertices are at the points $z = 0$, $z = 1$, $z = 1+i$ and $z = i$.

Solution. We want to evaluate $\oint_C (z+1)dz$, where C is the contour shown below:



Along OA, we have $y = 0$ so that $z = x + iy = x$ and $dz = dx$. Therefore

$$\int_{OA} f(z)\, dz = \int_0^1 (x+1)\, dx = \left[\frac{x^2}{2} + x\right]_0^1 = \frac{3}{2}.$$

Along AB, we have $x = 1$ and so $z = 1 + iy$ so that $dz = i\, dy$. Therefore

$$\int_{AB} f(z)dz = i\int_0^1 (2+iy)\, dy = i\left[2y + i\frac{y^2}{2}\right]_0^1$$

$$= i\left[2+\frac{i}{2}\right] = 2i - \frac{1}{2}.$$

Along BC, we have $y = 1$ so that $z = x + iy = x + i$ and $dz = dx$. Thus

$$\int_{BC} f(z)\, dz = \int_1^0 (x+i+1)\, dx = \left[\frac{x^2}{2} + (i+1)x\right]_1^0$$

$$= -\frac{1}{2} - (i+1) = -i - \frac{3}{2}.$$

Along CO, we have $x = 0$ so that $z = x + iy = iy$ and so $dz = i\, dy$. Thus

$$\int_{CO} f(z)dz = i\int_1^0 (iy+1)\, dy = \left[i\frac{y^2}{2} + y\right]_1^0$$

$$= -\frac{i}{2} - 1.$$

Hence

$$\int_C f(z)dz = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO} = \frac{3}{2} + 2i - \frac{1}{2} - i - \frac{3}{2} - \frac{i}{2} - 1$$

$$= \frac{i}{2} - \frac{3}{2} = \frac{1}{2}(i-3).$$

EXAMPLE 1.131

Evaluate $\int_C \log z\, dz$, where C is the circle $|z| = 1$.

Solution. Putting $z = e^{i\theta}$, we have $dz = ie^{i\theta} d\theta$. Therefore

$$\int_C \log z\, dz = \int_{|z|=1} \log e^{i\theta}(ie^{i\theta}) d\theta$$

$$= \int_0^{2\pi} i^2\theta\, e^{i\theta} d\theta = -\int_0^{2\pi} \theta\, e^{i\theta} d\theta$$

$$= -\left[\theta\frac{e^{i\theta}}{i} - \frac{e^{i\theta}}{i^2}\right]_0^{2\pi}$$

$$= -[(1-i\theta)e^{i\theta}]_0^{2\pi}$$

$$= -[(1-2\pi i)e^{2\pi i} - (1)] = 2i.$$

EXAMPLE 1.132

Evaluate the following integrals using Cauchy integral formula.

$\oint_C \frac{4-3z}{z(z-1)(z-2)}dz$,where C is $|z| = 3$.

Solution. Cauchy's integral formula states that "If $f(z)$ is analytic within and on any closed contour C and if a is a point within the contour C, then

$$f(a) = \frac{1}{2\pi i}\int \frac{f(z)}{z-a}\,dz \text{ ."}$$

The given integral is

$I = \oint_C \frac{4-3z}{z(z-1)(z-2)}dz$, where c is $|z| = 3$.

By partial fractions, we have

$$\frac{4-3z}{z(z-1)(z-2)} = \frac{2}{z} - \frac{1}{z-1} - \frac{1}{z-2}.$$

Therefore

$$\oint_C \frac{4-3z}{z(z-1)(z-2)}\,dz = 2\oint_C \frac{1}{z}\,dz - \oint_C \frac{1}{z-1}\,dz - \oint_C \frac{1}{z-2}\,dz.$$

The point 0 lies in $|z| = 3$. Therefore, by Cauchy's integral formula, we have

$$f(0) = \frac{1}{2\pi i}\oint_C \frac{dz}{z}.$$

Since $f(z) = 1$, $f(0) = 1$. Therefore

$$\oint_C \frac{dz}{z} = 2\pi i\, f(0) = 2\pi i.$$

The point $z = 1$ lies within C and so

$$f(1) = \frac{1}{2\pi i}\oint_C \frac{1}{z-1}dz,$$

that is,

$$\int_C \frac{1}{z-1}\,dz = 2\pi i\ f(1) = 2\pi i.$$

Similarly, the point $z = 2$ lies within C and so

$$f(2) = \frac{1}{2\pi i}\oint_C \frac{1}{z-2}\,dz.$$

Therefore

$$\oint_C \frac{1}{z-2}dz = 2\pi i\ f(2) = 2\pi i.$$

Hence

$$I = 2(2\pi i) - 2\pi i - 2\pi i = 0.$$

EXAMPLE 1.133

Evaluate $\int_C \frac{(z^3 - \sin 3z)dz}{\left(z-\frac{\pi}{2}\right)^3}$ with $C = |z| = 2$ using Cauchy's integral formula.

Solution. The singularity of the integrand is at $\frac{\pi}{2}$ which lies in $|z| = 2$. Therefore, by Cauchy's integral formula

$$\begin{aligned}\int_C \frac{(z^3 - \sin 3z)dz}{\left(z-\frac{\pi}{2}\right)^3} &= \frac{2\pi i}{2!} f''\left(\frac{\pi}{2}\right)\\ &= \frac{\pi i}{2}[z^3 - \sin 3z]''_{z=\frac{\pi}{2}}\\ &= \pi i[6z + 9\sin 3z]_{z=\frac{\pi}{2}}\\ &= \pi i\,[3\pi - 9] = 3\pi i\,(\pi - 3).\end{aligned}$$

EXAMPLE 1.134

Evaluate $\int_{(0,0)}^{(1,1)} (3x^2 + 4xy + ix^2)dz$ along $y = x^2$

Solution. We have $z = x + iy = x + ix^2$ so that $dz = (1 + 2ix)dx$. Hence

$$\begin{aligned}\int_{(0,0)}^{(1,1)} (3x^2 + 4xy + ix^2)dz &= \int_0^1 (3x^2 + 4x^3 + ix^2)(1 + 2ix)dx\\ &= \int_0^1 [(3x^2 + 4x^3 + ix^2) + (6ix^3 + 8ix^4 - 2x^3)]dx\\ &= \int_0^1 [3x^2 + 2x^3 + i(6x^3 + 8x^4 + x^2)]dx\\ &= \left[x^3 + \frac{x^4}{2} + i\left(\frac{3}{2}x^4 + \frac{8}{5}x^5 + \frac{x^3}{3}\right)\right]_0^1\\ &= 1 + \frac{1}{2} + i\left(\frac{3}{2} + \frac{8}{5} + \frac{1}{3}\right)\\ &= \frac{3}{2} + i\left(\frac{103}{30}\right).\end{aligned}$$

EXAMPLE 1.135

Using Cauchy's integral formula, evaluate $\int_C \frac{dz}{e^z(z-1)^3}$, where C is $|z| = 2$.

Solution. We have

$$\int_{|z|=2} \frac{dz}{e^z(z-1)^3} = \int_{|z|=2} \frac{e^{-z}}{(z-1)^3}dz.$$

The singularity z = 1 lies within $|z| = 2$ and so by Cauchy's integral formula, we have

$$\int_{|z|=2} \frac{dz}{e^2(z-1)^3} = \frac{2\pi i}{2!}f''(1), \text{where } f(z) = e^{-z}$$

$$= \pi i e^{-1} = \frac{\pi i}{e}.$$

EXAMPLE 1.136

Find $f(2)$ and $f(3)$ if $f(a) = \int_C \frac{2z^2 - z - 2}{z - a}dz$, where $|z| = 2.5$. Using Cauchy's Integral Formula.

Solution. We have $f(z) = 2z^2 - z - 2$ and $f(a) = \int_C \frac{2z^2 - z - 2}{z - a}dz$.
Therefore

$$f(2) = \int_C \frac{2z^2 - z - 2}{z - 2}dz$$

We note that a = 2 lies within the circle $|z| = 2.5$. Hence, by Cauchy's Integral Theorem,

$$\int_C \frac{2z^2 - z - 2}{z - a}dz = 2\pi i f(2)$$

$$= 2\pi i[2(2)^2 - 2 - 2] = 8\pi i.$$

Since a = 3 lies outside the circle $|z| = 2.5$, by Cauchy's integral Theorem, we have

$$f(3) = \int_C \frac{2z^2 - z - 2}{z - 3}dz = 0.$$

EXAMPLE 1.137

Find the Laurent's Series expansion of $\frac{z-1}{(z+2)(z+3)}$ valid in the region $2 < |z| < 3$.

Solution. We have

$$\frac{z-1}{(z+2)(z+3)} = \frac{-3}{z+2} + \frac{4}{z+3}.$$

For $|z| > 2$, we have

$$\frac{-3}{z+2} = \frac{-3}{z\left(1+\frac{2}{z}\right)} = \frac{-3}{z}\left[1 - \frac{2}{z} + \frac{4}{z^2} - \frac{8}{z^3} + \frac{16}{z^4} - \ldots\right]$$

$$= \frac{-3}{z} + \frac{6}{z^2} - \frac{12}{z^3} + \frac{24}{z^4} - \frac{48}{z^5} + \ldots$$

For $|z| < 3$, we have

$$\frac{4}{z+3} = \frac{4}{3\left(1+\frac{z}{3}\right)} = \frac{4}{3}\left[1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\frac{z^4}{81}-\ldots\right]$$

$$= \frac{4}{3}-\frac{4}{9}z+\frac{4}{27}z^2-\frac{4}{81}z^3+\frac{4}{243}z^4-\ldots$$

Hence the Laurent's series expansion is

$$f(z) = \ldots\ldots \frac{48}{z^5}+\frac{24}{z^4}\frac{12}{z^3}+\frac{6}{z^2}\frac{3}{z}+\frac{4}{3}\frac{4}{9}z+\frac{4}{27}z^2\frac{4}{81}z^3+\frac{4}{243}z^4-\ldots\ldots$$

EXAMPLE 1.138

Expand $f(z) = \dfrac{e^{2z}}{(z-1)^3}$ about z = 1 as a Laurent-series. Also find the region of convergence

Solution. Putting $z-1 = u,$ we have

$$\frac{e^{2z}}{(z-1)^2} = \frac{e^{2(u+1)}}{u^3} = \frac{e^2}{u^3}(e^{2u})$$

$$= \frac{e^2}{u^3}\left[1+2u+\frac{4u^2}{2!}+\frac{8u^3}{3!}+\frac{16u^4}{4!}+\ldots\right]$$

$$= e^2\left[\frac{1}{u^3}+\frac{2}{u^2}+\frac{2}{u}+\frac{4}{3}+4u+\ldots\right]$$

$$= e^2\left[\frac{1}{(z-1)^3}+\frac{2}{(z-1)^2}+\frac{2}{z-1}+\frac{4}{3}+4(z-1)+\ldots\right] \text{for } z \neq 1.$$

The region of convergence is $|z-1| > 1.$

EXAMPLE 1.139

Find Taylor's series for $f(z) = \dfrac{z}{z+2}$ about z = 1. Also determine the region of convergence.

Solution. The singularity of f(z) is $z = -2.$ If the centre of the circle is taken as z = 1, then the distance of the singularity $z = -2$ from the centre is 3 units. If a circle of radius 3 with centre at 1 is drawn, then f(z) is analytic within the circle $|z-1| = 3.$ Hence f(z) can be expanded in a Taylor's series. The region of convergence is the interior of the circle $|z-1| = 3.$

We have

$$f(z) = \frac{z}{z+2} = 1-\frac{2}{z+2}$$

$$= 1-\frac{2}{(z-1)+3}$$

$$= 1-\frac{2}{3\left(1+\frac{z-1}{3}\right)}$$

$$= 1-\frac{2}{3}\left(1+\frac{z-1}{3}\right)^{-1}$$

$$= 1 - \frac{2}{3}\left[1 - \frac{z-1}{3} + \left(\frac{z-1}{3}\right)^2 - \left(\frac{z-1}{3}\right)^3 + \ldots\ldots\right]$$

$$= 1 - \frac{2}{3}\left[1 - \frac{z-1}{3} + \frac{1}{9}(z-1)^2 - \frac{1}{27}(z-1)^3 + \ldots\right].$$

EXAMPLE 1.140

Obtain Laurent's series of the function $f(z) = \dfrac{7z-2}{(z+1)z(z+2)}$ about $z = -2$.

Solution. Substitute $z + 2 = u$. Then

$$\frac{7z-2}{z(z+1)(z+2)} = \frac{7(u-2)-2}{(u-2)(u-1)(u)} = \frac{7u-16}{u(u-1)(u-2)}$$

$$= \frac{7u-16}{u}\left[\frac{1}{(u-1)(u-2)}\right]$$

$$= \frac{7u-16}{u}\left[\frac{1}{u-2} - \frac{1}{u-1}\right]$$

$$= \frac{16-7u}{u}\cdot\frac{1}{2-u} - \frac{16-7u}{u}\cdot\frac{1}{1-u}$$

$$= \left(\frac{16}{u} - 7\right)\left[\frac{1}{2\left(1-\frac{u}{2}\right)}\right] - \left(\frac{16}{u} - 7\right)(1-u)^{-1}$$

$$= \frac{1}{2}\left(\frac{16}{u} - 7\right)\left(1 - \frac{u}{2}\right)^{-1} - \left(\frac{16}{u} - 7\right)(1-u)^{-1}$$

$$= \left(\frac{8}{u} - \frac{7}{2}\right)\left(1 + \frac{u}{2} + \frac{u^2}{4} + \frac{u^3}{8} + \frac{u^4}{16} + \ldots\right) - \left(\frac{16}{u} - 7\right)(1 + u + u^2 + u^3 + u^4)$$

$$= \left(\frac{8}{u} + \frac{1}{2} + \frac{u}{4} + \frac{1}{8}u^2 + \frac{1}{16}u^3 + \ldots\right) - \left(\frac{16}{u} + 9 + 9u + 9u^2 + 9u^3 + \ldots\right)$$

$$= -\frac{8}{u} - \frac{17}{2} - \frac{35}{4}u - \frac{71}{8}u^2 - \frac{143}{16}u^3 - \ldots$$

$$= -\frac{8}{z+7} - \frac{17}{2} - \frac{35}{4}(z+2) - \frac{71}{8}(z+2)^2 - \frac{143}{18}(z+2)^3 + \ldots$$

EXAMPLE 1.141

Find the Laurent's expansion of $\dfrac{1}{z^2 - 4z + 3}$ for $1 < |z| < 3$.

Solution. We have

$$\frac{1}{z^2 - 4z + 3} = \frac{1}{(z-3)(z-1)} = \frac{1}{2}\left[\frac{1}{z-3} - \frac{1}{z-1}\right]$$

Therefore for $1 < |z| < 3$, the Laurent's expansion is

$$\frac{1}{z^2 4z+3} = \frac{1}{2}\left[\frac{-1}{3-z} \frac{1}{z-1}\right]$$

$$= \frac{1}{2}\left[\frac{-1}{3\left(1-\frac{z}{3}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}\right]$$

$$= -\frac{1}{6}\left(1-\frac{z}{3}\right)^{-1} - \frac{1}{2z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{6}\left[1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}+...\right] - \frac{1}{2z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+...\right]$$

$$= -\frac{1}{6}\left[1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}+...\right] - \frac{1}{2}\left[\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+...\right].$$

EXAMPLE 1.142

Evaluate $\oint_C \frac{z^3+z+1}{z^3-3z+2}dz$, where C is the ellipse $4x^2+9y^2=1$.

Solution. We want to evaluate $\oint_C \frac{z^3+z+1}{z^3-3z+2}dz$, where C is the ellipse $\frac{x^2}{\left(\frac{1}{2}\right)^2}+\frac{y^2}{\left(\frac{1}{3}\right)^2}=1$. The poles of $z^2-3z+2=0$ are given by $z=\frac{3\pm\sqrt{9-8}}{2}=2,1.$

Both singularities z = 2 and z = 1 lie outside the contour C. Hence, by Cauchy – Gorsat Theorem,

$$\oint_C \frac{z^3+z+1}{z^3-3z+2}dz = 0.$$

EXAMPLE 1.143

Find the poles and the residues at the poles of $f(z)=\frac{z}{z^2+1}$.

Solution. The poles of the function $f(z)=\frac{z}{z^2+1}$ are given by $z^2+1=0$. Thus the poles are $z=\pm i$. The residues at these poles are

$$\text{Res(i)} = \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}(z-i)\frac{z}{(z-i)(z+i)}$$

$$= \lim_{z\to i}\frac{z}{z+i} = \frac{i}{2i} = \frac{1}{2}$$

and

$$\text{Res(–i)} = \lim_{z\to -i}(z+i)f(z) = \lim_{z\to -i}(z+i)\frac{z}{(z-i)(z+i)}$$

$$= \lim_{z\to -i}\frac{z}{z-i} = \frac{-i}{-2i} = \frac{1}{2}.$$

EXAMPLE 1.144

Find the residues of $f(z)=\dfrac{z\,e^z}{(z-a)^3}$.

Solution. We are given that

$$f(z)=\frac{ze^z}{(z-a)^3},\ a\neq 0.$$

The poles of $f(z)$ are a, a, a. Thus a is pole of order 3. Therefore

$$\text{Res}(a)=\frac{1}{(3-1)!}\lim_{z\to a}\frac{d^{(3-1)}}{dz^{(3-1)}}\left[(z-a)^3 f(z)\right]$$

$$=\frac{1}{2}\lim_{z\to a}\frac{d^2}{dz^2}\left[ze^z\right]$$

$$=\frac{1}{2}\lim_{z\to a}\frac{d}{dz}\left[e^z+ze^z\right]$$

$$=\frac{1}{2}\lim_{z\to a}\left[e^z+e^z+ze^z\right]$$

$$=\frac{1}{2}\left[2e^a+ae^a\right]=\frac{(2+a)e^a}{2}.$$

EXAMPLE 1.145

Find the poles and residue of each pole of

(i) $f(z)=\dfrac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^2}$ and (ii) $f(z)=\dfrac{z\,e^3}{(z-1)^3}$

Solution. (i) The function f(z) has a pole of order 2 at $z=\frac{\pi}{6}$. The residue at $z=\frac{\pi}{6}$ is

$$\text{Res}\left(\frac{\pi}{6}\right)=\lim_{z\to\frac{\pi}{6}}\left\{\frac{d}{dz}\left[\left(z-\frac{\pi}{6}\right)^2 f(z)\right]\right\}$$

$$=\lim_{z\to\frac{\pi}{6}}\left\{\frac{d}{dz}\sin^2 z\right\}=\lim_{z\to\frac{\pi}{6}}\{2\sin z\cos z\}$$

$$=\lim_{z\to\frac{\pi}{6}}\{\sin 2z\}=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}.$$

(ii) The function f(z) has a pole of order. 3 at z = 1.Then

$$\text{Res}\ (1)=\frac{1}{2\,!}\lim_{z\to 1}\left\{\frac{d^2}{dz^2}(z-1)^3 f(z)\right\}$$

$$=\frac{1}{2}\lim_{z\to 1}\left\{\frac{d^2}{dz^2}(ze^z)\right\}$$

$$= \frac{1}{2}\lim_{z\to 1}[(z+2)e^z] = \frac{3}{2}e.$$

EXAMPLE 1.146

Evaluate $\int_C \frac{z-2}{z(z-1)}dz$, where C is $|z| = 3$.

Solution. We have

$$I = \int_{|z|=3} \frac{z-2}{z(z-1)}dz.$$

The integrand has simple poles at $z = 0$ and $z = 1$. Both poles lie in $|z| = 3$. Further,

$$\text{Res}\,(0) = \lim_{z\to 0}(z-0)\frac{z-2}{z(z-1)} = \frac{-2}{-1} = 2$$

$$\text{Res}\,(1) = \lim_{z\to 1}(z-1)\frac{z-2}{z(z-1)} = \frac{-1}{1} = -1.$$

Hence, by Cauchy's Residue Theorem ,we have

$$I = 2\pi i[2-1] = 2\pi i.$$

EXAMPLE 1.147

Evaluate $\int_{|z|=\frac{3}{2}} \frac{\cos \pi z^2}{(z-1)(z-2)}dz$.

Solution. The integrand has simple poles at z = 1 and z = 2, out of which only z = 1 lies in the contour $|z| = \frac{3}{2}$. Therefore

$$\text{Res}(1) = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\frac{\cos \pi\, z^2}{z-2}$$

$$= \frac{\cos \pi}{-1} = \frac{-1}{-1} = 1.$$

Hence, by Cauchy's Residue Theorem, we have

$$\int_{|z|=\frac{3}{2}} \frac{\cos \pi\, z^2}{(z-1)(z-2)}dz = 2\pi\, i\; \Sigma\ (\text{Residues})$$

$$= 2\pi\, i\,(1) = 2\pi\, i.$$

EXAMPLE 1.148

Evaluate $\int_C \frac{ze^z}{z^2+9}dz$, where c is $|z| = 5$, by Cauchy's Residue Theorem.

Solution. The poles of the integrand are given by $z^2 + 9 = 0$ and so $z = \pm 3i$ are two simple poles of $f(z) = \frac{ze^z}{z^2+9}$. Both of these poles lie inside $|z| = 5$. Further,

$$\text{Res}(3i) = \lim_{z\to 3i}(z-3i)(f(z) = \lim_{z\to 3i}(z-3i)\frac{ze^z}{(z-3i)(z+3i)}$$

$$= \lim_{z\to 3i}\frac{ze^z}{z+3i} = \frac{3ie^{3i}}{6i} = \frac{1}{2}e^{3i}$$

and

$$\text{Res}(-3i) = \lim_{z\to -3i}(z+3i)(f(z) = \lim_{z\to -3i}(z+3i)\frac{ze^z}{(z-3i)(z+3i)}$$

$$= \lim_{z\to -3i}\frac{ze^z}{z-3i} = \frac{-3ie^{-3i}}{-6i} = \frac{1}{2}e^{-3i}.$$

Therefore, by Cauchy's Residue Theorem,

$$\int_{|z|=5}\frac{ze^z}{z^2+9}dz = 2\pi i\Sigma \text{ (Residues at the poles)}$$

$$= 2\pi i\left[\frac{1}{2}(e^{3i}+e^{-3i})\right] = 2\pi i\cos 3.$$

EXAMPLE 1.149

Evaluate $\int_0^{2\pi}\cos 2\theta/(5-4\cos\theta)d\theta$ using contour integration.

Solution. Similar to Example 14.90. Putting $z = e^{i\theta}$, we get

$$\int_0^{2\pi}\frac{e^{2i\theta}}{5-4\cos\theta}d\theta = \frac{-1}{2i}\int_{|z|=1}\frac{z^2}{\left(z-\frac{1}{2}\right)(z-2)}dz.$$

The integrand has simple poles at $z=\frac{1}{2}, 2$ and out of these, only $z=\frac{1}{2}$ lies in $|z|=1$. Now

$$\text{Res}\left(\frac{1}{2}\right) = \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{z^2}{\left(z-\frac{1}{2}\right)(z-2)} = -\frac{1}{6}.$$

Hence

$$\int_0^{2\pi}\frac{e^{2i\theta}}{5-4\cos\theta}d\theta = 2\pi i\left(\frac{-1}{2i}\right)\left(-\frac{1}{6}\right) = \frac{\pi}{6}.$$

Equating real and imaginary parts, we get

$$\int_0^{2\pi}\frac{\cos 2\theta}{5-4\cos\theta}d\theta = \frac{\pi}{6}$$

EXAMPLE 1.150

Evaluate $\int_0^{\infty} dx/\left(x^2+a^2\right)\left(x^2+b^2\right)\ a>0; b>0$ using contour integration.

Solution. Proceed as in Example 1.101. Here

$$\text{Res }(ai) = \lim_{z\to ai}(z-ai)\frac{1}{(z-ai)(z+ai)(z^2+b^2)} = \frac{1}{2ai(b^2-a^2)},$$

$$\text{Res }(bi) = \lim_{z\to bi}(z-bi)\frac{1}{(z^2+a^2)(z+bi)(z-bi)} = \frac{1}{2bi(a^2-b^2)}.$$

Therefore

$$\int_c \frac{1}{(z^2+a^2)(z^2+b^2)}dz = 2\pi i\left[\frac{1}{2bi(a^2-b^2)} - \frac{1}{2ai(a^2-b^2)}\right]$$

$$= \frac{\pi}{a^2-b^2}\left[\frac{1}{b}-\frac{1}{a}\right] = \frac{\pi}{ab(a+b)}.$$

EXAMPLE 1.151

Expand $\frac{e^z}{(z-1)^2}$ about $z = 1$.

Solution. The given function is

$$f(z) = \frac{e^z}{(z-1)^2}.$$

Substituting $z-1=u$, we get

$$f(z) = \frac{e^{u+1}}{u^2} = \frac{e}{u^2}.e^u$$

$$= \frac{e}{u^2}\left[1+u+\frac{u^2}{2!}+\frac{u^3}{3!}+\ldots\right]$$

$$= \frac{e}{u^2}+\frac{e}{u}+\frac{e}{2!}+\frac{eu}{3!}+\ldots$$

$$= \frac{e}{(z-1)^2}+\frac{e}{z-1}+\frac{e}{2!}+\frac{1}{3!}(z-1)+\ldots.$$

$$= e\left[\frac{1}{(z-1)^2}+\frac{1}{z-1}+\frac{1}{2!}+\frac{1}{3!}(z-1)+\ldots.\right].$$

EXAMPLE 1.152

Evaluate $\oint_C \frac{e^z}{\cos \pi z}dz$, where C is the unit circle $|z| = 1$. How many poles, the function $\frac{e^z}{\cos \pi z}$ has?

Solution. The simple poles of the integrand $f(z) = \frac{e^z}{\cos \pi z}$ are given by $\cos \pi z = 0$. Thus the poles are

$$z = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$$

Out of these simple poles only $z = \pm\frac{1}{2}$ lie inside the contour $|z| = 1$. Now

$$\text{Res}\left(\frac{1}{2}\right) = \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)f(z) = \lim_{z\to\frac{1}{2}}\frac{\left(z-\frac{1}{2}\right)e^z}{\cos\pi z}\left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{z\to\frac{1}{2}}\frac{(z-\frac{1}{2})e^z+e^z}{-\pi\sin\pi z}(L\ \ Hospital\ \ Rule)$$

$$= \frac{e^{\frac{1}{2}}}{-\pi},$$

and

$$\text{Res}\left(-\frac{1}{2}\right) = \lim_{z\to-\frac{1}{2}}\left(z+\frac{1}{2}\right)f(z) = \lim_{z\to-\frac{1}{2}}\frac{\left(z+\frac{1}{2}\right)e^z}{\cos\pi z}$$

$$= \lim_{z\to-\frac{1}{2}}\frac{\left(z+\frac{1}{2}\right)e^z+e^z}{-\pi\sin\pi z} = \frac{e^{-\frac{1}{2}}}{\pi}.$$

Hence, by Cauchy – Residue Theorem

$$\oint_C\frac{e^z}{\cos\pi z}dz = 2\pi i\sum R_i = -4i\left[\frac{e^{\frac{1}{2}}-e^{\frac{-1}{2}}}{2}\right] = -4i\sin h\frac{1}{2}.$$

EXAMPLE 1.153

Evaluate $\int_0^\infty \frac{x^2}{x^6+1}dx$ by residues.

Solution. Proceeding as in Example 14.98, the simple poles of $\frac{1}{z^6+1}$ are at

$$z = e^{\pi i/6}, e^{3\pi i/6}, e^{5\pi i/6}, e^{7\pi i/6}, e^{9\pi i/6}, e^{11\pi i/6}.$$

The first three poles lie in the upper half plane. Let α denote any of these three poles. Then

$$\text{Res}\,(\alpha) = \left[\frac{z^2}{\frac{d}{dz}\left(z^6+1\right)}\right]_{z=\alpha}$$

$$= \frac{\alpha^2}{6\alpha^5} = \frac{\alpha^3}{6\alpha^6} = -\frac{\alpha^3}{6}\text{ since }\alpha^6 = -1.$$

Therefore the sum of the residues at these poles is

$$\sum R_i = -\frac{1}{6}\left[e^{\pi i/2}+e^{3\pi i/2}+e^{5\pi i/2}\right]$$

$$= -\frac{1}{6}[i - i + i] = -\frac{i}{6}.$$

Hence

$$\int_0^{\infty} \frac{x^2}{x^6+1}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{x^6+1}\,dx = \frac{1}{2}.2\pi i \sum R_i$$

$$= \pi i\left(-\frac{i}{6}\right) = -\frac{\pi}{6}.$$

EXAMPLE 1.154

Using Residue Theorem, show that

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}} \quad , \ a > b > 0.$$

Solution. Putting $z = e^{i\theta}$ so that $d\theta = \frac{dz}{iz}$ and $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$, we get

$$I = \int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2}{b}\int_{|z|=1} \frac{dz}{z^2 + \frac{2iaz}{b} - 1}.$$

Suppose that the poles are α and β. Than $\alpha + \beta = -\frac{2ia}{b}$ and $|\alpha\,\beta| = 1$. Then α, which is less than β lies inside $|z| = 1$ and

$$\text{Res}\,(\alpha) = \lim_{z\to\alpha}[(z-1)(f(z)]$$

$$= \frac{1}{\alpha - \beta} = \frac{1}{\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}}$$

$$= \frac{1}{\sqrt{\left(-\frac{2ia}{b}\right)^2 - 4(-1)}} = \frac{b}{2i\sqrt{a^2 - b^2}}.$$

Hence, by Cauchy's Residue Theorem,

$$I = 2\pi\, i\left[\frac{2}{b}\left(\frac{b}{2i\sqrt{a^2 - b^2}}\right)\right] = \frac{2\pi}{\sqrt{a^2 - b^2}}.$$

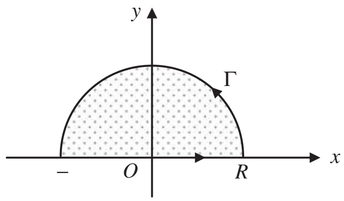
EXAMPLE 1.155

Using contour integration, evaluate $\int_0^{\infty} \frac{dx}{1+x^2}$.

Solution. Consider

$$I = \int_C \frac{1}{1+z^2}\,dz,$$

where C is the contour shown below:



The integrand has simple poles at $z = \pm i$ of which z = i lies in the contour. Therefore

$$\text{Res (i)} = \lim_{z\to i}(z-1)f(z) = \frac{1}{2i}\cdot$$

Hence, by Cauchy's Residue Theorem

$$\int_C \frac{dz}{z^2+1} = 2\pi\, i(\frac{1}{2i}) = \pi.$$

Thus

$$\int_{-R}^{R} \frac{dx}{x^2+1} + \int_{\Gamma} \frac{dz}{z^2+1} = \pi.$$

Since $\lim_{z\to\infty} zf(z) = \lim_{z\to\infty} \frac{z}{z^2+1} = 0$. Therefore

$$\int_{\Gamma} \frac{dz}{z^2+1} = 0 \text{ as } R \to \infty.$$

Hence

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \pi$$

or

$$\int_{0}^{\infty} \frac{dx}{x^2+1} = \frac{\pi}{2}.$$

EXAMPLE 1.156

Show that $\int_0^{\pi} \frac{\cos 2\theta}{1–2a\cos\theta + a^2} d\theta = \frac{\pi a^2}{1–a^2}(a^2 < 1).$

Solution. We have

$$I = \int_0^{\pi} \frac{\cos 2\theta}{1-2a\cos\theta + a^2} d\theta = \int_0^{\pi} \frac{\cos 2\theta(1+2a\cos\theta + a^2)}{1+a^2)^2 - 4a^2\cos^2\theta} d\theta$$

$$= \int_0^{\pi} \frac{(1+a^2)\cos 2\theta}{(1+a^2)^2 - 4a^2\cos^2\theta} d\theta + 2a\int_0^{\pi} \frac{\cos 2\theta\cos\theta}{(1+a^2)^2 - 4a^2\cos^2\theta} d\theta$$

$$= \int_0^{\pi} \frac{(1+a^2)\cos 2\theta}{(1+a^2) - 4a^2\cos^2\theta} d\theta + 0$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{(1+a^2)\cos\varphi\, d\varphi}{1+a^4 - 2a^2\cos\varphi}, \quad 2\theta = \phi$$

$$= \frac{1}{2i}\int_C \frac{(1+a^2)\frac{1}{z}\left(z+\frac{1}{z}\right)}{1+a^4-a^2\left(z+\frac{1}{z}\right)}\frac{dz}{z}, \quad z=e^{i\phi},\ c=|z|=1$$

$$= \frac{(1+a^2)}{4i}\int_C \frac{(z^2+1)dz}{z[(1+a^4)z-a^2z^2-a^2]}$$

$$= \frac{(1+a^2)i}{4a^2}\int \frac{(z^2+1)dz}{z\left[z^2-\left(a^2+\frac{1}{a^2}\right)z+1\right]}$$

$$= \frac{(1+a^2)i}{4a^2}\int_C \frac{(z^2+1)dz}{z(z-a^2)\left(z-\frac{1}{a^2}\right)}.$$

The integrand has simple poles at $z=0, z=a^2$ and $z=\frac{1}{a^2}$ of which the poles at $z=0$ and $z=a^2$ lie inside the circle $|z|=1$ since $a^2<1$. Sum of the residues at these pole is equal to

$$\frac{(1+a^2)i}{4a^2}\left[\lim_{z\to 0} zf(z)+\lim_{z\to a^2}(z-a^2)f(z)\right]$$

$$= \frac{(1+a^2)i}{4a^2}\left[2\frac{a^4}{a^4-1}\right] = \frac{a^2 i}{2(a^2-1)}.$$

Hence

$$I = 2\pi i\Sigma \text{ (Residues at the poles)}$$

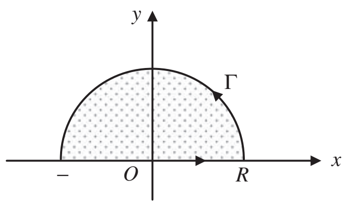
$$= 2\pi i\left(\frac{a^2 i}{2(a^2-1)}\right) = \frac{\pi a^2}{1-a^2}.$$

EXAMPLE 1.157

Show, by method of contour integration, that

$$\int_0^\infty \frac{\cos mx}{(a^2+x^2)^2} = \frac{\pi}{4a^3}(1+ma)e^{-ma}.$$

Solution. Consider $\int_C \frac{e^{mzi}}{(z^2+a^2)^2}dz$, where C is the contour shown below:



We have then

$$\int_C f(z)dz = \int_{-R}^{R} f(z)dz + \int_\Gamma f(z)dz = 2\pi\Sigma \text{ (Residues)}.$$

But, by Jordan Lemma, $\lim_{R\to\infty} \int_\Gamma f(z)dz = 0.$ Therefore as $R \to \infty$, we have

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi\, i\Sigma \text{ (Residues)}.$$

The integrand $f(z)$ has double pole at $z = \pm ai$ of which only $z = ai$ lies in the upper half-plane. But

$$\text{Res } (ai) = -\frac{e^{-ma}(am+1)}{4a^3}.$$

Therefore

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi\, i\left[-\frac{e^{-ma}(am+1)}{4a^3}i\right]$$

$$= \frac{\pi}{2a^3}(am+1)e^{-ma}.$$

Equating real and imaginary parts, we get

$$\int_{-\infty}^{\infty} \frac{\cos mx}{(x^2+a^2)^2}dx = \frac{\pi}{2a^3}(1+am)e^{-ma}$$

or

$$\int_{0}^{\infty} \frac{\cos mx}{(x^2+a^2)^2}dx = \frac{\pi}{4a^3}(1+am)e^{-ma}.$$

EXAMPLE 1.158

Show that the image of the hyperbole $x^2 - y^2 = 1$, under the transformation $\omega = \frac{1}{z}$, is $r^2 = \cos 2\theta$.

Solution. Let $z = r\,e^{i\theta}$ so that $x = r\cos\theta$, $y = r\sin\theta$. Let $\omega = \text{R}e^{i\phi}$. Then the inversion $\omega = \frac{1}{z}$ gives

$$\text{R}e^{i\phi} = \frac{1}{re^{i\theta}} \text{ and so } R = \frac{1}{r} \text{ and } \phi = -\theta.$$

The hyperbole $x^2 - y^2 = 1$, under this transformation becomes

$$r^2\cos^2\theta - r^2\sin^2\theta = 1$$

or

$$r^2(\cos^2\theta - \sin^2\theta) = 1$$

or

$$r^2\cos 2\theta = 1$$

or

$$\frac{1}{R^2}\cos(-2\phi) = 1$$

or

$$R^2 = \cos 2\phi.$$

Hence the hyperbole $x^2 - y^2 = 1$ transformation to the lemniscate $R^2 = \cos 2\phi$.

EXAMPLE 1.159

Show that the transformation $\omega = \dfrac{2z+3}{z-4}$ transform the circle $x^2 + y^2 - 4x = 0$ into a straight line.

Solution. Let $z = x + iy$ and $\omega = u + iv$. Then

$$x = \frac{z+\bar{z}}{2},\ x^2 + y^2 = z\bar{z} \text{ and } u = \frac{\omega + \bar{\omega}}{2}.$$

Therefore the equation of the given circle in z-plane reduces to

$$z\bar{z} - 2(z + \bar{z}) = 0 \tag{1.74}$$

The given transformation yields

$$\omega(z-4) = 2z + 3$$

or

$$z(\omega - 2) = 4\omega + 3$$

or

$$z = \frac{4\omega + 3}{\omega - 2}.$$

Therefore

$$\bar{z} = \frac{4\bar{\omega} + 3}{\bar{\omega} - 2}.$$

Therefore $\omega = \dfrac{2z+3}{z-4}$ transforms the circle (1.74) into

$$\left(\frac{4\omega+3}{\omega-2}\right)\left(\frac{4\bar{\omega}+3}{\bar{\omega}-2}\right) - 2\left(\frac{4\omega+3}{\omega-2} + \frac{4\bar{\omega}+3}{\bar{\omega}-2}\right) = 0$$

or

$$12(\omega + \bar{\omega}) + 16\omega\bar{\omega} + 9 - 2(8\omega\bar{\omega} - 5\omega - 5\bar{\omega} - 12) = 0$$

or

$$22(\omega + \bar{\omega}) + 33 = 0$$

or

$$2(\omega + \bar{\omega}) + 3 = 0$$

or

$$4u + 3 = 0,$$

which is a straight line in ω-plane

EXAMPLE 1.160

By the transformation $\omega = z^2$, show that the circle $|z - a| = c$ (a and c being real) in the z-plane corresponds to the binacon in the w-plane.

Solution. The equation of the given circle is

$$|z-a|=c$$

or

$$z-a=c\,e^{i\theta}$$

or

$$z=a+c\,e^{i\theta}.$$

Also then

$$\omega=z^2 \text{ implies } \omega-a^2=z^2-a^2=(z-a)(z+a)$$

$$=c\,e^{i\theta}(2a+ce^{i\theta}).$$

Therefore

$$\omega-(a^2-c^2)=c\,e^{i\theta}(2a+c\,e^{i\theta})+c^2$$

$$=c\,e^{i\theta}[2a+c\,e^{i\theta}+c\,e^{-i\theta}]$$

$$=c\,e^{i\theta}[2a+c(e^{i\theta}+e^{-i\theta})]$$

$$=c\,e^{i\theta}[2a+2c\cos\theta]$$

$$=2c\,e^{i\theta}[a+c\cos\theta].$$

If we take the pole (origin) at a^2-c^2, than we can take $\omega-(a^2-c^2)=R\,e^{i\phi}$ and so

$$R\,e^{i\phi}=2c\,e^{i\theta}(a+c\cos\theta),$$

which yields

$$R=2c(a+c\cos\phi) \text{ and } \phi=\theta$$

or

$$R=2c(a+c\cos\phi) \text{ (binacon in } w-\text{plane)}$$

Hence the circle $|z-a|=c$ is transformed into a binacon in the w-plane by the mapping $\omega=z^2$.

EXAMPLE 1.161

Find the bilinear transformation which maps the point $(-1,0,1)$ into the point $(0,i,3i)$.

Solution. Let the required transformation be

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}=\frac{(\omega-\omega_1)(\omega_2-\omega_3)}{(\omega-\omega_3)(\omega_2-\omega_1)}$$

We have $z_1=-1$, $z_2=0$, $z_3=1$, $\omega_1=0$, $\omega_2=i$ and $\omega_3=3i$.

Therefore

$$\frac{(z+1)(-1)}{(z-1)(1)} = \frac{\omega(i-3i)}{(\omega-3i)(i)}$$

or

$$\frac{z+1}{z-1} = \frac{2\omega}{\omega-3i}$$

or

$$z = \frac{3\omega-3i}{\omega+3i}$$

or

$$\omega = \frac{-3i(z+1)}{z-3},$$

which is the required bilinear transformation.

EXAMPLE 1.162

Find the image of infinite ship $0 < y < \frac{1}{2}$ under the transformation $\omega = \frac{1}{z}$.

Solution. We have

$$\omega = \frac{1}{z} \quad \text{or} \quad z = \frac{1}{\omega} = \frac{\overline{\omega}}{\omega\overline{\omega}} = \frac{\omega^2}{|\omega|^2} = \frac{u-iv}{u^2+v^2}.$$

or

$$x+iy = \frac{u-iv}{u^2+v^2} = \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$$

Equating real and imaginary parts, we have

$$x = \frac{u}{u^2+v^2} \quad \text{and} \quad y = \frac{-v}{u^2+v^2}.$$

Now $y = 0$ implies $v = 0$ and $y = \frac{1}{2}$ yields

$$u^2+v^2 = -2v \text{ or } u^2+v^2+2v = 0$$

or

$$u^2+(v+1)^2 = 1,$$

which is a circle with centre at $(0,-1)$ and radius 1 in the w-plane. It follows therefore that the line $y = 0$ (x-axis) is mapped into $v = 0$ (u-axis) and the line $y = \frac{1}{2}$ is transformed into the circle $u^2+(v+1)^2 = 1$. Thus the strip $0 < y < \frac{1}{2}$ in the z-plane is mapped into the region between the u-axis and the circle $u^2+(v+1)^2 = 1$ under the given inversion $\omega = \frac{1}{z}$.

EXERCISES

1. Solve the equation $e^{2z-1} = 1 + i$

   Hint: $1+i=\sqrt{2}\,e^{\frac{i\pi}{4}}$ (in exponential form).

   Therefore, $e^{2z-1}=\sqrt{2}\,e^{\frac{i\pi}{4}}$ and so $2z-1=\log\sqrt{2}+i\left(2n\pi+\frac{\pi}{4}\right)$.

   Hence $z=\frac{1}{2}+\frac{1}{4}\log 2+i\left(n+\frac{1}{8}\right)\pi$.

2. Find the values of $(1)^{\frac{1}{4}}$

   Ans. ± 1, ± i

3. Determine $(3z)^{\frac{1}{5}}$

   Ans. $2,\ 2\left(\cos\frac{2\pi}{5}\pm i\sin\frac{2\pi}{5}\right),\ 2\left(\cos\frac{4\pi}{5}\pm i\sin\frac{4\pi}{5}\right)$.

4. Express $\cos^8\theta$ in a series of cosines of multiples of θ.

   Ans. $2^{-7}[\cos 8\theta+8\cos 6\theta+28\cos 4\theta+56\cos 2\theta+35]$

5. Express $\sin^6\theta$ in a series of multiples of θ

   Ans. $-\frac{1}{2^5}[\cos 6\theta-6\cos 4\theta+15\cos 2\theta-10]$

6. Show that $\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$

7. Show that

$$i\log\frac{x-i}{x+i}=\pi-2\tan^{-1}x$$

8. Show that

$$\tan^{-1}x=\frac{1}{2i}\log\left(\frac{1+ix}{1-ix}\right).$$

   Hint: Substituting $1+ix=r(\cos\theta+i\sin\theta)$, we have $r\cos\theta=1$ and $r\sin\theta=x$ so that $\tan\theta=x$ or $\theta=\tan^{-1}x$ = L.H.S. Under the same substitution, we have R.H.S $=\frac{1}{2i}\cos\left(\frac{re^{i\theta}}{re^{-i\theta}}\right)=\frac{1}{2i}\cos(e^{2i\theta})=\frac{1}{2i}(2i\theta)=\theta$. Hence the result.

9. If $x + iy = \tan(A + iB)$, show that

$$x^2+y^2-2y\coth 2B+1=0$$

10. If $\sin(\theta + i\phi) = R(\cos\alpha + i\sin\alpha)$, show that

   (i) $R^2=\frac{1}{2}\{\cosh 2\phi-\cos 2\theta\}$

   (ii) $\tan\alpha=\tanh\phi\cot\theta$

Hint: $R(\cos a + i \sin a) = \sin \theta \cos i\phi + \cos \theta \sin i\phi = \sin \theta \cosh \phi + i \cos \theta \sinh \phi$. Therefore, equating real and imaginary parts, we get $R(\cos a = \sin \theta \cosh \phi$ and $R\sin a = \cos \theta \sin h\, \phi$. Squaring and adding, we get the result. Also dividing $R \sin a = \sin \theta \sinh a$ by $R \cos a = \sin \theta \cosh \phi$, we get the second result.

11. Separate log sin $(x + iy)$ into real and imaginary parts.

Ans. $\text{Re}[\log \sin(x+iy)] = \frac{1}{2}\log\left[\frac{1}{2}(\cosh 2y - \cos 2x)\right]$

$\text{Im}[\log \sin(x+iy)] = \tan^{-1}(\cot x \tanh y)$.

12. Show that $u = y^3 - 3x^2y$ is a harmonic function. Find its harmonic conjugate and the corresponding analytic function $f(z)$ in terms of z.

Ans. $v = -3xy^2 + x^3 + C, f(z) = iz^3 + Ci$

13. Show that the function $u = x^3 - 3xy^2$ is harmonic and find the corresponding analytic function.

Ans. $f(z) = z^3 + C$

14. If $f(z)$ an analytic function of z, prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |\text{Re} f(z)|^2 = 2|f'(z)|^2 .$$

Hint: As in Example 1.38, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\, \partial \bar{z}}$

$$|\text{Re } f(z)|^2 = |u|^2 = |\frac{1}{2}[f(z) + f(\bar{z})]|^2$$

Therefore,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |\text{Re} f(z)|^2 = \frac{\partial^2}{\partial z\, \partial \bar{z}} |f(z) + f(\bar{z})|^2$$

$$= \frac{\partial^2}{\partial z\, \partial \bar{z}} [(f(z) + f(\bar{z})\,(f(\bar{z}) + f(z))]$$

$$= \frac{\partial^2}{\partial z\, \partial \bar{z}} [f(z) + f(\bar{z})]^2 = \frac{\partial}{\partial z} .2[f(z) + f(\bar{z})] f'(z)$$

$$= 2f'(z) f'(\bar{z}) = 2|f'(z)|^2 .$$

15. Solve $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = x^2 - y^2$.

Hint: $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z\, \partial \bar{z}}, x^2 - y^2 = \frac{1}{2}(z^2 + \bar{z}^2)$.

Therefore, $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = x^2 - y^2$ implies $\frac{1}{2}(z^2 + \bar{z}^2) = 4\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial \bar{z}}\right)$.

Integrating w.r.t z, we get $\frac{\partial \phi}{\partial \bar{z}} = \frac{z^3}{24} + \frac{z\,\bar{z}^2}{8} + \phi_1(\bar{z})$

Integrating w.r.t. $\bar{z}$ now yields $\phi = \frac{z^3 \bar{z}}{24} + \frac{z\,\bar{z}^3}{24} + \phi_1(\bar{z}) + \phi_1(z)$

Replacing z by $x + iy$ and $\bar{z}$ by $x - iy$, we get

$$\phi = \frac{1}{12}[(x^4 - y^4) + \phi_1(x - iy) + \phi_2(x + iy)].$$

16. Find the analytic function $f(iz) = u + iv$, if $v = \left(r - \frac{1}{r}\right)\sin\theta$, $r \neq 0$.

Hint: By polar form of Cauchy-Riemann equation, $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$ (*). Thus $\frac{\partial u}{\partial r} = \frac{1}{r}\left[r - \frac{1}{r}\right]\cos\theta = \left(1 - \frac{1}{r^2}\right)\cos\theta.$

Integrating we get $u = \cos\theta\left(r + \frac{1}{r}\right) + \phi(\theta)$.

Then $\frac{\partial u}{\partial \theta} = -\sin\theta\left(r + \frac{1}{r}\right) + \phi'(0)$. But by (*) $\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} = -\left(r + \frac{1}{2}\sin\theta\right)$. Hence $\phi'(\theta) = 0$, which implies that $\phi(\theta)$ is constant. Hence $u = \cos\theta\left(r + \frac{1}{r}\right) + C$ and

$$\begin{aligned} f(z) &= u + iv \\ &= \cos\theta\left(r + \frac{1}{r}\right) + i\left(r - \frac{1}{r}\right)\sin\theta = C. \end{aligned}$$

17. If $u = x^2 - y^2$, find a function $f(z) = u + iv$ which is analytic.

Hint: $\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = -2y$ and so by Milne theorem, we have

$$\begin{aligned} f(z) &= \int [u_1(z,0) - iu_2(z,0)]\,dz = \int 2z \\ &= z^2 + C = x^2 - y^2 + i(2xy + C). \end{aligned}$$

18. If $f(z) = u + iv$ is an analytic function of z and $u - v = e^x(\cos y - \sin y)$, find $f(z)$.

Ans. $e^z + C$

19. Show that $f(z) = z + 2\bar{z}$ is not analytic anywhere in complex plane.
Hint: Cauchy-Riemann equations are not satisfied

20. Show that $\int \frac{dz}{z - a} = 2\pi i$, where C is the circle $|z - a| = r$.

21. Evaluate $\int_0^{3+i} z^2\,dz$ along $x = 3y^2$.

Ans. 4 + 3i

Hint: $z = x + iy = 3y^2 + iy$, $dz = (6y + i)dy$ and so the integral is $\int_0^1 (3y^2 + iy)(6y + i)\,dy$.

22. Evaluate $\int_{|z-1|=2} \frac{e^{2z}}{(z+1)^4}\,dz.$

Ans. $\frac{8\pi i}{3e^2}$

23. Evaluate $\int_{|z-1|=3} \frac{e^z}{(z+1)^4\,(z-2)}\,dz$

Ans. $\frac{2\pi i}{81}\left(e^2 - \frac{13}{e}\right)$

24. Evaluate $\int_{|z-i|=1} \frac{e^z}{z^2+1}\, dz.$

Hint: $\frac{e^z}{z^2+1}$ is analytic at all points except $\pm\, i$. The point $z = i$ lies inside $|z - i| = 1$. So, let $f(z) = \frac{e^z}{z+i}$. Then, by Cauchy integral formula, the given integral $= \pi\, e^i$.

25. Evaluate $I = \int_{|z|=2} \frac{z^3 - 2z + 1}{(z-i)^2}\, dz$.

Hint: By Cauchy integral formula, $I = 2\, i\, f'(i) = 2\pi i\, [3z^2 - 2]_{z=i} = 2\, i(-3 - 2) = -10\, \pi i.$

26. Expand $\log(1 + z)$ in a Taylor series about the point $z = 0$ and find the region of convergence of the series.

Ans. $f(z) = z - \frac{z^2}{2} + \frac{z^3}{3} + \ldots + (-1)^{n-1}\frac{z^n}{n} + \ldots$ This series converges for $|z| < 1$.

27. Expand $f(z) = \frac{z}{(z^2-1)(z^2+4)}$ as a Laurent series about $1 < |z| < 2$.

Hint: Use partial fraction and take cases of $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1$.

Ans. $\frac{1}{3}\left(\frac{1}{z^5} - \frac{1}{z^3} - \frac{z}{4} + \frac{z^3}{16} - \frac{z^5}{64} + \ldots\right)$

28. If $0 < |z| < 4$, show that $\frac{1}{4z - z^2} = \sum_{n=0}^{\infty} \frac{z^{n-1}}{4^{n+1}}$.

29. Find the singularities with their nature of the function $\frac{e^{\frac{c}{z-a}}}{e^{z/a} - 1}$.

Ans. Simple poles at $= 2\pi\, nia$, $n = 0, \pm 1, \pm 2, \ldots$

30. Find residues at each poles of $f(z) = \frac{z^2 - 2z}{(z+1)^2(z^2+4)}$.

Ans. $\text{Res}(1) = -\frac{14}{25}, \text{Res}(2i) = \frac{7+i}{25}, \ \text{Res}(-2i) = \frac{7-i}{25}$

31. Evaluate $I = \int_{|z|=4} \frac{e^z}{(z^2+\pi^2)^2}\, dz.$

Ans. $-\frac{i}{\pi}$

32. Evaluate $\int_{|z-2|=2} \frac{3z^2+2}{(z-1)(z^2+9)}\, dz.$

Ans. πi

33. Evaluate $\int_{|z|=2} \tan z\, dz$.

Ans. $-4\pi i$

34. Show that $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}, a > b > 0.$

35. Evaluate $\int_0^{2\pi} \frac{d\theta}{17-8\cos\theta}.$

Ans. $\frac{\pi}{15}$

36. Show that $\int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2} = \frac{5\pi}{32}.$

37. Show that $\int_0^{\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2} d\theta = \frac{\pi a^2}{1-a^2} (a^2 < 1).$

38. Show that $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}.$

39. Show that $\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}.$

40. Show that $\int_{-\infty}^{\infty} \frac{x^3 \sin mx}{x^4+a^4} = \frac{\pi}{2} e^{-ma/\sqrt{2}} \cos\left(\frac{ma}{\sqrt{2}}\right), m > 0, a > 0.$

Hint: Use Jordan lemma, the poles are $ae^{(2n+1)\pi i/4}$, poles $ae^{\pi i/4}$ and $ae^{i3\pi/4}$ lie in the upper half-plane.

41. Show that

$$\int_0^{\infty} \frac{xdx}{(x^2+1)(x^2+2x+2)} = \frac{-\pi}{5}.$$

42. Show that

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)} = \frac{\pi}{3}.$$

43. Show that

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}.$$

44. Evaluate

$$\int_0^{\infty} \frac{dx}{x^4+x^2+1}$$

Ans. $\pi(\sqrt{3}/6)$

45. Show that

$$\int_0^\infty \frac{d\theta}{1-2r\cos\theta+r^2} = \frac{\pi}{1-r^2}.$$

46. Show that

$$\int_0^\infty \frac{\sin \pi x}{x(1-x^2)}\,dx = \pi.$$

47. Discuss the transformation $w = \sqrt{z}$.
Hint: Letting $z = x + iy$, $w = u + iv$, we have $u^2 - v^2 = x$ and $2uv = y$. The lines $x = a$ and $y = b$ correspond to the rectangular hyperbolas $u^2 - v^2 = a$ and $2uv = b$, which are orthogonal to each other.

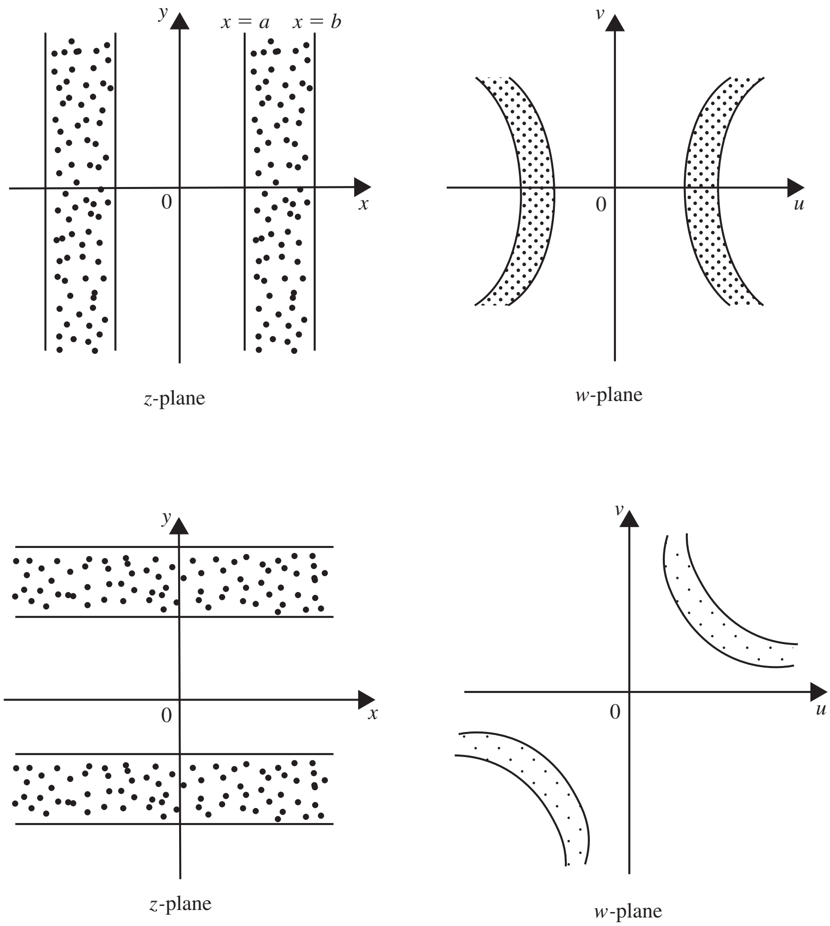


Figure 1.29

48. Discuss the mapping $w = \dfrac{z-1}{z+1}$.

Hint: $z = \frac{1+w}{1-w}$. Therefore, $x + iy = \frac{1+u+iv}{1-u-iv} \cdot \frac{1-u+iv}{1-u+iv} = \frac{1-u^2-v^2}{(1-u)^2+v^2}$. Hence $x = \frac{1-u^2-v^2}{(1-u)^2+v^2}, y = \frac{2u}{(-u)^2+v^2}$ and so on.

49. Find the fixed points of the mapping $\omega(z) = \frac{3z-4}{z-1}$.

Hint: Fixed points of the given mapping are given by $z = \frac{3z-4}{z-1}$, $z^2 - z = 3z - 4$, or $z^2 - 4z + 4 = 0$. Hence $z = \frac{4 \pm \sqrt{16-16}}{2} = 2$ is the fixed point of the mapping.

50. Find the bilinear transformation that maps the points $z = -1, 0, 1$ in the $z-$ plane on to the points $\omega = 0, i, 3i$ in the ω - plane.

Hint: The bilinear transformation is given by $\omega = \frac{az+b}{cz+d}, ad - bc \neq 0$. Therefore, we have $0 = \frac{a(-1)+b}{c(-1)+d}, i = \frac{a(0)+b}{c(0)+d}$, and $3i = \frac{a(1)+b}{c(1)+d}$. From the first equation we have $a = b$. Then second and the third equations hold $b = -ai$ and $c = \frac{ai}{3}$. Hence substituting these values in $\omega = \frac{az+b}{cz+d}$, we get $\omega = \frac{3(z+1)}{i(z-3)}$.

2 Elements of Statistics and Probability

Statistics is the science of assembling, analysing, characterizing, and interpreting the collection of data (information expressed numerically). The methods used for this purpose are called s*tatistical methods*. The general characteristics of data are:

1. Data shows a tendency to concentrate at certain values, usually somewhere in the centre of the distribution. Measures of this tendency are called *measures of central tendency* or *averages.*
2. The data varies about a measure of central tendency and the measures of deviation are called *measures of variability* or *dispersion.*
3. The data in a frequency distribution may fall into symmetrical or asymmetrical patterns. The measures of the degree of asymmetry are called the *measures of skewness.*
4. The measures of peakedness or flatness of the frequency curves are called *measures of kurtosis*.

If the figures in the original data are put into groups, then those groups are called *classes*. The difference between the upper and lower limits of a class is called the *width of the class* or simply the *class interval*. The number of observations in a class interval is called the *frequency.* The mid-point or the mid-value of the class is called the *class mark.* The table showing the classes and the corresponding frequencies is called a *frequency table*. The set of ungrouped data summarized by distributing it into a number of classes along with their frequencies is known as *frequency distribution*. The *cumulative frequency* (written as cum f) of the nth class in a frequency distribution is the sum of the frequencies beginning with the first and ending with the nth frequency. Thus

$$\text{Cum } f_n = \sum_{i=1}^{n} f_i.$$

For example, consider the following table:

Marks in physics (class)	Number of students (f)	Cum (f)
50–60	5	5
60–70	16	21
70–80	24	45
80–90	25	70
9–100	20	90
Total	**90**	

In this frequency table, the marks obtained by 90 students in physics have been divided into classes with class interval 10. The frequency for the interval 50–60 is 5 whereas it is 16 for the class interval 60–70. The cumulative frequency of the class interval 70–80 is 45.

2.1 MEASURES OF CENTRAL TENDENCY

The commonly used measures of central tendency are mean, median, and mode. We define these concepts one by one.

1. **The Mean:** The arithmetic mean $\overline{x}$ of a set of n values $x_1, x_2, ..., x_n$ of a variate is defined by the formula

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

The *weight* of a value of variate is a numerical multiplier assigned to indicate its relative importance. The *weighted arithmetic mean* of set of variates $x_1, x_2, \ldots, x_n$ with weights $w_1, w_2, \ldots, w_n$, respectively, is defined by

$$\overline{x} = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}.$$

Thus, in a frequency distribution, if $x_1, x_2, \ldots, x_n$ are the mid-values of the class intervals having frequencies $f_1, f_2, \ldots, f_n$, respectively, then

$$\overline{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}.$$

Let $d_i = x_i - A$. Then

$$\sum_{i=1}^{n} f_i d_i = \sum_{i=1}^{n} f_i x_i - A\sum_{i=1}^{n} f_i.$$

Therefore,

$$\frac{\sum_{i=1}^{n} f_i d_i}{\sum_{i=1}^{n} f_i} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i} - A = \overline{x} - A$$

or

$$\overline{x} = A + \frac{\sum_{i=1}^{n} f_i d_i}{\sum_{i=1}^{n} f_i}.$$

This formula, obtained by shifting the origin, is more convenient to find the mean.

2. **The Median:** Suppose that n values $x_1, x_2, \ldots, x_n$ of a variate have been arranged in the following order of magnitudes,

$$x_1 \le x_2 \le x_3 \le ... \le x_n.$$

Then the *median* of this ordered set of values is the value $x_{\frac{n+1}{2}}$ when n is odd, and the value $\frac{1}{2}\left(x_{\frac{n}{2}} + x_{\frac{n}{2}+1}\right)$ when n is even.

The median for the discrete frequency distribution is obtained as follows:

(i) Determine $\frac{1}{2}\sum f_i$.

(ii) Note the cumulative frequency just greater than $\frac{1}{2}\sum f_i$.

(iii) Find the value of x corresponding to the cumulative frequency obtained in step (ii). This value will be the median.

The median for the continuous frequency distribution is obtained as follows:

(i) Note the class corresponding to the cumulative frequency just greater than $\frac{1}{2}\sum f_i$. This class is known as *median class*.

(ii) Compute the value of median by the formula.

$$\text{Median} = L + \frac{h}{f}\left(\frac{1}{2}\sum f_i - c\right),$$

where

L is the lower limit of the median class

f is the frequency of the median class

h is the width of the median class

c is the cumulative frequency of the class preceeding the median class.

3. **The Mode:** The mode is defined as that value of a variate which occurs most frequently.

For example, in the frequency distribution

x:	1	2	3	4	5	6
f:	3	7	28	10	9	5

the value of x corresponding to the maximum frequency, namely, 28 is 3. Hence mode is 3.

For a grouped distribution, mode is given by

$$\text{Mode} = L + \frac{\Delta_1}{\Delta_1 + \Delta_2} h,$$

where

L = lower limit of the class containing the mode

Δ_1 = excess of modal frequency (maximum) over frequency of preceeding class

Δ_2 = excess of modal frequency over frequency of succeeding class

h = width of modal class.

The empirical relationship between mean, median, and mode of a frequency distribution is

$$\text{mean} - \text{mode} = 3\,(\text{mean} - \text{median}).$$

However, for a symmetrical distribution, the mean, median, and mode coincide.

For example, consider the following distribution

Class–interval:	0–10	10–20	20–30	30–40	40–80	50–60
Frequency:	6	8	14	26	17	10

The maximum frequency is 26 and $h = 10$. Further,

$$L = 30, \Delta_1 = 26 - 14 = 12, \Delta_2 = 26 - 17 = 9.$$

Therefore

$$\text{Mode} = 30 + \frac{12}{12+9}(10) = 30 + \frac{120}{21} = 35.714.$$

Apart from the above measures of central tendency, we consider now the following partition values of the frequency:

The *partition values* are those values which divide the series of frequencies into a number of equal parts.

The three values, which divide the series of the given frequencies into four equal parts are called *quartiles.* The *lower (first) quartile*, Q_1, is the value which exceeds 25% of the observations and is exceeded by 75% of the observations. The *second quartile*, Q_2, coincides with the mean whereas the *third quartile*, Q_3, is the value which exceeds 75% observations and has 25% observations after it. In fact, if $N = \sum_{i=1}^{n} f_i$, L the lower limit of the median class, h the magnitude of the median class, and f the frequency of the median class, then

$$Q_1 = L + \frac{\frac{N}{4} - Cum(f)}{f}.h, \text{ and}$$

$$Q_3 = L + \frac{\frac{3N}{4} - Cum(f)}{f}.h.$$

Similarly, the 9 values which divide the frequency series into 10 equal parts are called *deciles* whereas the 99 values which divide the frequency series into 100 equal parts are called *percentites*.

EXAMPLE 2.1

Determine the mean, median, and mode for the following data:

Mid value:	15	20	25	30	35	40	45	50	55
Frequency:	2	22	19	14	3	4	6	1	1
Cum f:	2	24	43	57	60	64	70	71	72

Solution. For the given frequency distribution, we have

$$\text{Mean } \bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$$

$$= \frac{2(15) + 22(20) + 19(25) + 14(30) + 3(35) + 4(40) + 6(45) + 1(50) + 1(55)}{2 + 22 + 19 + 14 + 3 + 4 + 6 + 1 + 1}$$

$$= \frac{2005}{72} = 27.85.$$

To compute the median, we note that $\frac{1}{2}\sum f_i = 36$. The median class (corresponding to cum frequency 43) is (20–30). Width of the median class is 10. Frequency of the median class is 19. The cumulative frequency of the class preceding to median class is 24. Therefore,

$$\begin{aligned}\text{Median} &= L + \frac{h}{f}\left(\frac{1}{2}\sum f_i - c\right)\\ &= 20 + \frac{10}{19}(36 - 24)\\ &= 23.32.\end{aligned}$$

To calculate the mode, we note that the maximum frequency is 22, that is, the modal frequency is 22. Then the modal class is (15–25). Therefore,

$$\begin{aligned}\text{Mode} &= L + \frac{\Delta_1}{\Delta_1 + \Delta_2}h,\\ &= 15 + \frac{22-2}{(22-2)+(22-19)}.10\\ &= 15 + \frac{200}{23} = 23.69.\end{aligned}$$

EXAMPLE 2.2

Obtain the median for the following distribution:

x:	1	2	3	4	5	6	7	8	9
f:	8	10	11	16	20	25	15	9	6

Solution. For the given discrete frequency distribution, we have $\frac{1}{2}\sum f_i = \frac{120}{2} = 60.$ The cumulative frequencies are

8, 18, 29, 45, 65, 90, 105, 114, 120

The cumulative frequency just greater then 60 is 65. The value of x corresponding to 65 is 5. Hence the median is 5.

EXAMPLE 2.3

Given that the median value is 46, find the missing frequencies for the following incomplete frequency distribution:

Class:	10–20	20–30	30–40	40–50	50–60	60–70	70–80	Total
f:	12	30	–	65	–	25	18	229

Solution. Suppose that the frequency of the class 30–40 be f_1 and that for the class 50–60 be f_2. Also $\Sigma f_i = 229$. Therefore,

$$f_1 + f_2 + (12 + 30 + 65 + 25 + 18) = 229$$

and so $f_1 + f_2 = 79$. Since the median is 46, the median class is 40–50. Therefore, using the formula

$$\text{Mode} = L + \frac{h}{f}\left(\frac{1}{2}\sum f_i - c\right),$$

we have

$$46 = 40 + \frac{10}{65}\left(\frac{229}{2} - c\right),$$

where c is the cumulative frequency of the class preceding the median class. Since the cumulative frequency are

$$12, 42, 42 + f_1, 107 + f_1, 132 + f_1 + f_2, 157 + f_1 + f_2, 175 + f_1 + f_2,$$

the value of c is $42 + f_1$. Hence

$$46 = 4 + \frac{10}{65}\left(\frac{229}{2} - (42 + f_1)\right),$$

which yields $f_1 = 33.5 \approx 34$. Then $f_2 = 79 - 34 = 45$. Hence the missing frequencies are 34 and 45.

2.2 MEASURES OF VARIABILITY (DISPERSION)

The measures of central tendency give us idea of the concentration of the observation about the central part of the distribution. They fail to give information whether the values are closely packed about the central value or widely scattered away from it. The two different distributions may have the same mean and same total frequency, yet they may differ in the sense that the individual values spread about the average differently. Thus, the measures of central tendency must be supplemented by some other measures to have the complete idea of distribution. One such measure is *dispersion.*

The degree to which numerical data tends to spread about an average value is called *variability* or *dispersion* of the data.

We now define some of the important measures of dispersion.

1. **Range:** The range is the difference of the greatest and the least values in the distribution. This is the simplest but a crude measure of dispersion.
2. **The Mean Deviation:** The mean deviation of a set of n values $x_1, x_2, \ldots, x_n$ of a variate is defined as the arithmetic mean of their absolute deviations from their average A (usually mean, median, or mode).

Thus if we consider the average as arithmetic mean of $x_1, x_2, \ldots, x_n$ then

$$\text{Mean deviation(M.D.)} = \frac{1}{n}\sum_{i=1}^{n} |x_i - \overline{x}|.$$

If $x_i|f_i$, $i = 1, 2, \ldots, n$ is the frequency distribution, the

$$\text{M.D.} = \frac{1}{N}\sum_{i=1}^{n} f_i\,|x_i - \overline{x}|, N = \sum_{i=1}^{n} f_i.$$

3. **The Variance:** Since mean deviation is based on all the observations, it is a better measure of dispersion than the range. But, in the definition, we have converted all minus signs to plus before averaging the deviations. Another method of eliminating minus sign is to square the deviations and then average these squares. This step gives rise to a most powerful measure of dispersion, called variance, defined as follows:

The *variance*, S^2, of a sample of n values $x_1, x_2, \ldots, x_n$ of a variate with arithmetic mean $\overline{x}$ is defined as the $\frac{1}{n}$ th of the sum of squares of their deviations from the mean. Thus

$$S^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i - \overline{x})^2.$$

If $x_i \mid f_i, i = 1, 2, ..., n$ is the frequency distribution, then

$$S^2 = \frac{1}{N}\sum_{i=1}^{n} f_i\,(x_i - \overline{x})^2,\, N = \sum_{i=1}^{n} f_i.$$

4. **The Standard Deviation:** It is defined as the positive square root of the variance. If is denoted by σ. Thus

$$\sigma = \left[\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2\right]^{\frac{1}{2}}.$$

In case of frequency distribution $x_i \mid f_i, i =$ 1, 2, …, n, we have

$$\sigma = \left[\frac{1}{N}\sum_{i=1}^{n} f_i(x_i - \overline{x})^2\right]^{\frac{1}{2}},\; N = \sum_{i=1}^{n} f_i.$$

5. **Quartile Deviation:** The quartile deviation Q is defined as

$$Q = \frac{1}{2}(Q_3 - Q_1),$$

where Q_1 and Q_3 are the first and third quartiles of the distribution, respectively.

Theorem 2.1. For the frequency distribution $x_i \mid f_i, i =$ 1, 2,…, n,

$$S^2 = \frac{1}{\sum f_i}\sum f_i x_i^2 - (\overline{x})^2.$$

Proof: We have

$$\begin{aligned} S^2 &= \frac{1}{\sum f_i}\sum f_i(x_i - \overline{x})^2 \\ &= \frac{1}{\sum f_i}\sum f_i[x_i^2 + (\overline{x})^2 - 2x_i\overline{x}] \\ &= \frac{1}{\sum f_i}\sum f_i x_i^2 + \frac{1}{\sum f_i}\sum f_i(\overline{x})^2 - \frac{2\overline{x}}{\sum f_i}\sum f_i x_i \\ &= \frac{1}{\sum f_i}\sum f_i x_i^2 + (\overline{x})^2 - 2\overline{x}\frac{\sum f_i x_i}{\sum f_i} \\ &= \frac{1}{\sum f_i}\sum f_i x_i^2 + (\overline{x})^2 - 2(\overline{x})^2 \\ &= \frac{1}{\sum f_i}\sum f_i x_i^2 - (\overline{x})^2. \end{aligned}$$

The ratio of the standard deviation to the mean is known as the *coefficient* of *variation.* Thus

$$\text{Coefficient of variation} = \frac{\sigma}{\overline{x}}.$$

Theorem 2.2. Variance and, hence, the standard deviation is independent of the change of origin.

Proof: From above, the variance is given by

$$S^2 = \frac{1}{\sum f_i}\sum f_i (x_i - \bar{x})^2.$$

Let $d_i = x_i - A$. Then

$$x_i - \bar{x} = (x_i - A) - (\bar{x} - A) = d_i - (\bar{x} - A)$$

and so

$$\begin{aligned}\sum f_i (x_i - \bar{x})^2 &= \sum f_i [d_i - (\bar{x} - A)]^2 \\ &= \sum f_i d_i^2 + (\bar{x} - A)^2 \sum f_i - 2(\bar{x} - A)\sum f_i d_i \\ &= \sum f_i d_i^2 - \frac{\left(\sum f_i d_i\right)^2}{\sum f_i}, \quad \bar{x} = A + \frac{\sum f_i d_i}{\sum f_i}.\end{aligned}$$

Therefore,

$$S^2 = \frac{\sum f_i d_i^2}{\sum f_i} - \left(\frac{\sum f_i d_i}{\sum f_i}\right)^2,$$

and

$$\sigma = \sqrt{\frac{\sum f_i d_i^2}{\sum f_i} - \left(\frac{\sum f_i d_i}{\sum f_i}\right)^2}.$$

Moments: *The rth moment about the mean* $\bar{x}$ of a distribution, denoted by μ_r, is defined by

$$\mu_r = \frac{1}{N}\sum_{i=1}^{n} f_i (x_i - \bar{x})^r, \quad N = \sum_{i=1}^{n} f_i.$$

The moment about any point a is defined by

$$\mu_r' = \frac{1}{N}\sum_{i=1}^{n} f_i (x_i - a)^r, \quad N = \sum_{i=1}^{n} f_i.$$

We note that

$$\begin{aligned}\mu_0 &= 1 = \mu_0', \\ \mu_1 &= \frac{1}{N}\sum_{i=1}^{n} f_i (x_i - \bar{x}) = \frac{1}{N}\sum f_i x_i - \frac{1}{N}\sum f_i \bar{x} \\ &= \bar{x} - \bar{x}\cdot\left(\frac{1}{N}\sum f_i\right) = \bar{x} - \bar{x} = 0, \\ \mu_1' &= \frac{1}{N}\sum_{I=1}^{n} f_i (x_i - a) \\ &= \frac{1}{N}\sum_{i=1}^{n} f_i x_i - \frac{1}{N}\sum_{i=1}^{n} f_i a = \bar{x} - a, \text{ and} \\ \mu_2 &= \frac{1}{N}\sum_{i=1}^{n} f_i (x_i - \bar{x})^2 = \sigma^2,\end{aligned}$$

where σ is the standard deviation.
If can be shown that

$$\mu_2 = \mu_2' - \mu_2'^2$$
$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$
$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

Consider the function

$$M_a(t) = \sum p_i e^{t(x_i - a)}.$$

This function is a function of the parameter t and is nothing but mean (expected value) of the probability distribution of $e^{t(x_i-a)}$. Expanding the exponential, we get

$$M_a(t) = \sum p_i[1 + t(x_i - a) + \frac{t^2}{2}(x_i - a)^2 + \ldots + \frac{t^r}{r!}(x_i - a)^r + \ldots]$$

$$= \sum p_i + t\sum p_i(x_i - a) + \frac{t^2}{2!}\sum p_i(x_i - a)^2 + \ldots + \frac{t^r}{r!}\sum p_i(x_i - a)^r + \ldots \quad (2.1)$$

$$= 1 + t\mu_1 + \frac{t^2}{2!}\mu_2 + \ldots + \frac{t^r}{r!}\mu_r + \ldots,$$

where μ^r is the moment of order r about a. Thus $M_a(t)$ generates moments and is, therefore, called the *moment generating function* of the discrete probability distribution of the variate X about the value $x = a$. Thus, the moment generating function of the discrete probability distribution of the variate X about $x = a$ is defined as the expected value of the function $e^{t(x-a)}$.

We observe that μ_r, the rth moment, is equal to the coefficient of $\frac{t^r}{r!}$ in the expansion of the moment generating function $M_a(t)$.

Alternately, μ_r can be obtained by differentiating (2.1) r times with respect to t and then putting $t = 0$. Thus

$$\mu_r = \left[\frac{d^r}{dt^r}M_a(t)\right]_{t=0}.$$

Also

$$M_a(t) = \sum p_i e^{t(x_i-a)} = e^{-at}\sum p_i e^{tx_i} = e^{-at}M_o(t).$$

Hence moment generating function about the value a is e^{-at} times the moment generating function about the origin.

2.3 MEASURES OF SKEWNESS

As pointed out earlier, the measure of skewness is the degree of asymmetry or the departure from the symmetry. Regarding skewness, we have

(i) *Pearson's coefficient of skewness, which is equal to* $\dfrac{mean - mode}{\sigma}$

(ii) Coefficient of skewness based on third moment is given by

$$\gamma_i = \sqrt{\beta_1},$$

where

$$\beta_1 = \frac{\mu_3^2}{\mu_2^2}.$$

Therefore, the *simplest measure of skewness* is $\gamma_i = \sqrt{\beta_1}$.

2.4 MEASURES OF KURTOSIS

The measures of peakness or flatness of the frequency curve, called the *measure of kurtosis*, is defined by

$$\beta_2 = \frac{\mu_4}{\mu_2^2}.$$

Further, $\gamma_2 = \beta_2 - 3$ yields the *excess of kurtosis*. The curves with $\gamma_2 > 0$ that is, $\beta_2 > 3$ are called *leptokurtic* and the curves with $\gamma_2 < 0$, that is, $\beta_2 < 3$ are called *platykurtic*. The curve (normal curve) for which $\gamma_2 = 0$, that is, $\beta_2 = 3$ is called *mesokurtic*. Thus, *the normal curve is symmetrical about its mean*.

EXAMPLE 2.4

The following table shows the marks obtained by 100 candidates in an examination. Calculate the mean, median, and standard deviation:

Marks obtained:	1–10	11–20	21–30	31–40	41–50	51–60
No. of candidates:	3	16	26	31	16	8

Solution. We form the table shown below:

Class	Mid-value x	Frequency f	Cum frequency	fx	fx^2
1–10	5.5	3	3	16.5	90.75
11–20	15.5	16	19	248	3844
21–30	25.5	26	45	663	16906.5
31–40	35.5	31	76	1100.5	39067.75
41–50	45.5	16	82	728	33124
51–60	55.5	8	90	444	24642
		100		**3200**	**117675**

Then

$$\text{Mean}(\bar{x}) = \frac{\sum f_i x_i}{\sum f_i} = \frac{3200}{100} = 32.$$

Since $\frac{1}{2}\sum f_i = 50$, the median class is corresponding to the cum frequency 76. Thus the median class is 31–40. Therefore,

$$\text{Median} = L + \frac{h}{f}\left(\frac{1}{2}\sum f_i - c\right)$$

$$= 31 + \frac{10}{31}(50–45) = 32.6.$$

Further, variance is given by

$$S^2 = \frac{1}{\sum f_i}\sum f_i x_1^2 - (\bar{x})^2$$

$$= \frac{117675}{100} - (32)^2 = 152.75$$

and so the standard deviation is

$$\sigma = \sqrt{S^2} = 12.36 \approx 12.4.$$

EXAMPLE 2.5

The score obtained by two batsmen A and B in 10 matches are follows:

A:	30	44	66	62	60	34	80	46	20	38
B:	34	46	70	38	55	48	60	34	45	30

Determine who is more efficient and consistent.

Solution. The mean $\bar{x}_A$ for the batsman A is

$$\bar{x}_A = \frac{1}{10}\sum x_i = \frac{480}{10} = 48.$$

The variance for the batsman A is

$$S_A^2 = \frac{1}{10}\sum (x_i - \bar{x}_A)^2$$

$$= \frac{1}{10}[(48-20)^2 + (48-44)^2 + (48-66)^2 + (48-62)^2 + (48-60)^2 + (48-34)^2$$
$$+ (48-80)^2 + (48-20)^2 + (48-38)^2]$$

$$= \frac{1}{10}[324 + 16 + 324 + 196 + 144 + 196 + 1024 + 784 + 100]$$

$$= 310.8.$$

The coefficient of variation $= \frac{\sigma_A}{\bar{x}_A} = \frac{\sqrt{310.8}}{48} = 0.37.$ On the other hand, the mean $\bar{y}_B$ for the batsman B is

$$\bar{y}_B = \frac{1}{10}\sum y_i = \frac{460}{10} = 46.$$

The variance for the batsman B is

$$S_B^2 = \frac{1}{10}\sum (y_i - \bar{y}_B)^2$$

$$= \frac{1}{10}[(46-34)^2 + (46-46)^2 + (46-70)^2 + (46-38)^2 + (46-55)^2 + (46-48)^2$$
$$+ (46-60)^2 + (46-34)^2 + (46-45)^2 + (46-30)^2]$$

$$= \frac{1}{10}(144 + 0 + 576 + 64 + 81 + 4 + 196 + 144 + 1 + 256]$$

$$= 146.6.$$

The coefficient of variation $= \frac{\sigma_B}{\overline{y}_B} = \frac{\sqrt{146.6}}{46} = 0.26$. Since the average of batsman A is greater than the average of B, we conclude that A is a better scorer and hence is more efficient. But the coefficient of variance of B is less than the coefficient of variance of A, therefore, it follows that B is more consistent than A.

EXAMPLE 2.6

The first three moments of a distribution about the value 2 of the variable are 1, 16, and – 40. Find the mean, variance, and third moment of the distribution about the value 2.

Solution. We are given that

$$\mu_1' = 1,\ \mu_2' = 16,\ \mu_3' = -40.$$

Since $N = \sum f_i$, we have

$$1 = \mu_1' = \frac{1}{N}\sum f_i(x_i - a) = \frac{1}{N}\sum f_i(x_i - 2),$$

$$= \frac{1}{N}\sum f_i x_i - \frac{2}{N}\sum f_i = \overline{x} - 2,$$

and so

$$\text{Mean}\,(\overline{x}) = 3.$$

The variance is

$$S^2 = \mu_2 = \mu_2' - \mu_1'^2 = 16 - 1 = 15.$$

The third moment μ_3 is given by

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = -40 - 3(16)(1) + 2(1)^3$$

$$= -40 - 48 + 2 = -86.$$

EXAMPLE 2.7

Determine Pearson’s coefficient of skewness for the data given below:

Class:	10–19	20–29	30–39	40–49
f:	5	9	14	20

Class:	50–59	60–69	70–79	80–89
f:	25	15	8	4

Solution. We form the table given below:

Class	Mid-value (x)	Frequency (f)	Cum. frequency	fx	fx^2
10–19	14.5	5	5	72.5	1051.25
20–29	24.5	9	14	220.5	5402.25
30–39	34.5	14	28	483	16663.5
40–49	44.5	20	48	890	39605
50–59	54.5	25	73	1362.5	74256.25

Class	Mid-value (x)	Frequency (f)	Cum. frequency	fx	fx^2
60–69	64.5	15	88	967.5	62403.75
70–79	74.5	8	96	596	44402
80–89	84.5	4	100	338	28561
		100		**4930**	**272345**

Then

$$\text{Mean } (\overline{x}) = \frac{4930}{100} = 49.3.$$

The maximum frequency is 25, that is, the modal frequency is 25. Therefore, the modal class is 50–59. Hence

$$\text{Mode} = L + \frac{\Delta_1}{\Delta_1 + \Delta_2} h = 50 + \frac{(25-20)9}{(25-20)+(25-15)}$$
$$= 50 + \frac{45}{15} = 53.0.$$

Also

$$\sigma^2 = \frac{1}{\sum f_i} \sum f_i x_i^2 - (\overline{x})^2 = 2723.45 - 2430.49$$
$$= 292.96,$$

and so $\sigma = 17.12$.

Pearson’s coefficient of skewness $= \dfrac{\text{mean} - \text{mode}}{\sigma} = \dfrac{49.3 - 53.0}{17.12} = -0.22.$

2.5 CURVE FITTING

Least Square Line Approximation

Suppose that we have an empirical data in the form of n pairs of values $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, where the experimental errors are associated with the functional values $y_1, y_2, \ldots, y_n$ only. Then we seek a linear function

$$y = f(x) = a + bx \tag{2.2}$$

fitting the given points as well as possible. Equation (2.2) will not in general be satisfied by any of the n pairs. Substituting in equation (2.1) each of the n pairs of values in turn, we get

$$\left.\begin{aligned} e_1 &= y_1 - a - bx_1 \\ e_2 &= y_2 - a - bx_2 \\ \cdots &\;\cdots \quad \cdots \\ \cdots &\;\cdots \quad \cdots \\ e_n &= y_n - a - bx_n \end{aligned}\right\}, \tag{2.3}$$

where e_k, $k = 1,\ldots,n$ are measurement errors, called residuals or deviations. To know how far the curve $y = f(x)$ lies from the given data, the following errors are considered:

(i) **Maximum error**

$$e_{\infty}(f) = \max_{1\le k\le n}\left\{\left|y_k - a - bx_k\right|\right\}$$

(ii) **Average error**

$$e_{A}(f) = \frac{1}{n}\sum_{k=1}^{n}\left|y_k - a - bx_k\right|$$

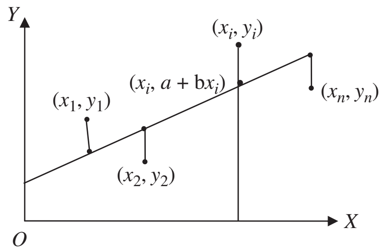
(iii) **Root mean square (RMS) error**

$$e_{\text{rms}}(f) = \left[\frac{e_1^2 + \ldots + e_n^2}{n}\right]^{1/2}.$$

The least square line $y = f(x) = a + bx$ is the line that minimizes the root mean square error $e_{\text{rms}}(f)$. But the quantity $e_{\text{rms}}(f)$ is minimum if and only if $\sum_{k=1}^{n}(y_k - a - bx_k)^2 = \sum_{k=1}^{n} e_k^2$ is minimum. Thus, in case of least square line we are looking for a linear function $a + bx$ as an approximation to a function $y = f(x)$ when we are given the values of y at the points $x_1,\ldots,x_n$. We aim at minimizing the sum of the squared errors

$$e(a,b) = \sum_{i=1}^{n}(y_i - a - bx_i)^2. \tag{2.4}$$

Geometrically, if d_i is the vertical distance from the data point (x_i, y_i) to the point $(x_i, a + bx_i)$ on the line, then $d_i = y_i - a - bx_i$ (see Figure below). We must minimize the sum of the squares of the vertical distances d_i, that is, the sum $\sum_{i=1}^{n} d_i^2$.



To minimize $e(a,b)$, we equate to zero the partial derivative of equation (2.4) with respect to a and with respect to b. Thus,

$$\frac{\partial e(a,b)}{\partial a} = \sum_{i=1}^{n} 2(y_i - a - bx_i) = 0$$

and

$$\frac{\partial e(a,b)}{\partial b} = \sum_{i=1}^{n} 2x_i(y_i - a - bx_i) = 0,$$

which are known as normal equations. We write these equations in the form

$$na + b\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \tag{2.5}$$

and

$$a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i. \tag{2.6}$$

The normal equations (2.5) and (2.6) can be solved for a and b using Cramer's rule or by some other method.

EXAMPLE 2.8

Show that, according to the principle of least squares, the best fitting linear function for the points (x_i, y_i), $i = 1, 2, ..., n$ may the expressed in the form

$$\begin{vmatrix} x & y & 1 \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} y_i & n \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} y_i^2 & \sum_{i=1}^{n} x_i \end{vmatrix} = 0.$$

Solution. Eliminating a and b from equations (2.4), (2.5), and $y = a + bx$, we get the required result.

We have supposed in the above derivation that the errors in x values can be neglected compared with the errors in the y values. Now we suppose that the x values as well as the y values are subject to errors of about the same order of magnitude. Now we minimize the sum of the squares of the perpendicular distances to the line. Thus, if $y = a + bx$ is the equation of the line, then

$$e(a,b) = \frac{1}{1+b^2}\sum_{i=1}^{n}(y_i - a - bx_i)^2.$$

For minimum, partial derivatives with respect to a and b should vanish. Thus,

$$\frac{\partial e(a,b)}{\partial a} = \frac{2}{1+b^2}\sum_{i=1}^{n}(y_i - a - bx_i) = 0$$

and

$$\frac{\partial e(a,b)}{\partial b} = -2(1+b^2)\sum_{i=1}^{n}(y_i - a - bx_i)x_i - 2b\sum_{i=1}^{n}(y_i - a - bx_i)^2 = 0, \text{ that is,}$$

$$\sum_{i=1}^{n}(y_i - a - bx_i) = 0 \tag{2.7}$$

and

$$(1+b^2)\sum_{i=1}^{n}(y_i - a - bx_i)x_i = b\sum_{i=1}^{n}(y_i - a - bx_i)^2. \tag{2.8}$$

From equation (2.7), we get

$$a = y_0 - bx_0, \tag{2.9}$$

where

$$x_0 = \frac{1}{n}\sum_{i=1}^{n} x_i \text{ and } y_0 = \frac{1}{n}\sum_{i=1}^{n} y_i.$$

After simplification, equation (2.8) yields

$$b^2 + \frac{A-C}{B}b - 1 = 0, \tag{2.10}$$

where

$$A = \sum_{i=1}^{n} x_i^2 - nx_0^2,$$

$$B = \sum_{i=1}^{n} x_i y_i - nx_0 y_0,$$

$$C = \sum_{i=1}^{n} y_i^2 - ny_0^2.$$

Finding the value of b from equation (2.10), we obtain the corresponding value of a from equation (2.9).

EXAMPLE 2.9

The points (2,2), (5,4), (6,6), (9,9), and (11,10) should be approximated by a straight line. Perform this assuming

(i) the error in x values can be neglected

(ii) that the errors in x and y values are of the same order of magnitude.

Solution. (i) The sum table for the given problem is

n	x	x^2	y	xy	y^2
1	2	4	2	4	4
1	5	25	4	20	16
1	6	36	6	36	36
1	9	81	9	81	81
1	11	121	10	110	100
5	**33**	**267**	**31**	**251**	**237**

Let the least square line be $y = a + bx$. Therefore, the normal equations are

$$5a + 33b = 31 \tag{2.11}$$

$$33a + 267b = 251. \tag{2.12}$$

Multiplying equation (2.11) by 33 and equation (2.12) by 5, we obtain

$$165a + 1089b = 1023$$

$$165a + 1335b = 1255.$$

Subtracting, we get

$$246b = 232 \text{ and so } b = \frac{116}{123} = 0.9431.$$

Then equation (2.11) yields

$$a = \frac{31 - 33(0.9431)}{5} = -0.0244.$$

Hence, the least square line is

$$y = 0.9431x - 0.0244.$$

(ii) We have

$$x_0 = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{33}{5}$$

$$y_0 = \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{31}{5}$$

$$A = \sum_{i=1}^{n} x_i^2 - nx_0^2 = 267 - 5\left(\frac{33}{5}\right)^2 = \frac{246}{5} = 49.2$$

$$B = \sum_{i=1}^{n} x_i y_i - nx_0 y_0 = 251 - \frac{5(33)(31)}{25} = 46.4$$

$$C = \sum_{i=1}^{n} y_i^2 - ny_0^2 = 237 - 5\left(\frac{31}{5}\right)^2 = 44.8.$$

Therefore, equation in b

$$b^2 + \frac{A - C}{B} b - 1 = 0$$

becomes

$$b^2 + \frac{4.4}{46.4} b - 1 = 0$$

or

$$b^2 + 0.0948b - 1 = 0.$$

Hence,

$$b = \frac{-0.948 \pm \sqrt{4.0089}}{2} = 0.9537 \text{ (+ve)}.$$

Then $a = y_0 - bx_0$ yields

$$a = -0.0944.$$

Hence,

$$y = 0.9537x - 0.0944$$

is the required least square line.

EXAMPLE 2.10

In the following data, x and y are subject to error of the same order of magnitude:

x:	1	2	3	4	5	6	7	8
y:	3	3	4	5	5	6	6	7

Find a straight line approximation using the least square method.

Solution. The sum table for the given problem is

n	x	x^2	y	xy	y^2
1	1	1	3	3	9
1	2	4	3	6	9
1	3	9	4	12	16
1	4	16	5	20	25
1	5	25	5	25	25
1	6	36	6	36	36
1	7	49	6	42	36
1	8	64	7	56	49
8	**36**	**204**	**39**	**200**	**205**

Let the equation be $y = a + bx$. Then

$$a = y_0 - bx_0, \tag{2.13}$$

where

$$x_0 = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{36}{8},$$

$$y_0 = \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{39}{8}.$$

Further,

$$A = \sum_{i=1}^{n} x_i^2 - nx_0^2 = 204 - 8\left(\frac{36}{8}\right)^2 = \frac{408 - 324}{2} = 42,$$

$$B = \sum_{i=1}^{n} x_i y_i - nx_0 y_0 = 200 - 8\frac{(36)(39)}{8^2} = \frac{400 - 351}{2} = \frac{49}{2} = 24.5,$$

$$C = \sum_{i=1}^{n} y_i^2 - ny_0^2 = 205 - 8\left(\frac{39}{8}\right)^2 = \frac{1640 - 1521}{8} = \frac{119}{8} = 14.87.$$

Then the value of b is given by

$$b^2 + \frac{A - C}{B}b - 1 = 0$$

or

$$b^2 + \frac{42.0 - 14.87}{24.5}b - 1 = 0$$

or

$$b^2 + 1.107b - 1 = 0,$$

which yields

$$b = \frac{-1.107 \pm \sqrt{5.225}}{2} = 0.5895 \quad (+\text{ve}).$$

Then equation (2.13) gives $a = 2.225$. Hence, the least square line is

$$y = 0.59x + 2.22.$$

The Power Fit $y = ax^m$

Suppose we require ax^m as an approximation to a function y, where m is a known constant. We must find the value of a such that the equation

$$y = ax^m \tag{2.14}$$

is satisfied as nearly as possible by each of the n pairs of observed values $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. Using least square technique, we should minimize the error function

$$e(a) = \sum_{i=1}^{n} (ax_i^m - y_i)^2. \tag{2.15}$$

For this purpose, partial derivative of equation (2.15) with respect to a must vanish. So, we have

$$0 = 2\sum_{i=1}^{n} (ax_i^m - y_i)(x_i^m)$$

and so

$$0 = a\sum_{i=1}^{n} x_i^{2m} - \sum_{i=1}^{n} x_i^m y_i,$$

which yields

$$a = \frac{\sum_{i=1}^{n} x_i^m y_i}{\sum_{i=1}^{n} x_i^{2m}}.$$

Putting the value of a in equation (2.14), we get the required equation.

Second method: Taking logarithms of both sides of equation (2.14) yields

$$\log y = \log a + m \log x,$$

which is of the form $Y = A + BX$, where $Y = \log y$, $A = \log a$, $B = m$, and $X = \log x$. Now the least square line can be found. Then a and m are found.

EXAMPLE 2.11

Find the gravitational constant g using the data below and the relation $h = \frac{1}{2}gt^2$, where h is distance in meters and t is the time in seconds.

t	0.200	0.400	0.600	0.800	1.000
h	0.1960	0.7850	1.7665	3.1405	4.9075

Solution. The sum table for the given problem is

t	h	$t^{2m} (m = 2)$	ht^2
0.200	0.1960	0.0016	0.00784
0.400	0.7850	0.0256	0.12560
0.600	1.7665	0.1296	0.63594
0.800	3.1405	0.4096	2.00992
1.000	4.9075	1.0000	4.90750
		1.5664	**7.68680**

Then using the formula $y = ax^m$ for power fit, we have

$$\frac{1}{2}g = \frac{\sum_{k=1}^{n} h_k t_k^m}{\sum_{k=1}^{n} t_k^{2m}} = \frac{7.68680}{1.5664} = 4.9073$$

and so the gravitational constant $g = 9.8146\,\text{m/sec}^2$.

EXAMPLE 2.12

Find the power fits $y = ax^2$ and $y = bx^3$ for the data given below and determine which curve fits best:

x	2.0	2.3	2.6	2.9	3.2
y	5.1	7.5	10.6	14.4	19.0

Solution. The sum table for the given problem is

x	x^2	x^3	x^4	x^6	y	yx^2	yx^3
2	4	8	16	64	5.1	20.4	40.8
2.3	5.29	12.167	27.984	148.035	7.5	39.675	91.252
2.6	2.76	17.576	45.698	308.918	10.6	71.656	182.306
2.9	8.41	24.389	70.729	594.831	14.4	121.104	351.202
3.2	10.24	32.768	104.858	1073.746	19.0	194.560	622.592
			265.269	**2189.530**		**447.395**	**1292.152**

Then for $y = ax^2$, we have

$$a = \frac{\sum y_i x_i^2}{\sum x_i^4} = \frac{447.395}{265.269} = 1.6866.$$

Hence, the power fit is

$$y = 1.6866x^2.$$

On the other hand, for $y = bx^3$, we have

$$b = \frac{\sum y_i x_i^3}{\sum x_i^6} = \frac{1292.152}{2189.530} = 0.5902.$$

Hence, the power fit is

$$y = 0.5902x^3.$$

To know which of these is best fit, we calculate the corresponding errors. For the first power fit, we have

$$e_{\text{rms}} = \left[\frac{1}{5}\left\{(ax_1^2 - y_1)^2 + (ax_2^2 - y_2)^2 + (ax_3^2 - y_3)^2 + (ax_4^2 - y_4)^2 + (ax_5^2 - y_5)^2\right\}\right]^{1/2}$$

$$= \left[\frac{1}{5}\left\{(1.646)^2 + (1.4330)^2 + (0.8014)^2 + (0.2157)^2 + (-1.7292)^2\right\}\right]^{1/2}$$

$$= \left[\frac{1}{5}(2.704 + 2.053 + 0.642 + 0.046 + 2.990\right]^{1/2} \approx 1.3.$$

Similarly for the second curve, we have

$$e_{\text{rms}} \approx 0.29.$$

Hence, the power fit curve $y = 0.5902x^3$ is the best.

EXAMPLE 2.13

By using the methods of least squares, find a relation of the form $y = ax^2$ that fits the data:

x	2	3	4	5
y	27.8	62.1	110	161

Solution. The sum table for the given problem is

x	x^2	x^4	y	yx^2
2	4	16	27.8	111.2
3	9	81	62.1	558.9
4	16	256	110	1760
5	25	625	161	4025
		978		**6455.1**

Then for $y = ax^2$, we have

$$a = \frac{\sum y_i x_i^2}{\sum x_i^4} = \frac{6455.1}{978} = 6.60.$$

Hence, the power fit is

$$y = 6.6x^2$$

Least Square Parabola (Parabola of Best Fit)

Suppose that we want to approximate a given function $y = f(x)$ by a quadratic $a + bx + cx^2$. We must find the values of a, b, and c such that the equation

$$y = a + bx + cx^2 \tag{2.16}$$

is satisfied as nearly as possible by each of the n pairs of observed values $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. The equation will not in general be satisfied exactly by any of the n pairs. Substituting in equation (2.16) each of the n pairs of values in turn, we get the following residual equations:

$$\begin{aligned} e_1 &= a + bx_1 + cx_1^2 - y_1 \\ e_2 &= a + bx_2 + cx_2^2 - y_2 \\ \ldots & \quad \ldots \quad \ldots \quad \ldots \\ \ldots & \quad \ldots \quad \ldots \quad \ldots \\ e_n &= a + bx_n + cx_n^2 - y_n \end{aligned}$$

The principle of least square says that the best values of the unknown constants a, b, and c are those which make the sum of the squares of the residuals a minimum, that is,

$$\sum_{i=1}^{n} e_i^2 = e_1^2 + e_2^2 + \ldots + e_n^2$$

must be minimum. Thus,

$$e(a,b,c)=\sum_{i=1}^{n}(a+bx_i+cx_i^2-y_i)^2$$

should be minimum. For this, the partial derivatives of $e(a,b,c)$ with respect to a, b, and c should be zero. We therefore have

$$\frac{\partial e(a,b,c)}{\partial a}=2(a+bx_1+cx_1^2-y_1)+2(a+bx_2+cx_2^2-y_2)+\ldots+2(a+bx_n+cx_n^2-y_n)=0$$

$$\frac{\partial e(a,b,c)}{\partial b}=2(a+bx_1+cx_1^2-y_1)x_1+2(a+bx_2+cx_2^2-y_2)x_2+\ldots+2(a+bx_n+cx_n^2-y_n)x_n=0$$

$$\frac{\partial e(a,b,c)}{\partial c}=2(a+bx_1+cx_1^2-y_1)x_1^2+2(a+bx_2+cx_2^2-y_2)x_2^2+\ldots+2(a+bx_n+cx_n^2-y_n)x_n^2=0.$$

Hence, the normal equations are

$$(a+bx_1+cx_1^2-y_1)+(a+bx_2+cx_2^2-y_2)+\ldots+(a+bx_n+cx_n^2-y_n)=0$$

$$(a+bx_1+cx_1^2-y_1)x_1+(a+bx_2+cx_2^2-y_2)x_2+\ldots+(a+bx_n+cx_n^2-y_n)x_n=0$$

$$(a+bx_1+cx_1^2-y_1)x_1^2+(a+bx_2+cx_2^2-y_2)x_2^2+\ldots+(a+bx_n+cx_n^2-y_n)x_n^2=0.$$

These equations can further be written as

$$na+b(x_1+x_2+\ldots+x_n)+c(x_1^2+x_2^2+\ldots+x_n^2)=y_1+y_2+\ldots+y_n,$$

$$a(x_1+x_2+\ldots+x_n)+b(x_1^2+x_2^2+\ldots+x_n^2)+c(x_1^3+x_2^3+\ldots+x_n^3)=x_1y_1+x_2y_2+x_3y_3,$$

$$a(x_1^2+x_2^2+\ldots+x_n^2)+b(x_1^3+x_2^3+\ldots+x_n^3)+c(x_1^4+x_2^4+\ldots+x_n^4)=x_1^2y_1+x_2^2y_2+\ldots+x_n^2y_n.$$

The above normal equations are solved by ordinary methods of algebra for solving simultaneous equations of first degree in two or more unknowns.

Remark 2.1. The number of normal equations is always the same as the number of unknown constants, whereas the number of residual equations is equal to the number of observations. The number of observations must always be greater than the number of undetermined constants if the method of least square is to be used in the solution.

EXAMPLE 2.14

Find the parabola of best fit with equation of the form $a+bx+cx^2$ for the data in the following table:

x	0	1	2	3	4
y	−2.1	−0.4	2.1	3.6	9.9

Solution. We establish the following sum table:

n	x	x^2	x^3	x^4	y	xy	x^2y
1	0	0	0	0	−2.1	0	0
1	1	1	1	1	−0.4	−0.4	−0.4
1	2	4	8	16	2.1	4.2	8.4
1	3	9	27	81	3.6	10.8	32.4
1	4	16	64	256	9.9	39.6	158.4
5	**10**	**30**	**100**	**354**	**13.1**	**54.2**	**198.8**

The normal equations are

$$5a + 10b + 30c = 13.1$$

$$10a + 30b + 100c = 54.2$$

$$30a + 100b + 354c = 198.8.$$

Solving these equations, we get

$$a = -1.80858,\ b = 0.45716 \text{ and } c = 0.5871$$

Hence, the parabola of best fit is

$$y = -1.80858 + 0.45716x + 0.58571x^2.$$

EXAMPLE 2.15

Find the least square polynomial of degree two for the following data:

x	0.78	1.56	2.34	3.12	3.81
y	2.50	1.20	1.12	2.25	4.28

Solution. Let the required polynomial be $a + bx + cx^2$. To make the calculations simple, we use the substitution

$$X = \frac{x - 2.34}{0.78}$$

making use of the equal spacing of the arguments. The sum table then becomes

n	X	X^2	X^3	X^4	y	Xy	X^2y
1	−2	4	−8	16	2.50	−5.00	10.00
1	−1	1	−1	1	1.20	−1.20	1.20
1	0	0	0	0	1.12	0	0
1	1	1	1	1	2.25	2.25	2.25
1	1.88	3.53	6.64	12.49	4.28	8.05	15.13
5	**−0.12**	**9.53**	**−1.36**	**30.49**	**11.35**	**4.10**	**28.58**

The normal equations are

$$5a - 0.12b + 9.53c = 11.35$$

$$-0.12a + 9.53b - 1.36c = 4.10$$

$$9.53a - 1.36b + 30.49c = 28.58.$$

Solving these equations by Cramer's rule, we get

$$a = 1.1155021,$$

$$b = 0.5316061,$$

$$c = 0.612401.$$

Hence, the parabola of best fit is

$$y = 1.1155 + 0.5316X + 0.6124X^2,$$

where $X = \dfrac{x - 2.34}{0.78}$.

EXAMPLE 2.16

Find the least square fit $y = a + bx + cx^2$ for the data

x	-3	-1	1	3
y	15	5	1	5

Solution. The sum table for the given problem is

n	x	x^2	x^3	x^4	y	xy	x^2y
1	-3	9	-27	81	15	-45	135
1	-1	1	-1	1	5	-5	5
1	1	1	1	1	1	1	1
1	3	9	27	81	5	15	45
4	**0**	**20**	**0**	**164**	**26**	**-34**	**186**

The normal equations are

$$4a + 20c = 26$$
$$20b = -34$$
$$20a + 164c = 186.$$

Solving these equations, we have

$$b = -\frac{34}{20} = -1.70,\ c = 0.875,\ a = 2.125.$$

Hence, the least square parabola is

$$y = 2.125 - 1.700x + 0.875x^2.$$

EXAMPLE 2.17

Fit a parabola to the following data

x	1	2	3	4
y	0.30	0.64	1.32	5.40

Solution. The sum table for the given problem is

n	x	x^2	x^3	x^4	y	xy	x^2y
1	1	1	1	1	0.30	0.30	0.30
1	2	4	8	16	0.64	1.28	2.56
1	3	9	27	81	1.32	3.96	11.88
1	4	16	64	256	5.40	21.60	82.40
4	**10**	**30**	**100**	**354**	**7.66**	**27.14**	**101.14**

The normal equations are

$$4a+10b+30c=7.66,$$

$$10a+30b+100c=27.14,$$

$$30a+100b+354c=101.14.$$

Solving these equations by Gauss elimination method or Cramer's rule, we get

$$a=-1.09,\ \ b=0.458,\ \ c=0.248.$$

Hence, the parabola of fit is

$$y=-1.09+0.458x+0.248x^2.$$

EXAMPLE 2.18

Fit a parabola $y = a + bx + x^2$ to the following data:

x	2	4	6	8	10
y	3.07	12.85	31.47	57.38	91.29

Solution. The sum table for the given problem is

n	x	x^2	x^3	x^4	y	xy	x^2y
1	2	4	8	16	3.07	6.14	12.28
1	4	16	64	256	12.85	51.4	205.6
1	6	36	216	1,296	31.47	188.82	1,132.92
1	8	64	512	4,096	57.38	459.04	3,672.32
1	10	100	1,000	1,000	91.29	912.9	9,129.00
5	**30**	**220**	**1,800**	**15,664**	**196.06**	**1,618.3**	**14,152.12**

The normal equations are

$$5a+30b+220c=196.06$$

$$30a+220b+1800c=1618.30$$

$$220a+1800b+15644c=14152.12.$$

These equations yield

$$40b+480c=44.94$$

and

$$480b+5984c=5525.48.$$

This last pair of equations give $b = -0.859$ and $c = 0.992$. Putting these values in the first normal equation, we get $a = 0.720$.
Hence, the least square parabola is

$$y=0.72-0.859x+0.992x^2$$

EXAMPLE 2.19

If x (km/hr) and y (kg/tonne) are related by a relation of the type $y = a + bx^2$, find by the method of least squares a and b with the help of the following table:

x	10	20	30	40	50
y	8	10	15	21	30

Solution. The normal equations for the curve fitting of the type $y = a + bx^2$ are

$$na + b(x_1^2 + x_2^2 + \cdots + x_n^2) = y_1 + y_2 + \cdots + y_n$$

$$a(x_1^2 + x_2^2 + \cdots + x_n^2) + b(x_1^4 + x_2^4 + \cdots + x_n^4) = x_1^2 y_1 + x_2^2 y_2 + \cdots + x_n^2 y_n.$$

So we establish the following table:

n	x	x^2	x^4	y	$x^2 y$
1	10	100	10,000	8	800
1	20	400	160,000	10	4,000
1	30	900	810,000	15	13,500
1	40	1,600	2,560,000	21	33,600
1	50	2,500	6,250,000	30	75,000
5	**150**	**5,500**	**9,790,000**	**84**	**126,900**

The normal equations are

$$5a + 5500b = 84.$$

and

$$5500a + 9790000b = 126900,$$

that is,

$$5a + 5500b = 84$$

and

$$55a + 97900b = 1269$$

Hence, $a = 6.76$, $b = 0.00924$ and the parabola of best fit is

$$y = 6.76 + 0.00924x^2$$

2.6 COVARIANCE

Suppose that the pair of random variable X and Y take n pairs of observations as follows:

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n).$$

The arithmetic means of the observed values of X and Y are, respectively

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \text{ and } \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i. \tag{2.17}$$

The deviations of the observed values of X and Y from their respective means are

$$x_1 - \bar{x},\ x_2 - \bar{x}, \ldots, x_n - \bar{x}$$

and

$$y_1 - \bar{y},\ y_2 - \bar{y}, \ldots, y_n - \bar{y}$$

respectively. The *covariance* of X and Y, denoted by Cov (X,Y) is defined by

$$\text{Cov}(X,Y) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}).$$

However, if $\bar{x}$ and $\bar{y}$ are not whole numbers, then the task of calculating Cov(X,Y) by this formula is time-consuming and cumbersome. A simplified expression for Cov (X,Y) can be derived as follows: Using (2.17) we get

$$\begin{aligned}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) &= \sum_{i=1}^{n}(x_i y_i - \bar{x}y_i - x_i\bar{y} + \bar{x}\,\bar{y}) \\ &= \sum_{i=1}^{n} x_i y_i - \bar{x}\sum_{i=1}^{n} y_i - \bar{y}\sum_{i=1}^{n} x_i + \bar{x}\,\bar{y}\sum_{i=1}^{n} 1 \\ &= \sum_{i=1}^{n} x_i y_i - \bar{x}(n\bar{y}) - \bar{y}(n\bar{x}) + n\bar{x}\,\bar{y} = \sum_{i=1}^{n} x_i y_i - n\bar{x}\,\bar{y}.\end{aligned}$$

Hence

$$\begin{aligned}\text{Cov}(X,Y) &= \frac{1}{n}\left[\sum_{i=1}^{n} x_i y_i - n\bar{x}\,\bar{y}\right] \\ &= \frac{1}{n}\sum_{i=1}^{n} x_i y_i - \bar{x}\,\bar{y}.\end{aligned}$$

It may be proved that *covariance is not affected by the change of origin but is affected by the change of scale.*

EXAMPLE 2.20

Find the covariance between x and y for the following data:

x:	3	4	5	8	7	9	6	2	1
y:	4	3	4	7	8	7	6	3	2

Solution. We have $n = 9$, $\sum x_i = 45$, $\sum y_i = 44$, $\bar{x} = \frac{1}{n}\sum x_i = \frac{45}{9} = 5$, $\bar{y} = \frac{1}{n}\sum y_i = \frac{44}{9}$, and $\sum x_i y_i = 263$.

Therefore,

$$\text{Cov}(X,Y) = \frac{1}{n}\sum_{i=1}^{n} x_i y_i - \bar{x}\,\bar{y} = \frac{263}{9} - \frac{5(44)}{9} = \frac{43}{9} = 4.78.$$

EXAMPLE 2.21

Calculate the covariance between height and weight of the following five persons:

Height in cm:	150	148	148	152	154
Weight in kg:	65	64	63	65	67

Solution. Since covariance is not affected by change of origin, we take $u_i = x_i - 148$ and $v_i = y_i - 65$ and get the following table:

x_i	y_i	$u_i = x_i - 148$	$v_i = y_i - 65$	$\sum u_i v_i$
150	65	2	0	0
148	64	0	–1	0
148	63	0	–2	0
152	65	4	0	0
154	67	6	2	12
		12	**–1**	**12**

Therefore,

$$\text{Cov}(X,Y) = \frac{1}{n}\sum u_i v_i - \bar{u}\,\bar{v}$$

$$= \frac{12}{5} - \frac{12}{5}\left(\frac{-1}{5}\right) = \frac{72}{25} = 2.88\,\text{cm kg.}$$

2.7 CORRELATION AND COEFFICIENT OF CORRELATION

The relation in which changes in one variable are associated or followed by changes in the other variable is called *correlation*. The data connecting such two variables is called *bivariate population*. For example, there is a correlation between the yield of a crop and the amount of rainfall.

A scale-free (numerical) measure for a relation between a pair of variable is called the *coefficient of correlation* or *correlation coefficient*.

The coefficient of correlation between two quantitative variables X and Y is defined by

$$\rho(X,Y) = \frac{\text{Cov}(X,Y)}{\sigma_x\,\sigma_y},$$

where

$$\sigma_x = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2} \text{ is the standard deviation for } X-\text{series}$$

$$\sigma_y = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2} \text{ is the standard deviation for } Y-\text{series.}$$

Since the dimensions of the numerator and denominator in the definition of $\rho(X,Y)$ are same, it follows that $\rho(X,Y)$ in non-dimensional quantity. $\rho(X,Y)$ *measures the degree of linear association* between the two variates. If two variates are not related, then $\rho(X,Y) = 0$. However, if $\rho(X,Y) = 0$ we cannot say that the two variables are not related.

We note that

$$\rho(X,Y) = \frac{\text{Cov}(X,Y)}{\sigma_x\, \sigma_y} = \frac{\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \overline{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \overline{y})^2}}.$$

But

$$\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y}) = n\,\text{Cov}\,(X,Y) = n\left[\frac{1}{n}\sum x_i y_i - \overline{x}\,\overline{y}\right]$$

$$= \sum x_i y_i - \frac{1}{n}\sum x_i \sum y_i,$$

$$\sum (x_i - \overline{x})^2 = \sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2, \text{ and}$$

$$\sum (y_i - \overline{y})^2 = \sum_{i=1}^{n} y_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} y_i\right)^2.$$

Therefore,

$$\rho(X,Y) = \frac{\sum x_i y_i - \frac{1}{n}\sum x_i \sum y_i}{\sqrt{\sum x_i^2 - \frac{1}{n}\left(\sum x_i\right)^2}\sqrt{\sum y_i^2 - \frac{1}{n}\left(\sum y_i\right)^2}}$$

$$= \frac{n\sum x_i y_i - \sum x_i \sum y_i}{\sqrt{n\sum x_i^2 - \left(\sum x_i\right)^2}\sqrt{n\sum y_i^2 - \left(\sum y_i\right)^2}},$$

which is called *Karl-Pearson's coefficient of correlation* or *product moment correlation coefficient*.

Remark 2.2.

(i) Since $\rho(X,Y) = \frac{\text{Cov}(X,Y)}{\sigma_x\, \sigma_y}$ and denominator contains positive square roots, it follows that the sign of $\rho(X,Y)$ is the same as that of Cov (X,Y)

(ii) $-1 \le \rho(X,Y) \le 1$.

(iii) If $\rho(X,Y) = 1$, then the variables X and Y are not only statistically related but also functionally related. There exists a linear relationship of the form

$$Y = a + bX, b \ge 0$$

or

$$X = c + dY, d \ge 0,$$

which are straight lines with positive slopes. In this case, the variables have *perfect positive correlation*.

(iv) If $\rho(X,Y) = -1$, then there exists a linear relationship of the form

$$Y = a - bX, b \ge 0$$

or

$$X = c - dY, d \geq 0$$

which are straight lines with negative slopes. In this case, the variables have *perfect negative correlations*.

(v) If $\rho(X,Y)$ is close to 1, there is a high degree of positive correlation and if it is close to −1, then there is a high degree of negative correlation.

(vi) If $\rho(X,Y)$ is close to 0 in magnitude, we cannot draw any conclusion about the existence of relation between the variables. To reach at some conclusion, in such a case, we have to draw scatter diagram.

EXAMPLE 2.22

Calculate the covariance and the coefficient of correlation between X and Y if

$$n = 10, \sum x = 60, \sum y = 60, \sum x^2 = 400, \sum y^2 = 580 \text{ and } \sum xy = 305.$$

Solution. For the given data

$$\text{Cov}(X,Y) = \frac{1}{n}\sum xy - \bar{x}\,\bar{y}$$

$$= \frac{1}{10}(305) - \left(\frac{60}{10}\right)\left(\frac{60}{10}\right) = -5.5,$$

$$\rho(X,Y) = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{\sqrt{n\sum x_i^2 - \left(\sum x_i\right)^2}\sqrt{n\sum y_i^2 - \left(\sum y_i\right)^2}}$$

$$= \frac{3050 - 3600}{\sqrt{4000 - 3600}\sqrt{5800 - 3600}}$$

$$= \frac{-550}{20\sqrt{2200}} = -\frac{11}{4\sqrt{22}} = 0.586.$$

EXAMPLE 2.23

Find the Karl Pearson coefficient of correlation between the industrial production and export using the following data:

Production (in crore tons):	55	56	58	59	60	60	62
Export (in crore tons):	35	38	38	39	44	43	45

Solution. Here n=7. Put $u_i = x_i - 60$, $v_i = y_i - 38$. Then we have the following table:

x	y	u	v	u^2	v^2	uv
55	35	−5	−3	25	9	15
56	38	−4	0	16	0	0
58	38	−2	0	4	0	0
59	39	−1	1	1	1	−1
60	44	0	6	0	36	0
60	43	0	5	0	25	0
62	45	2	7	4	49	14
		−10	**16**	**50**	**120**	**28**

Therefore,

$$\rho(X,Y) = \frac{n\sum u_i v_i - \sum u_i \sum v_i}{\sqrt{n\sum u_i^2 - \left(\sum u_i\right)^2}\sqrt{n\sum v_i^2 - \left(\sum v_i\right)^2}}$$

$$= \frac{196+160}{\sqrt{350-100}\sqrt{840-256}}$$

$$= \frac{356}{\sqrt{250}\sqrt{584}} = \frac{356}{382.08} = 0.93.$$

Since $\rho(X,Y)$ is close to 1, there is high degree of positive correlation.

2.8 REGRESSION

The value of the coefficient of correlation indicates whether statistical relationship exists between the variables X and Y. However, it does not give any expression for this statistical relationship. Regression analysis gives us a method for finding such expression.

Suppose that for a given value of x, we wish to determine the value of y. Thus we want to have an equation of the form

$$y = f(x). \tag{2.18}$$

The function f is called a *regression function* while equation (2.18) is called the *regression equation of Y on X*.

On the other hand, if for a given value of y we wish to find value of x, then we want to establish an equation of the form

$$x = g(y). \tag{2.19}$$

The function g is called *regression function* and equation (2.19) is called *regression equation of X on Y*.

We consider equation (2.18). Let (x_i, y_i), $i = 1, 2, \ldots, n$ be observed values in a given data. Then the estimate at x_i is $f(x_i)$, while the actual value is y_i. Thus, the error in the observed values are

$$y_i - f(x_1), y_2 - f(x_2), \ldots, y_n - f(x_n).$$

The regression equation is good if these errors are small. Here we consider the case of linear regression only. Thus we wish to express $f(x)$ and $g(y)$ in the form of linear polynomials of the form

$$f(x) = a + bx \text{ and } g(y) = c + dy.$$

We shall obtain these expressions using *least square approximation*.

Suppose we want to find the regression of Y on X. Let the approximation be

$$y = a + bx. \tag{2.20}$$

Let (x_i, y_i), $i = 1,2,\ldots,n$ be observed values. Then the errors of estimation are

$$y_1 - (a+bx_1), y_2 - (a+bx_2), \ldots, y_n - (a+bx_n).$$

Our aim is to find a and b such that the sum of squares of the errors is minimum. Thus we want to minimize $\sum_{i=1}^{n}[y_i - (a+bx_i)]^2$. With these values of a and b, $y = a + bx$ is called the *best approximation in the least square sense*. For minimizing $\sum_{i=1}^{n}[y_i - (a+bx_i)]^2$, its first derivatives with respect to a and b should be equal to zero. Thus we have

$$\sum_{i=1}^{n}[y_i - (a+bx_i)] = 0 \tag{2.21}$$

and

$$\sum_{i=1}^{n}[y_i-(a+bx_i)]x_i=0. \tag{2.22}$$

Simplifying (2.21) and (2.22), we get

$$na+b\sum_{i=1}^{n}x_i=\sum y_i$$

$$a\sum x_i+b\sum_{i=1}^{n}x_i^2=\sum x_i y_i.$$

Since $\overline{x}=\frac{1}{n}\sum_{i=1}^{n}x_i$, these equations reduces to

$$na+nb\overline{x}=n\overline{y} \tag{2.23}$$

$$na\overline{x}+b\sum_{i=1}^{n}x_i^2=\sum_{i=1}^{n}x_i y_i \tag{2.24}$$

Since

$$\text{Cov}(X,Y)=\frac{1}{n}\sum_{i=1}^{n}x_i y_i-\overline{x}\,\overline{y},$$

we get

$$\sum x_i y_i=n\text{Cov}(X,Y)+n\overline{x}\,\overline{y}.$$

Also

$$\sigma_x^2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^2=\frac{1}{n}\left[\sum_{i=1}^{n}x_i^2-n(\overline{x})^2\right],$$

which yields

$$\sum_{i=1}^{n}x_i^2=n\sigma_x^2+n(\overline{x})^2=n[\sigma_x^2+(\overline{x})^2].$$

Substituting these values in (2.24), we get

$$na\overline{x}+nb[\sigma_x^2+(\overline{x})^2]=n[\text{Cov}(X,Y)+\overline{x}\,\overline{y}],$$

that is,

$$a\overline{x}+b[\sigma_x^2+(\overline{x})^2]=\text{Cov}(X,Y)+\overline{x}\,\overline{y}. \tag{2.25}$$

Multiplying (2.23) by $\overline{x}$ and subtracting from (2.25), we get

$$b\sigma_x^2=\text{Cov}(X,Y)$$

and so

$$b=\frac{\text{Cov}(X,Y)}{\sigma_x^2}.$$

Then (2.23) yields

$$a = \overline{y} - \frac{\overline{x}\text{Cov}(X,Y)}{\sigma_x^2}.$$

Hence, the line of regression of Y on X is

$$y = a + bx = \overline{y} - \frac{\overline{x}\text{Cov}(X,Y)}{\sigma_x^2} + \frac{\text{Cov}(X,Y)}{\sigma_x^2}x$$

or

$$y - \overline{y} = \frac{\text{Cov}(X,Y)}{\sigma_x^2}(x - \overline{x}) = b_{yx}(x - \overline{x}), \tag{2.26}$$

where

$$b_{yx} = \frac{\text{Cov}(X,Y)}{\sigma_x^2} = \frac{\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^{n}(x_i - \overline{x})^2} \tag{2.27}$$

$$= \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

is called *regression coefficient* of Y on X. Since $\sigma_x^2 > 0$, the sign of b_{yx} is the same as that of $Cov(X,Y)$ or of $\rho(X,Y)$.

Similarly, the regression line of X on Y is

$$x - \overline{x} = b_{xy}(y - \overline{y}), \tag{2.28}$$

where

$$b_{xy} = \frac{\text{Cov}(X,Y)}{\sigma_y^2} = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} y_i\right)^2} \tag{2.29}$$

is the *regression coefficient of X* on *Y*.

We observe that

$$b_{xy}.b_{yx} = \frac{[\text{Cov}(X,Y)]^2}{\sigma_x^2\,\sigma_y^2} = \left[\frac{\text{Cov}(X,Y)}{\sigma_x\,\sigma_y}\right]^2$$

$$= [\rho(X,Y)]^2.$$

Hence, *the correlation coefficient is the geometric mean of the regression coefficients.*

Since $-1 \le \rho(X,Y) \le 1$, it follows that

$$b_{\text{xy}}\, b_{\text{yx}} = 1.$$

Remarks 2.3.

(i) The point of intersection of the two lines of regression obtained above is $(\overline{x}, \overline{y})$.
(ii) The regression coefficients are *independent of change of origin* but *not of scale.*

2.9 ANGLE BETWEEN THE REGRESSION LINES

The regression line of Y on X is

$$y-\overline{y}=b_{yx}(x-\overline{x}).$$

The slope of this line is

$$b_{yx}=\frac{\mathrm{Cov}(X,Y)}{\sigma_x^2}=\frac{\mathrm{Cov}(X,Y)}{\sigma_x\,\sigma_y}.\frac{\sigma_y}{\sigma_x}=\frac{\rho(X,Y)}{\sigma_x}\sigma_y.$$

The regression line of X on Y is

$$x-\overline{x}=b_{xy}(y-\overline{y}),$$

whose slop is

$$\frac{1}{b_{xy}}=\frac{\sigma_y^2}{\mathrm{Cov}(X,Y)}=\frac{\sigma_y}{\rho(X,Y)\sigma_x}.$$

Hence the angle θ between the lines of regression is given by

$$\tan\theta=\pm\frac{\frac{1}{b_{xy}}-b_{yx}}{1+\frac{1}{b_{xy}}b_{yx}}=\frac{\frac{\sigma_y}{\rho\,\sigma_x}-\frac{\rho\,\sigma_y}{\sigma_x}}{1+\frac{\sigma_y}{\rho\,\sigma_x}.\frac{\rho\,\sigma_y}{\sigma_x}}$$

$$=\pm\frac{(1-\rho^2)\sigma_x\,\sigma_y}{\rho(\sigma_x^2+\sigma_y^2)}. \qquad (2.30)$$

The angle θ is usually taken as the acute angle, that is, tan θ is taken as positive.

Deductions. If follows from (2.30) that

(i) if $\rho(X,Y)=\pm1$, then tan $\theta=0$and so $\theta=0$ or π. Hence the two lines of regression *coincides*.
(ii) if $\rho(X,Y)=0$, then tan $\theta=\infty$ which implies $\theta=90°$. Hence the lines are *perpendicular* in this case. The lines of regression in this case are $x=\overline{x}$ and $y=\overline{y}$, that is, they are parallel to the axes.

Least Square Error

(i) Least square error of prediction of Y on X is

$$\sum[y_i-(a+bx_i)]^2$$

which on simplification equals to

$$n\sigma_y^2\left\{1-[\rho(X,Y)]^2\right\}.$$

(ii) Least square error of prediction of X on Y is similarly

$$n\sigma_x^2\left\{1-[\rho(X,Y)]^2\right\}.$$

Clearly, if $\rho(X,Y) = \pm 1$, then the sum of least square of deviation (least square error) from either line of regression is 0. Hence each deviation is 0 and all the points lie on both lines of regression and so the lines coincide.

EXAMPLE 2.24

Find the regression of Y on X for the following data:

$$\sum x = \sum y = 15, \sum x^2 = \sum y^2 = 49,$$
$$\sum xy = 44, n = 5.$$

Solution. The regression of Y on X is given by

$$b_{yx} = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - \left(\sum x\right)^2}$$
$$= \frac{5(44) - 15(15)}{5(49) - (15)^2} = \frac{-5}{20} = -\frac{1}{4}.$$

Hence the regression line is

$$y - \overline{y} = b_{yx}(x - \overline{x})$$

or

$$y - \frac{15}{5} = -\frac{1}{4}\left(x - \frac{15}{5}\right)$$

or

$$y - 3 = -\frac{1}{4}(x - 3).$$

EXAMPLE 2.25

Find the equation of the lines of regression based on the following data:

x:	4	2	3	4	2
y:	2	3	2	4	4

Solution. For the given data, we have the following table:

x	y	xy	x^2	y^2
4	2	8	16	4
2	3	6	4	9
3	2	6	9	4
4	4	16	16	16
2	4	8	4	16
15	**15**	**44**	**49**	**49**

Since $n = 5$, we have

$$\overline{x} = \frac{\sum x}{5} = \frac{15}{5} = 3 \quad \text{and} \quad \overline{y} = \frac{\sum x}{5} = \frac{15}{5} = 3.$$

As in Example 2.24, the regression of Y on X is

$$y-3=-\frac{1}{4}(x-3) \text{ or } x+4y=15.$$

For the line of regression of X on Y, we have

$$b_{xy}=\frac{n\sum xy-\sum x\sum y}{n\sum y^2-\left(\sum y\right)^2}=\frac{5(44)-15(15)}{5(49)-(15)^2}=-\frac{1}{4}.$$

Hence the regression of X on Y is given by

$$x-3=-\frac{1}{4}(y-3) \text{ or } 4x+y=15.$$

Hence the lines of regression are

$$x+4y=15 \text{ and } 4x+y=15.$$

EXAMPLE 2.26

Out of the following two regression lines, find the regression line of Y on X:

$$x+4y=3 \text{ and } y+3x=15.$$

Solution. The line of regression of Y on X is

$$y=\overline{y}+b_{yx}(x-\overline{x})$$

and the line of regression of X on Y is

$$x=\overline{x}+b_{xy}(y-\overline{y}).$$

Suppose that the line of regression of Y on X is $x+4y=3$, that is, $y=-\frac{1}{4}x+\frac{3}{4}$. The other line is $x=-\frac{1}{3}y+5$. Hence $b_{yx}=-\frac{1}{4}$ and $b_{xy}=-\frac{1}{3}$. Therefore,

$$\rho^2=b_{yx}(b_{xy})=\frac{1}{12}<1.$$

Hence the required line of regression of Y on X is $x+4y=3$.

Remark 2.4. If we begin taking $y+3x=15$ as the line of regression of Y on X, then

$$y=-3x+5.$$

The other line is

$$x=-4y+3.$$

Thus

$$b_{yx}=-3,\quad b_{xy}=-4$$

and so

$$\rho^2=b_{yx}(b_{xy})=12,$$

which is absurd, since $\rho^2\leq 1$. Hence the line of regression is $x+4y=3$.

EXAMPLE 2.27

Two random variables have the regression lines with equation $3x+2y=26$ and $6x+y=31$. Find the mean values and the correlation coefficient between x and y. Also find the angle between these lines.

Solution. Since the point of intersection of the regression lines is $(\bar{x}, \bar{y})$, the mean $\bar{x}$ and $\bar{y}$ lie on the two regression lines. Thus we have

$$3\bar{x} + 2\bar{y} = 26 \text{ and } 6\bar{x} + \bar{y} = 31.$$

Solving these equations, we get $\bar{x} = 4, \bar{y} = 7$.

As in the above example, we can verify that $3x + 2y = 26$ is the line of regression of Y on X and $6x+y=31$ is the line of regression of X on Y. These lines can be written as

$$y = -\frac{3}{2}x + 13 \text{ and } x = -\frac{1}{6}y + \frac{31}{6}.$$

Therefore the regression coefficients are $b_{yx} = -\frac{3}{2}$ and $b_{xy} = -\frac{1}{6}$. Since $\rho^2 = b_{yx}.b_{xy}$, it follows that $\rho(x,y)$ is the geometric mean of these two regression coefficients. Hence

$$\rho(x, y) = \sqrt{b_{yx}.b_{xy}} = \sqrt{\frac{1}{4}} = -0.5,$$

the minus sign is taken because both of the regression coefficients b_{yx} and b_{xy} are negative.

The angle between the regression lines is given by

$$\tan\theta = \pm\frac{\frac{1}{b_{xy}} - b_{yx}}{1 + \frac{b_{yx}}{b_{xy}}} = \mp\frac{8}{15}.$$

Taking positive value, we get $\theta = \tan^{-1}\left(\frac{8}{15}\right)$.

2.10 PROBABILITY

Probability theory was developed in the seventeenth century to analyse games and so directly involved counting. It is a mathematical modelling of the phenomenon of chance or randomness. The measure of *chance* or *likelihood* for a statement to be true is called the *probability* of the statement. Thus, probability is an expression of an outcome of which we are not certain. For example, if we toss a coin, we cannot predict in advance whether a head or tail will show up. Similarly, if a dice (die) is thrown, then any one of the six faces can turn up. We cannot predict in advance which number (face) is going to turn up. Similarly, if we consider a pack of 52 playing cards, in which there are two colours, black and red, and four suits namely spades, hearts, diamonds, and clubs. Each suit has 13 cards. If we shuffle the pack of cards and draw a card from it, we are not sure to get a desired card.

An *experiment* is a process that yields an outcome. A *random experiment* or *experiment of chance* is an experiment in which (i) all the outcomes of the experiment are known in advance and (ii) the exact outcome of any specific performance of the experiment is not known in advance. For example, tossing of a fair coin is a random experiment. The possible outcomes of the experiment are head and tail. But we do not know in advance what the outcome will be on any performance of experiment.

The set of all the possible outcomes of a random experiment is called the *sample space* of that random experiment. It is denoted by S. An element of a sample space is called a *sample point*. An *event* is a subset of a sample space. An event may not contain any element. Such event is represented by ϕ and is called *impossible event.* An event may include the whole sample space S. Such event is called *sure* (*certain*) *event*. An event containing exactly one element is called a *simple event*.

For example, if we toss a fair coin, the sample space is

$$S_1 = \{T, H\},$$

where T stands for tail and H stands for head. Thus S_1 consists of $2^1 = 2$ sample points.

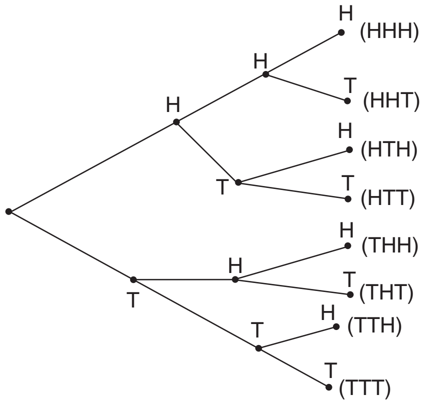
If the same coin is tossed twice, then

$$S_2 = \{TT, TH, HT, HH\}$$

consists of $2^2 = 4$ sample points.

Thus, in case of n toss, the sample space S_n shall have 2^n sample points.

The sample space of a random experiment can also be determined with the help of a *tree diagram*. For example, if a fair coin is tossed thrice, then the tree diagram for the sample space is as given below:



Thus,

$$S_3 = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Similarly, if an unbiased cubical dice is thrown, then

$$S_1 = \{1, 2, 3, 4, 5, 6\}.$$

If it is thrown again, then S_2 shall consists of $6^2 = 36$ sample points. These points can be determined in the following way:

	⚀	⚁	⚂	⚃	⚄	⚅
⚀	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
⚁	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
⚂	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
⚃	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
⚄	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
⚅	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

If two coins are tossed simultaneously, then the first coin may show up either H or T and the second coin may also show up either H or T. Therefore, the outcomes of the experiment are

$$S = \{HH, HT, TH, TT\}.$$

In general, when two random experiment having m outcomes $e_1, e_2, \ldots, e_m$ and n outcomes $p_1, p_2, \ldots, p_n$, respectively, are performed simultaneously, the sample space consists of mn sample points and so

$$S = \{(e_1, p_1), (e_1, p_2), \ldots, (e_1, p_n), \ldots, \\ (e_m, p_1), \ldots, (e_m, p_n)\}.$$

The *complement of an event* A with respect to the sample space S is the set of all elements of S which are not in A. It is denoted by $\overline{A}$ or by A′.

The intersection of two events A and B, denoted by A ∩ B, consists of all points that are common to A and B.

Thus A ∩ B denotes *simultaneous occurrence* of A and B.

Two events A and B are called *mutually exclusive* or *disjoint* if A ∩ B = ϕ.

The *union* of the two events A and B, denoted by A ∪ B, is the event containing all the elements that belong to A or to B or to both.

EXAMPLE 2.28

Let A be the event that a "sum of 6" appears on the dice when it is rolled *twice* and B denote the event that a "sum of 8" appears on the dice when rolled twice. Then

A = {(1,5), (2,4), (3,3), (4,2), (5,1)},
B = {(2,6), (3,5), (4,4), (5,3), (6,2)}.

We observe that A ∩ B = ϕ. Therefore, A and B cannot occur simultaneously and are mutually exclusive (disjoint).

The following combinations of events are usually needed in probability theory:

Combination	**Meaning**
$A \cup B$	Either A or B or both
$A \cap B$	Both A and B
$\overline{A}$ or A^c *or* A'	Not A
$A \cap B = \varphi$	Mutually exclusive events A and B
$A' \cap B'$ *or* $(A \cup B)'$	Neither A nor B
$A \cap B'$	Only A
$A' \cap B$	Only B
$(A \cap B') \cup (A' \cap B)$	Exactly one of A and B
$A \cap B \cup C$	At least one of A, B and C
$A \cap B \cap C$	All the three A, B and C

A collection of events $E_1, E_2, \ldots, E_n$ of a given sample space S is said to be *mutually exclusive* and *exhaustive system* of events if

(i) $E_i \cap E_j = \phi, i \neq j; i, j = 1, 2, \ldots, n$
(ii) $E_1 \cup E_2 \cup \ldots \cup E_n = S.$

A collection of events is said to be *equally likely* if all the outcomes of the sample space have the same chance of occurring.

If an event E_1 can occur in m ways and an event E_2 can occur in n ways, then E_1 or E_2 can occur in $m+n$ ways. This rule is called *Addition Rule*.

If an operation (task) is performed in 2 steps such that the first step can be performed in n_1 ways and the second step can be performed in n_2 ways (regardless of how the first step was performed), then the entire operation can be performed in $n_1\, n_2$ ways. This rule is called *Multiplication Rule*. The rule can be extended to k steps.

EXAMPLE 2.29

A coin is tossed thrice. If the event E denotes the "number of heads is odd" and event F denotes the "number of tails is odd", determine the cases favourable to $E \cap F$.

Solution. The coin is tossed thrice, therefore, the sample space is

$$S = \{HHH, HHT, HTH, HTT, THT, THH, TTH, TTT\}$$

The events E and F are

$$E = \{HHH, HTT, THT, TTH\} \text{ and}$$
$$F = \{HHT, HTH, THH, TTT\}$$

We note that $E \cap F = \phi$.

EXAMPLE 2.30

From a group of 2 men and 3 women, two persons are to be selected. Describe the sample space of the experiment. If E is the event in which a man and one woman are selected, determine the favourable cases to E.

Solution. Let M_1, M_2 and W_1, W_2, and W_3 be the men and women in the group. Then number of ways selecting two persons is equal to

$$\binom{5}{2} = \frac{5!}{3!\,2!} = 10.$$

The sample space is

$$S = \{M_1M_2, W_1W_2, W_2W_3, W_1W_3,$$
$$M_1W_1, M_1W_2, M_1W_3,$$
$$M_2W_1, M_2W_2, M_2W_3\}.$$

If E is the event where one man and one woman is selected, then

$$E = \{M_1W_1, M_1W_2, M_1W_3, M_2W_1, M_2W_2, M_2W_3\}$$

Thus, there are six favorable cases to the event E.

If S is a finite sample space having n mutually exclusive, equally likely and exhaustive outcomes out of which m are favourable to the occurrence of an event E, then the *probability* of occurrence of E, denoted by P(E), is

$$P(E) = \frac{\text{The number of favourable outcomes in E}}{\text{The total number of outcomes in S}} = \frac{|E|}{|S|}$$
$$= \frac{m}{n}.$$

It follows from the definition that

(i) The probability of the *sure event* is 1, that is, $P(S) = 1$

(ii) The probability of the impossible event is 0, that is, $P(\phi) = 0$.

(iii) Since $0 \le m \le n$, we have

$$0 \le \frac{m}{n} \le 1 \text{ or } 0 \le P(E) \le 1$$

This relation is called the *axiom of calculus of probability*.

(iv) The cases favourable to non-occurrence of event E is $n - m$. Therefore,

$$\mathrm{P(not\ E)} = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - \mathrm{P(E)},$$

that is,

$$\mathrm{P}(\overline{\mathrm{E}}) = 1 - \mathrm{P(E)} \text{ or } \mathrm{P(E)} + \mathrm{P}(\overline{\mathrm{E}}) = 1.$$

EXAMPLE 2.31

Three coins are tossed simultaneously. What is the probability that at least two tails are obtained?

Solution. The sample space consists of $2^3 = 8$ outcomes and

$$\mathrm{S} = \{\mathrm{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}\}.$$

Let E be the event obtaining at least 2 tails. Then

$$\mathrm{E} = \{\mathrm{HTT, THT, TTH, TTT}\}.$$

Thus, there are four favourable cases to the event E. Hence $\mathrm{P(E)} = \frac{4}{8} = \frac{1}{2}$.

EXAMPLE 2.32

In a single throw of two distinct dice, what is the probability of obtaining

(i) a total of 7?
(ii) a total of 13?
(iii) a total as even number?

Solution. The sample space shall consist of $6^2 = 36$ points. We list the total number of outcomes as given below:

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Let E_1 be the event in which a total of seven is obtained. Then

$$\mathrm{E}_1 = \{(6,1), (5,2), (4,3), (3,4), (2,5), (1,6)\}$$

and so number of favourable outcomes to the event E_1 is 6. Hence

$$P(E_1) = \frac{6}{36} = \frac{1}{6}.$$

(ii) Since the sum of outcomes on the two dices cannot exceed 6 + 6 = 12, there is no favourable outcome to an event E_2 having sum 13. Hence

$$P(E_2) = \frac{0}{36} = 0$$

(iii) Let E_3 be the event in which we get even number as the sum. Then

E_3 = {(1,1), (1,3), (1,5), (2,2), (2,4), (2,6), (3,1), (3,3), (3,5), (4,2), (4,4), (4,6), (5,1), (5,3),(5,5), (6,2), (6,4), (6,6)}

Thus, number of favourable outcomes to the event E_3 is 18. Hence

$$P(E_3) = \frac{18}{36} = \frac{1}{2}$$

EXAMPLE 2.33

What is the probability that

(i) a non-leap year will have 53 Sunday?
(ii) a leap year will have 53 Sunday?

Solution. (i) A non-leap year contains 365 days. So it has $\frac{365}{7} = 52$ complete weeks and one extra day. The extra day can be any one of seven days—Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday. Out of these seven possibilities, the first one is the only favourable to the event "53 Sundays". Therefore,

$$P(53 \text{ Sunday}) = \frac{1}{7}.$$

(ii) A leap year contains 366 days. So, it has 52 complete weeks and 2 extra days. These days can be any one of the following seven combinations

Sunday and Monday,	Monday and Tuesday
Tuesday and Wednesday,	Wednesday and Thursday
Thursday and Friday,	Friday and Saturday
Saturday and Sunday.	

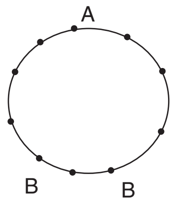
Out of these seven possibilities only two possibilities (enclosed in boxes) are favourable to the event "53 Sunday". Hence

$$P(53 \text{ Sunday in a leap year}) = \frac{2}{7}.$$

EXAMPLE 2.34

Ten persons among whom are A and B, sit down at random at a round table. Find the probability that there are three persons between A and B.

Solution. Let A occupy any seat at the round table. Then there are nine seats available to B. If there are three persons between A and B, then B has only two ways to sit as shown in the diagram below:



Thus, the probability of the required event is $\frac{2}{9}$.

EXAMPLE 2.35

Four microprocessors are randomly selected from a lot of 20 microprocessor among which five are defective. Find the probability of obtaining no defective microprocessor.

Solution. The sample space will consist of $\binom{20}{4}$ sample points since there are 20 C_4 ways to select 4 microprocessors out of 20 microprocessors. Further, since five microprocessors are defective, the number of favourable outcomes to the event "no defective microprocessor is obtained" is $\binom{15}{4}$. Hence

$$P\text{ (no defective microprocessor)} = \frac{\binom{15}{4}}{\binom{20}{4}} = \frac{15.14.13.12}{20.19.18.17} = \frac{32760}{116280} = 0.2817337$$

EXAMPLE 2.36

A bag contains 5 distinct white and 10 distinct black balls. Random samples of three balls are taken out without replacement. Find the probability that the sample contains

(i) exactly one white ball

(ii) no white ball.

Solution. The total number of ways of choosing 3 balls out of 15 balls is 15 C_3. Thus, the sample space consists of 15 C_3 points. Now

(i) The number of ways of choosing one white ball out of five white balls is 5 C_1. Similarly the number of ways of choosing 2 black balls out of 10 is 10 C_2. Therefore, by multiplication rule, the total number of outcomes for the event "sample consists exactly one white ball" is $5C_1$. 10 C_2. Hence

$$P\text{ (exactly one white ball)} = \frac{5C_1.10C_2}{15C_3}$$

$$= \frac{5.10.9.3.2}{2.15.14.13} = \frac{45}{91}.$$

(ii) The event "no white ball" means that all balls selected should be black. So we have to choose 3 balls out of 10 black balls. Hence the number of favourable outcomes to the event is $10C_3$. Therefore,

$$P\text{ (no white ball)} = \frac{10C_3}{15C_3} = \frac{24}{91}.$$

EXAMPLE 2.37

Given a group of four persons, find the probability that

(i) No two of them have their birthday on the same day

(ii) All of them have birthday on the same day.

Solution. Each of the four persons can have his birthday on any of 365 days. Thus, the sample space consists of 365^4 points. Now

(i) Since no two persons have their birthday on the same day, the number of favourable outcomes to this event is

$$365 \,.\, 364 \,.\, 363 \,.\, 362$$

Hence

$$\text{P (distinct birthday)} = \frac{365.364.363.362}{365^4}$$
$$= \frac{364.336.362}{365^3} = \frac{364\ P_3}{365^3}$$

(ii) If all the four persons have their birthday on the same day, then we have to choose just 1 day out of 365. Thus the number of favourable outcomes to the event is 365. Hence

$$\text{P (birthday on the same day)} = \frac{365}{365^4} = \frac{1}{365^3}$$

EXAMPLE 2.38

A bag contains n distinct white and n distinct red balls. Pair of balls are drawn *without replacement* until the bag is empty. Show that the probability that each pair consists of one white and one red ball is $\frac{2^{\text{n}}}{2nC_n}$.

Solution. The bag contains $2n$ distinct balls. Since the pairs are drawn without replacement, the total number of outcomes in the sample space is

$$\binom{2n}{2}\cdot\binom{2n-2}{2}\cdots\binom{4}{2}\cdot\binom{2}{2} = \frac{(2n)!}{2!(2n-2)!}\cdot\frac{(2n-2)!}{2!(2n-4)!}\cdots\frac{4!}{2!\,2!} = \frac{(2n)!}{2^n}.$$

Now, suppose that E is the event in which a pair of balls drawn consists of one white ball and one red ball. Then the first pair can be chosen in $n.n$ ways. Since there is no replacement, the second pair can be selected in $(n–1).\ (n–1)$ ways, and so on. Therefore, the number of favourable outcomes to the event is

$$n^2(n-1)^2(n-2)^2\ldots 2^2.1^2 = [n(n-1)(n-2)\ldots 2.1]^2 = (n!)^2.$$

Hence

$$P(E) = \frac{(n!)^2}{(2n)!}.2^n = \frac{2^n}{\frac{(2n)!}{n!\,n!}} = \frac{2^n}{\binom{2n}{n}}.$$

Theorem 2.3. If E and F are two mutually exclusive events of a random experiment, then

$$P(E \text{ or } F) = P(E \cup F) = P(E) + P(F).$$

Thus, the probability that at least one of the mutually exclusive event E or F occurs is the sum of their individual probabilities.

Proof: Suppose that a random experiment results in n mutually exclusive, equally likely, and exhaustive outcomes of which m_1 are favourable to the occurrence of the event E and m_2 to the occurrence of the event F. Then,

$$P(E) = \frac{m_1}{n} \text{ and } P(F) = \frac{m_2}{n}.$$

Since E and F are mutually exclusive, by addition rule, the number of favourable outcomes to the occurrence of E or F is $m_1 + m_2$. Hence

$$P(E \text{ or } F) = P(E \cup F) = \frac{m_1 + m_2}{n} = \frac{m_1}{n} + \frac{m_2}{n}$$
$$= P(E) + P(F).$$

Corollary (1). If $E_1, E_2, \ldots, E_n$ are n mutually exclusive events, then

$$P(E_1 \cup E_2 \cup \ldots E_n) = P(E_1) + P(E_2) + \ldots + P(E_n).$$

Proof: We shall prove the result by mathematical induction on n. By the above theorem,

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

Let the result be true for $n = k$, that is,

$$P(E_1 \cup E_2 \cup \ldots \cup E_k) = P(E_1) + P(E_2) + \ldots + P(E_k) \tag{2.31}$$

We put $E = E_1 \cup \ldots\ldots \cup E_k$. Then

$$P(E_1 \cup E_2 \cup \ldots \cup E_{k+1}) = P(E \cup E_{k+1}) = P(E_1)$$
$$+ P(E_2) + \ldots + P(Ek) + P(E_{k+1}) \text{ using (2.31).}$$

Hence, the result holds by mathematical induction.

Corollary (2). If $E_1, E_2, \ldots, E_n$ are n mutually exclusive and exhaustive events, then

$$P(E_1) + P(E_2) + \ldots + P(E_n) = 1.$$

Proof: Since $E_1, E_2, \ldots, E_n$ are mutually exclusive and exhaustive,

$$E_1 \cup E_2 \cup \ldots \cup E_n = S \text{ (sample space).}$$

Since P(S) = 1, we have

$$1 = P(S) = P(E_1 \cup E_2 \cup \ldots \cup E_n)$$
$$= P(E_1) + P(E_2) + \ldots + P(E_n).$$

Corollary (3). If E and F are two events, then

$$P(E \cap \overline{F}) = P(E) - P(E \cap F).$$

Proof: The events $E \cap \overline{F}$ and $E \cap F$ are mutually exclusive. Also

$$(E \cap \overline{F}) \cup (E \cap F) = E.$$

Hence, by the above theorem

$$P(E) = P(E \cup \overline{F}) + P(E \cap F) \text{ or } P(E \cap \overline{F}) = P(E) - P(E \cap F).$$

Corollary (4). If E and F are two events such that $E \subseteq F$, then P(E) ≤ P(F).

Proof: Since $E \subseteq F$, we have $F = E \cup (F - E)$. Also E ∩ (F − E) = φ. Hence, by Theorem 2.3, we have

$$P(F) = P(E) + P(F - E), \tag{2.32}$$

Since P(F\E) ≥ 0, it follows from (2.32) that P(F) ≥ P(E).

Theorem 2.4. (Addition Rule or Law of Addition of Probability). If *E* and *F* are any arbitrary events associated with a random experiment, then

$$P(E \text{ or } F) = P(E \cup F) = P(E) + P(F) - P(A \cap B).$$

Proof: The events $E \cap \overline{F}$ and F are two mutually exclusive events and

$$(E \cap \overline{F}) \cup F = E \cup F.$$

Hence

$$P(E \cap \overline{F}) + P(F) = P(E \cup F) \tag{2.33}$$

But $E \cap F$ and $E \cap \overline{F}$ are mutually exclusive, that is,

$$(E \cap \overline{F}) \cup (E \cap F) = E$$

and so

$$P(E \cap \overline{F}) + P(E \cap F) = P(E) \quad \text{or} \quad P(E \cap \overline{F}) = P(E) - P(E \cap F) \tag{2.34}$$

From (2.33) and (2.34), it follows that

$$P(E \cup F) = P(E) + P(F) - P(E \cap F).$$

Remark 2.5. If E and F are mutually exclusive, then E ∩ F = ϕ and P(ϕ) = 0, and so the above result reduces to

$$P(E \cup F) = P(E) + P(F),$$

an result proved already.

EXAMPLE 2.39

Two fair dices are rolled. Find the probability of getting doubles (two dices showing the same numbers) or the sum of 7.

Solution. The sample space S is given by

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

The total number of outcomes in S is 36. Let E_1 be the event "get doubles" and E_2 is the event "sum of 7". Then

E_1 = {(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)} and E_2 = {(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)}.

We notice that E_1 and E_2 are mutually exclusive. Therefore,

$$P(E_1 \text{or } E_2) = P(E_1 \cup E_2) = P(E_1) + P(E_2).$$

But

$$P(E_1) = \frac{\text{The number of favourable outcome in } E_1}{\text{The number of outcomes in S}}$$

$$= \frac{6}{36} = \frac{1}{6}.$$

Similarly,

$$P(E_2) = \frac{6}{36} = \frac{1}{6}.$$

Hence

$$P(E_1 \text{or } E_2) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

EXAMPLE 2.40

Two fair dices are thrown simultaneously. Find the probability of getting doubles or a multiple of 3 as the sum.

Solution. The sample space S consists of the points:

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
-----	-----	-----	-----	-----	-----
-----	-----	-----	-----	-----	-----
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Thus S consists of 36 outcomes. Let E_1 be the event of getting doubles. Then

$$E_1 = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$$

and so the number of favourable outcomes to the event E_1 is 6. So

$$P(E_1) = \frac{6}{36} = \frac{1}{6}.$$

Let E_2 be the event of getting a multiple of 3 as the sum. Then

$$E_2 = \{(1,2), (1,5), (2,1), (2,4), (3,3), (3,6), (4,2), (4,5), (5,1), (5,4), (6,3), (6,6)\}$$

and so the number of favourable outcomes to the event E_2 is 12. Thus

$$P(E_2) = \frac{12}{36} = \frac{1}{3}.$$

Further,

$$E_1 \cap E_2 = \{33, 66\}.$$

Thus

$$P(E_1 \cap E_2) = \frac{2}{36} = \frac{1}{18}.$$

Hence

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$
$$= \frac{1}{6} + \frac{1}{3} - \frac{1}{18} = \frac{4}{9}.$$

EXAMPLE 2.41

A bag contains five white, seven black, and eight red balls. A ball is drawn at random. What is the probability that it is a red ball or a white ball?

Solution. The number of outcomes in the sample space is

$$20\ C_1 = 20.$$

Let E_1 be the event where red ball is obtained and E_2 be the event where white ball is obtained. Then

$$P(E_1) = \frac{8C_1}{20} = \frac{8}{20} = \frac{2}{5} \text{ and}$$
$$P(E_2) = \frac{5C_1}{20} = \frac{5}{20} = \frac{1}{4}.$$

Also the events are mutually exclusive. Therefore,

$$P(E_1 \text{or } E_2) = P(E_1) + P(E_2)$$
$$= \frac{2}{5} + \frac{1}{4} = \frac{13}{20}$$

EXAMPLE 2.42

Let A and B be two mutually exclusive events of an experiment. If P(not A) = 0.65, $P(A \cup B) = 0.65$ and $P(B) = p$, find p.

Solution. We have

$$P(\text{not } A) = P(\bar{A}) = 0.65.$$

But

$$P(A) + P(\bar{A}) = 1 \text{ and so}$$
$$P(A) = 1 - P(\bar{A}) = 1 - 0.65 = 0.35.$$

Further, since A and B are mutually exclusive,

$$P(A \cup B) = P(A) + P(B) = P(A) + p$$

and so

$$p = P(A \cup B) - P(A) = 0.65 - 0.35 = 0.30.$$

2.11 CONDITIONAL PROBABILITY

Let E and F be events and let P(F) > 0. Then the conditional probability of E, given F, is defined as

$$P(E \backslash F) = \frac{P(E \cap F)}{P(F)}.$$

EXAMPLE 2.43

Let two fair dice be rolled. If the sum of 7 is obtained, find the probability that at least one of the dice shows 2.

Solution. Let E be the event "sum of 7 is obtained". Thus

$$E = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}.$$

Let F be the event "at least one dice shows 2". Then

$$F = \{(1,2), (2,2), (3,2), (4,2), (5,2), (6,2), (2,1), (2,3), (2,4), (2,5), (2,6)\}.$$

Since $E \cap F = \{25, 52\}$, by the definition of conditional probability, we have

$$P(F\backslash E) = \frac{P(E \cap F)}{P(E)} = \frac{\frac{2}{36}}{\frac{6}{36}} = \frac{1}{3}.$$

EXAMPLE 2.44

Weather records show that the probability of high barometric pressure is 0.82 and the probability of rain and high barometric pressure is 0.20. Find the probability of rain, given high barometric pressure?

Solution. Let E denote the event "rain" and F denote the event "high barometric pressure." Then

$$P(E\backslash F) = \frac{P(E \cap F)}{P(F)} = \frac{.20}{.82} = 0.2446.$$

Theorem 2.5. (Multiplication Law of Probability). Let $P(A\backslash B)$ denote the conditional probability of A when B has occurred. Then

$$P(A \cap B) = P(B)P(A\backslash B) = P(A)P(B\backslash A).$$

Proof: We know that

$$P(A\backslash B) = \frac{P(A \cap B)}{P(B)} \tag{2.35}$$

and

$$P(B\backslash A) = \frac{P(A \cap B)}{P(A)}. \tag{2.36}$$

From (2.35) and (2.36), we have

$$P(A \cap B) = P(B)\, P(A\backslash B) = P(A)\, P(B\backslash A).$$

EXAMPLE 2.45

A fair coin is tossed four times. Find the probability that they are all heads if the first two tosses results in head.

Solution. The sample space consists of $2^4 = 16$ outcomes. Let A be the event "all heads." Then A = {HHHH}. Let B be the event "first two heads". Then

$$B = \{HHHH, HHHT, HHTH, HHTT\}.$$

We notice that

$$A \cap B = \{HHHH\}.$$

Therefore,

$$P(B) = \frac{4}{16} = \frac{1}{4} \text{ and } P(A \cap B) = \frac{1}{16}$$

and so

$$P(A\backslash B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{16}}{\frac{1}{4}} = \frac{1}{4}.$$

2.12 INDEPENDENT EVENTS

Two events A and B are said to be *independent* if the occurrence or non-occurrence of one event does not affect the probability of the occurrence or non-occurrence of the other event. Mathematically, A and B are independent if and only if any one of the following conditions is satisfied.

$$\begin{aligned}
&P(A\backslash B) = P(A), P(\bar{A}\backslash B) = P(\bar{A}),\\
&P(A\backslash \bar{B}) = P(A), P(\bar{A}\backslash \bar{B}) = P(\bar{A}),\\
&P(B\backslash A) = P(B), P(\bar{B}\backslash A) = P(\bar{B}),\\
&P(B\backslash \bar{A}) = P(B), P(\bar{B}\backslash \bar{A}) = P(\bar{B}).
\end{aligned}$$

Thus, if A and B are independent events, then

$$P(A) = P(A\backslash B) = \frac{P(A \cap B)}{P(B)}$$

or

$$P(A \cap B) = P(A)P(B).$$

This relation is called *multiplication rule for independent events*.

Hence, we may also define independence of events as follows:

Events A and B are called *independent* if P (A ∩ B) = P(A) P(B).

EXAMPLE 2.46

A married couple (husband and wife) appear for an interview for two vacancies against the same post. The probability of husband's selection is $\frac{1}{6}$ and the probability of wife's selection is $\frac{2}{5}$. What is the probability that

(i) both of them will be selected
(ii) only one of them will be selected
(iii) none of them will be selected
(iv) at least one of them will be selected?

Solution. Let E be the event "husband is selected" and F denote the event "wife is selected". We are given that

$$P(E) = \frac{1}{6} \text{ and } P(F) = \frac{2}{5}.$$

Since there are two vacancies, selection of one does not affect the other. Hence E and F are independent events. Then

(i) P(both of them are selected) = P(E ∩ F)

$$\begin{aligned}
&= P(E)\,P(F) \text{ since E and F are independent}\\
&= \frac{1}{6}.\frac{2}{5} = \frac{1}{15}.
\end{aligned}$$

(ii) Since $E \cap \bar{F}$ and $\bar{E} \cap F$ are exclusive, we have

P (only one of them is selected)

$$= P(E \cap \overline{F}) \cup (\overline{E} \cap F)$$

$$= P(E \cap \overline{F}) + (\overline{E} \cap F) \quad \text{(exclusive events)}$$

$$= P(E)P(\overline{F}) + P(\overline{E})P(F), \quad \text{since E and F are independent}$$

$$= P(E)(1 - P(F)) + (1 - P(E))P(F)$$

$$= \frac{1}{6}\left(1 - \frac{2}{5}\right) + \left(1 - \frac{1}{6}\right)\frac{2}{5} = \frac{1}{10} + \frac{1}{3} = \frac{13}{30}.$$

(iii) We have

P(none of them is selected) = P(not E and F)

$$= P(\overline{E} \cap \overline{F}) = P(\overline{E})P(\overline{F}) \quad \text{since E and F are independent}$$

$$= (1 - P(E))(1 - P(F))$$

$$= \left(1 - \frac{1}{6}\right)\left(1 - \frac{2}{5}\right) = \frac{5}{6} \cdot \frac{3}{5} = \frac{1}{2}.$$

(iv) We have

P(at least one of them gets selected)

$$= P(E \text{ or } F) = P(E \cup F)$$

$$= P(E) + P(F) - P(E \cap F)$$

$$= \frac{1}{6} + \frac{2}{5} - \frac{1}{15}, \text{using(i)}$$

$$= \frac{1}{2}.$$

Second Method: $P(E \cup F) = 1 - P\left(\overline{E \cup F}\right) = 1 - P(\overline{E} \cap \overline{F}) = 1 - \frac{1}{2} = \frac{1}{2}.$

EXAMPLE 2.47

If $P(B) \neq 1$, show that

$$P(\overline{A} \backslash \overline{B}) = \frac{1 - P(A \cup B)}{P(\overline{B})}.$$

Solution. We have

$$P(\overline{A} \backslash \overline{B}) = \frac{P(\overline{A} \cap \overline{B})}{P(\overline{B})} = \frac{P(\overline{A \cup B})}{P(\overline{B})} = \frac{1 - P(A \cup B)}{P(\overline{B})}.$$

EXAMPLE 2.48

A problem in mathematics is given to three students whose chances of solving the problem are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. What is the probability that the problem is solved?

Solution. Let

A be the event "first student solves the problem"
B be the event "second student solves the problem"
C be the event "third student solves the problem"

It is given that

$$P(A) = \frac{1}{2}, \quad P(B) = \frac{1}{3}, \quad P(C) = \frac{1}{4}$$

and so

$$P(\overline{A}) = 1 - \frac{1}{2} = \frac{1}{2}, \quad P(\overline{B}) = 1 - \frac{1}{3} = \frac{2}{3},$$
$$P(\overline{C}) = 1 - \frac{1}{4} = \frac{3}{4}.$$

Hence
P(the problem is solved)

$$= P(A \text{ or } B \text{ or } C)$$
$$= P(A \cup B \cup C) = 1 - P[\overline{(A \cup B \cup C)}]$$
$$= 1 - P(\overline{A} \cap \overline{B} \cap \overline{C})$$
$$= 1 - P(\overline{A})P(\overline{B})P(\overline{C})$$

since A, B, and C are independent

$$= 1 - \frac{1}{2}.\frac{2}{3}.\frac{3}{4} = \frac{3}{4}.$$

Theorem 2.6. (Baye's Theorem). Let $A_1, A_2, \ldots\ldots, A_m$ be pairwise mutually exclusive and exhaustive random events, where $P(A_i) \geq 0$, $i = 1, 2, \ldots, m$. Then for any arbitrary event B of the random experiment,

$$P(A_i \setminus B) = \frac{P(A_i)P(B \setminus A_i)}{\sum_{i=1}^{m} P(A_i)P(B \setminus A_i)}.$$

Proof: Let S be the sample space of the random experiment. Since the events $A_1, A_2, \ldots\ldots, A_m$ are pairwise exclusive and exhaustive, we have

$$S = A_1 \cup A_2 \cup \ldots\ldots \cup A_m.$$

Therefore, we have

$$B = S \cap B = (A_1 \cup A_2 \cup \ldots\ldots \cup A_m) \cap B$$
$$= (A_1 \cap B) \cup (A_2 \cap B) \cup \ldots\ldots \cup (A_m \cap B)$$

Since $A_1 \cap B, A_2 \cap B, \ldots\ldots, A_m \cap B$ are mutually exclusive, it follows by addition law that

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \ldots.. + P(A_m \cap B)$$
$$= P(B \setminus A_1)P(A_1) + P(B \setminus A_2)P(A_2) + \ldots + P(B \setminus A_m)P(A_m).$$

This relation is called the "theorem on total probability." Using this relation, we have

$$P(A_i \setminus B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B \setminus A_i)P(A_i)}{P(B)} = \frac{P(A_i)P(B \setminus A_i)}{\sum_{i=1}^{m} P(A_i)P(B \setminus A_i)}.$$

EXAMPLE 2.49

A university purchased computers from three firms. The percentage of computer purchased and percentage of defective computers is shown in the table below:
Firm

	HCL	WIPRO	IBM
Percent purchase	45	25	30
Percent defective	2	3	1

Let A be the event "computer purchased from HCL"
B be the event "computer purchased from WIPRO"
C be the event "computer purchased from IBM"
D be the event "computer was defective".

Find P(A), P(B), P(C), P(D\A), P(D\B), P(D\C) and P(D).

Solution. We note that

$$P(A) = \frac{45}{45+25+30} = 0.45, \quad P(B) = \frac{25}{100} = 0.25,$$

$$P(C) = \frac{30}{100} = 0.30, \quad P(D \setminus A) = \frac{2}{100} = 0.02,$$

$$P(D \setminus B) = \frac{3}{100} = 0.03, \quad P(D \setminus C) = \frac{1}{100} = 0.01.$$

$$\begin{aligned} P(D) &= P(D \setminus A)P(A) + P(D \setminus B)P(B) + P(D \setminus C)P(C) \\ &= (0.02)(0.45) + (0.03)(0.25) + (0.01)(0.30) \\ &= 0.0090 + 0.0075 + 0.0030 = 0.0195. \end{aligned}$$

EXAMPLE 2.50

In a test, an examinee either guesses, or copies or knows the answer to multiple choice questions with four choices. The probability that he makes a guess is $\frac{1}{3}$ and the probability that he copies the answer is $\frac{1}{6}$. The probability that his answer is correct, given that he copied it is $\frac{1}{8}$. Find the probability that he knew the answer to the question given that he correctly answered.

Solution. Let us consider the following events:

A: the examinee guesses the answer
B: the examinee copies the answer
C: the examinee knows the answer
D: the examinee answers correctly.
It is given that

$$P(A) = \frac{1}{3}, \quad P(B) = \frac{1}{6} \text{ and } P(D \setminus B) = \frac{1}{8}.$$

Also, the hypothesis that examinee either guesses or copies or knows the answer implies that

$$P(C) = 1 - P(A) - P(B) = 1 - \frac{1}{3} - \frac{1}{6} = \frac{1}{2}.$$

Further,

$$P(D \backslash C) = 1 \text{ since he knows the answer correctly.}$$

$$P(D \backslash A) = \frac{1}{4} \quad \text{(since if he guesses, he can tick any one of the four choices).}$$

Then, by Baye's law,

$$P(C \backslash D) = \frac{P(D \backslash C)P(C)}{P(D \backslash A)P(A) + P(D \backslash B)P(B) + P)(D \backslash C)P(C)}$$

$$= \frac{1.\frac{1}{2}}{\frac{1}{4}\cdot\frac{1}{3} + \frac{1}{8}\cdot\frac{1}{6} + 1\cdot\frac{1}{2}} = \frac{24}{29}.$$

EXAMPLE 2.51

The following observations were made at a clinic where HIV virus test was performed.

(i) 15% of the patients at the clinic have HIV virus
(ii) among those who have HIV virus, 95% test positive on the ELISA test
(iii) among those that do not have HIV virus, 2% test positive on the ELISA test.

Find the probability that a patient has the HIV virus if the ELISA test is positive.

Solution. We consider the following events:

A: "has the HIV virus"
B: "does not have the HIV virus"
C: "test positive".
We are given that

$$P(A) = \frac{15}{100} = 0.15.$$

Therefore,

$$P(B) = P(\bar{A}) = 1 - P(A) = 1 - 0.15 = 0.85$$

$$P(C \backslash A) = \frac{95}{100} = 0.95 \text{ and } P(C \backslash B) = \frac{2}{100} = 0.02.$$

We want to find P (A\C). By Baye's theorem, we have

$$P(A \backslash C) = \frac{P(C \backslash A)P(A)}{P(C \backslash A)P(A) + P(C \backslash B)P(B)}$$

$$= \frac{(0.95)(0.15)}{(0.95)(0.15) \times (.02)(.085)} = 0.89.$$

EXAMPLE 2.52

An item is manufactured by three factories F_1, F_2, and F_3. The number of units of the item produced by F_1, F_2, and F_3 are $2x$, x, and x, respectively. It is known that 2% of the items produced by F_1 and F_2 are defective and 4% of the items produced by F_3 are defective. All units produced by these factories are put together in one stockpile and one unit is chosen at random. It is found that this item is defective. What is the probability that this defective unit came from (i) factory F_1, (ii) factory F_2, or (iii) factory F_3?

Solution. Consider the events:

A: "the unit is defective"
B: "the defective unit came from F_1"
C: "the defective unit came from F_2"
D: "the defective unit came from F_3"
We have then, as per given hypothesis,

$$P(B) = \frac{2x}{4x} = \frac{1}{2}, \quad P(C) = \frac{x}{4x} = \frac{1}{4}, \quad P(D) = \frac{x}{4x} = \frac{1}{4}$$

$$P(A \setminus B) = \frac{2}{100} = 0.02, \quad P(A \setminus C) = \frac{2}{100} = 0.02,$$

$$P(A \setminus D) = \frac{4}{100} = 0.04$$

Then the theorem on total probability implies that

$$P(A) = P(A \setminus B)P(B) + P(A \setminus C)P(C) + P(A \setminus D)P(D)$$

$$= (0.02)\left(\frac{1}{2}\right) + (0.02)\left(\frac{1}{4}\right) + (0.04)\left(\frac{1}{4}\right) = 0.025.$$

We then have, by Baye's theorem,

$$P(B \setminus A) = \frac{P(A \setminus B)P(B)}{P(A)} = \frac{(0.02)\left(\frac{1}{2}\right)}{0.025} = 0.4,$$

$$P(C \setminus A) = \frac{P(A \setminus C)P(C)}{P(A)} = \frac{(0.02)\left(\frac{1}{4}\right)}{0.025} = 0.2,$$

$$P(D \setminus A) = \frac{\overline{P}(\overline{A} \setminus \overline{D})\overline{P}(\overline{D})}{\overline{P}(\overline{A})} = \frac{(0.04)\left(\frac{1}{4}\right)}{0.025} = 0.4.$$

2.13 PROBABILITY DISTRIBUTION

Let S be a sample space of an random experiment. A *random variable X* is a function of the possible events of S which assigns a numerical value of each outcome in S. A random variable is also called a *Variate*.

Let a random variable X assume the values $x_1, x_2, \ldots, x_n$ corresponding to various outcomes of a random experiment. If the probability of x_i is $P(x_i) = p_i$, $1 \le i \le n$ such that $p_1 + p_2 + \ldots + p_n = 1$, then the function $P(X)$ is called the *probability function* of the random variable X and the set $\{P(x_i)\}$ is called the *probability distribution* of X. Since random variable X takes a finite set of values, it is called discrete variate and $\{P(x_i)\}$ is called the *discrete probability distribution*.

The probability distribution of X is denoted by the table:

$$\begin{array}{ccccc} X & x_1 & x_2 & x_3 \ldots . x_n \\ P(X) & p_1 & p_2 & p_3 \ldots . p_n \end{array}$$

If x is an integer, then the function F defined by

$$F(X) = P(X \le x) = \sum_{i=1}^{x} p(x_i)$$

is called the *distribution function* or *cumulative distribution function* of the discrete variate X.

If a variate X takes every value in an interval, the number of events is infinitely large and so the probability for an event to occur is practically zero. In such a case, the probability of x falling in a small interval is determined. The function f defined by

$$P\left(x-\frac{1}{2}dx \le x \le x+\frac{1}{2}dx\right) = f(x)dx$$

is called the *probability density function* and the continuous curve $y = f(x)$ is called the *probability curve.*

If the range of x is finite, we may consider it as infinite by supposing the density function f to be zero outside the given range. Thus if $f(x) = \phi(x)$ be the density function for x in $[a,b]$, then we take

$$f(x) = \begin{cases} 0, & x < a \\ \phi(x), & x \in [a,b] \\ 0, & x > b. \end{cases}$$

Further, the density function f is always positive and $\int_{-\infty}^{\infty} f(x)dx = 1$, that is, the total area under the probability curve and the x-axis is unity, This fulfills the requirement that the total probability of the occurrence of an event is 1.

If X is continuous variate, then the function F defined by

$$F(x) = P(X \le x) = \int_{-\infty}^{\infty} f(x)dx$$

is called the *cumulative distribution function of the continuous variate X.*

The cumulative distribution function F has the following important properties:

(i) $F'(x) = f(x) \ge 0$ and so F is non-decreasing function.

(ii) $F(-\infty) = 0$ and $F(\infty) = 1$.

(iii) $$\begin{aligned} P(a \le x \le b) &= \int_a^b f(x)dx \\ &= \int_{-\infty}^{b} f(x)dx - \int_{-\infty}^{a} f(x)dx \\ &= F(b) - F(a). \end{aligned}$$

2.14 MEAN AND VARIANCE OF A RANDOM VARIABLE

Let X be a random variable which takes the values $x_1, x_2, \ldots, x_m$ with corresponding probabilities $p_1, p_2, \ldots, p_m$. Then the *mean* (also called *expectation*) and *variance* of the random variables are defined by

$$\text{Mean: } \mu = \frac{\sum_{i=1}^{m} p_i\, x_i}{\sum_{i=1}^{m} p_i} = \sum_{i=1}^{m} p_i\, x_i \quad \text{since} \sum_{i=1}^{m} p_i = 1$$

$$\text{Variance: } \sigma^2 = \sum_{i=1}^{m} (\mathrm{x_i} - \mu)^2 \, p_{\mathrm{i}}$$

$$= \sum_{i=1}^{m} (x_i^2 - 2\mu x_i + \mu^2) p_i$$

$$= \sum_{i=1}^{m} p_i x_i^2 - 2\mu \sum_{i=1}^{m} p_i x_i + \mu^2 \sum_{i=1}^{m} p_i$$

$$= \sum_{i=1}^{m} p_i x_i^2 - 2\mu^2 + \mu^2$$

$$\text{since} \sum_{i=1}^{m} p_i x_i = \mu \quad \text{and} \sum_{i=1}^{m} p_i = 1$$

$$= \sum_{i=1}^{m} p_i x_i^2 - \mu^2,$$

where σ is the *standard deviation of the distribution.*

In case of continuous probability distribution, the mean (expected value) and variations are defined by

$$\mu = \int_{-\infty}^{\infty} x f(x) dx,$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx.$$

EXAMPLE 2.53

A random variable x has the following probability function:

x:	–2	–1	0	1	2	3
$p(x)$:	0.1	k	0.2	2k	0.3	k

Find the value of k and calculate the mean and variance.

Solution. Since $\Sigma p_i = 1$, we have

$$0.1 + k + 0.2 + 2k + 0.3 + k = 1,$$

which yields $k = 0.1$. Further,

$$\text{Mean: } \mu = \sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{6} p_i x_i$$

$$= -2(0.1) + (-1)(0.1) + 0(0.2) + 2(0.1) + 2(0.3) + 3(0.1) = 0.8$$

and

$$\text{Variance: } \sigma^2 = \sum_{i=1}^{n} (x_i - \mu)^2 p_i = \sum_{i=1}^{n} p_i x_i^2 - \mu^2$$

$$= 0.4 + 0.1 + 0 + 0.2 + 1.2 + 0.9 - 0.69 = 2.16.$$

EXAMPLE 2.54

A die is tossed thrice. A success is "getting 1 or 6" on a toss. Find the mean and variance of the number of successes.

Solution. We have $n = 3$. Let X denote the number of success. Then

$$\text{Probability of success} = \frac{2}{6} = \frac{1}{3},$$
$$\text{Probability of failure} = 1 - \frac{1}{3} = \frac{2}{3}$$

and

$$P(X=0) = P(\text{no success}) = \text{P(all failures)}$$
$$= \frac{2}{3}.\frac{2}{3}.\frac{2}{3} = \frac{8}{27}$$
$$P(X=1) = P(1 \text{ success and 2 failures})$$
$$= \left({}^3C_1 \times \frac{1}{3}\right)\frac{2}{3}.\frac{2}{3} = \frac{4}{9},$$
$$P(X=2) = P(2 \text{ success and 1 failure})$$
$$= \left({}^3C_2 \times \frac{1}{3}\right)\frac{1}{3} \times \frac{2}{3} = \frac{2}{9},$$
$$P(X=3) = P(3 \text{ success}) = \frac{1}{3}.\frac{1}{3}.\frac{1}{3} = \frac{1}{27}.$$

Therefore, the probability distribution is

$$\begin{array}{ccccc} X: & 0 & 1 & 2 & 3 \\ P(X): & \frac{8}{27} & \frac{4}{9} & \frac{2}{9} & \frac{1}{27} \end{array}$$

Now

$$\text{Mean}\,(\mu) = \sum p_i x_i = \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1,$$
$$\text{Variance}\left(\sigma^2\right) = \sum p_i x_i^2 - \mu^2 = \frac{4}{9} + \frac{8}{9} + \frac{9}{27} - 1 = \frac{2}{3}.$$

EXAMPLE 2.55

Find the standard deviation for the following discrete distribution:

$$\begin{array}{cccccc} X: & 8 & 12 & 16 & 20 & 24 \\ P(X): & \frac{1}{8} & \frac{1}{6} & \frac{3}{8} & \frac{1}{4} & \frac{1}{12} \end{array}$$

Solution. For the given discrete distribution, $n = 5$ and the mean

$$\mu = \sum_{i=1}^{5} p_i x_i = 1 + 2 + 6 + 5 + 2 = 16.$$

Thus the variance is

$$\sigma^2 = \sum_{i=1}^{5} p_i x_i^2 - \mu^2$$
$$= 8 + 24 + 96 + 100 + 48 - 256 = 20.$$

Hence, the standard deviation is

$$\sigma = \sqrt{20} = 2\sqrt{5}.$$

EXAMPLE 2.56

The diameter x of an electric cable is assumed to be a continuous variate with possible probability density function $f(x) = 6x(1-x)$, $0 \le x \le 1$. Verify whether f is a probability density function. Also find the mean and variance.

Solution. The given function is non-negative and $\int_0^1 f(x)dx = \int_0^1 6x dx - \int_0^1 6x^2 dx = 1$. Hence f is a probability density function. Further

$$\text{Mean}(\mu) = \int_0^1 x f(x)dx = \int_0^1 6x^2 dx - \int_0^1 6x^3 dx$$
$$= 6\left[\frac{x^3}{3}\right]_0^1 - 6\left[\frac{x^4}{4}\right]_0^1 = 2 - \frac{3}{2} = \frac{1}{2},$$
$$\text{Variance}(\sigma^2) = \int_0^1 (x-\mu)^2 f(x)dx$$
$$= \int_0^1 \left(x - \frac{1}{2}\right)^2 [6x(1-x)]dx$$
$$= \int_0^1 \left(-6x^4 + 12x^3 - \frac{15}{2}x^2 + \frac{3}{2}x\right)dx$$
$$= -\frac{6}{5} + 3 - \frac{15}{6} + \frac{3}{4} = \frac{1}{20}.$$

EXAMPLE 2.57

The probability density $p(x)$ of a continuous random variable is given by

$$p(x) = y_0 e^{-|x|}, -\infty < x < \infty.$$

Prove that $y_0 = \frac{1}{2}$. Find the mean and variance of distribution.

Solution. Since $e^{-|x|}$ is an even function of a, we have

$$\int_{-\infty}^{\infty} p(x)dx = y_0 \int_{-\infty}^{\infty} e^{-|x|}dx = 2y_0 \int_0^{\infty} e^{-|x|}dx$$
$$= 2y_0 \int_0^{\infty} e^{-x}dx = 2y_0 \left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 2y_0.$$

But, $p(x)$ being probability density function, we have

$$\int_{-\infty}^{\infty} p(x)dx = 1.$$

Therefore $2y_0 = 1$, which yields $y_0 = \frac{1}{2}$. Further, since $xe^{-|x|}$ is an odd function, we have

$$\text{Mean}(\mu) = \int_{-\infty}^{\infty} xp(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} xe^{-[x]}dx = 0.$$

Since $x^2e^{-|x|}$ is an even function, we have

$$\text{Variance}(\sigma^2) = \int_{-\infty}^{\infty} (x-\mu)^2 p(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} x^2e^{-|x|}dx$$

$$= \frac{2}{2}\int_{0}^{\infty} x^2e^{-|x|}dx = \int_{0}^{\infty} x^2e^{-x}dx = \Gamma(3) = 2! = 2.$$

EXAMPLE 2.58

Show that the function f defined by

$$f(x) = \begin{cases} \frac{3+2x}{18}, & 2 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

is a density function. Find mean, variance, standard deviation, and mean deviation from the mean of the distribution.

Solution. The function f is non-negative and

$$\int_{-\infty}^{\infty} f(x)dx = \frac{1}{18}\int_{2}^{4}(3+2x)dx = \frac{1}{18}\left|3x+\frac{x^2}{2}\right|_2^4 = \frac{1}{18}(28-10) = 1.$$

Hence f is a density function. Also

$$\text{Mean}(\mu) = \int_{-\infty}^{\infty} xf(x)dx = \frac{1}{18}\int_{2}^{4}(3x+2x^2)\,dx = \frac{1}{18}\left[\frac{3x^2}{2}+2\frac{x^3}{3}\right]_2^4 = \frac{83}{27},$$

$$\text{Variance}(\sigma^2) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = \frac{1}{18}\int_{2}^{4}\left(x-\frac{83}{27}\right)^2(3+2x)dx = \frac{239}{729}.$$

Therefore,

$$\text{Standard deviation} = \sqrt{\sigma^2} = \sqrt{\frac{239}{729}} = 0.57.$$

$$\text{Mean deviation} = \int_{-\infty}^{\infty} |\,x-\mu\,|\,f(x)dx = \int_{2}^{4}\left|x-\frac{83}{27}\right|\left(\frac{3+2x}{18}\right)dx$$

$$= \int_{2}^{\frac{83}{27}}\left(\frac{83}{27}-x\right)\left(\frac{3+2x}{18}\right)dx + \int_{\frac{83}{27}}^{4}\left(x-\frac{83}{27}\right)\left(\frac{3+2x}{18}\right)dx = 0.49.$$

EXAMPLE 2.59

Two cards are drawn successively *with replacement* from a well-shuffled pack of 52 playing cards. Find the probability distribution of the number of aces.

Solution. Let X be the random variable that is the number of aces obtained in the draw of two cards. There are three possibilities: (i) there is no ace, (ii) there is one ace, and (iii) there are two aces. Thus, the random variable takes the values 0, 1, 2. Then

$$\text{P(no ace is drawn)} = P(X = 0) = \frac{48}{52}.\frac{48}{52} = \frac{144}{169}$$

P(one ace is drawn) = P(X = 1)

= P(one ace is drawn in the first draw and no ace is drawn in the second draw) + P(no ace is drawn in the first draw and one ace is drawn in the second draw)

$$= \frac{4}{52}.\frac{48}{52} + \frac{48}{52}.\frac{4}{52} = \frac{24}{169}.$$

$$\text{P(two aces are drawn)} = P(X = 2) = \frac{4}{52}.\frac{4}{52} = \frac{1}{169}.$$

Hence the probability distribution is

X:	0	1	2
$P(X)$:	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

EXAMPLE 2.60

Find the probability distribution of the number of green balls drawn when three balls are drawn one by one *without replacement* from a bag containing three greens and five white balls.

Solution. Let X be the random variable which is the number of green balls drawn when three balls are drawn without replacement. The random variable takes the values 0, 1, 2, 3.

We represent green ball by G and white ball by W. Then we have

$$\text{P(no green ball is drawn)} = P(X = 0) = P(WWW) = \frac{5}{8}.\frac{4}{7}.\frac{3}{6} = \frac{5}{28}$$

P(one green ball is drawn) = P(X = 1)

= P(GWW)+P(WGW)+P(WWG)

$$= \frac{3}{8}.\frac{5}{7}.\frac{4}{6} + \frac{5}{8}.\frac{3}{7}.\frac{4}{6} + \frac{5}{8}.\frac{4}{7}.\frac{3}{6} = \frac{15}{28}.$$

P(two green balls are drawn) = P(X = 2)

= P(GGW) + P(GWG) + P(WGG)

$$= \frac{3}{8}.\frac{2}{7}.\frac{5}{6} + \frac{3}{8}.\frac{5}{7}.\frac{2}{6} + \frac{5}{8}.\frac{3}{7}.\frac{2}{6} = \frac{15}{56}.$$

P(three green balls are drawn) = P(GGG)

$$= \frac{3}{8}.\frac{2}{7}.\frac{1}{6} = \frac{1}{56}.$$

Therefore, the probability distribution is

X:	0	1	2	3
$p(X)$:	$\frac{5}{28}$	$\frac{15}{28}$	$\frac{15}{86}$	$\frac{1}{56}$

2.15 BINOMIAL DISTRIBUTION

Let S be a sample space for a random experiment. Let A be an event associated with a subset of S and let P(A) = p, then we know that P($\overline{A}$) = 1–p. If we denote P($\overline{A}$) = q, then $p + q = 1$.

If we call the occurrence of the event A as "success" and non-occurrence of the event A as a "failure", then

$$\text{P(failure)} = 1 - \text{P(success) and so}$$
$$\text{P(failure)} + \text{P(success)} = 1.$$

Suppose that X is a random variable on the sample space as the "number of success." Then the probability distribution associated with the above random experiment is

X:	0	1
$p(X)$:	q	p

If the experiment is conducted two times, then the possible outcomes are success success, success failure, failure success, and failure failure. Since the trials are independent, we have

$$\begin{aligned}\text{P(success success)} &= \text{P(both success)}\\ &= \text{P(success)P(success)}\\ &= p.p = p^2,\\ \text{P(success failure)} &= \text{P(success)P(failure)} = pq,\\ \text{P(failure success)} &= \text{P(failure)P(success)} = qp,\\ \text{P(failure failure)} &= \text{P(failure)P(failure)} = q^2.\end{aligned}$$

Thus, in term of random variable, we have

$$\begin{aligned}&\text{P}(X = 0) = \text{P(failure, failure)} = q^2,\\ &\text{P}(X = 1) = pq + qp = 2pq,\\ &\text{P}(X = 2) = p\text{(success, success)} = p^2.\end{aligned}$$

Also we note that

$$\begin{aligned}&P(X = 0) + P(X = 1) + P(X = 2)\\ &\quad = p^2 + q^2 + 2pq = (p + q)^2 = (1)^2 = 1.\end{aligned}$$

Thus, the probability distribution associated with the two experiments is

X:	0	1	2
$P(X)$:	q^2	$2pq$	p^2

The term of P(X) are the terms in the binomial expansion of $(q + p)^2$.

Similarly, the probability distribution associated with the three experiments is

X:	0	1	2	3
$P(X)$:	q^3	$3q^2p$	$3qp^2$	p^3

Thus probabilities are the terms in the binomial expansions of $(q+p)^3$. If the experiment is repeated n times, then the probability distribution is

$$X: 0 \quad 1 \quad 2 \quad r\ldots\ldots\ldots \quad n$$
$$P(X): q^n \quad n_{c_1} q^{n-1} p \quad n_{c_2} q^{n-2} p^2 \quad n_{c_r} q^{n-r} p^r \ldots\ldots p^n$$

Clearly the probabilities are terms in the binomial expansion of $(q+p)^n$.

This probability distribution is called the *binomial distribution* and X is called a *binomial random variable*.

Further, *mean of the binomial distribution* is given by

$$\begin{aligned}\text{Mean: } \mu &= \sum_{r=0}^{n} r\mathrm{P}(r) = \mathrm{P}(1) + 2\mathrm{P}(2) + \ldots\ldots + n\mathrm{P}(n)\\ &= n_{c_1} q^{n-1} p + 2n_{c_2} q^{n-2} p^2 + \ldots\ldots + n_{c_n} p^n\\ &= npq^{n-1} + \frac{2n(n-1)}{2!} p^2 q^{n-2} + \ldots\ldots + n\mathrm{p}^n\\ &= np[q^{n-1} + (n-1)pq^{n-2} + \ldots\ldots + p^{n-1}]\\ &= np[(q+p)^{n-1}] = np, \text{ since } q+p=1.\end{aligned}$$

The *variance of the binomial distribution* is

$$\text{Variance: } \sigma^2 = \sum_{r=0}^{n} r^2\mathrm{P}(r) - \mu^2. \tag{2.37}$$

Now

$$\begin{aligned}\sum_{r=0}^{n} r^2\mathrm{P}(r) &= \sum_{r=0}^{n} [r + r(r-1)]\mathrm{P}(r) = \sum_{r=0}^{n} r\mathrm{P}(r) + \sum_{r=0}^{n} r(r-1)\mathrm{P}(r)\\ &= \mu + \sum_{r=0}^{n} r(r-1)\mathrm{P}(r) = np + \sum_{r=2}^{n} r(r-1)\mathrm{P}(r) \text{ since } \mu = np.\\ &= np + \sum_{r=2}^{n} r(r-1) n_{C_r} p^r q^{n-r} = np + n(n-1)p^2(q+p)^{n-2}\\ &= np + n(n-1)p^2 \text{ since } (q+p) = 1\end{aligned}$$

Hence

$$\begin{aligned}\sigma^2 &= np + n(n-1)p^2 - \mu^2\\ &= np + n(n-1)p^2 - n^2p^2 \text{ since } \mu = np\\ &= np + n^2p^2 - np^2 - n^2p^2\\ &= np(1-p) = npq.\end{aligned}$$

Thus the variance of the binomial distribution is

$$\sigma^2 = npq$$

and the standard deviation of the binomial distribution is

$$\sigma = \sqrt{npq}.$$

To derive a recurrence formula for the binomial distribution, we note that

$$P(r) = {}^nC_r q^{n-r} p^r = \frac{n!}{r!(n-r)!} q^{n-r} p^r$$

and so

$$P(r+1) = {}^nC_{r+1} q^{n-(r+1)} p^{r+1} = \frac{n!}{(r+1)!(n-r-1)!} q^{n-r-1} p^{r+1}.$$

Then

$$\frac{P(r+1)}{P(r)} = \frac{n-r}{r+1} \cdot \frac{p}{q}.$$

Hence

$$P(r+1) = \frac{n-r}{r+1} \cdot \frac{p}{q} P(r),$$

which is the required *recurrence formula*. Thus, if $P(0)$ is known, we can determine $P(1), P(2), P(3), \ldots$.

2.16 PEARSON'S CONSTANTS FOR BINOMIAL DISTRIBUTION

We know that moment generating function about the origin is

$$M_0(t) = \sum p_i e^{t(x_i - 0)} = \sum p_i e^{tx_i}.$$

Thus, for binomial distribution,

$$\begin{aligned} M_0(t) &= \sum_{i=0}^{n} {}^nC_i q^{n-i} p^i e^{ti} = {}^n C_i (pe^t)^i q^{n-i} \\ &= (q + pe^t)^n. \end{aligned} \tag{2.38}$$

Differentiating with respect to t and then putting $t = 0$, we get

$$\begin{aligned} \left[\frac{d}{dt}(q + pe^t)^n\right]_{t=0} &= [n(q + pe^t)^{n-1}.pe^t]_{t=0} = n(q+p)^{n-1}.p \\ &= np, \text{since } p + q = 1. \end{aligned}$$

Thus the mean $(\mu) = np$. Further

$$M_a(t) = e^{-at} M_0(t) \text{ or } 1 + t\mu_1 + \frac{t^2}{2!}\mu_2 + \frac{t^3}{3!}\mu_3 + \frac{t^4}{4!}\mu_4 + \ldots = e^{-at} M_0(t) \tag{2.39}$$

If we take $a = \mu = np$, then (2.39) reduces to

$$\begin{aligned} & 1 + t\mu_1 + \frac{t^2}{2!}\mu_2 + \frac{t^3}{3!}\mu_3 + \frac{t^4}{4!}\mu_4 + \ldots = e^{-npt} M_0(t) \\ &= e^{-npt}(q + pet)^n, \text{ using (2.38)} \\ &= (qe^{-pt} + pe^{(1-p)t})^n = (qe^{-pt} + pe^{qt})^n \\ &= \left[1 + pq\frac{t^2}{2!} + pq(q^2 - p^2)\frac{t^3}{3!} + pq(q^3 + p^3)\frac{t^4}{4!} + \ldots\right]^n \\ &= 1 + npq\frac{t^2}{2!} + npq(q-p)\frac{t^3}{4!} + npq[1 + 3(n-2)pq]\frac{t^4}{4!} + \ldots \end{aligned}$$

Comparing the coefficients of the power of t on both sides, we get

$$\text{Variance } (\mu_2) = npq, \mu_3 = npq(q-p),$$
$$\mu_4 = npq[1+3(n-2)pq].$$

Therefore, Pearson's constants for binomial distributions are

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(q-p)^2}{npq} = \frac{(1-2p)^2}{npq},$$
$$\beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{1-6pq}{npq},$$
$$\gamma_1 = \sqrt{\beta_1} = \frac{q-p}{\sqrt{npq}} = \frac{1-2p}{\sqrt{npq}},$$
$$\gamma_2 = \beta_2 - 3 = \frac{1-6pq}{\sqrt{npq}}.$$

Hence

$$\text{Mean}(\mu) = np,$$
$$\text{Variance}(\sigma^2) = npq,$$
$$\text{Standard deviation}(\sigma) = \sqrt{npq},$$
$$\text{Skewness}\left(\sqrt{\beta_1}\right) = \frac{1-2p}{\sqrt{npq}},$$
$$\text{Kurtosis}(\beta_2) = 3 + \frac{1-6pq}{npq}.$$

We observe that

(i) skewness of the binomial distribution is 0 for $p = \frac{1}{2}$,

(ii) skewness is positive for $p < \frac{1}{2}$,

(iii) skewness is negative for $p > \frac{1}{2}$.

EXAMPLE 2.61

The incidence of occupational disease in an industry is such that the workers have a 20% chance of suffering from it. What is the probability that out of six workers chosen at random, four or more will suffer from the disease?

Solution. We are given that $p = \frac{20}{100} = \frac{1}{5}$. Therefore, $q = 1 - p = \frac{4}{5}$. Let $P(X > 3)$ denote the probability that out of six workers chosen four or more will suffer from the disease. Then

$$P(X > 3) = P(X = 4) + P(X = 5) + P(X = 6)$$
$$= {}^6C_4 q^{6-4} p^4 + {}^6C_5 q^{6-5} p^5 + {}^6C_6 q^0 p^6$$
$$= {}^6C_4 q^2 p^4 + 6q\,p^5 + p^6 \;= \frac{15 \times 16}{25 \times 625} + \frac{6 \times 4}{5 \times 3125} + \frac{1}{25 \times 625}$$
$$= \frac{240 + 24 + 1}{25 \times 625} = \frac{265}{25 \times 625} = \frac{53}{3125}.$$

EXAMPLE 2.62

The probability that a bomb dropped from a plane will strike the target is $\frac{1}{5}$. If six bombs are dropped, find the probability that (i) exactly two will strike the target and (ii) at least two will strike the target.

Solution. The probability to strike the target is $p = \frac{1}{5}$. Therefore, $q = 1 - \frac{1}{p} = \frac{4}{5}$. Then

(i) $P(X = 2) = {}^6C_2 q^{6-2} p^2 = {}^6C_2 q^4 p^2 = \frac{15 \times 256 \times 1}{625 \times 25} = 0.24576.$

(ii) $$P(X \geq 2) = P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6)$$
$$= 1 - [P(X = 0) + P(X = 1)] = 1 - [q^6 + 6q^5 p]$$
$$= 1 - \left[\frac{4096}{15625} - \frac{6144}{15625}\right] = 0.34478.$$

EXAMPLE 2.63

The probability that a pen manufactured by a company will be defective is $\frac{1}{10}$. If 12 such pens are manufactured, find the probability that

(i) exactly two pens will be defective
(ii) at least two pens will be defective
(iii) none will be defective.

Solution. We have $p = \frac{1}{10}$ and so, $q = 1 - \frac{1}{10} = \frac{9}{10}$. Then since $n = 12$, we have

(i) $$P(X = 2) = {}^{12}C_2 q^{10} p^2 = 66(0.1)^2 (0.9)^{10}$$
$$= 0.2301.$$

(ii) $$P(X \geq 2) = 1 - [P(X = 0) + P(X = 1)]$$
$$= 1 - [q^{12} + {}^{12}C_1 q^{11} p]$$
$$= 1 - [(0.9)^{12} + 12(0.9)^{11}(0.1)]$$
$$= 0.3412.$$

(iii) $P(X = 0) = q^{12} = (0.9)^{12} = 0.2833.$

EXAMPLE 2.64

Out of 800 families with 5 children each, how many families would be expected to have

(i) Three boys and two girls
(ii) Two boys and three girls
(iii) One girl
(iv) At the most two girls, under the assumption that probabilities for boys and girls are equal.

Solution. We have $n = 5$. Further

$$p = \text{probability to have a boy} = \frac{1}{2}$$

$$q = \text{probability to have a girl} = \frac{1}{2}.$$

Then

(i) The expected number of families to have three boys and two girls is

$$800\left[{}^5C_3q^{5-3}p^3\right] = 800\left[10\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3\right] = 250.$$

(ii) The expected number of families to have two boys and three girls is

$$800\,[{}^5C_2q^{5-2}p^2] = 800\left[10\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2\right] = 250.$$

(iii) The expected number of families to have no girls that is to have five boys is

$$800\,[{}^5C_2q^0p^5] = 800\left(\frac{1}{2}\right)^5 = \frac{800}{32} = 25.$$

(iv) The expected number of families to have at the most two girls, that is, at least three boys is

$$\begin{aligned}&800[P(X=3)+P(X=4)+P(X=5)]\\&= 800[{}^5C_3q^{5-3}p^3 + {}^5C_4q^{5-4}p^4 + {}^5C_5q^0p^5]\\&= 800\left[10\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 + 5\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5\right]\\&= 800\left[\frac{10}{32}+\frac{5}{32}+\frac{1}{32}\right] = 400.\end{aligned}$$

EXAMPLE 2.65

The following data shows the number of seeds germinating out of 10 on damp filter for 80 sets of seeds. Fit a binomial distribution to this data:

x:	0	1	2	3	4	5	6	7	8	9	10
f:	6	20	28	12	8	6	0	0	0	0	0

Solution. We note that $n = 10$ and

$$\sum f_i = 6+20+28+12+8+6 = 80.$$

Therefore, the mean of the binomial distribution μ is given by

$$\begin{aligned}\mu &= \frac{\sum f_i x_i}{\sum f_i} = \frac{20+56+36+32+30}{80} = \frac{174}{80}\\&= 2.175.\end{aligned}$$

But $\mu = np$. Therefore,

$$p = \frac{\mu}{n} = \frac{2.175}{10} = 0.2175 \text{ and } q = 1-p = 0.7825.$$

Therefore, the probability distribution is

x:	0	1	2	3	……	9	10
$p(x)$:	q^{10}	${}^{10}C_1q^9p$	${}^{10}C_2q^8p^2$	${}^{10}C_3q^7p^3$	……	${}^{10}C_9qp^9$	p^{10}

Hence the frequencies are given by $f = 80p(x)$. Putting the values of p and q, we get

x:	0	1	2	3	4	5	6	7	8	9	10
f:	6.9	19.1	24.0	17.8	8.6	2.9	0.7	0.1	0	0	0

EXAMPLE 2.66

If the chance that one of the 10 telephone lines is busy at an instant is 0.2, then (i) what is the chance that five of the lines are busy? and (ii) what is the probability, that all lines are busy?

Solution. Here $n = 10$, $p = 0.2$, and so $q = 1 - 0.2 = 0.8$. Then

(i) Probability of five lines to be busy is

$$\begin{aligned} P(X=5) &= {}^{10}C_5 q^{10-5} p^5 \\ &= {}^{10}C_5 q^5 p^5 = 252(0.8)^5(0.2)^5 \\ &= 252(0.32768)(0.00032) = 0.0264. \end{aligned}$$

(ii) Probability that all the lines are busy is

$$\begin{aligned} P(X=10) &= {}^{10}C_{10} p^{10} = (0.2)^{10} \\ &= 1024 \times 10^{-10}. \end{aligned}$$

EXAMPLE 2.67

In sampling a large number of parts manufactured by a machine, the mean number of defective parts in a sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain at least 3 defective parts.

Solution. We are given that $n = 20$ and $\mu = np = 2$ and so

$$p = \frac{2}{n} = \frac{2}{20} = \frac{1}{10},\ q = 1 - \frac{1}{10} = \frac{9}{10}.$$

Then

$$\begin{aligned} \mathrm{P(X > 2)} &= 1 - [\mathrm{P(X=0)} + \mathrm{P(X=1)} + \mathrm{P(X=2)}] \\ &= 1 - [{}^{20}C_0 q^{20} + {}^{20}C_1 q^{19} p + {}^{20}C_2 q^{18} p^2] \\ &= 1 - [(0.9)^{20} + 20(0.9)^{19}(0.1)] + 190(0.9)^{18}(0.9)^2 = 0.323. \end{aligned}$$

Hence the number of sample having atleast three defective parts out of the 1000 samples is $1000 \times 0.323 = 323$.

EXAMPLE 2.68

Fit a binomial distribution to the following data and compare the theoretical frequencies with the actual ones

x:	0	1	2	3	4	5
f:	2	14	20	34	22	8

Solution. We have $n = 5$, $\Sigma f_i = 100$. Therefore,

$$\mu = \frac{\sum f_i x_i}{\sum f_i} = \frac{14 + 40 + 102 + 88 + 40}{100} = 2.84.$$

But for binomial distribution, $\mu = np$. Therefore,

$$p = \frac{\mu}{n} = \frac{2.84}{5} = 0.568 \text{ and } q = 1 - p = 0.432.$$

Therefore, the probability distribution is

x:	0	1	2	3	4	5
$P(x)$:	q^5	${}^5C_1 q^4 p$	${}^5C_2 q^3 p^2$	${}^5C_3 q^2 p^3$	${}^5C_4 q p^4$	p^5

Therefore the expected (theoretical) frequencies are

$$100(0.432)^5, 500(0.432)^4(0.568), 10^3(0.432)^3(0.568)^2,$$
$$10^3(0.432)^2(0.568)^3, 500(0.432)(0.568)^4, 100(0.568)^5.$$

After computation, we get the theoretical frequencies as

1.504, 9.891, 26.010, 34.199, 22.483, 5.918.

EXAMPLE 2.69

Find the probability of number 4 turning up at least once in *two* tosses of a fair dice.

Solution. Let X denote the number of times the number 4 turn up. We note that

$$\text{P(4 turns up)} = \text{p} = \frac{1}{6} \text{ and so q} = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}.$$

Thus the probability distribution is

X:	0	1	2
$P(X)$:	q^2	$2pq$	p^2

Hence

$$\text{P(4 turns up at least once)}$$
$$= P(X = 1) + \text{P}(X = 2) = 2pq + p^2$$
$$= 2.\frac{1}{6}.\frac{5}{6} + \left(\frac{1}{6}\right)^2 = \frac{11}{36}.$$

EXAMPLE 2.70

A coin is tossed five times. What is the probability of getting at least three heads?

Solution. Let X denote the "number of heads obtained". We know that

$$p = \text{P(head obtained)} = \frac{1}{2}.$$

Therefore,

$$q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}.$$

The random variable X takes the values 0, 1, 2, 3, 4, 5, and $n = 5$. Hence

P(at least three heads) = P($X \geq 3$)

$$= P(X=3)+P(X=4)+P(X=5) = 5_{C_3}\, p^3 q^2 + 5_{C_4}\, p^4 q + 5_{C_5}\, p^5$$

$$= 10\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 + 5\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^5 = \frac{10}{32}+\frac{5}{32}+\frac{1}{32} = \frac{1}{2}.$$

EXAMPLE 2.71

The mean and variance of a binomial variable X are 2 and 1, respectively. Find the probability that X takes a value greater than 1.

Solution. Suppose n is the number of independent trials. Since X is a binomial variate, we have

$$\text{Mean} = np = 2 \quad \text{(given)} \tag{2.40}$$

$$\text{Variance} = np\,q = 1 \quad \text{(given)} \tag{2.41}$$

Dividing (2.41) by (2.40), we get $q = \frac{1}{2}$, which yields $p = 1 - q = \frac{1}{2}$. Also then (2.40) gives $n = 4$. Hence

$$P(X>1) = 1 - [P(X=0)+P(X=1)]$$

$$= 1 - [{}^4C_0 q^4 + {}^4C_1 q^3 p]$$

$$= 1 - [q^4 - 4p\,q^3] = 1 - \left[\left(\frac{1}{2}\right)^4 - 4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3\right]$$

$$= 1 - \frac{5}{16} = \frac{11}{16}.$$

2.17 POISSON DISTRIBUTION

The *Poisson distribution* is a limiting case of binomial distribution when n is very large and p is very small in such a way that mean np remains constant. To derive Poisson distribution, we assume that when n is large and p is very small, then $np = \lambda$ (constant). In the binomial distribution, the probability of r successes is given by

$$P(r) = {}^nC_r q^{n-r} p^r$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}(1-p)^{n-r} p^r$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}\left(1-\frac{\lambda}{n}\right)^{n-r} \cdot \left(\frac{\lambda}{n}\right)^r$$

$$= \frac{\lambda^r}{r!} \cdot \frac{n(n-1)(n-2)\ldots(n-r+1)}{n^r} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^r}$$

$$= \frac{\lambda^r}{r!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{r-1}{n}\right) \frac{\left[\left(1-\frac{\lambda}{n}\right)^{-\frac{n}{\lambda}}\right]^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^r}.$$

Therefore, the Poisson distribution is given by

$$\lim_{n\to\infty} P(r) = \frac{\lambda^r}{r!} e^{-\lambda} \, (r = 0,1,2,3,...),$$

where $\lambda = np$ is called *parameter of the Poisson distribution*. Thus, the probabilities of $0,1,2,\ldots, r,\ldots$ of successes in a Poisson distribution are

$$e^{-\lambda}, \lambda e^{-\lambda}, \frac{\lambda^2}{2!} e^{-\lambda}, \ldots, \frac{\lambda^r}{r!} e^{-\lambda}, \ldots$$

The sum of the probabilities $P(r)$, $r = 0, 1, 2,\ldots$ is

$$e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2}{2!} e^{-\lambda} + \ldots + \frac{\lambda^r}{r!} e^{-\lambda} + \ldots$$

$$= e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \ldots + \frac{\lambda^r}{r!} + \ldots\right) = e^{-\lambda}.e^{\lambda} = 1.$$

Further, for the Poisson distribution,

$$\frac{P(r+1)}{P(r)} = \frac{\lambda^{r+1} e^{-\lambda}}{(r+1)!} \cdot \frac{r!}{\lambda^r e^{-\lambda}} = \frac{\lambda}{r+1}$$

and so

$$P(r+1) = \frac{\lambda}{r+1} P(r),$$

which is the *recurrence formula* for the Poisson distribution. Some examples of Poisson distribution are

(i) The number of defective screws per box of 100 screws
(ii) The number of fragments from a shell hitting a target
(iii) Number of typographical error per page in typed material
(iv) Mortality rate per thousand.

2.18 CONSTANTS OF THE POISSON DISTRIBUTION

The constants of the Poisson distribution can be derived from the corresponding constants of the binomial distribution by letting $n \to \infty$ and $p \to 0$. Since $q = 1 - p$, $p \to 0$ if $q \to 1$. Therefore, mean (μ), variation (σ^2), standard deviation σ, skewness $(\sqrt{\beta_1})$, and kurtosis (β_2) are given by

$$\mu = \lim_{n\to\infty} np = \lambda, \text{since } np = \lambda\,(\text{constant}),$$

$$\sigma^2 = \mu_2 = \lim_{\substack{p\to 0\\ n\to\infty}} npq = \lim_{q\to 1} \lambda q = \lambda,$$

$$\sigma = \sqrt{\lambda}, \quad \mu_3 = \lambda, \quad \mu_4 = 3\lambda^2 + \lambda,$$

$$\text{Skewness}\left(\sqrt{\beta_1}\right) = \sqrt{\frac{\mu_3^2}{\mu_2^3}} = \sqrt{\frac{\lambda^2}{\lambda^3}} = \sqrt{\frac{1}{\lambda}},$$

$$\text{Kurtosis } (\beta_2) = \frac{\mu_4}{\mu_2^2} = 3 + \frac{1}{\lambda}.$$

EXAMPLE 2.72

A certain screw making machine produces an average of 2 defective screws out of 100 and pack them in boxes of 500. Find the probability that a box contains 15 defective screws.

Solution. The probability of occurrence is $\frac{2}{100} = 0.02$ and, therefore, it follows a Poisson distribution. Now $n = 500$, and $p = 0.02$. Therefore,

$$\lambda = \text{mean} = np = 10.$$

Now the probability that a box contains 15 defective screws is

$$\frac{\lambda^{15}}{15!} e^{-\lambda} = \frac{10^{15}}{15!} e^{-10} = 0.035.$$

EXAMPLE 2.73

A book of 520 pages has 390 typographical errors. Assuming Poisson law for the number of errors per page, find the probability that a random sample of five pages will contain no error.

Solution. The average number of typographical error per page is given by

$$\lambda = \frac{390}{520} = 0.75.$$

Therefore, probability of zero error per page is

$$P(X = 0) = e^{-\lambda} = e^{-0.75}.$$

Hence, required probability that a random sample of five pages contains no error is

$$[P(X = 0)]^5 = (e^{-0.75})^5 = e^{-3.75}.$$

EXAMPLE 2.74

Fit a Poisson distribution to the following:

x:	0	1	2	3	4
f:	46	38	22	9	1

Solution. The mean of the Poisson distribution is

$$\lambda = \frac{\sum f_i x_i}{\sum f_i} = \frac{0 + 38 + 44 + 27 + 4}{116} = \frac{113}{116}$$
$$= 0.974.$$

Therefore, frequencies are

$$116e^{-\lambda}, 116\lambda e^{-\lambda}, 116\frac{\lambda^2}{2}e^{-\lambda}, 116\frac{\lambda^3}{3!}e^{-\lambda}, 116\frac{\lambda^4}{4!}e^{-\lambda}.$$

Since $e^{-\lambda} = e^{-0.974} = 0.3776$, the required Poisson distribution is

x:	0	1	2	3	4
y:	44	43	21	7	1

EXAMPLE 2.75

An insurance company insures 6,000 people against death by tuberculosis (TB). Based on the previous data, the rates were computed on the assumption that 5 persons in 10,000 die due to TB each year. What is the probability that more then two of the insured policy will get refund in a given year?

Solution. Here $n = 6000$ is large and the probability p of death due to TB $\frac{5}{10000} = 0.0005$ is (small). Therefore, the data follows Poisson distribution. The parameter of the distribution is

$$\lambda = np = 6000 \times 0.0005 = 3.0.$$

The required probability that more than two of the insured policies will get refunded is

$$P(X > 2) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

$$= 1 - \left[e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2}{2} e^{-\lambda} \right] = 1 - e^{-\lambda} \left[1 + \lambda + \frac{\lambda^2}{2} \right]$$

$$= 1 - 0.04979 \left[1 + 3 + \frac{9}{2} \right], \text{since } e^{-\lambda} = 0.4979$$

$$= 1 - 0.4232 = 0.5768.$$

EXAMPLE 2.76

Fit a Poisson distribution to the following data

x:	0	1	2	3	4
f:	122	60	15	2	1

Solution. If the above distribution is approximated by a Poisson distribution, then the parameter of the Poisson distribution is given by

$$\lambda = \text{Mean} = \frac{\sum f_i x_i}{\sum f_i} = \frac{0 + 60 + 30 + 6 + 4}{200} = 0.5.$$

Therefore, the theoretical frequencies are

$$Ne^{-\lambda}, N\lambda e^{-\lambda}, N\frac{\lambda^2}{2!}e^{-\lambda}, N\frac{\lambda^3}{3!}e^{-\lambda}, N\frac{\lambda^4}{4!}e^{-\lambda},$$

where $N = 200$. Also $e^{-\lambda} = e^{-0.5} = 0.6065$.
Therefore, the required Poisson distribution is

x:	0	1	2	3	4
f:	121	61	15	2	0

EXAMPLE 2.77

A car hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as a Poisson distribution with mean 1.5. Calculate the proportion of days (i) on which there is no demand and (ii) on which demand is refused:

Solution. We are given that $\lambda = 1.5$. When there is no demand, the probability is

$$e^{-\lambda} = e^{-1.5} = 0.2231.$$

When demand is refused, then the probability of number of demands exceeds 2. Therefore, the probability for this event is

$$1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

$$= 1 - \left[e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2}{2!} e^{-\lambda} \right] = 1 - e^{-\lambda} \left[1 + \lambda + \frac{\lambda^2}{2!} \right]$$

$$= 1 - 0.2231 \left[1 + 1.5 + \frac{(1.5)^2}{2} \right] = 1 - 0.80874 = 0.19126.$$

EXAMPLE 2.78

The mortality rate for a certain disease is 6 per 1000. What is the probability for just four deaths from that disease in a group of 400?

Solution. The parameter of the Poisson distribution is given by

$$\lambda = np = 400p.$$

But

$$p = \frac{6}{1000} = 0.0006.$$

Therefore,

$$\lambda = 400 \times 0.006 = 2.4$$

and so

$$P(X = 4) = \frac{\lambda^4}{4!}e^{-\lambda} = \frac{(2.4)^4}{4!}e^{-2.4}$$
$$= \frac{(2.4)^4}{4!}(0.09072) = 0.1254.$$

EXAMPLE 2.79

Find the probability that at most 5 defective diodes will be found in a pack of 600 diodes if previous data shows that 3% of such diodes are defective.

Solution. Here $n = 600$, $p = 0.03$. Therefore, parameter of Poisson distribution is

$$\lambda = np = 600(0.03) = 6.$$

Therefore,

$$P(X \le 5) = e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2}{2!}e^{-\lambda} + \frac{\lambda^3}{3!}e^{-\lambda} + \frac{\lambda^4}{4!}e^{-\lambda} + \frac{\lambda^5}{5!}e^{-\lambda}$$
$$= e^{-\lambda}\left[1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \frac{\lambda^5}{5!}\right]$$
$$= e^{-6}\left[1 + 6 + \frac{6^2}{2} + \frac{6^3}{6} + \frac{6^4}{24} + \frac{6^5}{120}\right]$$
$$= 0.00248[179.8] = 0.4459.$$

2.19 NORMAL DISTRIBUTION

The normal distribution is a continuous distribution, which can be regarded as the limiting form of the Binomial distribution when n, the number of trials, is very large but neither p nor q is very small. The limit approach more rapidly if p and q are nearly equal, that is, if p and q are close to $\frac{1}{2}$. In fact, using Stirling's formula, the following theorem can be proved:

A binomial probability density function

$$P(x) = {}^n C_x q^{n-x} p^x,$$

in which n becomes infinitely large, approaches as a limit to the so-called normal probability density function

$$f(x) = \frac{1}{\sqrt{2\pi npq}} e^{-\frac{(x-np)^2}{2npq}}.$$

Since for a Binomial distribution, the mean and standard deviations are given by

$$\mu = np \quad \text{and} \quad \sigma = \sqrt{npq},$$

the normal frequency function becomes

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2},$$

where the variable x can assume all values from $-\infty$ to ∞.

The graph of the normal frequency function is called the *normal curve.* The normal curve is bell-shaped and is symmetrical about the mean μ. This curve is unimodal and its mode coincide with its mean μ. The two tails of the curve extend to $+\infty$ and $-\infty$, respectively, towards the positive and negative directions of x-axis, approaching the x-axis without ever meeting it. Thus the curve is asymptotic to the x-axis. Since the curve is symmetrical about $x = \mu$, its mean, median, and mode are the same. Its points of inflexion are found to be $x = \mu \pm \sigma$, that is, the points are equidistant from the mean on either side. As we shall prove, the total area under the normal curve above the x-axis is unity. Thus the graph of the normal frequency curve is as shown in the Figure 2.1.

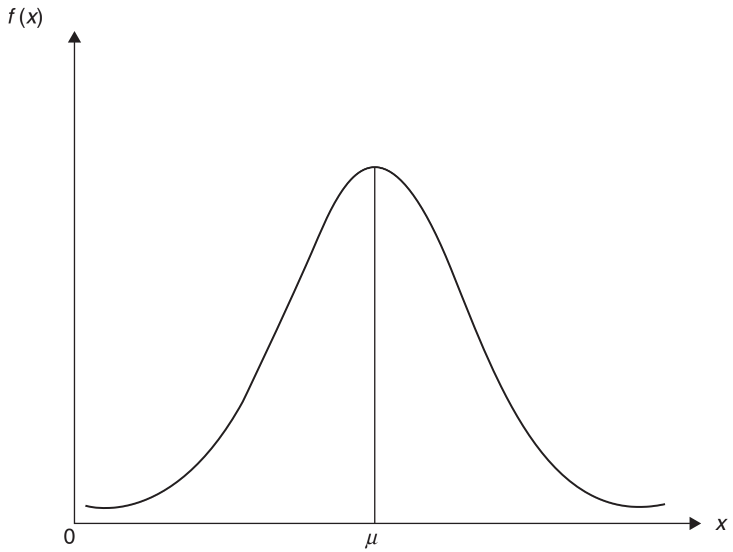


Figure 2.1

The parameters μ and σ determines the position and relative proportions of the normal curve. If two populations defined by normal frequency functions have different means μ_1 and μ_2 but identical standard deviations $\sigma_1 = \sigma_2$, then their graphs appear as shown in the Figure 2.2.

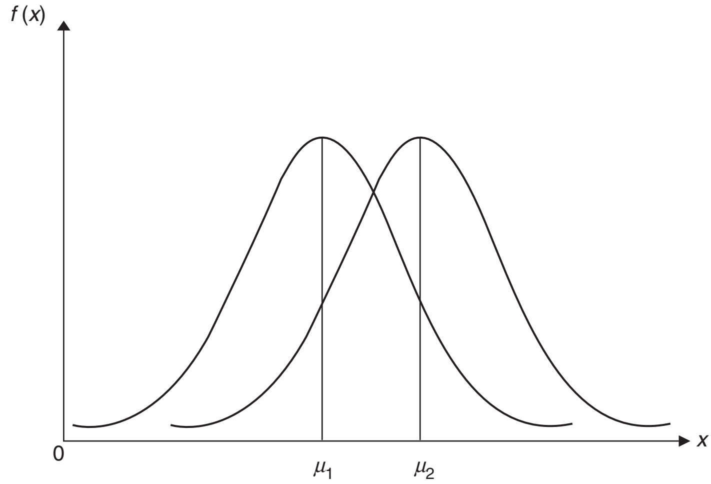


Figure 2.2

On the other hand, if the two populations have identical means $\mu_1 = \mu_2$ and different standard deviations σ_1 and σ_2, then their graphs would appear as shown in the Figure 2.3.

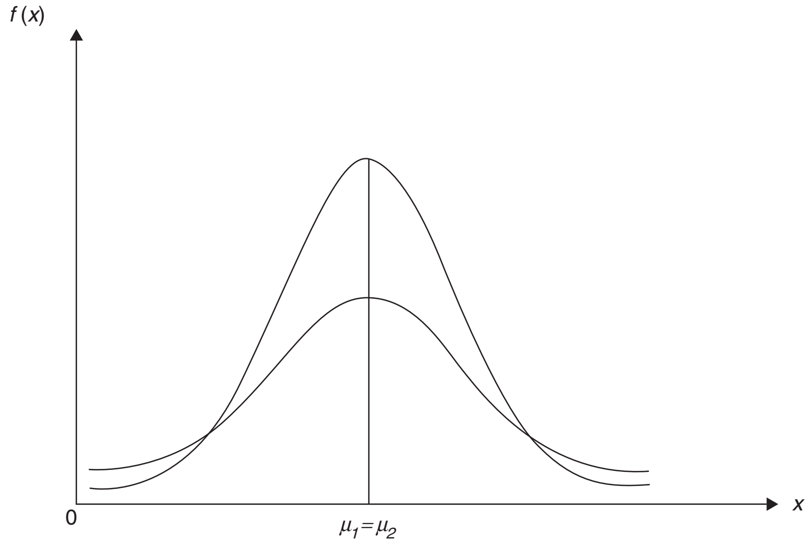


Figure 2.3

2.20 CHARACTERISTICS OF THE NORMAL DISTRIBUTION

The normal distribution has the following properties:

1. **Normal distribution is a continuous distribution:** The probability density function of the normal distribution is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

Therefore, area under the normal curve is equal to

$$\int_{-\infty}^{\infty} f(x)dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

Putting $\frac{x-\mu}{\sigma\sqrt{2}} = t$, we have $dx = \sigma\sqrt{2}\, dt$, and so

$$\int_{-\infty}^{\infty} f(x)dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2}\sigma\sqrt{2}dt = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$= \frac{1}{\sqrt{\pi}}\sqrt{\pi} = 1.$$

Thus $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$. Hence f is a continuous distribution.

2. **Mean, mode, and median of the normal distribution coincide. Hence the distribution is symmetrical**
   (i) **Mean*:*** The general form of the normal curve is

$$y = f(x) = \frac{N}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

Therefore,

$$\text{Mean} = \frac{1}{N}\int_{-\infty}^{\infty} yxdx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

Put $\frac{x-\mu}{\sigma\sqrt{2}} = t$ so that $x = \mu + t\sigma\sqrt{2}$ and $dx = \sigma\sqrt{2}\, dt$. Hence

$$\text{Mean} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + t\sigma\sqrt{2})e^{-t^2}.\sigma\sqrt{2}dt$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\mu + t\sigma\sqrt{2})e^{-t^2} dt$$

$$= \frac{\mu}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt + \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} te^{-t^2} dt$$

$$= \frac{\mu}{\sqrt{\pi}}\sqrt{\pi} = \mu,$$

because $\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$ and $\int_{-\infty}^{\infty} te^{-t^2} dt = 0$ due to oddness of te^{-t^2} .

   (ii) **Mode:** Mode is the value of x for which f is maximum. In other words, mode is the solution of $f'(x) = 0$ and $f''(x) < 0$.

For normal distribution, we have

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

Taking log, we get

$$\log f(x) = \log\left(\frac{1}{\sigma\sqrt{2\pi}}\right) - \frac{1}{2\sigma^2}(x-\mu)^2.$$

Differentiating with respect to x, we get

$$\frac{f'(x)}{f(x)} = -\frac{1}{\sigma^2}(x-\mu)$$

and so $f'(x) = \frac{-1}{\sigma^2}(x-\mu) f(x)$. Then

$$f''(x) = -\frac{1}{\sigma^2}[f(x) + (x-\mu) f'(x)]$$

$$= -\frac{f(x)}{\sigma^2}\left[1 + \frac{(x-\mu)}{\sigma^2}\right].$$

New $f'(x) = 0$ implies $x = \mu$. Also at $x = \mu$, we have

$$f''(\mu) = -\frac{1}{\sigma^2} \cdot \frac{1}{\sigma\sqrt{2\pi}} < 0.$$

Hence $x = \mu$ *is mode of the normal distribution*.

(iii) **Median:** We know that

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

Therefore, if M is the median of the normal distribution, we must have

$$\int_{-\infty}^{M} f(x)dx = \frac{1}{2}.$$

Therefore,

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{M} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$

or

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\mu} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx + \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{M} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}.$$

But

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\mu} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{t^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{2}} = \frac{1}{2}.$$

Therefore,

$$\frac{1}{2} + \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{M} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2},$$

which implies

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{M} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 0.$$

Consequently, $M = \mu$.

Thus, mean, mode, and median coincide for the normal distribution. Hence the normal curve is symmetrical.

3. **The variance of the normal distribution is σ^2 and so the standard deviation is σ:** In fact, we have

$$\begin{aligned}
\text{Variance} &= \int_{-\infty}^{\infty} (x-\mu)^2 \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sigma^2 t^2 . e^{-t^2} \sigma\sqrt{2}\, dt, \quad \frac{x-\mu}{\sigma\sqrt{2}} = t, \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^2 e^{-t^2} dt = \frac{4\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} t^2 e^{-t^2} dt \\
&= \frac{4\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} z\, e^{-z} \frac{dz}{2\sqrt{z}}, \quad t^2 = z \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} z^{\frac{1}{2}} e^{-z} dz = \frac{2\sigma^2}{\sqrt{\pi}} \int_{0}^{\infty} z^{\left(\frac{3}{2}-1\right)} e^{-z} dz \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2}\sqrt{\pi} = \sigma^2.
\end{aligned}$$

Therefore,

$$\text{Standard deviation} = \sqrt{\text{variance}} = \sigma.$$

4. **Points of inflexion of normal curve:** At the point of inflexion of the normal curve, we should have $f''(x) = 0$ and $f'''(x) \neq 0$. As we have seen, for normal distribution

$$f''(x) = -\frac{f(x)}{\sigma^2}\left[1 - \frac{(x-\mu)^2}{\sigma^2}\right].$$

Therefore, $f''(x) = 0$ yields

$$1 - \frac{(x-\mu)^2}{\sigma^2} = 0 \quad \text{or} \quad (x-\mu)^2 = \sigma^2.$$

Hence $x = \mu \pm \sigma$. Further, at $x = \mu \pm \sigma$, we have $f'''(x) \neq 0$. Thus the normal curve has two points of inflexion given by $x = \mu - \sigma$ and $x = \mu + \sigma$. Clearly, the points of inflexions are equidistant (at a distance σ) from the mean.

5. **Mean deviation about the mean:** The mean deviation from the mean is given by

$$\text{Mean deviation (about mean)} = \int_{-\infty}^{\infty} |x-\mu|\, f(x)dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |x-\mu|\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |t|\, e^{-\frac{t^2}{2}} dt, \quad \frac{x-\mu}{\sigma} = t$$

$$= \frac{2\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |t|\, e^{-\frac{t^2}{2}} dt,$$

since the integral is an even function of z. Since $|t| = t$ for $t \in [0, \infty]$, we have

$$\text{Mean deviation} = \sqrt{\frac{2}{\pi}}\sigma \int_0^{\infty} t\, e^{-\frac{t^2}{2}} dt$$

$$= \sqrt{\frac{2}{\pi}}\sigma \int_0^{\infty} e^{-z} dt, \quad \frac{t^2}{2} = z$$

$$= \sqrt{\frac{2}{\pi}}\sigma \left[\frac{e^{-z}}{-1}\right]_0^{\infty} = \sqrt{\frac{2}{\pi}}\sigma$$

$$= \frac{4}{5}\sigma, \text{ approximately.}$$

Thus for the normal distribution, the mean deviation is approximately $\frac{4}{5}$ times the standard deviation.

2.21 NORMAL PROBABILITY INTEGRAL

If X is a normal random variable with mean μ and variance σ^2, then the probability that random value of X will lie between $X = \mu$ and $X = x_1$ is given by

$$P(\mu < X < x_1) = \int_{\mu}^{x_1} f(x)dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{x_1} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx.$$

Put $\frac{X-\mu}{\sigma} = Z$. Then $X - \mu = \sigma Z$. Therefore, when $X = \mu$, $Z = 0$ and when $X = x_1$, $Z = \frac{x_1 - \mu}{\sigma} = z_1$, say.
Therefore,

$$P(\mu < X < x_1) = P(0 < Z < z_1)$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-\frac{z^2}{2}} dz = \int_0^{z_1} \phi(z)dz,$$

where $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$ is the probability function of the *standard normal variate* $Z = \frac{X-\mu}{\sigma}$. The definite integral $\int_0^{z_1} \phi(z)dz$ is called the *normal probability integral* which gives the area under the standard normal curve (Figure 2.4) between the ordinates at $z = 0$ and $z = z_1$.

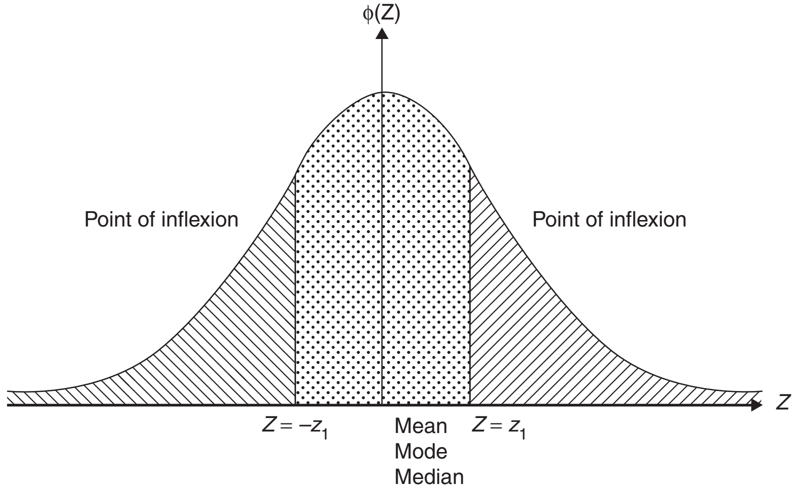


Figure 2.4

The standard normal curve

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

is *symmetrical with respect to* $\phi(x)$ − axis since $\phi(z)$ remain unchanged if z is replaced by $-z$. Thus arithmetic mean and the median of a normal frequency distribution coincide at the centre of it. The exponent of e in $\phi(z)$ is negative, $-\frac{z^2}{2}$. Hence $\phi(z)$ is maximum when $z = 0$. All other values of z make $\phi(z)$ smaller since $e^{-\frac{z^2}{2}} = \frac{1}{e^{\frac{z^2}{2}}}$. Thus the maximum value of $\phi(z)$ is

$$\phi(0) = \frac{1}{\sqrt{2\pi}} = 0.3989.$$

As z increases numerically, $e^{-\frac{z^2}{2}}$ decreases and approaches zero when z becomes infinite. Thus the standard normal curve is asymptotic to the z-axis in both the positive and negative directions. Differentiating $\phi(z)$with respect to z, we get

$$\phi'(z) = z\phi(z), \text{ and}$$
$$\phi''(z) = -\phi(z) - z\phi'(z) = -\phi(z) + z^2\phi(z)$$
$$= (z^2 - 1)\phi(z).$$

Therefore, $\phi''(z) = 0$ implies $z = \pm 1$. Thus the points of inflexion (at which the curve changes from concave downward to concave upward) are situated at a unit distance from the $\phi(z)$–axis.

2.22 AREAS UNDER THE STANDARD NORMAL CURVE

The equation of the standard normal curve is

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}.$$

The area under this curve is given by

$$\int_{-\infty}^{\infty} \phi(z)dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-\frac{z^2}{2}} dz$$

$$= \sqrt{\frac{1}{\pi}} \int_0^{\infty} t^{-\frac{1}{2}} e^{-t} dt, \frac{z^2}{2} = t$$

$$= \sqrt{\frac{1}{\pi}} \Gamma\left(\frac{1}{2}\right) = \frac{1}{\sqrt{\pi}}. \sqrt{\pi} = 1.$$

If follows, therefore, that the *area under* $\phi(z)$ from $z = z_1$ to $z = z_2$, that is, $\int_{z_1}^{z_2} \phi(z)dz$ *is always less than* 1, where z_1 and z_2 are finite.

Further, because of the symmetry of the curve with respect to $\phi(z)$-axis, the area from any $z = z_1$ to $+\infty$ is equal to the area from $-\infty$ to $-z_1$. Thus

$$\int_{z_1}^{\infty} \phi(z)dz = \int_{-\infty}^{-z_1} \phi(z)dz.$$

Since the area under the curve from $z = 1$ to $z = \infty$ is 0.1587, we have

$$\int_{-1}^{1} \phi(z)dz = \int_{-\infty}^{\infty} \phi(z)dz - \int_{-\infty}^{-1} \phi(z)dz - \int_{1}^{\infty} \phi(z)dz = \int_{-\infty}^{\infty} \phi(z)dz - 2\int_{1}^{\infty} \phi(z)dz$$

$$= 1 - 2(0.1587) = 0.6826.$$

In term of statistics, this means that 68% of the normal variates deviate from their mean by less than one standard deviation. Similarly,

$$\int_{-2}^{2} \phi(z)dz = 0.9544, \quad \int_{-3}^{3} \phi(z)dz = 0.9974 \cdot$$

Thus, over 95% of the area is included between the limits –2 and 2 and over 99% of the area is included between –3 and 3 as shown in the Figure 2.5.

2.23 FITTING OF NORMAL DISTRIBUTION TO A GIVEN DATA

The equation of the normal curve fitted to a given data is

$$y = f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty.$$

Therefore, first calculate the mean μ and the standard deviation σ. Then find the standard normal variate $Z = \frac{X - \mu}{\sigma}$ corresponding to the lower limits of each of the class interval, that is, determine $z_1' = \frac{x_1' - \mu}{\sigma}$, where x_1' is the lower limit of the ith class. The third step is to calculate the area under the normal curve to the left of the ordinate $Z = z_1$, say $\Phi(z_i)$, from the tables. Then areas for the successive class intervals are obtained by subtraction, $viz, \Phi\,(z_i + 1) - \Phi(z_i), i = 1, 2, 3\ldots$ Then

Expected frequency $= N[\Phi(z_{i+1}) - \Phi(z_i)]$.

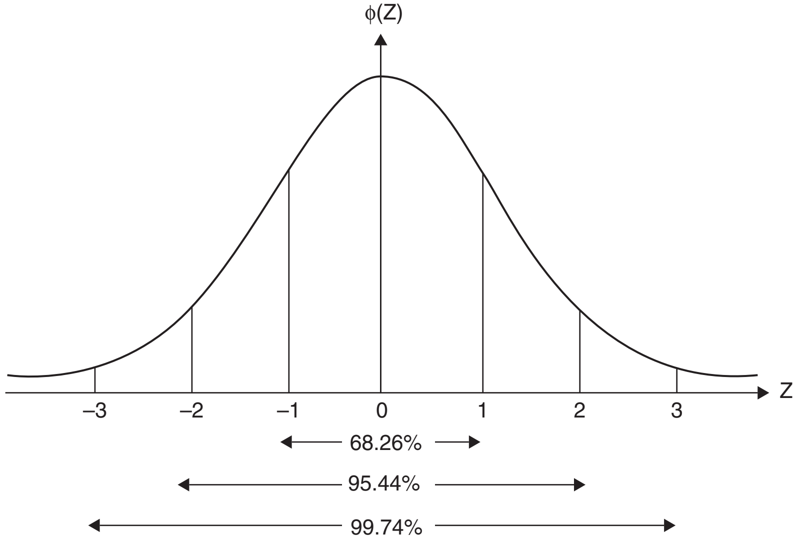


Figure 2.5

EXAMPLE 2.80

The scores in a competitive examination is normally distributed with mean 400 and standard deviation 80. Out of 10,000 candidates appeared in the examination, it is desired to pass 350 candidates. What should be the lowest score permitted for passing the examination?

Solution. The fraction of the passing candidate is $\frac{350}{10000} = 0.035$. Thus the fraction of the failing candidates is 0.965. The passing fraction is shown in the right—tail area of the Figure 2.6 of normal curve. Thus the area of the standard normal curve is

$$\int_z^{\infty} \phi(z)dz = 0.035$$

as shown in the Figure 2.7.

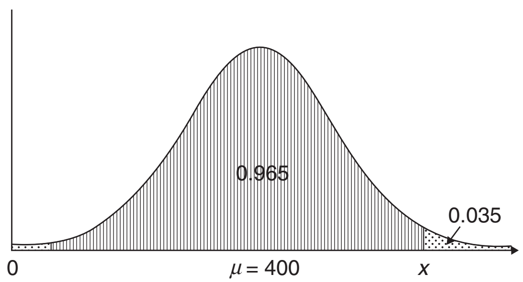


Figure 2.6

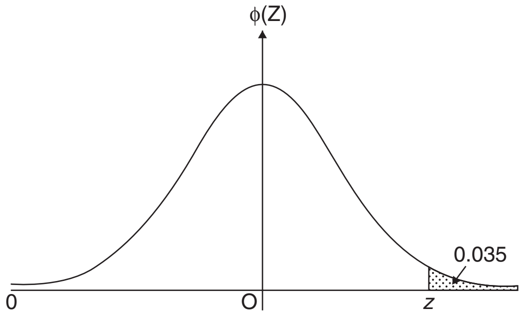


Figure 2.7

Consulting the table for area under the normal curve, we have $z = 1.81$. Therefore, the relation $Z = \frac{x-\mu}{\sigma}$ yields

$$1.81 = \frac{x-400}{80} \text{ or } x = 400 + 80(1.81) = 545.$$

Thus the candidates having scores of 545 or above will be declared pass.

EXAMPLE 2.81

In an examination taken by 500 candidates the average and standard deviation of marks obtained (normally distributed) are 40% and 10%. Find approximately

(i) How many will pass if 50 is fixed as a minimum?
(ii) What should be minimum score if 350 candidates are to be declared as pass?
(iii) How many candidates have scored marks above 60%?

Solution. We are given that

$$N = 500,\ \mu = 40, \text{ and } \sigma = 10.$$

Then

(i) $Z = \dfrac{50-40}{10} = 1$.

Therefore, consulting table for standard normal curve, we have

$$P(X \geq 50) = P(Z \geq 1) = 0.1587.$$

Hence the number of candidates passed, if 50 is fixed as minimum, is

$$N \times 0.1587 = 500 \times 0.1587 = 79.35 \approx 79$$

(ii) Fraction of passing students $= \dfrac{350}{500} = 0.7$

Fraction of failing students $= 1 - 0.7 = 0.3$.

Thus

$$\int_{-\infty}^{z_1} \phi(z)dz = \int_{-z_1}^{\infty} \phi(z)dz = 0.3$$

which yields $-z_1 = 0.52$. Hence

$$-0.52 = \frac{x-40}{10}$$

and so

$$x = 40 - 5.2 = 34.8 \approx 35\%.$$

(iii) We have

$$Z = \frac{60-40}{10} = 2.$$

Therefore, from the standard normal curve table

$$P(X \geq 60) = P(Z \geq 2) = 0.0288.$$

Hence number of candidates scoring more than $60\% = 500 \times 0.0288 \approx 11$.

EXAMPLE 2.82

For a normally distributed variate X with mean 1 and standard deviation 3, find out the probability that

(i) $3.43 \le x \le 6.19$

(ii) $-1.43 \le x \le 6.19$.

Solution. We have $\mu = 1$ and $\sigma = 3$.

(i) When $x = 3.43$,

$$Z = \frac{x-\mu}{\sigma} = \frac{3.43-1}{3} = 0.81$$

and when $x = 6.19$

$$Z = \frac{6.19-1}{3} = 1.73.$$

Therefore,

$$\begin{aligned} P(3.43 \le x \le 6.19) &= P(0.81 \le Z \le 1.73) \\ &= P(Z \ge 0.81) - P(Z \ge 1.73) \\ &= 0.2090 - 0.0418 = 0.1672. \end{aligned}$$

(ii) When $x = -1.43$,

$$Z = \frac{-1.43-1}{3} = -\frac{2.43}{3} = -0.81$$

and when $x = 6.19$,

$$Z = \frac{6.19-1}{3} = 1.73.$$

Therefore,

$$\begin{aligned} P(-1.43 \le x \le 6.19) &= P(-0.81 \le Z \le 1.73) \\ &= P(-0.81 \le Z \le 0) + P(0 \le Z \le 1.73) \\ &= P(0 \le Z \le 0.81) + P(0 \le Z \le 1.73) \quad \text{(by symmetry)} \\ &= 0.2910 + 0.4582 = 0.7492. \end{aligned}$$

EXAMPLE 2.83

The mean height of 500 students is 151 cm and the standard deviation is 15 cm. Assuming that the heights are normally distributed, find the number of students whose heights lie between 120 and 155 cm.

Solution. We have $N = 500$, $\mu = 151$, $\sigma = 15$. If $x = 120$, then

$$Z = \frac{120-\mu}{\sigma} = \frac{120-151}{15} = -\frac{-31}{15} = -2.07.$$

If $x = 155$, then

$$Z = \frac{155-151}{15} = \frac{4}{15} = 0.27.$$

Therefore,

$$\begin{aligned} P(120 \le x \le 150) &= P(-2.07 \le Z \le 0.27) \\ &= P(-2.07 \le Z \le 0) + P(0 \le Z \le 0.27) \\ &= P(0 \le Z \le 2.07) + P(0 \le Z \le 0.27) \quad \text{(By symmetry)} \\ &= 0.4808 + 0.1064 = 0.5872 = 293.60 \approx 294. \end{aligned}$$

EXAMPLE 2.84

Fit a normal curve to the following data:

Class:	1–3	3–5	5–7	7–9	9–11
Frequency:	1	4	6	4	1

Also obtain the expected normal frequency.

Solution. The class marks (mid-values) are 2, 4, 6, 8, 10. Therefore, for the given data, we have

$$\text{Mean}(\mu) = \frac{\sum fx}{\sum f} = \frac{2\times 1 + 4\times 4 + 6\times 6 + 8\times 4 + 10\times 1}{1+4+6+4+1} = \frac{96}{16} = 6,$$

$$\text{Standard deviation } (\sigma) = \sqrt{\frac{\sum fx^2}{\sum f} - \mu^2} = \sqrt{40-36} = 2.$$

Hence the equation of the normal curve fitted to the given data is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{8}(x-6)^2}.$$

To calculate the expected frequency, we note that the area under $f(x)$ in (z_1, z_2) is

$$\Delta\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_0^{z_2} e^{-\frac{z^2}{2}} dz - \frac{1}{\sqrt{2\pi}}\int_0^{z_1} e^{-\frac{z^2}{2}} dz, \quad z = \frac{x-\mu}{\sigma} = \frac{x-6}{2}.$$

Thus, the theoretic normal frequencies $N\Delta\Phi(z)$ are given by the following table:

Class interval	Mid-value	(z_1,z_2)	$\Delta\Phi(z) = \Phi(z+1) - \Phi(z)$	Expected frequency
1–3	2	(–2.5, –1.5)	0.4938 – 0.4332 = 0.606	16(0.606) = 0.97 ≈ 1
3–5	4	(–1.5, –0.5)	0.4332 – 0.1915 = 0.2417	16(0.2417) = 3.9 ≈ 4
5–7	6	(–0.5, 0.5)	0.1915 + 0.1915 = 0.383	16(0.383) = 6.1 ≈ 6
7–9	8	(0.5, 1.5)	0.4332 – 0.1915 = 0.2417	16(0.2417) = 3.9 ≈ 4
9–11	10	(1.5, 2.5)	0.4938 – 0.4332 = 0.606	16(0.606) = 0.97 ≈ 1

Thus, the expected frequencies agree with the observed frequencies. Hence the normal curve obtained above is a proper fit to the given data.

EXAMPLE 2.85

In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the means and the standard deviation of the distribution.

Solution. When $x = 45$, we have

$$z_1 = \frac{x-\mu}{\sigma} = \frac{45-\mu}{\sigma}.$$

When $x = 64$, we have

$$z_2 = \frac{64-\mu}{\sigma}.$$

Further

$$\int_{-\infty}^{z_1} \phi(z)dz = 0.31 \quad \text{and} \quad \int_{z_2}^{\infty} \phi(z)dz = 0.08,$$

that is,

$$\int_{-z_1}^{\infty} \phi(z)dz = 0.31 \quad \text{and} \quad \int_{z_2}^{\infty} \phi(z)dz = 0.08.$$

Hence $-z_1 = 0.5$ or $z_1 = -0.5$ and $z_2 = 1.4$. Thus

$$45-\mu = -0.5\sigma \quad \text{and} \quad 64-\mu = 1.45.$$

Solving these questions, we get $\sigma = 10$ and $\mu = 50$.

EXAMPLE 2.86

The marks obtained by the number of students for a certain subject are assumed to be approximately distributed with mean value 65 and with a standard deviation of 5. If three students are taken at random from this set of students, what is the probability that exactly two of them will have marks over 70?

Solution. We are given that $\mu = 65$ and $\sigma = 5$. If $x = 70$, we have

$$Z = \frac{x-\mu}{\sigma} = \frac{70-65}{5} = 1.$$

Thus

$$P(X > 70) = P(Z > 1)$$
$$= 0.1587 \text{ (using the table).}$$

Since this probability is the same for each student, the required probability that out of three students selected at random, exactly two will get marks over 70 is

$$^3C_2 p^2 q, \quad \text{where } p = 0.1587 \text{ and } q = 1-p = 0.8413,$$

which is equal to $3(0.1587)^2(0.8413) = 0.06357$.

2.24 SAMPLING

A *population* or *universe* is an aggregate of objects, animate, or inanimate, under study. More precisely, a population consists of numerical values connected with these objects. A population containing a finite number of objects is called a *finite population*, while a population with infinite number of objects is called an *infinite population*.

For any statistical investigation, complete enumeration of the infinite population is not practicable. For example, to calculate average per capita income of the people of a country, we have to enumerate all the earning individuals in the country, which is a very difficult task. So we take the help of sampling in such a case.

A *sample* is a finite subset of statistical individual of a population. The number of individual in a sample is called the *sample size*. A sample is said to be *large* if the number of objects in the sample is at least 30, otherwise it is called *small*. The process of selecting a sample from a population is called *sampling*.

A sampling in which the objects are chosen in such a manner that one object has as good chance of being selected as another is called a *random sampling*. This sample obtained in a random sampling is called a *random sample*.

The error involved in approximation by sampling technique is known as *sampling error* and is inherent and unavoidable in any and every sampling scheme. But sampling results in considerable gains, especially in time and cost.

The statistical constants of the population, namely, mean, variance etc., are denoted by, μ, σ^2, etc., respectively, and are called *parameters* whereas the statistical measures computed from the sample observations alone, namely, mean, variance, etc., are denoted by $\bar{x}$, s^2, etc., and are called *statistics.*

Suppose that we draw possible samples of size n from a population at random. For each sample, we compute the mean. The means of the samples are not identical. The frequency distribution obtained by grouping the different means according to their frequencies is called *sampling distribution of the mean*. Similarly, the frequency distribution obtained by grouping different variances according to their frequency is called *sampling distribution of the variance.*

The sampling of large samples is assumed to be normal. The standard deviation of the sampling distribution of a statistics is called *standard error of that statistics.* The standard error of the sampling distribution of means is called *standard error of means*. Similarly, standard error of the sampling distribution of variances is called *standard error of the variances*. The standard error is used to assess the difference between the expected and observed values. The reciprocal of the standard error is called *precision*.

Certain assumptions about the population are made to reach decisions about populations based on sample information. Such assumptions, true or false, are called *statistical hypothesis*.

A hypothesis which is a definite statement about the population parameter is called *null hypothesis* and is denoted by H_0. In fact, the *null hypothesis is that which is tested for possible rejection under the assumption that it is true.* For example, let us take the hypothesis that a coin is unbiased (true). Thus H_0 is that $p = \frac{1}{2}$, where p is probability for head. We toss this coin 10 times and observe the number of times a head appears. If head appears too often or too seldom, we shall reject the hypothesis H_0 and, thus, decide that the coin is biased, otherwise we shall decide that the penny is a fair one.

A hypothesis which is complementary to the null hypothesis is called the *alternative hypothesis*, which is denoted by H_1. For example, if $H_0\colon p = \frac{1}{2}$, then the alternative hypothesis H_1 can be

(i) $H_1\colon p \neq \frac{1}{2}$,

(ii) $H_1\colon p > \frac{1}{2}$,

(iii) $H_1\colon p < \frac{1}{2}$.

The alternative hypothesis in (i) is called a *two-tailed alternative,* in (ii) it is called *right tailed alternative,* and in (iii) it is known as *left-tailed alternative*.

If a hypothesis is rejected while it should have been accepted, we say that a *type I error* is committed. If a hypothesis is accepted while it should have been rejected, we say that the *type II error* has been committed.

2.25 LEVEL OF SIGNIFICANCE AND CRITICAL REGION

The probability level, below which we reject the hypothesis, is called the *level of significance*. A region in the sample space where hypothesis is rejected is called the *critical region* or *region of rejection*. The levels of significance, usually employed in testing of hypothesis, are 5% and 1%.

We know that for large n,

$$Z = \frac{x - np}{\sqrt{npq}}$$

is distributed as a standard normal variate. Thus, the shaded area is the standard normal curve shown in Figure 2.8 corresponds to 5% level of significance.

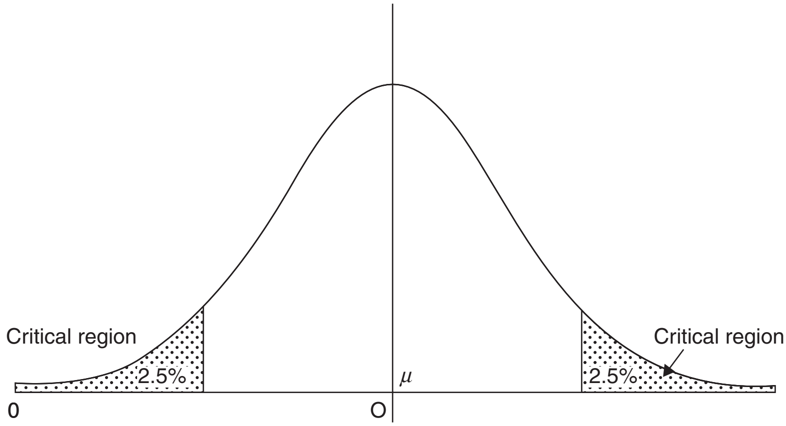


Figure 2.8

The probability of the value of the variate falling in the critical region is the level of significance.

We use a single-tail test or double-tail test to estimate for the significance of a result. In a double-tail test, the areas of both the tails of the curve representing the sampling distribution are taken into account whereas in the single-tail test, only the area on the right of an ordinate is taken into account. For example, we should use double-tail test to test whether a coin is biased or not because a biased coin gives either more number of heads than tails (right tail) or more number of tails than heads (left-tail).

The procedure which enables us to decide whether to accept or reject a hypothesis is called the *test of significance*. The procedure usually consists in assuming or accepting the hypothesis as correct and then calculating the probability of getting the observed or more extreme sample. If this probability is less than a certain pre-assigned value, the hypothesis is rejected, since samples with small probabilities should be rare and we assume that a rare event has not happened.

2.26 TEST OF SIGNIFICANCE FOR LARGE SAMPLES

We know that for large number of trials, the binomial and Poisson distributions are very closely approximated by normal distribution. Therefore, for large samples we apply the *normal test*, which is based on the area property of normal probability curve. In standard normal curve, the standard normal variate Z is given by

$$Z = \frac{X - \mu}{\sigma}.$$

Then

$$P(-3 \le Z \le 3) = P(-3 \le Z \le 0) + P(0 \le Z \le 3)$$
$$= P(0 \le Z \le 3) + P(0 \le Z \le 3) \text{ (by symmetry)}$$
$$= 2P(0 \le Z \le 3) = 2(0.4987) = 0.9974$$

and so

$$P(|Z| > 3) = 1 - 0.9974 = 0.0026.$$

If follows therefore that, in all probability, we should expect a *standard normal variate to lie between –3 and 3.* Further,

$$P(-1.96 \le Z \le 1.96) = P(-1.96 \le Z \le 0) + P(0 \le Z \le 1.96)$$
$$= 2P(0 \le Z \le 1.96)$$
$$= 2(0.4750) = 0.9500$$

and so

$$P(|Z| > 1.96) = 1 - 0.95 = 0.05.$$

If follows that the *significant value of Z at 5% level of significance for a two-tailed test is 1.96.*

Also, we note that

$$P(-2.58 \le Z \le 2.58) = P(-2.58 \le Z \le 0) + P(0 \le Z \le 2.58)$$
$$= 2P(0 \le Z \le 2.58)$$
$$= 2(0.4951) = 0.9902$$

and so

$$P(|Z| > 2.58) = 0.01.$$

Hence the *significant value of Z at 1% level of significance for a two-tailed test is 2.58.*

Now we find value of Z for single-tail test. From normal probability tables, we note that

$$P(Z > 1.645) = 0.5 - P(0 \le Z \le 1.645)$$
$$= 0.5 - 045 = 0.05$$
$$P(Z > 2.33) = 0.5 - P(0 \le Z \le 2.33)$$
$$= 0.5 - 0.49 = 0.01.$$

Hence, *significant value of Z at 5% level of significance of a single-tail test is 1.695, whereas the significant value of Z at 1% level of significance is 2.33.*

As a consequence of the above discussion, the steps to be used in the normal test are:

(i) Compute the test statistic Z under the null hypothesis
(ii) If $|Z| > 3$, H_0 is always rejected
(iii) If $|Z| \le 3$, we test its level of significance at 5% or 1% level.
(iv) For a two-tailed test, if $|Z| > 1.96$, H_0 is rejected at 5% level of significance. If $|Z| > 2.58$, H_0 is rejected at 1% level of significance and if $|Z| \le 2.58$, H_0 may be accepted at 1% level of significance
(v) For a single-tailed test, if $|Z| > 1.645$, then H_0 is rejected at 5% level and if $|Z| > 2.33$, then H_0 is rejected at 1% level of significance.

The following theorem of statistics helps us to determine sample mean $\overline{x}$ and sample variance S^2 in terms of population mean μ and population variance σ^2.

Theorem 2.7. (The Central Limit Theorem). The mean $\overline{x}$ of a sample of size N drawn from any population (continuous or discrete) with mean μ and finite variance σ^2 will have a distribution that approaches the normal distribution as $N \to \infty$, with mean μ and variance $\dfrac{\sigma^2}{N}$.

The quantity $\dfrac{\sigma}{\sqrt{N}}$ *is called the standard error of the mean.*

2.27 CONFIDENCE INTERVAL FOR THE MEAN

For the standard normal distribution, let Z_α be a point on the z-axis for which the area under the density function $\phi(z)$ to its right is equal to α (see Figure 2.9a). Thus

$$P(Z > z_\alpha) = \alpha,$$

or equivalently,

$$P(Z < z_\alpha) = 1 - \alpha = \int_{-\infty}^{z_\alpha} \phi(z)dz.$$

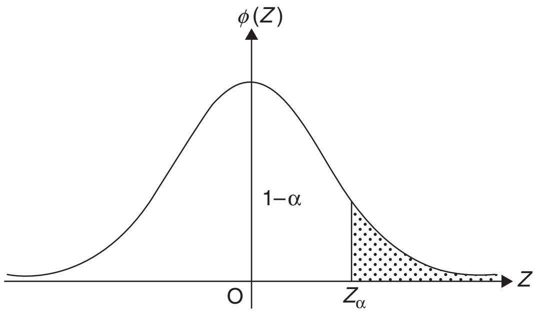


Figure 2.9a

Since standard normal curve is symmetrical about $\phi(z)$-axis, we have

$$P(-z_{\frac{\alpha}{2}} < Z < z_{\frac{\alpha}{2}}) = 1 - \alpha \text{ (see Figure 2.9b).}$$

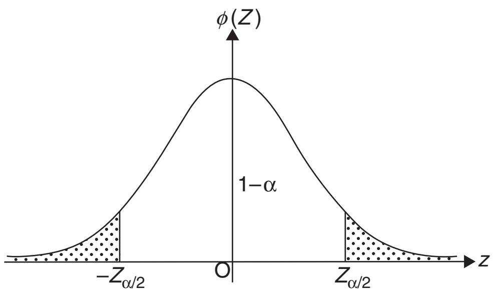


Figure 2.9b

But, assuming normality of the sample average, the Central Limit theorem yields

$$Z = \frac{\overline{x} - \mu}{\frac{\sigma}{\sqrt{n}}}.$$

Therefore,

$$P\left(-z_{\frac{\alpha}{2}} < \frac{\overline{x}-\mu}{\frac{\sigma}{\sqrt{n}}} < z_{\frac{\alpha}{2}}\right) = 1-\alpha$$

and so cross multiplication and change of sign yields

$$P\left(\overline{x} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} < \mu < \overline{x} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right) = 1-\alpha.$$

The interval defined by

$\left(\overline{x} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \overline{x} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right)$ is called a 100(1 – α)% *confidence interval for the mean* with variance (σ) known. Thus if α is specif ed, the upper and lower limit of this interval can be calculated from the sample average.

We know that if $\alpha = 0.05$, then $z_{0.05} = 1.645$ and then the 95% confidence interval for single-tailed test is

$$\left(\overline{x} - 1.645\frac{\sigma}{\sqrt{n}}, \overline{x} + 1.645\frac{\sigma}{\sqrt{n}}\right).$$

EXAMPLE 2.87

The temperature, in degree celsius, at 12 points chosen at random in New Delhi is measured. The observations at these points are:

25°	23°	22.5°	26.5°	27°	27.5°
23.5°	22.5°	26°	24°	24.5°	25.5°

The past experience shows that the standard deviation of temperature in Delhi is 1°C. Find a 95% confidence interval for the mean temperature in the city.

Solution. We note that

$$\mu = \frac{297.5}{12} = 24.792.$$

Since $z_{0.05} = 1.645$, the 95% confidence interval is

$$\left(24.79 \pm 1.645\left(\frac{1}{\sqrt{12}}\right)\right) = (24.79 \pm 0.4748)$$
$$= (24.32, 25.26).$$

EXAMPLE 2.88

For all children taking an examination, the mean mark was 60% with a standard deviation of 8%. A particular class of 30 children achieved an average of 63%. Is this unusual?

Solution. Let H_0 be null hypothesis that the achievement is usual. We have

$$N = 30,\ \mu = 60, \sigma = 8.$$

Since the sample is large, the distribution tends to normal distribution. The standard normal variate is given by

$$Z = \frac{x-\mu}{\frac{\sigma}{\sqrt{N}}} = \frac{63-60}{\frac{8}{\sqrt{30}}} = 2.0539.$$

Since $|Z| < 2.55$, H_0 is accepted at 5% level of significance and rejected at 1% level of significance since $|Z| > 1.96$.

EXAMPLE 2.89

A coin was tossed 400 times and the head turned up 216 times. Test the hypothesis that the coin is unbiased.

Solution. The null hypothesis is

$$H_0\text{: The coin is unbiased, that is, } p \text{ (head)} = \frac{1}{2}.$$

The number of trial (n) = 400. Therefore,

$$\text{Expected number of success} = np = \frac{1}{2}\times 400 = 200.$$

$$\text{Observed number of success} = 216.$$

Further $p = \frac{1}{2}$ implies $q = 1-p = \frac{1}{2}$. Therefore,

$$\sigma = \sqrt{npq} = \sqrt{400\times\frac{1}{2}\times\frac{1}{2}} = 10.$$

Hence, standard normal variate is

$$Z = \frac{x-np}{\sigma} = \frac{216-200}{10} = 1.6.$$

Since $|Z| = 1.6 < 1.96$, H_0, is accepted at 5% level of significance. We conclude that the coin is unbiased.

EXAMPLE 2.90

In IIT joint entrance test, the score showed $\mu = 64$ and $\sigma = 8$. How large a sample of candidates appearing in the test must be taken in order that there be a 10% chance that its mean score is less than 62%?

Solution. We are given that $\mu = 64$, $\sigma = 8$, and $\overline{x} = 62$

$$P(\overline{x} < 62) = \frac{10}{100} = 0.1.$$

Therefore,

$$0.1 = \int_{-\infty}^{z_1} \phi(z)dz = \int_{-z_1}^{\infty} \phi(z)dz.$$

The table of areas under the normal curve yield

$$-z_1 = 1.28 \text{ and so } z_1 = -1.28.$$

Hence

$$-1.28 = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{N}}} = \frac{(\bar{x}-\mu)\sqrt{N}}{\sigma} = \frac{(62-64)}{8}\sqrt{N},$$

which yields $N = 26.21$. Hence, we must take a sample of size 26.

EXAMPLE 2.91

If the mean breaking strength of copper wire is 575 kg with a standard deviation of 8.3 kg, how large a sample must be used so that there be one chance in 100 that the mean breaking strength of the sample is less than 572 kg?

Solution. We are given that

$$\mu = 575\,kg, \quad \sigma = 8.3\,kg, \quad \text{and} \quad \bar{x} = 572$$

and

$$P(\bar{x} < 572) = \frac{1}{100} = 0.01.$$

Therefore,

$$0.01 = \int_{-\infty}^{z_1} \phi(z)dz = \int_{-z_1}^{\infty} \phi(z)dz.$$

The table of areas under normal curve yields

$$-z_1 = 2.33, \text{ that is } z_1 = -2.33,$$

Therefore,

$$-2.33 = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{N}}} = \frac{572-575}{8.3}\sqrt{N},$$

which gives $N = 41.602$. Hence, we must take a sample of size 42.

EXAMPLE 2.92

A normal population has a mean of 6.8 and standard deviation of 1.5. A sample of 400 members gave a mean of 6.75. Is the difference between the means significant?

Solution. Let the null and alternative hypothesis be

H_0: there is no significant difference : between $\bar{x}$ and μ,
H_1: there is significant difference between $\bar{x}$ and μ.

It is given that $\mu = 6.8$, $\sigma = 1.5$, $N = 400$, and $\bar{x} = 6.75$. Therefore, the standard normal variate is given by

$$Z = \frac{\bar{x}-\mu}{\sigma}\sqrt{N} = \frac{6.75-6.8}{1.5}\sqrt{400}$$
$$= -0.666 \approx -0.67.$$

Since $|Z| = 0.67 < 1.96$, H_0 is accepted at 5% level of significance and so there is no significant difference between $\bar{x}$ and μ.

EXAMPLE 2.93

A research worker wishes to estimate mean of a population by using sufficiently large sample. The probability is 95% that sample mean will not differ from the true mean by more than 25% of the standard deviation. How large a sample should be taken?

Solution. We are given that

$$P\left(\left| \bar{x} - \mu \right| < 0.25\sigma \right) = 0.95$$

Also

$$P\left(\left| Z \right| \le 1.96 \right) = 0.95,$$

that is,

$$P\left(\frac{\bar{x} - \mu}{\sigma} \sqrt{n} \le 1.96 \right) = 0.95$$

or

$$P\left(\left| \bar{x} - \mu \right| \le 1.96 \left(\frac{\sigma}{\sqrt{n}} \right) \right) = 0.95.$$

Therefore,

$$1.96 \left(\frac{\sigma}{\sqrt{n}} \right) < 0.25\sigma$$

or

$$n > \left(\frac{1.96}{0.25} \right)^2 = (7.84)^2 = 61.47.$$

Therefore, the sample should be of the size 62.

EXAMPLE 2.94

As an application of Central Limit theorem, show that if E is such that $P\left(\left| \bar{x} - \mu \right| < E \right) > 0.95$, then the minimum sample size n is given by $n = \frac{(1.96)^2 \sigma^2}{E^2}$, where μ and σ^2 are the mean and variance, respectively, of the population and $\bar{x}$ is the mean of the random variable.

Solution. By Central Limit theorem $Z = \frac{\bar{x} - \mu}{\sigma} \sqrt{n}$ is a standard normal variate and, therefore, $P(|Z| \le 1.96) = 0.95$ implies

$$P\left\{ \left| \frac{\bar{x} - \mu}{\sigma} \sqrt{n} \right| \le 1.96 \right\} = 0.95$$

or

$$P\left\{ \left| \bar{x} - \mu \right| \le 1.96 \frac{\sigma}{\sqrt{n}} \right\} = 0.95.$$

Also, it is given that

$$P\left\{ \left| \bar{x} - \mu \right| < E \right\} > 0.95.$$

Thus

$$E > \frac{1.96}{\sqrt{n}} \sigma \text{ or } n > \frac{(1.96)^2 \sigma^2}{E^2} = \frac{3.84\sigma^2}{E^2}.$$

Hence, minimum sample size is given by $n = \frac{3.84\sigma^2}{E^2}$.

2.28 TEST OF SIGNIFICANCE FOR SINGLE PROPORTION

Let X be the number of successes in n independent trials with probability p of success for each trial. Then

$$E(X) = np, \text{variation}\,(X) = npq, q = 1 - p.$$

Let $P = \frac{X}{n}$ be called the observed proportion of success. Then

$$\text{Variance (P)} = \text{Variation}\left(\frac{X}{n}\right)$$

$$= n.\frac{p}{n}.\frac{q}{n} = \frac{pq}{n}$$

Standard error (P) $= \sqrt{\frac{pq}{n}}$

and

$$Z = \frac{P - E(P)}{S.E(P)} = \frac{P - p}{\sqrt{\frac{pq}{n}}},$$

where Z is test statistics used to test the significant difference of sample and population proportion. Further, the limit for p at the level of significance is given by $P \pm z_{\alpha}\sqrt{\frac{PQ}{n}}$. In particular,

$$95\% \text{ confidence limits for } p \text{ are given by } P \pm 1.96\sqrt{\frac{PQ}{n}}$$

$$99\% \text{ confidence limits for } p \text{ are given by } P \pm 2.58\sqrt{\frac{PQ}{n}}.$$

EXAMPLE 2.95

Solve Example 2.89 using significance for single proportion.

Solution. Let the null and alternative hypothesis be

$$H_0\text{: The coin is unbiased, that is, } p = \frac{1}{2} = 0.5.$$

$$H_1\text{: The coin is biased, that is, } p \neq 0.5.$$

We are given that $n = 400$ and number of successes $(X) = 216$. Therefore,

Proportion of success in the sample $(P) = \frac{X}{n} = \frac{216}{400} = 0.54.$

Further, population proportion $= p = 0.5$ and so $q = 1 - p = 1 - 0.5 = 0.5.$

Hence

$$\text{Test statistics}(Z) = \frac{P - p}{\sqrt{\frac{pq}{n}}} = \frac{0.54 - 0.50}{\sqrt{0.25}}\sqrt{400}$$

$$= \frac{0.04 \times 20}{0.5} = 1.6.$$

Since $|Z| = 1.6 < 1.96$, H_0 is accepted at 5% level of significance. Hence the coin is unbiased.

EXAMPLE 2.96

In an opinion poll conducted with a sample of 2,000 people chosen at random, 40% people told that they support a certain political party. Find a 95% confidence interval for the actual proportion of the population who support this party.

Solution. The required 95% confidence interval is

$$0.4 \pm 1.96\sqrt{\frac{(0.4)(0.6)}{2000}} = 0.4 \pm 1.96(0.01095)$$
$$= 0.4 \pm 0.0214 = (0.3786, 0.4214).$$

This shows that a variation of about 4% either way is expected when conducting opinion poll with sample size of this order.

EXAMPLE 2.97

A random sample of 400 mangoes was taken from a large consignment out of which 80 were found to be rotten. Obtain 99% confidence limits for the percentage of rotten mangoes in the consignment.

Solution. We have $n = 400$ and proportion of rotten mangoes in the sample $(P) = \frac{80}{400} = 0.2$. Since significant value of Z at 99% confidence coefficient (level of significance 1%) is 2.58, the 99% confidence limits are

$$P \pm 2.58\sqrt{\frac{PQ}{n}} = 0.2 \pm 2.58\sqrt{\frac{0.2 \times 0.8}{400}}$$
$$= 0.2 \pm 2.58\sqrt{\frac{0.16}{400}}$$
$$= 0.2 \pm 2.58(0.02)$$
$$= (0.148, 0.252).$$

Hence, 99% confidence limits for percentage of rotten mangoes in the consignments are (14.8, 25.2).

EXAMPLE 2.98

A die was thrown 9,000 times and a throw of 3 or 4 was observed 3,240 times. Show that the die cannot be regarded as an unbiased one.

Solution. Let the null and alternative hypothesis be

$$H_0: \text{die is unbiased,}$$
$$H_1: \text{die is biased.}$$

Further,

$$p = \text{probability of success (getting 3 or 4)}$$
$$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3} = 0.333$$
$$q = \text{probability of failure} = 1 - \frac{1}{3} = \frac{2}{3}.$$
$$\text{P} = \text{proportion of success in the sample} = \frac{3240}{9000} = 0.360.$$

Therefore, the test statistics Z is given by

$$Z = \frac{P-p}{\sqrt{\frac{pq}{n}}} = \frac{0.360-0.333}{\sqrt{\frac{2}{9}}}\sqrt{9000} = 5.37.$$

Since $|Z| = 5.37 > 3$, the null hypothesis is rejected. So, we conclude that the die is almost certainly biased.

EXAMPLE 2.99

Out of 650 truck drivers, 40 were found to have consumed alcohol more than the legal limit. Find 95% confidence interval for the true proportion of drivers who were over the limit during the time of the tests.

Solution. The observed proportion of the sample is

$$P = \frac{40}{650} = \frac{4}{65}.$$

Therefore,

$$Q = 1 - P = 1 - \frac{4}{65} = \frac{61}{65}.$$

The 95% confidence interval of the proportion is

$$\begin{aligned} P \pm 1.96\sqrt{\frac{PQ}{n}} &= \frac{4}{65} \pm 1.96\sqrt{\frac{\frac{4}{65}\cdot\frac{61}{65}}{650}} \\ &= \frac{4}{65} \pm 1.96(0.009426) \\ &= 0.00615 \pm 0.01847 \\ &= (0.0431, 0.0800). \end{aligned}$$

This mean, 4% to 8% of the drivers were over the limit during the tests.

2.29 TEST OF SIGNIFICANCE FOR DIFFERENCE OF PROPORTION

Let X_1 and X_2 be the number of persons possessing the given attribute A in random samples of sizes n_1 and n_2 from two populations, respectively. The sample proportions are given by.

$$P_1 = \frac{X_1}{n_1}, \quad P_2 = \frac{X_2}{n_2}.$$

Then

$$E(P_1) = E\left(\frac{X_1}{n_1}\right) = \frac{1}{n_1}E(X_1) = \frac{1}{n_1}(n_1 p_1) = p_1$$

$$V(P_1) = \frac{p_1 q_1}{n_1}, \quad V(P_2) = \frac{p_2 q_2}{n_2}.$$

Since for large samples P_1 and P_2 (the probability of success) are independent and normally distributed, $P_1 - P_2$ is also normally distributed. Therefore, the standard normal variate corresponding to the difference $P_1 - P_2$ is given by

$$Z = \frac{(P_1 - P_2) - E(P_1 - P_2)}{\sqrt{V(P_1 - P_2)}}.$$

Let H_0: $P_1 = P_2$, that is, the population are similar be the null hypothesis. Then

$$E(P_1 - P_2) = E(P_1) - E(P_2) = p_1 - p_2 = 0.$$

Also $V(P_1 - P_2) = V(P_1) + V(P_2)$

$$= \frac{p_1q_1}{n_1} + \frac{p_2q_2}{n_2} = pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right),$$

because under H_0, $p_1 = p_2 = p$, say. Therefore,

$$Z = \frac{P_1 - P_2}{\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}},$$

where an unbiased pooled estimate of proportion is taken as

$$p = \frac{n_1P_1 + n_2P_2}{n_1 + n_2} = \frac{X_1 + X_2}{n_1 + n_2}.$$

If $|Z| > 1.96$, H_0 is rejected at 5% level of significance. If $|Z| < 2.58$, H_0 is accepted at 1% level of significance.

EXAMPLE 2.100

In a sample of 600 men from a certain city, 450 are found to be smokers. In another sample of 900 men from another city, 450 are smokers. Does the data indicate the habit of smoking among men?

Solution. We have

$$\text{Proportion } P_1 = \frac{450}{600} = \frac{3}{4},$$

$$\text{Proportion } P_2 = \frac{450}{900} = \frac{1}{2}.$$

Then the test statistics is given by

$$Z = \frac{P_1 - P_2}{\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

where

$$p = \frac{X_1 + X_2}{n_1 + n_2} = \frac{450 + 450}{900 + 600} = \frac{900}{1500} = \frac{3}{5},$$

$$q = 1 - p = \frac{2}{5}.$$

Therefore,

$$Z = \frac{\frac{3}{4} - \frac{1}{2}}{\sqrt{\frac{6}{25}\left(\frac{1}{600} + \frac{1}{900}\right)}} = \frac{1}{4(0.0258)} = 9.68.$$

and hence the cites are significantly different.

EXAMPLE 2.101

A drug manufacturer claims that the proportion of patients exhibiting side effects to their new anti-arthritis drug is at least 8% lower than for the standard brand X. In a controlled experiment, 31 out of 100 patients receiving the new drug exhibited side effects, as did 74 out of 150 patients receiving brand X. Test the manufacturer's claim using 95% confidence for the true proportion.

Solution. We have $n_1 = 100$, $n_2 = 150$, and

$$\text{Proportion } (P_1) \text{ for new drug} = \frac{31}{100},$$

$$\text{Proportion } (P_2) \text{ for the standard drug } X = \frac{74}{150}.$$

The test statistics is

$$Z = \frac{P_1 - P_2}{\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}},$$

where

$$p = \frac{X_1 + X_2}{n_1 + n_2} = \frac{31 + 74}{100 + 150} = \frac{21}{50},$$

$$q = 1 - p = 1 - \frac{21}{50} = \frac{29}{50}.$$

Therefore,

$$Z = \frac{\frac{31}{100} - \frac{74}{150}}{\sqrt{\frac{609}{2500}\left(\frac{5}{300}\right)}} = \frac{-11}{60\sqrt{\frac{203}{150000}}} = -4.984.$$

Thus $|Z| = 4.984 > 1.96$. Thus the difference between the two brands are significant at 5% level of significance.

Also, the 95% confidence interval is

$$P_1 - P_2 \pm 1.96\sqrt{\frac{P_1Q_1}{n_1} + \frac{P_2Q_2}{n_2}}$$

$$= \frac{-11}{60} \pm 1.96\sqrt{\frac{31 \times 69}{(100)^3} + \frac{74 \times 26}{(150)^3}}$$

$$= -0.1833 \pm 0.1020 = (-0.2853, -0.0813).$$

Since 0 does not lie within the interval, the difference is significant. Further, the claim of the manufacturer is accepted as it lies within the confidence interval.

EXAMPLE 2.102

In two large populations, there are 30% and 25%, respectively, of fair-haired people. Is this difference likely to be hidden in samples of 1,200 and 900, respectively, from the two populations?

Solution. Let

$$P_1 = \text{proportion of fair – haired people in first population}$$

$$= \frac{30}{100} = 0.30$$

$$P_2 = \text{proportion of fair} - \text{haired people in second population}$$
$$= \frac{25}{100} = 0.25.$$

Accordingly,

$$Q_1 = 1 - 0.3 = 0.7, \quad Q_2 = 1 - 0.25 = 0.75.$$

Let the null and alternative hypothesis be

H_0: Sample proportion are equal, that is, $P_1 = P_2$.

H_1: $P_1 \neq P_2$.

Then the test statistics is given by

$$Z = \frac{P_1 - P_2}{\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}},$$

where pooled estimate of proportion is

$$p = \frac{X_1 + X_2}{n_1 + n_2} = \frac{1200(0.3) + 900(0.25)}{1200 + 900}$$
$$= \frac{360 + 225}{2100} = 0.2786.$$

Therefore,

$$q = 1 - p = 0.7214.$$

Hence

$$Z = \frac{0.3 - 0.25}{\sqrt{(0.2786)(0.7214)\left(\frac{1}{1200} + \frac{1}{900}\right)}} = \frac{0.05}{0.019768} = 2.53.$$

Since $|Z| = 2.53 > 1.96$, the proportions are significantly different and so H_0 is rejected. The differences are unlikely to be hidden.

EXAMPLE 2.103

Random samples of 400 men and 600 women were asked whether they would like to have a flyover near their residence. Two hundred men and 325 women were in favour of the proposal. Test the hypothesis that proportions of men and women in favour of the proposal are same against that they are not, at 5% level.

Solution. The null hypothesis is

H_0: $P_1 = P_2$, that is, no significant difference between the opinion of men and women as far as the proposal of flyover is concerned.
We have

$$n_1 = 400, \quad X_1 = 200,$$
$$n_2 = 600, \quad X_2 = 325,$$
$$P_1 = \frac{200}{400} = 0.5 \quad \text{and} \quad P_2 = \frac{325}{600} = 0.541.$$

Therefore, the test statistics is

$$Z = \frac{P_1 - P_2}{\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}},$$

where

$$p = \frac{X_1 + X_2}{n_1 + n_2} = \frac{200 + 325}{400 + 600} = \frac{525}{1000} = 0.525$$

and

$$q = 1 - p = 0.475.$$

Hence

$$Z = \frac{0.5 - 0.541}{\sqrt{(0.525)(0.475)\left(\frac{1}{400} + \frac{1}{600}\right)}} = \frac{-0.041}{0.3223} = -1.272.$$

Since $|z| = 1.272 < 1.96$, H_0 may be accepted at 5% level of significance, that is, men and women do not differ significantly in their opinions.

EXAMPLE 2.104

In a referendum submitted to the student body at a university, 850 men and 560 women voted. Out of these 500 men and 320 women voted "yes". Does this indicate a significant difference of opinion between men and women on the matter at 1% level of significance?

Solution. We have $n_1 = 850$, $n_2 = 560$, $X_1 = 500$, $X_2 = 320$.
Let the null hypothesis be
H_0: there is no significant difference in voting pattern, that is, $P_1 = P_2$,
where

$$\text{Proportion } (P_1) = \frac{500}{850} = \frac{10}{17} = 0.588,$$

$$\text{Proportion } (P_2) = \frac{32}{560} = \frac{4}{7} = 0.571.$$

Then the test statistics is

$$Z = \frac{P_1 - P_2}{\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}},$$

where

$$p = \frac{X_1 + X_2}{n_1 + n_2} = \frac{820}{1410} = 0.582 \quad \text{and}$$

$$q = 1 - p = 0.418.$$

Therefore,

$$Z = \frac{0.588 - 0.571}{\sqrt{(0.582)(0.418)(0.1765 + 0.1786)}} = \frac{0.017}{0.294} = 0.578.$$

Since $|Z| = 0.578 < 2.58$, the hypothesis H_0 is accepted at 1% level of significance.

EXAMPLE 2.105

Suppose that 10 year ago 500 people were working in a factory, and 180 of them were exposed to a material which is now suspected as being carcinogenic. Of these 180, 30 have developed cancer, whereas 32 of the other workers, who were not exposed, have also developed cancer. Obtain 95% confidence interval for the difference between the proportions with cancer among those exposed and not exposed, and assess whether the material should be considered carcinogenic on this evidence.

Solution. According to the given data

$$\text{Total No. of workers} = 500,$$

n_1 = No. of people exposed to materials = 180,
n_2 = No. of people not exposed to materials = 320,
X_1 = No. of people out of n_1, who suffered with cancer = 30,
X_2 = No. of people out of n_2, who suffered with cancer = 32,

$$\text{Proportion } (P_1) = \frac{30}{180} = 0.167,$$

$$\text{Proportion } (P_2) = \frac{32}{320} = 0.100.$$

Therefore, a 95% confidence interval for the difference between the true proportions is

$$P_1 - P_2 \pm 1.96\sqrt{\frac{P_1Q_1}{n_1} + \frac{P_2Q_2}{n_2}}$$

$$= 0.067 \pm 1.96(0.0325)$$

$$= 0.07 \pm 0.0637 = (0.033, 0.131).$$

On the other hand, the test statistics is given by

$$Z = \frac{P_1 - P_2}{\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}},$$

where

$$p = \frac{X_1 + X_2}{n_1 + n_2} = \frac{62}{500} = 0.124 \quad \text{and}$$

$$q = 1 - p = 0.876.$$

Therefore,

$$Z = \frac{0.167 - 0.100}{\sqrt{(0.124)(0.876)\left(\frac{1}{180} + \frac{1}{320}\right)}} = \frac{0.067}{\sqrt{0.000942}}$$

$$= \frac{0.067}{0.031} = 2.16.$$

Since $|Z| > 1.96$, the difference is significant at 5% level and so the material should be considered carcinogenic on this evidence.

2.30 TEST OF SIGNIFICANCE FOR DIFFERENCE OF MEANS

Let $\bar{x}_1$ be the mean of a sample of size n_1 from a population with mean μ_1 and variance σ_1^2 and let $\bar{x}_2$ be the mean of sample of size n_2 from another population with mean μ_2 and variance σ_2^2. Then $\bar{x}_1$ and $\bar{x}_2$ are two independent normal variates. Therefore, $\bar{x}_1 - \bar{x}_2$ is also a normal variate. The value of the standard normal variate Z corresponding to $\bar{x}_1 - \bar{x}_2$ is given by

$$Z = \frac{(\bar{x}_1 - \bar{x}_2) - E(\bar{x}_1 - \bar{x}_2)}{\text{Standard error of } (\bar{x}_1 - \bar{x}_2)}.$$

If H_0: $\mu_1 = \mu_2$, that is, there is no significant difference between the sample means is the null hypothesis, then

$$E(\bar{x}_1 - \bar{x}_2) = E(\bar{x}_1) - E(\bar{x}_2) = \mu_1 - \mu_2 = 0$$

$$V(\bar{x}_1 - \bar{x}_2) = V(\bar{x}_2) + V(\bar{x}_2) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2},$$

since the covariance term vanishes due to independence of $\bar{x}$ and $\bar{x}_2$. Therefore, under the null hypothesis H_0, the test statistics Z is given by

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}.$$

If $\sigma_1^2 = \sigma_2^2$, that is, if the samples have been drawn from the same population, then the test statistics reduces to

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}.$$

If σ is not known, then its estimate $\hat{\sigma}$ based on sample variance is used and

$$(\hat{\sigma})^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{(n_1 + n_2 - 2)}.$$

If σ_1, σ_2, are known and $\sigma_1^2 \neq \sigma_2^2$ then they are estimated from sample values and we have

$$Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}.$$

EXAMPLE 2.106

A sample of 100 electric bulbs produced by a manufacture A showed a mean life time of 1,190 hours and a standard deviation of 90 hours. A sample of 75 bulbs produced by manufacturer B showed a mean life time of 1,230 hours with a standard deviation of 120 hours. Is there a difference between the mean life time of the two brands at significance levels of 5% and 1%?

Solution. We have

$$n_1 = 100, \bar{x}_1 = 1190, \sigma_1 = 90$$
$$n_2 = 75, \bar{x}_2 = 1230, \sigma_2 = 120.$$

Therefore, the test statistics is

$$Z = \frac{\overline{x}_1 - \overline{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{1190 - 1230}{\sqrt{\frac{90^2}{100} + \frac{120^2}{75}}}$$

$$= -\frac{40}{\sqrt{81 + 192}} = -\frac{40}{16.523} = -2.42.$$

Since $|Z| = 2.42 > 1.96$, there is a difference between the mean life time of the two brands at a significant level of 5%.

On the other hand $|Z| = 2.42 < 2.58$, therefore, there is no difference between the mean life time of the two brands at a significant level of 1%.

EXAMPLE 2.107

The means of simple samples of sizes 1,000 and 2,000 are 67.5 and 68.0, respectively. Can the samples be regarded as drawn from the same population of standard deviation 2.5?

Solution. We are given that

$$n_1 = 1000, \overline{x}_1 = 67.5, \quad n_2 = 2000, \overline{x}_2 = 68.0, \quad \sigma = 2.5.$$

Therefore, the test statistics is

$$Z = \frac{\overline{x}_1 - \overline{x}_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{67.5 - 68.0}{2.5\sqrt{\frac{1}{1000} + \frac{1}{2000}}} = -\frac{0.5}{2.5(0.03873)} = -5.16$$

Since $|Z| = 5.16 > 1.96$, the difference between the mean is very significant. Therefore, the samples cannot be regarded drawn from the same population.

EXAMPLE 2.108

The mean height of 50 male students who showed above average participation in college athletics was 68.2 inches with a standard deviation of 2.5 inches, whereas 50 male students who showed no interest in such participation had a mean height of 67.5 inches with a standard deviation of 2.8 inches. Test the hypothesis that male students who participle in college athletics are taller than other male students.

Solution. It is given that

$$n_1 = 50,\ \overline{x}_1 = 68.2,\ s_1 = 2.5,$$
$$n_2 = 50,\ \overline{x}_2 = 67.5, s_2 = 2.8.$$

Let the null and alternative hypothesis be

Null hypothesis: $\mu_1 = \mu_2$,

Alternative hypothesis: $\mu_1 > \mu_2$ (right tailed).

The test statistics is

$$Z = \frac{\overline{x}_1 - \overline{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{68.2 - 67.5}{\sqrt{\frac{(2.5)^2}{50} + \frac{(2.8)^2}{50}}} = \frac{0.7}{\sqrt{0.282}} = \frac{0.7}{0.53} = 1.32.$$

Since $Z = 1.32 < 1.645$ (critical value of Z at 5% level of significance). Therefore, it is not significant at 5% level of significance. Hence, the null hypothesis is accepted. Hence the students who participate in college athletics are not taller than other students.

2.31 TEST OF SIGNIFICANCE FOR THE DIFFERENCE OF STANDARD DEVIATIONS

Let s_1 and s_2 be the standard deviations of two independent samples of size n_1 and n_2, respectively. Let the null hypothesis be that the sample standard deviation does not differ significantly. Then the statistics of the hypothesis is

$$Z = \frac{s_1 - s_2}{S.E(s_1 - s_2)}.$$

For large samples,

$$S.E(s_1 - s_2) = \sqrt{\frac{\sigma_1^2}{2n_1} + \frac{\sigma_2^2}{2n_2}}$$

and so

$$Z = \frac{s_1 - s_2}{\sqrt{\frac{\sigma_1^2}{2n_1} + \frac{\sigma_2^2}{2n_2}}}.$$

If σ_1^2 and σ_2^2 are unknown, then s_1^2 and s_1^2 are used in place of them. Hence, in that case, we have

$$Z = \frac{s_1 - s_2}{\sqrt{\frac{s_1^2}{2n_1} + \frac{s_2^2}{2n_2}}}.$$

EXAMPLE 2.109

The yield of wheat in a random sample of 1,000 farms in a certain area has a standard deviation of 192 kg. Another random sample of 1,000 farms gives a standard deviation of 224 kg. Are the standard deviations significantly different?

Solution. We are given that

$$n_1 = 1000, s_1 = 192, \quad n_2 = 1000, s_2 = 224.$$

Therefore, the test statistics for the null hypothesis that standard deviations are same is

$$Z = \frac{s_1 - s_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{192 - 224}{\sqrt{\frac{(192)^2}{1000} + \frac{(224)^2}{1000}}} = \frac{-32}{\sqrt{36.864 + 50.176}} = \frac{-32}{9.33} = -3.43.$$

Since $|Z| = 3.43 > 1.96$. Hence the null hypothesis is rejected and so the standard deviations are significantly different.

2.32 SAMPLING WITH SMALL SAMPLES

In large sample theory, the sampling distribution approaches a normal distribution. But in case of small size, the distributions of the various statistics like $Z = \frac{\bar{x} - \mu}{\sigma}\sqrt{n}$ are far from normality and as such normal test cannot be applied to such samples.

The problem of testing the significance of the deviation of a sample mean from a given population mean when sample size is small and only the sample variance is known was first solved by W.S. Gosset, who wrote under the pen-name "student." Later on R.A. Fisher modified the method given by Gosset. The test discovered by them is known as *Students Fisher t-test.*

Let $x_1, x_2, \ldots, x_n$ be a random small sample of size n drawn from a normal population with mean μ and variance σ. The statistics t is defined as

$$t = \frac{\overline{x} - \mu}{S}\sqrt{n},$$

where $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is the sample mean and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x}_i)^2$ is an unbiased estimate of the population variance σ^2. If we calculate t for each sample, we obtain a distribution for t, known as *Student Fisher t-distribution,* defined by

$$y = f(t) = C(1+t^2)^{-\frac{v+1}{2}},$$

where the parameter $v = n-1$ is called the *number of degrees of freedom* and C is a constant, depending upon v, such that the area under the curve is unity.

The curve $y = f(t)$ is symmetrical about y-axis like the normal curve. But it is more peaked than the normal curve with the same standard deviation. Further, this curve (Figure 2.10) approaches the horizontal t-axis less rapidly than the normal curve. If attains its maximum value at $t = 0$ and so its mode coincides with the mean.

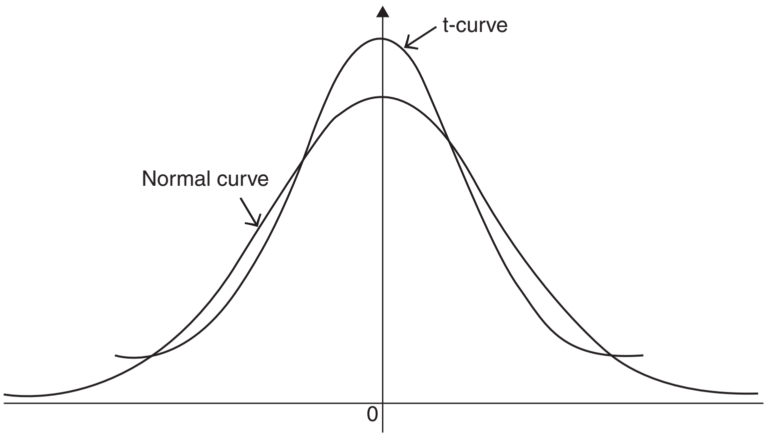


Figure 2.10

If $v \to \infty$, we have

$$y = Ce^{-\frac{t^2}{2}},$$

which is a normal curve. Hence t is normally distributed for large samples.

The probability p that the value of t will exceed t_0 is given by

$$P = \int_{t_0}^{\infty} y\,dx.$$

Fisher tabulated the values of t corresponding to various levels of significance for different values of v. For example for $v = 10$ and $p = 0.022$ we note that $t = 2.76$. Thus

$$P(t > 2.76) = P(t < -2.76) = 0.02$$

or

$$P(|t| > 2.76) = 0.02.$$

If the calculated value of t is greater than $t_{0.05}$ (the tabulated value), then the difference between $\bar{x}$ and μ is said to be significant at 5% level of significance. Similarly if $t > t_{0.01}$, then the difference between $\bar{x}$ and μ is said to be significant at 1% level of significance.

Since the probability P that $t > t_{0.05}$ is 0.95, the 95% confidence limits for μ are given by

$$\left| \frac{\bar{x} - \mu}{S} \sqrt{n} \right| \le t_{0.05}$$

or

$$|\bar{x} - \mu| \le \frac{S}{\sqrt{n}} t_{0.05}.$$

Thus 95% confidence interval for μ is

$$\left(\bar{x} - \frac{S}{\sqrt{n}} t_{0.05}, \bar{x} + \frac{S}{\sqrt{n}} t_{0.05} \right).$$

EXAMPLE 2.110

A random sample of 10 boys had the following IQ:

70, 120, 110, 101, 88, 83, 95, 98, 107, 100.

Do these data support the assumption of population mean IQ of 100 at 5% level of significance?

Solution. The statistics t is defined by

$$t = \frac{\bar{x} - \mu}{\frac{S}{\sqrt{n}}}$$

So, we first find $\bar{x}$ and S. we have

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i = \frac{972}{10} = 97.2.$$

To calculate S, we use the following table:

x:	70	120	110	101	88
$x - \bar{x}$:	–27.2	22.8	12.80	3.80	–9.2
$(x - \bar{x})^2$:	739.84	519.84	163.84	14.44	84.64

x:	83	95	98	107	100
$x - \bar{x}$:	–14.2	–2.2	0.8	9.80	2.80
$(x - \bar{x})^2$:	201.64	4.84	0.64	96.04	7.84

We have

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{10}(x - x_i)^2 = \frac{1}{9}(1833.96) = 203.773.$$

Therefore, $S = 14.275$. Then

$$t = \frac{97.2 - 100}{14.275}\sqrt{10} = -\frac{2.80}{14.275}(3.1623) = -0.620.$$

But

$$t_{0.05} = 2.26 \text{ for } v = 10 - 1 = 9.$$

Since $|t| = 0.62 < 2.26$, the value of t is not significant at 5% level of significance. Therefore, the data supports the population mean 100.
Further, 95% confidence interval is

$$\overline{x} \pm t_{0.05}\frac{S}{\sqrt{n}} = 97.2 \pm 2.26\left(\frac{14.275}{\sqrt{10}}\right) = 97.2 \pm 10.20 = (87, 107.4).$$

Since 100 lies within this interval, the data support the population mean.

EXAMPLE 2.111

A certain stimulus administered to each of 12 patients resulted in the following change in blood pressure:

5, 2, 8, –1, 3, 0, –2, 1, 5, 0, 4, 6.

Can it be concluded that the stimulus will increase the blood pressure?

Solution. The mean of sample is

$$\overline{x} = \frac{1}{12}\sum_{i=1}^{12} x_i = \frac{31}{12} = 2.583.$$

Therefore,

$$\begin{aligned}
S^2 &= \frac{1}{n-1}\sum_{i=1}^{12}(x_i - \overline{x})^2 \\
&= \frac{1}{11}[(5-2.583)^2 + (2-2.583)^2 + (8-2.583)^2 + (-1-12.583)^2 \\
&\quad + (3-2.583)^2 + (0-2.583)^2 + (-2-2.583)^2 + (1-2.583)^2 \\
&\quad + (5-2.583)^2 + (0-2.583)^2 + (4-2.583)^2 + (6-2.583)^2] \\
&= \frac{1}{11}[5.842 + 0.340 + 29.344 + 12.838 + 0.174 + 6.672 + 21.004 + 2.506 \\
&\quad + 5.842 + 6.672 + 2.008 + 11.676] \\
&= \frac{104.918}{11} = 9.538.
\end{aligned}$$

Therefore, $S = 3.088$ and so the 95% confidence interval for the mean is

$$\left(\bar{x} - \frac{S}{\sqrt{n}} t_{0.05}, \bar{x} + \frac{S}{\sqrt{n}} t_{0.05}\right)$$
$$= \left(2.583 - \frac{3.088(2.2)}{\sqrt{12}}, 2.583 + \frac{3.088(2.2)}{\sqrt{12}}\right)$$
$$= (2.583 - 1.910, 2.583 + 1.910) = (0.673, 4.493).$$

Since the average change in blood pressure of the population (μ) is positive, the stimulus will increase the blood pressure.

EXAMPLE 2.112

The measured lifetime of a sample of 15 electronic components gave an average of 750 hours with a sample standard deviation of 85 hours. Find a 95% confidence interval for the mean life time of the population and test the hypothesis that the mean is 810 hours.

Solution. We have

$$n = 15, \quad \bar{x} = 750, S = 85.$$

The table value $t_{0.05}$ for $v = 14$ is 2.14. Therefore, 95% confidence interval is

$$750 \pm \frac{2.14(85)}{\sqrt{15}} = (750 - 46.97, 750 + 46.97)$$
$$= (703.03, 796.97).$$

Since 810 is not included in this interval, the hypothesis that the mean is 810 hours is rejected at 5% significance level. The same conclusion is reached by evaluating the test statistics:

$$Z = \frac{\bar{x} - \mu}{S}\sqrt{n} = \frac{750 - 810}{85}\sqrt{15} = -\frac{60}{85}(3.873) = -2.73.$$

Since $|Z| = 2.73 > t_{0.05}$ (for $v = 14$), the difference is significant at 5% level of significance.

2.33 SIGNIFICANCE TEST OF DIFFERENCE BETWEEN SAMPLE MEANS

Let $x_1, x_2, \ldots, x_{n_1}$ and $y_1, y_2, \ldots, y_{n_2}$ and be two independent samples with means $\bar{x}$ and $\bar{y}$ and standard deviation S_1 and S_2, respectively, from a normal population with the same variance. The test hypothesis is that the means are the same. The test statistics is

$$t = \frac{\bar{x} - \bar{y}}{S\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}},$$

where

$$\bar{x} = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i, \ \bar{y} = \frac{1}{n_2}\sum_{i=1}^{n_2} y_i.$$
$$S^2 = \frac{1}{n_1 + n_2 - 2}[(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2]$$
$$= \frac{1}{n_1 + n_2 - 2}\left[\sum_{i=1}^{n_1}(x_i - \bar{x})^2 + \sum_{i=1}^{n_2}(y_1 - \bar{y})^2\right].$$

The variate t defined above follow the t-distribution with $n_1 + n_2 - 2$ degree of freedom.

If $t > t_{0.05}$, the difference between the sample means is significant at 5% level of significance. If $t < t_{0.05}$, the data is consistent with the hypothesis that the means are the same.

Similarly if $t > t_{0.01}$, the difference between the sample means is significant at 1% level of significance. If $t < t_{0.01}$, the data is consistent with the hypothesis that the means are the same.

If $n_1 = n_2$, that is, if the samples are of the same size and the data are paired, then the test statistics is given by

$$t = \frac{\bar{d}}{S}\sqrt{n},$$

where

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(d_i - \bar{d})^2, d_i = x_i - y_i, \bar{d} = \frac{\sum_{i=1}^{n} d_i}{n},$$

No. of degree of freedom $= n - 1$

EXAMPLE 2.113

A group of 10 boys fed on a diet A and another group of 8 boys fed on a different diet B, recorded the following increase in weights (in kg):

Diet A:	5	6	8	1	12	4	3	9	6	10
Diet B:	2	3	6	8	10	1	2	8		

Does it show the superiority of diet A over that of B?

Solution. We have

$$\bar{x} = \frac{1}{n_1}\sum_{i=1}^{n_1} x_i = \frac{1}{10}(64) = 6.4,$$

$$\bar{y} = \frac{1}{n_2}\sum_{i=1}^{n_2} y_2 = \frac{1}{8}(40) = 5.0,$$

$$S^2 = \frac{1}{n_1 + n_2 - 2}\left[\sum_{i=1}^{n_1}(x_i - \bar{x})^2 + \sum_{i=1}^{n_2}(y_i - \bar{y})^2\right]$$

$$= \frac{1}{16}\left[\sum_{i=1}^{10}(x_i - \bar{x})^2 + \sum_{i=1}^{8}(y_i - \bar{y})^2\right]$$

$$= \frac{1}{16}[(1.4)^2 + (0.4)^2 + (1.6)^2 + (5.4)^2 + (5.6)^2 + (2.4)^2 + (3.4)^2$$

$$+ (2.6)^2 + (0.4)^2 + (3.6)^2 + 3^2 + 2^2 + 1^2 + 3^2 + 5^2 + 4^2 + 3^2 + 3^2]$$

$$= \frac{1}{16}[1.96 + 0.16 + 2.56 + 2.32 + 3.14 + 5.76 + 11.56 + 6.76 + 0.16 + 12.96$$

$$+ 9 + 4 + 1 + 9 + 25 + 16 + 9 + 9] = \frac{129.34}{16},$$

which yields $S = 2.843$. Then the test statistics is

$$t = \frac{\bar{x} - \bar{y}}{S\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} = \frac{6.4 - 5.0}{2.843\sqrt{\left(\frac{1}{10} + \frac{1}{8}\right)}} = \frac{1.4}{2.843(0.474)} = 1.038.$$

From the table, $t_{0.05}$ for $v = n_1 + n_2 - 2 = 16$ is 2.12. Since calculated t is less than $t_{0.05}$, we conclude that the difference between sample mean is not significant. Hence, there is no superiority of diet A over the diet B.

EXAMPLE 2.114

Eleven school boys were given a test in drawing. They were given a month's further tuition and a second test of equal difficulty was held at the end of the month. Do the marks give evidence that the students have been benefited by extra coaching?

Marks in 1st test:	23	20	19	21	18	20	18	17	23	16	19
Marks in 2nd test:	24	19	22	18	20	22	20	20	23	20	17

Solution. We have $n_1 = n_2 = 11$. Representing marks in second test by x_i and that of first test by y_i we have the differences $d_i = x_i - y_i$ as

$$1, \quad -1, \quad 3, \quad -3, \quad 2, \quad 2, \quad 2, \quad 3, \quad 0, \quad 4, \quad -2.$$

Therefore,

$$\bar{d} = \frac{\sum d_i}{n} = \frac{\sum (x_i - y_i)}{11} = \frac{11}{11} = 1,$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{11}(d_i - \bar{d})^2$$

$$= \frac{1}{10}[0^2 + (-2)^2 + 2^2 + (-4)^2 + 1^2 + 1^2 + 1^2 + 2^2 + (-1)^2 + 3^2 + (-3)^2] = 5$$

and so $\quad S = \sqrt{5} = 2.24.$

The test statistics for equal sample means is

$$t = \frac{\bar{d}}{\frac{S}{\sqrt{n}}} = \frac{1}{2.24}\sqrt{11} = 1.481.$$

The tabular value of $t_{0.05}$ for $v = 10$ is 2.228. Thus the calculated value of t is less than $t_{0.05}$.

Therefore, the hypothesis that the mean are same, is accepted. Hence, the data provides no evidence that the students have benefited by extra coaching.

EXAMPLE 2.115

A group of boys and girls were given an intelligent test. The mean score, standard deviations, and number in each group are as follows:

	Boys	Girls
Mean	124	121
S:D	12	10
N	18	14

Is the mean score of boys significantly different from that of girls?

Solution. We have

$$n_1 = 18, \qquad n_2 = 14,$$
$$S_1 = 12, \qquad S_2 = 10,$$
$$\bar{x} = 124, \qquad \bar{y} = 121.$$

Therefore,

$$S^2 = \frac{1}{n_1+n_2-2}[(n_1-1)S_1^2+(n_2-2)S_2^2] = \frac{1}{20}[17(144)+13(100)] = 187.40$$

and so $S = 13.69$. Therefore, the test statistics for the hypothesis that the mean are same is

$$t = \frac{\bar{x}-\bar{y}}{S\sqrt{\left(\frac{1}{18}+\frac{1}{14}\right)}} = \frac{124-121}{13.69(0.350)} = \frac{3}{4.792} = 0.626.$$

From the table, $t_{0.05}$ for $v = n_1 + n_2 - 2 = 20$ is 2.09. Since calculated value of t is less than the tubular value of $t_{0.05}$ for $v = 20$, the difference in mean is not significant.

EXAMPLE 2.116

A manufacturer claims that the lifetime of a particular electronic component is unaffected by temperature variation within the range 0–60° *C*. Two samples of these components were tested and their measured lifetimes are (in hours) recorded as follows:

0°*C*	7050	6970	7370	7910	6790	6850	7280	7830
60°*C*	7030	7270	6510	6700	7350	6770	6220	7230

Solution. The sample sizes are equal, that is, $n_1 = n_2 = 8$. Representing the lifetimes at 0°*C* by x_i and the lifetimes at 60°*C* by y_i, we get the differences $d_i = x_i - y_i$ as

20, –300, 860, 1210, –560, 80, 1060, 600.

Therefore,

$$\bar{d} = \frac{\sum d_i}{n} = \frac{2970}{8} = 371.25,$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{8}(d_i-\bar{d})^2$$

$$= \frac{1}{7}[123376.56+450576.56+238876.56+703501.56+867226.56+84826.56$$
$$+474376.56+52326.56] = 427869.56$$

and so $S = 654.12$. The test statistics for equal sample mean is

$$t = \frac{\bar{d}}{S}\sqrt{n} = \frac{371.25}{654.12}\sqrt{8} = 1.61.$$

The tabular value of $t_{0.05}$ for $v = 7$ is 2.36. Since the calculated values of t is less than $t_{0.05}$, the difference in the mean is not significant at 5% level of significance. Hence, the manufacture claims is accepted at 5% level of significance.

If we calculate the 95% confidence interval, we get

$$\bar{d} \pm 2.36\left(\frac{S}{\sqrt{8}}\right) = \bar{d} \pm \frac{2.36(654.12)}{\sqrt{8}}$$
$$= (371.25-545.87, 371.25+545.82)$$
$$= (-174.62, 917.12).$$

Since zero lies within the 95% confidence interval, the difference in mean is not significant and so the manufacturer's claim is accepted.

EXAMPLE 2.117

Two kinds of photographic films were tested for sharpness of definition in the same camera under varying conditions. Each pair of readings given below was produced under the same conditions except for difference of film. Is there any unusual difference between the sharpness of the definition of the two films?

Film X:	27	30	30	32	24	26	40	35
Film Y:	25	28	30	30	27	28	37	28

Solution. The sample sizes are $n_1 = n_2 = 8$. The null and alternative hypotheses are

H_0: Mean μ of the population of difference is zero

H_1: Mean μ of the population of difference is not zero.

We shall test under 5% level of significance. We have

$$\bar{d} = \frac{\sum d_i}{n} = \frac{\sum (x_i - y_i)}{n} = \frac{2+2+0+2-3-2+3+7}{8} = 1.38$$

and

$$S^2 = \frac{\sum d_i^2}{n} - \left(\bar{d}\right)^2 = \frac{4+4+0+4+9+4+9+49}{8} - (1.38)^2 = 8.47.$$

Thus S = 2.910. Therefore, test statistics is given by

$$t = \frac{\bar{d} - \mu}{S}\sqrt{n} = \frac{1.38}{2.91}\sqrt{8} = 1.34.$$

From the table, for $v = 7$, we have $t_{0.005} = 2.36$. Thus, the calculated value of t is less than the tabulated $t_{0.005}$. Therefore, the difference is not significant at 5% level of confidence. Hence H_0 is accepted and consequently there is no unusual difference between the sharpness of definitions of the two films.

2.34 CHI-SQUARE DISTRIBUTION

Let f_{o_i} and f_{e_i} be the observed and expected frequencies of a class interval, then χ^2 is defined by the relation

$$\chi^2 = \sum_{i=1}^{n} \frac{\left(f_{o_i} - f_{e_i}\right)^2}{f_{e_i}}.$$

where summation extends to all class intervals.

Note that χ^2 describes the magnitude of discrepancy between the observed and expected frequencies.

For large sample sizes, the sampling distribution of χ^2 can be closely approximated by a continuous curve known as χ^2-distribution. Thus χ^2-distribution is defined by means of the function

$$y = C\, e^{-\frac{\chi^2}{2}} (\chi^2)^{\frac{v-1}{2}},$$

where v is the degree of freedom and C is a constant. In the case of binomial distribution, the degree of freedom is $n - 1$. In case of Poisson distribution, the degree of freedom is $n - 2$ whereas in case of normal distribution, the degree of freedom is $n - 3$ In fact, if we have $s \times t$ contingency table, then the degree of freedom is $(s-1)(t-1)$. If $v = 1$, the χ^2-curve reduces to $y = C\,e^{-\frac{\chi^2}{2}}$, which is right half of a normal curve as shown in Figure 2.11.

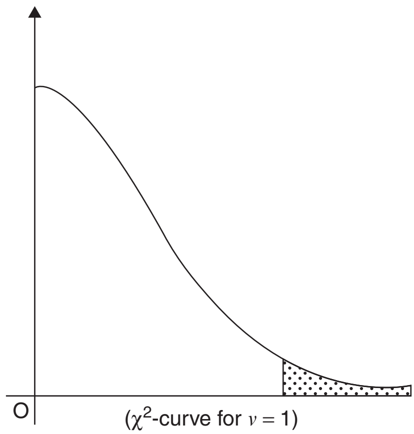


Figure 2.11

If $v > 1$, the χ^2-curve is tangential to the x-axis at the origin, as shown in Figure 2.12.

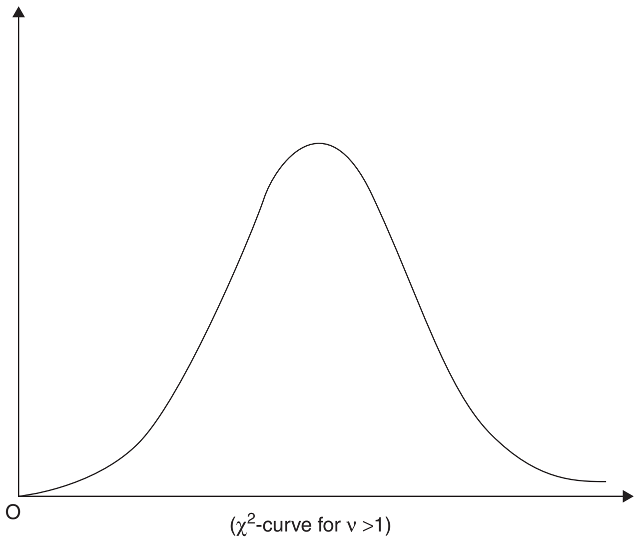


Figure 2.12

As v increases, the curve becomes more symmetrical. If $v > 30$, the χ^2-curve approximates to the normal curve and in such case the sample is of large size and we should refer to normal distribution table.

The probability P that the value of χ^2 from a random sample will exceed χ_0^2 is given by

$$P = \int_{\chi_0^2}^{\infty} y\,dx.$$

The values of χ^2 for degree of freedom from $v = 1$ to $v = 30$ have been tabulated for various convenient probability values. The table yields the values for the probability P that χ^2 exceeds a given value, χ_0^2.

We observe that the χ^2-test depends only on the set of observed and expected frequencies and on the degree of freedom. The χ^2-curve does not involve any parameter of the population and so the χ^2-distribution? does not depend on the form of the population. That is why, χ^2-test is called *non-parametric test* or *distribution-free test.*

2.35 χ^2-TEST AS A TEST OF GOODNESS-OF-FIT

The χ^2-test is used to test whether the deviation of the observed frequencies from the expected (theoretical) frequencies are significant or not. Thus, this test tells us how a set of observations fits a given distribution. Hence χ^2-test provides a test of goodness-of-fit for Binomial distribution, Poisson distribution, Normal distribution, etc. If the calculated values of χ^2 is greater than the tabular value, the fit is considered to be poor.

To apply χ^2-test, we first calculate χ^2. Then consulting χ^2-table, we find the probability P corresponding to this calculated value of χ^2 for the given degree of freedom. If

(i) $P < 0.005$, the observed value of χ^2 is significant at 5% level of significance
(ii) $P < 0.01$, the observed value of χ^2 is significant at 1% level of significance
(iii) $P > 0.05$, it is good fit and the value of χ^2 is not significant.

This mean that we accept the hypothesis if calculated χ^2 is less than the tabulated value, otherwise reject it.

Conditions for the validity of χ^2-test: In 2.34, we pointed out that χ^2-test is used for large sample size. For the validity of χ^2-test as a test of goodness-of-fit regarding significance of the deviation of the observed frequencies from the expected (theoretical) frequencies, the following conditions must be satisfied:

(i) The sample observations should be independent.
(ii) The total frequency (the sum of the observed frequencies or the sum of expected frequency) should be larger than 50.
(iii) No theoretical frequency should be less than 5 because χ^2-distribution cannot maintain continuity character if frequency is less than 5.
(iv) Constraints on the frequencies, if any, should be linear.

EXAMPLE 2.118

Fit a binomial distribution to the data

x :	0	1	2	3	4	5
y :	38	144	342	287	164	25

and test for goodness-of-fit at the level of significance 0.05.

Solution. We have $n = 5$, $\Sigma f_i = 1000$. Therefore,

$$\mu = \frac{\sum f_i x_i}{\sum f_i} = \frac{0+144+684+861+656+125}{1000} = 2.470.$$

But, for a binomial distribution,

$$\mu = np \text{ and so } p = \frac{\mu}{n} = \frac{2.470}{5} = 0.494,\ q = 1 - p = 0.506.$$

Therefore, the binomial distribution to be fitted is

$$1000(0.506+0.494)^5 = 1000[{}^5C_0(0.506)^5 + {}^5C_1(0.506)^4(0.494) + {}^5C_2(0.506)^3(0.494)^2 + {}^5C_3(0.506)^2(0.494)^3 + {}^5C_4(0.506)(0.494)^4 + {}^5C_5(0.494)^5]$$

$$= 1000\,[0.0332+0.1619+0.3161+0.3086+0.1507+0.02942]$$

$$= 33.2+161.9+316.1+308.6+150.7+29.42.$$

Thus the theoretical frequencies are

x:	0	1	2	3	4	5
y:	33.2	161.9	316.1	308.6	150.7	29.42

Therefore,

$$\chi^2 = \frac{(38-33.2)^2}{33.2} + \frac{(144-161.9)^2}{161.9} + \frac{(342-316.1)^2}{316.1} + \frac{(287-308.6)^2}{308.6} + \frac{(164-150.7)^2}{150.7} + \frac{(25-29.42)^2}{29.42}$$

$$= 0.6940+1.9791+2.1222+1.5119+1.1738+0.6640 = 8.145.$$

The number of degree of freedom is 6 – 1 = 5. For $v=5$, $\chi^2_{0.05} = 11.07$. Thus the calculated value of χ^2 is less than $\chi^2_{0.05}$ and so the binomial distribution gives a good fit at 5% level of significance.

EXAMPLE 2.119

The following table gives the frequency of occupancy of digits 0, 1, 2,…,9 in the last place in four logarithms of numbers 10–99. Examine if there is any peculiarity.

Digits:	0	1	2	3	4	5	6	7	8	9
Frequency:	6	16	15	10	12	12	3	2	9	5

Solution. Let the null hypothesis be

H_0: frequency of occupance of digits is equal, that is, there is no significant difference between the observed and the expected frequency.

Therefore under the null hypothesis, the expected frequency is $f_e = \frac{90}{10} = 9$. Then

$$\chi^2 = \frac{\sum(f_{o_i} - f_{e_i})^2}{f_{e_i}} = \frac{9+49+36+1+9+9+36+49+0+16}{9} = 23.777.$$

Number of degree of freedom is 10 –1 = 9. The tabulated value of $\chi^2_{0.05}$ for $v = 9$ is 16.92. Since the calculated value of χ^2 is greater than the tabulated value of $\chi^2_{0.05}$, the hypothesis is rejected and so there is a significant difference between the observed and expected frequency.

EXAMPLE 2.120

In a locality, 100 persons were randomly selected and asked about their academic qualifications. The results are as given below:

	Education			
Sex	Middle standard	High school	Graduation	Total
Male:	10	15	25	50
Female:	25	10	15	50
Total	**35**	**25**	**40**	**100**

Can you say that education depends on sex?

Solution. Let the null hypothesis be

H_0: Education does not depend on sex.
On this hypothesis the expected frequencies are (taking averages).

Sex	Middle standard	High school	Graduation	Total
Male:	17.5	12.5	20	50
Female:	17.5	12.5	20	50
Total	**35**	**25**	**40**	**100**

Therefore,

$$\chi^2 = \frac{(10-17.5)^2}{-17.5} + \frac{(15-12.5)^2}{12.5} + \frac{(25-20)^2}{20} + \frac{(25-17.5)^2}{17.5} + \frac{(10-12.5)^2}{12.5} + \frac{(15-20)^2}{20}$$
$$= 9.93.$$

Further, the number of degree of freedom (v) = $(s-1)(t-1) = (3-1)(2-1) = 2$.

From χ^2-table, $\chi^2_{0.05}$ for $v = 2$ is 5.99. Thus the calculated value of χ^2 is greater than the tabulated value of χ^2. Hence H_0 is rejected and so the education depends on sex.

EXAMPLE 2.121

Fit a Poisson distribution to the following data and test for its goodness-of-fit at 5% level of significance.

x:	0	1	2	3	4
f:	419	352	154	56	19

Solution. If the given distribution is approximated by a Poisson distribution, then the parameter of the Poisson distribution is given by

$$\lambda = \frac{\sum f_i x_i}{\sum f_i} = \frac{0+352+308+168+76}{1000} = 0.904.$$

Therefore, the theoretical frequencies are

$$1000\,e^{-\lambda}, 1000\lambda e^{-\lambda}, 1000\frac{\lambda^2}{2}e^{-\lambda}, 1000\frac{\lambda^3}{3!}e^{-\lambda}, 1000\frac{\lambda^4}{4!}e^{-\lambda}.$$

Also $e^{-\lambda} = e^{-0.904} = 0.4049$. Therefore, the theoretical frequencies are

x:	0	1	2	3	4	Total
f:	404:9	366	165:4	49:8	11:3	997:4
	406				12:8	

To make the total of frequencies 1000, we take the first frequency as 406 and the last frequency as 12.8. Then

$$\chi^2 = \frac{(419-406)^2}{406} + \frac{(352-366)^2}{366} + \frac{(154-165.4)^2}{165.4} + \frac{(56-49.8)^2}{49.8} + \frac{(19-12.8)^2}{12.8}$$
$$= 0.416 + 0.536 + 0.786 + 0.772 + 3.003 = 5.513.$$

The number of degree of freedom in case of Poisson distribution is $n - 2 = 5 - 2 = 3$. Therefore, the tabular value of χ^2 for $v = 3$ is 7.82. Thus the calculated value of χ^2 is less than the tabulated value of $\chi^2_{0.05}$. Therefore, the Poisson distribution provides a good fit to the data.

EXAMPLE 2.122

Obtain the equation of the normal curve that may be fitted to the data given below and test the goodness-of-fit.

x:	4	6	8	10	12	14	16	18	20	22	24
y:	1	7	15	22	35	43	38	20	13	5	1

Solution. For the given data, we have

x	x^2	f	fx	fx^2
4	16	1	4	16
6	36	7	42	252
8	64	15	120	960
10	100	22	220	2200
12	144	35	420	5040
14	196	43	602	8428
16	256	38	608	9728
18	324	20	360	6480
20	400	13	260	5200
22	484	5	110	2420
24	576	1	24	576
		200	**2770**	**41300**

Therefore,

$$\text{Mean } (\mu) = \frac{\sum fx}{\sum f} = \frac{2770}{200} = 13.85.$$

$$\text{Standard deviation } (\sigma) = \sqrt{\frac{\sum fx^2}{\sum f} - \mu^2} = \sqrt{\frac{41300}{200} - (13.85)^2} = \sqrt{14.678} = 3.8311.$$

Hence, the equation of the normal curve fitted to the given data is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{1}{13.85\sqrt{2\pi}} e^{-\frac{1}{29.36}(x-13.85)^2}.$$

To calculate the theoretical normal frequencies, we note that the area under $f(x)$ in (z_1, z_2) is

$$\Delta\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^{z_2} e^{-\frac{z^2}{2}} dz - \frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-\frac{z^2}{2}} dz,$$

where $z = \frac{x-\mu}{\sigma} = \frac{x - 13.85}{3.83}$. Thus, the expected normal frequencies are given by

Class interval	Mid-value	(z_1, z_2)	$\Delta\Phi(z)$	Expected frequency $N\Delta\Phi(z)$
3–5	4	(–2.83, –2.31)	0.4977 – 0.4896 = 0.0081	200(0.0081) = 1.62
5–7	6	(–2.83, –1.79)	0.4896 – 0.4633 = 0.0263	200(0.0263) = 5.26
7–9	8	(–1.79, –1.27)	0.4633 – 0.3980 = 0.0653	200(0.0653) = 13.06
9–11	10	(–1.27, –0.74)	0.3980 – 0.2704 = 0.1276	200(0.1276) = 25.52
11–13	12	(–0.74, –0.22)	0.2704 – 0.0871 = 0.1833	200(0.1833) = 36.66
13–15	14	(–0.22, 0:30)	0.1179) 0.0871 = 0.2050	200(0.2050) = 41.00
15–17	16	(0.30, 0.82)	0.2939 – 0.1179 = 0.1760	200(0.1760) = 35.20
17–19	18	(0.82, 1.34)	0.4099 – 0.2939 = 0.1160	200(0.1160) = 23.20
19–21	20	(1.34, 1.86)	0.4686 – 0.4099 = 0.0587	200(0.0587) = 11.74
21–23	22	(1.86, 2.38)	0.4913 – 0.4686 = 0.0227	200(0.0227) = 4.54
23–25	24	(2.38, 2.91)	0.4982 – 0.4913 = 0.0069	200(0.0069) = 1.38

Therefore,

$$\chi^2 = \frac{(1-1.62)^2}{1.62} + \frac{(7-5.26)^2}{5.26} + \frac{(15-13.06)^2}{13.06} + \frac{(22-25.52)^2}{25.52}$$
$$+\frac{(35-36.66)^2}{36.66} + \frac{(43-41)^2}{41} + \frac{(38-35.20)^2}{35.20}$$
$$+\frac{(20-23.20)^2}{23.20} + \frac{(13-11.74)^2}{11.74} + \frac{(5-4.54)^2}{4.54} + \frac{(1-1.38)^2}{1.38}$$
$$= 0.0912 + 0.5756 + 0.2882 + 0.4855 + 0.0752 + 0.0976$$
$$+ 0.2227 + 0.4414 + 0.1352 + 0.0466 + 0.1046 = 2.56.$$

The number of degree of freedom is $n - 3 = 11 - 3 = 8$ and $\chi^2_{0.005}$ at $v = 8$ is 15.51. Therefore, the normal distribution provides a good fit.

2.36 SNEDECOR'S F-DISTRIBUTION

Let $x_1, x_2, \ldots x_{n_1}$ and $y_1, y_2, \ldots y_{n2}$ be the values of two independent random samples drawn from two normal populations with equal variance σ^2. Let $\overline{x}$ and $\overline{y}$ be the sample means and let

$$S_1^2 = \frac{1}{n_1 - 1}\sum_{i=1}^{n_1}(x_i - \overline{x})^2,$$

$$S_2^2 = \frac{1}{n_2 - 1}\sum_{i=1}^{n_2}(y_i - \overline{y})^2.$$

Then we define the statistics F by the relation

$$F = \frac{S_1^2}{S_2^2}.$$

The Snedecor's F-distribution is defined by the function

$$y = C\,F^{\frac{v_1-2}{2}}\left(1+\frac{v_1}{v_2}F\right)^{-\frac{v_1+v_2}{2}},$$

where the constant C depends on v_1 and v_2 and is so chosen that area under the curve is unity. The F-distribution is independent of the population variance σ^2 and depends only on v_1 and v_2, the numbers of degree of freedom of the samples. The F-curve is bell-shaped for $v_1 > 2$, as shown in Figure 2.13.

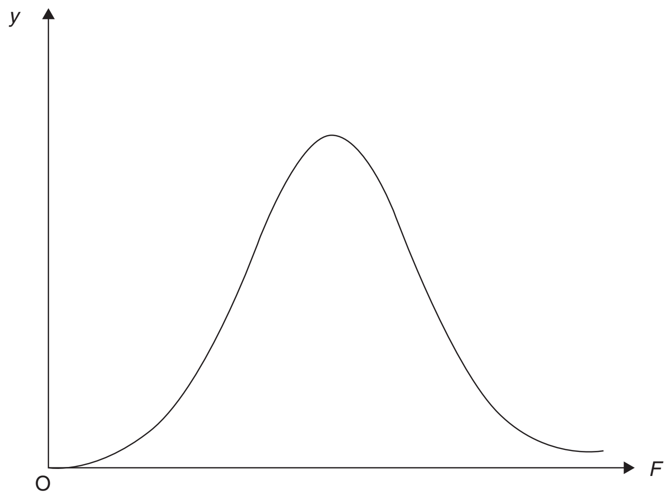


Figure 2.13

Significant test is performed by means of Snedecor's F-tables which provides 5% and 1% of points of significance for F. Five percent points of F means that area under the F-curve, to the right of the ordinate at a value of F is 0.05. Further the F-tables give only single tail test. However, if we are testing the hypothesis that the population variances are same, then we should use both tail areas under the F-curve and in that case F-table will provide 10% and 2% levels of significance.

2.37 FISHER'S Z-DISTRIBUTION

Putting $F = e^{2z}$ in the F-distribution, we get

$$y = Ce^{v_1 z}(v_1 e^{2z} + v_2),$$

which is called the *Fisher's z-distribution*, where C is a constant depending upon v_1 and v_2 such that area under the curve is unity. The curve for this distribution is more symmetrical than F-distribution.

Significance test are performed from the z-table in a similar way as in the case of F-distribution.

EXAMPLE 2.123

In testing for percent of ash content, 17 tests from one shipment of coal shows $S^2 = 7.08$ percent and 21 tests from a second shipment shows $S^2 = 20.70$. Can these samples be regarded as drawn from the same shipment?

Solution. We have $n_1 = 21$, $n_2 = 17$, $S_1^2 = 20.70$ and $S_2^2 = 7.08$. Therefore, the test statistics is

$$F(v_1, v_2) = F(20,16) = \frac{20.70}{7.08} = 2.92.$$

From the F-table, we have $F_{0.05}$ (20, 16) = 2.18. Since $F(v_1, v_2)$ is greater than $F_{0.05}$, the population variances are significantly different.

EXAMPLE 2.124

Two independent samples of sizes 7 and 6 have the following values:

Sample A:	28	30	32	33	33	29	34
Sample B:	29	30	30	24	27	29	

Examine whether the samples have been drawn from normal populations having the same variance.

Solution. The means for the sample A and B are, respectively

$$\overline{x} = \frac{219}{7} = 31.285 \quad \text{and} \quad \overline{y} = \frac{169}{6} = 28.166.$$

Then

$$\begin{aligned} S_1^2 &= \frac{1}{n_1 - 1}\sum (x_i - \overline{x})^2 \\ &= \frac{1}{6}[(28-31.285)^2 + (30-31.285)^2 + (32-31.285)^2 + (33-31.285)^2 \\ &\quad + (33-31.285)^2 + (29-31.285)^2 + (34-31.285)^2] \\ &= \frac{1}{6}[10.791 + 1.651 + 0.511 + 2.941 + 2.941 + 5.221 + 7.371] \\ &= 5.238 \end{aligned}$$

and

$$\begin{aligned} S_2^2 &= \frac{1}{n_2 - 1}\sum (y_i - \overline{y})^2 \\ &= \frac{1}{5}[(29-28.166)^2 + (30-28.166)^2 + (30-28.166)^2 + (24-28.166)^2 \\ &\quad + (27-28.166)^2 + (29-28.166)^2] \\ &= \frac{1}{5}[0.695 + 3.364 + 3.364 + 17.355 + 1.359 + 0.695] \\ &= 5.366. \end{aligned}$$

Therefore, the test statistics is given by

$$F = \frac{S_1^2}{S_2^2} = \frac{5.238}{5.366} = 0.976.$$

Further, since numbers of degree of freedom are 6 and 5, we have

$$F_{0.05}\ (6,5) = 4.95.$$

Thus, the calculated value of F is less than the tabular value. Hence the samples have been drawn from normal population having the same variance.

EXAMPLE 2.125

Two samples of sizes 9 and 8 give the sum of squares of deviations from their respective means equal to 160 and 91, respectively. Examine, whether the samples have been drawn from normal population having the same variance.

Solution. We have

$$\sum_{i=1}^{9}(x_i-\bar{x})^2=160 \quad \text{and} \quad \sum_{i=1}^{8}(y_i-\bar{y})^2=91.$$

Therefore, their variances are

$$S_1^2=\frac{1}{8}(160)=20, \quad \text{and} \quad S_2^2=\frac{1}{7}(91)=13.$$

Their test statistics for F-test is

$$F=\frac{S_1^2}{S_2^2}=\frac{20}{13}=1.54.$$

From the F-table, we have

$$F_{0.05}\ (8,7)=3.73.$$

Since the calculated value of F is less than $F_{0.05}$ (8,7), the population variances are not significantly different. So the samples can be regarded as drawn from the populations having the same variance.

EXAMPLE 2.126

The nicotine content (in mg) of two samples of tobacco were found to be as follows:

Sample A:	24	27	26	21	25	
Sample B:	27	30	28	31	22	36

Can it be said that the two samples came from the same population?

Solution. Suppose that $\bar{x}$ be the sample mean for the sample B and $\bar{y}$ be the sample mean of the sample A. Then

$$\bar{x}=\frac{174}{6}=29 \text{ and } \bar{y}=\frac{123}{5}=24.6$$

$$S_1^2=\frac{1}{n_1-1}\sum(x_1-\bar{x})^2$$

$$=\frac{1}{5}[(27-29)^2+(30-29)^2+(28-29)^2+(31-29)^2+(22-29)^2+(36-29)^2]$$

$$=\frac{1}{5}[4+1+1+4+49+49]=21.6,$$

$$S_2^2=\frac{1}{n_2-1}\sum(y_1-\bar{y})^2$$

$$=\frac{1}{4}[(24-24.6)^2+(27-24.6)^2+(26-24.6)^2+(21-24.6)^2+(25-24.6)^2]$$

$$=\frac{1}{4}[0.36+5.76+1.96+12.96+0.16]=5.3.$$

Therefore, the statistics for F-test is

$$F=\frac{S_1^2}{S_2^2}=\frac{21.6}{5.3}=4.08.$$

But tabular value of $F_{0.05}$ (5,4) is 6.26. The calculated value of F is less than the tabular value. So there is no significant difference. Hence the two samples may be considered to come from the same population.

2.38 ANALYSIS OF VARIANCE (ANOVA)

Fisher developed a powerful statistical method to test the significance of the difference between more than two sample means and to infer whether samples are drawn from the populations having the same mean. This method is known as *Analysis of Variance*.

Consider a set of N values of the variate x arranged in r rows and k columns. Let

(i) x_{ij} represent the value of the entry in the ith row and jth column.
(ii) $\overline{x}_{..}$ represent the general mean of x_{ij} , $1 \le i \le r$, $1 \le j \le k$.
(iii) $\overline{x}_{.j}$ represent the mean of the jth column.
(iv) $\overline{x}_{i.}$ represent the mean of the ith row.

Then

$$V = \sum_i \sum_j (x_{ij} - \overline{x}_{..})^2$$

$$= \sum_i \sum_j x_{ij}^2 - 2\overline{x}_{..} \sum_i \sum_j x_{ij} + N\overline{x}_{..}^2$$

$$= \sum_i \sum_j x_{ij}^2 - \frac{2}{N}\left(\sum_i \sum_j x_{ij}\right)^2 + \frac{N}{N^2}\left(\sum_i \sum_j x_{ij}\right)^2$$

$$= \sum_i \sum_j x_{ij}^2 - \frac{1}{N}\left(\sum_i \sum_j x_{ij}\right)^2$$

is called *"Overall" sum of squares* or *total variation. The* quantity $\frac{1}{N}\left(\sum_i \sum_j x_{ij}\right)^2$ is called the *correction term.*

Further,

$$\overline{x}_{.j} = \frac{1}{r}\sum_{i=1}^{r} x_{ij} .$$

Therefore

$$\sum_{j=1}^{k} \left(\overline{x}_{.j} - \overline{x}_{..}\right)^2 = \sum_{j=1}^{k} \left[\overline{x}_{.j}^2 + \overline{x}_{..}^2 - 2\overline{x}_{.j}\overline{x}_{..}\right]$$

$$= \sum_{j=1}^{k} \overline{x}_{.j}^2 + \sum_{j=1}^{k} \overline{x}_{..}^2 - 2\sum_{j=1}^{k} \overline{x}_{.j}\overline{x}_{..}$$

$$= \frac{1}{r^2}\sum_{j=1}^{k} \left(\sum_{i=1}^{r} x_{ij}\right)^2 + \sum_{j=1}^{k} \overline{x}_{..}^2$$

$$-2\sum_{j=1}^{k} \left[\frac{1}{r}\sum_{i=1}^{r} x_{ij}\right]\overline{x}_{..}$$

$$= \frac{1}{r^2}\sum_{j=1}^{k} \left(\sum_{i=1}^{r} x_{ij}\right)^2 + \frac{k}{N^2}\left(\sum_i \sum_j x_{ij}\right)^2$$

$$-\frac{2}{r}\left[\sum_{j=1}^{k} \sum_{i=1}^{r} x_{ij}\right]\left[\frac{1}{N}\sum_i \sum_j x_{ij}\right]$$

$$= \frac{1}{r^2}\sum_{j=1}^{k}\left(\sum_{i=1}^{r} x_{ij}\right)^2 + \frac{k}{(rk)^2}\left(\sum_i \sum_j x_{ij}\right)^2$$

$$- \frac{2}{r(rk)}\left[\sum_i \sum_j x_{ij}\right]^2$$

$$= \frac{1}{r^2}\sum_{j=1}^{k}\left(\sum_{i=1}^{r} x_{ij}\right)^2 - \frac{1}{rN}\left[\sum_i \sum_j x_{ij}\right]^2.$$

Then

$$V_c = r\sum_{j=1}^{k} (\overline{x}_{.j} - \overline{x}_{..})^2$$

$$= \frac{1}{r}\sum_{j=1}^{k}\left(\sum_{i=1}^{r} x_{ij}\right)^2 - \frac{1}{N}\left[\sum_i \sum_j x_{ij}\right]^2$$

is called *"between column sum of squares"* or *between column variation.* Thus *"between column sum of squares is* $\frac{1}{r}$ *th the sum of the squares of the column sums minus the correction term"*.

Further,

$$R = \sum_i \sum_j x_{ij}^2 - \frac{1}{r}\sum_{j=1}^{k}\left(\sum_{i=1}^{r} x_{ij}\right)^2$$

is called the *"within column sum of squares" or residual.* We notice that

$$V = V_c + R.$$

Thus, we say that the total variation has two components : a between sample variation and a residual variation.

The degree of freedom for V_c is k – 1, whereas the degree of freedom for R is N – k. Thus the sum of degrees of freedom for V_c and R is N – 1, which is the degree of freedom for the total variation. Therefore estimated variance is defined as

$$F = \frac{\frac{V_c}{k-1}}{\frac{R}{N-k}}$$

with $n_1 = k-1$ and $n_2 = N-k$. For significance at 5% level, we compare it with $F_{0.05}(n_1, n_2)$ from the F – distribution table.

EXAMPLE 2.127

Yields of 4 varieties of wheat in 3 blocks are given below:

Block \ Variety	I	II	III	IV
1	10	7	8	5
2	9	7	5	4
3	8	6	4	4

Is the difference between varieties significant?

Solution. We have

Block \ Variety	I	II	III	IV
1	10	7	8	5
2	9	7	5	4
3	8	6	4	4
$\sum x$	**27**	**20**	**17**	**13**
$\sum x^2$	**245**	**134**	**105**	**57**
$\left(\sum x\right)^2$	**729**	**400**	**289**	**169**

Then

$$V_C = \frac{1}{r}\sum_{j=1}^{k}\left(\sum_{i=1}^{r} x_{ij}\right)^2 - \frac{1}{n}\left(\sum_i \sum_j x_{ij}\right)^2$$

$$= \frac{1}{3}[729+400+289+169] - \frac{1}{12}(77)^2$$

$$= 529 - 494.08 = 34.92$$

and

$$R = (245+134+105+57) - 529 = 12\,.$$

Therefore estimated variance is

$$F = \frac{\dfrac{V_c}{k-1}}{\dfrac{R}{N-k}} = \frac{\dfrac{34.92}{3}}{\dfrac{12}{8}} = \frac{11.4}{1.5} = 7.6\,.$$

But

$$F_{0.05}(x_1, x_2) = F_{0.05}(3,8) = 4.07.$$

Therefore difference between the varieties is highly significant.

EXAMPLE 2.128

Three groups of 4 rats each were injected with commercial Intocostrin and the number of minutes that elapsed before a reaction took place were recorded with the following results. Is the difference between groups significant?

	A	*B*	*C*
1	11	12	2
2	8	10	5
3	7	17	2
4	6	7	7

Solution. We have

	A	B	C
1	11	12	2
2	8	10	5
3	7	17	2
4	6	7	7
$\sum x$	**32**	**36**	**16**
$\sum x^2$	**270**	**582**	**82**
$\left(\sum x\right)^2$	**1024**	**1296**	**256**

Therefore

$$V_c = \frac{1}{4}[1024+1296+256] - \frac{1}{12}(84)^2$$

$$= 644 - 588 = 56$$

and

$$R = (270+582+82) - 644 = 290\,.$$

Therefore

$$F = \frac{\dfrac{V_c}{k-1}}{\dfrac{R}{N-k}} = \frac{\dfrac{56}{2}}{\dfrac{290}{9}} = \frac{252}{290} = 0.89.$$

But

$$F_{0.05}(n_1, n_2) = F_{0.05}(2,9) = 4.26.$$

Hence there is no significant difference between the groups and the sample is taken from the same population.

EXAMPLE 2.129

The result of testing the lifetime of three electric bulbs, measured in hundreds of hours, of each of the four brands A, B, C, D is shown below. Can we infer that there is no significant difference between the lifetime of different brands of bulbs?

Brand			
A	B	C	D
20	25	24	23
19	23	20	20
21	21	22	20

Solution. We make the following table:

A	B	C	D
20	25	24	23
19	23	20	20
21	21	22	20
$\sum x$ **60**	**69**	**66**	**63**
$\sum x^2$ **1202**	**1595**	**1460**	**1329**
$(\sum x)^2$ **3600**	**4761**	**4356**	**3969**

Therefore "between columns Variance" is given by

$$V_c = \frac{1}{r}\sum_{j=1}^{k}\left(\sum_{i=1}^{r} x_{ij}\right)^2 - \frac{1}{N}\left[\sum_i \sum_j x_{ij}\right]^2$$

$$= \frac{1}{3}[3600 + 4761 + 4356 + 3969] - \frac{1}{12}[60 + 69 + 66 + 63]^2$$

$$= \frac{16686}{3} - \frac{(258)^2}{12} = 5562 - 5547 = 15,$$

and the "residual" is given by

$$R = \sum_i \sum_j x_{ij}^2 - \frac{1}{r}\sum_{j=1}^{k}\left(\sum_{i=1}^{r} x_{ij}\right)^2$$

$$= (1202 + 1595 + 1460 + 1329) - 5562$$

$$= 5586 - 5562 = 24\,.$$

Therefore (one way) variation table is

Source of variation	Sum of square	Degree of freedom	Mean square
Between column	$V_c = 15$	$k - 1 = 3$	$\frac{15}{3}$
Residual	$R = 24$	$N - k = 8$	$\frac{24}{8}$

Hence the test statistics is $F = \dfrac{\frac{V_c}{k-1}}{\frac{R}{N-k}} = \dfrac{\frac{15}{3}}{\frac{24}{8}} = \dfrac{5}{3} = 1.67.$

But from the table, the tabular value $F_{0.05}(n_1, n_2)$ is

$$F_{0.05}(n_1, n_2) = F_{0.05}(3, 8) = 4.0662.$$

Hence the difference is insignificant and the inference is that the lifetimes of different brands of bulbs are equal

EXAMPLE 2.130

The data given below shows the yields in quintal per acre of a certain variety of onion grown in a given type of soil treated with fertilizers A, B, or C. Is the difference in yields for all treatments significant?

A	B	C
48	47	49
49	49	51
50	48	50
49	48	50

Solution. Since the sum of squares is not affected by change of origin, we subtract a suitable number say 46, from all entries of the yields and get the following table for analysis of variance.

	A	B	C
	2	1	3
	3	3	5
	4	2	4
	3	2	4
$\sum x$	**12**	**8**	**16**
$\sum x^2$	**38**	**18**	**66**
$(\sum x)^2$	**144**	**64**	**256**

Therefore "between columns variance" V_c is given by

$$V_c = \frac{1}{4}[144+64+256] - \frac{1}{12}(36)^2$$
$$= 116 - 108 = 8$$

and the "residual" R is given by

$$R = (38+18+66) - 116 = 122 - 116 = 6.$$

Therefore the table for analysis of variation is as shown below:

Source of variation	Sum of square	Degree of freedom	Mean square
Between column	$V_c = 8$	$3-1=2$	$\frac{8}{2}$
Residual	$R = 6$	$12-3=9$	$\frac{6}{9}$

Hence

$$F = \frac{\frac{V_c}{k-1}}{\frac{R}{N-k}} = \frac{\frac{8}{2}}{\frac{6}{9}} = 6.$$

But, from the F-distribution table, we have

$$F_{0.05}(n_1, n_2) = F_{0.05}(2,9) = 4.2565 .$$

Since $F > F_{0.05}$, the difference in yields is significant.

2.39 FORECASTING AND TIME SERIES ANALYSIS

The process of estimating future conditions on a systematic basis is called *forecasting* and the figure or statement obtained as a result of this process is called *forecast*. Thus forecasting is concerned mainly with handling of uncertainty about the future. Some important methods of forecasting are:

(i) Field Survey and Opinion Poll
(ii) Historical Analog Method
(iii) Extrapolation
(iv) Regression Analysis
(v) Exponential Smoothing
(vi) Time series Analysis

Out of these methods, the time series analysis is the most popular method of business forecasting. While forecasting, we usually deal with statistical data collected at successive intervals of time. Such data are called *time series.* A time series pairs one set of variates with intervals of time. Thus a time series is a set of values $y_1, y_2,\cdots$ of a variable y at time $t_1, t_2,...$ As such y is a function of t. Analyzing the past behaviour, the time series analysis helps us in forecasting the future behaviour. For example, the time series:

Time (minutes):	0	1	2	3	4	5	6
Temperature:	70	77	92	118	136	143	155

shows that the temperature is rising with the passage of time.

The variations observed in a time series over a period of time are of the following four types, called the *components* (or *elements*) *of the time series:*

(i) *Secular Trend or simply Trend*. The general tendency of the functional values in a time series to *grow* or *decline* over a long period of time is called secular trend or simply trend. For example, time series for population show secular trend since the population goes on increasing with passage of time. Similarly, time series for death-rate in India show a secular trend since the death-rate is decreasing with the passage of time.

(ii) *Seasonal Variations:* The variations in a time series occurred due to climate/weather change; customs, habits and traditions change are called seasonal variations.

(iii) *Cyclical Variations:* The variations in a time series due to cyclic variations like *prosperity* and *recession* in business are called cyclical variations. In recession, the buyers wait for lower prices and that amounts to decrease functional values in the time series.

(iv) *Irregular Variations*. The variations in a time series caused by isolated special occurrences like earthquakes, floods, wars and strikes are called irregular variations.

Methods to Determine Trends

In what follows, we discuss generally used methods to determine trends.

1. **The Semi-Average Method.** In this method, the given data is separated in two parts (preferably equal) and then average the data in each part. By doing so, we get two (average) points on the graph of the time series. A trend line is drawn between these two points on the graph of the time series. Extending the trend line downwards or upwards, we get intermediate values and so can predict future values.

The method of semi-average is of course simple but can be applied only for linear trends.

EXAMPLE 2.131

Fit a trend line to the following data by the method of semi-average and predict the sales for the year 2006.

Year	Sales in lakh tonnes
2001	90
2002	110
2003	100
2004	120
2005	90

Solution. We are given a time interval of 5 years, therefore we omit the middle year 2003. Then average of the first two years (2001 and 2002) is $\frac{90+110}{2}=100$ and average of the last two years (2004 and 2005) is $\frac{120+90}{2}=105$. Thus we get two values 100 and 105. We plot 100 corresponding to middle of 2001 and 2002 and similarly plot 105 corresponding to the middle of 2004 and 2005. Joining these two points, we shall get the trend line as shown below:

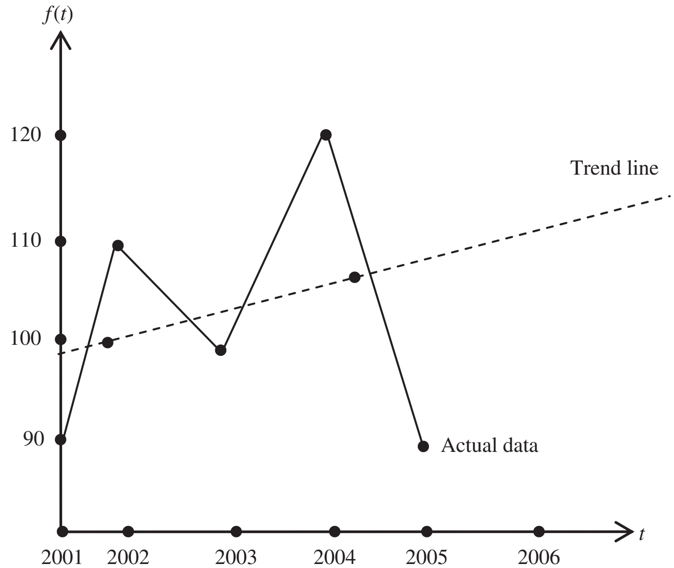


Figure 2.14

The expected sale for the year 2006 is about 107.

EXAMPLE 2.132

Fit a trend line to the following data by the method of semi-average. Predict the earning during the year 2004.

Year :	1996	1997	1998	1999	2000	2001	2002	2003
Earning in lakhs:	38	40	65	72	69	60	87	95

Solution. Dividing the given data into two parts, the average of the first four years is $\frac{38+40+65+72}{4} = 53.5$, while the average for the last four years is $\frac{69+60+87+95}{4} = 77.75$. We plot 53.5 against the middle of the years 1997 and 1998 and the point 77.75 against the middle of the year 2001-2002. Joining the two points obtained, we get the trend line as shown below:

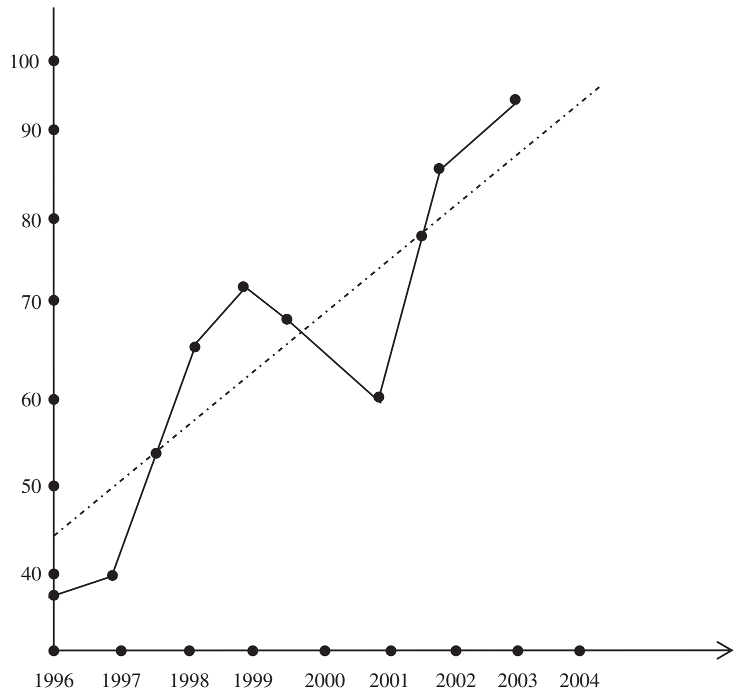


Figure 2.15

The predicted earning for the year 2004 is 98.

2. Least Square Line Method. While finding least square line approximation to a given data, we found that the normal equations for the line $y = a + bx$ are:

$$n\,a + b\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \tag{2.42}$$

and

$$a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_1^2 = \sum_{i=1}^{n} x_i y_i \tag{2.43}$$

Solving these equations, we get values of a and b. Putting these values of a and b in the equation $y = a + bx$, the required line is obtained.

The calculations are simplified when the midpoint in time is taken as the origin. Thus we take $\sum_{i=1}^{n} x_i = 0.$ By doing so, the normal equations reduce to

$$na = \sum_{i=1}^{n} y_i$$

and

$$b\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i .$$

These last two equations yield

$$a = \frac{\sum_{i=1}^{n} y_1}{n} \quad \text{and} \quad b = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_1^2} .$$

It may be mentioned here that in case of odd number of years, there is no difficulty in selecting the origin. But in case of even number of years , $\sum_{i=1}^{n} x_i$ will be equal to zero if the origin is placed midway between the two middle years. However, in such a case, it in better to solve the original normal equations (2.42) and (2.43) .

EXAMPLE 2.133

Fit a straight line trend by the method of least squares to the data of Example 2.120. Now predict the earning during the year 2004.

Solution. The given data (with deviations from the year 1999) is

Year	Earning in lakhs (Y)	Deviation(X)	XY	X^2
1996	38	–3	–114	9
1997	40	–2	–80	4
1998	65	–1	–65	1
1999	72	0	0	0
2000	69	1	69	1
2001	60	2	120	4
2002	87	3	261	9
2003	95	4	380	16
$n = 8$	$\Sigma Y = 526$	$\Sigma X = 4$	$\Sigma XY = 571$	$\Sigma X^2 = 44$

Here the number of years is even, so we have to solve the normal equations

$$8a + 4b = 526$$
$$4a + 44b = 571 .$$

Solving these equations, we get $a = 62.09, b = 7.33$. Hence the line of best fit is $y = 62.09 + 7.33x$.

For the year 2004, $x = 5$ and therefore the earning predicted for that year is

$y = 62.09 + 5(7.33) = 98.74$ Lakhs.

EXAMPLE 2.134

Fit a straight line trend to determine sale for the year 2004 to the following time series.

Year:	1998	1999	2000	2001	2002
Sales:	100	110	130	125	160

Solution. The tabular values are:

Year	Sale (Y)	Deviation from the year 200(X)	XY	X^2
1998	100	–2	–200	4
1999	110	–1	–110	1
2000	130	0	0	0
2001	125	1	125	1
2002	160	2	320	4
$n = 5$	$\Sigma Y = 625$	$\Sigma X = 0$	$\Sigma XY = 135$	$\Sigma X^2 = 10$

Here $\Sigma X = 0$. Therefore, for the line $y = a + bx$, we have

$$a = \frac{\sum_{i=1}^{n} x_i y_i}{n} = \frac{625}{5} = 125,$$

and

$$b = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2} = \frac{135}{10} = 13.5.$$

Therefore, the line of best fit is

$$y = a + bx = 125 + 13.5x.$$

For the year 2004, we have $x = 4$. Therefore the prediction of sale for that year is

$$y = 125 + 4(13.5) = 179 .$$

3. Method of Moving Average. This is a method for measuring non-linear trend. If $y_1, y_2, \ldots$ is a set of numbers, then the sequence of arithmetic means

$$\frac{y_1 + y_2 + \ldots + y_n}{n}, \frac{y_2 + y_3 + \ldots + y_{n+1}}{n}, \frac{y_3 + y_4 + \ldots y_{n+2}}{n}, \ldots$$

is called the *Moving Average of order n.* The sums in the numerator of the above sequence are called *moving totals of order n.*

For example, let 3, 5, 1, 2, 5, 6 be a set of numbers. Then

$$\frac{3+5+1+2}{4}, \frac{5+1+2+5}{4}, \frac{1+2+5+6}{4}$$

or

$$\frac{11}{4}, \frac{13}{4}, \frac{14}{4}$$

is a moving average of order 4, while 11, 13, 14 are moving totals of order 4.

We notice that we started in this example by six numbers, but with moving average of order 4, we arrived at 3 numbers.

A two-period moving average of the moving average is called *centered moving average.*

EXAMPLE 2.135

The table below shows the number of crimes in a city for the year 1999–2008. Construct a five year moving average and five year centered moving average.

Year:	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008
No. of crimes	22	19	20	24	23	27	29	26	25	30

Solution. The five year moving average table is

Year	No. of crimes	Five year moving total	Five year moving average	Five year centered moving average
1999	22	–	–	
2000	19	–	–	
2001	20	108	21.60	
2002	24	113	22.60	22.10
2003	23	123	24.60	23.60
2004	27	129	25.80	25.20
2005	29	130	26.00	25.90
2006	26	137	27.40	26.70
2007	25	–	–	
2008	30	–	–	

4. Least Square Parabola: We know that the normal equations for the parabola $y = a + bx + cx^2$ of best fit are

$$na + b\sum_{i=1}^{n} x_i + c\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} y_i,$$

$$a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 + c\sum_{i=1}^{n} x_i^3 = \sum_{i=1}^{n} x_i y_i,$$

$$a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i^3 + c\sum_{i=1}^{n} x_i^4 = \sum_{i=1}^{n} x_i^2 y_i.$$

Solving these normal equation for a, b and c and putting in $y = a + bx + cx^2$, we get the required parabola of best fit.

EXAMPLE 2.136

Fit a second degree parabola $y = a + bx + cx^2$ is the following population data of a city and estimate the population in 2007.

Year:	1999	2000	2001	2002	2003	2004	2005	2006
Population: in lakhs	6	8	10	13	16	18	20	23

Solution. The table for fitting of second degree parabola for the given data is

Year	Population Y	Deviation from the year 2002 (X)	XY	X^2	X^2y	X^3	X^4
1999	6	–3	–18	9	54	–27	81
2000	8	–2	–16	4	32	–8	16
2001	10	–1	–10	1	10	–1	1
2002	13	0	0	0	0	0	0
2003	16	1	16	1	16	1	1
2004	18	2	36	4	72	8	16
2005	20	3	60	9	180	27	81
2006	23	4	92	16	368	64	256
$n = 8$	$\Sigma Y = 114$	$\Sigma X = 4$	$\Sigma XY = 160$	$\Sigma X^2 = 44$	$\Sigma X^2 Y = 732$	$\Sigma X^3 = 64$	$\Sigma X^4 = 452$

Therefore the normal equations are

$$8a + 4b + 44c = 114$$

$$4a + 44b + 64c = 160$$

$$44a + 64b + 452c = 732.$$

Solving these equations for a, b and c, we get

$$a = 12.96,\ b = 2.44 \text{ and } c = 0.012.$$

Therefore the required parabola of best fit is

$$y = 12.96 + 2.44x + 0.012x^2.$$

The estimate of population in the year 2007 is

$$y = 12.96 + 5(2.44) + 25(0.012) = 25.46 \text{ Lakh}.$$

2.40 STATISTICAL QUALITY CONTROL

The statistical analysis of the inspection data based on the concept of sampling and normal curve principles to maintain and improving quality standards of products is called *statistical quality control.* This is a statistical method to determine the extant to which quality goals are achieved without necessarily checking every item produced and to indicate whether the variations occurring are exceeding expectations.

To locate and eliminate special causes of variations, statistical devices called, control charts are used. A *control chart* consists of a *center line* and two control limits, known as *upper control limit* (UCL) and *lower control limit* (LCL). A process is said to be out of statistical control if point on the control chart of that process falls outside the control limits. Thus the outline of a control chart is as shown below:

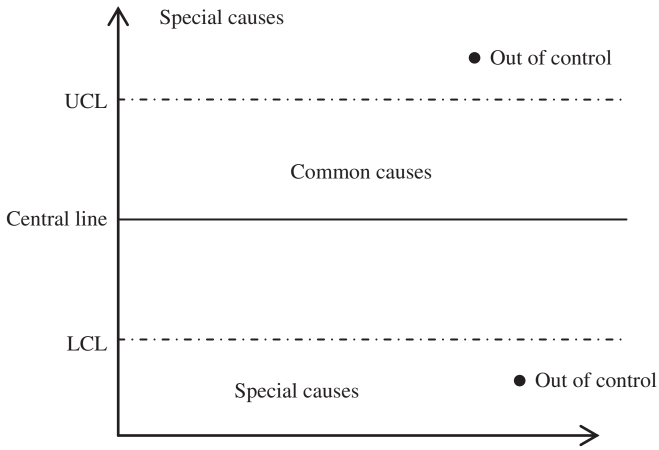


Figure 2.16

Types of Control Charts

There are two types of control charts:

1. **Variable Control Charts.** This type of charts are used to control causes of variations in variables (quality characteristic of a product) like time. weight, volume, length, pressure change, concentration, area etc.

Two important variable control charts along with their statistics plotted and control limits are given in the following table ($\bar{\bar{x}}$ denotes mean of sample means and $\bar{R} = \frac{\Sigma R}{n}$):

Type of chart	Statistics plotted	Control limits
$\bar{X}$ and σ chart	Average and standard deviations of subgroups of variable data	Centre line: μ UCL: $\mu + 3\left(\frac{\sigma}{\sqrt{n}}\right)$ LCL: $\mu - 3\left(\frac{\sigma}{\sqrt{n}}\right)$
$\bar{X}$ and R chart	Averages and Ranges of m subgroups of variable data	Center line: $\bar{X}$ UCL: $\bar{X} + A_2\bar{R}$ LCL: $\bar{X} - A_2\bar{R}$

2. Attribute Control Charts. These type of charts are used to control the number of defects associated with a particular type of items. The generally used attribute control charts are given in the following table:

Type of chart	Statistics plotted	Control limits
p-chart	Ratio of defective items to total number inspected	Central line: $\overline{p} = \dfrac{no.\ of\ defective}{total\ number\ inspected}$ UCL: $\overline{p} + 3\sqrt{\dfrac{\overline{p}(1-\overline{p})}{n}}$ LCL: $\overline{p} - 3\sqrt{\dfrac{\overline{p}(1-\overline{p})}{n}}$
np- chart	Actual number of defective items	Central line: np UCL: $n\overline{p} + 3\sqrt{n\overline{p}(1-\overline{p})}$ LCL: $n\overline{p} - 3\sqrt{n\overline{p}(1-\overline{p})}$
c-chart	Number of defects per item for a constant sample size	Control line: $\overline{c}$ = mean of defects in 25 units of item $UCL = \overline{c} + 3\sqrt{\overline{c}}$ $LCL = \overline{c} - 3\sqrt{\overline{c}}$

EXAMPLE 2.137

Samples of 50 pens are drawn randomly from daily production of large out put of pens and number of defective pens are noted in the form of the table given below: Draw conclusion using p-chart.

Sample No.	No. inspected	No. of defectives	Sample No.	No. inspected	No. of defectives
1	50	2	11	50	3
2	50	3	12	50	1
3	50	1	13	50	2
4	50	4	14	50	5
5	50	3	15	50	3
6	50	2	16	50	1
7	50	2	17	50	2
8	50	3	18	50	3
9	50	1	19	50	2
10	50	3	20	50	4

Solution. We have for the p-chart

$$\overline{p} = \frac{total\ defective}{total\ number\ inspected} = \frac{50}{1000} = 0.05$$

$$UCL = \overline{p} + 3\sqrt{\frac{\overline{p}(1-\overline{p})}{n}} = 0.05 + 3\sqrt{\frac{0.05(1-0.05)}{50}}.$$

$$= 0.05 + 0.0925 = 0.1425$$

$$LCL = 0.05 - 0.0925 = -0.0425. \text{ (which we take as 0)}$$

The fraction defectives are

0.04 , 0.06 , 0.02 , 0.08 , 0.06 , 0.04 , 0.04 , 0.06 , 0.02 , 0.06

0.06 , 0.02 , 0.04 , 0.10 , 0.06 , 0.02 , 0.04 , 0.06 , 0.04 , 0.08

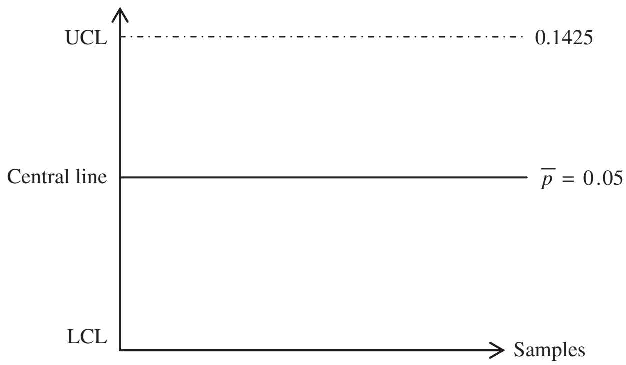


Figure 2.17

We note that all points (showing fractional defective) lie in the control limits. Therefore the process of production is in a state of control.

EXAMPLE 2.138

15 Computer sets being produced by a manufacturing company were examined for quality control test. The number of defects for each computer are recorded as given below. Prepare a C-chart and draw conclusion from it.

No. of defects:	1	3	2	2	4	5	3	2
	1	3	3	2	1	2	4	

Solution. Since number of defects in a particular item are given, we shall use C-chart. We have

Central line $\overline{c}$ = mean of defects in 15 computer $= \frac{36}{15} = 2.4$.

$$UCL = \overline{c} + 3\sqrt{\overline{c}} = 2.4 + 3\sqrt{2.4} = 2.400 + 3(1.549) = 7.048$$

$$LCL = \overline{c} - 3\sqrt{\overline{c}} = 2.400 - 3(1.549) = -2.247 \text{ (to be taken as 0).}$$

Therefore the c-chart is as shown below:

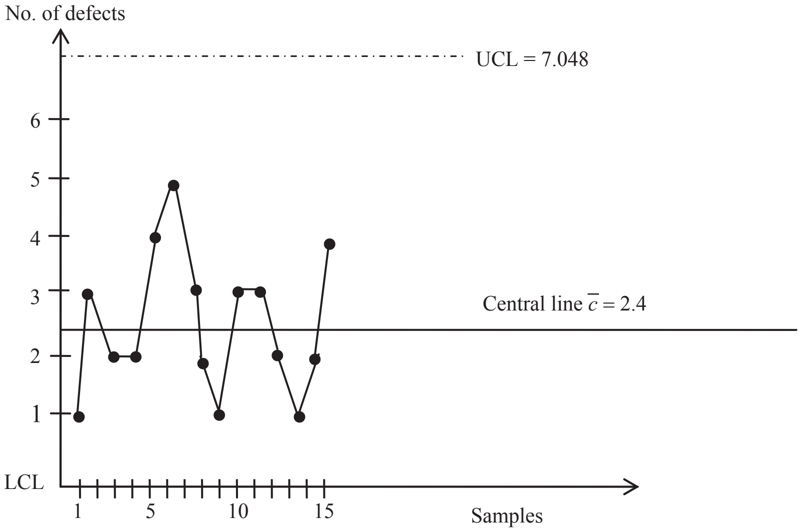


Figure 2.18

Since all points lie within the control limits, the process is in a state of control.

EXAMPLE 2.139

The data given below shows the mean and range for ten samples of size 5 each. Construct the $\overline{X}$ and Range charts:

Sample No:	1	2	3	4	5	6	7	8	9	10
$\overline{X}$:	11.2	11.8	10.8	11.6	11.0	9.6	10.4	9.6	10.6	10
R:	7	4	8	5	7	4	8	4	7	9

Solution. We have samples of size 5. Also

$$\overline{\overline{X}} = \frac{106.6}{10} = 10.66 \text{ (central line)}$$

$$\overline{R} = \frac{\Sigma R}{10} = \frac{63}{10} = 6.3$$

$$UCL = \overline{\overline{X}} + A_2\overline{R} \text{ (corresponding to n = 5)}$$

$$= 10.66 + 0.577(6.3) \text{ since from the table } A_2 = 0.577$$

$$= 14.2951.$$

$$LCL = \overline{\overline{X}} - A_2\overline{R} = 10.66 - (0.579)(6.3)$$

$$= 10.66 - 3.635 = 7.025.$$

Since all means lie within the control limits, the process in a state of control.

2.41 MISCELLANEOUS EXAMPLES

EXAMPLE 2.140

An insurance company insured 2,000 scooter drivers, 4,000 car drivers and 6,000 truck drivers. The probability of accident is 0.01, 0.03 and 0.15 respectively. One of the insured persons meets an accident. What is the probability that he is a scooter driver?

Solution. Consider the events

A: Accident takes place
B: Person met with accident is a scooter driver
C: Person met with accident is a car driver
D: Person met with accident is a truck driver.

Then

$$P(B) = \frac{2000}{12000} = \frac{1}{6},\ P(C) = \frac{4000}{12000} = \frac{1}{3},$$

$$P(D) = \frac{6000}{12000} = \frac{1}{2}.$$

Further, it is given that

$$P(A \setminus B) = 0.01,\ P(A \setminus C) = 0.03,\ P(A \setminus D) = 0.15.$$

Then, by theorem on total probability, we have

$$\begin{aligned} P(A) &= P(A \setminus B)P(B) + P(A \setminus C)P(C) + P(A \setminus D)P(D) \\ &= 0.01\left(\frac{1}{6}\right) + 0.03\left(\frac{1}{3}\right) + 0.15\left(\frac{1}{2}\right) \\ &= \frac{1}{6}[0.01 + 0.06 + 0.45] = \frac{0.52}{6}. \end{aligned}$$

Now, Baye's Theorem implies

$$P(B \setminus A) = \frac{P(A \setminus B)P(B)}{P(B)} = \frac{0.01\left(\frac{1}{6}\right)}{0.\frac{52}{6}} = \frac{1}{52}.$$

EXAMPLE 2.141

If A, B, C are mutually exclusive and exhaustive events associated with a random experiment and $P(B) = 0.6P(A)$ and $P(C) = 0.2P(A)$. Then find $P(A)$.

Solution. Since A, B and C are mutually exclusive and exchaustive events, we have

$$P(A) + P(B) + P(C) = 1.$$

Since $P(B) = 0.6\ P(A)$ and $P(C) = 0.2\ P(A)$, we get

$$(1 + 0.6 + 0.2)\,P(A) = 1$$

or

$$P(A) = \frac{1}{1.8} = \frac{5}{9}.$$

EXAMPLE 2.142

A factory is manufacturing electric bulbs, there is a chance of 1/500 for any bulb to be defective. The bulbs are packed in packets of 50. Calculate the approximate number of packets containing no defective, one, two and three defective bulbs in a consignment of 10,000 packets.

Solution. We are given that $p = \frac{1}{150}$ and $n = 50$. Therefore parameter λ of the Poisson distribution is

$$\lambda = np = 0.1.$$

Then

$$p(r)=\frac{e^{-\lambda}\lambda^r}{r!}=\frac{e^{-0.1}(0.1)^r}{r!}.$$

Thus

(i) $p(\text{no defective}) = p\ (x = 0)\ = \frac{e^{-0.1}(0.1)^0}{0!} = e^{-0.1} = 0.90483.$ Therefore number of packets containing no defective bulb is $10000\times 0.90483 \approx 9048.$

(ii) $P(\text{one defective})\ = P(x=1) = \frac{e^{-0.1}(0.1)}{1} = (0.9048)\ (0.1)$
$\approx 0.090483.$

Therefore the number of packets containing one defective bulb is $10000\times 0.090483 \approx 904$

(iii) $P\ (\text{two defective})\ = p(x=2) = \frac{e^{-0.1}(0.1)^2}{2} = \frac{(0.90483)\ (0.01)}{2}$
$\approx 0.00452.$

Therefore the number of packets containing two defective bulbs is $10000\times 0.00452 \approx 45$

(iv) $P(\text{three defective})\ = P(x=3) = \frac{e^{-0.1}(0.1)^3}{3!} = \frac{(0.90483)(0.001)}{6} \approx 0.00015\ .$

Therefore the number of packets containing three defective bulbs is $10000\times 0.00015 \approx 2.$

EXAMPLE 2.143

Two random samples have the following values:

Sample 1	15	22	28	26	18	17	29	21	24
Sample 2	8	12	9	16	15	10			

Test the difference of the estimates of the population variances at 5% level of significance (Given that $F_{0.05}$ for $v_1 = 8$ and $v_2 = 5$ is 4.82).

Solution. The means for samples 1 and 2 are respectively

$$\bar{x} = 22.22 \text{ and } \bar{y} = 11.66.$$

Then for $n_1 = 9, n_2 = 6,$ we have

$$S_1^2 = \frac{1}{n_1-1}\Sigma(x_i-\bar{x})^2$$
$$= \frac{171.10364}{8} = 21.387955,$$
$$S_2^2 = \frac{1}{n_2-1}\Sigma(y_i-\bar{y})^2$$
$$= \frac{53.3336}{5} = 10.66672.$$

Therefore the test statistics is given by

$$F = \frac{S_1^2}{S_2^2} = \frac{21.387955}{10.66672} = 2.005.$$

Further, the numbers of degree of freedom are 8 and 5.
Therefore, we have

$$F_{0.05}\,(8,5) = 4.82 \text{ (given)}.$$

Thus the calculated value of F is less than the tabulated value. Hence the samples have been drawn from normal population having the same variance, that is, there is no significant difference between the population variances.

EXERCISES

1. Find the mean, median, and mode of the following data relating to weight of 120 articles.

Weight in gm :	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60
No. of articles:	14	17	22	26	23	18

Ans. Mean: 32.58, Median: 32.6 Mode: 35.1

2. Determine the mean and standard deviation for the following data

Size of item:	6	7	8	9	10	11	12
Frequency:	3	6	9	13	18	5	4

Ans. Mean: 9, S.D:1.61

3. Find (i) mean $\bar{x}$ and $\bar{y}$ (ii) regression coefficients b_{yx}, and b_{xy} (iii) coefficient of correlation between x and y for the two regression lines $2x + 3y - 10 = 0$ and $4x + y - 5 = 0$

Ans. $\bar{x} = \frac{1}{2}, \bar{y} = 3, b_{yx} = -\frac{2}{3}, b_{xy} = -\frac{1}{4}, \rho = -\frac{1}{\sqrt{6}}$

4. Out of the following two regression lines, find the regression line of Y on X:

$$3x + 12y = 9,\ \ 3x + 9x = 46.$$

Ans. 3x + 12y = 9

5. Calculate the coefficient of correlation between X and Y from the following data:

x:	43	44	46	40	44	42	45	50
y:	29	31	19	18	19	27	27	22

Ans. – 0.057

6. In a single throw of two distinct dice, what is the probability of getting a total of 11?

Ans. $\frac{1}{18}$

7. Find the probability that a randomly chosen three-digit integer is divisible by 5.

Ans. $\frac{1}{5}$

8. Show that the number of distinguishable words that can be formed from the letters of MISSISSIPPI is 34650.
9. A certain defective dice is tossed. The probabilities of getting the faces 1 to 6 are respectively

$$p_1 = \frac{2}{18}, p_2 = \frac{3}{18}, p_3 = \frac{4}{18}, p_4 = \frac{3}{18},$$
$$p_5 = \frac{4}{18}, p_6 = \frac{2}{18}.$$

What is the probability that a prime number is on the top?

Ans. $\frac{11}{18}$

10. Let A and B be two events such that P(A) = 0.4, P(B) = p and P(A ∪ B) = 6. Find p so that A and B are independent.

Ans. $\frac{1}{3}$

11. A bag contains 3 red and 5 black balls and a second bag contains 6 red 11. and 4 black balls. A ball is drawn from each bag. Find the probability that one ball is red and the other is black.

Ans. $\frac{21}{40}$

12. The probability of a man hitting a target is $\frac{1}{3}$. If he fires six times, what is the probability that he hits the target
(i) at least twice
(ii) at most twice

Ans. $\frac{473}{729}, \frac{496}{729}$

13. A candidate takes on 20 questions, each with four multiple choices. One of the choice in every question is incorrect. The candidate makes guess of the remaining choices. Find the expected number of correct answers and the standard deviation.

Ans. $\frac{20}{3}, \sqrt{\frac{40}{9}}$

14. A random variable X has the following probability function:

x:	0	1	2	3	4	5	6	7
y:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Find k, evaluate $P(X < 6)$, $P(X \geq 6)$, $P(3 < X \leq 6)$ and find the minimum value of x so that $P(X \leq x) > \frac{1}{2}$.

Ans. $k = 10,\ P(X < 6) = \frac{81}{100},\ P(X \geq 6) = \frac{19}{100},\ P(3 < X \leq 6) = \frac{33}{100},\ x = 4.$

15. A die is tossed twice. Getting a number greater than 4 is considered a success. Find the variance of the probability distribution of the number of successes.

Ans. $\frac{4}{9}$

16. The frequency distribution of a measurable characteristic varying between 0 and 2 is as follows:

$$f(x) = \begin{cases} x^3, & 0 \leq x \leq 1 \\ (2-x)^3, & 1 \leq x \leq 2. \end{cases}$$

Calculate the standard deviation and the mean deviation about the mean.

Hint:

$$\mu = \frac{1}{2}\left[\int_0^2 x f(x) dx\right]$$

$$= \frac{1}{2}\left[\int_0^1 x^4 dx + \int_0^1 x(2-x)^3 dx\right] = 1,$$

$$\sigma^2 = \frac{1}{2}\left[\int_0^2 (x-1)^2 f(x)dx\right]$$

$$= \frac{1}{15} \text{ and so } \sigma = \frac{1}{\sqrt{15}}$$

Mean deviation for the mean

$$= \frac{1}{2}\left[\int_0^2 |x-\mu| f(x)dx\right] = \frac{1}{5}.$$

17. The diameter X of an electric cable is assumed to be a continuous random variable with probability density function $f(x) = 6x(1-x)$, $0 \le x \le 1$. Determine a number k such that $P(X<k) = P(x>k)$.

Hint:

$$P(X<k) = P(X>k) \Rightarrow \int_0^k f(x)dx = \int_k^1 f(x)$$

$$\Rightarrow 6\int_0^k x(1-x)dx = 6\int_k^1 x(1-x)dx$$

$$\Rightarrow 3k^2 - 2k^3 = 1 - 3k^2 + 2k^3$$

$$\Rightarrow 4k^3 - 6k^2 + 1 = 0 \Rightarrow k = \frac{1 \pm \sqrt{3}}{2} \text{ and } k = \frac{1}{2}.$$

$k = \frac{1}{2}$ lies between 0 and 1

Ans. $\frac{1}{2}$

18. In a precision bombing attack there is a 50% chance that any bomb will strike the target. Two direct hits are required to destroy the target completely. How many bombs must be dropped to give a 99% chance or better to completely destroy the target?

Hint:

$$p = \frac{1}{2}, q = 1 - \frac{1}{2} = \frac{1}{2}$$

$$P(X=r) = \binom{n}{r}\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{n-r} = \binom{n}{r}\left(\frac{1}{2}\right)^n$$

We should have $P(X \ge 2) \ge 0.99$ or $[1 - p(X \le 1)] \ge 0.99$

or $[1 - p(0) - p(1)] \ge 0.99$

$$\text{or}\left[1 - \left\{\binom{n}{0} + \binom{n}{1}\right\}\left(\frac{1}{2}\right)^n\right] \ge 0.99$$

$$\text{or } 0.01 \ge \frac{1+n}{2^n} \text{ or } 2^n \ge 100 + 100n.$$

Note that $n = 11$ satisfies this equation.

19. If, on an average 1 vessel in every 10 is wrecked, find the probability that out of 5 vessel's expected to arrive, at least 4 will arrive safely.

Hint: $p = \frac{1}{10}, q = \frac{9}{10}$. P (at the most one will be wrecked).

Therefore,

$$P(X \le 1) = P(0) + P(1) = nc_0 q^n + nc_1 q^{n-1} p.$$

$$= \left(\frac{9}{10}\right)^5 + 5\left(\frac{9}{10}\right)^4 \left(\frac{1}{10}\right)$$

$$= \left(\frac{9}{10}\right)^4 \left[\frac{9}{10} + \frac{5}{10}\right] = \frac{9^4(7)}{10^5} = \frac{45927}{50000}.$$

20. Fit a binomial distribution to the following frequency distribution:

x:	0	1	2	3	4	5	6
f:	13	25	52	58	32	16	4

Ans. $200(0.554 + 0.446)^6$

21. Six dice are thrown 729 times. How many times do you expect at least three dice to show a five or six?

Hint: Calculate $P(X \ge 3)$.

Ans. 233

Poisson's Distribution

22. In a certain factory turning razor blades, there is a small chance of 0.002 for any blade to be defective. The blades are supplied in packets, of 10. Use Poisson's distribution to calculate the approximate number of packets containing no defective, one defective, and two defective blades, respectively, in a consignment of 10,000 blades.

Ans. 9802,196,2

23. Show that in a Poisson distribution with unit mean, mean deviation about mean is $\frac{2}{e}$ times the standard deviation.

Hint: $P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$, Here $\lambda = 1$. Therefore,

$$P(X = x) = \frac{e^{-1}}{x!}$$

Mean deviation about mean 1 is $E(|X-1|) = \sum |x-1| P(X = x)$

$$= e^{-1}\left(1 + \frac{1}{2!} + \frac{2}{3!} + \ldots\right)$$

$$= e^{-1}\left\{1 + \left(1 - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{3!} - \frac{1}{4!}\right) + \ldots\right\}$$

$$= e^{-1}(1+1) = \frac{2}{e} \times 1 = \frac{2}{e} \times \text{standard deviation.}$$

24. Fit a Poisson distribution to the following data:

x :	0	1	2	3	4
y :	419	352	154	56	19

Ans.

0	1	2	3	4
404.9	366	165.4	49.8	11.3

25. If the probability of a bad reaction from a certain injection is 0.001, determine the chance that out of 2,000 individuals more than 2 will get a bad reaction.

Hint: $\lambda = np = 2000(0.001)$, Probability $= 1 - \left(e^{-\lambda} + \frac{\lambda e^{-\lambda}}{1} + \frac{\lambda^2 e^{-\lambda}}{2!}\right) = 0.32$.

26. If a random variable has a Poisson distribution such that $P(1) = P(2)$, find (i) mean of the distribution (ii) $P(4)$

Hint:

$$P(1) = P(2) \Rightarrow \lambda e^{-\lambda} = \frac{\lambda^2 e^{-\lambda}}{2} \Rightarrow \lambda = 2$$

$$P(4) = \frac{\lambda^4 e^{-\lambda}}{4!} = \frac{2^4 e^{-2}}{4!} = \frac{2}{3} e^{-2}.$$

27. Fit a Poisson distribution to the following data:

x:	0	2	2	3	4
y:	192	100	24	3	1

Hint: $\lambda = \frac{\sum f_i x_i}{\sum f_i} = 0.503$, then the frequencies are $320\left[\frac{e^{0.503}(0.503)^r}{r!}\right]$.

28. The incidence of occupational disease in an industry is such that the workmen have a 10% chance of suffering from it. What is probability that in a group of 7, five, or more will suffer from the disease?

Ans. 0.0008

Normal Distribution

29. The mean yield of a crop for one-acre plot is 662 kg with a standard deviation 32 kg. Assuming normal distribution how many one-acre plots in a batch of 1,000 plots would you expect to have yield over 700 kg?

Hint:

$$\mu = 662, \sigma = 32, z = \frac{x - \mu}{32} = 1.19$$

$$P(z > 1.19) = 0.1170.$$

$$\text{No. of plots} = 1000 \times 0.117 = 117.$$

30. The mean and standard deviation of the marks obtained by 1,000 students in an examination are respectively, 34.4 and 16.5. Assuming the normality of the distribution, find the approximate number of students expected to obtain marks between 30 and 60.

Hint:

$$z_1 = \frac{30-34.4}{16.5} = -0.266,$$

$$z_2 = \frac{60-34.4}{16.5} = 1.552$$

$$P(-0.266 \le z \le 1.552)$$

$$= P(-0.27 \le z \le 0) + P(0 \le z \le 1.56)$$

$$= P(0 \le z \le 0.27) + P(0 \le z \le 1.56)$$

$$= 0.1064 + 0.4406 = 0.5470.$$

Therefore, number of students = 1000 × 0.5470 = 547.

31. Fit a normal curve to the following data:

x:	0	1	2	3	4	5
frequency :	13	23	24	15	11	4

Hint:

$$\mu = \frac{\sum fx}{\sum f} = \frac{23+68+45+44+20}{100} = 2$$

$$\sigma = \sqrt{\frac{\sum fx}{\sum f} - \mu^2} = \sqrt{5.70-4} = 1.304$$

$$\text{Normal curve is } y = \frac{100}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{100}{2\sqrt{2\pi}} e^{-\frac{(x-2)^2}{3.4}}$$

32. If is known from past-experience that the number of telephone calls made daily in a certain community between 3 pm and 4 pm have a mean of 352 and a standard deviation of 31. What percentage of the time will there be for more than 400 telephone calls made in this community between 3 pm and 4pm?

Ans. 6% approx.

33. If X is a normal variate with mean 30 and standard deviation 5, find the probability that $|X-5|>5$.

Hint:

$$P(|X-5| \le 5) = P(25 \le X \le 35)$$

$$= P(-1 \le z \le 1)$$

$$= 2P(0 \le z \le 1)$$

$$= 2(0.3413) = 0.6826$$

Therefore, $$P(|x-5| > 5) = 1 - 0.6826 = 0.3174.$$

34. In a normal distribution, 10.03% of the items are under 25 kg weight and 89.97% of the items are under 70 kg weight. Find the mean and standard deviation of the distribution.

Ans. $\mu = 47.5kg$, $\sigma = 17.578kg$

Significance for Means

35. A sample of 900 members has a mean 3.4 cm and standard deviation 2.61 cm. Is this a sample from a large population of mean 3.25 and standard deviation 2.61 cm?

Ans. $z = \frac{\bar{x} - \mu}{\sigma}\sqrt{n} = 1.73$. Also 95% conf dence interval: (3.2295, 3.5705).

The mean 3.25 lies in the interval.

36. A sample of 30 pieces of a semi-conduction metrical gave an average of resistivity of 73.2 units with a sample standard deviation of 5.4 units. Obtain a 95% confidence interval for the resistivity of the material and test the hypothesis that this is 75 units.

Hint: $\bar{x} \pm 1.96\frac{5}{\sqrt{n}}$.

Ans. (71.2, 75.2), accepted.

37. The mean of a certain normal population is equal to the standard error of the mean of the samples of 100 from that distribution. Find the probability that the mean of the sample of 25 from the distribution will be negative.

Hint:

$$\mu = \frac{\sigma}{\sqrt{100}} = \frac{\sigma}{10}, z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\bar{x} - \frac{\sigma}{10}}{\frac{\sigma}{\sqrt{n}}}$$

$$= \frac{5\bar{x}}{\sigma} - \frac{1}{2}$$

Since $\bar{x}$ is –ve, $z < -\frac{1}{2}$. Therefore,

$$P\left(z < -\frac{1}{2}\right) = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{1}{2}} e^{-\frac{z^2}{2}} dz$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\frac{1}{2}}^{\infty} e^{-\frac{z^2}{2}} dz = 0.3085$$

38. A sample of height of 6,400 soldiers has a mean of 67.85 inches and a standard deviation of 2.56 inches whereas a simple sample of heights of 1,600 sailors has a mean of 68.55 inches and a standard deviation of 2.52 inches. Do the data indicate that the sailors are on the average taller than soldiers?

Ans. Yes

39. A sample of 400 individuals is found to have a mean height of 67.47 inches. Can it be reasonably regarded as a sample from a large population with mean height of 67.39 inches and standard deviation 1.30 inches?

Hint: $\mu = 69.39$, $\sigma = 1.30$, $\bar{x} = 67.47$, $n = 400$, $z = 1.23$, Yes.

40. If 60 new entrants in a given university are found to have a mean height of 68.60 inches and 50 seniors a mean height of 69.51 inches, can we conclude that the mean height of the senior is greater than that of new entrants. Assume the standard deviation of height to be 2.48 inches.

Ans. No

41. Two kinds of a new plastic material are to be compared for strength. From tensile strength, measurement of 10 similar pieces of each type, the sample average and standard deviations were found as follows:

$\bar{x}_1 = 78.3$, $S_1 = 5.6$, $\bar{x}_2 = 84.2$, $S_2 = 6.3$ compare the mean strength, assuming normal data.

Hint: σ is not known, so calculate

$$\sigma^2 = \frac{n_1S_1^2 + n_2S_2^2}{n_1 + n_2} = \frac{10}{20}[(5.6)^2 + (6.3)^2] = 35.525$$

$$\therefore \sigma = 5.96 \text{(pooled estimate of standard deviation)}$$

Then

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{\frac{1}{2}}} = \frac{78.3 - 84.2}{\frac{5.96}{\sqrt{5}}} = -2.21$$

$| z | = 2.21 > 1.96$ implies that the difference is significant. Also 95% confidence interval is

$$\bar{x}_1 - \bar{x}_2 \pm 1.96\left[\sigma\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}\right] = 78.3 - 84.2 \pm 1.96\left[\frac{5.96}{\sqrt{5}}\right] = -5.90 \pm 4.95 = (-10.85, -0.95).$$

Since 0 does not lie within the interval, the difference is significant.

42. An examination was given to 50 students of a college A and to 60 students of college B. For A, the mean grade was 75 with standard deviation of 9 and for B, the mean grade was 79 with standard deviation of 7. Is there any significant difference between the performance of the students of college A and those of college B?

Ans. No

43. The mean yield and standard deviation of a set of 40 plots are 1258 kg and 34 kg whereas mean yield and standard deviation of another set of 60 plots are 1243 kg and 28 kg. Is the difference in the mean yields of two sets of plots significant?

Ans. $z = 2.3$, Yes at 5% level of conf dence

Significance for Single Proportion

44. A random sample of 500 apples was taken from a large consignment and 60 were found to be bad. Obtain 98% confidence limits far the percentage of bad apples in the consignment.

Ans. (0.086, 0.154), that is, 8.6% to 15.4%

45. A bag contains defective articles, the exact number of which is not known. A sample of 100 from the bag gives 10 defective articles. Find the limits for the proportion of defective articles in the bag.

Ans. $0.1 \pm 1.96\sqrt{\frac{0.1(0.9)}{100}} = (0.0412, 0.1589)$

46. A sample of 1,000 days is taken from meteorological records of a certain district and 120 of them are found to be foggy. What are the probable limits to percentage of foggy days in the district?

Ans. 8.91% to 15.07%

Significance for Difference of Proportion

47. Before an increase in excise duty on tea, 800 persons out of a sample of 1,000 persons were found to be tea drinkers. After an increase in duty, 800 people were the drinkers in a sample of 1,200 people. Using standard error of proportion, state whether there is a significant decrease in the consumption of tea after the increase in excise duty.

Ans. $z = 6.84$, signif cant decrease

48. One type of aircraft is found to develop engine trouble in 5 flights out of a total of 100 and another type in 7 flights out of 200 flights. Is there a significant difference in the two types of aircrafts so far as defects are concerned?

Ans. Difference is not signif cant

49. In a random sample of 400 students of the university teaching departments, it was found that 300 students failed in the examination. On another random sample of 500 students of the affiliated colleges, the number of failures in the same examination was found to be 300. Find out whether the proportion of failures in the university teaching departments is significantly greater than the proportion of failures in the university teaching departments and affiliated colleges taken together.

Ans. $z = 4.08$

50. In a random sample of 100 men taken from village A, 60 were found to be consuming alcohol. In another sample of 200 men taken from village B, 100 were found to be consuming alcohol. Do the two villages differ significantly of the proportion of men who consume alcohol?

Ans. $z = 1.64$

51. 500 articles from a factory are examined and found to be 2% defective. 800 similar articles from another factory are found to be only 1.5% defective. Can we conclude that the products of the first factory are inferior to those of the second?

Ans. $z = 0.68$, No

Significance for Difference of Standard Deviations

52. Random samples drawn from two countries A and B gave the following data regarding the heights (in inches) of the adult males

	Country A	Country B
Mean height	67:42	67:25
Standard deviation	2:58	2:50
Number in sample	1000	1200

Is the difference between the standard deviations significant?

Ans. $z = \dfrac{S_1 - S_2}{\sqrt{\dfrac{S_1^2}{2n_1} + \dfrac{S_2^2}{2n_2}}} = 1.03$

53. In Exercise 44, examine whether the difference in the variability in yields is significant.

Ans. $z = 1.31$, Difference not signif cant at 5% level of signif cance.

t-Distribution

54. A random sample of eight envelops is taken from letter box of a post office. The weights in grams are found to be 12.1, 11.9, 12.4, 12.3, 11.9, 12.1, 12.4, and 12.1. Find 99% confidence limits for the mean weight of the envelopes received at the post office.

Hint:

$$\bar{x} = 12.15, S = 0.2, \bar{x} \pm t_{0.05} \cdot \frac{S}{\sqrt{n}}$$

$$= 12.15 \pm 2.35 \frac{0.2}{\sqrt{8}}$$

$$= (11.984, 12.316).$$

55. The nine items of a sample have the following values: 45, 47, 50, 52, 48, 47, 49, 53, 51. Does the mean of these differ significantly from the assumed mean of 47.5?

Ans. Not signif cant at 5% level of signif cances

56. Two horses A and B were tested according to the time (in seconds) to run a particular track with the following results:

Horse A:	28	30	32	33	33	29	34
Horse B:	29	30	30	24	27	29	

Test whether the two horses have the same running capacity (use *t*-test)

Ans. t = 2.5, Yes

57. A sample of 10 measurements of the diameter of a sphere gave a mean of 12 cm and a standard deviation of 0.15 cm. Find 95% confidence limit for the actual diameter.

Ans. (11.887, 12.113)

58. For a random sample of 10 pigs fed on diet A, the increase in weight in a certain period were 16, 6, 16, 17, 13, 12, 8, 14, 15, 9 kg. For another random sample of 12 pigs fed on diet B, the increases in the same period were 7, 13, 22, 15, 12, 14, 18, 8, 21, 23, 10, 17 kg. Are these two samples significantly different regarding the effect of diet?

Ans. $t = 1.51$, Sample mean do not differ signif cantly

59. A car company has to decide between two brands A and B of tyre for its car. A trial is conducted using 12 of each brand, run until they tear out. The sample average and standard deviations of running distance (in km) are, respectively, 36,300 and 5,000 for A, and 39100 and 6100 for B. Obtain a 95% confidence interval for the difference in means assuming the distribution to be normal and test the hypothesis that brand B tyres outrun brand A tyres.

Hint: $S^2 = \frac{1}{n_1 + n_2 - 2}[(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2]$.

Here $n_1 = n_2 = 12$.

Degree of freedom $= n_1 + n_2 - 2 = 24 - 2 = 22$, $t_{0.05}$ at $v = 22$ is 1.71.

95% confidence interval is $36300 - 39100 \pm 1.71\left[S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right]$.

Also $t = \frac{36300 - 39100}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$. Find t and compare with $t_{0.05}$.

χ^2–Distribution

60. The following figures show the distribution of digits in numbers chosen at random from a telephone directory:

Digits:	0	1	2	3	4	5	6	7	8	9	Total
Frequency:	1026	1107	997	966	1075	933	1107	972	964	853	10000

Test whether the digits may be taken to occur equally and frequently in the directory.

Ans. $\chi^2 = 58.542, \chi^2_{0.05}(9) = 16.92$

61. A set of five similar coins is tossed 320 times and the result is

No. of heads:	0	1	2	3	4	5
Frequency:	6	27	72	112	71	32

Test the hypothesis that the data follow a binomial distribution.

Ans. $\chi^2 = 78.68, \chi^2_{0.05}(5) = 11.07$, hypothesis rejected

62. Fit a normal distribution to the data given below and test the goodness-of-fit.

x :	50	55	60	65	70	75	80	85	90	95	100
f:	2	3	5	9	10	12	7	2	3	1	0

Ans. good f t.

63. The following table gives the number of aircraft accidents occurred during the days of the week. Find whether the accidents are uniformly distributed over the week.

Days :	Monday	Tuesday	Wednesday	Thursday	Friday	Satuday
No. of accidents:	14	18	12	11	15	14

Ans. $\chi^2 = 2.14$

64. During proof reading 392 pages of a book of 1,200 pages were read. The distribution of printing mistakes were found to be as follows:

No. of mistakes in page (x)	0	1	2	3	4	5	6
No. of page (f)	275	72	30	7	5	2	1

Fit a Poisson distribution to the above data and test the goodness-of-fit.
Hints: The expected (theoretical) frequencies are 242.1, 116.7, 28.1, 4.5, 0.5, 0.1, 0. Further, $\chi^2 = 40.937, \chi^2_{0.05}(2) = 5.99$. Not a good f t.

65. A survey of 800 families with four children were taken. Each revealed the following distribution:

No. of boys:	0	1	2	3	4
No. of girls:	4	3	2	1	0
No. of families:	32	178	290	236	64

Test the hypothesis that male and female births are equally possible.

Hint: Probability for boy's birth $(p) = \frac{1}{2}$, *so* $q = \frac{1}{2}$. Fit binomial distribution to male birth, which is 50, 200, 300, 200, 50. Then proceed to find χ^2, which is 19.63 and $\chi^2_{0.05}(4) = 9.488$. Hypothesis rejected.

F-Distribution

66. Two samples of sizes 8 and 10, respectively, give the sum of the squares of deviations from their respective means equals to 84.4 and 102.6, respectively. Examine whether the samples have been drawn from normal population having the same variance.

Ans. $F = 1.057$, $F_{0.05}(7, 9) = 3.29$ Hypothesis accepted at 5% level of signif cance.

67. Two random samples from two normal populations are given below Do the estimates of population variance differ significantly?

Sample I:	16	26	27	23	24	22
Sample II :	33	42	35	32	28	31

Ans. F = 1.49, Do not differ signif cantly

68. The following are the values in thousands of an inch obtained by two engineers in 10 successive measurements with the same micrometer. Is one engineer significantly more consistent than the other?

Engineer A:	503	505	497	505	495	502	499	493	510	501
Engineer B:	502	497	492	498	499	495	497	496	498	

Ans. $F = 2.4$ $F_{0.05}(9,8) = 5.47$ Equally consistent.

ANOVA

69. Four machines A,B,C and D are used to produce a certain kind of cotton fabrics. Samples of size 4 with each unit as 100 square meters are selected from the outputs of the machines at random and the number of flaws in each 100 square meters are counted with following result.

A	*B*	*C*	*D*
8	6	14	20
9	8	12	22
11	10	18	25
12	4	9	23.

Is there significance difference in the performance of the four machines?
Hint: $V_c = 540.69$, $n_1 = 3$ and $R = 85.75$, $n_2 = 12$

Ans. Signif cant difference is there

70. Ten varieties of wheat are grown in 3 plots each and the following yields in quintal per acre obtained.

Variety / plot	1	2	3	4	5	6	7	8	9	10
	7	7	14	11	9	6	9	8	12	9
	8	9	13	10	9	7	13	13	11	11
	7	6	16	11	12	5	12	11	11	11

Test the significance of differences between variety yields

Ans. F = 8.22

71. The following data contains the results regarding the measures of I Q of male students of tall, short and medium stature. Is there significant difference in the I Q score relative to the height difference.

Tall	110	105	118	112	90
Short	95	103	115	107	96
Medium	108	112	93	104	97

Ans. Difference is in signif cant.

72. Fit a straight line trend for the following data and estimate the sale for the year 2001.

Year:	1999	2000	2001	2002	2003	2004	2005
Sale in Rs. Lakhs	33	35	60	67	68	82	90

Ans. $62.141 + 9.746x$ prediction for 2011 is Rs 149.80 Lakhs

73. Fit a least square line to following time series and predict the sale for the year 2005.

Year:	1994	1995	1996	1997	1998	1999	2000	2001	2002	2003
Sale: in Rs.	42	50	61	75	92	111	120	127	140	138

Ans. $y = 95.61 + 6x$ Prediction for 2005 is 174.

74. Find a trend line for the time series in Exercise 72 using semi average method.

75. Construct a four-year moving average and four-year centered moving average for the following time series.

Year:	1985	1986	1987	1988	1989	1990	1991	1992	1993
Production: in tons.	20	18	19	21	17	20	22	18	20

76. The following are the number of defects observed in the final inspection of 30 bales of woolen cloth:

0	3	1	4	2	2	1	3	5	0
2	0	0	1	2	4	3	0	0	0
1	2	4	5	0	9	4	10	3	6

Compute the values for an appropriate control chart. Is the process in a state of control?

Hint: The question is of c-chart,

$\bar{c} = 2.57$, $UCL = 7.38$

$LCL = -2.24$ (taken as 0)

Ans. Since some points lie onside the control limits, the process is not under control.

77. The values of sample means and range for the sample of size 5 each are given below. Construct $\bar{X}$ and R chart and comment.

Sample No.:	1	2	3	4	5	6	7	8	9	10
$\bar{X}$:	43	49	37	4	5	37	51	46	43	47
R:	5	6	5	7	7	4	8	6	4	6

Ans. Central line: 44.2

UCL = 47.564

LCL = 40.836

Not in a state of control.

78. The average number of defectives in 22 sampled lots of 2000 rubber belts each, was found to be 16%. Determine control limits for the p-chart.

3 Non-Linear Equations

The aim of this chapter is to discuss the most useful methods for finding the roots of any equation having numerical coefficients. Polynomial equations of degree ≤ 4 can be solved by standard algebraic methods. But no general method exists for finding the roots of the equations of the type $a \log x + bx = c$ or $ae^{-x} + b \tan x = 4$, etc. in terms of their coefficients. These equations are called transcendental equations. Therefore, we take help of numerical methods to solve such type of equations.

Let f be a continuous function. Any number ξ for which $f(\xi) = 0$ is called a root of the equation $f(x) = 0$. Also, ξ is called a zero of function $f(x)$.
A zero ξ is called of multiplicity p, if we can write

$$f(x) = (x - \xi)^p g(x),$$

where $g(x)$ is bounded at ξ and $g(\xi) \neq 0$. If $p = 1$, then ξ is said to be simple zero and if $p > 1$, then ξ is called a multiple zero.

3.1 CLASSIFICATION OF METHODS

The methods for finding roots numerically may be classified into the following two types:

1. **Direct Methods.** These methods require no knowledge of an initial approximation and are used for solving polynomial equations. The best known method is Graeffe's root squaring method.
2. **Iterative Methods.** There are many such methods. We shall discuss some of them in this chapter. In these methods, successive approximations to the solution are used. We begin with the first approximation and successively improve it till we get result to our satisfaction. For example, Newton–Raphson method is an iterative method.

Let $\{x_i\}$ be a sequence of approximate values of the root of an equation obtained by an iteration method and let x denote the exact root of the equation. Then the iteration method is said to be convergent if and only if

$$\lim_{n \to \infty} \left| x_n - x \right| = 0 \cdot$$

An iteration method is said to be of order p, if p is the smallest number for which there exists a finite constant k such that

$$\left| x_{n+1} - x \right| \le k \left| x_n - x \right|^p.$$

3.2 APPROXIMATE VALUES OF THE ROOTS

Let

$$f(x) = 0 \tag{3.1}$$

be the equation whose roots are to be determined. If we take a set of rectangular co-ordinate axes and plot the graph of

$$y = f(x), \tag{3.2}$$

then the values of x where the graph crosses the x-axis are the roots of the given equation (3.1), because at these points y is zero and therefore equation (3.1) is satisfied.

However, the following fundamental theorem is more useful than a graph.

Theorem 3.1. If f is continuous on $[a,b)$ and if $f(a)$ and $f(b)$ are of opposite signs, then there is at least one real root of $f(x) = 0$ between a and b.

In many cases, the approximate values of the real roots of $f(x) = 0$ are found by writing the equation in the form

$$f_1(x) = f_2(x) \tag{3.3}$$

and then plotting the graphs, on the same axes, of two equations $y_1 = f_1(x)$ and $y_2 = f_2(x)$. The abscissas of the point of intersection of these two curves are the real roots of the given equation because at these points $y_1 = y_2$ and therefore $f_1(x) = f_2(x)$. Hence, equation (3.3) is satisfied and consequently $f(x) = 0$ is satisfied.

For example, consider the equation $x\log_{10}x = 1.2$. We write the equation in the form

$$f(x) = x\log_{10} x - 1.2 = 0.$$

It is obvious from the table given below that $f(2)$ and $f(3)$ are of opposite signs:

x	:	1	2	3	4
$f(x)$	:	−1.2	−0.6	0.23	1.21

Therefore, a root lies between $x = 2$ and $x = 3$ and this is the only root.

The approximate value of the root can also be found by writing the equation in the form

$$\log_{10} x = \frac{1.2}{x}$$

and then plotting the graphs of $y_1 = \log_{10} x$ and $y_2 = 1.2/x$. The abscissa of the point of intersection of these graphs is the desired root.

3.3 BISECTION METHOD (BOLZANO METHOD)

Suppose that we want to find a zero of a continuous function f. We start with an initial interval $[a_0,b_0]$, where $f(a_0)$ and $f(b_0)$ have opposite signs. Since f is continuous, the graph of f will cross the x-axis at a root $x = \xi$ lying in $[a_0,b_0]$. Thus, the graph shall be as shown in Figure 3.1.

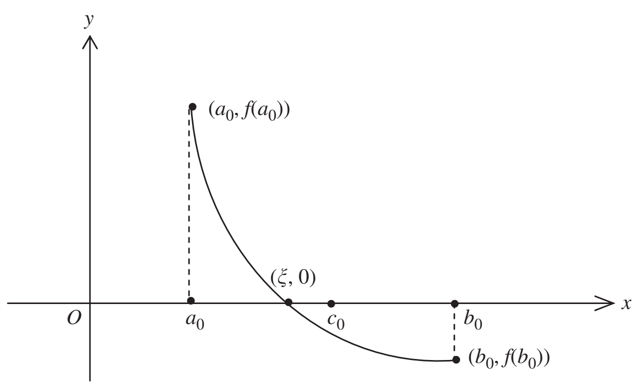


Figure 3.1

The bisection method systematically moves the endpoints of the interval closer and closer together until we obtain an interval of arbitrary small width that contains the root. We choose the midpoint $c_0 = (a_0 + b_0)/2$ and then consider the following possibilities:

(i) If $f(a_0)$ and $f(c_0)$ have opposite signs, then a root lies in $[a_0, c_0]$.

(ii) If $f(c_0)$ and $f(b_0)$ have opposite signs, then a root lies in $[c_0, b_0]$.

(iii) If $f(c_0) = 0$, then $x = c_0$ is a root.

If (iii) happens, then nothing to proceed as c_0 is the root in that case. If anyone of (i) and (ii) happens, let $[a_1, b_1]$ be the interval (representing $[a_0, c_0]$ or $[c_0, b_0]$) containing the root, where $f(a_1)$ and $f(b_1)$ have opposite signs. Let $c_1 = (a_1 + b_1)/2$ and $[a_2, b_2]$ represent $[a_1, c_1]$ or $[c_1, b_1]$ such that $f(a_2)$ and $f(b_2)$ have opposite signs. Then the root lies between a_2 and b_2. Continue with the process to construct an interval $[a_{n+1}, b_{n+1}]$, which contains the root and its width is half that of $[a_n, b_n]$. In this case $[a_{n+1}, b_{n+1}] = [a_n, c_n]$ or $[c_n, b_n]$ for all n.

Theorem 3.2. Let f be a continuous function on $[a,b]$ and let $\xi \in [a,b]$ be a root of $f(x) = 0$. If $f(a)$ and $f(b)$ have opposite signs and $\{c_n\}$ represents the sequence of the midpoints generated by the bisection process, then

$$|\xi - c_n| \le \frac{b-a}{2^{n+1}}, \quad n = 0,1,2,\ldots$$

and hence $\{c_n\}$ converges to the root $x = \xi$, that is, $\lim_{n\to\infty} c_n = \xi$.

Proof. Since both the root ξ and the midpoint c_n lie in $[a_n, b_n]$, the distance from c_n to ξ cannot be greater than half the width of $[a_n, b_n]$ as shown in Figure 3.2.

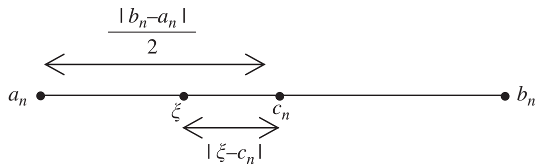


Figure 3.2

Thus,

$$|\xi - c_n| \le \frac{|b_n - a_n|}{2} \quad \text{for all } n.$$

But, we note that

$$|b_1 - a_1| = \frac{|b_0 - a_0|}{2},$$

$$|b_2 - a_2| = \frac{|b_1 - a_1|}{2} = \frac{|b_0 - a_0|}{2^2},$$

$$|b_3 - a_3| = \frac{|b_2 - a_2|}{2} = \frac{|b_0 - a_0|}{2^3}$$

$$\ldots$$

$$|b_n - a_n| = \frac{|b_{n-1} - a_{n-1}|}{2} = \frac{|b_0 - a_0|}{2^n}.$$

Hence,

$$|\xi - c_n| \le \frac{|b_0 - a_0|}{2^{n+1}} \quad \text{for all } n$$

and so $\lim_{n\to\infty} |\xi - c_n| = 0$ or $\lim_{n\to\infty} c_n = \xi$.

EXAMPLE 3.1

Find a real root of the equation $x^3 + x^2 - 1 = 0$ using bisection method.

Solution. Let

$$f(x) = x^3 + x^2 - 1.$$

Then $f(0) = -1, f(1) = 1$. Thus, a real root of $f(x) = 0$ lies between 0 and 1. Therefore, we take $x_0 = 0.5$. Then $f(0.5) = (0.5)^3 + (0.5)^2 - 1 = 0.125 + 0.25 - 1 = -0.625$.

This shows that the root lies between 0.5 and 1, and we get

$$x_1 = \frac{1 + 0.5}{2} = 0.75.$$

Then $f(x_1) = (0.75)^3 + (0.75)^2 - 1 = 0.421875 + 0.5625 - 1 = -0.015625$.

Hence, the root lies between 0.75 and 1. Thus, we take

$$x_2 = \frac{1 + 0.75}{2} = 0.875$$

and then

$$f(x_2) = 0.66992 + 0.5625 - 1 = 0.23242 \text{ (+ve)}.$$

It follows that the root lies between 0.75 and 0.875. We take

$$x_3 = \frac{0.75 + 0.875}{2} = 0.8125$$

and then

$$f(x_3) = 0.53638 + 0.66015 - 1 = 0.19653 \text{ (+ve)}.$$

Therefore, the root lies between 0.75 and 0.8125. So, let

$$x_4 = \frac{0.75 + 0.8125}{2} = 0.781,$$

which yields

$$f(x_4) = (0.781)^3 + (0.781)^2 - 1 = 0.086 \text{ (+ve)}.$$

Thus, the root lies between 0.75 and 0.781. We take

$$x_5 = \frac{0.750 + 0.781}{2} = 0.765$$

and note that

$$f(0.765) = 0.0335 \text{ (+ve)}.$$

Hence, the root lies between 0.75 and 0.765. So, let

$$x_6 = \frac{0.750 + 0.765}{2} = 0.7575$$

and then

$$f(0.7575) = 0.4346 + 0.5738 - 1 = 0.0084\ (+\text{ve}).$$

Therefore, the root lies between 0.75 and 0.7575.

Proceeding in this way, the next approximations shall be

$$x_7 = 0.7538, \quad x_8 = 0.7556, \quad x_9 = 0.7547,$$
$$x_{10} = 0.7551, \quad x_{11} = 0.7549, \quad x_{12} = 0.75486,$$

and so on.

EXAMPLE 3.2

Find a root of the equation $x^3 - 3x - 5 = 0$ by bisection method.

Solution. Let $f(x) = x^3 - 3x - 5$. Then we observe that $f(2) = -3$ and $f(3) = 13$. Thus, a root of the given equation lies between 2 and 3. Let $x_0 = 2.5$. Then

$$f(2.5) = (2.5)^3 - 3(2.5) - 5 = 3.125\ (+\text{ve}).$$

Thus, the root lies between 2.0 and 2.5. Then

$$x_1 = \frac{2 + 2.5}{2} = 2.25.$$

We note that $f(2.25) = -0.359375\ (-\text{ve})$. Therefore, the root lies between 2.25 and 2.5. Then we take

$$x_2 = \frac{2.25 + 2.5}{2} = 2.375$$

and observe that $f(2.375) = 1.2715$ (+ve). Hence, the root lies between 2.25 and 2.375. Therefore, we take

$$x_3 = \frac{2.25 + 2.375}{2} = 2.3125.$$

Now $f(2.3125) = 0.4289\ (+\text{ve})$. Hence, a root lies between 2.25 and 2.3125. We take

$$x_4 = \frac{2.25 + 2.3125}{2} = 2.28125.$$

Now

$$f(2.28125) = 0.0281\ (+\text{ve}).$$

We observe that the root lies very near to 2.28125. Let us try 2.280. Then

$$f(2.280) = 0.0124.$$

Thus, the root is 2.280 approximately.

3.4 REGULA–FALSI METHOD

The Regula–Falsi method, also known as method of false position, chord method or secant method, is the oldest method for finding the real roots of a numerical equation. We know that the root of the equation $f(x) = 0$ corresponds to abscissa of the point of intersection of the curve $y = f(x)$ with the x-axis. In Regula–Falsi method, we replace the curve by a chord in the interval, which contains a root of the equation $f(x) = 0$. We take the point of intersection of the chord with the x-axis as an approximation to the root.

Suppose that a root $x = \xi$ lies in the interval (x_{n-1}, x_n) and that the corresponding ordinates $f(x_{n-1})$ and $f(x_n)$ have opposite signs. The equation of the straight line through the points $P(x_n, f(x_n))$ and is $Q(x_{n-1}, f(x_{n-1}))$

$$\frac{f(x) - f(x_n)}{f(x_{n-1}) - f(x_n)} = \frac{x - x_n}{x_{n-1} - x_n}. \tag{3.4}$$

Let this straight line cut the x-axis at x_{n+1}. Since $f(x) = 0$ where the line (3.4) cuts the x-axis, we have, $f(x_{n+1}) = 0$ and so

$$x_{n+1} = x_n - \frac{x_{n-1} - x_n}{f(x_{n-1}) - f(x_n)} f(x_n). \tag{3.5}$$

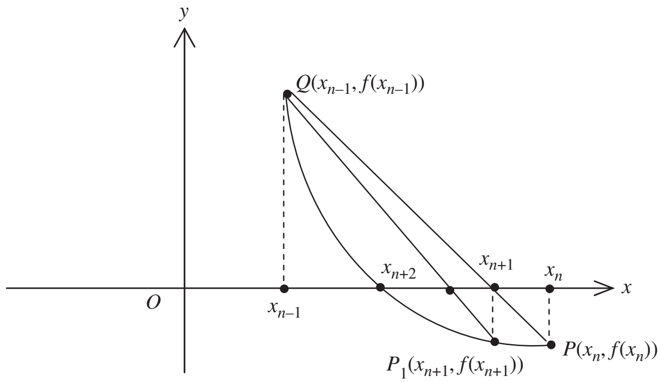


Figure 3.3

Now $f(x_{n-1})$ and $f(x_{n+1})$ have opposite signs. Therefore, it is possible to apply the approximation again to determine a line through the points Q and P_1. Proceeding in this way we find that as the points approach ξ, the curve becomes more nearly a straight line. Equation (3.5) can also be written in the form

$$x_{n+1} = \frac{x_n f(x_{n-1}) - x_{n-1} f(x_n)}{f(x_{n-1}) - f(x_n)}, \quad n = 1, 2, \ldots. \tag{3.6}$$

Equation (3.5) or (3.6) is the required formula for Regula–Falsi method.

3.5 CONVERGENCE OF REGULA–FALSI METHOD

Let ξ be the actual root of the equation $f(x) = 0$. Thus, $f(\xi) = 0$. Let $x_n = \xi + \varepsilon_n$, where ε_n is the error involved at the nth step while determining the root. Using

$$x_{n+1} = \frac{x_n f(x_{n-1}) - x_{n-1} f(x_n)}{f(x_{n-1}) - f(x_n)}, \quad n = 1, 2, \ldots,$$

we get

$$\xi + \varepsilon_{n+1} = \frac{(\xi + \varepsilon_n) f(\xi + \varepsilon_{n-1}) - (\xi + \varepsilon_{n-1}) f(\xi + \varepsilon_n)}{f(\xi + \varepsilon_{n-1}) - f(\xi + \varepsilon_n)}$$

and so

$$\varepsilon_{n+1} = \frac{(\xi+\varepsilon_n)f(\xi+\varepsilon_{n-1})-(\xi+\varepsilon_{n-1})f(\xi+\varepsilon_n)}{f(\xi+\varepsilon_{n-1})-f(\xi+\varepsilon_n)} - \xi$$

$$= \frac{\varepsilon_n f(\xi+\varepsilon_{n-1})-\varepsilon_{n-1}f(\xi+\varepsilon_n)}{f(\xi+\varepsilon_{n-1})-f(\xi+\varepsilon_n)}.$$

Expanding the right-hand side by Taylor's series, we get

$$\varepsilon_{n+1} = \frac{\varepsilon_n\left[f(\xi)+\varepsilon_{n-1}f'(\xi)+\frac{1}{2}\varepsilon_{n-1}^2 f''(\xi)+\cdots\right]-\varepsilon_{n-1}\left[f(\xi)+\varepsilon_n f'(\xi)+\frac{1}{2}\varepsilon_n^2 f''(\xi)+\ldots\right]}{f(\xi)+\varepsilon_{n-1}f'(\xi)+\frac{1}{2}\varepsilon_{n-1}^2 f''(\xi)+\cdots-f(\xi)-\varepsilon_n f'(\xi)-\frac{1}{2}\varepsilon_n^2 f''(\xi)-\cdots}$$

that is,

$$\varepsilon_{n+1} = k\varepsilon_{n-1}\varepsilon_n + O(\varepsilon_n^2), \tag{3.7}$$

where

$$k = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}.$$

We now try to determine some number m such that

$$\varepsilon_{n+1} = A\varepsilon_n^m \tag{3.8}$$

and

$$\varepsilon_n = A\varepsilon_{n-1}^m \quad \text{or} \quad \varepsilon_{n-1} = A^{-\frac{1}{m}}\varepsilon_n^{\frac{1}{m}}.$$

From equations (3.7) and (3.8), we get

$$\varepsilon_{n+1} = k\varepsilon_{n-1}\varepsilon_n = kA^{-\frac{1}{m}}\varepsilon_n^{\frac{1}{m}}\varepsilon_n$$

and so

$$A\varepsilon_n^m = kA^{-\frac{1}{m}}\varepsilon_n^{\frac{1}{m}}\varepsilon_n = kA^{-\frac{1}{m}}\varepsilon_n^{1+\frac{1}{m}}.$$

Equating powers of ε_n on both sides, we get

$$m = \frac{m+1}{m} \quad \text{or} \quad m^2 - m - 1 = 0,$$

which yields $m = \frac{1\pm\sqrt{5}}{2} = 1.618\,(\text{+ve value})$. Hence,

$$\varepsilon_{n+1} = A\varepsilon_n^{1.618}.$$

Thus, Regula–Falsi method is of order 1.618.

EXAMPLE 3.3

Find a real root of the equation $x^3 - 5x - 7 = 0$ using Regula–Falsi method.

Solution. Let $f(x) = x^3 - 5x - 7 = 0$. We note that $f(2) = -9$ and $f(3) = 5$. Therefore, one root of the given equation lies between 2 and 3. By Regula–Falsi method, we have

$$x_{n+1} = \frac{x_n f(x_{n-1}) - x_{n-1}f(x_n)}{f(x_{n-1}) - f(x_n)}, \quad n = 1, 2, 3, \ldots.$$

We start with $x_0 = 2$ and $x_1 = 3$. Then

$$x_2 = \frac{x_1 f(x_0) - x_0 f(x_1)}{f(x_0) - f(x_1)} = \frac{3(-9) - 2(5)}{-9-5} = \frac{37}{14} \approx 2.6.$$

But $f(2.6) = -2.424$ and $f(3) = 5$. Therefore,

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)} = \frac{(2.6)\ 5 + 3\ (2.424)}{5 + 2.424} = 2.73.$$

Now $f(2.73) = -0.30583$. Since we are getting close to the root, we calculate $f(2.75)$ which is found to be 0.046875. Thus, the next approximation is

$$x_4 = \frac{2.75 f(2.73) - (2.73) f(2.75)}{f(2.73) - f(2.75)}$$
$$= \frac{2.75(-0.303583) - 2.73(0.0468675)}{-0.303583 - 0.0468675} = 2.7473.$$

Now $f(2.747) = -0.0062$. Therefore,

$$x_5 = \frac{2.75 f(2.747) - 2.747 f(2.75)}{f(2.747) - f(2.75)}$$
$$= \frac{2.75(-0.0062) - 2.747(0.046875)}{-0.0062 - 0.046875} = 2.74724.$$

Thus, the root is 2.747 correct up to three places of decimal.

EXAMPLE 3.4

Solve $x \log_{10} x = 1.2$ by Regula–Falsi method.

Solution. We have $f(x) = x \log_{10} x - 1.2 = 0$. Then $f(2) = -0.60$ and $f(3) = 0.23$. Therefore, the root lies between 2 and 3. Then

$$x_2 = \frac{x_1 f(x_0) - x_0 f(x_1)}{f(x_0) - f(x_1)} = \frac{3(-0.6) - 2(0.23)}{-0.6 - 0.23} = 2.723.$$

Now $f(2.72) = 2.72 \log(2.72) - 1.2 = -0.01797$. Since we are getting closer to the root, we calculate $f(2.75)$ and have

$$f(2.75) = 2.75 \log\ (2.75) - 1.2 = 2.75\ (0.4393) - 1.2 = 0.00816.$$

Therefore,

$$x_3 = \frac{2.75\ (-0.01797) - 2.72\ (0.00816)}{-0.01797 - 0.00816} = \frac{-0.04942 - 0.02219}{-0.02613} = 2.7405.$$

Now $f(2.74) = 2.74 \log(2.74) - 1.2 = 2.74(0.43775) - 1.2 = -0.00056.$

Thus, the root lies between 2.74 and 2.75 and it is more close to 2.74. Therefore,

$$x_4 = \frac{2.75\ (-0.00056) - 2.74\ (0.00816)}{-0.00056 - 0.00816} = 2.7408.$$

Thus the root is 2.740 correct up to three decimal places.

EXAMPLE 3.5

Find by Regula–Falsi method the real root of the equation $\log x - \cos x = 0$ correct to four decimal places.

Solution. Let

$$f(x) = \log x - \cos x.$$

Then

$$f(1) = 0 - 0.54 = -0.54(-\text{ve})$$

$$f(1.5) = 0.176 - 0.071 = 0.105(+\text{ve}).$$

Therefore, one root lies between 1 and 1.5 and it is nearer to 1.5.
We start with $x_0 = 1$, $x_1 = 1.5$. Then, by Regula–Falsi method,

$$x_{n+1} = \frac{x_n f(x_{n-1}) - x_{n-1} f(x_n)}{f(x_{n-1}) - f(x_n)}$$

and so

$$x_2 = \frac{x_1 f(x_0) - x_0 f(x_1)}{f(x_0) - f(x_1)} = \frac{1.5(-0.54) - 1(0.105)}{-0.54 - 0.105} = 1.41860 \approx 1.42.$$

But, $f(x_2) = f(1.42) = 0.1523 - 0.1502 = 0.0021$. Therefore,

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)} = \frac{1.42(0.105) - 1.5(0.0021)}{0.105 - 0.0021} = 1.41836 \approx 1.4184.$$

Now $f(1.418) = 0.151676 - 0.152202 = -0.000526.$

Hence, the next iteration is

$$x_4 = \frac{x_3 f(x_2) - x_2 f(x_3)}{f(x_2) - f(x_3)} = \frac{1.418(0.0021) - (1.42)(-0.000526)}{0.0021 + 0.000526} = 1.41840.$$

EXAMPLE 3.6

Find the root of the equation $\cos x - xe^x = 0$ by secant method correct to four decimal places.

Solution. The given equation is

$$f(x) = \cos x - x e^x = 0\cdot$$

We note that $f(0) = 1, f(1) = \cos 1 - e = 0 - e = -e$ (–ve). Hence, a root of the given equation lies between 0 and 1. By secant method, we have

$$x_{n+1} = x_n - \frac{x_{n-1} - x_n}{f(x_{n-1}) - f(x_n)} f(x_n).$$

So taking initial approximation as
$x_0 = 0, x_1 = 1, f(x_0) = 1$ and $f(x_1) = -e = -2.1780$, we have

$$x_2 = x_1 - \frac{x_0 - x_1}{f(x_0) - f(x_1)} f(x_1) = 1 - \frac{-1}{1 + 2.178}(-2.178) = 0.3147.$$

Further, $f(x_2) = f(0.3147) = 0.5198$. Therefore,

$$x_3 = x_2 - \frac{x_1 - x_2}{f(x_1) - f(x_2)} f(x_2) = 0.3147 - \frac{1 - 0.3147}{-2.178 - 0.5198}(0.5198) = 0.4467.$$

Further, $f(x_3) = f(0.4467) = 0.2036.$ Therefore

$$x_4 = x_3 - \frac{x_2 - x_3}{f(x_2) - f(x_3)} f(x_3) = 0.4467 - \frac{0.3147 - 0.4467}{0.5198 - 0.2036}(0.2036) = 0.5318,$$

$$f(x_4) = f(0.5318) = -0.0432.$$

Therefore,

$$x_5 = x_4 - \frac{x_3 - x_4}{f(x_3) - f(x_4)} f(x_4) = 0.5318 - \frac{0.4467 - 0.5318}{0.20.36 + 0.0432}(-0.0432) = 0.5168,$$

and

$$f(x_5) = f(0.5168) = 0.0029.$$

Now

$$x_6 = x_5 - \frac{x_4 - x_5}{f(x_4) - f(x_5)} f(x_5) = 0.5168 - \frac{0.5318 - 0.5168}{-0.0432 - 0.0029}(0.0029) = 0.5177,$$

and

$$f(x_6) = f(0.5177) = 0.0002.$$

The sixth iteration is

$$x_7 = x_6 - \frac{x_5 - x_6}{f(x_5) - f(x_6)} f(x_6) = 0.5177 - \frac{0.5168 - 0.5177}{0.0029 - 0.0002}(0.0002) = 0.51776.$$

We observe that $x_6 = x_7$ up to four decimal places. Hence, $x = 0.5177$ is a root of the given equation correct to four decimal places.

3.6 NEWTON–RAPHSON METHOD

If the derivative of a function f can be easily found and is a simple expression, then the real roots of the equation $f(x) = 0$ can be computed rapidly by Newton–Raphson method.

Let x_0 denote the approximate value of the desired root and let h be the correction which must be applied to x_0 to give the exact value of the root x. Thus, $x = x_0 + h$ and so the equation $f(x) = 0$ reduces to $f(x_0 + h) = 0$. Expanding by Taylor's Theorem, we have

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0 + \theta h), \; 0 < \theta < 1.$$

Hence,

$$f(x_0) + xf'(x_0) + \frac{h^2}{2} f''(x_0 + \theta h) = 0.$$

If h is relatively small, we may neglect the term containing h^2 and have

$$f(x_0) + hf'(x_0) = 0.$$

Hence,

$$h = -\frac{f(x_0)}{f'(x_0)}$$

and so the improved value of the root becomes

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

If we use x_1 as the approximate value, then the next approximation to the root is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general, the $(n + 1)$th approximation is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0,1,2,3,\ldots. \tag{3.9}$$

Formula (3.9) in called Newton–Raphson method.

The expression $h = -\frac{f(x_0)}{f'(x_0)}$ is the fundamental formula in Newton–Raphson method. This formula tells us that the larger the derivative, the smaller is the correction to be applied to get the correct value of the root. This means, when the graph of f is nearly vertical where it crosses the x-axis, the correct value of the root can be found very rapidly and with very little labor. On the other hand, if the value of $f'(x)$ is small in the neighborhood of the root, the value of h given by the fundamental formula would be large and therefore the computation of the root shall be a slow process. Thus, Newton–Raphson method should not be used when the graph of f is nearly horizontal where it crosses the x-axis. Further, the method fails if $f'(x) = 0$ in the neighborhood of the root.

EXAMPLE 3.7

Find the smallest positive root of $x^3 - 5x + 3 = 0$.

Solution. We observe that there is a root between –2 and –3, a root between 1 and 2, and a (smallest) root between 0 and 1. We have

$$f(x) = x^3 - 5x + 3, \; f'(x) = 3x^2 - 5.$$

Then taking $x_0 = 1$, we have

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{(-1)}{-2} = 0.5,$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.5 + \frac{5}{34} = 0.64,$$

$$x_3 = 0.64 + \frac{0.062144}{3.7712} = 0.6565,$$

$$x_4 = 0.6565 + \frac{0.000446412125}{3.70702325} = 0.656620,$$

$$x_5 = 0.656620 + \frac{0.00000115976975}{3.70655053} = 0.656620431.$$

We observe that the convergence is very rapid even though x_0 was not very near to the root.

EXAMPLE 3.8

Find the positive root of the equation

$$x^4 - 3x^3 + 2x^2 + 2x - 7 = 0$$

by Newton–Raphson method.

Solution. We have $f(0) = -7, f(1) = -5, f(2) = -3, f(3) = 17$. Thus, the positive root lies between 2 and 3. The Newton–Raphson formula becomes

$$x_{n+1} = x_n - \frac{x_n^4 - 3x_n^3 + 2x_n^2 + 2x_n - 7}{4x_n^3 - 9x_n^2 + 4x_n + 2}.$$

Taking $x_0 = 2.1$, the improved approximations are

$$x_1 = 2.39854269,$$
$$x_2 = 2.33168543,$$
$$x_3 = 2.32674082,$$
$$x_4 = 2.32671518,$$
$$x_5 = 2.32671518.$$

Since $x_4 = x_5$, the Newton–Raphson formula gives no new values of x and the approximate root is correct to eight decimal places.

EXAMPLE 3.9

Use Newton–Raphson method to solve the transcendental equation $e^x = 5x$.

Solution. Let $f(x) = e^x - 5x = 0$. Then $f'(x) = e^x - 5$. The Newton–Raphson formula becomes

$$x_{n+1} = x_n - \frac{e^{x_n} - 5x_n}{e^{x_n} - 5}, \quad n = 0, 1, 2, 3, \ldots \ .$$

The successive approximations are

$$x_0 = 0.4, x_1 = 0.2551454079, x_2 = 0.2591682786,$$
$$x_3 = 0.2591711018, x_4 = 0.2591711018.$$

Thus, the value of the root is correct to 10 decimal places.

EXAMPLE 3.10

Find by Newton–Raphson method, the real root of the equation $3x = \cos x + 1$.

Solution. The given equation is

$$f(x) = 3x - \cos x - 1 = 0.$$

We have

$$f(0) = -2\,(-\text{ve}) \text{ and } f(1) = 3 - 0.5403 - 1 = 1.4597\,(+\text{ve}).$$

Hence, one of the roots of $f(x) = 0$ lies between 0 and 1. The values at 0 and 1 show that the root is nearer to 1. So let us take $x = 0.6$. Further,

$$f'(x) = 3 + \sin x.$$

Therefore, the Newton–Raphson formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}$$
$$= \frac{3x_n + x_n \sin x_n - 3x_n + \cos x_n + 1}{3 + \sin x_n} = \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n}.$$

Hence,

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} = \frac{0.6(0.5646) + 0.8253 + 1}{3 + 0.5646} = 0.6071,$$

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1} = \frac{(0.6071)(0.5705) + 0.8213 + 1}{3 + 0.5705} = 0.6071.$$

Hence the required root, correct to four decimal places, is 0.6071.

EXAMPLE 3.11

Using Newton–Raphson method, find a root of the equation $f(x) = x\sin x + \cos x = 0$ correct to three decimal places, assuming that the root is near to $x = \pi$.

Solution. We have

$$f(x) = x\sin x + \cos x = 0.$$

Therefore,

$$f'(x) = x\cos x + \sin x - \sin x = x\cos x.$$

Since the root is nearer to π, we take $x_0 = \pi$. By Newton–Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n \sin x_n + \cos x_n}{x_n \cos x_n}$$

$$= \frac{x_n^2 \cos x_n - x_n \sin x_n - \cos x_n}{x_n \cos x_n}$$

Thus,

$$x_1 = \frac{x_0^2 \cos x_0 - x_0 \sin x_0 - \cos x_0}{x_0 \cos x_0}$$

$$= \frac{\pi^2 \cos \pi - \pi \sin \pi - \cos \pi}{\pi \cos \pi} = \frac{1 - \pi^2}{\pi} = \frac{1 - 9.87755}{-3.142857} = 2.824,$$

$$x_2 = \frac{x_1^2 \cos x_1 - x_1 \sin x_1 - \cos x_1}{x_1 \cos x_1}$$

$$= \frac{(7.975)(-0.95) - (2.824)(0.3123) + (0.95)}{(2.824)(-0.95)}$$

$$= \frac{-7.576 - 0.8819 + 0.95}{-2.6828} = \frac{7.5179}{2.6828} = 2.8022,$$

$$x_3 = \frac{7.8512(-0.9429) - (2.8022)(0.3329) + 0.9429}{(2.8022)(-0.9429)}$$

$$= \frac{-7.4029 - 0.93285 + 0.9429}{-2.6422} = \frac{7.39285}{2.6422} = 2.797.$$

Calculate x_4 and x_5 similarly.

3.7 SQUARE ROOT OF A NUMBER USING NEWTON–RAPHSON METHOD

Suppose that we want to find the square root of N. Let

$$x = \sqrt{N} \quad \text{or} \quad x^2 = N.$$

We have

$$f(x) = x^2 - N = 0.$$

Then, Newton–Raphson method yields

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n}$$

$$= \frac{1}{2}\left[x_n + \frac{N}{x_n}\right],\ n = 0,1,2,3,\ldots.$$

For example, if $N = 10$, taking $x_0 = 3$ as an initial approximation, the successive approximations are

$$x_1 = 3.166666667,\ x_2 = 3.162280702,$$
$$x_3 = 3.162277660,\ x_4 = 3.162277660$$

correct up to nine decimal places.

However, if we take $f(x) = x^3 - Nx$ so that if $f(x) = 0$, then $x = \sqrt{N}$. Now $f'(x) = 3x^2 - N$ and so the Newton–Raphson method gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - Nx_n}{3x_n^2 - N} = \frac{2x_n^3}{3x_n^2 - N}.$$

Taking $x_0 = 3$, the successive approximations to $\sqrt{10}$ are

$$x_1 = 3.176,\ x_2 = 3.1623,\ x_3 = 3.16227,\ x_4 = 3.16227$$

correct up to five decimal places.

Suppose that we want to find the pth root of N. Then consider $f(x) = x^p - N$. The Newton–Raphson formula yields

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^p - N}{px_n^{p-1}}$$

$$= \frac{(p-1)x_n^p + N}{px_n^{p-1}},\quad n = 0,1,2,3,\ldots.$$

For $p = 3$, the formula reduces to

$$x_{n+1} = \frac{2x_n^3 + N}{3x_n^2} = \frac{1}{3}\left(2x_n + \frac{N}{x_n^2}\right).$$

If $N = 10$ and we start with the approximation $x_0 = 2$, then

$$x_1 = \frac{1}{3}\left(4 + \frac{10}{4}\right) = 2.16666,\ x_2 = 2.154503616,$$

$$x_3 = 2.154434692,\ x_4 = 2.154434690,\ x_5 = 2.154434690$$

correct up to eight decimal places.

3.8 ORDER OF CONVERGENCE OF NEWTON–RAPHSON METHOD

Suppose $f(x) = 0$ has a simple root at $x = \xi$ and let ε_n be the error in the approximation. Then $x_n = \xi + \varepsilon_n$. Applying Taylor's expansion of $f(x_n)$ and $f'(x_n)$ about the root ξ, we have

$$f(x_n) = \sum_{r=1}^{\infty} a_r \varepsilon_n^r \quad \text{and} \quad f'(x_n) = \sum_{r=1}^{\infty} r a_r \varepsilon_n^{r-1},$$

where $a_r = \dfrac{f^{(r)}(\xi)}{r!}$. Then

$$\frac{f(x_n)}{f'(x_n)} = \varepsilon_n - \frac{a_2}{a_1}\varepsilon_n^2 + O(\varepsilon_n^3).$$

Therefore, Newton–Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

gives

$$\xi + \varepsilon_{n+1} = \xi + \varepsilon_n - \left[\varepsilon_n - \frac{a_2}{a_1}\varepsilon_n^2 + O(\varepsilon_n^3)\right]$$

and so

$$\varepsilon_{n+1} = \frac{a_2}{a_1}\varepsilon_n^2 = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\varepsilon_n^2 .$$

If $\dfrac{1}{2}\dfrac{f''(\xi)}{f'(\xi)} < 1$, then

$$\varepsilon_{n+1} < \varepsilon_n^2. \tag{3.10}$$

It follows therefore that Newton–Raphson method has a quadratic convergence (or second order convergence) if $\dfrac{1}{2}\dfrac{f''(\xi)}{f'(\xi)} < 1$.

The inequality (3.10) implies that if the correction term $\dfrac{f(x_n)}{f'(x_n)}$ begins with n zeros, then the result is correct to about 2^n decimals. Thus, in Newton–Raphson method, the number of correct decimal roughly doubles at each stage.

3.9 FIXED POINT ITERATION

Let f be a real-valued function $f : \Re \to \Re$. Then a point $x \in \Re$ is said to be a fixed point of f if $f(x) = x$.

For example, let $I : \Re \to \Re$ be an identity mapping. Then all points of $\Re$ are fixed points for I since $I(x) = x$ for all $x \in \Re$. Similarly, a constant map of $\Re$ into $\Re$ has a unique fixed point.

Consider the equation

$$f(x) = 0. \tag{3.11}$$

The fixed point iteration approach to the solution of equation (3.11) is that it is rewritten in the form of an equivalent relation

$$x = \phi(x). \tag{3.12}$$

Then any solution of equation (3.11) is a fixed point of the iteration function ϕ. Thus, the task of solving the equation is reduced to find the fixed points of the iteration function ϕ.

Let x_0 be an initial solution (approximate value of the root of equation (3.11) obtained from the graph of f or otherwise). We substitute this value of x_0 in the right-hand side of equation (3.12) and obtain a better approximation x_1 given by

$$x_1 = \phi(x_0).$$

Then the successive approximations are

$$\begin{aligned} x_2 &= \phi(x_1), \\ x_3 &= \phi(x_2), \\ \ldots &\;\;\ldots \;\;\ldots \\ \ldots &\;\;\ldots \;\;\ldots \\ x_{n+1} &= \phi(x_n), \quad n = 0,1,2,3,\ldots. \end{aligned}$$

The iteration

$$x_{n+1} = \phi(x_n), \quad n = 0,1,2,3,\ldots$$

is called fixed point iteration.

Obviously, Regula–Falsi method and Newton–Raphson method are iteration processes.

3.10 CONVERGENCE OF ITERATION METHOD

We are interested in determining the condition under which the iteration method converges, that is, for which x_{n+1} converges to the solution of $x = \phi\ (x)$ as $n \to \infty$. Thus, if $x_{n+1} = x$ up to the number of significant figures considered, then x_n is a solution to that degree of approximation. Let ξ be the true solution of $x = \phi(x)$, that is,

$$\xi = \phi(\xi). \tag{3.13}$$

The first approximation is

$$x_1 = \phi(x_0). \tag{3.14}$$

Subtracting equation (3.14) from equation (3.13), we get

$$\begin{aligned} \xi - x_1 &= \phi(\xi) - \phi(x_0) \\ &= (\xi - x_0)\phi'(\xi_0), \quad x_0 < \xi_0 < \xi, \end{aligned}$$

by Mean Value Theorem. Similar equations hold for successive approximations so that

$$\begin{aligned} \xi - x_2 &= (\xi - x_1)\phi'(\xi_1), \quad x_1 < \xi_1 < \xi \\ \xi - x_3 &= (\xi - x_2)\phi'(\xi_2), \quad x_2 < \xi_2 < \xi \\ \ldots \quad & \quad \ldots \quad\quad \ldots \\ \xi - x_{n+1} &= (\xi - x_n)\phi'(\xi_n), \quad x_n < \xi_n < \xi. \end{aligned}$$

Multiplying together all the equations, we get

$$\xi - x_{n+1} = (\xi - x_0)\phi'(\xi_0)\phi'(\xi_1)\ldots\phi'(\xi_n)$$

and so

$$|\xi - x_{n+1}| = |\xi - x_0||\phi'(\xi_0)|\ldots|\phi'(\xi_n)|.$$

If each of $|\phi'(\xi_0)|,\ldots|\phi'(\xi_n)|$ is less than or equal to $k < 1$, then

$$|\xi - x_{n+1}| \le |\xi - x_0|k^{n+1} \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, the error $\xi - x_{n+1}$ can be made as small as we please by repeating the process a sufficient number of times. Thus, the condition for convergence is

$$|\phi'(x)| < 1$$

in the neighborhood of the desired root.

Consider the iteration formula $x_{n+1} = \phi(x_n), n = 0,1,2,.$ If ξ is the true solution of $x = \phi(x)$, then $\xi = \phi(\xi)$. Therefore,

$$\begin{aligned}\xi - x_{n+1} &= \phi(\xi) - \phi(x_n) = (\xi - x_n)\phi'(\xi)\\ &= (\xi - x_n)k, \quad |\phi'(\xi)| \le k < 1,\end{aligned}$$

which shows that the iteration method has a linear convergence. This slow rate of convergence can be accelerated in the following way: we write

$$\xi - x_{n+1} = (\xi - x_n)k$$

$$\xi - x_{n+2} = (\xi - x_{n+1})k.$$

Dividing, we get

$$\frac{\xi - x_{n+1}}{\xi - x_{n+2}} = \frac{\xi - x_n}{\xi - x_{n+1}}$$

or

$$(\xi - x_{n+1})^2 = (\xi - x_{n+2})(\xi - x_n)$$

or

$$\xi = x_{n+2} - \frac{(x_{n+2} - x_{n+1})^2}{x_{n+2} - 2x_{n+1} + x_n} = x_{n+2} - \frac{(\Delta x_{n+1})^2}{\Delta^2 x_n}. \qquad (3.15)$$

Formula (3.15) is called the Aitken's Δ^2-method.

3.11 SQUARE ROOT OF A NUMBER USING ITERATION METHOD

Suppose that we want to find square root of a number, say N. This is equivalent to say that we want to find x such that $x^2 = N$, that is, $x = \frac{N}{x}$ or $x + x = x + \frac{N}{x}$. Thus,

$$x = \frac{x + \frac{N}{x}}{2}.$$

Thus, if x_0 is the initial approximation to the square root, then

$$x_{n+1} = \frac{x_n + \frac{N}{x_n}}{2}, \quad n = 0,1,2,\ldots .$$

Suppose N = 13. We begin with the initial approximation of $\sqrt{13}$ found by bisection method. The solution lies between 3.5625 and 3.625. We start with $x_0 = \frac{3.5625 + 3.6250}{2} \approx 3.59375$. Then, using the above iteration formula, we have

$$x_1 = 3.6055705, \; x_2 = 3.6055513, \; x_3 = 3.6055513$$

correct up to seven decimal places.

3.12 SUFFICIENT CONDITION FOR THE CONVERGENCE OF NEWTON–RAPHSON METHOD

We know that an iteration method $x_{n+1} = \phi(x_n)$ converges if $|\phi'(x)| < 1$. Since Newton–Raphson method is an iteration method, where $\phi(x) = x - \dfrac{f(x)}{f'(x)}$ and therefore it converges if $|\phi'(x)| < 1$, that is, if

$$\left| 1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2} \right| < 1,$$

that is, if

$$|f(x)f''(x)| < (f'(x))^2,$$

which is the required sufficient condition for the convergence of Newton–Raphson method.

EXAMPLE 3.12

Derive an iteration formula to solve $f(x) = x^3 + x^2 - 1 = 0$ and solve the equation.

Solution. Since $f(0)$ and $f(1)$ are of opposite signs, there is a root between 0 and 1. We write the equation in the form

$$x^3 + x^2 = 1, \text{ that is, } x^2(x+1) = 1, \text{ or } x^2 = \frac{1}{x+1},$$

or equivalently,

$$x = \frac{1}{\sqrt{1+x}}.$$

Then

$$x = \phi(x) = \frac{1}{\sqrt{1+x}}, \ \phi'(x) = -\frac{1}{2(1+x)^{\frac{3}{2}}}$$

so that

$$|\phi'(x)| < 1 \quad \text{for } x < 1.$$

Hence, this iteration method is applicable. We start with $x_0 = 0.75$ and obtain the next approximations to the root as

$$x_1 = \phi(x_0) = \frac{1}{\sqrt{1+x_0}} \approx 0.7559, x_2 = \phi(x_1) \approx 0.7546578, x_3 \approx 0.7549249,$$

$$x_4 \approx 0.7548674, \ x_5 \approx 0.754880, \ x_6 \approx 0.7548772, \ x_7 \approx 0.75487767$$

correct up to six decimal places.

EXAMPLE 3.13

Find, by the method of iteration, a root of the equation $2x - \log_{10} x = 7$.

Solution. The fixed point form of the given equation is

$$x = \frac{1}{2}(\log_{10} x + 7).$$

From the intersection of the graphs $y_1 = 2x - 7$ and $y_2 = \log_{10} x$, we find that the approximate value of the root is 3.8. Therefore,

$$x_0 = 3.8,\ x_1 = \frac{1}{2}(\log 3.8 + 7) \approx 3.78989,$$

$$x_2 = \frac{1}{2}(\log 3.78989 + 7) \approx 3.789313,$$

$$x_3 = \frac{1}{2}\ (\log 3.789313 + 7) \approx 3.78928026,$$

$$x_4 \approx 3.789278,\ x_5 \approx 3.789278$$

correct up to six decimal places.

EXAMPLE 3.14

Use iteration method to solve the equation $e^x = 5x$.

Solution. The iteration formula for the given problem is

$$x_{n+1} = \frac{1}{5}e^{x_n}.$$

We start with $x_0 = 0.3$ and get the successive approximations as

$$x_1 = \frac{1}{5}(1.34985881) = 0.269972,\ x_2 = 0.26198555,$$

$$x_3 = 0.25990155,\ x_4 = 0.259360482,$$

$$x_5 = 0.259220188,\ x_6 = 0.259183824,$$

$$x_7 = 0.259174399,\ x_8 = 0.259171956,$$

$$x_9 = 0.259171323,\ x_{10} = 0.259171159,$$

correct up to six decimal places.

If we use Aitken's Δ^2-method, then

$$x_3 = x_2 - \frac{(\Delta x_1)^2}{\Delta^2 x_0} = x_2 - \frac{(x_2 - x_1)^2}{x_2 - 2x_1 + x_0} = 0.26198555 - \frac{0.000063783}{0.02204155} = 0.259091$$

and so on.

3.13 NEWTON'S METHOD FOR FINDING MULTIPLE ROOTS

If ξ is a multiple root of an equation $f(x) = 0$, then $f(\xi) = f'(\xi) = 0$ and therefore the Newton–Raphson method fails. However, in case of multiple roots, we proceed as follows:

Let ξ be a root of multiplicity m. Then

$$f(x) = (x - \xi)^m A(x) \tag{3.16}$$

We make use of a localized approach that in the immediate vicinity (neighborhood) of $x = \xi$, the relation (3.16) can be written as

$$f(x) = A(x - \xi)^m,$$

where $A = A(\xi)$ is effectively constant. Then

$$f'(x) = mA(x - \xi)^{m-1}$$

$$f''(x) = m(m-1)A(m - \xi)^{m-2}, \text{ and so on.}$$

We thus obtain

$$\frac{f'(x)}{f(x)} = \frac{m}{x-\xi}$$

or

$$\xi = x - \frac{mf(x)}{f'(x)},$$

where x is close to ξ, which is a modification of Newton's rule for a multiple root. Thus, if x_1 is in the neighborhood of a root ξ of multiplicity m of an equation $f(x) = 0$, then

$$x_2 = x_1 - m\frac{f(x_1)}{f'(x_1)}$$

is an even more close approximation to ξ. Hence, in general, we have

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}. \tag{3.17}$$

Remark 3.1. **(i)** The case $m = 1$ of equation (3.17) yields Newton–Raphson method.
(ii) If two roots are close to a number, say x, then

$$f(x+\varepsilon) = 0 \text{ and } f(x-\varepsilon) = 0,$$

that is,

$$f(x)+\varepsilon f'(x)+\frac{\varepsilon^2}{2!}f''(x)+\cdots = 0,\ f(x)-\varepsilon f'(x)+\frac{\varepsilon^2}{2!}f''(x)-\cdots = 0.$$

Since ε is small, adding the above expressions, we get

$$0 = 2f(x)+\varepsilon^2 f''(x) = 0$$

or

$$\varepsilon^2 = -2\frac{f(x)}{f''(x)}$$

or

$$\varepsilon = \pm\sqrt{\frac{-2f(x)}{f''(x)}}.$$

So in this case, we take two approximations as $x+\varepsilon$ and $x-\varepsilon$ and then apply Newton–Raphson method.

EXAMPLE 3.15

The equation $x^4 - 5x^3 - 12x^2 + 76x - 79 = 0$ has two roots close to $x = 2$. Find these roots to four decimal places.

Solution. We have

$$f(x) = x^4 - 5x^3 - 12x^2 + 76x - 79$$

$$f'(x) = 4x^3 - 15x^2 - 24x + 76$$

$$f''(x) = 12x^2 - 30x - 24.$$

Thus

$$f(2) = 16 - 40 - 48 + 152 - 79 = 1$$

$$f''(2) = 48 - 60 - 24 = -36.$$

Therefore,

$$\varepsilon = \pm\sqrt{\frac{-2f(2)}{f''(2)}} = \pm\sqrt{\frac{-2}{-36}} = \pm 0.2357.$$

Thus, the initial approximations to the roots are

$$x_0 = 2.2357 \quad \text{and} \quad y_0 = 1.7643.$$

The application of Newton–Raphson method yields

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.2357 + 0.00083 = 2.0365.$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.2365 + 0.000459 = 2.24109.$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.24109 - 0.00019 = 2.2410.$$

Thus, one root, correct to four decimal places is 2.2410. Similarly, the second root correct to four decimal places will be found to be 1.7684.

EXAMPLE 3.16

Find a double root of the equation

$$x^3 - 5x^2 + 8x - 4 = 0$$

near 1.8.

Solution. We have

$$f(x) = x^3 - 5x^2 + 8x - 4$$

$$f'(x) = 3x^2 - 10x + 8$$

and $x_0 = 1.8$. Therefore,

$$f(x_0) = f(1.8) = 5.832 - 16.2 + 14.4 - 4 = 0.032$$

$$f'(x_0) = 9.72 - 18 + 8 = -0.28.$$

Hence,

$$x_1 = x_0 - 2\frac{f(x_0)}{f'(x_0)} = 1.8 - 2\frac{f(1.8)}{f'(1.8)} = 1.8 - 2\frac{0.032}{-0.28} = 2.02857.$$

We take $x_1 = 2.028$. Then

$$f(x_1) = 8.3407 - 20.5639 + 16.224 - 4 = 0.0008$$

$$f'(x_1) = 12.3384 - 20.28 + 8 = 0.0584.$$

Therefore,

$$x_2 = x_1 - 2\frac{f(x_1)}{f'(x_1)}$$

$$= 2.028 - \frac{2(0.0008)}{0.0584} = 2.0006,$$

which is quite close to the actual double root 2.

EXAMPLE 3.17

Find the double root of $x^3 - x^2 - x + 1 = 0$ close to 0.8.

Solution. We have

$$f(x) = x^3 - x^2 - x + 1 = 0$$

$$f'(x) = 3x^2 - 2x - 1.$$

We choose $x_0 = 0.8$. Then

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}$$

and so

$$x_1 = x_0 - 2\frac{f(0.8)}{f'(0.8)} = 0.8 - 2\left(\frac{(0.8)^3 - (0.8)^2 - 0.8 + 1}{3(0.8)^2 - 2(0.8) - 1}\right) = 1.01176$$

$$x_2 = x_1 - \frac{2f(1.0118)}{f'(1.0118)} = 1.0118 - 0.0126 = 0.9992,$$

which is very close to the actual double root 1.

3.14 NEWTON–RAPHSON METHOD FOR SIMULTANEOUS EQUATIONS

We consider the case of two equations in two unknowns. So let the given equations be

$$\phi(x, y) = 0, \tag{3.18}$$

$$\psi(x, y) = 0 \tag{3.19}$$

Now if x_0, y_0 be the approximate values of a pair of roots and *h*, *k* be the corrections, we have

$$x = x_0 + h \quad \text{and} \quad y = y_0 + k.$$

Then equations (3.18) and (3.19) become

$$\phi(x_0 + h, y_0 + k) = 0 \tag{3.20}$$

$$\psi(x_0 + h, y_0 + k) = 0. \tag{3.21}$$

Expanding equations (3.20) and (3.21) by Taylor's Theorem for a function of two variables, we have

$$\phi(x_0 + h, y_0 + k) = \phi(x_0, y_0) + h\left(\frac{\partial \phi}{\partial x}\right)_{x=x_0} + k\left(\frac{\partial \phi}{\partial y}\right)_{y=y_0} + \cdots = 0,$$

$$\psi(x_0 + h, y_0 + k) = \psi(x_0, y_0) + h\left(\frac{\partial \psi}{\partial x}\right)_{x=x_0} + k\left(\frac{\partial \psi}{\partial y}\right)_{y=y_0} + \cdots = 0.$$

Since *h* and *k* are relatively small, their squares, products, and higher powers can be neglected. Hence,

$$\phi(x_0, y_0) + h\left(\frac{\partial \phi}{\partial x}\right)_{x=x_0} + k\left(\frac{\partial \phi}{\partial y}\right)_{y=y_0} = 0 \tag{3.22}$$

$$\psi(x_0, y_0) + h\left(\frac{\partial \psi}{\partial x}\right)_{x=x_0} + k\left(\frac{\partial \psi}{\partial y}\right)_{y=y_0} = 0. \tag{3.23}$$

Solving the equations (3.22) and (3.23) by Cramer's rule, we get

$$h = \frac{\begin{vmatrix} -\phi(x_0, y_0) & \left(\dfrac{\partial \phi}{\partial y}\right)_{y=y_0} \\ -\psi(x_0, y_0) & \left(\dfrac{\partial \psi}{\partial y}\right)_{y=y_0} \end{vmatrix}}{D},$$

$$k = \frac{\begin{vmatrix} \left(\dfrac{\partial \phi}{\partial x}\right)_{x=x_0} & -\phi(x_0, y_0) \\ \left(\dfrac{\partial \psi}{\partial x}\right)_{x=x_0} & -\psi(x_0, y_0) \end{vmatrix}}{D},$$

where

$$D = \begin{vmatrix} \left(\dfrac{\partial \phi}{\partial x}\right)_{x=x_0} & \left(\dfrac{\partial \phi}{\partial y}\right)_{y=y_0} \\ \left(\dfrac{\partial \psi}{\partial x}\right)_{x=x_0} & \left(\dfrac{\partial \psi}{\partial y}\right)_{y=y_0} \end{vmatrix}$$

Thus,

$$x_1 = x_0 + h, \;\; y_1 = y_0 + k.$$

Additional corrections can be obtained by repeated application of these formulae with the improved values of x and y substituted at each step.

Proceeding as in Section 3.10, we can prove that the iteration process for solving simultaneous equations $\phi(x, y) = 0$ and $\psi(x, y) = 0$ converges if

$$\left|\frac{\partial \phi}{\partial x}\right| + \left|\frac{\partial \psi}{\partial x}\right| < 1 \quad \text{and} \quad \left|\frac{\partial \phi}{\partial y}\right| + \left|\frac{\partial \psi}{\partial y}\right| < 1 .$$

Remark 3.2. The Newton–Raphson method for simultaneous equations can be used to find complex roots. In fact the equation $f(z) = 0$ is $u(x, y) + iv(x, y) = 0$. So writing the equation as

$$u(x, y) = 0$$
$$v(x, y) = 0,$$

we can find x and y, thereby yielding the complex root.

EXAMPLE 3.18

Solve by Newton–Raphson method

$$x + 3\log_{10} x - y^2 = 0,$$
$$2x^2 - xy + 5x + 1 = 0.$$

Solution. On plotting the graphs of these equations on the same set of axes, we find that they intersect at the points (1.4,–1.5) and (3.4, 2.2). We shall compute the second set of values correct to four decimal places. Let

$$\phi(x, y) = x + 3\log_{10} x - y^2,$$
$$\psi(x, y) = 2x^2 - xy - 5x + 1.$$

Then

$$\frac{\partial \phi}{\partial x} = 1 + \frac{3M}{x}, \quad M = 0.43429$$

$$= 1 + \frac{1.30287}{x}$$

$$\frac{\partial \phi}{\partial y} = -2y$$

$$\frac{\partial \psi}{\partial x} = 4x - y - 5$$

$$\frac{\partial \psi}{\partial y} = -x.$$

Now $x_0 = 3.4, y_0 = 2.2.$ Therefore,

$$\phi(x_0, y_0) = 0.1545, \quad \psi(x_0, y_0) = 0.72,$$

$$\left(\frac{\partial \phi}{\partial x}\right)_{x=x_0} = 1.383, \quad \left(\frac{\partial \phi}{\partial y}\right)_{y=y_0} = 4.4,$$

$$\left(\frac{\partial \psi}{\partial x}\right)_{x=x_0} = 6.4, \quad \left(\frac{\partial \psi}{\partial y}\right)_{y=y_0} = -3.1.$$

Putting these values in

$$\phi(x_0, y_0) + h_1\left(\frac{\partial \phi}{\partial x}\right)_{x=x_0} + k_1\left(\frac{\partial \phi}{\partial y}\right)_{y=y_0} = 0,$$

$$\psi(x_0, y_0) + h_1\left(\frac{\partial \psi}{\partial x}\right)_{x=x_0} + k_1\left(\frac{\partial \psi}{\partial y}\right)_{y=y_0} = 0,$$

we get

$$0.1545 + h_1(1.383) + k_1(4.4) = 0$$

$$-0.72 + h_1(6.4) + k_1(-3.1) = 0.$$

Solving these for h_1 and k_1, we get

$$h_1 = 0.157 \quad \text{and} \quad k_1 = 0.085.$$

Thus,

$$x_1 = 3.4 + 0.517 = 3.557,$$

$$y_1 = 2.2 + 0.085 = 2.285.$$

Now

$$\phi(x_1, y_1) = 0.011, \quad \psi(x_1, y_1) = 0.3945,$$

$$\left(\frac{\partial \phi}{\partial x}\right)_{x=x_1} = 1.367, \quad \left(\frac{\partial \phi}{\partial y}\right)_{y=y_1} = -4.57,$$

$$\left(\frac{\partial \psi}{\partial x}\right)_{x=x_1} = 6.943, \quad \left(\frac{\partial \psi}{\partial y}\right)_{y=y_1} = -3.557.$$

Putting these values in

$$\phi(x_1, y_1) + h_2\left(\frac{\partial\phi}{\partial x}\right)_{x=x_1} + k_2\left(\frac{\partial\phi}{\partial y}\right)_{y=y_1} = 0,$$

$$\psi(x_1, y_1) + h_2\left(\frac{\partial\psi}{\partial x}\right)_{x=x_1} + k_2\left(\frac{\partial\psi}{\partial y}\right)_{y=y_1} = 0$$

and solving the equations so obtained, we get

$$h_2 = -0.0685,\ k_2 = -0.0229.$$

Hence,

$$x_2 = x_1 + h_2 = 3.4885 \quad \text{and} \quad y_2 = y_1 + k_2 = 2.2621.$$

Repeating the process, we get

$$h_3 = -0.0013,\ k_3 = -0.000561.$$

Hence, the third approximations are

$$x_3 = 3.4872 \quad \text{and} \quad y_3 = 2.26154.$$

Finding the next approximation, we observe that the above approximation is correct to four decimal places.

EXAMPLE 3.19

Find the roots of $1 + z^2 = 0$, taking initial approximation as $(x_0, y_0) = \left(\frac{1}{2}, \frac{1}{2}\right)$.

Solution. We have

$$f(z) = 1 + (x + iy)^2 = 1 + x^2 - y^2 + 2ixy = u + iv,$$

where

$$u(x, y) = 1 + x^2 - y^2,$$
$$v(x, y) = 2xy.$$

Then

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y,$$
$$\frac{\partial v}{\partial x} = 2y, \quad \frac{\partial v}{\partial y} = -2x.$$

Taking initial approximation as $(x_0, y_0) = \left(\frac{1}{2}, \frac{1}{2}\right)$, we have

$$u(x_0, y_0) = u\left(\frac{1}{2}, \frac{1}{2}\right) = 1 + \frac{1}{4} - \frac{1}{4} = 1,$$

$$v(x_0, y_0) = v\left(\frac{1}{2}, \frac{1}{2}\right) = 2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{2},$$

$$u_x(x_0, y_0) = u_x\left(\frac{1}{2}, \frac{1}{2}\right) = 2\left(\frac{1}{2}\right) = 1,$$

$$u_y(x_0, y_0) = u_y\left(\frac{1}{2}, \frac{1}{2}\right) = -2\left(\frac{1}{2}\right) = -1,$$

$$v_x(x_0, y_0) = v_x\left(\frac{1}{2}, \frac{1}{2}\right) = 2\left(\frac{1}{2}\right) = 1,$$

$$v_y(x_0, y_0) = v_y\left(\frac{1}{2}, \frac{1}{2}\right) = 2\left(\frac{1}{2}\right) = 1.$$

Putting these values in

$$u(x_0, y_0) + h_1 u_x(x_0, y_0) + k_1 u_y(x_0, y_0) = 0$$

and

$$v(x_0, y_0) + h_1 v_x(x_0, y_0) + k_1 v_y(x_0, y_0) = 0,$$

we get

$$1 + h_1 - k_1 = 0 \text{ and } \frac{1}{2} + h_1 - k_1 = 0.$$

Solving these equations for h_1 and k_1, we get $h_1 = -\frac{3}{4}, k_1 = \frac{1}{4}$. Hence,

$$x_1 = x_0 + h_1 = \frac{1}{2} - \frac{3}{4} = -\frac{1}{4},$$

$$y_1 = y_0 + k_1 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

Now

$$u(x_1, y_1) = 1 + \frac{1}{16} - \frac{9}{16} = \frac{1}{2},$$

$$v(x_1, y_1) = 2\left(-\frac{1}{4}\right)\left(\frac{3}{4}\right) = -\frac{3}{8},$$

$$u_x(x_1, y_1) = 2\left(-\frac{1}{4}\right) = -\frac{1}{2},$$

$$u_y(x_1, y_1) = -2\left(\frac{3}{4}\right) = -\frac{3}{2},$$

$$v_x(x_1, y_1) = 2\left(\frac{3}{4}\right) = \frac{3}{2},$$

$$v_y(x_1, y_1) = 2\left(-\frac{1}{4}\right) = -\frac{1}{2}.$$

Putting these values in

$$u(x_1, y_1) + h_2 u_x(x_1, y_1) + k_2 u_y(x_1, y_1) = 0$$

and

$$v(x_1, y_1) + h_2 v_x(x_1, y_1) + k_2 v_y(x_1, y_1) = 0,$$

we get

$$\frac{1}{2} - \frac{1}{2}h_2 - \frac{3}{2}k_2 = 0 \quad \text{and} \quad -\frac{3}{8} + \frac{3}{2}h_2 - \frac{1}{2}k_2 = 0.$$

Solving these equations, we get $h_2 = \frac{13}{40}$, $k_2 = \frac{9}{40}$. Hence,

$$x_2 = x_1 + h_2 = -\frac{1}{4} + \frac{13}{40} = \frac{3}{40} = 0.075,$$

$$y_2 = y_1 + k_2 = \frac{3}{4} + \frac{9}{40} = \frac{39}{40} = 0.975.$$

Proceeding in the same fashion, we get

$$x_3 = -0.00172 \text{ and } y_3 = 0.9973.$$

EXERCISES

1. Find the root of the equation $x - \cos x = 0$ by bisection method.

Ans. 0.739.

2. Find a positive root of equation $xe^x = 1$ lying between 0 and 1 using bisection method.

Ans. 0.567.

3. Solve $x^3 - 4x - 9 = 0$ by Bolzano method.

Ans. 2.706.

4. Use Regula–Falsi method to solve $x^3 + 2x^2 + 10x - 20 = 0$.

Ans. 1.3688.

5. Use the method of false position to obtain a root of the equation $x^3 - x + 4 = 0$.

Ans. 1.796.

6. Solve $e^x \sin x = 1$ by Regula–Falsi method.

Ans. 0.5885.

7. Using Newton–Raphson method find a root of the equation $x \log_{10} x = 1.2$.

Ans. 2.7406.

8. Use Newton–Raphson method to obtain a root of $x - \cos x = 0$.

Ans. 0.739.

9. Solve $\sin x = 1 + x^3$ by Newton–Raphson method.

Ans. −1.24905.

10. Find the real root of the equation $3x = \cos x + 1$ using Newton–Raphson method.

Ans. 0.6071.

11. Derive the formula $x_{i+1} = \frac{1}{2}\left(x_i + \frac{N}{x_i}\right)$ to determine square root of N. Hence calculate the square root of 2.

Ans. 1.414214.

12. Find a real root of the equation $\cos x = 3x - 1$ correct to three decimal places using iteration method.

Hint: Iteration formula is $x_n = \frac{1}{3}(1 + \cos x_n)$.

Ans. 0.607.

13. Using iteration method, find a root of the equation $x^3 + x^2 - 100 = 0$.

Ans. 4.3311.

14. Find the double root of the equation $x^2 - x^2 - x + 1 = 0$ near 0.9.

Ans. 1.0001.

15. Use Newton's method to solve

$$x^2 - y^2 = 4, \quad x^2 + y^2 = 16$$

taking the starting value as (2.828, 2.828).

Ans. $x = 3.162$, $y = 2.450$.

16. Use Newton's method to solve

$$x^2 - 2x - y + 0.5 = 0,$$
$$x^2 + 4y^2 - 4 = 0,$$

taking the starting value as (2.0, 0.25).

Ans. $x = 1.900677$, $y = 0.311219$.

4 Linear Systems of Equations

In this chapter, we shall study direct and iterative methods to solve linear system of equations. Among the direct methods, we shall study Gauss elimination method and its modification by Jordan, Crout, and triangularization methods. Among the iterative methods, we shall study Jacobi and Gauss–Seidel methods.

4.1 DIRECT METHODS

Matrix Inversion Method

Consider the system of n linear equations in n unknowns:

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = a_2 \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ &a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}n_n = b_n \end{aligned} \tag{4.1}$$

The matrix form of the system (4.1) is

$$\mathbf{AX} = \mathbf{B}, \tag{4.2}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_n \end{bmatrix}, \quad \text{and } \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ \cdots \\ b_n \end{bmatrix}.$$

Suppose **A** is non-singular, that is, det $\mathbf{A} \neq 0$. Then $\mathbf{A}^{-1}$ exists. Therefore, premultiplying (4.2) by $\mathbf{A}^{-1}$, we get

$$\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B}$$

or

$$\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}.$$

Thus, finding $\mathbf{A}^{-1}$ we can determine **X** and so $x_1, x_2, \ldots, x_n$.

EXAMPLE 4.1

Solve the equations

$$\begin{aligned} x + y + 2z &= 1 \\ x + 2y + 3z &= 1 \\ 2x + 3y + z &= 2. \end{aligned}$$

Solution. We have

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{vmatrix} = -4 \neq 0.$$

Also

$$\mathbf{A}^{-1} = \frac{1}{4}\begin{bmatrix} 7 & -5 & 1 \\ -5 & 3 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

Hence,

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{A}^{-1}\mathbf{B} = \frac{1}{4}\begin{bmatrix} 7 & -5 & 1 \\ -5 & 3 & 1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and so $x = 1, y = 0, z = 0$.

Gauss Elimination Method

This is the simplest method of step-by-step elimination and it reduces the system of equations to an equivalent upper triangular system, which can be solved by back substitution.

Let the system of equations be

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots . \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{array}$$

The matrix form of this system is

$$\mathbf{AX} = \mathbf{B},$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & \ldots & a_{nn} \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ \ldots \\ x_n \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ \ldots \\ b_n \end{bmatrix}.$$

The augmented matrix is

$$[\mathbf{A}:\mathbf{B}]=\left[\begin{array}{cccccc|c} a_{11} & a_{12} & \cdots & \cdots & a_{1n} & & b_1 \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} & & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} & & b_n \end{array}\right].$$

The number a_{rr} at position (r, r) that is used to eliminate x_r in rows $r + 1, r + 2, \ldots, n$ is called the rth pivotal element and the rth row is called the pivotal row. Thus, the augmented matrix can be written as

$$\begin{array}{r} \text{pivot}\rightarrow \\ m_{2,1}=a_{21}/a_{11} \\ \\ \\ m_{n,1}=a_{n1}/a_{11} \end{array}\left[\begin{array}{ccccc|c} \underline{a_{11}} & a_{12} & \cdots & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} & b_n \end{array}\right]\begin{array}{l} \leftarrow \text{pivotal row} \\ \\ \\ \\ \\ \end{array}$$

The first row is used to eliminate elements in the first column below the diagonal. In the first step, the element a_{11} is pivotal element and the first row is pivotal row. The values $m_{k,1}$ are the multiples of row 1 that are to be subtracted from row k for $k = 2, 3, 4, \ldots, n$. The result after elimination becomes

$$\begin{array}{r} \\ \text{pivot}\rightarrow \\ m_{3,2}=c_{32}/c_{22} \\ \\ m_{n,2}=c_{n2}/c_{22} \end{array}\left[\begin{array}{ccccc|c} a_{11} & a_{12} & \cdots & \cdots & a_{1n} & b_1 \\ & \underline{c_{22}} & \cdots & \cdots & c_{2n} & d_2 \\ & c_{32} & \cdots & \cdots & c_{3n} & d_3 \\ & \cdots & \cdots & \cdots & \cdots & \cdots \\ & c_{n2} & \cdots & \cdots & c_{nn} & d_n \end{array}\right]\begin{array}{l} \\ \leftarrow \text{pivotal row.} \\ \\ \\ \\ \end{array}$$

The second row (now pivotal row) is used to eliminate elements in the second column that lie below the diagonal. The elements $m_{k,2}$ are the multiples of row 2 that are to be subtracted from row k for $k = 3, 4, \ldots, n$.

Continuing this process, we arrive at the matrix:

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & \cdots & \cdots & a_{1n} & b_1 \\ & c_{22} & \cdots & \cdots & c_{2n} & d_2 \\ & \cdots & \cdots & \cdots & \cdots & \cdots \\ & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & h_{nn} & p_n \end{array}\right].$$

Hence, the given system of equation reduces to

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ c_{22}x_2 + \ldots + c_{2n}x_n &= d_2 \\ \cdots \quad \cdots \quad & \cdots \\ \cdots \quad \cdots \quad & \cdots \\ h_{nn}c_n &= p_n. \end{aligned}$$

In the above set of equations, we observe that each equation has one lesser variable than its preceding equation. From the last equation, we have $x_n = \frac{p_n}{h_{nn}}$. Putting this value of x_n in the preceding equation, we can find x_{n-1}. Continuing in this way, putting the values of $x_2, x_3, \ldots, x_n$ in the first equation, x_1 can be determined. The process discussed here is called back substitution.

Remark 4.1. It may occur that the pivot element, even if it is different from zero, is very small and gives rise to large errors. The reason is that the small coefficient usually has been formed as the difference between two almost equal numbers. This difficulty is overcome by suitable permutations of the given equations. It is recommended therefore that the pivotal equation should be the equation which has the largest leading coefficient.

EXAMPLE 4.2

Express the following system in augmented matrix form and find an equivalent upper triangular system and the solution:

$$2x_1 + 4x_2 - 6x_3 = 4$$
$$x_1 + 5x_2 + 3x_3 = 10$$
$$x_1 + 3x_2 + 2x_3 = 5.$$

Solution. The augmented matrix for the system is

$$\begin{matrix} \text{pivot} \rightarrow \\ m_{2,1} = 0.5 \\ m_{3,1} = 0.5 \end{matrix} \left[\begin{array}{ccc|c} \underline{2} & 4 & -6 & -4 \\ 1 & 5 & 3 & 10 \\ 1 & 3 & 2 & 5 \end{array}\right] \begin{matrix} \leftarrow \text{pivotal row} \\ \\ \\ \end{matrix}$$

The result after first elimination is

$$\begin{matrix} \\ \text{pivot} \rightarrow \\ m_{3,2} = 1/3 \end{matrix} \left[\begin{array}{ccc|c} 2 & 4 & -6 & -4 \\ 0 & \underline{3} & 6 & 12 \\ 0 & 1 & 5 & 7 \end{array}\right] \begin{matrix} \\ \leftarrow \text{pivotal row} \\ \\ \end{matrix}$$

The result after second elimination is

$$\left[\begin{array}{ccc|c} 2 & 4 & -6 & -4 \\ 0 & 3 & 6 & 12 \\ 0 & 0 & 3 & 3 \end{array}\right].$$

Therefore, back substitution yields

$$3x_3 = 3 \qquad \text{and so } x_3 = 1,$$
$$3x_2 + 6x_3 = 12 \qquad \text{and so } x_2 = 2,$$
$$2x_1 + 4x_3 - 6x_3 = -4 \qquad \text{and so } x_1 = -3.$$

Hence, the solution is $x_1 = -3$, $x_2 = 2$, and $x_3 = 1$.

EXAMPLE 4.3

Solve by Gauss elimination method:

$$10x - 7y + 3z + 5u = 6$$
$$-6x + 8y - z - 4u = 5$$
$$3x + y + 4z + 11u = 2$$
$$5x - 9y - 2z + 4u = 7.$$

Solution. The augmented matrix for the given system is

$$\begin{matrix} \text{pivot} \rightarrow \\ m_{2,1} = -0.6 \\ m_{3,1} = 0.3 \\ m_{4,1} = 0.5 \end{matrix} \left[\begin{array}{cccc|c} \underline{10} & -7 & 3 & 5 & 6 \\ -6 & 8 & -1 & -4 & 5 \\ 3 & 1 & 4 & 11 & 2 \\ 5 & -9 & -2 & 4 & 7 \end{array}\right] \leftarrow \text{pivotal row}$$

The first elimination yields

$$\begin{matrix} \\ \text{pivot} \rightarrow \\ m_{3,2} = 0.81579 \\ m_{4,2} = 1.4474 \end{matrix} \left[\begin{array}{cccc|c} 10 & -7 & 3 & 5 & 6 \\ 0 & \underline{3.8} & 0.8 & -1 & 8.6 \\ 0 & 3.1 & 3.1 & 9.5 & 0.2 \\ 0 & -5.5 & -3.5 & 1.5 & 4 \end{array}\right] \begin{matrix} \\ \leftarrow \text{pivotal row} \\ \\ \\ \end{matrix}$$

The result after second elimination is

$$\begin{matrix} \\ \\ \text{pivot} \rightarrow \\ m_{4,3} = -0.957 \end{matrix} \left[\begin{array}{cccc|c} 10 & -7 & 3 & 5 & 6 \\ 0 & 3.8 & 0.8 & -1 & 8.6 \\ 0 & 0 & \underline{2.4474} & 10.3158 & -6.8158 \\ 0 & 0 & -2.3421 & 0.0526 & 16.44764 \end{array}\right] \begin{matrix} \\ \\ \leftarrow \text{pivotal row} \\ \\ \end{matrix}$$

The result after third elimination is

$$\left[\begin{array}{cccc|c} 10 & -7 & 3 & 5 & 6 \\ 0 & 3.8 & 0.8 & -1 & 8.6 \\ 0 & 0 & 2.4474 & 10.3158 & -6.8158 \\ 0 & 0 & 0 & 9.9248 & 9.9249 \end{array}\right].$$

Therefore, back substitution yields

$$9.9248u = 9.9249 \text{ and so } u \approx 1$$
$$2.4474z + 10.3158u = -6.8158 \text{ and so } z = -6.9999 \approx -7$$
$$3.8y + 0.8z - u = 8.6 \text{ and so } y = 4$$
$$10x - 7y + 3z + 5u = 6 \text{ and so } x = 5.$$

Hence, the solution of the given system is $x = 5$, $y = 4$, $z = -7$, and $u = 1$.

EXAMPLE 4.4

Solve the following equations by Gauss elimination method:

$$2x + y + z = 10,\ 3x + 2y + 3z = 18,\ x + 4y + 9z = 16.$$

Solution. The given equations are

$$2x + y + z = 10,\ 3x + 2y + 3z = 18,\ x + 4y + 9z = 16.$$

The augmented matrix for given system of equations is

$$\begin{matrix} \text{pivot} \rightarrow \\ m_{2,1} = 3/2 \\ m_{3,1} = 1/2 \end{matrix} \left[\begin{array}{cccc} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right] \leftarrow \text{pivotal row}$$

The result of first Gauss elimination is

$$\begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & \frac{1}{2} & \frac{3}{2} & 3 \\ 0 & \frac{7}{2} & \frac{17}{2} & 11 \end{bmatrix} \begin{matrix} \\ \leftarrow \text{pivotal row} \\ \\ \end{matrix}$$

The second elimination yields

$$\begin{bmatrix} 2 & 1 & 1 & 10 \\ 0 & \frac{1}{2} & \frac{3}{2} & 3 \\ 0 & 0 & -2 & -10 \end{bmatrix}$$

Thus, the given system equations reduces to

$$\begin{aligned} 2x + y + z &= 10 \\ 0.5y + 1.5z &= 3 \\ -2z &= -10 \end{aligned}$$

Hence, back substitution yields

$$z = 5, y = -9, x = 7.$$

Jordan Modification to Gauss Method

Jordan modification means that the elimination is performed not only in the equation below but also in the equation above the pivotal row so that we get a diagonal matrix. In this way, we have the solution without further computation.

Comparing the methods of Gauss and Jordan, we find that the number of operations is essentially $\frac{n^3}{3}$ for Gauss method and $\frac{n^3}{2}$ for Jordan method. Hence, Gauss method should usually be preferred over Jordan method.

To illustrate this modification we reconsider Example 4.2. The result of first elimination is unchanged and we have

$$\begin{matrix} m_{1,2} = 4/3 \\ \text{pivot} \rightarrow \\ m_{3,2} = 1/3 \end{matrix} \left[\begin{array}{ccc|c} 2 & 4 & -6 & -4 \\ 0 & \underline{3} & 6 & 12 \\ 0 & 1 & 5 & 7 \end{array}\right] \begin{matrix} \\ \leftarrow \text{pivotal row} \\ \\ \end{matrix}$$

Now, the second elimination as per Jordan modification yields

$$\begin{matrix} m_{1,3} = -14/3 \\ m_{2,3} = 2 \\ \text{pivot} \rightarrow \end{matrix} \left[\begin{array}{ccc|c} 2 & 0 & -14 & -20 \\ 0 & 3 & 6 & 12 \\ 0 & 0 & \underline{3} & 3 \end{array}\right] \begin{matrix} \\ \\ \leftarrow \text{pivotal row} \end{matrix}$$

The third elimination as per Jordan modification yields

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & -6 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 3 & 3 \end{array}\right].$$

Hence,

$$2x_1 = -6 \quad \text{and so } x_1 = -3,$$

$$3x_2 = 6 \quad \text{and so } x_2 = 2,$$

$$3x_3 = 3 \quad \text{and so } x_3 = 1.$$

EXAMPLE 4.5

Solve

$$\begin{aligned} x + 2y + z &= 8 \\ 2x + 3y + 4z &= 20 \\ 4x + 3y + 2z &= 16 \end{aligned}$$

by Gauss–Jordan method.

Solution. The augmented matrix for the given system of equations is

$$\begin{matrix} \text{pivot} \rightarrow \\ m_{2,1} = 2 \\ m_{3,1} = 4 \end{matrix} \left[\begin{array}{ccc|c} \underline{1} & 2 & 1 & 8 \\ 2 & 3 & 4 & 20 \\ 4 & 3 & 2 & 16 \end{array}\right] \leftarrow \text{pivotal row}$$

The result of first elimination is

$$\begin{matrix} m_{1,2} = -2 \\ \text{pivot} \rightarrow \\ m_{3,2} = 5 \end{matrix} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & \underline{-1} & 2 & 4 \\ 0 & -5 & -2 & -16 \end{array}\right] \begin{matrix} \\ \leftarrow \text{pivotal row} \\ \\ \end{matrix}$$

The second Gauss–Jordan elimination yields

$$\begin{matrix} m_{1,3} = -5/12 \\ m_{2,3} = -1/6 \\ \text{pivot} \rightarrow \end{matrix} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 16 \\ 0 & -1 & 2 & 4 \\ 0 & 0 & \underline{-12} & -36 \end{array}\right] \begin{matrix} \\ \\ \leftarrow \text{pivotal row} \end{matrix}$$

The third Gauss–Jordan elimination yields

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & -12 & -36 \end{array}\right].$$

Therefore, $x = 1$, $y = 2$, and $z = 3$ is the required solution.

EXAMPLE 4.6

Solve

$$\begin{aligned} 10x + y + z &= 12 \\ x + 10y + z &= 12 \\ x + y + 10z &= 12 \end{aligned}$$

by Gauss–Jordan method.

Solution. The augmented matrix for the given system is

$$\begin{matrix} \text{pivot} \rightarrow \\ m_{2,1} = 1/10 \\ m_{3,1} = 1/10 \end{matrix} \left[\begin{array}{ccc|c} \underline{10} & 1 & 1 & 12 \\ 1 & 10 & 1 & 12 \\ 1 & 1 & 10 & 12 \end{array}\right] \begin{matrix} \leftarrow \text{pivotal row} \\ \\ \\ \end{matrix}$$

The first Gauss–Jordan elimination yields

$$\begin{matrix} m_{1,2} = 10/99 \\ \text{pivot} \rightarrow \\ m_{3,2} = 1/11 \end{matrix} \left[\begin{array}{ccc|c} 10 & 1 & 1 & 12 \\ 0 & \underline{99/10} & 9/10 & 108/10 \\ 0 & 9/10 & 99/10 & 108/10 \end{array}\right] \begin{matrix} \\ \leftarrow \text{pivotal row} \\ \\ \end{matrix}$$

Now the Gauss–Jordan elimination gives

$$\begin{matrix} m_{1,3} = 10/108 \\ m_{2,3} = 11/120 \\ \text{pivot} \rightarrow \end{matrix} \left[\begin{array}{ccc|c} 10 & 0 & 10/11 & 120/11 \\ 0 & 99/10 & 9/10 & 108/10 \\ 0 & 0 & \underline{108/11} & 108/11 \end{array}\right] \begin{matrix} \\ \\ \leftarrow \text{pivotal row} \end{matrix}$$

The next Gauss–Jordan elimination yields

$$\left[\begin{array}{ccc|c} 10 & 0 & 0 & 10 \\ 0 & 99/10 & 0 & 99/10 \\ 0 & 0 & 108/11 & 108/11 \end{array}\right].$$

Hence, the solution of the given system is $x = 1, y = 1, z = 1$.

EXAMPLE 4.7

Solve by Gauss–Jordan method

$$\begin{aligned} x + y + z &= 9 \\ 2x - 3y + 4z &= 13 \\ 3x + 4y + 5z &= 40. \end{aligned}$$

Solution. The augmented matrix for the given system is

$$\begin{matrix} \\ m_{21} = 2 \\ m_{31} = 3 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 9 \\ 2 & -3 & 4 & 13 \\ 3 & 4 & 5 & 40 \end{bmatrix} \begin{matrix} \leftarrow \text{pivotal row} \\ \\ \\ \end{matrix}$$

The first Gauss–Jordan elimination yields

$$\begin{matrix} m_{12} = -\dfrac{1}{5} \\ \\ m_{32} = -\dfrac{1}{5} \end{matrix} \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & -5 & 2 & -5 \\ 0 & 1 & 2 & 13 \end{bmatrix} \begin{matrix} \\ \leftarrow \text{pivotal row.} \\ \\ \end{matrix}$$

The second Gauss elimination yields

$$\begin{matrix} m_{13} = 7/12 \\ m_{23} = 10/12 \\ \\ \end{matrix} \begin{bmatrix} 1 & 0 & \frac{7}{5} & 8 \\ 0 & -5 & 2 & -5 \\ 0 & 0 & \frac{12}{5} & 12 \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \text{pivotal row} \end{matrix}$$

The third Gauss elimination yields

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -5 & 0 & -15 \\ 0 & 0 & \frac{12}{5} & 12 \end{bmatrix}.$$

Thus, we have attained the diagonal form of the system. Hence, the solution is

$$x = 1\,,\ y = \frac{15}{5} = 3\,,\ z = \frac{12(5)}{12} = 5.$$

Triangularization (Triangular Factorization) Method

We have seen that Gauss elimination leads to an upper triangular matrix, where all diagonal elements are 1. We shall now show that the elimination can be interpreted as the multiplication of the original coefficient matrix **A** by a suitable lower triangular matrix. Hence, in three dimensions, we put

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}.$$

In this way, we get nine equations with nine unknowns (six l elements and three u elements).

If the lower and upper triangular matrices are denoted by **L** and **U**, respectively, we have

$$\mathbf{LA} = \mathbf{U}$$

or

$$\mathbf{A} = \mathbf{L}^{-1}\mathbf{U}.$$

Since $\mathbf{L}^{-1}$ is also a lower triangular matrix, we can find a factorization of **A** as a product of one lower triangular matrix and one upper triangular matrix. Thus, a non-singular matrix **A** is said to have a triangular factorization if it can be expressed as a product of a lower triangular matrix **L** and an upper triangular matrix **U**, that is, if **A**= **LU**. For the sake of convenience, we can choose $l_{ii} = 1$ or $u_{ii} = 1$. Thus, the system of equations **AX** = **B** is resolved into two simple systems as follows:

$$\mathbf{AX} = \mathbf{B}$$

or

$$\mathbf{LUX} = \mathbf{B}$$

or

$$\mathbf{LY} = \mathbf{B} \text{ and } \mathbf{UX} = \mathbf{Y}.$$

Both the systems can be solved by back substitution.

EXAMPLE 4.8

Solve the following system of equations by triangularization method:

$$x_1 + 2x_2 + 3x_3 = 14$$
$$2x_1 + 5x_2 + 2x_3 = 18$$
$$3x_1 + x_2 + 5x_3 = 20.$$

Solution. The matrix form of the given system is

$$\mathbf{AX} = \mathbf{B},$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 14 \\ 18 \\ 20 \end{bmatrix}.$$

Let

$$\mathbf{A} = \mathbf{LU},$$

that is,

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

and so we have

$1 = u_{11}$

$2 = l_{21}u_{11}$ and so $l_{21} = 2$

$3 = l_{31}u_{11}$ and so $l_{31} = 3$

$2 = u_{12}$

$5 = l_{21}u_{12} + u_{22} = 2(2) + u_{22}$ and so $u_{22} = 1$

$1 = l_{31}u_{12} + l_{32}\, u_{22} = 3(2) + l_{32}(1)$ and so $l_{32} = 5$

$3 = u_{13}$

$2 = l_{21}u_{13} + u_{23} = 2(3) + u_{23}$ and so $u_{23} = -4$

$5 = l_{31}u_{13} + l_{32}u_{23} + u_{33} = 3(3) + (-5)(-4) + u_{33}$ and so $u_{33} = -24$.

Hence,

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{U} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & -24 \end{bmatrix}.$$

Now we have

$$\mathbf{AX} = \mathbf{B}$$

or

$$\mathbf{LUX} = \mathbf{B}$$

or

$$\mathbf{LY} = \mathbf{B} \text{ where } \mathbf{UX} = \mathbf{Y}.$$

But $\mathbf{LY} = \mathbf{B}$ yields

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 18 \\ 20 \end{bmatrix}$$

and we have

$$y_1 = 14,$$

$$2y_1 + y_2 = 18 \text{ and so } y_2 = -10,$$

$$3y_1 - 5y_2 + y_3 = 20 \text{ and so } y_3 = -72.$$

Then **UX** = **Y** yields

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & -24 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -10 \\ -72 \end{bmatrix}$$

and so

$$-24x_3 = -72 \text{ which yields } x_3 = 3,$$

$$x_2 - 4x_3 = -10 \text{ which yields } x_2 = 2,$$

$$x_1 + 2x_2 + x_3 = 14 \text{ which yields } x_1 = 1.$$

Hence, the required solution is $x_1 = 1$, $x_2 = 2$, and $x_3 = 3$.

EXAMPLE 4.9

Use Gauss elimination method to find triangular factorization of the coefficient matrix of the system

$$x_1 + 2x_2 + 3x_3 = 14$$
$$2x_1 + 5x_2 + 2x_3 = 18$$
$$3x_1 + x_2 + 5x_3 = 20$$

and hence solve the system.

Solution. In matrix form, we have

$$\mathbf{AX} = \mathbf{B},$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{bmatrix}, \ \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 14 \\ 18 \\ 20 \end{bmatrix}.$$

Write

$$\mathbf{A} = \mathbf{IA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{1} & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{bmatrix} \begin{matrix} \leftarrow \text{pivotal row} \\ m_{2,1} = 2 \\ m_{3,1} = 3 \end{matrix}$$

The elimination in the second member on the right-hand side is done by Gauss elimination method while in the first member l_{21} is replaced by m_{21} and l_{31} is replaced by m_{31}. Thus, the first elimination yields

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & \underline{1} & -4 \\ 0 & -5 & -4 \end{bmatrix} \begin{matrix} \\ \leftarrow \text{pivotal row} \\ m_{3,2} = -5 \end{matrix}$$

Then the second elimination gives the required triangular factorization as

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & -24 \end{bmatrix}$$

$$= \mathbf{LU},$$

where

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{U} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & -24 \end{bmatrix}.$$

The solution is then obtained as in Example 4.8.

EXAMPLE 4.10

Solve

$$\begin{aligned} 2x_1 + 4x_2 - 6x_3 &= -4 \\ x_1 + 5x_2 + 3x_3 &= 10 \\ x_1 + 3x_2 + 2x_3 &= 5. \end{aligned}$$

Solution. Write

$$\mathbf{A} = \mathbf{IA},$$

that is,

$$\begin{bmatrix} 2 & 4 & -6 \\ 1 & 5 & 3 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{2} & 4 & -6 \\ 1 & 5 & 3 \\ 1 & 3 & 2 \end{bmatrix} \begin{matrix} \leftarrow \text{pivotal row} \\ m_{2,1} = 1/2 \\ m_{3,1} = 1/2 \end{matrix}$$

Using Gauss elimination method, discussed in Example 4.9, the first elimination yields

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -6 \\ 0 & \underline{3} & 6 \\ 0 & 1 & 5 \end{bmatrix} \begin{matrix} \\ \leftarrow \text{pivotal row} \\ m_{3,2} = 1/3 \end{matrix}$$

The second elimination yields

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -6 \\ 0 & 3 & 6 \\ 0 & 0 & 3 \end{bmatrix} = \mathbf{LU}.$$

Therefore, **AX** = **B** reduces to **LUX** = **B** or **LY** = **B**, **UX** = **Y**.
Now **LY** = **B** gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 10 \\ 5 \end{bmatrix}$$

and so

$$y_1 = -4$$

$$\frac{1}{2} y_1 + y_2 = 10 \text{ which yields } y_2 = 12,$$

$$\frac{1}{2} y_1 + \frac{1}{3} y_2 + y_3 = 5 \text{ which yields } y_3 = 3.$$

Then **UX** = **Y** implies

$$\begin{bmatrix} 2 & 4 & -6 \\ 0 & 3 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 12 \\ 3 \end{bmatrix}$$

and so

$$3x_3 = 3 \text{ which yields } x_3 = 1,$$
$$3x_2 + 6x_3 = 12 \text{ which yields } x_2 = 2,$$
$$2x_1 + 4x_2 - 6x_3 = -4 \text{ which yields } x_1 = -3.$$

Hence, the solution of the given system is $x_1 = -3$, $x_2 = 2$, and $x_3 = 1$.

EXAMPLE 4.11

Solve

$$x + 3y + 8z = 4$$
$$x + 4y + 3z = -2$$
$$x + 3y + 4z = 1$$

by the method of factorization.

Solution. The matrix form of the system is $\mathbf{AX} = \mathbf{B}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}, \ \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}.$$

Write

$$\mathbf{A} = \mathbf{IA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{1} & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{matrix} \leftarrow \textit{pivotal row} \\ m_{2,1} = 1 \\ m_{3,1} = 1 \end{matrix}$$

Applying Gauss elimination method to the right member and replacing l_{21} by m_{21} and l_{31} by m_{31} in the left member, we get

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 8 \\ 0 & \underline{1} & -5 \\ 0 & 0 & -4 \end{bmatrix} \leftarrow \text{pivotal row}$$

$$= \mathbf{LU}.$$

Then $\mathbf{AX} = \mathbf{B}$ reduces to $\mathbf{LUX} = \mathbf{B}$ or $\mathbf{LY} = \mathbf{B}$ and $\mathbf{UX} = \mathbf{Y}$. Now $\mathbf{LY} = \mathbf{B}$ gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

and so

$$y_1 = 4, y_2 = -6, y_1 + y_3 = 1 \text{ which implies } y_3 = -3.$$

Then $\mathbf{UX} = \mathbf{Y}$ gives

$$\begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & -5 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ -3 \end{bmatrix}.$$

Hence, the required solution is $x = \frac{19}{4}, \ y = -\frac{9}{4}, \ z = \frac{3}{4}$.

Triangularization of Symmetric Matrix

When the coefficient matrix of the system of linear equations is symmetric, we can have a particularly simple triangularization in the form

$$\mathbf{A} = \mathbf{L}\mathbf{L}^{\mathrm{T}}$$

or

$$\begin{bmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{12} & a_{22} & \dots & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & \dots & \dots & 0 \\ l_{21} & l_{22} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & l_{n-1,n-1} & 0 \\ l_{n1} & l_{n2} & \dots & \dots & l_{nn} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & \dots & \dots & l_{n1} \\ 0 & l_{22} & \dots & \dots & l_{n2} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & l_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & \dots & \dots & l_{11}l_{n1} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & \dots & \dots & l_{21}l_{n1} + l_{22}l_{n2} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ l_{n1}l_{11} & l_{n1}l_{21} + l_{n2}l_{22} & \dots & \dots & l_{n1}^2 + l_{n2}^2 + \dots + l_{nn}^2 \end{bmatrix}$$

Hence,

$$\begin{array}{llll} l_{11}^2 = a_{11}, & l_{21}l_{31} + l_{22}l_{32} = a_{23}, & l_{21}^2 + l_{22}^2 = a_{22} \\ l_{11}l_{21} = a_{12}, & \dots, & \dots \\ \dots \quad \dots & \dots & \dots \\ l_{11}l_{n1} = l_{1n}, & l_{21}l_{n1} + l_{22}l_{n2} = a_{2n}, & l_{n1}^2 + \dots + l_{nn}^2 = a_{nn} \end{array}$$

However, it may encounter with some terms which are purely imaginary but this does not imply any special complications. The matrix equation **AX** = **B** reduces to $\mathbf{L}\mathbf{L}^{\mathrm{T}}\mathbf{X} = \mathbf{B}$ or **LZ** = **B** and $\mathbf{L}^{\mathrm{T}}\mathbf{X} = \mathbf{Z}$.

This method is known as the square root method and is due to Banachiewicz and Dwyer.

EXAMPLE 4.12

Solve by square root method:

$$\begin{aligned} 4x - y + 2z &= 12 \\ -x + 5y + 3z &= 10 \\ 2x + 3y + 6z &= 18. \end{aligned}$$

Solution. The matrix form of the given system is

$$\mathbf{AX} = \mathbf{B},$$

where

$$\mathbf{A} = \begin{bmatrix} 4 & -1 & 2 \\ -1 & 5 & 3 \\ 2 & 3 & 6 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 12 \\ 10 \\ 18 \end{bmatrix}.$$

The matrix **A** is symmetric. Therefore, we have triangularization of the type $\mathbf{A} = \mathbf{LL}^{\mathrm{T}}$, that is,

$$\begin{bmatrix} 4 & -1 & 2 \\ -1 & 5 & 3 \\ 2 & 3 & 6 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$= \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{11}l_{31} & l_{21}l_{31} + l_{22}l_{32} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}.$$

Hence,

$$l_{11}^2 = 4 \text{ and so } l_{11} = 2,$$

$$l_{11}l_{21} = -1 \text{ and so } l_{21} = -\frac{1}{2},$$

$$l_{11}l_{31} = 2 \text{ and so } l_{31} = 1,$$

$$l_{21}^2 + l_{22}^2 = 5 \text{ and so } l_{22} = \sqrt{5 - \frac{1}{4}} = \sqrt{\frac{19}{4}},$$

$$l_{21}l_{31} + l_{22}l_{32} = 3 \text{ and so } -\frac{1}{2} + \sqrt{\frac{19}{4}}l_{32} = 3 \text{ or } l_{32} = \frac{7}{\sqrt{19}}.$$

$$l_{31}^2 + l_{32}^2 + l_{33}^2 = 6 \text{ and so } 1 + \frac{49}{19} + l_{33}^2 = 6 \text{ or } l_{33} = \sqrt{\frac{46}{19}}.$$

Thus,

$$\mathbf{L} = \begin{bmatrix} 2 & 0 & 0 \\ -\frac{1}{2} & \sqrt{\frac{19}{4}} & 0 \\ 1 & \frac{7}{\sqrt{19}} & \sqrt{\frac{46}{19}} \end{bmatrix}.$$

Then, **LZ** = **B** yields

$$\begin{bmatrix} 2 & 0 & 0 \\ -\frac{1}{2} & \sqrt{\frac{19}{4}} & 0 \\ 1 & \frac{7}{\sqrt{19}} & \sqrt{\frac{46}{19}} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 10 \\ 18 \end{bmatrix}$$

and so

$$z_1 = 6$$

$$-3 + \sqrt{\frac{19}{4}}z_2 = 10 \text{ which yields } z_2 = \frac{26}{\sqrt{19}}.$$

$$6+\frac{7}{\sqrt{19}}\times\frac{26}{\sqrt{19}}+\sqrt{\frac{46}{19}}z_3=18, \text{ which yields } z_3=\sqrt{\frac{46}{19}}.$$

Now $\mathbf{L}^{\mathrm{T}}\mathbf{X}=\mathbf{Z}$ gives

$$\begin{bmatrix} 2 & -\frac{1}{2} & 1 \\ 0 & \sqrt{\frac{19}{4}} & \frac{7}{\sqrt{19}} \\ 0 & 0 & \sqrt{\frac{46}{19}} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 6 \\ \frac{26}{\sqrt{19}} \\ \sqrt{\frac{46}{19}} \end{bmatrix}.$$

Hence,

$$z = 1,$$

$$\sqrt{\frac{19}{4}}\,y+\frac{7}{\sqrt{19}}z=\frac{26}{\sqrt{19}} \text{ or } y=\sqrt{19}\times\frac{\sqrt{4}}{\sqrt{19}}=2,$$

$$2x-\frac{1}{2}y+z=6 \text{ which gives } x = 3.$$

Hence, the solution is $x = 3$, $y = 2$, and $z = 1$.

Crout's Method

Crout suggested a technique to determine systematically the entries of the lower and upper triangles in the factorization of a given matrix **A**. We describe the scheme of the method stepwise.

Let the matrix form of the system (in three dimensions) be **AX** = **B**, where

$$\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \ \mathbf{X}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ \mathbf{B}=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

The augmented matrix is

$$[\mathbf{A}:\mathbf{B}]=\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right].$$

The matrix of the unknowns (in factorization of **A**), called the derived matrix or auxiliary matrix, is

$$\begin{bmatrix} l_{11} & u_{12} & u_{13} & y_1 \\ l_{21} & l_{22} & u_{23} & y_2 \\ l_{31} & l_{32} & l_{33} & y_3 \end{bmatrix}.$$

The entries of this matrix are calculated as follows:

Step 1. The first column of the auxiliary matrix is identical with the first column of the augmented matrix $[\mathbf{A}:\mathbf{B}]$.

Step 2. The first row to the right of the first column of the auxiliary matrix is obtained by dividing the corresponding elements in $[\mathbf{A}:\mathbf{B}]$ by the leading diagonal element a_{11}.

Step 3. The remaining entries in the second column of the auxiliary matrix are l_{22} and l_{32}. These entries are equal to corresponding element in $[\mathbf{A}:\mathbf{B}]$ minus the product of the first element in that row and in that column. Thus,

$$l_{22} = a_{22} - l_{21}u_{12},$$
$$l_{32} = a_{32} - l_{31}u_{12}.$$

Step 4. The remaining elements of the second row of the auxiliary matrix are equal to: [corresponding element in $[\mathbf{A}:\mathbf{B}]$ minus the product of the first element in that row and first element in that column]/leading diagonal element in that row. Thus,

$$u_{23} = \frac{a_{23} - l_{21}u_{13}}{l_{22}}$$

$$y_2 = \frac{b_2 - l_{21}y_1}{l_{22}}.$$

Step 5. The remaining elements of the third column of the auxiliary matrix are equal to: corresponding element in $[\mathbf{A}:\mathbf{B}]$ minus the sum of the inner products of the previously calculated elements in the same row and column. Thus

$$l_{33} = a_{33} - (l_{31}u_{13} + l_{32}u_{23}).$$

Step 6. The remaining elements of the third row of the auxiliary matrix are equal to: [corresponding element in $[\mathbf{A}:\mathbf{B}]$ minus the sum of inner products of the previously calculated elements in the same row and column]/leading diagonal element in that row. Thus,

$$y_3 = \frac{b_3 - (l_{31}y_1 + l_{32}y_2)}{l_{33}}.$$

Following this scheme, the upper and lower diagonal matrices can be found and then using

$$\mathbf{UX} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

we can determine x_1, x_2, x_3.

EXAMPLE 4.13

Solve by Crout's method:

$$x_1 + 2x_2 + 3x_3 = 1$$
$$3x_1 + x_2 + x_3 = 0$$
$$2x_1 + x_2 + x_3 = 0.$$

Solution. The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 3 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right].$$

Let the derived matrix be

$$\mathbf{M} = \begin{bmatrix} l_{11} & u_{12} & u_{13} & y_1 \\ l_{21} & l_{22} & u_{23} & y_2 \\ l_{31} & l_{32} & l_{33} & y_3 \end{bmatrix}.$$

Then

$$\mathbf{M} = \begin{bmatrix} 1 & \dfrac{2}{1} & \dfrac{3}{1} & \dfrac{1}{1} \\ 3 & 1-3(2) & \dfrac{1-3(3)}{-5} & \dfrac{0-3(1)}{-5} \\ 2 & 1-2(2) & 1-\left[3(2)+(-3)\left(\dfrac{8}{5}\right)\right] & \dfrac{0-[2(1)+(-3) \rightleftarrows (3/5)]}{1-[6-(24/5)]} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 1 \\ 3 & -5 & 8/5 & 3/5 \\ 2 & -3 & -1/5 & 1 \end{bmatrix}$$

Now **UX** = **Y** gives

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 8/5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3/5 \\ 1 \end{bmatrix}.$$

Hence,

$$x_3 = 1$$

$$x_2 + \frac{8}{5}x_3 = \frac{3}{5} \text{ and so } x_2 = \frac{3}{5} - \frac{8}{5} = -1$$

$$x_1 + 2x_2 + 3x_3 = 1 \text{ and so } x_1 = 1 - 2x_2 - 3x_3 = 1 + 2 - 3 = 0.$$

Hence, the solution is $x_1 = 0$, $x_2 = -1$, and $x_3 = 1$.

EXAMPLE 4.14

Solve by Crout's method:

$$2x + y + 4z = 12$$
$$8x - 3y + 2z = 20$$
$$4x + 11y - z = 33.$$

Solution. The augmented matrix for the given system of equations is

$$\left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 8 & -3 & 2 & 20 \\ 4 & 11 & -1 & 33 \end{array}\right].$$

Let the derived matrix be

$$\mathbf{M} = \begin{bmatrix} l_{11} & u_{12} & u_{13} & y_1 \\ l_{21} & l_{22} & u_{23} & y_2 \\ l_{31} & l_{32} & l_{33} & y_3 \end{bmatrix}.$$

Then

$$\mathbf{M}=\begin{bmatrix} 2 & \frac{1}{2} & \frac{4}{2} & \frac{12}{2} \\ 8 & -3-8\left(\frac{1}{2}\right) & \frac{2-[8(2)]}{-7} & \frac{20-[8(6)]}{-7} \\ 4 & 11-4\left(\frac{1}{2}\right) & -1-[4(2)+9(2)] & \frac{33-[6(4)+9(4)]}{-27} \end{bmatrix}$$

$$=\begin{bmatrix} 2 & 1/2 & 2 & 6 \\ 8 & -7 & 2 & 4 \\ 4 & 9 & -27 & 1 \end{bmatrix}.$$

Now **UX** = **Y** gives

$$\begin{bmatrix} 1 & 1/2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}.$$

By back substitution, we get

$$z = 1,$$
$$y + 2z = 4 \text{ and so } y = 4 - 2z = 2,$$
$$x+\frac{1}{2}y+2z = 6 \text{ and so } x = 6-2z-\frac{1}{2}y = 3.$$

Hence, the required solution is $x = 3, y = 2, z = 1$.

EXAMPLE 4.15

Using Crout's method, solve the system

$$x + 2y - 12z + 8v = 27$$
$$5x + 4y + 7z - 2v = 4$$
$$-3x + 7y + 9z + 5v = 11$$
$$6x - 12y - 8z + 3v = 49.$$

Solution. The augmented matrix of the given system is

$$\left[\begin{array}{cccc|c} 1 & 2 & -12 & 8 & 27 \\ 5 & 4 & 7 & -2 & 4 \\ -3 & 7 & 9 & 5 & 11 \\ 6 & -12 & -8 & 3 & 49 \end{array}\right].$$

Then the auxiliary matrix is

$$\mathbf{M}=\begin{bmatrix} 1 & 2 & -12 & 8 & 27 \\ 5 & -6 & -67/6 & 7 & 131/6 \\ -3 & 13 & 709/6 & -372/709 & -1151/709 \\ 6 & -24 & -204 & 11319/709 & 5 \end{bmatrix}.$$

The solution of the equation is given by **UX** = **Y**, that is,

$$\begin{bmatrix} 1 & 2 & -12 & 8 \\ 0 & 1 & -67/6 & 7 \\ 0 & 0 & 1 & -372/709 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ v \end{bmatrix} = \begin{bmatrix} 27 \\ 131/6 \\ -1151/709 \\ 5 \end{bmatrix}$$

or

$$x + 2y - 12z + 8v = 27$$
$$y - \frac{67}{6}z + 7v = \frac{131}{6}$$
$$z - \frac{372}{709}v = \frac{1,151}{709}$$
$$v = 5.$$

Back substitution yields

$$x = 3, y = -2, z = 1, v = 5.$$

4.2 ITERATIVE METHODS FOR LINEAR SYSTEMS

We have seen that the direct methods for the solution of simultaneous linear equations yield the solution after an amount of computation that is known in advance. On the other hand, in case of iterative or indirect methods, we start from an approximation to the true solution and, if convergent, we form a sequence of closer approximations repeated till the required accuracy is obtained. The difference between direct and iterative method is therefore that in direct method the amount of computation is fixed, while in an iterative method, the amount of computation depends upon the accuracy required.

In general, we prefer a direct method for solving system of linear equations. But, in case of matrices with a large number of zero elements, it is economical to use iterative methods.

Jacobi Iteration Method

Consider the system

$$\begin{aligned} a_{11}x + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + \ldots + a_{3n}x_n &= b_3 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad & \ldots \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad & \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned} \tag{4.3}$$

in which the diagonal coefficients a_{ii} do not vanish. If this is not the case, the equations should be rearranged so that this condition is satisfied. Equations (4.3) can be written as

$$\begin{aligned} x_1 &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2 - \ldots - \frac{a_{1n}}{a_{11}}x_n \\ x_2 &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1 - \ldots - \frac{a_{2n}}{a_{22}}x_n \\ \ldots \quad & \ldots \quad \ldots \quad \ldots \\ x_n &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}}x_1 - \ldots - \frac{a_{n-n}}{a_{nn}}x_{n-1} \end{aligned} \tag{4.4}$$

Suppose $x_1^{(1)},\ x_2^{(1)},\ldots,x_n^{(1)}$ are first approximation to the unknowns $x_1, x_2, ..., x_n$. Substituting in the right side of equation (4.4), we find a system of second approximations:

$$\begin{aligned}
x_1^{(2)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(1)} - \ldots - \frac{a_{1n}}{a_{11}} x_n^{(1)} \\
x_2^{(2)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(1)} - \ldots - \frac{a_{2n}}{a_{22}} x_n^{(1)} \\
\ldots &\quad \ldots \qquad \ldots \ \ldots \\
\ldots &\quad \ldots \qquad \ldots \ \ldots \\
x_n^{(2)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(1)} - \ldots - \frac{a_{n,n-1}}{a_{nn}} x_{n-1}^{(1)}
\end{aligned}.$$

In general, if $x_1^{(n)},\ x_2^{(n)},\ldots,x_n^{(n)}$ is a system of nth approximations, then the next approximation is given by the formula

$$\begin{aligned}
x_1^{(n+1)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(n)} - \ldots - \frac{a_{1n}}{a_{11}} x_n^{(n)} \\
x_n^{(n+1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(n)} - \ldots - \frac{a_{2n}}{a_{22}} x_n^{(n)} \\
\ldots &\quad \ldots \qquad \ldots \ \ldots \\
x_n^{(n+1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(n)} - \ldots - \frac{a_{n,n-1}}{a_{nn}} x_n^{(n)}
\end{aligned}.$$

This method, due to Jacobi, is called the method of simultaneous displacements or Jacobi method.

Gauss–Seidel Method

A simple modification of Jacobi method yields faster convergence. Let $x_1^{(1)}, x_2^{(1)},\ldots,x_n^{(1)}$ be the first approximation to the unknowns $x_1, x_2, ..., x_n$. Then the second approximations are given by:

$$\begin{aligned}
x_1^{(2)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(1)} - \ldots - \frac{a_{1n}}{a_{11}} x_n^{(1)} \\
x_2^{(2)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(2)} - \frac{a_{23}}{a_{22}} x_3^{(1)} - \ldots - \frac{a_{2n}}{a_{22}} x_n^{(1)} \\
x_3^{(2)} &= \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}} x_1^{(2)} - \frac{a_{32}}{a_{33}} x_2^{(2)} - \ldots - \frac{a_{3n}}{a_{33}} x_n^{(1)} \\
\ldots &\quad \ldots \qquad \ldots \qquad \ldots \ \ldots \\
x_n^{(2)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(2)} - \frac{a_{n2}}{a_{nn}} x_2^{(2)} - \ldots - \frac{a_{n,n-1}}{a_{nn}} x_{n-1}^{(2)}
\end{aligned}.$$

The entire process is repeated till the values of $x_1, x_2, ..., x_n$ are obtained to the accuracy required. Thus, this method uses an improved component as soon as available and so is called the method of successive displacements or Gauss–Seidel method.

Introducing the matrices

$$\mathbf{A}_1 = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \text{ and } \mathbf{A}_2 = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 0 & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

it can be shown that the condition for convergence of Gauss–Seidel method is that the absolutely largest eigenvalue of $\mathbf{A}_1^{-1}\mathbf{A}_2$ must be absolutely less than 1. In fact, we have convergence if for $i = \mathbf{1}, \mathbf{2}, \ldots, n$, $|a_{ii}| > S_i$, where $S_i = \sum_{k \neq i} |a_{ik}|$. Thus, for convergence, the coefficient matrix should have a clear diagonal dominance.

It may be mentioned that Gauss–Seidel method converges twice as fast as the Jacobi's method.

EXAMPLE 4.16

Starting with $(x_0, y_0, z_0) = (0, 0, 0)$ and using Jacobi method, find the next five iterations for the system

$$5x - y + z = 10$$
$$2x + 8y - z = 11$$
$$-x + y + 4z = 3.$$

Solution. The given equations can be written in the form

$$x = \frac{y - z + 10}{5}, \quad y = \frac{-2x + z + 11}{8}, \text{ and } z = \frac{x - y + 3}{4}, \text{ respectively.}$$

Therefore, starting with $(x_0, y_0, z_0) = (0, 0, 0)$, we get

$$x_1 = \frac{y_0 - z_0 + 10}{5} = 2$$

$$y_1 = \frac{-2x_0 + z_0 + 11}{8} = 1.375$$

$$z_1 = \frac{x_0 - y_0 + 3}{4} = 0.75.$$

The second iteration gives

$$x_2 = \frac{y_1 - z_1 + 10}{5} = \frac{1.375 - 0.75 + 10}{5} = 2.125$$

$$y_2 = \frac{-2x_1 + z_1 + 11}{8} = \frac{-4 + 0.75 + 11}{8} = 0.96875$$

$$z_2 = \frac{x_1 - y_1 + 3}{4} = \frac{2 - 1.375 + 3}{4} = 0.90625.$$

The third iteration gives

$$x_3 = \frac{y_2 - z_2 + 10}{5} = \frac{0.96875 - 0.90625 + 10}{5} = 2.0125$$

$$y_3 = \frac{-2x_2 + z_2 + 11}{8} = \frac{-4.250 + 0.90625 + 11}{8} = 0.95703125$$

$$z_3 = \frac{x_2 - y_2 + 3}{4} = \frac{2.125 - 0.96875 + 3}{4} = 1.0390625.$$

The fourth iteration yields

$$x_4 = \frac{y_3 - z_3 + 10}{5} = \frac{0.95703125 - 1.0390625 + 10}{5} = 1.98359375$$

$$y_4 = \frac{-2x_3 + z_3 + 11}{8} = \frac{-4.0250 + 1.0390625 + 11}{4} = 0.8767578$$

$$z_4 = \frac{x_3 - y_3 + 3}{4} = \frac{2.0125 - 0.95703125 + 3}{4} = 1.0138672,$$

whereas the fifth iteration gives

$$x_5 = \frac{y_4 - z_4 + 10}{5} = 1.9725781$$

$$y_5 = \frac{-2x_4 + z_4 + 11}{8} = \frac{-3.9671875 + 1.0138672 + 11}{8} = 1.005834963$$

$$z_5 = \frac{x_4 - y_4 + 3}{4} = \frac{1.98359375 - 0.8767578 + 3}{4} == 1.02670898.$$

One finds that the iterations converge to (2, 1, 1).

EXAMPLE 4.17

Using Gauss–Seidel iteration and the first iteration as (0, 0, 0), calculate the next three iterations for the solution of the system of equations given in Example 4.16.

Solution. The first iteration is (0, 0, 0). The next iteration is

$$x_1 = \frac{y_0 - z_0 + 10}{5} = 2$$

$$y_1 = \frac{-2x_1 + z_0 + 11}{8} = \frac{-4 + 0 + 11}{8} = 0.875$$

$$z_1 = \frac{x_1 - y_1 + 3}{4} = \frac{2 - 0.875 + 3}{4} = 1.03125.$$

Then

$$x_2 = \frac{y_1 - z_1 + 10}{5} = \frac{0.875 - 1.03125 + 10}{5} = 1.96875$$

$$y_2 = \frac{-2x_2 + z_1 + 11}{8} = \frac{-3.9375 + 1.03125 + 11}{8} = 1.01171875$$

$$z_2 = \frac{x_2 - y_2 + 3}{4} = \frac{1.96875 - 1.01171875 + 3}{4} = 0.989257812.$$

Further,

$$x_3 = \frac{y_2 - z_2 + 10}{5} = \frac{1.01171875 - 0.989257812 + 10}{5} = 2.004492188$$

$$y_3 = \frac{-2x_3 + z_2 + 11}{8} = \frac{-4.008984376 + 0.989257812 + 11}{8} = 0.997534179$$

$$z_3 = \frac{x_3 - y_3 + 3}{4} = \frac{2.004492188 - 0.997534179 + 3}{4} = 1.001739502 \cdot$$

The iterations will converge to (2, 1, 1).

Remark 4.2. It follows from Examples 4.16 and 4.17 that Gauss–Seidel method converges rapidly in comparison to Jacobi's method.

EXAMPLE 4.18

Solve

$$54x + y + z = 110$$
$$2x + 15y + 6z = 72$$
$$-x + 6y + 27z = 85$$

by Gauss–Seidel method.

Solution. From the given equations, we have

$$x = \frac{110 - y - z}{54}, \; y = \frac{72 - 2x - 6z}{15}, \text{ and } z = \frac{85 + x - 6y}{27}.$$

We take the initial approximation as $x_0 = y_0 = z_0 = 0$. Then the first approximation is given by

$$x_1 = \frac{110}{54} = 2.0370$$

$$y_1 = \frac{72 - 2x_1 - 6z_0}{15} = 4.5284$$

$$z_1 = \frac{85 + x_1 - 6y_1}{27} = 2.2173.$$

The second approximation is given by

$$x_2 = \frac{110 - y_1 - z_1}{54} = 1.9122$$

$$y_2 = \frac{72 - 2x_2 - 6z_1}{15} = 3.6581$$

$$z_2 = \frac{85 + x_2 - 6y_2}{27} = 2.4061.$$

The third approximation is

$$x_3 = \frac{110 - y_2 - z_2}{54} = 1.9247$$

$$y_3 = \frac{72 - 2x_3 - 6z_2}{15} = 3.5809$$

$$z_3 = \frac{85 + x_3 - 6y_3}{27} = 2.4237.$$

The fourth approximation is

$$x_4 = \frac{110 - y_3 - z_3}{54} = 1.9258$$

$$y_4 = \frac{72 - 2x_4 - 6z_3}{15} = 3.5738$$

$$z_4 = \frac{85 + x_4 - 6y_4}{27} = 2.4253.$$

The fifth approximation is

$$x_5 = \frac{110 - y_4 - z_4}{54} = 1.9259$$

$$y_5 = \frac{72 - 2x_5 - 6z_4}{15} = 3.5732$$

$$z_5 = \frac{85 + x_5 - 6y_5}{27} = 2.4254.$$

Thus, the required solution, correct to three decimal places, is

$$x = 1.926, y = 3.573, z = 2.425.$$

EXAMPLE 4.19

Solve

$$28x + 4y - z = 32$$
$$2x + 17y + 4z = 35$$
$$x + 3y + 10z = 24$$

by Gauss–Seidel method.

Solution. From the given equations, we have

$$x = \frac{32 - 4y + z}{28}, \quad y = \frac{35 - 2x - 4z}{17}, \text{ and } z = \frac{24 - x - 3y}{10}.$$

Taking first approximation as $x_0 = y_0 = z_0 = 0$, we have

$x_1 = 1.1428571$, $y_1 = 1.9243697$, $z_1 = 1.7084034$
$x_2 = 0.9289615$, $y_2 = 1.5475567$, $z_2 = 1.8428368$
$x_3 = 0.9875932$, $y_3 = 1.5090274$, $z_3 = 1.8485325$
$x_4 = 0.9933008$, $y_4 = 1.5070158$, $z_4 = 1.8485652$
$x_5 = 0.9935893$, $y_5 = 1.5069741$, $z_5 = 1.8485488$
$x_6 = 0.9935947$, $y_6 = 1.5069774$, $z_6 = 1.8485473$.

Hence the solution, correct to four decimal places, is

$$x = 0.9935, y = 1.5069, z = 1.8485.$$

EXAMPLE 4.20

Solve the equation by Gauss–Seidel method:

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25.$$

Solution. The given equation can be written as

$$x = \frac{1}{20}[17 - y + 2z]$$

$$y = \frac{1}{20}[-18 - 3x + z]$$

$$z = \frac{1}{20}[25 - 3x + 3y].$$

Taking the initial rotation as $(x_0, y_0, z_0) = (0,0,0)$, we have by Gauss–Seidal method,

$$x_1 = \frac{1}{20}[17 - 0 + 0] = 0.85$$

$$y_1 = \frac{1}{20}[-18 - 3(0.85) + 1] = -1.0275$$

$$z_1 = \frac{1}{20}[25 - 2(0.85) - 3(1.0275)] = 1.0108$$

$$x_2 = \frac{1}{20}[17 + 1.0275 + 2(1.0108)] = 1.0024$$

$$y_2 = \frac{1}{20}[-18 - 3(1.0024) + 1.0108] = -0.9998$$

$$z_2 = \frac{1}{20}[25 - 2(1.0024) + 3(-0.9998)] = 0.9998$$

$$x_3 = \frac{1}{20}[17 + 0.9998 + 2(0.9998)] = 0.99997$$

$$y_3 = \frac{1}{20}[-18 - 3(0.99997) + 0.9998] = -1.00000$$

$$z_3 = \frac{1}{20}[25 - 2(0.99997) + 3(-1.0000)] = 1.00000.$$

The second and third iterations show that the solution of the given system of equations is $x = 1$, $y = -1$, $z = 1$.

Convergence of Iteration Method

(A) Condition of Convergence of Iteration Methods

We know (see Section 3.14) that conditions for convergence of the iteration process for solving simultaneous equations $f(x, y) = 0$ and $g(x, y) = 0$ is

$$\left|\frac{\partial f}{\partial x}\right| + \left|\frac{\partial g}{\partial x}\right| < 1$$

and

$$\left|\frac{\partial f}{\partial y}\right| + \left|\frac{\partial g}{\partial y}\right| < 1.$$

This result can be extended to any finite number of equations. For example, consider the following system of three equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3. \end{aligned}$$

Then, in the fixed-point form, we have

$$\begin{aligned} x_1 = f(x_1, x_2, x_3) &= \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3) \\ &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2 - \frac{a_{13}}{a_{11}}x_3, \end{aligned} \tag{4.5}$$

$$\begin{aligned} x_2 = g(x_1, x_2, x_3) &= \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3) \\ &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1 - \frac{a_{23}}{a_{33}}x_3, \end{aligned} \tag{4.6}$$

$$\begin{aligned} x_3 = h(x_1, x_2, x_3) &= \frac{1}{a_{33}}(b_3 - a_{31}x_1 - a_{32}x_3) \\ &= \frac{b_3}{a_{33}} - \frac{a_{31}}{s_{33}}x_1 - \frac{a_{32}}{a_{33}}x_2. \end{aligned} \tag{4.7}$$

Then the conditions for convergence are

$$\left|\frac{\partial f}{\partial x_1}\right| + \left|\frac{\partial g}{\partial x_1}\right| + \left|\frac{\partial h}{\partial x_1}\right| < 1, \tag{4.8}$$

$$\left|\frac{\partial f}{\partial x_2}\right| + \left|\frac{\partial g}{\partial x_2}\right| + \left|\frac{\partial h}{\partial x_2}\right| < 1, \tag{4.9}$$

$$\left|\frac{\partial f}{\partial x_3}\right| + \left|\frac{\partial g}{\partial x_3}\right| + \left|\frac{\partial h}{\partial x_3}\right| < 1. \tag{4.10}$$

But partial differentiation of equations (4.5), (4.6), and (4.7) yields

$$\frac{\partial f}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} = -\frac{a_{12}}{a_{11}}, \quad \frac{\partial f}{\partial x_3} = -\frac{a_{13}}{a_{11}},$$

$$\frac{\partial g}{\partial x_1} = -\frac{a_{21}}{a_{22}}, \quad \frac{\partial g}{\partial x_2} = 0, \quad \frac{\partial g}{\partial x_3} = -\frac{a_{23}}{a_{22}},$$

$$\frac{\partial h}{\partial x_1} = -\frac{a_{31}}{a_{33}}, \quad \frac{\partial h}{\partial x_2} = -\frac{a_{32}}{a_{33}}, \quad \frac{\partial h}{\partial x_3} = 0.$$

Putting these values in inequalities (4.8), (4.9), and (4.10), we get

$$\left|\frac{a_{21}}{a_{22}}\right| + \left|\frac{a_{31}}{a_{33}}\right| < 1, \tag{4.11}$$

$$\left|\frac{a_{12}}{a_{11}}\right| + \left|\frac{a_{32}}{a_{33}}\right| < 1, \tag{4.12}$$

$$\left|\frac{a_{13}}{a_{11}}\right| + \left|\frac{a_{23}}{a_{22}}\right| < 1. \tag{4.13}$$

Adding the inequalities (4.11), (4.12), and (4.13), we get

$$\left|\frac{a_{21}}{a_{22}}\right| + \left|\frac{a_{31}}{a_{33}}\right| + \left|\frac{a_{12}}{a_{11}}\right| + \left|\frac{a_{32}}{a_{33}}\right| + \left|\frac{a_{13}}{a_{11}}\right| + \left|\frac{a_{23}}{a_{22}}\right| < 3$$

or

$$\left[\left|\frac{a_{12}}{a_{11}}\right| + \left|\frac{a_{13}}{a_{11}}\right|\right] + \left[\left|\frac{a_{21}}{a_{22}}\right| + \left|\frac{a_{23}}{a_{22}}\right|\right] + \left[\left|\frac{a_{31}}{a_{33}}\right| + \left|\frac{a_{32}}{a_{33}}\right|\right] < 3 \tag{4.14}$$

We note that inequality (4.14) is satisfied by the conditions

$$\left|\frac{a_{12}}{a_{11}}\right| + \left|\frac{a_{13}}{a_{22}}\right| < 1 \quad \text{or} \quad |a_{22}| > |a_{12}| + |a_{13}|$$

$$\left|\frac{a_{21}}{a_{22}}\right| + \left|\frac{a_{23}}{a_{22}}\right| < 1 \quad \text{or} \quad |a_{22}| > |a_{21}| + |a_{23}|$$

$$\left|\frac{a_{31}}{a_{33}}\right| + \left|\frac{a_{32}}{a_{33}}\right| < 1 \quad \text{or} \quad |a_{33}| > |a_{31}| + |a_{32}|.$$

Hence, the condition for convergence in the present case is

$$|a_{ii}| > \sum_{j=1}^{3} |a_{ij}|, \quad i = 1, 2, 3; \; i \neq j.$$

For a system of n equations, the condition reduces to

$$|a_{ii}| > \sum_{j=1}^{n} |a_{ij}|, \quad i = 1, 2, \ldots n; \; i \neq j. \tag{4.15}$$

Thus, the process of iteration (Jacobi or Gauss–Seidel) will converge if in each equation of the system, the absolute value of the largest coefficient is greater than the sum of the absolute values of all the remaining coefficients in that equation.

A system of equations satisfying condition (4.15) is called diagonally dominated system.

(B) Rate of Convergence of Iteration Method

In view of equations (4.5), (4.6), and (4.7), the $(k+1)$th iteration is given by

$$x_1^{(k+1)} = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(k)} - \frac{a_{13}}{a_{11}} x_3^{(k)}, \tag{4.16}$$

$$x_2^{(k+1)} = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(k+1)} - \frac{a_{23}}{a_{22}} x_3^{(k)}, \tag{4.17}$$

$$x_3^{(k+1)} = \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}} x_1^{(k+1)} - \frac{a_{32}}{a_{33}} x_2^{(k+1)}. \tag{4.18}$$

Putting the value of $x_1^{(k+1)}$ from equations (4.16) in (4.17), we get

$$x_2^{(k+1)} = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}\left[\frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(k)} - \frac{a_{13}}{a_{11}} x_3^{(k)}\right] - \frac{a_{23}}{a_{22}} x_3^{(k)}$$

$$= \frac{b_2}{a_{22}} - \frac{a_{21}b_1}{a_{22}a_{11}} + \frac{a_{21}a_{12}}{a_{11}a_{22}} x_2^{(k)} + \frac{a_{21}a_{13}}{a_{22}a_{11}} x_3^{(k)} - \frac{a_{23}}{a_{22}} x_3^{(k)}.$$

Then

$$x_2^{(k+2)} = \frac{b_2}{a_{22}} - \frac{a_{21}b_1}{a_{22}a_{11}} + \frac{a_{21}a_{12}}{a_{11}a_{22}} x_2^{(k+1)} + \frac{a_{21}a_{13}}{a_{22}a_{11}} x_3^{(k)} - \frac{a_{23}}{a_{22}} x_3^{(k)}.$$

Hence,

$$x_2^{(k+2)} - x_2^{(k+1)} = \frac{a_{21}a_{12}}{a_{11}a_{22}} (x_2^{(k+1)} - x_2^{(k)}). \tag{4.19}$$

In terms of errors, equation (4.19) yields

$$e_2^{(k+1)} = \frac{a_{21}a_{12}}{a_{11}a_{22}} e_2^{(k)}.$$

Therefore, the error will decrease if $\frac{a_{12}a_{21}}{a_{11}a_{22}} < 1$.

4.3 ILL-CONDITIONED SYSTEM OF EQUATIONS

System of equations, where small changes in the coefficient result in large deviations in the solution is said to be ill-conditioned system. Such systems of equations are very sensitive to round-off errors.

For example, consider the system

$$3x_1 + x_2 = 9$$

$$3.015x_1 + x_2 = 3.$$

The solution of this system is

$$x_1 = \frac{9-3}{3-3.015} = -400 \text{ and } x_2 = \frac{9-9(3.015)}{3-3.015} = 1209.$$

Now, we round off the coefficient of x_1 in the second equation to 4.02. Then the solution of the system is

$$x_1 = \frac{9-3}{3-3.02} = -300 \text{ and } x_2 = \frac{9-9(3.02)}{3-3.02} = 909.$$

Putting these values of x_1 and x_2 in the given system of equations, we have the residuals as

$$r_1 = -900 + 909 - 9 = 0 \text{ and } r_2 = 3.015(-300) + 909 - 3 = 1.5.$$

Thus, the first equation is satisfied exactly whereas we get a residual for the second equation. This happened due to rounding off the coefficient of x_1 in the second equation. Hence, the system in question is ill-conditioned.

Let $\mathbf{A} = \left[a_{ij} \right]$ be an $n \times n$ coefficient matrix of a given system. If $\mathbf{C} = \mathbf{AA}^{-1}$ is close to identity matrix, then the system is well-conditioned, otherwise it is ill-conditioned. If we define norm of the matrix **A** as

$$\|A\| = \max_{1 \le i \le n} \sum_{j=1}^{n} a_{ij} ,$$

then the number $\|A\| \|A^{-1}\|$ is called the condition number, which is the measure of the ill-conditionedness of the system. The larger the condition number, the more is the ill-conditionedness of the system.

EXERCISES

1. Solve the system

$$2x + y + z = 10$$
$$3x + 2y + 3z = 18$$
$$x + 4y + 9z = 16$$

by Gauss elimination method.

Ans. $x = 7, y = -9, z = 5$

2. Solve the following system of equations by Gauss elimination method:

$$x_1 + 2x_2 - x_3 = 3$$
$$3x_1 - x_2 + 2x_3 = 1$$
$$2x_1 - 2x_2 + 3x_3 = 2$$

Ans. $x_1 = -1, x_2 = 4, x_3 = 4$

3. Solve the following system of equations by Gauss elimination method:

$$2x + 2y + z = 12$$
$$3x + 2y + 2z = 8$$
$$5x + 10y - 8z = 10.$$

Ans. $x = -12.75, y = 14.375, z = 8.75$

4. Solve the following system of equations by Gauss–Jordan method:

$$5x - 2y + z = 4$$
$$7x + y - 5z = 8$$
$$3x + 7y + 4z = 10.$$

Ans. $x = 11.1927, y = 0.8685, z = 0.1407$

5. Solve by Gauss–Jordan method:

$$2x_1 + x_2 + 5x_3 + x_4 = 5$$
$$x_1 + x_2 - 3x_3 + 4x_4 = -1$$
$$3x_1 + 6x_2 - 2x_3 + x_4 = 8$$
$$2x_1 + 2x_2 + 2x_3 - 3x_4 = 2.$$

Ans. $x_1 = 2,\ x_2 = \frac{1}{5}, x_3 = 0,\ x_4 = \frac{4}{5}$

6. Solve by Gauss–Jordan method:

$$x + y + z = 9$$
$$2x - 3y + 4z = 13$$
$$3x + 4y + 5z = 40.$$

Ans. $x = 1, y = 3, z = 5$

7. Solve by Gauss–Jordan method:

$$2x - 3y + z = -1$$
$$x + 4y + 5z = 25$$
$$3x - 4y + z = 2.$$

Ans. $x = 8.7, y = 5.7, z = -1.3$

8. Solve Exercise 4 by factorization method.

9. Solve the following system of equations by factorization method:

$$2x + 3y + z = 9$$
$$x + 2y + 3z = 6$$
$$3x + y + 2z = 8.$$

Ans. $x = 1.9444, y = 1.6111, z = 0.2777$

10. Solve the following system of equations by Crout's method:

$$3x + 2y + 7z = 4$$
$$2x + 3y + z = 5$$
$$3x + 4y + z = 7.$$

Ans. $x = \frac{7}{8},\ y = \frac{9}{8},\ z = -\frac{1}{8}$

11. Use Crout's method to solve

$$2x - 6y + 8z = 24$$
$$5x + 4y - 3z = 2$$
$$3x + y + 2z = 16.$$

Ans. $x = 1, y = 3, z = 5$

12. Solve by Crout's method:

$$10x + y + z = 12$$
$$2x + 10y + z = 13$$
$$2x + 2y + 10z = 14.$$

Ans. $x = 1, y = 1, z = 1$

13. Use Jacobi's iteration method to solve

$$5x + 2y + z = 12$$
$$x + 4y + 2z = 15$$
$$x + 2y + 5z = 20.$$

Ans. $x = 1.08, y = 1.95, z = 3.16$

14. Solve by Jacobi's iteration method

$$10x + 2y + z = 9$$
$$2x + 20y - 2z = -44$$
$$-2x + 3y + 10z = 22.$$

Ans. $x = 1, y = -2, z = 3$

15. Solve by Jacobi's method

$$5x - y + z = 10$$
$$2x + 4y = 12$$
$$x + y + 5z = -1.$$

Ans. $x = 2.556, y = 1.722, z = -1.055$

16. Use Gauss–Seidel method to solve

$$54x + y + z = 110$$
$$2x + 15y + 6z = 72$$
$$-x + 6y + 27z = 85$$

Ans. $x = 1.926, y = 3.573, z = 2.425$

17. Find the solution, to three decimal places, of the system using Gauss–Seidel method.

$$83x + 11y - 4z = 95$$
$$7x + 52y + 13z = 104$$
$$3x + 8y + 29z = 71$$

Ans. $x = 1.052,\ y = 1.369,\ z = 1.962$

18. Solve Exercise 14 by Gauss–Seidel method.

19. Show that the following systems of equations are ill-conditioned:

(i) $2x_1 + x_2 = 25$
$2.001x_1 + x_2 = 25.01$

(ii) $y = 2x + 7$
$y = 2.01 + 3$

5 Finite Differences and Interpolation

Finite differences play a key role in the solution of differential equations and in the formulation of interpolating polynomials. The interpolation is the art of reading between the tabular values. Also the interpolation formulae are used to derive formulae for numerical differentiation and integration.

5.1 FINITE DIFFERENCES

Suppose that a function $y = f(x)$ is tabulated for the equally spaced arguments $x_0, x_0 + h, x_0 + 2h, \ldots, x_0 + nh$ giving the functional values $y_0, y_1, y_2, \ldots, y_n$. The constant difference between two consecutive values of x is called the interval of differencing and is denoted by h.

The operator Δ defined by

$$\begin{aligned}
\Delta y_0 &= y_1 - y_0, \\
\Delta y_1 &= y_2 - y_1, \\
&\ldots\ldots\ldots\ldots\ldots, \\
&\ldots\ldots\ldots\ldots\ldots, \\
\Delta y_{n-1} &= y_n - y_{n-1}.
\end{aligned}$$

is called the Newton's forward difference operator. We note that the first difference $\Delta y_n = y_{n+1} - y_n$ is itself a function of x. Consequently, we can repeat the operation of differencing to obtain

$$\begin{aligned}
\Delta^2 y_0 &= \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0, \\
&= y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0,
\end{aligned}$$

which is called the second forward difference. In general, the nth difference of f is defined by

$$\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r.$$

For example, let

$$f(x) = x^3 - 3x^2 + 5x + 7.$$

Taking the arguments as 0, 2, 4, 6, 8, 10, we have $h = 2$ and

$$\begin{aligned}
\Delta f(x) &= (x+2)^3 - 3(x+2)^2 + 5(x+2) + 7 - (x^3 - 3x^2 + 5x + 7) = 6x^2 + 6, \\
\Delta^2 f(x) &= \Delta(\Delta f(x)) = \Delta(6x^2 + 6) = 6(x+2)^2 + 6 - (6x^2 + 6) = 24x + 24, \\
\Delta^3 f(x) &= 24(x+2) + 24 - (24x + 24) = 48, \\
\Delta^4 f(x) &= \Delta^5 f(x) = \ldots = 0.
\end{aligned}$$

In tabular form, we have

Difference Table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0	7					
		6				
2	13		24			
		30		48		
4	43		72		0	
		102		48		0
6	145		120		0	
		222		48		
8	367		168			
		390				
10	757					

Theorem 5.1. If $f(x)$ is a polynomial of degree n, that is,

$$f(x) = \sum_{i=0}^{n} a_i x^i,$$

then $\Delta^n f(x)$ is constant and is equal to $n!a_n h^n$

Proof: We shall prove the theorem by induction on n. If $n = 1$, then $f(x) = a_1 x + a_0$ and $\Delta f(x) = f(x+h) - f(x) = a_1 h$ and so the theorem holds for $n = 1$. Assume now that the result is true for all degrees $1, 2, \ldots, n-1$. Consider

$$f(x) = \sum_{i=0}^{n} a_i x^i.$$

Then by the linearity of the operator Δ, we have

$$\Delta^n f(x) = \sum_{i=0}^{n} a_i \Delta^n x^i.$$

For $i < n$, $\Delta^n x^i$ is the nth difference of a polynomial of degree less than n and hence must vanish, by induction hypothesis. Thus,

$$\begin{aligned}\Delta^n f(x) &= a_n \Delta^n x^n = a_n \Delta^{n-1}(\Delta x^n)\\ &= a_n \Delta^{n-1}[(x+h)^n - x^n]\\ &= a_n \Delta^{n-1}[nhx^{n-1} + g(x)]\end{aligned}$$

where $g(x)$ is a polynomial of degree less than $n-1$. Hence, by induction hypothesis,

$$\Delta^n f(x) = a_n \Delta^{n-1}(nhx^{n-1}) = a_n (hn)(n-1)!h^{n-1} = a_n n!h^n.$$

Hence, by induction, the theorem holds.

Let $y_0, y_1, \ldots, y_n$ be the functional values of a function f for the arguments $x_0, x_0 + h, x_0 + 2h, \ldots, x_0 + nh.$ Then the operator ∇ defined by

$$\nabla y_r = y_r - y_{r-1}$$

is called the Newton's backward difference operator.

The higher-order backward differences are

$$\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}$$
$$\nabla^3 y_r = \nabla^2 y_r - \nabla^2 y_{r-1}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\nabla^n y_r = \nabla^{n-1} y_r - \nabla^{n-1} y_{r-1}.$$

Thus, the backward difference table becomes

x	y	1st difference	2nd difference	3rd difference
x_0	y_0			
		∇y_1		
x_1	y_1		$\nabla^2 y_2$	
		∇y_2		$\nabla^3 y_3$
x_2	y_2		$\nabla^2 y_3$	
		∇y_3		
x_3	y_3			

EXAMPLE 5.1

Form the table of backward differences for the function

$$f(x) = x^3 - 3x^2 + 5x - 7$$

for $x = -1, 0, 1, 2, 3, 4$, and 5.

Solution.

x	y	1st difference	2nd difference	3rd difference	4th difference
−1	−16				
		9			
0	−7		−6		
		3		6	
1	−4		0		0
		3		6	
2	−1		6		0
		9		6	
3	8		12		0
		21		6	
4	29		18		
		39			
5	68				

An operator E, known as enlargement operator, displacement operator or shifting operator, is defined by

$$Ey_r = y_{r+1}.$$

Thus, shifting operator moves the functional value $f(x)$ to the next higher value $f(x+h)$. Further,

$$E^2 y_r = E(Ey_r) = E(y_{r+1}) = y_{r+2}$$
$$E^3 y_r = E(E^2 y_r) = E(y_{r+2}) = y_{r+3}.$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$E^n y_r = y_{r+n}$$

Relations between Δ, ∇, and *E*

We know that

$$\Delta y_r = y_{r+1} - y_r = Ey_r - y_r = (E - I)y_r,$$

where I is the identity operator. Hence,

$$\Delta = E - I \text{ or } E = I + \Delta. \tag{5.1}$$

Also, by definition,

$$\nabla y_r = y_r - y_{r-1} = y_r - E^{-1}y_r = y_r(I - E^{-1}),$$

and so

$$\nabla = I - E^{-1} \text{ or } E^{-1} = I - \nabla$$

or

$$E = \frac{I}{I - \nabla}. \tag{5.2}$$

From equations (5.1) and (5.2), we have

$$I + \Delta = \frac{1}{I - \nabla} \tag{5.3}$$

or

$$\Delta = \frac{I}{I - \nabla} - I = \frac{\nabla}{I - \nabla}. \tag{5.4}$$

From equations (5.3) and (5.4)

$$\nabla = I - \frac{I}{I + \Delta} = \frac{\Delta}{1 + \Delta} \tag{5.5}$$

Theorem 5.2. $f_{x+nh} = \sum_{k=0}^{\infty} \binom{n}{k} \Delta^k f_x$

Proof: We shall prove our result by mathematical induction. For $n = 1$, the theorem reduces to $f_{x+h} = f_x + \Delta f_x$ which is true. Assume now that the theorem is true for $n - 1$. Then

$$f_{x+nh} = E^n f_x = E(E^{n-1} f_x) = E\sum_{i=0}^{\infty} \binom{n-1}{i} \Delta^i f_x \text{ by induction hypothesis.}$$

But $E = I + \Delta$. So

$$
\begin{aligned}
E^n f_x &= (I+\Delta)E^{n-1} f_x = E^{n-1} f_x + \Delta E^{n-1} f_x \\
&= \sum_{i=0}^{\infty} \binom{n-1}{i} \Delta^i f_x + \sum_{i=0}^{\infty} \binom{n-1}{i} \Delta^{i+1} f_x \\
&= \sum_{i=0}^{\infty} \binom{n-1}{i} \Delta^i f_x + \sum_{j=1}^{\infty} \binom{n-1}{j-1} \Delta^j f_x
\end{aligned}
$$

The coefficient of $\Delta^k f_x (k = 0, 1, 2, \ldots, n)$ is given by

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}.$$

Hence,

$$f_{x+nh} = E^n f_x = \sum_{k=0}^{\infty} \binom{n}{k} \Delta^k f_x,$$

which completes the proof of the theorem.

As a special case of this theorem, we get

$$f_x = E^x f_0 = \sum_{k=0}^{\infty} \binom{x}{k} \Delta^k f_0,$$

which is known as Newton's advancing difference formula and expresses the general functional value f_x in terms of f_0 and its differences.

Let h be the interval of differencing. Then the operator δ defined by

$$\delta f_x = f_{x+\frac{h}{2}} - f_{x-\frac{h}{2}}$$

is called the central difference operator.

We note that

$$\delta f_x = f_{x+\frac{h}{2}} - f_{x-\frac{h}{2}} = E^{\frac{1}{2}} f_x - E^{-\frac{1}{2}} f_x = \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) f_x.$$

Hence,

$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}. \tag{5.6}$$

Multiplying both sides by $E^{\frac{1}{2}}$, we get

$$E - \delta E^{\frac{1}{2}} - 1 = 0 \text{ or } \left(E^{\frac{1}{2}} - \frac{\delta}{2}\right)^2 - \frac{\delta^2}{4} - I = 0$$

or

$$E^{\frac{1}{2}} - \frac{\delta}{2} = \sqrt{I + \frac{\delta^2}{4}} \text{ or } E^{\frac{1}{2}} = \frac{\delta}{2} + \sqrt{I + \frac{\delta^2}{4}}$$

or

$$E = \frac{\delta^2}{4} + I + \frac{\delta^2}{4} + \delta\left(1 + \frac{\delta^2}{4}\right)^{\frac{1}{2}} = I + \frac{\delta^2}{2} + \delta\sqrt{I + \frac{\delta^2}{4}}. \tag{5.7}$$

Also, using equation (5.7), we note that

$$\Delta = E - I = \frac{\delta^2}{2} + \delta\sqrt{I + \frac{\delta^2}{4}} \tag{5.8}$$

$$\nabla = I - \frac{I}{E} = I - \left(I + \frac{\delta^2}{2} + \delta\sqrt{I + \frac{\delta^2}{4}} \right)^{-1}$$

$$= -\frac{\delta^2}{2} + \delta\sqrt{I + \frac{\delta^2}{4}}. \tag{5.9}$$

Conversely,

$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}} = (I + \Delta)^{\frac{1}{2}} - \frac{I}{(I+\Delta)^{1/2}}$$

$$= \frac{I + \Delta - I}{\sqrt{I + \Delta}} = \frac{\Delta}{\sqrt{I + \Delta}} \tag{5.10}$$

and

$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}} = \frac{I}{\sqrt{1 - \nabla}} - \sqrt{1 - \nabla} = \frac{\nabla}{\sqrt{I - \nabla}}. \tag{5.11}$$

Let h be the interval of differencing. Then the operator μ defined by

$$\mu f_x = \frac{1}{2}\left[f_{x+\frac{h}{2}} + f_{x-\frac{h}{2}} \right]$$

is called the mean value operator or averaging operator. We have

$$\mu f_x = \frac{1}{2}\left[f_{x+\frac{h}{2}} + f_{x-\frac{h}{2}} \right] = \frac{1}{2}\left[E^{\frac{1}{2}} f_x + E^{-\frac{1}{2}} f_x \right].$$

Hence,

$$\mu = \frac{1}{2}\left[E^{\frac{1}{2}} + E^{-\frac{1}{2}} \right] \tag{5.12}$$

or

$$2\mu = E^{\frac{1}{2}} + E^{-\frac{1}{2}}. \tag{5.13}$$

Also, we know that

$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}. \tag{5.14}$$

Adding equations (5.13) and (5.14), we get

$$2\mu + \delta = 2E^{\frac{1}{2}} \text{ or } E^{\frac{1}{2}} = \mu + \frac{\delta}{2}. \tag{5.15}$$

Also,

$$E^{\frac{1}{2}} = \frac{\delta}{2} + \sqrt{I + \frac{\delta^2}{4}}$$

Hence,

$$\mu + \frac{\delta}{2} = \frac{\delta}{2} + \sqrt{I + \frac{\delta^2}{4}} \text{ or } \mu = \sqrt{I + \frac{\delta^2}{4}} \tag{5.16}$$

The relation equation (5.16) yields

$$I + \frac{\delta^2}{4} = \mu^2 \text{ or } \delta = 2\sqrt{\mu^2 - I}. \tag{5.17}$$

Multiplying equation (5.13) throughout by $E^{\frac{1}{2}}$, we get

$$E + I = 2\mu E^{\frac{1}{2}} \text{ or } E - 2\mu E^{\frac{1}{2}} + I = 0$$

or

$$\left(E^{\frac{1}{2}} - \mu\right)^2 - \mu^2 + I = 0 \text{ or } E^{\frac{1}{2}} - \mu = \sqrt{\mu^2 - I}$$

or

$$E^{\frac{1}{2}} = \mu + \sqrt{\mu^2 - I} \text{ or } E = 2\mu^2 - I + 2\mu\sqrt{\mu^2 - I}. \tag{5.18}$$

Then

$$\Delta = E - I = 2\mu^2 - 2I + 2\mu\sqrt{\mu^2 - I} \tag{5.19}$$

and

$$\nabla = I - \frac{I}{E} = I - \left(2\mu^2 + 2\mu\sqrt{\mu^2 - I} - I\right)^{-1}$$

$$= \frac{2\mu\left(\mu + \sqrt{\mu^2 - 1}\right) - 2I}{2\mu^2 + 2\mu\sqrt{\mu^2 - I} - I} \tag{5.20}$$

The differential operator D is defined by

$$\Delta f(x) = f'(x).$$

By Taylor's Theorem, we have

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \ldots$$

$$= f(x) + hDf(x) + \frac{h^2}{2!} D^2 f(x) + \ldots$$

$$= \left(1 + hD + \frac{h^2}{2!} D^2 + \ldots\right) f(x)$$

and so

$$Ef(x) = f(x+h) = \left(1 + hD + \frac{h^2}{2!} D^2 + \ldots\right) f(x).$$

Hence,

$$E = 1 + hD + \frac{h^2}{2!} D^2 + \ldots = e^{hD} = e^U, \text{where } U = hD. \tag{5.21}$$

Then

$$\Delta = E - I = e^{U} - I \text{ and } \nabla = I - e^{-U}.$$

We note that

$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}} = e^{\frac{U}{2}} - e^{-\frac{U}{2}}$$

$$= 2\sinh\frac{U}{2} \tag{5.22}$$

$$\mu = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right) = \frac{1}{2}\left(e^{\frac{U}{2}} + e^{-\frac{U}{2}}\right). \tag{5.23}$$

Conversely

$$e^{\frac{U}{2}} + e^{-\frac{U}{2}} = 2\mu$$

or

$$e^{U} + 1 = 2\mu e^{\frac{U}{2}} \left(\text{quadratic in } e^{\frac{U}{2}}\right)$$

or

$$e^{\frac{U}{2}} = \mu + \sqrt{\mu^2 - I}$$

or

$$U = \log\left(2\mu^2 + I + 2\mu\sqrt{\mu^2 - I}\right). \tag{5.24}$$

Since, by equation (5.22),

$$\delta = 2\sinh\frac{U}{2},$$

it follows that

$$U = \sinh^{-1}\frac{\delta}{2}. \tag{5.25}$$

From the above discussion, we obtain the following table for the relations among the finite difference operators:

	Δ	∇	δ	E	$U = hD$
Δ	Δ	$(I-\nabla)^{-1} - I$	$\frac{\delta^2}{2} + \delta\sqrt{I + \frac{\delta^2}{4}}$	$E - I$	$e^{U} - I$
∇	$I - \frac{I}{\Delta + I}$	∇	$-\frac{\delta^2}{2} + \delta\sqrt{I + \frac{\delta^2}{4}}$	$I - \frac{1}{E}$	$I - e^{-U}$
δ	$\frac{\Delta}{\sqrt{I+\Delta}}$	$\frac{\nabla}{\sqrt{I-\nabla}}$	δ	$E^{\frac{1}{2}} - E^{-\frac{1}{2}}$	$2\sinh\frac{U}{2}$
E	$I+\Delta$	$\frac{I}{I-\nabla}$	$I + \frac{\delta^2}{2} + \delta\sqrt{I + \frac{\delta^2}{4}}$	E	e^{U}
$U = hD$	$\log(I+\Delta)$	$\log\frac{I}{I-\nabla}$	$2\sinh\frac{\delta}{2}$	$\log E$	U

EXAMPLE 5.2

The expression δy_0 cannot be computed directly from a difference scheme. Find its value expressed in known central differences.

Solution. We know that

$$\mu = \left(I + \frac{\delta^2}{4}\right)^{\frac{1}{2}} \text{ or } \mu\left(I + \frac{\delta^2}{4}\right)^{-\frac{1}{2}} = I$$

or

$$\delta y_0 = \mu\delta\left(1 + \frac{\delta^2}{4}\right)^{-\frac{1}{2}} y_0$$

$$= \mu\delta\left[1 - \frac{\delta^2}{8} + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2!}\frac{\delta^4}{16} + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}\frac{\delta^8}{64} + \dots\right] y_0$$

$$= \mu\delta\left[y_0 - \frac{\delta^2 y_0}{8} + \frac{3}{128}\delta^4 y_0 - \dots\right]. \tag{5.26}$$

But

$$\mu\delta y_0 = \delta\mu y_0 = \delta\left(\frac{y_{\frac{1}{2}} + y_{-\frac{1}{2}}}{2}\right) = \frac{1}{2}\left(\delta y_{\frac{1}{2}} + \delta y_{-\frac{1}{2}}\right)$$

$$= \frac{1}{2}(y_1 - y_0 + y_0 - y_{-1}) = \frac{1}{2}(y_1 - y_{-1}).$$

Hence, equation (5.26) reduces to

$$\delta y_0 = \frac{1}{2}[y_1 - y_0] - \frac{1}{16}[\delta^2 y_1 - \delta^2 y_{-1}] + \frac{3}{256}[\delta^4 y_1 - \delta^4 y_{-1}] - \dots$$

which is the required form.

5.2 SOME MORE EXAMPLES OF FINITE DIFFERENCES

EXAMPLE 5.3

Find the missing term in the following table:

x	0	1	2	3	4
$f(x)$	1	3	9	–	81

Solution. Since four entries y_0, y_1, y_2, y_3, y_4 are given, the given function can be represented by a third degree polynomial. The difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	1				
		2			
1	3		4		
		6		$y_3 - 19$	
2	9		$y_3 - 15$		$124 - 4y_3$
		$y_3 - 9$		$105 - 3y_3$	
3	y_3		$90 - 2y_3$		
		$81 - y_3$			
4	81				

Since polynomial is of degree 3, $\Delta^4 f(x) = 0$ and so $124 - 4y_3 = 0$ and hence $y_3 = 31$.

EXAMPLE 5.4

If $y_0 = 3, y_1 = 12, y_2 = 81, y_3 = 2000$, and $y_4 = 100$, determine $\Delta^4 y_0$.

Solution. The difference table for the given data is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	3				
		9			
1	12		60		
		69		1790	
2	81		1850		−7459
		1919		−5669	
3	2000		−3819		
		−1900			
4	100				

From the table, we have $\Delta^4 y_0 = -7459$.

EXAMPLE 5.5

Establish the relations

(i) $\Delta\nabla = \nabla\Delta = \Delta - \nabla = \delta^2$;

(ii) $\mu\delta = \frac{1}{2}(\Delta + \nabla)$;

(iii) $\Delta = E\nabla = \nabla E = \delta E^{\frac{1}{2}}$.

Solution. (i) We know that

$$\Delta = E - I \text{ and } \nabla = I - \frac{I}{E}$$

Therefore,

$$\Delta\nabla = (E - I)\left(I - \frac{I}{E}\right) = E + \frac{I}{E} - 2I = \nabla\Delta \tag{5.27}$$

and

$$\Delta - \nabla = E + \frac{I}{E} - 2I \tag{5.28}$$

Furthermore,

$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}} = E^{\frac{1}{2}} - \frac{I}{E^{1/2}}$$

and so

$$\delta^2 = E + \frac{I}{E} - 2I. \tag{5.29}$$

The result follows from equations (5.27), (5.28), and (5.29).

(ii) We have

$$\mu = \frac{1}{2}(E^{1/2} + E^{-1/2}), \delta = E^{1/2} - E^{-1/2}.$$

Therefore,

$$\begin{aligned}\mu\delta &= \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2})\\ &= \frac{1}{2}(E - E^{-1}) = \frac{1}{2}\left(E - \frac{I}{E}\right)\\ &= \frac{1}{2}\left(E - I + I - \frac{I}{E}\right) = \frac{1}{2}(\Delta + \nabla).\end{aligned}$$

(iii) We have

$$\begin{aligned}E\nabla &= E\left(I - \frac{I}{E}\right) = E - I = \Delta\\ \nabla E &= \left(I - \frac{1}{E}\right)E = E - I = \Delta\\ \delta E^{1/2} &= (E^{1/2} - E^{-1/2})E^{1/2} = E - I = \Delta.\end{aligned}$$

Hence,

$$E\nabla = \nabla E = \delta E^{\frac{1}{2}} = \Delta.$$

EXAMPLE 5.6

Show that

$$E^r = \frac{E^{r+1} - E^{-t}}{E - E^{-1}} = \frac{\sinh 2r\theta}{\sinh 2\theta} E + \frac{\sinh 2t\theta}{\sinh 2\theta},$$

where $t = 1 - r$ and $\theta = \dfrac{hD}{2}$.

Solution. We have

$$\frac{E^{r+1}-E^{-t}}{E-E^{-1}}=\frac{E^{r+1}-E^{r-1}}{E-E^{-1}}=E^{r}\left(\frac{E-E^{-1}}{E-E^{-1}}\right)=E^{r}.$$

Also,

$$E-E^{-1}=e^{hD}-e^{-hD}=2\sinh 2\theta.$$

Therefore,

$$\frac{E^{r+1}-E^{-t}}{E-E^{-1}}=\frac{\frac{1}{2}(E^{r+1}-E^{r-1})}{\frac{1}{2}(E-E^{-1})}=\frac{\frac{1}{2}[EE^{r}-E^{r-1}]}{\sinh 2\theta}$$

$$=\frac{\frac{1}{2}[E(E^{r}-E^{-r})+E^{-(r-1)}-E^{r-1}]}{\sinh 2\theta}$$

$$=\frac{\frac{1}{2}[E(E^{r}-E^{-r})]-\frac{1}{2}(E^{1-r}-E^{r-1})}{\sinh 2\theta}$$

$$=\frac{\frac{1}{2}[E(E^{r}-E^{-r})]-\frac{1}{2}(E^{1-r}-E^{-(1-r)})}{\sinh 2\theta}$$

$$=\frac{E\sinh 2r\theta+\sinh 2t\theta}{\sinh 2\theta}$$

$$=\frac{\sinh 2r\theta}{\sinh 2\theta}E+\frac{\sinh 2t\theta}{\sinh 2\theta}.$$

EXAMPLE 5.7

Show that

$$\sum_{k=0}^{n-1}\Delta^{2}f_{k}=\Delta f_{n}-\Delta f_{0}.$$

Solution. We have

$$\Delta^{2}f_{0}=\Delta(\Delta f_{0})=\Delta(f_{1}-f_{0})=\Delta f_{1}-\Delta f_{0}$$
$$\Delta^{2}f_{1}=\Delta(\Delta f_{1})=\Delta(f_{2}-f_{1})=\Delta f_{2}-\Delta f_{1}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\Delta^{2}f_{n-1}=\Delta(\Delta f_{n-1})=\Delta(f_{n}-f_{n-1})=\Delta f_{n}-\Delta f_{n-1}.$$

Adding we get

$$\sum_{k=0}^{n-1}\Delta^{2}f_{k}=(\Delta f_{1}-\Delta f_{0})+(\Delta f_{2}-\Delta f_{1})+\ldots+(\Delta f_{n}-\Delta f_{n-1})$$
$$=\Delta f_{n}-\Delta f_{0}.$$

EXAMPLE 5.8

Show that $\sum_{k=0}^{n-1} \delta^2 f_{2k+1} = \tanh\left(\frac{U}{2}\right)(f_{2n} - f_0)$.

Solution. We have

$$\delta^2 = (E^{\frac{1}{2}} - E^{-\frac{1}{2}})^2 = \frac{(E^{\frac{1}{2}} - E^{-\frac{1}{2}})^2(E^{\frac{1}{2}} + E^{-\frac{1}{2}})}{E^{\frac{1}{2}} + E^{-\frac{1}{2}}} = \frac{(E^{\frac{1}{2}} - E^{-\frac{1}{2}})(E - E^{-1})}{E^{\frac{1}{2}} + E^{-\frac{1}{2}}}$$

$$= \frac{(e^{\frac{U}{2}} - e^{-\frac{U}{2}})(E - E^{-1})}{e^{\frac{U}{2}} + e^{-\frac{U}{2}}} = \tanh\left(\frac{U}{2}\right)(E - E^{-1}).$$

Thus,

$$\delta^2 f_{2k+1} = \tanh\left(\frac{U}{2}\right)[Ef_{2k+1} - E^{-1} f_{2k+1}] = \tanh\left(\frac{U}{2}\right)[f_{2k+2} - f_{2k}].$$

Therefore,

$$\sum_{k=0}^{n-1} \delta^2 f_{2k+1} = \tanh\left(\frac{U}{2}\right)[(f_2 - f_0) + (f_4 - f_2) + \ldots + (f_{2n} - f_{2n-2})] = \tanh\left(\frac{U}{2}\right)[f_{2n} - f_0].$$

EXAMPLE 5.9

Find the cubic polynomial $f(x)$ which takes on the values $f_0 = -5$, $f_1 = 1$, $f_2 = 9$, $f_3 = 25$, $f_4 = 55$, $f_5 = 105$.

Solution. The difference table for the given function is given below:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	−5				
		6			
1	1		2		
		8		6	
2	9		8		0
		16		6	
3	25		14		0
		30		6	
4	55		20		
		50			
5	105				

Now,

$$f_x = E^x f_0 = (I + \Delta)^x f_0$$

$$= \left[1 + x\Delta + \frac{x(x-1)}{2!}\Delta^2 + \frac{x(x-1)(x-2)}{3!}\Delta^3\right] f_0$$

$$= f_0 + x\Delta f_0 + \frac{x^2 - x}{2}\Delta^2 f_0 + \frac{x^3 - 3x^2 + 2x}{6}\Delta^3 f_0$$

$$= -5 + 6x + \frac{x^2 - x}{2}(2) + \frac{x^3 - 3x^2 + 2x}{6}(6) = x^3 - 2x^2 + 7x - 5,$$

which is the required cubic polynomial.

EXAMPLE 5.10

Determine

$$\Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)].$$

Solution. We have

$$\Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)]$$
$$=\Delta^{10}[abcd\ x^{10}+Ax^9+Bx^8+\cdots+1].$$

The polynomial in the square bracket is of degree 10. Therefore, $\Delta^{10}f(x)$ is constant and is equal to $a_n n!h^n$. In this case, we have $a_n = abcd$, $n = 10$, $h = 1$. Hence,

$$\Delta^{10} f(x) = abcd(10)!$$

EXAMPLE 5.11

Show that

$$\delta^2 y_5 = y_1 - 2y_5 + y_4.$$

Solution. We know that

$$\delta^2 = \Delta - \nabla.$$

Therefore,

$$\delta^2 y_5 = (\Delta - \nabla)y_5 = \Delta y_5 - \nabla y_5 = y_6 - y_5 - (y_5 - y_4)$$
$$= y_6 - 2y_5 + y_4.$$

EXAMPLE 5.12

Show that

$$e^x = \left(\frac{\Delta^2}{E}\right)e^x \cdot \frac{Ee^x}{\Delta^2 e^x},$$

the interval of differencing being h.

Solution. We note that

$$Ee^x = e^{x+h},\ \Delta e^x = e^{x+h} - e^x = e^x(e^h - 1)$$
$$\Delta^2 e^x = e^x(e^h - 1)^2$$

and

$$\left(\frac{\Delta^2}{E}\right)e^x = \Delta^2 E^{-1}(e^x) = \Delta^2 e^{x-h} = e^{-h}\Delta^2 e^x = e^{-h}e^x(e^h - 1)^2.$$

Hence,

$$\left(\frac{\Delta^2}{E}\right)e^x \cdot \frac{Ee^x}{\Delta^2 e^x} = e^{-h}e^x(e^h - 1)^2 \frac{e^{x+h}}{e^x(e^h - 1)^2} = e^x.$$

EXAMPLE 5.13

Show that

(i) $\delta^n y_x = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} y_{x+\frac{n}{2}-k}$

(ii) $\Delta \binom{n}{i+1} = \binom{n}{i}$.

Solution. (i) We have

$$\delta^n = \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)^n = E^{-\frac{n}{2}}(E-I)^n$$

Therefore,

$$\delta^n y_x = (E-I)^n y_{x-\frac{n}{2}} = [E^n \binom{n}{1} E^{n-1} + \binom{n}{2} E^{n-2} - \ldots + (-1)^n] y_{x-\frac{n}{2}}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} E^{n-k} y_{x-\frac{n}{2}}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} y_{x+n-k-\frac{n}{2}}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} y_{x+\frac{n}{2}-k.}$$

(ii) We have $\binom{n}{i+1} = \frac{n!}{(i+1)!(n-i-1)!}$.

Now,

$$\Delta \binom{n}{i+1} = \binom{n+1}{i+1} - \binom{n}{i+1} = \frac{n!(i+1)}{(i+1)!(n-i)!} = \frac{n!}{i!(n-1)!} = \binom{n}{i}.$$

EXAMPLE 5.14

Assuming that the following values of y belong to a polynomial of degree 4, find the missing values in the table:

x	0	1	2	3	4	5	6	7
y	1	−1	1	−1	1	–	–	–

Solution. The difference table of the given data is shown below:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		−2			
1	−1		4		
		2		−8	
2	1		−4		16
		−2		8	
3	−1		4		$\Delta^3 y_2 - 8$
		2		$\Delta^3 y_2$	
4	1		$\Delta^2 y_3$		$\Delta^3 y_3 - \Delta^3 y_2$
		Δy_4		$\Delta^3 y_3$	
5	y_5		$\Delta^2 y_4$		$\Delta^3 y_4 - \Delta^3 y_3$
		Δy_5		$\Delta^3 y_4$	
6	y_6		$\Delta^2 y_5$		
		Δy_6			
7	y_7				

Since the polynomial of the data is of degree 4, $\Delta^4 y$ should be constant. One of $\Delta^4 y$ is 16. Hence, all of the fourth differences must be 16. But then

$$\Delta^3 y_2 - 8 = 16 \text{ giving } \Delta^3 y_2 = 24$$

$$\Delta^3 y_2 - \Delta^3 y_2 = 16 \text{ giving } \Delta^3 y_2 = 40$$

$$\Delta^3 y_4 - \Delta^3 y_3 = 16 \text{ giving } \Delta^3 y_4 = 56$$

$$\Delta^2 y_3 - 4 = \Delta^3 y_2 = 24 \text{ and so } \Delta^2 y_3 = 28$$

$$\Delta^2 y_4 - \Delta^2 y_3 = \Delta^3 y_3 = 40 \text{ and so } \Delta^2 y_4 = 68$$

$$\Delta^2 y_5 - \Delta^2 y_4 = \Delta^3 y_4 = 56 \text{ and so } \Delta^2 y_5 = 124$$

$$\Delta y_4 - 2 = \Delta^2 y_3 = 28 \text{ and so } \Delta y_4 = 30$$

$$\Delta y_5 - \Delta y_4 = \Delta^2 y_4 = 68 \text{ and so } \Delta y_5 = 98$$

$$\Delta y_6 - \Delta y_5 = \Delta^2 y_5 = 124 \text{ and so } \Delta y_6 = 222$$

Hence,
$$y_5 - 1 = \Delta y_4 = 30 \text{ which gives } y_5 = 31$$

$$y_6 - y_5 = \Delta y_5 = 98 \text{ which gives } y_6 = 129$$

$$y_7 - y_6 = \Delta y_6 = 222 \text{ which yields } y_7 = 351.$$

Hence, the missing terms are

$$y_5 = 31, y_6 = 129, y_7 = 351.$$

5.3 ERROR PROPAGATION

Let $y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8$ be the values of the function f at the arguments $x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8$, respectively. Suppose an error ε is committed in y_4 during tabulation. To study the error propagation, we use the difference table. For the sake of convenience, we construct difference table up to fourth difference only. If the error in y_4 is ε, then the value of the function f at x_4 is $y_4 + \varepsilon$. The difference table of the data is as shown below.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0 + \varepsilon$
		Δy_2		$\Delta^3 y_1 + \varepsilon$	
x_3	y_3		$\Delta^2 y_2 + \varepsilon$		$\Delta^4 y_1 - 4\varepsilon$
		$\Delta y_3 + \varepsilon$		$\Delta^3 y_2 - 3\varepsilon$	
x_4	$y_4 + \varepsilon$		$\Delta^2 y_3 - 2\varepsilon$		$\Delta^4 y_2 + 6\varepsilon$
		$\Delta y_4 - \varepsilon$		$\Delta^3 y_3 + 3\varepsilon$	
x_5	y_5		$\Delta^2 y_4 + \varepsilon$		$\Delta^4 y_3 - 4\varepsilon$
		Δy_5		$\Delta^3 y_4 - \varepsilon$	
x_6	y_6		$\Delta^2 y_5$		$\Delta^4 y_4 + \varepsilon$
		Δy_6		$\Delta^3 y_5$	
x_7	y_7		$\Delta^2 y_6$		
		Δy_7			
x_8	y_8				

We note that

(i) Error propagates in a triangular pattern (shown by fan lines) and grows quickly with the order of difference.

(ii) The coefficients of the error ε in any column are the binomial coefficients of $(1-\varepsilon)^n$ with alternating signs. Thus, the errors in the third column are $\varepsilon, -3\varepsilon, 3\varepsilon, -\varepsilon$.

(iii) The algebraic sum of the errors in any difference column is zero.

(iv) If the difference table has even differences, then the maximum error lies on the same horizontal line on which the tabular value in error lies.

EXAMPLE 5.15

One entry in the following table of a polynomial of degree 4 is incorrect. Correct the entry by locating it

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
y	1.0000	1.5191	2.0736	2.6611	3.2816	3.9375	4.6363	5.3771	6.1776	7.0471	8.0

Solution. The difference table for the given data is shown below. Since the degree of the polynomial is four, the fourth difference must be constant. But we note that the fourth differences are oscillating for the larger values of x. The largest numerical fourth difference 0.0186 is at $x = 1.6$. This suggests that the error in the value of f is at $x = 1.6$. Draw the fan lines as shown in the difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.0	1.0000				
		0.5191			
1.1	1.5191		0.0354		
		0.5545		−0.0024	
1.2	2.0736		0.0330		0.0024
		0.5875		0	
1.3	2.6611		0.0330		0.0024
		0.6205		0.0024	
1.4	3.2816		0.0354		0.0051 Fan line
		0.6559		0.0075	
1.5	3.9375		0.0429		−0.0084
		0.6988		−0.0009	
1.6	4.6363		0.0420		0.0186
		0.7408		0.0177	
1.7	5.3771		0.0597		−0.0084
		0.8005		0.0093	
1.8	6.1776		0.0690		0.0051
		0.8695		0.0144	
1.9	7.0471		0.0834		
		0.9529			
2.0	8.0000				

Then taking 1.6 as x_0, we have

$$\Delta^4 f_{-4} + \varepsilon = 0.0051$$

$$\Delta^4 f_{-3} - 4\varepsilon = -0.0084$$

$$\Delta^4 f_{-2} + 6\varepsilon = 0.0186$$

$$\Delta^4 f_{-1} - 4\varepsilon = -0.0084$$

$$\Delta^4 f_0 + \varepsilon = 0.0051.$$

We want all fourth differences to be alike. Eliminating $\Delta^4 f$ between any two of the compatible equations and solving for ε will serve our purpose. For example, subtracting the second equation from the first, we get

$$5\varepsilon = 0.0135 \text{ and so } \varepsilon = 0.0027.$$

Putting this value of ε in the above equations, we note that all the fourth differences become 0.0024. Further,

$$f(1.6) + \varepsilon = 4.6363,$$

which yields

$$f(1.6) = 4.6363 - \varepsilon = 4.6363 - 0.0027 = 4.6336.$$

Thus, the error was a transposing error, that is, writing 63 instead of 36 while tabulation.

EXAMPLE 5.16

Find and correct the error, by means of differences, in the data:

x	0	1	2	3	4	5	6	7	8	9	10
y	2	5	8	17	38	75	140	233	362	533	752

Solution. The difference table for the given data is shown below. The largest numerical fourth difference −12 is at $x = 5$. So there is some error in the value $f(5)$. The fan lines are drawn and we note from the table that

$$\Delta^4 f_{-4} + \varepsilon = -2$$

$$\Delta^4 f_{-3} - 4\varepsilon = 8$$

$$\Delta^4 f_{-2} + 6\varepsilon = -12$$

$$\Delta^4 f_{-1} - 4\varepsilon = 8$$

$$\Delta^4 f_{0} + \varepsilon = -2$$

and

$$\Delta^3 f_{-3} + \varepsilon = 4$$

$$\Delta^3 f_{-2} - 3\varepsilon = 12$$

$$\Delta^3 f_{-1} + 3\varepsilon = 0$$

$$\Delta^3 f_{0} - \varepsilon = 8.$$

Subtracting second equation from the first (for both sets shown above), we get

$$5\varepsilon = -10 \text{ (for the first set) and } 4\varepsilon = -8 \text{ (for the second set).}$$

Hence, $\varepsilon = -2$.

Difference table

x	y	Δ	Δ^2	Δ^3	Δ^4
0	2				
		3			
1	5		0		
		3		6	
2	8		6		0
		9		6	
3	17		12		−2 Fan line
		21		4	
4	38		16		8
		37		12	
5	75		28		−12
		65		0	
6	140		28		8
		93		8	
7	233		36		−2
		129		6	
8	362		42		0
		171		6	
9	533		48		
		219			
10	752				

We now have

$$f(5) + \varepsilon = 75 \text{ and so } f(5) = 75 - \varepsilon = 75 - (-2) = 77.$$

Therefore, the true value of $f(5)$ is 77.

5.4 NUMERICAL UNSTABILITY

Subtraction of two nearly equal numbers causes a considerable loss of significant digits and may magnify the error in the later calculations. For example, if we subtract 63.994 from 64.395, which are correct to five significant figures, their difference 0.401 is correct only to three significant figures.

A similar loss of significant figures occurs when a number is divided by a small divisor. For example, we consider

$$f(x) = \frac{1}{1-x^2},\ x = 0.9.$$

Then true value of $f(0.9)$ is 0.526316×10. If x is approximated to $x^* = 0.900005$, that is, if some error appears in the sixth figure, then $f(x^*) = 0.526341 \times 10$. Thus, an error in the sixth place has caused an error in the fifth place in $f(x)$.

We note therefore that every arithmetic operation performed during computation gives rise to some error, which once generated may decay or grow in subsequent calculations. In some cases, error may grow so large as to make the computed result totally redundant. We call such a process (procedure) numerically unstable.

Adopting the calculation procedure that avoids subtraction of nearly equal numbers or division by small numbers or retaining more digits in the mantissa may avoid numerical instability.

EXAMPLE 5.17 (WILKINSON): CONSIDER THE POLYNOMIAL

$$P_{20}(x) = (x-1)(x-2)(x-20)$$
$$= x^{20} - 210x^{19} + \quad +(20)!$$

The zeros of this polynomial are 1, 2,..., 20. Let the coefficient of x^{19} be changed from 210 to $(210+2^{-23})$. This is a very small absolute change of magnitude 10^{-7} approximately. Most computers, generally, neglect this small change which occurs after 23 binary bits. But we note that smaller zeros of the new polynomial are obtained with good efficiency while the large roots are changed by a large amount. The largest change occurs in the roots 16 and 17. For example, against 16, we get $16.73\ldots \pm i2.81$ where magnitude is 17 approximately.

5.5 INTERPOLATION

Interpolation is the process of finding the value of a function for any value of argument (independent variable) within an interval for which some values are given.

Thus, interpolation is the art of reading between the lines in a given table.

Extrapolation is the process of finding the value of a function outside an interval for which some values are given.

We now discuss interpolation processes for equal spacing.

(A) Newton's Forward Difference Formula

Let $\cdots, f_{-2}, f_{-1}, f_0, f_1, f_2, \cdots$ be the values of a function for $\ldots, x_0-2h, x_0-h, x_0, x_0+h, x_0+2h, \ldots$

Suppose that we want to compute the function value f_p for $x = x_0 + ph$, where in general $-1 < p < 1$. We have

$$f_p = f(x_0+ph) \text{ and } p = \frac{x-x_0}{h},$$

where h is the interval of differencing. Then using shift operator and Binomial Theorem, we have

$$\begin{aligned} f_x &= E^p f_0 = (I+\Delta)^p f_0 \\ &= \left[I + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{(p(p-1)(p-2)}{3!}\Delta^3 + \ldots \right] f_0 \\ &= f_0 + \binom{p}{1}\Delta f_0 + \binom{p}{2}\Delta^2 f_0 + \binom{p}{3}\Delta^3 f_0 + \ldots \end{aligned} \tag{5.30}$$

The expression (5.30) is called Newton's forward difference formula for interpolation.

(B) Newton's Backward Difference Formula

Let... $f_{-2}, f_{-1}, f_0, f_1, f_2, \ldots$ be the values of a function for... $x_0-2h, x_0-h, x_0, x_0+h, x_0+2h, \ldots$. Suppose that we want to compute the function value f_p for $x = x_0 + ph$, $-1 < p < 1$. We have

$$f_p = f(x_0 + ph),\ p = \frac{x - x_0}{h}.$$

Using Newton's backward differences, we have

$$\begin{aligned} f_x &= E^p f_0 = (I - \nabla)^{-p} f_0 \\ &= \left[I + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \ldots \right] f_0 \\ &= f_0 + p\nabla f_0 + \frac{p(p+1)}{2!}\nabla^2 f_0 + \frac{p(p+1)(p+2)}{3!}\nabla^3 f_0 + \ldots, \end{aligned}$$

which is known as Newton's backward difference formula for interpolation.

Remark 5.1. It is clear from the differences used that

(i) Newton's forward difference formula is used for interpolating the values of the function near the beginning of a set of tabulated values.

(ii) Newton's backward difference formula is used for interpolating the values of the function near the end of a set of tabulated values.

EXAMPLE 5.18

Calculate approximate value of sin x for $x = 0.54$ and $x = 1.36$ using the following table:

x	0.5	0.7	0.9	1.1	1.3	1.5
sin$x$	0.47943	0.64422	0.78333	0.89121	0.96356	0.99749

Solution.

We take

$$x_0 = 0.50,\ x_p = 0.54, \text{ and } p = \frac{0.54 - 0.50}{0.2} = 0.2.$$

Using Newton's forward difference method, we have

$$\begin{aligned} f_p &= f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 f_0 \\ &\quad + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 f_0 + \frac{p(p-1)(p-2)(p-3)(p-4)}{5!}\Delta^5 f_0 \\ &= 0.47943 + 0.2(0.16479) + \frac{0.2(0.2-1)}{2}(\ 0.0268) \\ &\quad + \frac{0.2(0.2-1)(0.2-2)}{6}(-0.00555) + \frac{0.2(0.2-1)(0.2-2)(0.2-3)}{4!}(0.00125) \\ &\quad + \frac{0.2(0.2-1)(0.2-2)(0.2-3)(0.2-4)}{5!}(0.00016) \approx 0.51386. \end{aligned}$$

Difference table

x	sin *x*	1st difference	2nd difference	3rd difference	4th difference	5th difference
0.5	0.47943					
		0.16479				
0.7	0.64422		−0.02568			
		0.13911		−0.00555		
0.9	0.78333		−0.03123		0.00125	
		0.10788		−0.00430		0.00016
1.1	0.89121		−0.03553		0.00141	
		0.07235		−0.00289		
1.3	0.96356		−0.03842			
		0.03393				
1.5	0.99749					

Further, the point $x = 1.36$ lies toward the end of the tabulated values. Therefore, to find the value of the function at $x = 1.36$, we use Newton's backward differences method. We have

$$x_p = 1.36, \quad x_0 = 1.3, \text{ and } p = \frac{1.36 - 1.30}{0.2} = 0.3,$$

$$f_p = f_0 + p\nabla f_0 + \frac{p(p+1)}{2!}\nabla^2 f_0 + \frac{p(p+1)(p+2)}{3!}\nabla^3 f_0 + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 f_0$$

$$= 0.96356 + 0.3(0.07235) + \frac{0.3(0.3+1)}{2}(0.03553)$$

$$+ \frac{0.3(0.3+1)(0.3+2)}{6}(-0.00430) + \frac{0.3(0.3+1)(0.3+2)(0.3+3)}{24}(0.00125)$$

$$= 0.96356 + 0.021705 - 0.006128 - 0.000642 + 0.000154 \approx 0.977849.$$

EXAMPLE 5.19

Find the cubic polynomial $f(x)$ which takes on the values $f(0) = -4$, $f(1) = -1$, $f(2) = 2$, $f(3) = 11$, $f(4) = 32$, $f(5) = 71$. Find $f(6)$ and $f(2.5)$.

Solution. The difference table for the given data is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	−4			
		3		
1	−1		0	
		3		6
2	2		6	
		9		6
3	11		12	
		21		6
4	32		18	
		39		
5	71			

Using Newton's forward difference formula, we have

$$
\begin{aligned}
f_x &= f_0 + x\Delta f_0 + \frac{x(x-1)}{2!}\Delta^2 f_0 + \frac{x(x-1)(x-2)}{3!}\Delta^3 f_0 \\
&= -4 + x(3) + \frac{x^2 - x}{2}(0) + \frac{x^3 - 3x^2 + 2x}{6}(6) \\
&= x^3 - 3x^2 + 2x + 3x - 4 \\
&= x^3 - 3x^2 + 5x - 4,
\end{aligned}
$$

which is the required cubic polynomial. Therefore,

$$
\begin{aligned}
f(6) &= 6^3 - 3(6^2) + 5(6) - 4 \\
&= 216 - 108 + 30 - 4 = 134.
\end{aligned}
$$

On the other hand, if we calculate $f(6)$ using Newton's forward difference formula, then take $x_0 = 0$, $p = \frac{x - x_0}{h} = \frac{6-0}{1} = 6$ and have

$$
f(6) = f_6 = f_0 + 6\Delta f_0 + \frac{(6)(5)}{2}\Delta^2 f_0 + \frac{(6)(5)(4)}{6}\Delta^3 f_0
$$

$$
= 4 + 6(3) + 15(0) + 20(6) = 134 \text{ (exact value of } f(6)).
$$

Again taking $x_0 = 2$, we have $p = \frac{x - x_0}{h} = 2.5 - 2.0 = 0.5$. Therefore,

$$
f(2.5) = f_0 + pf_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{6}\Delta^3 f_0
$$

$$
= 2 + 0.5(9) + \frac{(0.5)(0.5-1)}{2}(12) + \frac{0.5(0.5-1)(0.5-2)}{6}(6)
$$

$$
= 2 + 4.50 - 1.50 + 0.375
$$

$$
= 6.875 - 1.500 = 5.375 \text{ (exact value of } f(2.5)).
$$

EXAMPLE 5.20

Find a cubic polynomial in x for the following data:

x	0	1	2	3	4	5
y	–3	3	11	27	57	107

Solution. The difference table for the given data is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	–3			
		6		
1	3		2	
		8		6
2	11		8	
		16		6
3	27		14	
		30		6
4	57		20	
		50		
5	107			

Using Newton's forward difference formula, we have

$$f_x = f_0 + x\Delta f_0 + \frac{x(x-1)}{2!}\Delta^2 f_0 + \frac{x(x-1)(x-2)}{3!}\Delta^3 f_0$$

$$= 3 + 6x + \frac{x^2 - x}{2}(2) + \frac{x^3 - 3x^2 + 2x}{6}(6)$$

$$= x^3 - 3x^2 + 2x + x^2 - x + 6x - 3$$

$$= x^3 - 2x^2 + 7x - 3.$$

EXAMPLE 5.21

The area A of a circle of diameter d is given for the following values:

d	80	85	90	95	100
A	5026	5674	6362	7088	7854

Calculate the area of a circle of diameter 105.

Solution. The difference table for the given data is

d	A				
80	5026				
		648			
85	5674		40		
		688		−12	
90	6362		28		32
		716		22	
95	7088		50		
		766			
100	7854				

Letting $x_p = 105, x_0 = 100$, and $p = \frac{105-100}{5} = 1$, we shall use Newton's backward difference formula

$$f_p = f_0 + p\nabla f_0 + \frac{p(p+1)}{2}\nabla^2 f_0 + \frac{p(p+1)(p+2)}{3!}\nabla^3 f_0 + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 f_0.$$

Therefore,

$$f(105) = 7854 + 766 + 50 + 22 + 32 = 8724.$$

Remark 5.2. We note (in the above example) that if a tabulated function is a polynomial, then interpolation and extrapolation would give exact values.

(C) Central Difference Formulae

Let $..., f_{-2}, f_{-1}, f_0, f_1, f_2, ...$ be the values of a function f for $..., x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h, ...$ Then

$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

and so

$$\delta E^{\frac{1}{2}} = E - I = \Delta, \quad \Delta^2 = \delta^2 E \text{ and } \Delta^3 = \delta^3 E^{\frac{3}{2}}.$$

Thus,

$$\Delta f_{-2} = \delta E^{\frac{1}{2}} f_{-2} = \delta f_{-\frac{3}{2}}$$

$$\Delta f_{-1} = \delta E^{\frac{1}{2}} f_{-1} = \delta f_{-\frac{1}{2}}$$

$$\Delta f_0 = \delta E^{\frac{1}{2}} f_0 = \delta f_{\frac{1}{2}}$$

$$\Delta f_1 = \delta E^{\frac{1}{2}} f_1 = \delta f_{\frac{3}{2}}$$

and so on. Similarly,

$$\Delta^2 f_{-2} = \delta^2 E f_{-2} = \delta^2 f_{-1}$$

$$\Delta^2 f_{-1} = \delta^2 E f_{-1} = \delta^2 f_0$$

$$\Delta^2 f_0 = \delta^2 E f_0 = \delta^2 f_1$$

and so on. Further,

$$\Delta^3 f_{-2} = \delta^3 E^{\frac{3}{2}} f_{-2} = \delta^3 f_{-\frac{1}{2}}$$

$$\Delta^3 f_{-1} = \delta^3 E^{\frac{3}{2}} f_{-1} = \delta^3 f_{\frac{1}{2}}$$

$$\Delta^3 f_0 = \delta^3 E^{\frac{3}{2}} f_0 = \delta^3 f_{\frac{3}{2}}$$

and so on. Hence, central difference table is expressed as

x	$f(x)$	δf_x	$\delta^2 f_x$	$\delta^3 f_x$	$\delta^4 f_x$
x_{-2}	f_{-2}				
		$\delta f_{-\frac{3}{2}}$			
x_{-1}	f_{-1}		$\delta^2 f_{-1}$		
		$\delta f_{-\frac{1}{2}}$		$\delta^3 f_{-\frac{1}{2}}$	
x_0	f_0		$\delta^2 f_0$		$\delta^4 f_0$
		$\delta f_{\frac{1}{2}}$		$\delta^3 f_{\frac{1}{2}}$	
x_1	f_1		$\delta^2 f_1$		
		$\delta f_{\frac{3}{2}}$			
x_2	f_2				

Now we are in a position to develop central difference interpolation formula.

(C_1) Gauss's Forward Interpolating Formula:

Let $\ldots, f_{-2}, f_{-1}, f_0, f_1, f_2, \ldots$ be the values of the function f at ... $x_0 - 2h$, $x_0 - h$, x_0, $x_0 + h$, $x_0 + 2h$,.... Suppose that we want to compute the function value for $x = x_0 + ph$. In Gauss's forward formula, we use the differences $\delta f_{\frac{1}{2}}, \delta^2 f_0, \delta^3 f_{\frac{1}{2}}, \delta^4 f_0, \ldots$ as shown by **boldface** letters in the table given below. The value f_p can be written as

$$f_p = f_0 + g_1 \delta f_{\frac{1}{2}} + g_2 \delta^2 f_0 + g_3 \delta^3 f_{\frac{1}{2}} + g_4 \delta^4 f_0 + \ldots$$

where $g_1, g_2, g_3, \ldots$ are the constants to be determined. The above equation can be written as

$$E^p f_0 = f_0 + g_1 \delta E^{\frac{1}{2}} f_0 + g_2 \delta^2 f_0 + g_3 \delta^3 E^{\frac{1}{2}} f_0 + g_4 \delta^4 f_0 + \ldots$$

Difference table

x	$f(x)$	δf_x	$\delta^2 f_x$	$\delta^3 f_x$	$\delta^4 f_x$
x_{-2}	f_{-2}				
		$\delta f_{-\frac{3}{2}}$			
x_{-1}	f_{-1}		$\delta^2 f_{-1}$		
		$\delta f_{-\frac{1}{2}}$		$\delta^3 f_{-\frac{1}{2}}$	
x_0	$\boldsymbol{f_0}$		$\boldsymbol{\delta^2 f_0}$		$\delta^4 f_0$
		$\boldsymbol{\delta f_{\frac{1}{2}}}$		$\boldsymbol{\delta^3 f_{\frac{1}{2}}}$	
x_1	f_1		$\delta^2 f_1$		
		$\delta f_{\frac{3}{2}}$			
x_2	f_2				

Hence,

$$E^p = I + g_1 \delta E^{\frac{1}{2}} + g_2 \delta^2 + g_3 \delta^3 E^{\frac{1}{2}} + g_4 \delta^4 + \ldots$$

or

$$(1+\Delta)^p = 1 + g_1 \Delta + g_2 \frac{\Delta^2}{1+\Delta} + g_3 \frac{\Delta^3}{1+\Delta} + g_4 \frac{\Delta^4}{(1+\Delta)^2} + \ldots$$

$$= I + g_1 \Delta + g_2 \Delta^2 (1 - \Delta + \Delta^2 - \ldots) + g_3 \Delta^3 (1 - \Delta + \Delta^2 - \ldots) + g_4 \Delta^4 (1 - 2\Delta - \ldots).$$

The left-hand side equals

$$1 + p\Delta + \frac{p(p-1)}{2!} \Delta^2 + \frac{p(p-1)(p-2)}{3!} \Delta^3 + \frac{p(p-1)(p-2)(p-3)}{4!} + \ldots.$$

Comparing coefficients of the powers of Δ on both sides, we get

$$g_1 = p, g_2 = \frac{p(p-1)}{2!}, \; g_3 - g_2 = \frac{p(p-1)(p-2)}{6}$$

and so

$$g_3 = \frac{p(p-1)(p-2)}{6} + \frac{p(p-1)}{2} = \frac{p(p-1)(p-2) + 3p(p-1)}{6}$$

$$= \frac{p(p-1)(p-2+3)}{6} = \frac{(p+1)p(p-1)}{3!},$$

$$g_4 - g_3 + g_2 = \frac{p(p-1)(p-2)(p-3)}{4!},$$

and so

$$g_4 = \frac{p(p-1)(p-2)(p-3)}{4!} + g_3 - g_2$$

$$= \frac{p(p-1)(p-2)(p-3)}{4!} + \frac{(p+1)p(p-1)}{3!} - \frac{p(p+1)}{2!}$$

$$= \frac{p(p-1)[(p-2)(p-3)+4(p+1)-12]}{4!}$$

$$= \frac{p(p-1)[p^2-p-2]}{4!} = \frac{(p+1)p(p-1)(p-2)}{4!},$$

and so on. Hence,

$$f_p = f_0 + p\delta f_{\frac{1}{2}} + \frac{p(p-1)}{2!}\delta^2 f_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 f_{\frac{1}{2}} + \frac{(p+1)p(p-1)(p-2)}{4!}\delta^4 f_0 + \ldots$$

$$= f_0 + \binom{p}{1}\delta f_{\frac{1}{2}} + \binom{p}{2}\delta^2 f_0 + \binom{p+1}{3}\delta^3 f_{\frac{1}{2}} + \binom{p+1}{4}\delta^4 f_0 + \binom{p+2}{5}\delta^5 f_{\frac{1}{2}} + \ldots.$$

(C_2) Gauss's Backward Interpolation Formula:

The central difference table for this formula is shown below:

x	$f(x)$	δf_x	$\delta^2 f_x$	$\delta^3 f_x$	$\delta^4 f_x$
x_{-2}	f_{-2}				
		$\delta f_{-\frac{3}{2}}$			
x_{-1}	f_{-1}		$\delta^2 f_{-1}$		
		$\boldsymbol{\delta f_{-\frac{1}{2}}}$		$\boldsymbol{\delta^3 f_{-\frac{1}{2}}}$	
x_0	$\boldsymbol{f_0}$		$\boldsymbol{\delta^2 f_0}$		$\boldsymbol{\delta^4 f_0}$
		$\delta f_{\frac{1}{2}}$		$\delta^3 f_{\frac{1}{2}}$	
x_1	f_1		$\delta^2 f_1$		
		$\delta f_{\frac{3}{2}}$			
x_2	f_2				

In Gauss's backward interpolation formula, we use the differences

$\delta f_{-\frac{1}{2}}, \delta^2 f_0, \delta^3 f_{-\frac{1}{2}}, \delta^4 f_0, \ldots$. Thus, f_p can be written as

$$f_p = f_0 + g_1 \delta f_{-\frac{1}{2}} + g_2 \delta^2 f_0 + g_3 \delta^3 f_{-\frac{1}{2}} + g_4 \delta^4 f_0 + \ldots$$

where $g_1, g_2, g_3, \ldots$ are the constants to be determined. The above equation can be written as

$$E^p f_0 = f_0 + g_1 \delta E^{-\frac{1}{2}} f_0 + g_2 \delta^2 f_0 + g_3 \delta^3 E^{-\frac{1}{2}} f_0 + g_4 \delta^4 f_0 + \ldots$$

and so

$$E^p = I + g_1 \delta E^{-\frac{1}{2}} + g_2 \delta^2 + g_3 \delta^3 E^{-\frac{1}{2}} + g_4 \delta^4 + \ldots$$

or

$$(I + \Delta)^p = I + g_1 \frac{\Delta}{1+\Delta} + g_2 \frac{\Delta^2}{1+\Delta} + g_3 \frac{\Delta^3}{(1+\Delta)^2} + g_4 \frac{\Delta^4}{(1+\Delta)^2} + \ldots$$

$$= 1 + g_1 \Delta (1 - \Delta + \Delta^2 - \Delta^3 + \ldots) + g_2 \Delta^2 (1 - \Delta + \Delta^2 - \Delta^3 + \ldots)$$
$$+ g_3 \Delta^3 (1 - 2\Delta + \ldots) + g_4 \Delta^4 (1 - 2\Delta + \ldots).$$

But

$$(1+\Delta)^p = 1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3$$
$$+ \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 + \ldots.$$

Therefore, comparing coefficients of the powers of Δ, we have

$$g_1 = p, \ \ g_2 - g_1 = \frac{p(p-1)}{2!} \text{ and so } g_2 = \frac{p(p-1)}{2!} + g_1 = \frac{(p+1)p}{2!},$$

$$g_3 - g_2 + g_1 = \frac{p(p-1)(p-2)}{3!} \text{ and so } g_3 = \frac{(p+1)p(p-1)}{3!}.$$

Hence,

$$f_p = f_0 + p\delta f_{-\frac{1}{2}} + \frac{(p+1)p}{2!}\delta^2 f_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 f_{-\frac{1}{2}} + \ldots$$

$$= f_0 + \binom{p}{1}\delta f_{-\frac{1}{2}} + \binom{p+1}{2}\delta^2 f_0 + \binom{p+1}{3}\delta^3 f_{-\frac{1}{2}} + \binom{p+2}{4}\delta^4 f_0 + \ldots.$$

(C_3) Stirling's Interpolation Formula:

The central differences table for this formula is shown below. In this formula, we use $f_0, \delta f_{-\frac{1}{2}}, \delta f_{\frac{1}{2}}, \delta^2 f_0$, $\delta^3 f_{-\frac{1}{2}}, \delta^3 f_{\frac{1}{2}}, \delta^4 f_0, \ldots$

Difference table

x	$f(x)$	δf_x	$\delta^2 f_x$	$\delta^3 f_x$	$\delta^4 f_x$
x_{-2}	f_{-2}				
		$\delta f_{-\frac{3}{2}}$			
x_{-1}	f_{-1}		$\delta^2 f_{-1}$		
		$\boldsymbol{\delta f_{-\frac{1}{2}}}$		$\boldsymbol{\delta^3 f_{-\frac{1}{2}}}$	
x_0	$\boldsymbol{f_0}$		$\boldsymbol{\delta^2 f_0}$		$\boldsymbol{\delta^4 f_0}$
		$\boldsymbol{\delta f_{\frac{1}{2}}}$		$\boldsymbol{\delta^3 f_{\frac{1}{2}}}$	
x_1	f_1		$\delta^2 f_1$		
		$\delta f_{\frac{3}{2}}$			
x_2	f_2				

By Gauss's forward interpolation formula, we have

$$f_p = f_0 + \binom{p}{1}\delta f_{\frac{1}{2}} + \binom{p}{2}\delta^2 f_0 + \binom{p+1}{3}\delta^3 f_{\frac{1}{2}} + \binom{p+1}{4}\delta^4 f_0 + \binom{p+2}{5}\delta^5 f_{\frac{1}{2}} + \dots \tag{5.31}$$

and by Gauss's backward interpolation formula, we have

$$f_p = f_0 + \binom{p}{1}\delta f_{-\frac{1}{2}} + \binom{p+1}{2}\delta^2 f_0 + \binom{p+1}{3}\delta^3 f_{-\frac{1}{2}} + \binom{p+2}{4}\delta^4 f_0 + \dots. \tag{5.32}$$

Adding equations (5.31) and (5.32), we get

$$f_p = f_0 + \frac{1}{2}\binom{p}{1}\left[\delta f_{\frac{1}{2}} + \delta f_{-\frac{1}{2}}\right] + \frac{1}{2}\left[\binom{p}{2} + \binom{p+1}{2}\right]\delta^2 f_0 + \frac{1}{2}\binom{p+1}{3}\left[\delta^3 f_{\frac{1}{2}} + \delta^3 f_{-\frac{1}{2}}\right] + \frac{1}{2}\left[\binom{p+1}{4} + \binom{p+2}{4}\right]\delta^4 f_0 + \dots$$

$$= f_0 + \frac{p}{2}\left(\delta f_{\frac{1}{2}} + \delta f_{-\frac{1}{2}}\right) + \frac{p^2}{2}\delta^2 f_0$$

$$+ \frac{(p+1)p(p-1)}{2(3!)}\left(\delta^3 f_{\frac{1}{2}} + \delta^3 f_{-\frac{1}{2}}\right)$$

$$+ \frac{p(p+1)p(p-1)}{4(3!)}\delta^4 f_0 + \ldots$$

$$= f_0 + \binom{p}{1}\mu\delta f_0 + \frac{p}{2}\binom{p}{1}\delta^2 f_0 + \binom{p+1}{3}\mu\delta^3 f_0,$$

$$+ \frac{p}{4}\binom{p+1}{3}\delta^4 f_0 + \binom{p+2}{5}\mu\delta^5 f_0 + \ldots$$

which is the required Stirling's formula.

Second Method: We have

$$f_p = f_0 + S_1\left(\delta f_{\frac{1}{2}} + \delta f_{-\frac{1}{2}}\right) + S_2\delta^2 f_0 + S_3\left(\delta^3 f_{\frac{1}{2}} + \delta^3 f_{-\frac{1}{2}}\right) + S_4\delta^4 f_0 + \ldots, \tag{5.33}$$

where $S_1, S_2 \ldots$ are the constants to be determined. Expression (5.33) can be written as

$$E^p f_0 = f_0 + S_1(\delta E^{\frac{1}{2}} f_0 + \delta E^{-\frac{1}{2}} f_0 + S_2\delta^2 f_0 + S_3(\delta^3 E^{\frac{1}{2}} f_0 + \delta^3 E^{-\frac{1}{2}} f_0) + S_4\delta^4 f_0 + \ldots$$

$$= \left(I + S_1\left(\Delta + \frac{\Delta}{1+\Delta}\right) + S_2\frac{\Delta^2}{1+\Delta} + S_3\left(\frac{\Delta^3}{1+\Delta} + \frac{\Delta^3}{(1+\Delta)^2}\right) + S_4\frac{\Delta^4}{(1+\Delta)^4} + \ldots\right) f_0.$$

Therefore, expression (5.33) gives

$$E^p = (I+\Delta)^p = I + S_1[\Delta + \Delta(I - \Delta + \Delta^2 - \ldots) + S_2\Delta^2(1 - \Delta + \Delta^2 - \ldots)]$$
$$+ S_3[\Delta^3(I - \Delta + \Delta^2 - \ldots) + \Delta^3(1 - 2\Delta + \ldots)] + S_4\Delta^4(1 - 4\Delta + \ldots).$$

The left-hand side is

$$(I+\Delta)^p = 1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 + \ldots.$$

Comparing coefficients of the powers of Δ, we get

$$S_1 = \frac{p}{2},$$

$$S_2 = S_1 + \frac{p(p-1)}{2} = \frac{p^2}{2},$$

$$S_3 = \frac{p(p-1)(p+1)}{2(3!)},$$

$$S_4 = \frac{p}{4}\frac{(p+1)^2 p(p-1)}{3!}.$$

Thus,

$$f_p = f_0 + \frac{p}{2}\left(\delta f_{\frac{1}{2}} + \delta f_{-\frac{1}{2}}\right) + \frac{p^2}{2}\delta^2 f_0 + \frac{(p+1)p(p-1)}{2(3!)}\left(\delta^3 f_{\frac{1}{2}} + \delta^3 f_{-\frac{1}{2}}\right)$$
$$+\frac{p}{4}\frac{(p+1)p(p-1)}{3!}\delta^4 f_0 + \ldots,$$

which is the required Stirling's formula.

(C_4) Bessel's Interpolation Formula

Let $\ldots, f_{-2}, f_{-1}, f_0, f_1, f_2, \ldots$ be the values of a function at $\ldots, x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h, \ldots$. Suppose that we want to compute the function value f_p for $x = x_0 + ph$. In Bessel's formula, we use the differences as indicated in the table below. In this method $f_0, \delta f_{\frac{1}{2}}, \delta^2 f_0, \delta^2 f_1, \delta^3 f_{\frac{1}{2}}, \delta^4 f_0, \delta^4 f_1, \ldots$ are used to approximate f_p. These values are shown in this difference table in boldface. Therefore, f_p can be written in the form

$$f_p = f_0 + B_1 \delta f_{\frac{1}{2}} + B_2(\delta^2 f_0 + \delta^2 f_1) + B_3 \delta^3 f_{\frac{1}{2}} + B_4(\delta^4 f_0 + \delta^4 f_1) + \ldots$$

where $B_1, B_2, \ldots$ are the constants to be determined.

x	$f(x)$	δf_x	$\delta^2 f_x$	$\delta^3 f_x$	$\delta^4 f_x$
x_{-2}	f_{-2}				
		$\delta f_{-\frac{3}{2}}$			
x_{-1}	f_{-1}		$\delta^2 f_{-1}$		
		$\delta f_{-\frac{1}{2}}$		$\delta^3 f_{-\frac{1}{2}}$	
x_0	$\boldsymbol{f_0}$		$\boldsymbol{\delta^2 f_0}$		$\boldsymbol{\delta^4 f_0}$
		$\boldsymbol{\delta f_{\frac{1}{2}}}$		$\boldsymbol{\delta^3 f_{\frac{1}{2}}}$	
x_1	f_1		$\boldsymbol{\delta^2 f_1}$		$\boldsymbol{\delta^4 f_1}$
		$\delta f_{\frac{3}{2}}$			
x_2	f_2				

The above equation can be written as

$$E^p f_0 = f_0 + B_1 \delta E^{\frac{1}{2}} f_0 + B_2(\delta^2 f_0 + \delta^2 E f_0) + B_3 \delta^3 E^{\frac{1}{2}} f_0 + B_4(\delta^4 f_0 + \delta^4 E f_0) + \ldots$$

or

$$E^p = I + B_1 \delta E^{\frac{1}{2}} + B_2(\delta^2 + \delta^2 E) + B_3 \delta^3 E^{\frac{1}{2}} + B_4(\delta^4 + \delta^4 E) + \ldots$$

or

$$(I+\Delta)^p = I + B_1\Delta + B_2\left(\frac{\Delta^2}{1+\Delta} + \Delta^2\right) + B_3\frac{\Delta^3}{1+\Delta} + B_4[\Delta^2(I+\Delta)^{-2} + \Delta^4(I+\Delta)] + \ldots \qquad (5.34)$$

The left-hand side equals

$$I + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 + \dots \quad (5.35)$$

Comparing coefficients of the powers of Δ in equations (5.34) and (5.35), we have

$$B_1 = p,$$

$$2B_2 = \frac{p(p-1)}{2!} \text{ and so } B_2 = \frac{1}{2}\frac{p(p-1)}{2!},$$

$$-B_2 + B_3 = \frac{p(p-1)(p-2)}{3!},$$

and so

$$B_3 = \frac{p(p-1)(p-2)}{3!} + \frac{1}{2}\frac{p(p-1)}{2} = \frac{p(p-1)\left(p-2+\frac{3}{2}\right)}{3!} = \frac{p(p-1)\left(p-\frac{1}{2}\right)}{3!}$$

and

$$B_2 - B_3 + 2B_4 = \frac{p(p-1)(p-2)(p-3)}{4!},$$

which yields

$$B_4 = \frac{1}{2}\left[\frac{p(p-1)(p-2)(p-3)}{4!} + \frac{p(p-1)(p-2)}{3!}\right]$$

$$= \frac{1}{2}\left[\frac{p(p-1)(p-2)}{4!}(p-3+4)\right] = \frac{1}{2}\frac{(p+1)p(p-1)(p-2)}{4!}.$$

Similarly

$$B_5 = \frac{(p+1)p\left(p-\frac{1}{2}\right)(p-1)(p-2)}{5!}$$

and so on. Therefore,

$$f_p = f_0 + p\delta f_{\frac{1}{2}} + \frac{p(p-1)}{2!}\left[\frac{\delta^2 f_0 + \delta^2 f_1}{2}\right] + \frac{p\left(p-\frac{1}{2}\right)(p-1)}{3!}\delta^3 f_{\frac{1}{2}}$$
$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\left[\frac{\delta^4 f_0 + \delta^4 f_1}{2}\right] + \dots$$

$$= f_0 + p\delta f_{\frac{1}{2}} + \frac{p(p-1)}{2!}\mu\delta^2 f_{\frac{1}{2}} + \frac{p\left(p-\frac{1}{2}\right)(p-1)}{3!}\delta^3 f_{\frac{1}{2}}$$
$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\mu\delta^4 f_{\frac{1}{2}} + \ldots$$

$$= f_0 + p\delta f_{\frac{1}{2}} + \binom{p}{2}\mu\delta^2 f_{\frac{1}{2}} + \frac{p\left(p-\frac{1}{2}\right)(p-1)}{3!}\delta^3 f_{\frac{1}{2}} + \binom{p+1}{4}\mu\delta^4 f_{\frac{1}{2}} + \ldots$$

$$= f_0 + \frac{1}{2}\delta f_{\frac{1}{2}} + \left(p-\frac{1}{2}\right)\delta f_{\frac{1}{2}} + \binom{p}{2}\mu\delta^2 f_{\frac{1}{2}} + \frac{p\left(p-\frac{1}{2}\right)(p-1)}{3!}\delta^3 f_{\frac{1}{2}}$$
$$+ \binom{p+1}{4}\mu\delta^4 f_{\frac{1}{2}} + \ldots$$

$$= f_0 + \frac{1}{2}(f_1 - f_0) + \left(p-\frac{1}{2}\right)\delta f_{\frac{1}{2}} + \binom{p}{2}\mu\delta^2 f_{\frac{1}{2}} + \frac{p\left(p-\frac{1}{2}\right)(p-1)}{3!}\delta^3 f_{\frac{1}{2}}$$
$$+ \binom{p+1}{4}\mu\delta^4 f_{\frac{1}{2}} + \ldots$$

$$= \frac{1}{2}(f_0 + f_1) + \left(p-\frac{1}{2}\right)\delta f_{\frac{1}{2}} + \binom{p}{2}\mu\delta^2 f_{\frac{1}{2}} + \frac{p\left(p-\frac{1}{2}\right)(p-1)}{3!}\delta^3 f_{\frac{1}{2}}$$
$$+ \binom{p+1}{4}\mu\delta^4 f_{\frac{1}{2}} + \ldots$$

$$= \mu f_{\frac{1}{2}} + \left(p-\frac{1}{2}\right)\delta f_{\frac{1}{2}} + \binom{p}{2}\mu\delta^2 f_{\frac{1}{2}} + \frac{p\left(p-\frac{1}{2}\right)(p-1)}{3!}\delta^3 f_{\frac{1}{2}}$$
$$+ \binom{p+1}{4}\mu\delta^4 f_{\frac{1}{2}} + \ldots$$

$$= \binom{p}{0}\mu f_{\frac{1}{2}} + \left(p-\frac{1}{2}\right)\delta f_{\frac{1}{2}} + \binom{p}{2}\mu\delta^2 f_{\frac{1}{2}} + \frac{p\left(p-\frac{1}{2}\right)(p-1)}{3!}\delta^3 f_{\frac{1}{2}}$$
$$+ \binom{p+1}{4}\mu\delta^4 f_{\frac{1}{2}} + \ldots$$

$$= \binom{p}{0}\mu f_{\frac{1}{2}} + \left(p-\frac{1}{2}\right)\delta f_{\frac{1}{2}} + \binom{p}{2}\mu\delta^2 f_{\frac{1}{2}} + \frac{1}{3}\left(p-\frac{1}{2}\right)\binom{p}{2}\delta^3 f_{\frac{1}{2}}$$
$$+\binom{p+1}{4}\mu\delta^4 f_{\frac{1}{2}} + \ldots$$

If we put $p=\frac{1}{2}$, we get

$$f_{\frac{1}{2}} = \mu f_{\frac{1}{2}} - \frac{1}{8}\mu\delta^2 f_{\frac{1}{2}} + \frac{3}{128}\mu\delta^4 f_{\frac{1}{2}} - \ldots,$$

which is called formula for interpolating to halves or formula for halving an interval. It is used for computing values of the function midway between any two given values.

(C_5) Everett's Interpolation Formula

Let $\ldots, f_{-2}, f_{-1}, f_0, f_1, f_2, \ldots$ be the values of the function f at $\ldots, x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h, \ldots$. Suppose that we want to compute the function value for $x = x_0 + ph$. In Everett's formula, we use differences of even order only. Thus, we use the values $f_0, f_1, \delta^2 f_0, \delta^2 f_1, \delta^4 f_0, \delta^4 f_1, \ldots$, which have been shown in boldface in the difference table below:

x	$f(x)$	δf_x	$\delta^2 f_x$	$\delta^3 f_x$	$\delta^4 f_x$
x_{-2}	f_{-2}		$\delta f_{-\frac{3}{2}}$		
x_{-1}	f_{-1}		$\delta^2 f_{-1}$		
		$\delta f_{-\frac{1}{2}}$		$\delta^3 f_{-\frac{1}{2}}$	
x_0	$\boldsymbol{f_0}$		$\boldsymbol{\delta^2 f_0}$		$\boldsymbol{\delta^4 f_0}$
		$\delta f_{\frac{1}{2}}$		$\delta^3 f_{\frac{1}{2}}$	
x_1	$\boldsymbol{f_1}$		$\boldsymbol{\delta^2 f_1}$		$\boldsymbol{\delta^4 f_1}$
		$\delta f_{\frac{3}{2}}$			
x_2	f_2				

By Bessel's formula, we have

$$f_p = f_0 + p\delta f_{\frac{1}{2}} + \frac{p(p-1)}{2!}\left[\frac{\delta^2 f_0 + \delta^2 f_1}{2}\right] + \frac{p\left(p-\frac{1}{2}\right)(p-1)}{3!}\delta^3 f_{\frac{1}{2}}$$
$$+\frac{(p+1)p(p-1)p(p-2)}{4!}\left[\frac{\delta^4 f_0 + \delta^4 f_1}{2}\right] + \ldots$$

Since Everett's formula expresses f_p in terms of even differences lying on the horizontal lines through f_0 and f_1, therefore we convert $\delta f_{\frac{1}{2}}, \delta^3 f_{\frac{1}{2}}, \ldots$ in the Bessel's formula into even differences.

By doing so, we have

$$f_p = f_0 + p(f_1 - f_0) + \frac{p(p-1)}{2!}\left[\frac{\delta^2 f_0 + \delta^2 f_1}{2}\right] + \frac{p\left(p-\frac{1}{2}\right)(p-1)}{3!}[\delta^2 f_1 - \delta^2 f_0]$$

$$+\frac{(p+1)p(p-1)(p-2)}{4!}\left[\frac{\delta^4 f_0 + \delta^4 f_1}{2}\right]+\dots$$

$$= (1-p)f_0 - \frac{p(p-1)(p-2)}{3!}\delta^2 f_0 - \frac{(p+1)p(p-1)(p-2)(p-3)}{5!}\delta^4 f_0 - \dots$$

$$+pf_1 + \frac{(p+1)p(p-1)}{3!}\delta^2 f_1 + \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\delta^4 f_1 + \dots$$

$$= (1-p)f_0 + \binom{2-p}{3}\delta^2 f_0 + \binom{3-p}{5}\delta^4 f_0 + \dots$$

$$+pf_1 + \binom{p+1}{3}\delta^2 f_1 + \binom{p+2}{5}\delta^4 f_1 + \dots$$

$$= qf_0 + \binom{q+1}{3}\delta^2 f_0 + \binom{q+2}{5}\delta^4 f_0 + \dots$$

$$+pf_1 + \binom{p+1}{3}\delta^2 f_1 + \binom{p+2}{5}\delta^4 f_1 + \dots,$$

where $q = 1 - p$.

Second Method: Let $\dots, f_{-2}, f_{-1}, f_0, f_1, f_2, \dots$ be the values of a function f for $\dots, x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h, \dots$ We want to compute f_p, where $x = x_0 + ph$. We use the even differences lying on the horizontal lines through f_0 and f_1. So let

$$f_p = E_0 f_0 + E_2\delta^2 f_0 + E_4\delta^4 f_0 + \dots$$
$$+F_0 f_1 + F_2\delta^2 f_1 + F_4\delta^4 f_1 + \dots.$$

Therefore,

$$E^p f_0 = E_0 f_0 + E_2\delta^2 f_0 + E_4\delta^4 f_0 + \dots$$
$$+F_0 Ef_0 + F_2\delta^2 Ef_0 + F_4\delta^4 Ef_0 + \dots$$

or

$$(I+\Delta)^p = E_0 + E_2\delta^2 + E_4\delta^4 + \dots$$
$$+F_0 E + F_2\delta^2 E + F_4\delta^4 + \dots$$
$$= E_0 + E_2\frac{\Delta^2}{1+\Delta} + E_4\frac{\Delta^4}{(1+\Delta)^2} + \dots$$
$$+F_0(1+\Delta) + F_2\Delta^2 + F_4\frac{\Delta^4}{I+\Delta} + \dots. \quad (5.36)$$

The left-hand side of equation (5.36) is

$$I + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 + \ldots,$$

whereas the right-hand side is

$$E_0 + E_2[\Delta^2(I - \Delta + \Delta^2 - \Delta^3 + \ldots)] + E_4[\Delta^4(I - 2\Delta + \ldots)] + \ldots$$

$$+F_0(I + \Delta) + F_2\Delta^2 + F_4[\Delta^4(I - \Delta + \Delta^2 - \Delta^3 + \ldots)] + \ldots$$

Comparing coefficients of $\Delta, \Delta^2, \Delta^3, \ldots$ on both sides, we get

$$p = F_0, 1 = E_0 + F_0 \text{ and so } E_0 = 1 - p,$$

$$-E_2 = \frac{p(p-1)(p-2)}{3!}, E_2 + F_2 = \frac{p(p-1)}{2} \text{ and so } F_2 = \binom{p+1}{3}.$$

Similarly, other coefficients are obtained. Hence,

$$f_p = (1-p)f_0 + \binom{2-p}{3}\delta^2 f_0 + \binom{3-p}{5}\delta^4 f_0 + \ldots$$

$$+pf_0 + \binom{p+1}{3}\delta^2 f_1 + \binom{p+2}{5}\delta^4 f_1 + \ldots$$

$$= qf_0 + \binom{q+1}{3}\delta^2 f_0 + \binom{q+2}{5}\delta^4 f_0 + \ldots$$

$$+pf_0 + \binom{p+1}{3}\delta^2 f_0 + \binom{p+2}{5}\delta^4 f_1 + \ldots,$$

where $q = 1 - p$.

Remark 5.3. The Gauss's forward, Gauss's backward, Stirling's, Bessel's, Everett's, Newton's forward and Newton's backward interpolation formulae are called classical formulae and are used for equal spacing.

EXAMPLE 5.22

The function y is given in the table below:

x	0.01	0.02	0.03	0.04	0.05
y	98.4342	48.4392	31.7775	23.4492	18.4542

Find y for $x = 0.0341$.

Solution. The central difference table is

x	y	δ	δ^2	δ^3	δ^4
0.01	98.4342				
		−49.9950			
0.02	48.4392		33.3333		
		−16.6617		−24.9999	
0.03	31.7775		8.3334		19.9998
		−8.3283		−5.0001	
0.04	23.4492		3.3333		
		− 4.9950			
0.05	18.4542				

Letting $x_0 = 0.03$, we have $p = \dfrac{x - x_0}{h} = \dfrac{0.0341 - 0.030}{0.01} = 0.41$. Using Bessel's formula, we have

$$f(0.0341) = f_0 + p\delta f_{\frac{1}{2}} + \frac{p(p-1)}{4}(\delta^2 f_0 + \delta^2 f_1) + \frac{p\left(p - \frac{1}{2}\right)(p-1)}{3!}\delta^3 f_{\frac{1}{2}}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\left(\frac{\delta^4 f_0 + \delta^4 f_1}{2}\right) + \ldots$$

$$= 31.7775 + 0.41(8.3283) + \frac{11.6667}{4}(0.2419)$$

$$= 27.475924 \text{ approximately.}$$

EXAMPLE 5.23

If third differences are constant, prove that

$$y_{x+\frac{1}{2}} = \frac{1}{2}(y_x + y_{x+1}) - \frac{1}{16}(\Delta^2 y_{x-1} + \Delta^2 y_x).$$

Solution. The Bessel's formula in this case becomes

$$y_p = \frac{y_0 + y_1}{2} + \left(p - \frac{1}{2}\right)\Delta y_0 + \frac{p(p-1)}{2!}\frac{[\Delta^2 y_{-1} + \Delta^2 y_0]}{2}$$

$$+ \frac{p\left(p - \frac{1}{2}\right)(p-1)}{3!}\Delta^3 y_{-1},$$

because the higher differences than that of third order will be equal to zero by the hypothesis. Putting $p = \dfrac{1}{2}$, we get

$$y_{\frac{1}{2}} = \frac{y_0 + y_1}{2} - \frac{1}{16}(\Delta^2 y_{-2} + \Delta^2 y_0).$$

Changing the origin to x, we have

$$y_{x+\frac{1}{2}} = \frac{1}{2}(y_x + y_{x+1}) - \frac{1}{16}(\Delta^2 y_{x-1} + \Delta^2 y_x).$$

EXAMPLE 5.24

Given $y_0, y_1, y_2, y_3, y_4, y_5$ (fifth difference constant), prove that

$$y_{\frac{5}{2}} = \frac{c}{2} + \frac{25(c-b) + 3(a-c)}{256},$$

where $a = y_0 + y_5, b = y_1 + y_4, c = y_2 + y_3$.

Solution. Putting $p = \dfrac{1}{2}$ in the Bessel's formula, we have

$$y_{\frac{1}{2}} = \frac{y_0 + y_1}{2} - \frac{1}{8}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{3}{128}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right).$$

Shifting the origin to 2, we obtain

$$y_{\frac{5}{2}} = \frac{1}{2}(y_2 + y_3) - \frac{1}{16}(\Delta^2 y_1 + \Delta^2 y_2) + \frac{3}{256}(\Delta^4 y_0 + \Delta^4 y_1). \tag{5.37}$$

But

$$\Delta^2 y_1 = y_3 - 2y_2 + y_1,\ \Delta^4 y_0 = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 \text{ etc.}$$

Substituting these values in equation (5.37), we get the required result.

5.6 USE OF INTERPOLATION FORMULAE

We know that the Newton formulae with forward and backward differences are most appropriate for calculation near the beginning and the end, respectively, of tabulation, and their use is mainly restricted to such situations.

The Gaussian forward and backward formulae terminated with an even difference are equivalent to each other and to the Stirling's formula terminated with the same difference. The Gaussian forward formula terminated with an odd difference is equivalent to the Bessel formula terminated with the same difference. The Gaussian backward formula launched from x_0 and terminating with an odd difference is equivalent to the Bessel's formula launched from x_{-1} and terminated with the same difference. Thus, in place of using a Gaussian formula, we may use an equivalent formula of either Stirling or Bessel for which the coefficients are extensively tabulated.

To interpolate near the middle of a given table, Stirling's formula gives the most accurate result for $-\frac{1}{4} \le p \le \frac{1}{4}$ and Bessel's formula is most efficient near $p = \frac{1}{2}$, say $\frac{1}{4} \le p \le \frac{3}{4}$. When the highest difference to be retained is odd, Bessel's formula is recommended and when the highest difference to be retained is even, then Stirling's formula is preferred.

In case of Stirling's formula, the term containing the third difference, viz.,

$$\frac{p(p^2-1)}{6}\delta^3 f_{-\frac{1}{2}}$$

may be neglected if its contribution to the interpolation is less than half a unit in the last place. This means that

$$\left|\frac{p(p^2-1)}{6}\delta^3 f_{-\frac{1}{2}}\right| < \frac{1}{2} \text{ for all } p \text{ in the range } 0 \le p \le 1.$$

But the maximum value of $\frac{p(p^2-1)}{6}$ is 0.064 and so we have

$$\left|0.064\delta^3 f_{-\frac{1}{2}}\right| < \frac{1}{2} \text{ or } \left|\delta^3 f_{-\frac{1}{2}}\right| < 8.$$

If we consider Bessel's formula, the contribution from the term containing the third difference will be less than half a unit in the last place, provided that

$$\left|\frac{p\left(p-\frac{1}{2}\right)(p-1)}{6}\delta^3 f_{\frac{1}{2}}\right| < \frac{1}{2}.$$

But the maximum value of $\frac{p\left(p-\frac{1}{2}\right)(p-1)}{6}$ is 0.008 and so

$$\left|\delta^3 f_{\frac{1}{2}}\right| < 60.$$

Thus, if we neglect the third differences, Bessel's formula is about eight times more accurate than Stirling's formula. If the third differences need to be retained (when they are more than 60 in magnitude), then Everett's formula may be gainfully employed since Everett's formula with second difference is equivalent to Bessel's formula with third differences.

5.7 INTERPOLATION WITH UNEQUAL-SPACED POINTS

The classical polynomial interpolating formulae discussed so far are limited to the case in which intervals of independent variables were equally spaced. We shall now discuss interpolation formulae with unequally spaced values of the argument.

(A) Divided Differences

Let $f(x_0), f(x_1),..., f(x_n)$ be the values of a function f corresponding to the arguments $x_0, x_1,..., x_n$ where the intervals $x_1 - x_0, x_2 - x_1,..., x_n - x_{n-1}$ are not necessarily equally spaced. Then the first divided differences of f for the arguments $x_0, x_1, x_2,...$ are defined by

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0},$$

$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

and so on. The second divided difference (divided difference of order 2) of f for three arguments x_0, x_1, x_2 is defined by

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

and similarly, the divided difference of order n is defined by

$$f(x_0, x_1, \ldots, x_n) = \frac{f(x_1, x_2, \ldots, x_n) - f(x_0, x_1, \ldots, x_{n-1})}{x_n - x_0}.$$

Remark 5.4. Even if the arguments are equal, the divided difference may still have a meaning. For example, if we set $x_1 = x_0 + \varepsilon$, then

$$f(x_0, x_1) = f(x_0, x_0 + \varepsilon) = \frac{f(x_0 + \varepsilon) - f(x_0)}{\varepsilon}$$

and in the limit when $\varepsilon \to 0$, we have

$$f(x_0, x_0) = f'(x_0) \text{ if } f \text{ is derivable.}$$

Similarly,

$$f(x_0, x_0, \ldots, x_0) = \frac{f^{(r)}(x_0)}{r!} \text{ for } r + 1 \text{ equal arguments } x_0.$$

Further, we observe that

$$f(x_0,x_1)=\frac{f(x_1)-f(x_0)}{x_1-x_0}=\frac{f(x_0)-f(x_1)}{x_0-x_1}=f(x_1,x_0)$$

$$f(x_0,x_1,x_2)=\frac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0},$$

$$=\frac{1}{x_2-x_0}\left[\frac{f(x_2)-f(x_1)}{x_2-x_1}-\frac{f(x_1)-f(x_0)}{x_1-x_0}\right]$$

$$=\frac{f(x_0)}{(x_0-x_1)(x_0-x_2)}+\frac{f(x_1)}{(x_1-x_0)(x_1-x_2)}+\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)}$$

and in general,

$$f(x_0,x_1,\ldots,x_n)=\frac{f(x_0)}{(x_0-x_1)\ldots(x_0-x_n)}+\frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}+$$

$$\cdots+\frac{f(x_n)}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})},$$

Hence, the divided differences are symmetrical in their arguments. It follows therefore that for any function f, the value of the divided difference remains unaltered when any of the arguments involved are interchanged. Thus, the value of the divided difference depends only on the value of the arguments involved and not on the order in which they are taken. Thus,

$$f(x_0,x_1)=f(x_1,x_0)$$

$$f(x_0,x_1,x_2)=f(x_2,x_1,x_0)=f(x_1,x_0,x_2).$$

Theorem 5.3. The nth divided differences of a polynomial of the nth degree are constant.

Proof: Consider the function $f(x)=x^n$. The first divided difference

$$f(x_r,x_{r+1})=\frac{f(x_{r+1})-f(x_r)}{x_{r+1}-x_r}=\frac{x_{r+1}^n-x_r^n}{x_{r+1}-x_r}$$

$$=x_{r+1}^{n-1}+x_r x_{r+1}^{n-2}+\ldots+x_r^{n-2}x_{r+1}\ldots+x_r^{n-1}$$

is a homogeneous polynomial of degree $n-1$ in x_r, x_{r+1}.

Similarly, it can be shown that second divided differences are homogeneous polynomials of degree $n-2$. Proceeding by mathematical induction, it can be shown that divided difference of nth order is a polynomial of degree $n-n=0$ and so is a constant.

For a polynomial of the nth degree with leading term a_0x^n, the nth divided difference of all terms except the leading term are zero. So the nth divided differences of this polynomial are constant and of value a_0.

Remark 5.5. Let the arguments be equally spaced so that $x_1-x_0=x_2-x_1=\ldots=x_n-x_{n-1}=h.$ Then

$$f(x_0,x_1)=\frac{f(x_1)-f(x_0)}{h}=\frac{\Delta f_0}{h}$$

$$f(x_0,x_1,x_2)=\frac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0}=\frac{1}{2h}\left[\frac{\Delta f_1}{h}-\frac{\Delta f_0}{h}\right]$$

$$=\frac{1}{2h^2}\Delta^2 f_0=\frac{1}{2!}\frac{1}{h^2}\Delta^2 f_0$$

and, in general,

$$f(x_0, x_1, \ldots, x_n) = \frac{1}{n!}\frac{1}{h^n}\Delta^n f_0.$$

If the tabulated function is a polynomial of nth degree, then $\Delta^n f_0$ would be constant and hence the nth divided difference would also be a constant.

5.8 NEWTON'S FUNDAMENTAL (DIVIDED DIFFERENCE) FORMULA

Let $f(x_0), f(x_1), \ldots, f(x_n)$ be the values of a function f corresponding to the arguments $x_0, x_1, \ldots, x_n$, where the intervals $x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}$ are not necessarily equally spaced. By the definition of divided differences, we have

$$f(x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

and so

$$f(x) = f(x_0) + (x - x_0) f(x, x_0). \tag{5.38}$$

Further,

$$f(x, x_0, x_1) = \frac{f(x, x_0) - f(x_0, x_1)}{x - x_1},$$

which yields

$$f(x, x_0) = f(x_0, x_1) + (x - x_1) f(x, x_0, x_1). \tag{5.39}$$

Similarly,

$$f(x, x_0, x_1) = f(x_0, x_1, x_2) + (x - x_2) f(x, x_0, x_1, x_2) \tag{5.40}$$

and in general,

$$f(x, x_0, \ldots, x_{n-1}) = f(x_0, x_1, \ldots, x_n) + (x - x_n) f(x, x_0, x_1, \ldots, x_n). \tag{5.41}$$

Multiplying equation (5.39) by $(x - x_0)$ (5.40) by $(x - x_0)\,(x - x_1)$, and so on and finally the last term (5.41) by $(x - x_0)\,(x - x_1)\ldots(x - x_{n-1})$ and adding, we obtain

$$\begin{aligned} f(x) = f(x_0) &+ (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) \\ &+ \ldots + (x - x_0)(x - x_1)\ldots(x - x_{n-1}) f(x_0, x_1, \ldots, x_n) + R \end{aligned}$$

where

$$R = (x - x_0)(x - x_1)\ldots(x - x_n) f(x, x_0, \ldots, x_n).$$

This formula is called Newton's divided difference formula. The last term R is the remainder term after $(n + 1)$ terms.

Remark 5.6. If we consider the case of equal spacing, then we have

$$f(x_0, x_1, \ldots, x_n) = \frac{1}{h^n n!}\Delta^n f_0$$

and so

$$
\begin{aligned}
f(x) &= f(x_0) + \frac{x - x_0}{h}\Delta f_0 + \frac{(x - x_0)(x - x_1)}{h^2 2!}\Delta^2 f_0 + \ldots \\
&= f_0 + \frac{x_0 + ph - x_0}{h}\Delta f_0 + \frac{(x_0 + ph - x_0)(x_0 + ph - x_1)}{h^2 2!}\Delta^2 f_0 + \ldots \\
&= f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \ldots,
\end{aligned}
$$

which is nothing but Newton’s forward difference formula.

EXAMPLE 5.25

Find a polynomial satisfied by (−4, 1245), (−1, 33), (0, 5), (2, 9), and (5, 1335).

Solution. The divided difference table based on the given nodes is shown below:

x	y				
−4	1245				
		− 404			
−1	33		94		
		−28		−14	
0	5		10		3
		2		13	
2	9		88		
		442			
5	1335				

In fact,

$$f(x_0, x_1) = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = \frac{1245 - 33}{-3} = -404,$$

$$f(x_1, x_2) = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = \frac{28}{-1} = -28,$$

$$f(x_2, x_3) = \frac{f(x_2) - f(x_3)}{x_2 - x_3} = \frac{5 - 9}{-2} = 2,$$

$$f(x_3, x_4) = \frac{f(x_3) - f(x_4)}{x_3 - x_4} = \frac{9 - 1335}{-3} = 442,$$

$$f(x_0, x_1, x_2) = \frac{f(x_0, x_1) - f(x_1, x_2)}{x_0 - x_2} = \frac{-404 + 28}{-4} = 94,$$

$$f(x_1, x_2, x_3) = \frac{f(x_1, x_2) - f(x_2, x_3)}{x_1 - x_3} = \frac{-28 - 2}{-3} = 10,$$

$$f(x_2, x_3, x_4) = \frac{f(x_2, x_3) - f(x_3, x_4)}{x_2 - x_4} = \frac{2 - 442}{-5} = 88,$$

$$f(x_0,x_1,x_2,x_3)=\frac{f(x_0,x_1,x_2)-f(x_1,x_2,x_3)}{x_0-x_3}=\frac{94-10}{-6}=-14,$$

$$f(x_1,x_2,x_3,x_4)=\frac{f(x_1,x_2,x_3)-f(x_2,x_3,x_4)}{x_1-x_4}=\frac{10-88}{-6}=13,$$

$$f(x_0,x_1,x_2,x_3,x_4)=\frac{f(x_0,x_1,x_2,x_3)-f(x_1,x_2,x_3,x_4)}{x_0-x_4}=\frac{-14-13}{-9}=3.$$

Putting these values in Newton's fundamental formula, we have

$$\begin{aligned}f(x)&=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)\\&\quad+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)\\&\quad+(x-x_0)(x-x_1)(x-x_2)(x-x_3)f(x_0,x_1,x_2,x_3,x_4)\\&=1245-404(x+4)+94(x+4)(x+1)-14(x+4)(x+1)x\\&\quad+3(x+4)(x+1)x(x-2)\\&=3x^4-5x^3+6x^2-14x+5.\end{aligned}$$

EXAMPLE 5.26

Using the table given below, find $f(x)$ as a polynomial in x.

x	−1	0	3	6	7
$f(x)$	3	−6	39	822	1611

Solution. The divided difference table for the given data is shown below

	x	$f(x)$				
x_0	−1	3				
			−9			
x_1	0	−6		6		
			15		5	
x_2	3	39		41		1
			261		13	
x_3	6	822		132		
			789			
x_4	7	1611				

Putting these values in the Newton's divided difference formula, we have

$$\begin{aligned}f(x)&=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)\\&\quad+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)\\&\quad+(x-x_0)(x-x_1)(x-x_2)(x-x_3)f(x_0,x_1,x_2,x_3,x_4)\\&=3-9(x+1)+6(x+1)x+5(x+1)x(x-3)\\&\quad+1(x+1)x(x-3)(x-6)=x^4-3x^3+5x^2-6.\end{aligned}$$

EXAMPLE 5.27

By means of Newton's divided difference formula, find the value of $f(8)$ and $f(15)$ from the following table:

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

Solution. The divided difference table is

	x	$f(x)$					
x_0	4	48					
			52				
x_1	5	100		15			
			97		1		
x_2	7	294		21		0	
			202		1		0
x_3	10	900		27		0	
			310		1		
x_4	11	1210		33			
			409				
x_5	13	2028					

Using the formula

$$f(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2)$$
$$+(x - x_0)(x - x_1)(x - x_2)f(x_0, x_1, x_2, x_3),$$

we obtain

$$f(8) = 48 + (8-4)(52) + (8-4)(8-5)15 + (8-4)(8-5)(8-7)(1) = 448 \text{ and}$$

$$f(15) = 48 + (15-4)(52) + (15-4)(15-5)(15) + (15-4)(15-5)(15-7)(1) = 3150.$$

5.9 ERROR FORMULAE

Let $f(x)$ be approximated by a polynomial $p(x)$ of degree n by Newton's divided difference formula. Then $f(x)$ and $p(x)$coincide at $(n + 1)$ distinct point $x_0, x_1, \ldots, x_n$ and the error $E(x) = f(x) - p(x)$ is given by

$$E(x) = \prod(x) f(x, x_0, x_1, \ldots, x_n), \tag{5.42}$$

where

$$\prod(x) = (x - x_0)(x - x_1)\ldots(x - x_n) \tag{5.43}$$

is a polynomial of degree $n + 1$.

Assume that f possesses $n + 1$ continuous derivatives in the relevant interval. Consider a linear combination of $f(x)$, $p(x)$, and $\prod(x)$ as

$$F(x) = f(x) - p(x) - K\prod(x), \tag{5.44}$$

where K is a constant to be determined in such a way that $F(x)$ vanishes not only at the $n + 1$ points but also at an arbitrarily chosen point X which differs from all these points.

Let $\bar{I}$ constitute the closed interval limited by the smallest and largest of $n+2$ values $x_0, x_1, \ldots, x_n, X$. Then F vanishes at least $n + 2$ times in the closed interval $\bar{I}$. By Rolle's Theorem $F'(x)$ vanishes at least $n + 1$ times in $\bar{I}$, $F''(x)$ at least n times, and finally $F^{(n+1)}(x)$ vanishes at least once inside $\bar{I}$. Let ξ be the one such point. It follows from equation (5.44) that

$$0 = f^{(n+1)}(\xi) - p^{(n+1)}(\xi) - K\prod^{(n+1)}(\xi). \tag{5.45}$$

But since $p(x)$ is a polynomial of degree n, its $(n + 1)$th derivative vanishes identically. Also, by equation (5.43), we have $\prod^{(n+1)}(x) = (n+1)!$. Hence, equation (5.45) yields $K = \frac{1}{(n+1)!} f^{(n+1)}(\xi)$, and relation $F(x) = 0$ becomes

$$f(X) - p(X) = \frac{\prod(x) f^{(n+1)}(\xi)}{(n+1)!}, \quad \xi \in \bar{I}.$$

Even if X is taken any of the arguments $x_0, x_1, \ldots, x_n$, both sides of this relation vanishes. Since X is arbitrary, we have

$$E(x) = f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod(x) \tag{5.46}$$

for some $\xi \in I$, where ξ is in the interval limited by the largest and smallest of the numbers $x_0, x_1, \ldots, x_n, x$. Since equations (5.42) and (5.46) must be equivalent, we have

$$f(x, x_0, x_1, \ldots, x_n) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)$$

for some argument in the interval I.

EXAMPLE 5.28

Find the maximum error in interpolating to find sin x for any x within the range of the table given below:

x	0°	15°	30°	45°	60°	75°	90°
sin$x$	0	0.25882	0.5	0.70711	0.86603	0.96593	1.0

Solution. We have

$$f(x) = \sin x \text{ and } n + 1 = 7.$$

Then $f^{(7)}(x) = -\cos x$. The formula for error will not yield the maximum error because we know nothing about ξ except that it lies in the range $0° - 90°$. But since cos x is bounded in that interval, the formula will give us an upper bound on the size of the error. Thus,

$$\left| f(x) - p(x) \right| \le \frac{1}{7!} \left| (x - 0)\left(x - \frac{\pi}{12}\right)\left(x - \frac{2\pi}{12}\right) \ldots \left(x - \frac{6\pi}{12}\right) \right|.$$

For example, if we compute the value of $\sin \frac{5\pi}{24}$, then

$$\left| f\left(\frac{5\pi}{24}\right) - p\left(\frac{5\pi}{24}\right) \right| \le \frac{1}{5040}(5)(3)(1)(1)(3)(5)(7)\frac{\pi^7}{(24)^7} = (2.06)10^{-7}.$$

EXAMPLE 5.29

The function $y = f(x)$ is supposed to be differentiable three times. Show that

$$f(x) = -\frac{(x-x_1)(x-2x_0+x_1)}{(x_1-x_0)^2} f(x_0) + \frac{(x-x_0)(x-x_1)}{x_0-x_1} f'(x_0) + \frac{(x-x_0)^2}{(x_1-x_0)^2} f(x_1) + R(x),$$

where

$$R(x) = \frac{1}{6}(x-x_0)^2 (x-x_1) f'''(\xi), x_0 < x, \quad \xi < x_1.$$

Solution. We apply Newton's divided difference formula to three points x_0, x_0, x_1 and have

$$f(x) = f(x_0) + (x-x_0) f(x_0,x_0) + (x-x_0)(x-x_0) f(x_0,x_0,x_1) + R(x),$$

where

$$R(x) = \frac{f'''(\xi)}{3!} \prod(x) = \frac{f'''(\xi)}{6}(x-x_0)(x-x_0)(x-x_1)$$
$$= \frac{f'''(\xi)}{6}(x-x_0)^2 (x-x_1).$$

But $f(x_0,x_0) = f'(x_0)$. Therefore,

$$f(x) = f(x_0) + (x-x_0) f'(x_0) + (x-x_0)^2 \frac{f(x_0,x_0) - f(x_0,x_1)}{x_0-x_1} + R(x)$$
$$= f(x_0) + (x-x_0) f'(x_0) + (x-x_0)^2 \frac{f'(x_0) - [f(x_0) - f(x_1)]\big/(x_0-x_1)}{x_0-x_1} + R(x)$$
$$= f(x_0) + (x-x_0) f'(x_0) + \frac{(x-x_0)^2 f'(x_0)}{x_0-x_1} - \frac{(x-x_0)^2 f(x_0)}{(x_0-x_1)^2} + \frac{(x-x_0)^2 f(x_1)}{(x_0-x_1)^2} + R(x)$$
$$= -f(x_0)\left[\frac{(x-x_0)^2}{(x_0-x_1)^2} - 1\right] + f'(x_0)\left[\frac{(x-x_0)(x_0-x_1+x-x_0)}{x_0-x_1}\right] + \frac{(x-x_0)^2}{(x_0-x_1)^2} f(x_1) + R(x)$$
$$= -\frac{(x-x_0)(x-2x_0+x_1)}{(x_1-x_0)^2} f(x_0) + \frac{(x-x_0)(x-x_1)}{x_0-x_1} f'(x_0) + \frac{(x-x_0)^2}{(x_0-x_1)^2} f(x_1) + R(x).$$

EXAMPLE 5.30

Find the missing term in the following table:

x	0	1	2	3	4
y	1	3	9	–	81

Explain, why the result differs from $3^3 = 27$.

Solution. The divided difference table is

x	y				
x_0	0	1			
			2		
x_1	1	3		2	
			6		2
x_2	2	9		10	
			36		
x_3	4	81			

Therefore, using Newton's divided difference formula, we have

$$\begin{aligned} f(3) &= f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3) \\ &= 1+(3-0)(2)+(3-0)(3-1)(2)+(3-0)(3-1)(3-2)(2) \\ &= 1+6+12+12=31. \end{aligned}$$

It differs from $3^3 = 27$ because of the error $E(x)=\dfrac{f^{(n+1)}(\xi)}{(n+1)!}\prod(x)$.

Remark 5.7. Using Newton's divided difference formula, the polynomial satisfying the given data in the above example is

$$\begin{aligned} f(x) &= 1+2x+2x(x-1)+2x(x-1)(x-2) \\ &= 2x^3-4x^2+4x+1. \end{aligned}$$

5.10 LAGRANGE'S INTERPOLATION FORMULA

Let f be continuous and differentiable $(n+1)$ times in an interval (a,b) and let $f_0, f_1, f_2, \ldots, f_n$ be the values of f at $x_0, x_1, x_2, \ldots, x_n$ where $x_0, x_1, x_2, \ldots, x_n$ are not necessarily equally spaced. We wish to find a polynomial of degree n, say $P_n(x)$ such that

$$P_n(x_i) = f(x_i) = f_i,\ i = 0,1,\ldots,n. \tag{5.47}$$

Let

$$P_n(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n, \tag{5.48}$$

be the desired polynomial. Substituting the condition (5.47) in equation (5.48), we obtain the following system of equations:

$$\left.\begin{aligned} f_0 &= a_0 + a_1x_0 + a_2x_0^2 + \ldots + a_nx_0^n \\ f_1 &= a_0 + a_1x_1 + a_2x_1^2 + \ldots + \boldsymbol{a}_nx_1^n \\ f_2 &= a_0 + a_1x_2 + a_2x_2^2 + \ldots + a_nx_2^n \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{5.49}$$

$$f_n = a_0 + a_1x_n + a_2x_n^2 + \ldots + a_nx_n^n$$

This set of equations will have a solution if the determinant

$$\begin{vmatrix} 1 & x_0 & x_0^2 \ldots\ldots\ldots\ldots x_0^n \\ 1 & x_1 & x_1^2 \ldots\ldots\ldots\ldots x_1^n \\ 1 & x_2 & x_2^2 \ldots\ldots\ldots\ldots x_2^n \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots \\ 1 & x_n & x_n^2 \ldots\ldots\ldots\ldots x_n^n \end{vmatrix} \neq 0.$$

The value of this determinant, called Vandermonde's determinant, is $(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)(x_1 - x_2)$ $(x_1 - x_3)\ldots(x_1 - x_n)\ldots(x_{n-1} - x_n)$. Eliminating $a_0, a_1, \ldots, a_n$ from equations (5.48) and (5.49), we obtain

$$\begin{vmatrix} P_n(x) & 1 & x & x^2 \ldots\ldots\ldots\ldots x^n \\ f_0 & 1 & x_0 & x_0^2 \ldots\ldots\ldots\ldots x_0^n \\ f_1 & 1 & x_1 & x_1^2 \ldots\ldots\ldots\ldots x_1^n \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots\ldots \\ f_n & 1 & x_n & x_n^2 \ldots\ldots\ldots\ldots x_n^n \end{vmatrix} = 0, \tag{5.50}$$

which shows that $P_n(x)$ is a linear combination of $f_0, f_1, \ldots, f_n$. Hence, we write

$$P_n(x) = \sum_{i=0}^{n} L_i(x) f_i, \tag{5.51}$$

where $L_i(x)$ are polynomials in x of degree n. But $P_n(x_j) = f_j$ for $j = 0, 1, 2, \ldots, n$. Therefore, equation (5.51) yields

$$\left.\begin{aligned} L_i(x_j) &= 0 \text{ for } i \neq j \\ L_i(x_j) &= 1 \text{ for } i = j \end{aligned}\right\} \text{ for all } j. \tag{5.52}$$

Hence, we may take $L_i(x)$ as

$$L_i(x) = \frac{(x - x_0)(x - x_1)\ldots(x - x_{i-1})(x - x_{i+1})\ldots(x - x_n)}{(x_i - x_0)(x_i - x_1)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_n)} \tag{5.53}$$

which clearly satisfies the condition (5.52). Let

$$\Pi(x) = (x - x_0)(x - x_1)\ldots(x - x_{i-1})(x - x_i)(x - x_{i+1})\ldots(x - x_n). \tag{5.54}$$

Then

$$\Pi'(x_i) = \left[\frac{d}{dx}\Pi(x)\right]_{x=x_i} = (x_i - x_0)(x_i - x_1)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_n)$$

and so equation (5.53) becomes

$$L_i(x) = \frac{\Pi(x)}{(x - x_i)\Pi'(x_i)}.$$

Hence, equation (5.51) becomes

$$P_n(x) = \sum_{i=0}^{n} \frac{\Pi(x)}{(x - x_i)\Pi'(x_i)} f_i, \tag{5.55}$$

which is called Lagrange's interpolation formula. The coefficients $L_i(x)$ defined in equation (5.53) are called Lagrange's interpolation coefficients.

Interchanging x and y in equation (5.55), we get the formula

$$P_n(y) = \sum_{i=0}^{n} \frac{\Pi(y)}{(y - y_i)\Pi'(y_i)} x_i, \tag{5.56}$$

which is useful for inverse interpolation.

Second Method: Let $f(x_0), f(x_1), \ldots, f(x_n)$ be the values of the function f corresponding to the arguments $x_0, x_1, \ldots, x_n$, not necessarily equally spaced. We wish to find a polynomial $P_n(x)$ in x of degree n such that

$$P_n(x_0) = f(x_0), P_n(x_1) = f(x_1), \ldots, P_n(x_n) = f(x_n).$$

Suppose that

$$\begin{aligned} P_n(x) = {} & A_0(x - x_1)(x - x_2)\ldots(x - x_n) + A_1(x - x_0)(x - x_2)\ldots(x - x_n) \\ & + A_2(x - x_0)(x - x_1)(x - x_3)\ldots(x - x_n) + \ldots \\ & + A_n(x - x_0)(x - x_1)\ldots(x - x_{n-1}). \end{aligned} \tag{5.57}$$

where $A_1, A_2, \ldots, A_n$ are the constants to be determined.

To determine A_0, we put $x = x_0$ and $P_n(x_0) = f(x_0)$ and have

$$f(x_0) = A_0(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)$$

and so

$$A_0 = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)}.$$

Similarly, putting $x = x_1, x_2, \ldots, x_n$. we get

$$A_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)\ldots(x_1 - x_n)}$$

$$A_2 = \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)\ldots(x_2 - x_n)}$$

....

....

$$A_n = \frac{f(x_n)}{(x_1 - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})}.$$

Substituting these values in equation (5.57), we get

$$P_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i),$$

with

$$L_i(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}{(x_i-x_0)(x_i-x_1)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)},$$

which is Lagrange's interpolation formula.
Clearly,

$$L_i(x_j) = 0 \text{ for } i \neq j, \text{ and } L_i(x_j) = 1 \text{ for } i = j.$$

Remark 5.8. If *f* takes same value, say *k*, at each of the points $x_0, x_1, \dots, x_n$, we have

$$P_n(x) = \sum_{i=0}^{n} L_i(x)k = k\sum_{i=0}^{n} L_i(x).$$

This yields

$$\sum_{i=0}^{n} L_i(x) = 1,$$

which is an important check during calculations.

Further, dividing both sides of Lagrange's interpolation formula by $(x-x_0)(x-x_1)\dots(x-x_n)$, we obtain

$$\begin{aligned}\frac{P_n(x)}{(x-x_0)(x-x_1)\dots(x-x_n)} &= \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)}\cdot\frac{1}{x-x_0} \\ &\quad + \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)}\cdot\frac{1}{x-x_1} \\ &\quad + \dots + \frac{f(x_n)}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})}\frac{1}{x-x_n}.\end{aligned}$$

Thus, $\dfrac{P_n(x)}{(x-x_0)(x-x_1)\dots(x-x_n)}$ has been expressed as the sum of partial fractions.

EXAMPLE 5.31

Use Lagrange's formula to express the function $\dfrac{x^2+6x-1}{(x-1)(x+1)(x-4)(x-6)}$ as a sum of partial fractions.

Solution. We have

$$P_n(x) = x^2 + 6x - 1,$$

and so

$$P_n(x_0) = f(x_0) = f(1) = 6$$

$$P_n(x_1) = f(x_1) = f(-1) = -6$$

$$P_n(x_2) = f(x_2) = f(4) = 39$$

$$P_n(x_3) = f(x_3) = f(6) = 71.$$

Therefore,

$$\frac{x^2+6x-1}{(x-1)(x+1)(x-4)(x-6)} = \frac{6}{(x-1)(2)(-3)(-5)} + \frac{-6}{(x+1)(-2)(-5)(-7)} + \frac{39}{(x-4)(3)(5)(-2)} + \frac{71}{(x-6)(5)(7)(2)}$$

$$= \frac{1}{5(x-1)} + \frac{3}{35(x+1)} - \frac{13}{10(x-4)} + \frac{71}{70(x-6)}.$$

EXAMPLE 5.32

Use Lagrange's interpolation formula to express the function

$$\frac{x^2+x-3}{x^3-2x^2-x+2}$$

as sum of partial functions.

Solution. We have

$$\frac{x^2+x-3}{x^3-2x^2-x+2} = \frac{x^2+x-3}{(x-1)(x+1)(x-2)}.$$

Let

$$P_n(x) = x^2+x-3,$$

and let $x_0 = 1, x_1 = -1, x_2 = 2$. Then

$$P_n(x_0) = f(x_0) = f(1) = -1$$
$$P_n(x_1) = f(x_1) = f(-1) = -3$$
$$P_n(x_2) = f(x_2) = f(2) = 3.$$

Therefore,

$$\frac{x^2+x-3}{(x-1)(x+1)(x-2)} = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)}\frac{1}{(x-x_0)} + \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)}\frac{1}{(x-x_1)} + \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)}\frac{1}{(x-x_2)}$$

$$= \frac{-1}{(x-1)(2)(-1)} + \frac{-3}{(x+1)(-2)(-3)} + \frac{3}{(x-2)(1)(3)}$$

$$= \frac{1}{2(x-1)} - \frac{1}{2(x+1)} + \frac{1}{(x-2)}.$$

EXAMPLE 5.33

Using Lagrange's interpolation formula, prove that

$$32f(1) = -3f(-4) + 10f(-2) + 30f(2) - 5f(4).$$

Solution. We have

$$x_0 = -4, x_1 = -2, x_2 = 2, x_3 = 4 \text{ and } x = 1.$$

Then

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(1+2)(1-2)(1-4)}{(-4+2)(-4-2)(-4-4)} = -\frac{3}{32}.$$

Similarly

$$L_1(x) = \frac{5}{16}, L_2(x) = \frac{15}{16}, \ L_3(x) = -\frac{5}{32}.$$

We observe that $\sum_{i=0}^{3} L_i(x) = 1$. Therefore,

$$f(x) = \sum_{i=0}^{3} L_i(x) f(x_i)$$

or

$$f(1) = \frac{5}{16} f_1 + \frac{15}{16} f_2 - \frac{5}{32} f_3 - \frac{3}{32} f_0$$

or

$$32f(1) = -3f(-4) + 10f(-2) + 30f(2) - 5f(4).$$

EXAMPLE 5.34

The function $y = f(x)$ is given in the points (7,3), (8,1), (9,1), and (10,9). Find the value of y for $x = 9.5$ using Lagrange's interpolation formula.

Solution. We have

	x	$y = f(x)$
x_0	7	3
x_1	8	1
x_2	9	1
x_3	10	9

By Lagrange's formula, we have

$$f(x) \approx P_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i),$$

where

$$L_i(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}.$$

In the present problem, $x = 9.5$ and we have

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(9.5-8)(9.5-9)(9.5-10)}{(7-8)(7-9)(7-10)}$$

$$= \frac{0.375}{6} = 0.06250,$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(9.5-7)(9.5-9)(9.5-10)}{(8-7)(8-9)(8-10)}$$

$$= -\frac{0.625}{2} = -0.3125,$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(9.5-7)(9.5-8)(9.5-10)}{(9-7)(9-8)(9-10)}$$

$$= \frac{1.875}{2} = 0.9375,$$

$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(9.5-7)(9.5-8)(9.5-9)}{(10-7)(10-8)(10-9)}$$

$$= \frac{1.875}{6} = 0.3125.$$

We observe that $L_0(x)+L_1(x)+L_2(x)+L_3(x)=1$ and therefore, so far, our calculations are correct. Hence,

$$P(x) = P(9.5) = \sum_{i=0}^{3} L_i(x) f(x_i)$$

$$= L_0 f_0 + L_1 f_1 + L_2 f_2 + L_3 f_3$$

$$= (0.06250)(3) - 0.3125(1) + 0.9395(1) + 0.3125(9)$$

$$= 0.1875 - 0.3125 + 0.9375 + 2.8125 = 3.625.$$

EXAMPLE 5.35

Find the interpolating polynomial for (0, 2), (1, 3), (2, 12), and (5, 147).

Solution. The given data is

x	0	1	2	5
$f(x)$	2	3	12	147

The Lagrange's formula reads

$$P_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i),$$

where

$$L_i(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)}.$$

Thus,

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}$$

$$= -\frac{1}{10}(x^3 - 8x^2 + 17x - 10)$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}$$

$$= \frac{1}{4}(x^3 - 7x^2 + 10x)$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}$$

$$= -\frac{1}{6}(x^3 - 6x^2 + 5x)$$

$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}$$

$$= \frac{1}{60}(x^3 - 3x^2 + 2x).$$

Putting these values in Lagrange's formula, we have

$$P(x) = \sum_{i=0}^{3} L_i(x) f(x_i) = -\frac{2}{10}(x^3 - 8x^2 + 17x - 10) + \frac{3}{4}(x^3 - 7x^2 + 10x)$$

$$-\frac{12}{6}(x^3 - 6x^2 + 5x) + \frac{147}{60}(x^3 - 3x^2 + 2x) = x^3 + x^2 - x + 2.$$

EXAMPLE 5.36

Use Lagrange's interpolation formula to find the value of y when $x = 5$, if the following values of x and y are given:

x	1	2	3	4	7
y	2	4	8	16	128

Solution. Let $y_i = f(x_i)$ be the value of a function at x_i, $0 \le i \le n$. Then Lagrange's interpolating polynomial $P_n(x)$ is given by

$$P_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i),$$

where

$$L_i(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}.$$

In the given problem, $x = 5$ and we have

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} = \frac{(5-2)(5-3)(5-4)(5-7)}{(1-2)(1-3)(1-4)(1-7)}$$
$$= -\frac{1}{3}$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} = \frac{(5-1)(5-3)(5-4)(5-7)}{(2-1)(2-3)(2-4)(2-7)}$$
$$= \frac{8}{5}$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} = \frac{(5-1)(5-2)(5-4)(5-7)}{(3-1)(3-2)(3-4)(3-7)}$$
$$= -3$$

$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} = \frac{(5-1)(5-2)(5-3)(5-7)}{(4-1)(4-2)(4-3)(4-7)}$$
$$= \frac{8}{3}$$

$$L_4(x) = \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} = \frac{(5-1)(5-2)(5-3)(5-4)}{(7-1)(7-2)(7-3)(7-4)}$$
$$= \frac{1}{15}.$$

We note that $\sum_{i=0}^{n} L_i(x) = 1$. Hence, our calculations are correct up to this stage. By Lagrange's formula

$$P(x) = P(5) = \sum_{i=0}^{n} L_i(x) f(x_i)$$
$$= -\frac{1}{3}(2) + \frac{8}{5}(4) + (-3)(8) + \frac{8}{3}(16) + \frac{1}{15}(128)$$
$$= -\frac{2}{3} + \frac{32}{5} - 24 + \frac{128}{3} + \frac{128}{15} \approx 32.933.$$

5.11 ERROR IN LAGRANGE'S INTERPOLATION FORMULA

The error in this case is the difference between $f(x)$ and the Lagrange's polynomial $P_n(x)$ at a given point. Let $f(x) = P_n(x)$ at the $n + 1$ points $x_0, x_1, \ldots, x_n$. Suppose that the point X lies in the closed interval I bounded by the extreme points of $(x_0, x_1, \ldots, x_n)$ and further that $X \neq x_k$, $k = 0, 1, \ldots, n$. Also, we assume that f can be differentiated $n+1$ times. We define the function

$$F(x) = f(x) - P_n(x) - R\prod(x),$$

where $\prod(x) = (x - x_0)(x - x_1)...(x - x_n)$ and R is a constant to be determined such that $F(X) = 0$. Obviously, $F(x) = 0$ for $x = x_0, x_1, \ldots, x_n$. Using Rolle's Theorem repeatedly, we conclude that $F^{(n+1)}(\xi) = 0$, where $\xi \in I$. Since $P_n(x)$ is of degree n, we have

$$F^{(n+1)}(\xi) = f^{(n+1)}(\xi) - R(n+1)!$$

and so $F^{(n+1)}(\xi) = 0$ implies $R = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}$. Thus, $F(X) = 0$ implies

$$f(X) - P_n(X) = R\prod(X) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod(X).$$

Replacing X by x, we get

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod(x)$$

or

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod(x)$$

$$= \sum_{i=0}^{n} L_i(x) f(x_i) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\ldots(x - x_n).$$

5.12 INVERSE INTERPOLATION

Inverse interpolation is the process of finding the value of the argument to a given value of the function when the latter is intermediate between two tabulated values.

(A) Inverse Interpolation Using Newton's Forward Difference Formula

Let $\ldots, f_{-3}, f_{-2}, f_{-1}, f_0, f_1, f_2, f_3, \ldots$ be the functional values of a function f at $\ldots, x_{-3}, x_{-2}, x_{-1}, x_0, x_1, x_2, x_3, \ldots$. Then Newton's forward difference formula reads as

$$f_p = f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 f_0 + \ldots \tag{5.58}$$

We want to find value of x between x_0 and $x_1 = x_0 + h$ such that $f(x) = f_p$, where f_p is a given value.

As before, we denote $\dfrac{x - x_0}{h} = p$, that is, $x = x_0 + ph$. Thus, our aim is to find p to get the value of x. From equation (5.58), we have

$$p = \frac{1}{\Delta f_0}\left[f_p - f_0 - \frac{p(p-1)}{2!}\Delta^2 f_0 - \frac{p(p-1)(p-2)}{3!}\Delta^3 f_0 - \ldots\right].$$

We use iteration technique to find p. So, we first neglect the second and higher differences and find first approximation to p as

$$p_1 = \frac{1}{\Delta f_0}(f_p - f_0).$$

Now the approximate value p_1 of p is inserted in the second difference term to get

$$p_2 = \frac{1}{\Delta f_0}\left[f_p - f_0 - \frac{p_1(p_1-1)}{2!}\Delta^2 f_0 \right].$$

Next retaining the term with third difference, we get

$$p_3 = \frac{1}{\Delta f_0}\left[f_p - f_0 - \frac{p_2(p_2-1)}{2!}\Delta^2 f_0 - \frac{p_2(p_2-1)(p_2-2)}{3!}\Delta^3 f_0 \right].$$

The process is carried out till two successive approximations of p agree with each other up to desired accuracy. Then

$$x = x_0 + p_n h.$$

EXAMPLE 5.37

Find the value of x for $f(x) = 10$ using the following table:

x	2	3	4	5
$f(x)$	8	27	64	125

Solution. The difference table for the given data is

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
2	8			
		19		
3	27		18	
		37		6
4	64		24	
		61		
5	125			

We are given that $f_p = 10,\ h = 1,\ f_0 = 8,\ x_0 = 2,\ \frac{x - x_0}{h} = p.$ Using Newton's forward differences, the first approximation to p is

$$p_1 = \frac{1}{\Delta f_0}[f_p - f_0] = \frac{1}{19}(10-8) = 0.1.$$

The second approximation to p is

$$p_2 = \frac{1}{\Delta f_0}\left[f_p - f_0 - \frac{p_1(p_1-1)}{2!}\Delta^2 f_0 \right]$$

$$= \frac{1}{19}\left[10 - 8 - \frac{0.1(0.1-1)}{2}(18) \right] = 0.15$$

The third approximation to p is

$$p_3 = \frac{1}{\Delta f_0}\left[f_p - f_0 - \frac{p_2(p_2-1)}{2!}\Delta^2 f_0 - \frac{p_2(p_2-1)(p_2-2)}{3!}\Delta^3 f_0 \right]$$

$$= \frac{1}{19}\left[10 - 8 - \frac{0.15(0.15-1)}{2}(18) - \frac{0.15(0.15-1)(0.15-2)}{6}(6) \right]$$

$$= 0.1532.$$

The fourth approximation (using the same available differences but replacing p_2 by p_3) is

$$p_4 = \frac{1}{\Delta f_0}\left[f_p - f_0 - \frac{p_3(p_3-1)}{2}\Delta^2 f_0 - \frac{p_3(p_3-1)(p_3-2)}{6}\Delta^3 f_0 \right]$$

$$= \frac{1}{4}\left[10 - 8 - \frac{0.1532(0.1532)}{2}(18) - \frac{0.1532(0.1532-1)(0.1532-2)}{6}(6) \right]$$

$$= 0.1541.$$

The next approximation is

$$p_5 = 0.1542.$$

Thus, $p = 0.154$ correct to three decimal places.
Hence,

$$x = x_0 + ph = 2 + 0.154(1) = 2.154.$$

(B) Inverse Interpolation Using Everett's Formula

Let a function f be tabulated with $\delta^2 f$ and $\delta^4 f$. We want to find a value x between x_0 and $x_1 = x_0 + h$ such that $f(x) = f_p$ where f_p is a given value. If $p = \frac{x - x_0}{h}$, then by Everett's formula, we have

$$f_p = pf_1 + \binom{p+1}{3}\delta^2 f_1 + \binom{p+2}{5}\delta^4 f_1$$
$$+ (1-p)f_0 + \binom{2-p}{3}\delta^2 f_0 + \binom{3-p}{5}\delta^4 f_0.$$

To determine the first approximation to p, we have

$$p_1 f_1 + (1 - p_1) f_0 = f_p.$$

This approximated value is inserted into $\delta^2 f$ terms and we have second approximation p_2 given by

$$p_2 f_1 + (1 - p_2) f_0 = f_p - \binom{p_1+1}{3}\delta^2 f_1 - \binom{2-p_1}{3}\delta^2 f_0.$$

Next, we obtain p_3 from

$$p_3 f_1 + (1 - p_3) f_0 = f_p - \binom{p_2+1}{3}\delta^2 f_1 - \binom{2-p_1}{3}\delta^2 f_0$$
$$- \binom{p_2+2}{5}\delta^4 f_1 - \binom{3-p_2}{5}\delta^4 f_0.$$

If necessary, the process is repeated until we get value to the required accuracy.

EXAMPLE 5.38

The function $y = \log(x\,!)$ has a minimum between 0 and 1. Find the abscissa from the data below:

x	$\frac{d}{dx}\log(x!)$	δ^2	δ^4
0.46	−0.0015805620	−0.0000888096	−0.0000000396
0.47	0.0080664890	−0.0000872716	−0.0000000383

Solution. The problem is clearly of inverse interpolation. We are provided with even differences and therefore Everett's formula is to be used. The relevant terms in the Everett's formula are

$$f_p = (1-p)f_0 + \binom{2-p}{3}\delta^2 f_0 + \binom{3-p}{5}\delta^4 f_0$$
$$+pf_1 + \binom{p+1}{3}\delta^2 f_1 + \binom{p+2}{5}\delta^4 f_1.$$

For minimum, $\frac{d}{dx}\log(x!) = 0$. We choose $x_0 = 0.46, x_1 = 0.47, f_0 = -0.0015805620$, and $f_1 = 0.0080664890$. Therefore,

$$0 = (1-p)f_0 + pf_1 + \binom{2-p}{3}\delta^2 f_0 + \binom{p+1}{3}\delta^2 f_1$$
$$+\binom{3-p}{5}\delta^4 f_0 + \binom{p+2}{5}\delta^4 f_1.$$

First we determine a value p_1 from the equation

$$p_1 f_1 + (1-p_1)f_0 = 0$$

and get

$$p_1 = -\frac{f_0}{f_1 - f_0} = \frac{0.0015805620}{0.0096470510}$$
$$= 0.16383887677.$$

This value is inserted in the $\delta^2 f$ terms while $\delta^4 f$ terms are neglected. Then we obtain a value p_2 from

$$p_2 f_1 + (1-p_2)f_0 = -\binom{p_1+1}{3}\delta^2 f_1 - \binom{2-p_1}{3}\delta^2 f_0$$

and so

$$p_2 = \frac{1}{0.0096470510}\left[0.0015805620 - \frac{1}{6}(p_1^3 - p_1)\delta^2 f_0 + \frac{1}{6}(-p_1^3 + 3p_1^2 - 2p_1)\delta^2 f_0\right]$$
$$= 0.163219205537.$$

Next inserting the value of p_2 in $\Delta^4 f$ terms, we obtain $p_3 = 0.16321441$. The value is correct to five decimal places. We have $h = 0.01$. Hence,

$$x = x_0 + ph = 0.46 + 0.00163321 = 0.46163321.$$

(C) Inverse Interpolation Using Lagrange's Interpolation Formula

While deriving Lagrange's formula, we observed that it is a relation between two variables either of which may be taken as independent variables. Therefore, interchanging f and x in the Lagrange's formula, we have

$$x = \sum_{i=0}^{n} L_i(f)x_i,$$

where

$$L_i(f) = \frac{(f-f_0)(f-f_1)\ldots(f-f_{i-1})(f-f_{i+1})\ldots(f-f_n)}{(f_i-f_0)(f_i-f_1)\ldots(f_i-f_{i-1})(f_i-f_{i+1})\ldots(f_i-f_n)}.$$

EXAMPLE 5.39

Apply Lagrange's formula inversely to obtain the root of the equation $f(x) = 0$ given that

$$f(30) = -30, f(34) = -13, f(38) = 3 \text{ and } f(42) = 180.$$

Solution. Since $f(34) = -13$ and $f(38) = 3$, the root lies between 34 and 38. We have

	(D) x	(E) $f(x)$
x_0	30	−30
x_1	34	−13
x_2	38	3
x_3	42	18

In the present case $f(x) = 0$. Therefore,

$$L_0(f) = \frac{(f-f_1)(f-f_2)(f-f_3)}{(f_0-f_1)(f_0-f_2)(f_0-f_3)} = \frac{(0+13)(0-3)(0-18)}{(-30+13)(-30-3)(-30-18)}$$

$$= \frac{702}{-26928} = -0.0261$$

$$L_1(f) = \frac{(f-f_0)(f-f_2)(f-f_3)}{(f_1-f_0)(f_1-f_2)(f_1-f_3)} = \frac{(30)(-3)(-18)}{(17)(-16)(-31)} = \frac{1620}{8432} = 0.192$$

$$L_2(f) = \frac{(f-f_0)(f-f_1)(f-f_3)}{(f_2-f_0)(f_2-f_1)(f_2-f_3)} = \frac{(30)(13)(-18)}{(33)(16)(-15)} = \frac{7020}{7920} = 0.8864$$

$$L_3(f) = \frac{(f-f_0)(f-f_1)(f-f_2)}{(f_3-f_0)(f_3-f_1)(f_3-f_2)} = \frac{(30)(13)(-3)}{(48)(31)(15)} = -\frac{1170}{22320} = -0.0524.$$

Therefore,

$$x = L_0x_0 + L_1x_1 + L_2x_2 + L_3x_3 = -0.7830 + 6.5314 + 33.6832 - 2.2008 = 37.231.$$

EXAMPLE 5.40

A function f is known in three points x_1, x_2, and x_3 in the vicinity of an extreme point x_0. Show that

$$x_0 \approx \frac{x_1 + 2x_2 + x_3}{4} - \frac{f(x_1,x_2) + f(x_2,x_3)}{4f(x_1,x_2,x_3)}.$$

Use this formula to find x_0 when the following values are known:

x	3.00	3.6	3.8
f	0.13515	0.83059	0.26253

Solution. By Newton's divided difference formula, we have

$$f(x) = f_1 + (x - x_1)f(x_1, x_2) + (x - x_1)(x - x_2)f(x_1, x_2, x_3).$$

Now x_0 is given to be an extreme point, therefore derivative at x_0 vanishes. Therefore,

$$0 = [f'(x)]_{x=x_0} = [0 + f(x_1, x_2) + 2xf(x_1, x_2, x_3) - (x_1 + x_2)f(x_1, x_2, x_3)]_{x=x_0}$$

$$= f(x_1, x_2) + 2x_0 f(x_1, x_2, x_3) - (x_1 + x_2)f(x_1, x_2, x_3)$$

and so

$$2x_0 f(x_1, x_2, x_3) = (x_1 + x_2)f(x_1, x_2, x_3) - f(x_1, x_2)$$

which yields

$$x_0 = \frac{x_1 + x_2}{2} - \frac{f(x_1, x_2)}{2f(x_1, x_2, x_3)}$$

$$= \frac{x_1 + 2x_2 + x_3}{4} - \frac{f(x_1, x_2)}{2f(x_1, x_2, x_3)} - \frac{x_3 - x_1}{4}$$

$$= \frac{x_1 + 2x_2 + x_3}{4} - \frac{2f(x_1, x_2) + f(x_1, x_2, x_3)(x_3 - x_1)}{4f(x_1, x_2, x_3)}$$

$$= \frac{x_1 + 2x_2 + x_3}{4} - \frac{2f(x_1, x_2) + (x_3 - x_1)\left[\dfrac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1}\right]}{4f(x_1, x_2, x_3)}$$

$$= \frac{x_1 + 2x_2 + x_3}{4} - \frac{f(x_1, x_2) + f(x_2, x_3)}{4f(x_1, x_2, x_3)}. \tag{5.59}$$

Further, we have

$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{0.83059 - 0.13515}{0.6} = 1.15906$$

$$f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = \frac{0.26253 - 0.83059}{0.2} = -2.84030$$

$$f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1} = \frac{-2.84030 - 1.15906}{0.8} = -4.99920.$$

Putting these values in equation (5.59), we have

$$x_0 = 3.4915925.$$

EXAMPLE 5.41

The equation $x^3 - 15x + 4 = 0$ has a root close to 0.3. Obtain this root with six decimal places using inverse interpolation (for example, with Bessel's interpolation formula).

Solution. Taking h to be 0.02, we tabulate the values as below:

	x	$f(x)$	δ	δ^2	δ^3	δ^4
x_{-2}	0.22	0.710648				
			−0.296824			
x_{-1}	0.24	0.413824		0.000576		
			−0.296248		0.000048	
x_0	0.26	0.117576		0.000624		0
			−0.295624		0.000048	
x_1	0.28	−0.178048		0.000672		0
			−0.294952		0.000048	
x_2	0.30	−0.47300		0.000720		0
			−0.294232		0.000048	
x_3	0.32	−0.767232		0.000768		0
			−0.293464		0.000048	
x_4	0.34	−1.060696		0.000816		0
			−0.292648		0.000048	
x_5	0.36	−1.353344		0.000864		
			−0.291784			
x_6	0.38	−1.645128				

It is visible from the table that the root lies between 0.26 and 0.28. Therefore, we take 0.26 to be x_0. The Bessel's formula reads as

$$f_p = f_0 + p\delta f_{\frac{1}{2}} + \frac{p(p-1)}{2!}\left(\frac{\delta^2 f_0 + \delta^2 f_1}{2}\right) + \frac{p\left(p-\frac{1}{2}\right)(p-1)}{3!}\delta^3 f_{\frac{1}{2}} + \ldots$$

where $p = \dfrac{x - x_0}{h}$. Using first order difference, we get the first approximation p_1 from the equation

$$0 = f_0 + p_1\delta f_{\frac{1}{2}}$$

which yields

$$p_1 = -\frac{f_0}{\delta f_{\frac{1}{2}}} = \frac{0.117576}{0.295624} = 0.39772.$$

This value is inserted in $\delta^2 f$ terms and p_2 is obtained from the equation

$$0 = f_0 + p_2\delta f_{\frac{1}{2}} + \frac{p_1^2 - p_1}{4}\left(\frac{\delta^2 f_0 + \delta^2 f_1}{2}\right)$$

$$= 0.117576 + p_2(-0.295624) + \frac{0.001296}{4}(-0.23915)$$

and so

$$p_2 = \frac{-0.117576 + 0.00007755}{-0.295624} = 0.39746$$

Inserting the value of p_2 in $\delta^3 y$ term, we have the next approximation p_3 given by

$$0 = f_0 + p_3\delta f_{\frac{1}{2}} + \frac{p_2^2 - p_2}{4}(-0.001296)$$
$$+\frac{(p_2^2 - p_2)\left(p_2 - \frac{1}{2}\right)}{3!}(0.000048)$$

which yields

$$p_3 = 0.39753.$$

Thus,

$$x = x_0 + ph = 0.26 + 0.3975(0.02) = 0.26 + 0.00796 = 0.267950.$$

EXERCISES

1. Evaluate
   (i) $\Delta^2 \cos 2x$ (ii) $\Delta^n\left(\frac{1}{x}\right)$.

   Ans. (i) $-4\sin^2 h\cos(2x+2h)$

   (ii) $\dfrac{(-1)^n n!}{x(x+1)(x+2)\dots(x+n)}$

2. Show that $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$.

3. Show that $\Delta^3 y_i = y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i$.

4. Find the function whose first difference is $9x^2 + 11x + 5$.

   Ans. $3x^3 + x^2 + x + k$

5. Find the missing values in the following data:

x	45	50	55	60	65
y	3.0	–	2.0	–	−2.4

   Ans. $f(50) = 2.925$, $f(60) = 0.225$

6. Express $3x^4 - 4x^3 + 6x^2 + 2ix + 1$ as a factorial polynomial and find fourth order difference

   Ans. $3[x]^4 + 14[x]^3 + 15[x]^2 + 7[x] + 1$, $\Delta^4 y = 72$

7. Form a difference table to fourth differences

x	1	2	3	4	5	6	7	8
f_x	7.93	10.05	12.66	15.79	19.47	23.73	28.60	34.11

   Repeat the procedure for the same table when $f_5 = 19.47 + \in$, where $\in$ represents an error. How many $\Delta^n f_x$ are affected?

8. If $f(x)$ is a cubic polynomial, use the difference table to locate and correct the error in the data:

x	0	1	2	3	4	5	6	7
$f(x)$	25	21	18	18	27	45	76	123

   Ans. $f(3)$ is in error, true value is 19

9. If $f(x)$ is a polynomial of degree 4, locate and correct the error in the table

x	1	2	3	4	5	6	7	8
y	3010	3424	3802	4105	4472	4771	5051	5315

10. The function y is given in the table below:

x	20	24	28	32
y	2854	3162	3544	3992

Find y for $x = 25$ using Bessel's interpolation formula.

Ans. 3250.875 approx.

11. Evaluate $f(3.75)$ from the table

x	2.5	3.0	3.5	4.0	4.5	5.0
y	24.145	22.043	20.225	18.644	17.262	16.047

(Hint: Use Gauss's forward formula).

Ans. 19.40746093

12. Use Stirling's interpolation formula to find $f(35)$ from the table

x	20	30	40	50
y	512	439	346	243

Ans. 395

13. Using Newton's divided difference formula find $f(x)$ as a polynomial in x for the table:

x	0	1	2	4	5	6
y	1	14	15	5	6	19

Ans. $x^3 - 9x^2 + 21x + 1$

14. Let $f(x) = x^3 - 4x$. Construct the divided difference table based on the nodes $x_0 = 1, x_1 = 2, \ldots, x_5 = 6$ and find the Newton's polynomial $P_3(x)$ based on x_0, x_1, x_2, x_3.

Ans. $P_3(x) = 3 + 3(x-1) + 6(x-1)(x-2) + (x-1)(x-2)(x-3)$

15. Using Lagrange's interpolation formula, find the value of t for $A = 85$ using the table

t	2	5	8	14
A	94.8	87.9	81.3	68.7

Ans. 6.5928

16. Use Lagrange's interpolation formula to find the value of y for $x = 10$ using the table given below:

x	5	6	9	11
y	12	13	14	16

Ans. 14.3

17. Find the Lagrange's interpolating polynomial for (1, −3), (3, 9), (4, 30), and (6, 132).

Ans. $x^3 - 3x^2 + 5x - 6$

6 Numerical Differentiation

Let $p(x)$ be an interpolation polynomial approximating satisfactorily a given function $f(x)$ over a certain interval I. We may hope that the result of differentiating $p(x)$ will also satisfactorily approximate the corresponding derivative of $f(x)$. However, if we observe a curve representing the polynomial approximating and oscillating about the curve representing $f(x)$, we may anticipate the fact that even though the deviation between $p(x)$ and $f(x)$ be small throughout the interval, still the slope of the two curves representing them may differ quite appreciably. Also it is seen that the round-off errors of alternating sign in consecutive ordinates could affect the calculation of the derivative quite strongly if those ordinates were fairly closed spaced. That is why, numerical differentiation is considered the weakest concept in the subject of numerical analysis.

6.1 CENTERED FORMULA OF ORDER $O(h^2)$

Let f be a function defined in $[a,b]$. The derivative of f is defined by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Suppose further that f has continuous derivatives of order 1, 2, and 3 and that $x-h, x, x+h \in [a, b]$. Then, by Taylor's expansion, we have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(c_1) \tag{6.1}$$

and

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(c_2). \tag{6.2}$$

Subtracting equation (6.2) from equation (6.1), we get

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{h^3}{3!}[f'''(c_1) + f'''(c_2)].$$

Since $f'''(x)$ is continuous, by intermediate value theorem, there exits a value c such that

$$\frac{f'''(c_1) + f'''(c_2)}{2} = f'''(c).$$

Therefore,

$$f(x+h) - f(x-h) = 2hf'(x) + 2\frac{h_3}{3!}[f'''(c)]$$

and so

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{f'''(c)h^2}{3!}$$

$$= \frac{f(x+h) - f(x-h)}{2h} + E_{\text{trunc}}(f, h) \tag{6.3}$$

where

$$E_{\text{trunc}}(f, h) = \frac{-h^2}{6} f'''(c) = O(h^2)$$

is called truncation error. Expression (6.3) for the derivative of *f* is called the centered formula of order $O(h^2)$.

If the third derivative $f'''(c)$ does not change too rapidly, that is, $f'''(c)$ is bounded, then the truncation error in equation (6.3) tends to zero along with h^2.

6.2 CENTERED FORMULA OF ORDER $O(h^4)$

It is not desirable to choose *h* too small when computer is used for calculation of derivative. For this reason, a formula for approximating $f'(x)$ and having a truncation error term of order $O(h^4)$ is used.

Suppose *f* has continuous derivatives of order 1, 2, 3, 4, 5 and $x - 2h$, $x - h$, x, $x + h$, $x + 2h$ be the points in (a, b). Then, by fifth degree Taylor's expansion, we have

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{2f'''(x)h^3}{3!} + \frac{2f^{(v)}(c_1)h^5}{5!}. \tag{6.4}$$

If we use step size $2h$ instead of h, then

$$f(x+2h) - f(x-2h) = 4hf'(x) + \frac{16f'''(x)h^3}{3!} + \frac{64f^{(v)}(c_2)h^5}{5!}. \tag{6.5}$$

Multiplying both sides of equation (6.4) by 8 and subtracting equation (6.5) from it, we get

$$-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h) = 12hf'(x) + \frac{\left[16f^{(v)}(c_1) - 64f^{(v)}(c_2)\right]h^5}{120}.$$

If the sign and magnitude of $f^{(v)}(x)$ does not change rapidly, we can find a value *c* in $[x - 2h, x + 2h]$ so that

$$16f^{(v)}(c_1) - 64f^{(v)}(c_2) = -48f^{(v)}(c)$$

and so

$$f'(x) \approx \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h} + \frac{f^{(v)}(c)}{30}h^4$$

$$= \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h} + E_{\text{trunc}}(f, h),$$

where

$$E_{\text{trunc}}(f, h) = \frac{f^{(v)}(c)}{30}h^4 = O(h^4)$$

is the truncation error.

EXAMPLE 6.1

Approximate the derivative of $f(x) = \sin x$ at $x = 0.50$ by

(i) centered formula of order $O(h^2)$

(ii) centered formula of order $O(h^4)$

and determine which of the two yields a better approximation.

Solution. (i) Using centered formula of order $O(h^2)$ and taking spacing $h = 0.01$, we get

$$f'(0.50) \approx \frac{f(0.51) - f(0.49)}{2(0.01)} = \frac{0.488177 - 0.470626}{0.02} = 0.87755.$$

(ii) Using centered formula of order $O(h^4)$, we have

$$f'(0.50) \approx \frac{-f(0.52) + 8f(0.51) - 8f(0.49) + f(0.48)}{0.12}$$

$$= \frac{-0496880 + 3.905418 - 3.765007 + 0.461779}{0.12} = 0.877583.$$

Since $f'(x) = \cos x,$ we have

$$f'(0.50) = 0.877582.$$

Hence, formula of $O(h^4)$ yields better result.

6.3 ERROR ANALYSIS

(A) Error for Centered Formula of Order $O(h^2)$

Suppose $f(x_0 - h)$ and $f(x_0 + h)$ are approximated by y_{-1} and y_1, and e_{-1} and e_1 are the associated round-off errors, respectively. Then

$$f'(x_0) = \frac{y_1 - y_{-1}}{2h} + E(f, h),$$

where the total error $E(f, h)$ is given by

$$E(f, h) = e_{\text{round}}(f, h) + E_{\text{trunc}}(f, h)$$

$$= \frac{e_1 - e_{-1}}{2h} - \frac{h^2}{6} f'''(c).$$

If $|e_{-1}| \le \varepsilon$ and $|e_1| \le \varepsilon$ and $M = \max_{x \in [a,b]} \{|f'''(x)|\}$, then

$$|E(f, h)| \le \frac{\varepsilon}{h} + \frac{Mh^2}{6} = \frac{6\varepsilon + Mh^3}{6h}. \tag{6.6}$$

The derivative of equation (6.6) is

$$\frac{h(3Mh^2) - (6\varepsilon + Mh^3)}{h^2}.$$

Equating this derivative to zero, we get $2Mh^3 = 6\varepsilon$ and so the value of h that minimizes the right-hand side of equation (6.6) is

$$h = \left(\frac{3\varepsilon}{M}\right)^{1/3}.$$

(B) Error for Centered Formula of Order $O(h^4)$

If $f(x_0 + kh) = y_k + e_k$, then

$$f'(x_0) = \frac{-y_2 + 8y_1 - 8y_{-1} + y_{-2}}{12h} + E(f,h),$$

where

$$\mathrm{E}(f,h) = \frac{-e_2 + e_1 - e_{-1} + e_{-2}}{12h} + \frac{h^4}{30} f^{(v)}(c).$$

If $|e_k| \le \varepsilon$ and $M = \max_{x\in[a,b]}\{|f^{(v)}(x)|\}$, then

$$|E(f,h)| \le \frac{3\varepsilon}{2h} + \frac{Mh^4}{30} \tag{6.7}$$

and the value of h that minimizes the right-hand side of equation (6.7) is $h = \left(\frac{45\varepsilon}{4M}\right)^{1/5}$.

6.4 RICHARDSON'S EXTRAPOLATION

The method of obtaining a formula for $f'(x_0)$ of higher order from a formula of lower order is called Richardson's extrapolation.

Let $D_0(h)$ and $D_0(2h)$ denote the approximations to $f'(x_0)$ obtained from centered formula of order $O(h^2)$ with step size h and $2h$, respectively. Then

$$f'(x_0) \approx D_0(h) + ch^2 \tag{6.8}$$

and

$$f'(x_0) \approx D_0(2h) + 4ch^2. \tag{6.9}$$

Multiplying equation (6.8) by 4 and subtracting equation (6.9) from the product, we get

$$\begin{aligned} 3f'(x_0) &\approx 4D_0(h) - D_0(2h) \\ &= 4\frac{f_1 - f_{-1}}{2h} - \frac{f_2 - f_{-2}}{4h} \end{aligned}$$

and so

$$\begin{aligned} f'(x_0) &\approx \frac{4D_0(h) - D_0(2h)}{3} \\ &= \frac{-f_2 + 8f_1 - 8f_{-1} + f_{-2}}{12h}, \end{aligned}$$

which is nothing but centered formula of order $O(h^4)$.

Similarly, if $D_1(h)$ and $D_1(2h)$ denote the approximation to $f'(x_0)$, obtained from centered formula of order $O(h^4)$ with step size h and $2h$, respectively, then

$$f'(x_0) = \frac{-f_2 + 8f_1 - 8f_{-1} + f_{-2}}{12h} + \frac{h^4}{30} f^{(\text{v})}(c_1) \tag{6.10}$$
$$\approx D_1(h) + ch^4$$

and

$$f'(x_0) = \frac{-f_4 + 8f_2 - 8f_{-2} + f_{-4}}{12h} + \frac{16h^4}{30} f^{(\text{v})}(c_2) \tag{6.11}$$
$$\approx D_1(2h) + 16ch^4.$$

Multiplying equation (6.10) by 16 and subtracting equation (6.11) from the product, we get

$$f'(x_0) = \frac{16D_1(h) - D_1(2h)}{15}.$$

In general, if two approximations of order $O(h^{2k})$ for $f'(x_0)$ are $D_{k-1}(h)$ and $D_{k-1}(2h)$ and if

$$f'(x_0) = D_{k-1}(h) + c_1 h^{2k} + c_2 h^{2k+2} + \ldots$$

and

$$f'(x_0) = D_{k-1}(2h) + 4^k c_1 h^{2k} + 4^{k+1} c_2 h^{2k+2} + \ldots,$$

then an improved approximation is of the form

$$f'(x_0) = D_k(h) + O(h^{2k+2})$$
$$= \frac{4^k D_{k-1}(h) - D_{k-1}(2h)}{4^k - 1} + O(h^{2k+2}).$$

This result is known as Richardson's extrapolation.

EXAMPLE 6.2

The voltage $E(t)$ in an electrical circuit obeys the equation

$$E(t) = L\frac{\mathrm{d}I}{\mathrm{d}t} + RI(t),$$

where L is the inductance and R is the resistance. If $L = 0.05$, $R = 2$ and $I(t)$ at time t is given by the table

t:	1.0	1.1	1.2	1.3	1.4
$I(t)$:	8.2277	7.2428	5.9908	4.5260	2.9122

Find $I'(1.2)$ by numerical differentiation and compute $E(1.2)$.

Solution. Using centered formula of order $O(h^2)$, we have

$$I'(1.2) \approx \frac{I(x+h) - I(x-h)}{2h} = \frac{4.5260 - 7.2428}{2(0.1)}$$
$$= -13.5840$$

and then

$$E(1.2) \approx 0.05(-13.5840) + 2(5.9908)$$
$$= -0.6792 + 11.9816 = 11.3024.$$

If we use centered formula of order $O(h^4)$, then

$$I'(1.2) \approx \frac{-I(x+2h)+8I(x+h)-8I(x-h)+I(x-2h)}{12h}$$
$$= \frac{-2.9122+8(4.5260)-8(7.2428)+8.2277}{12(0.1)}$$
$$= \frac{-2.9122+36.2080-57.9424+8.2277}{1.2}$$
$$= \frac{-60.8546+49.4357}{1.2} = -13.6824$$

and so

$$E(1.2) \approx 0.05(-13.6824)+2(5.9908)$$
$$= -0.6841+11.9816 = 11.2975.$$

EXAMPLE 6.3

Find $I'(1.2)$ in the Example 6.2 using Richardson's extrapolation.

Solution. We have

$$D_0(h) \approx \frac{I(x+h)-I(x-h)}{2h} = \frac{I(1.3)-I(1.1)}{2(0.1)}$$
$$= \frac{4.5260-7.2428}{0.2} = -13.5840$$
$$D_0(2h) \approx \frac{I(x+2h)-I(x-2h)}{4h} = \frac{I(1.4)-I(1.0)}{0.4}$$
$$= \frac{2.9122-8.227}{0.4} = -13.28875.$$

Therefore,

$$I'(1.2) \approx \frac{4D_0(h)-D_0(2h)}{3}$$
$$= \frac{4(-13.5840)+13.28875}{3}$$
$$= \frac{-54.3360+13.28875}{3} = -13.6824.$$

We observe that this value is exactly that we found by centered formula of order $O(h^4)$.

EXAMPLE 6.4

From the following table, find $f'(1.4)$.

x:	1.2	1.3	1.4	1.5	1.6
$f(x)$:	1.5095	1.6984	1.9043	2.1293	2.3756

Solution. Using centered formula of order (h^4) and proceeding as in Example 6.2, we have

$$f'(1.4) \approx \frac{-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h)}{12h}$$

$$= \frac{-2.3756+8(2.1293)-8(1.6984)+1.5095}{12(0.1)}$$

$$= \frac{-2.3756+17.0344-13.5872+1.5095}{1.2} = 2.1509.$$

6.5 CENTRAL DIFFERENCE FORMULA OF ORDER $O(h^4)$ FOR $f''(x)$

By Taylor's expansion

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4 f^{(\text{iv})}(x)}{4!} + \ldots \tag{6.12}$$

and

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(\text{iv})}(x) + \ldots. \tag{6.13}$$

Adding equations (6.12) and (6.13), we have

$$f(x+h) + f(x-h) = 2f(x) + \frac{2h^2}{2!}f''(x) + \frac{2h^4}{4!}f^{(\text{iv})}(x) + \ldots,$$

which yields

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - \frac{2h^2}{24}f^{(\text{iv})}(x) - \ldots.$$

Truncating at the fourth derivative, we get

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - \frac{h^2}{12}f^{(\text{iv})}(c)$$

and hence the desired formula is

$$f''(x_0) \approx \frac{f_1 - 2f_0 + f_{-1}}{h^2}. \tag{6.14}$$

Let $f_k = y_k + e_k$, where e_k is the error in computing. Then the total error in equation (6.14) is

$$E(f,h) = \frac{e_1 - 2e_0 + e_{-1}}{h^2} - \frac{h^2}{12}f^{(\text{iv})}(c).$$

If $|e_k| \le \varepsilon$ and $\left|f^{(\text{iv})}(c)\right| \le M$, then

$$\left|E(f,h)\right| \le \frac{4\varepsilon}{h^2} + \frac{Mh^2}{12}.$$

Differentiating the right-hand side and equating the differential to zero gives

$$h = \left(\frac{48\varepsilon}{M}\right)^{1/4}$$

for minimum error.

The central difference formulae of order $O(h^2)$ for $f'''(x_0)$ and $f^{(\text{iv})}(x_0)$ are, respectively,

$$f'''(x_0) \approx \frac{f_2 - 2f_1 + 2f_{-1} - f_{-2}}{2h^3},$$

$$f^{(\text{iv})}(x_0) \approx \frac{f_2 - 4f_1 + 6f_0 - 4f_{-1} + f_{-2}}{h^4}.$$

On the other hand, central difference formulae of order $O(h^4)$ for $f''(x_0)$, $f'''(x_0)$, and $f^{(\text{iv})}(x_0)$ are, respectively,

$$f''(x_0) \approx \frac{-f_2 + 16f_1 - 30f_0 + 16f_{-1} - f_{-2}}{12h^2},$$

$$f'''(x_0) \approx \frac{-f_3 + 8f_2 - 13f_1 + 13f_{-1} - 8f_2 + f_{-3}}{8h^3},$$

$$f^{(\text{iv})}(x_0) \approx \frac{-f_3 + 12f_2 - 39f_1 + 56f_0 - 39f_{-1} + 12f_{-2} - f_{-3}}{6h^4}.$$

6.6 GENERAL METHOD FOR DERIVING DIFFERENTIATION FORMULAE

Suppose the function f is analytic and tabulated at equidistant points. We know that

$$e^{hD} = E = I + \Delta.$$

Therefore,

$$hD = \log(I + \Delta) = \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots$$

and so

$$D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots\right].$$

Hence,

$$f'(x) = \frac{1}{h}\left[\Delta f(x) - \frac{\Delta^2}{2} f(x) + \frac{\Delta^3}{3} f(x) - \frac{\Delta^4}{4} f(x) + \ldots\right]. \quad (6.15)$$

To find second derivative, we have

$$h^2 D^2 = (\log E)^2 = (\log(1 + \Delta))^2$$
$$= \left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots\right]^2$$

and so

$$D^2 = \frac{1}{h^2}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots\right]^2.$$

Hence,

$$f''(x) = \frac{1}{h^2}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots\right]^2 f(x). \quad (6.16)$$

In terms of central differences, we know that

$$U = hD = 2\sinh^{-1}\left(\frac{\delta}{2}\right).$$

If we put $f(x) = \sinh^{-1} x,$ then

$$f'(x) = (1+x^2)^{\frac{-1}{2}} = 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \ldots,$$

which on integration yields

$$f(x) = x - \frac{x^3}{6} + \frac{3}{40}x^5 - \ldots.$$

Thus,

$$f\left(\frac{\delta}{2}\right) = \sinh^{-1}\left(\frac{\delta}{2}\right) = \frac{\delta}{2} - \frac{\delta^3}{48} + \frac{3}{1280}\delta^5 - \ldots$$

and so

$$hD = \delta - \frac{\delta^3}{24} + \frac{3}{640}\delta^5 - \frac{5}{7168}\delta^7 + \ldots. \tag{6.17}$$

Hence,

$$D = \frac{1}{h}\left[\delta - \frac{\delta^3}{24} + \frac{3}{640}\delta^5 - \frac{5}{7168}\delta_3 + \ldots\right]$$

and we have

$$f'(x) = \frac{1}{h}\left[\delta f(x) - \frac{\delta^3}{24}f(x) + \frac{3}{640}\delta^5 f(x) - \frac{5}{7168}\delta^7 f(x) + \ldots\right]. \tag{6.18}$$

Squaring equation (6.17), we have

$$h^2D^2 = \delta^2 - \frac{1}{12}\delta^4 + \frac{1}{90}\delta^6 - \ldots$$

or

$$D^2 = \frac{1}{h^2}\left[\delta^2 - \frac{1}{12}\delta^4 + \frac{1}{90}\delta^6 - \ldots\right]$$

and so we have

$$f''(x) = \frac{1}{h^2}\left[\delta^2 f(x) - \frac{1}{12}\delta^4 f(x) + \frac{1}{90}\delta^6 f(x) - \frac{1}{560}\delta^8 f(x) + \ldots\right]. \tag{6.19}$$

The above derived formulae (6.15), (6.16), (6.18), and (6.19) yield derivatives at the nodes. We now seek derivatives at interior points. Let $x = x_0 + ph.$ Then d$x = h$dp. Therefore,

$$\frac{df}{dx} = \frac{df}{h dp}.$$

Newton's forward difference formula states that

$$f_p = f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \frac{p(p-1)}{3!}\Delta^3 f_0 + \cdots\cdots$$

Therefore,

$$\frac{df_p}{dx} = \frac{df_p}{h dp} = \frac{1}{h}\left[\Delta f_0 + \frac{2p-1}{2!}\Delta^2 f_0 + \frac{3p^2-6p+2}{3!}\Delta^3 f_0 + \ldots\right]. \tag{6.20}$$

Usually, we are interested in the derivative at a tabular point x_0 or at a mid-interval $x_{1/2}$. These are obtained by putting $p = 0$ and $\frac{1}{2}$, respectively. Thus, we get

$$f_0' = \frac{1}{h}\left[\Delta f_0 - \frac{1}{2}\Delta^2 f_0 + \frac{1}{3}\Delta^3 f_0 + \ldots\right] \tag{6.21}$$

and

$$f_{1/2}' = \frac{1}{h}\left[\Delta f_0 + \frac{1}{12}\Delta^3 f_0 + \ldots\right]. \tag{6.22}$$

When the point is midway the table, then we use central differences formulae. For example, if we take Bessel's formula

$$f_p = f_0 + p\delta f_{1/2} + \frac{p(p-1)}{2!}\left(\frac{\delta^2 f_0 + \delta^2 f_1}{2}\right) + \frac{p\left(p-\frac{1}{2}\right)(p-1)}{3!}\delta^3 f_{1/2} + \ldots$$

then

$$\frac{df_p}{dx} = \frac{df_p}{h dp} = \frac{1}{h}\left[\delta f_{1/2} + \frac{2p-1}{(2)2!}(\delta^2 f_0 + \delta^2 f_1) + \frac{3p^2-3p+\frac{1}{2}}{3!}\delta^3 f_{1/2} + \ldots\right]. \tag{6.23}$$

If we put $p = 0$, then $x = x_0$ and we have

$$f_0' = \frac{1}{h}\left[\delta f_{1/2} - \frac{1}{4}(\delta^2 f_0 + \delta^2 f_1) + \frac{1}{12}\delta^3 f_{1/2} + \ldots\right]. \tag{6.24}$$

If we put $p = \frac{1}{2}$, the coefficients of even differences become zero and we have

$$f_{1/2}' = \frac{1}{h}\left[\delta f_{1/2} - \frac{1}{24}\delta^3 f_{1/2} + \frac{3}{640}\delta^5 f_{1/2} - \ldots\right]. \tag{6.25}$$

If we use Everett's formula, then we have

$$f_p = (1-p)f_0 - \frac{p(p-1)(p-2)}{3!}\delta^2 f_0 + \cdots + pf_1 + \frac{p(p-1)(p-2)}{3!}\delta^2 f_1 + \cdots$$

Therefore,

$$f'_p = \frac{1}{h}\left[f_1 - f_0 + \frac{3p^2-1}{3!}\delta^2 f_1 - \frac{3p^2-6p+2}{3!}\delta^2 f_0 + \ldots \right]. \tag{6.26}$$

Similarly, Stirling's formula reads that

$$f_p = f_0 + p\mu\delta f_0 + \frac{p^2}{2}\delta^2 f_0 + \frac{(p+1)p(p-1)}{3!}\mu\delta^3 f_0 + \cdots + \frac{p^2(p+1)(p-1)}{(4)3!}\delta^4 f_0 + \cdots.$$

Therefore,

$$f'_p = \frac{1}{h}\left[\mu\delta f_0 + p\delta^2 f_0 + \frac{3p^2-1}{6}\mu\delta^3 f_0 + \frac{4p^3-2p}{24}\delta^4 f_0 + \ldots \right]. \tag{6.27}$$

Putting $p = 0$, we get

$$\begin{aligned} f'_0 &= \frac{1}{h}\left[\mu\delta f_0 - \frac{1}{6}\mu\delta^3 f_0 + \frac{1}{30}\mu\delta^5 f_0 - \ldots \right] \\ &= \frac{1}{2h}\left[(f_1 - f_{-1}) - \frac{1}{6}(\delta^2 f_1 - \delta^2 f_{-1}) + \frac{1}{30}(\delta^4 f_1 - \delta^4 f_{-1}) - \ldots \right]. \end{aligned} \tag{6.28}$$

Also,

$$f''_p = \frac{1}{h^2}\left[\delta^2 f_0 + p\mu\delta^3 f_0 + \frac{p^2-1}{12}\delta^4 f_0 + \ldots \right] \tag{6.29}$$

Putting $p = 0$, we have

$$f''_0 = \frac{1}{h^2}\left[\delta^2 f_0 - \frac{1}{12}\delta^4 f_0 + \frac{1}{90}\delta^6 f_0 - \ldots \right]. \tag{6.30}$$

From equation (6.30), we have

$$h^2 D^2 = \delta^2 = \left(1 - \frac{1}{12}\delta^2 + \frac{1}{90}\delta^4 - \ldots \right)$$

or

$$\begin{aligned} \delta^2 &= h^2 \left(1 - \frac{\delta^2}{12} + \frac{1}{90}\delta^4 - \ldots \right)^{-1} D^2 \\ &= h^2 \left(1 + \frac{\delta^2}{12} - \frac{1}{240}\delta^4 + \ldots \right) D^2 \end{aligned}$$

and so

$$\delta^2 f_0 = h^2 = \left(f''_0 + \frac{1}{12}\delta^2 f_0'' - \frac{1}{240}\delta^4 f''_0 + \ldots \right),$$

which expresses the second difference of f in terms of second and higher differences of f''.

EXAMPLE 6.5

The function $y = \sin x$ is tabulated below. Find the derivative at the point $x = 1$.

x:	0.7	0.8	0.9	1.0	1.1	1.2	1.3
y:	0.644218	0.717356	0.783327	0.841471	0.891207	0.932039	0.963558

Solution. The difference table for the given data is

x	y	δ	δ^2	δ^3	δ^4
0.7	0.644218				
		0.073138			
0.8	0.717356		−0.007167		
		0.065971		−0.00660	
0.9	0.783327		−0.007827		0.000079
		0.058144		−0.00581	
1.0	0.841471		−0.008408		0.000085
		0.049736		−0.00496	
1.1	0.891207		−0.008904		0.000087
		0.040832		−0.00409	
1.2	0.932039		−0.009313		
		0.031519			
1.3	0.963558				

Since $x = 1$ is tabulated argument, we have $p = 0$ and it will be better to use Stirling's formula. Using

$$f'_p = \frac{1}{h}\left[\mu\delta f_0 - \frac{1}{6}\mu\delta^3 f_0 + \frac{1}{30}\mu\delta^5 f_0 - \ldots\right]$$

$$= \frac{1}{2h}\left[\left(\delta f_{\frac{1}{2}} + \delta f_{-\frac{1}{2}}\right) - \frac{1}{6}\left(\delta^3 f_{\frac{1}{2}} + \delta^3 f_{-\frac{1}{2}}\right) + \frac{1}{30}\left(\delta^5 f_{\frac{1}{2}} + \delta^5 f_{-\frac{1}{2}}\right) + \ldots\right]$$

$$= \frac{1}{2h}\left[(f_1 - f_{-1}) - \frac{1}{6}(\delta^2 f_1 - \delta^2 f_{-1}) + \frac{1}{30}(\delta^4 f_1 - \delta^4 f_{-1}) + \ldots\right]$$

$$\approx \frac{1}{0.2}\left[(0.891207 - 0.783327) - \frac{1}{6}(0.008904 - 0.007827) + \frac{1}{30}(0.000087 - 0.000079)\right]$$

$$= \frac{1}{0.2}[0.107880 - 0.0001795 + 0.0000003] = 0.538504.$$

The tabulated value of cos 1 is 0.540302. Thus, the computed value of the derivative is in good agreement with the tabulated value.

EXAMPLE 6.6

The function $y = f(x)$ has a minimum in the interval $0.2 < x < 1.4$. Find the x coordinate of the minimum point.

x:	0.2	0.4	0.6	0.8	1.0	1.2	1.4
$y = f(x)$:	2.10022	1.98730	1.90940	1.86672	1.85937	1.88737	1.95063

Solution.

x	y	δ	δ^2	δ^3	δ^4	δ^5
0.2	2.10022					
		−0.11292				
0.4	1.98730		0.03502			
		−0.07790		0.00020		
0.6	1.90940		0.03522		−0.00009	
		−0.04268		0.00011		0
0.8	1.86672		0.03533		−0.00009	
		−0.00735		0.00002		−0.00002
1.0	1.85937		0.03535		−0.00011	
		0.02800		−0.00009		
1.2	1.88737		0.03526			
		0.06326				
1.4	1.95063					

Taking $x_0 = 0.80$, we shall use Everett's formula

$$f_p = (1-p)f_0 - \frac{p(p-1)(p-2)}{3!}\delta^2 f_0 + \ldots + pf_1 + \frac{(p-1)p(p-2)}{3!}\delta^2 f_1 + \ldots$$

and obtain

$$f'_p = \frac{1}{h}\left[f_1 - f_0 + \frac{3p^2-1}{3!}\delta^2 f_1 - \frac{3p^2-6p+2}{3!}\delta^2 f_0 + \ldots\right]$$

$$= \frac{1}{h}\left[-0.00735 + \frac{3p^2-1}{3!}(0.03535) - \frac{3p^2-6p+2}{3!}(0.03533) + \ldots\right]$$

$$\approx \frac{1}{h}[-0.00735 + \frac{0.03535}{6}(3p^2 - 1 - 3p^2 + 6p - 2)]$$

since $0.03533 \approx 0.03535$. Now for minimum, $f'_p = 0$ and so we get

$$-0.00735 + \frac{0.03535}{6}(6p-3) = 0,$$

which yields

$$p = \frac{0.025025}{0.03535} = 0.707921.$$

Therefore,

$$x = x_0 + ph = 0.80 + (0.707921)(0.2)$$
$$= 0.80 + 0.1415842 = 0.9416.$$

EXAMPLE 6.7

y is a function of x satisfying the differential equation $xy'' + ay' + (x-b)y = 0$, where a and b are known to be integers. Find the constants a and b from the table below:

x:	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y:	1.73036	1.95532	2.19756	2.45693	2.73309	3.02549	3.33334	3.65563

Solution.

x	y	δ	δ^2	δ^3	δ^4
0.8	1.73036				
		0.22496			
1.0	1.95532		0.01728		
		0.24224		−0.00015	
1.2	2.19756		0.01713		−0.00019
		0.25937		−0.00034	
1.4	2.45693		0.01679		−0.00021
		0.27616		−0.00055	
1.6	2.73309		0.01624		−0.00024
		0.29240		−0.00079	
1.8	3.02549		0.01545		−0.00022
		0.30785		−0.00101	
2.0	3.33334		0.01444		
		0.32229			
2.2	3.65563				

We have to find two constants a and b and so we require two equations. So, we evaluate y' and y'' at two points say 1.4 and 1.6. Thus, $p = 0$ in this case and the formula to be used will be that derived from Stirling's formula. We have

$$y'_p = \frac{1}{2h}\left[y_1 - y_{-1} - \frac{1}{6}(\delta^2 y_1 - \delta^2 y_{-1}) + \frac{1}{30}(\delta^4 y_1 - \delta^4 y_{-1}) + \dots\right]$$

and

$$y''_p = \frac{1}{h^2}\left[\delta^2 y_0 - \frac{1}{12}\delta^4 y_0 + \frac{1}{90}\delta^6 y_0 + \dots\right].$$

Thus,

$$\begin{aligned}
y'(1.4) &\approx \frac{1}{0.4}\left[(2.73309 - 2.19756) - \frac{1}{6}(0.01624 - 0.01713) + \frac{1}{30}(-0.00024 + 0.00019) + \cdots\right] \\
&= \frac{1}{0.4}[0.53553 + 0.000148 - 0.000002] \\
&= 1.33919,
\end{aligned}$$

$$y''(1.4) \approx \frac{1}{0.04}\left[(0.01679 - \frac{1}{12}(-0.00021)\right] = 0.419325,$$

$$\begin{aligned}
y'(1.6) &\approx \frac{1}{0.4}\left[(3.02549 - 2.45693) - \frac{1}{6}(0.01545 - 0.01679) + \frac{1}{30}(-0.00022 + 0.00021)\right] \\
&= \frac{1}{0.4}[0.56856 + 0.00022 - 0.0000003] = 1.42195,
\end{aligned}$$

$$y''(1.6) = \frac{1}{0.04}\left[0.01624 - \frac{1}{12}(-0.00024)\right] = 0.4065.$$

Thus, we have two equations

$$(1.4)(0.419325) + a(1.33919) + (1.4 - b)(2.45693) = 0$$
$$(1.6)(0.40650) + a(1.42195) + (1.6 - b)(2.73309) = 0$$

or

$$1.33919a - 2.45693b = -4.026757$$
$$1.42195a - 2.73309b = -5.02\,3344.$$

We shall use Cramer's rule to find a and b. We have

$$\Delta = 3.6601268 + 3.493632 = 0.1664948$$
$$\Delta_1 = 11.005489 - 12.34200 = 1.336515$$
$$\Delta_2 = -1.001365$$

and so

$$a = \frac{\Delta_1}{\Delta} = 8.0273 \text{ and } b = \frac{\Delta_2}{\Delta} = 6.0143.$$

The true values of a and b are 8 and 6, respectively.

EXAMPLE 6.8

A function $y = f(x)$ is given in the table below. The function is a solution of the equation $x^2 y'' + xy' + (x^2 - n^2)y = 0$, where n is a positive integer. Find n.

x:	85	85.01	85.02	85.03	85.04
y:	0.0353878892	0.0346198696	0.033849002	0.0330753467	0.032298975

Solution. The difference table for the given data is

x	y	δ	δ^2	δ^3	δ^4
85.0	0.0353878892				
		−0.0007680196			
85.01	0.0346198696		−0.0000028498		
		−0.0007708694		0.0000000657	
85.02	0.0338490002		−0.0000027841		$\frac{2}{10^9}$
		−0.0007736535	−0.0000027182	0.0000000659	
85.03	0.0330753467		−0.0000027182		
		−0.0007763717			
85.04	0.0322989750				

Differentiating Stirling's formula, the value of f' at $p = 0$ and the value of f'' at $p = 0$ are

$$f_0' = \frac{1}{2h}\left[(f_1 - f_{-1}) - \frac{1}{6}(\delta^2 f_1 - \delta^2 f_{-1}) + \frac{1}{30}(\delta^4 f_1 - \delta^4 f_{-1}) - \dots\right]$$
$$f_0'' = \frac{1}{h^2}\left[\delta^2 f_0 - \frac{1}{12}\delta^4 f_0 + \dots\right].$$

We calculate y_0' and y_0'' at $x = 85.02$. We have

$$y_0' \approx \frac{1}{0.02} 0.0330753467 - 0.0346198676 - \frac{1}{6}\{(-0.0000027182) + 0.0000028498\} + \cdots$$

$$= \frac{1}{0.02}[-0.0015445229 - 0.0000002193] = -0.077227,$$

$$y_0'' = \frac{1}{0.001}\left[\delta^2 f_0 - \frac{1}{12}\delta^4 f_0\right] = -0.02784116.$$

Putting these values in $x^2 y'' + xy' + (x^2 - n^2)y = 0,$ we have

$$0.033849002n^2 = -201.238667136 - 6.56583954 + 224.6741116.$$

Thus, $n^2 = 1089.23$ and so $n \approx \pm 33.003$. Hence, $n = 33$ is the required value.

EXAMPLE 6.9

Given that

x:	1.0	1.1	1.2	1.3	1.4	1.5
y:	7.989	8.403	8.781	9.129	9.451	9.750

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.6$.

Solution. Differentiating Newton's backward formula, we get

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \ldots\right]$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right].$$

The difference table is

x	y						
1.0	7.989						
		0.414					
			−0.036				
1.1	8.403			0.006			
		0.378					
			−0.030		−0.002		
1.2	8.781			0.004		0.001	
		0.348					
					−0.001		0.002
			−0.026				
1.3	9.129			0.003		0.003	
		0.322					
			−0.023		0.002		
1.4	9.451						

x	y						
		0.299		0.005			
			−0.018				
1.5	9.750						
		0.281					
1.6	10.031						

Therefore, for the given spacing $h = 0.1$, we have

$$\left(\frac{dy}{dx}\right)_{x=1.6} = \frac{1}{0.1}\left[0.281 + \frac{1}{2}(-0.018) + \frac{1}{3}(0.005) + \frac{1}{4}(0.002)\right] = 2.7416.$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{x=1.6} = \frac{1}{0.01}\left[-0.018 + 0.005 + \frac{11}{12}(0.02)\right] = -1.117.$$

6.7 DIFFERENTIATION OF A FUNCTION TABULATED IN UNEQUAL INTERVALS

Let f be a function continuously differentiable in the interval $[c,d]$. If $x_0, x_1, \ldots, x_n$ are distinct points in $[c,d]$, then

$$f(x) = P_n(x) + f[x_0, x_1, \cdots, x_n, x]\Psi_n(x), \tag{6.31}$$

where $P_n(x)$ is a polynomial of degree $\le n$ which interpolates $f(x)$ at $x_0, x_1, \ldots, x_n$, and

$$\Psi_n(x) = \prod_{i=0}^{n}(x - x_i).$$

Also,

$$\frac{d}{dx} f[x_0, x_1, \ldots, x_n, x] = f[x_0, x_1, \ldots, x_n, x, x].$$

Therefore, differentiating equation (6.31), we get

$$f'(x) = P_p'(x) + f[x_0, x_1, \ldots, x_n, x, x]\psi_n(x) + f[x_0, x_1, \ldots, x_n, x]\psi_n'(x).$$

Thus, if $a \in [c,d]$, then

$$f'(a) = P_n'(a) + f[x_0, x_1, \ldots, x_n, a, a]\psi_n(a) + f[x_0, x_1, \ldots, x_n, a]\psi_n'(a)$$

and so if we approximate $f''(a)$ by $P_n'(a)$, the error in the approximation is

$$\begin{aligned} E(f) &= f[x_0, x_1, \ldots, x_n, a, a]\psi_n(a) + f[x_0, x_1, \ldots, x_n, a]\psi_n'(a) \\ &= \frac{f^{(n+2)}(\xi)\psi_n(a)}{(n+2)!} + \frac{f^{(n+1)}(\eta)\psi_n'(a)}{(n+1)!} \end{aligned}$$

for some $\xi, \eta \in [c,d]$.

In light of the above discussion, we can derive derivative formulae differentiating Lagrange's polynomial.

6.8 DIFFERENTIATION OF LAGRANGE'S POLYNOMIAL

Consider the Lagrange's interpolation polynomial for $f(x)$ based on three points x_0, x_1, and x_2. We have

$$f(x) \approx f_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}.$$

Differentiating, we get

$$f'(x_0) \approx f_0 \frac{(x-x_1)+(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f_1 \frac{(x-x_0)+(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f_2 \frac{(x-x_0)+(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

and so

$$f'(x_0) \approx f_0 \frac{(x_0-x_1)+(x_0-x_2)}{(x_0-x_1)(x_0-x_2)} + f_1 \frac{(x_0-x_0)+(x_0-x_2)}{(x_1-x_0)(x_1-x_2)} + f_2 \frac{(x_0-x_0)+(x_0-x_1)}{(x_2-x_0)(x_2-x_1)}.$$

But $x_1 - x_0 = h$, $x_2 - x_0 = 2h$. Therefore,

$$\begin{aligned} f'(x_0) &\approx f_0 \frac{(-h)+(-2h)}{(-h)(-2h)} + f_1 \frac{(-2h)}{(h)(-h)} + f_2 \frac{(-h)}{(2h)(h)} \\ &= \frac{f_0}{2h^2}(-3h) + f_1\left(\frac{2h}{h^2}\right) + f_2 \frac{(-1)}{2h} \\ &= \frac{-3f_0 + 4f_1 - f_2}{2h} \end{aligned} \tag{6.32}$$

which is first order differential formula.

If we consider Lagrange's interpolation polynomial for $f(x)$ based on four points x_0, x_1, x_2, and x_3, then

$$\begin{aligned} f(x) \approx{}& f_0 \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + f_1 \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \\ &+ f_2 \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + f_3 \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}. \end{aligned}$$

Differentiating twice, we get

$$\begin{aligned} f''(x) \approx{}& f_0 \frac{2[(x-x_1)+(x-x_2)+(x-x_3)]}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + f_1 \frac{2[(x-x_0)+(x-x_2)+(x-x_3)]}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \\ &+ f_2 \frac{2[(x-x_0)+(x-x_1)+(x-x_3)]}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + f_3 \frac{2[(x-x_0)+(x-x_1)+(x-x_2)]}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}. \end{aligned}$$

Putting $x = x_0$ and taking $x_i - x_j = (i-j)h$, we get

$$\begin{aligned} f''(x_0) \approx{}& f_0 \frac{2[(-h)+(-2h)+(-3h)]}{(-h)(-2h)(-3h)} + f_1 \frac{2[0+(-2h)+(-3h)]}{h(-h)(-2h)} + f_2 \frac{2[0+(-h)+(-3h)]}{(2h)h(-h)} \\ &+ f_3 \frac{2[0+(-h)+(-2h)]}{(3h)(2h)h} \\ ={}& f_0\left(\frac{-12h}{-6h^3}\right) + f_1\left(\frac{-10h}{2h^3}\right) + f_2\left(\frac{-8h}{-2h^3}\right) + f_3\left(\frac{-6h}{6h^3}\right) \\ ={}& \frac{2f_0 - 5f_1 + 4f_2 - f_3}{h^2} \end{aligned} \tag{6.33}$$

which is second order derivative formula.

EXAMPLE 6.10

Find the approximations to $f'(x_n)$ of order $O(h^2)$ at $x = 0$ and $x = 0.1$ in the following table:

x:	0.0	0.1	0.2	0.3
$f(x)$:	0.989992	0.999135	0.998295	0.987480

Solution. Using first order differential formula (6.32), we have

$$\begin{aligned} f'(0) &\approx \frac{-3(0.989992) + 4(0.999135) - 0.998295}{2(0.1)} \\ &= \frac{-2.969976 + 3.99654 - 0.998295}{0.2} \\ &= \frac{3.99654 - 3.968271}{0.2} = \frac{0.028269}{0.2} \\ &= 0.141345 \end{aligned}$$

and

$$\begin{aligned} f'(0.1) &\approx \frac{-3(0.999135) + 4(0.998295) - 0.987480}{0.2} \\ &= \frac{-2.997405 + 3.99318 - 0.987480}{0.2}. \\ &= 0.041475. \end{aligned}$$

6.9 DIFFERENTIATION OF NEWTON POLYNOMIAL

Consider the Newton polynomial $P(x)$ based on the three nodes x_0, x_1, and x_2. We have

$$P(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1), \tag{6.34}$$

where

$$a_0 = f(x_0), \quad a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0},$$

$$a_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}.$$

Differentiating equation (6.34) with respect to x, we get

$$P'(x) = a_1 + a_2[(x - x_0) + (x - x_1)] \tag{6.35}$$

and so

$$P'(x_0) = a_1 + a_2(x_0 - x_1).$$

Thus,

$$f'(x_0) \approx P'(x_0) = a_1 + a_2(x_0 - x_1). \tag{6.36}$$

If we set $x_0 = x$, $x_1 = x + h$, and $x_2 = x_0 + 2h$, then

$$a_1 = \frac{f(x_1) - f(x)}{x_1 - x} = \frac{f(x+h) - f(x)}{h},$$

$$a_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x)}{x_1 - x}}{x_2 - x}$$

$$= \frac{\dfrac{f(x+2h) - f(x+h)}{h} - \dfrac{f(x+h) - f(x)}{h}}{2h}$$

$$= \frac{f(x) - 2f(x+h) + f(x+2h)}{2h^2},$$

and so equation (6.35) becomes

$$P'(x) = \frac{f(x+h) - f(x)}{h} + \frac{f(x) - 2f(x+h) + f(x+2h)}{2h^2}(x - x_1)$$

$$= \frac{f(x+h) - f(x)}{h} + \frac{f(x) - 2f(x+h) + f(x+2h)}{2h^2}(-h)$$

$$= \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h}$$

and so

$$f'(x) \approx P'(x) = \frac{3f(x) + 4f(x+h) - f(x+2h)}{2h},$$

which is second order forward difference formula for $f'(x)$ (see equation 6.32).

EXAMPLE 6.11

Find the maximum value of $f(x)$ using the table given below:

x:	−1	1	2	3
$f(x)$:	−21	15	12	3

Solution. We note that the arguments given are not equispaced. Therefore, we shall use Newton's divided difference formula. The divided difference table is

x	$f(x)$	$f(x_0, x_1)$	$f(x_0, x_1, x_2)$	$f(x_0, x_1, x_2, x_3)$
−1	−21			
		18		
1	15		−7	
		−3		1
2	12		−3	
3	3	−9		

The divided difference formula yields

$$f(x) = f_0 + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2)f(x_0, x_1, x_2, x_3)$$

$$= -21 + (x+1)(18) + (x+1)(x-1)(-7) + (x+1)(x-1)(x-2)(1)$$

$$= x^3 - 9x^2 + 17x + 6.$$

Therefore,

$$f'(x) = 3x^2 - 18x + 17.$$

For maximum value, we should have $3x^2 - 18x + 17 = 0$. This equation yields $x = 4.8257$ or 1.1743. The value 1.1743 is the value in the given range. Then

$$\text{Maximum value of } f(x) = (1.1743)^3 - 9(1.1743)^2 + 17(1.1743) + 6 \approx 15.1716.$$

EXAMPLE 6.12

Evaluate the first derivative at $x = -3$ and $x = 0$ from the following table:

x:	−3	−2	−1	0	1	2	3
y:	−33	−12	−3	0	3	12	33

Solution. The difference table for the given problem is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
−3	−33				
		21			
−2	−12		−12		
		9		6	
−1	−3		−6		0
		3		6	
0	0		0		0
		3		6	
1	3		6		0
		9		6	
2	12		12		
		21			
3	33				

We know that (see formula (6.15) or (6.21))

$$f'(x) = \frac{1}{h}\left[\Delta f(x) - \frac{\Delta^2}{2} f(x) + \frac{\Delta^3}{3} f(x) - \frac{\Delta^4}{4} f(x) + \ldots\right].$$

Therefore,

$$f'(-3) = \frac{1}{1}\left[21 - \frac{1}{2}(-12) + \frac{1}{3}(6)\right] = 29$$

and

$$f'(0) = \frac{1}{1}\left[3 - \frac{1}{2}(6) + \frac{1}{3}(6)\right] = 2.$$

EXAMPLE 6.13

Find the first and second derivatives of $f(x)$ at $x = 1.5$ using the following data:

x:	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$:	3.375	7.000	13.625	24.000	38.875	59.000

Solution. The difference table for the given problem is

x	$f(x)$					
1.5	3.375					
		3.625				
2.0	7.000		3			
		6.625		0.750		
2.5	13.625		3.750		0	
		10.375		0.750		0
3.0	24.000		4.500		0	
		14.875		0.750		
3.5	38.875		5.250			
		20.125				
4.0	59.000					

Since the tabular point x = 1.5 lies in the beginning of the table, we use the differentiation formula obtained by differentiating Newton's forward difference formula. Thus,

$$f'(x_0) = \frac{1}{h}\left[\Delta f(x_0) - \frac{1}{2}\Delta^2 f(x_0) + \frac{1}{3}\Delta^3 f(x_0) - \frac{1}{4}\Delta^4 f(x_0) + \ldots\right].$$

Here $h = 0.5$. Therefore, we have

$$f'(1.5) = \frac{1}{0.5}[3.625 - 1.5 + 0.250] = 4.750.$$

For the second derivative, we have

$$f''(x_0) = \frac{1}{h^2}\left[\Delta^2 f(x_0) - \Delta^3 f(x_0) + \frac{11}{12}\Delta^4 f(x_0)\right],$$

which implies

$$f''(1.5) = \frac{1}{0.25}[3 - 0.750 + 0] = 9.$$

EXAMPLE 6.14

Find $f'(10)$ from the following table:

x:	3	5	11	27	34
$f(x)$:	−13	23	899	17315	35606

Solution. Since the spacing is unequal, we use differentiation formula derived from Newton's divided difference formula. The formula is (see Section 6.9, expression 6.35)

$$f'(x) \approx P'(x) = a_1 + a_2[(x - x_0) + (x - x_1)], \tag{6.37}$$

where

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

and

$$a_2 = \frac{\dfrac{f(x_2)-f(x_1)}{x_2-x_1} - \dfrac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2-x_0}.$$

In the given data, we have

$$x_0 = 5, x_1 = 11, x_2 = 27, x = 10.$$

Therefore,

$$a_1 = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{899-23}{11-5} = 146,$$

$$a_2 = \frac{\dfrac{f(x_2)-f(x_1)}{x_2-x_1} - \dfrac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2-x_0} = \frac{\dfrac{17315-899}{16} - \dfrac{899-23}{6}}{22}$$

$$= \frac{1026-146}{22} = 40.$$

Therefore equation (6.37) yields

$$f'(10) = 146 + 40[(10-5)+(10-11)] = 306.$$

EXERCISES

1. Approximate the derivative of $f(x) = \cos x$ at $x = 0.1$ by central formula of order $O(h^2)$.

Ans. −0.93050

2. The data given below give the distance covered by a body at a specified period. Calculate the velocity of the body at 0.3 second using Stirling's formula

t:	0	0.1	0.2	0.3	0.4	0.5	0.6
x:	30.13	31.62	32.87	33.64	33.95	33.81	33.24

Ans. −5.33 units

3. Find the value of $\frac{dy}{dx}$ at $x = 2.03$ using the following table:

x:	1.96	1.98	2.0	2.02	2.04
y:	0.7825	0.7739	0.7651	0.7563	0.7473

Ans. − 0.06

4. Find $f'(x)$ at $x = 1.5$ using the following table:

x:	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$:	3.375	7.000	13.625	24.000	38.875	59.000

Hint: Taking $x_0 = 1.5$, use $= f_0' = \frac{1}{h}\left[\Delta f_0 - \frac{1}{2}\Delta^2 f_0 + \frac{1}{3}\Delta^3 f_0 + \ldots\right].$

Ans. 4.75

5. Find the value of cos 1.74 using the table given below:

x:	1.70	1.74	1.78	1.82	1.86
sin x:	0.9916	0.9857	0.9781	0.9691	0.9584

Ans. 0.175

6. Using the table given below, determine the value of x for which y is maximum. Also find the maximum value of y.

x:	1.2	1.3	1.4	1.5	1.6
y:	0.9320	0.9636	0.9855	0.9975	0.9996

Hint: Use Everett's formula, put the derivative f_p' equal to zero and find p. Then $x_p = x_0 + ph$. Everett's formula then gives the maximum value.

Ans. $x = 1.58$, $y \approx 1.00$

7. Using the table given below, find the value of x for which y is maximum:

x:	3	4	5	6	7	8
y:	0.205	0.240	0.259	0.262	0.250	0.224

Ans. $x = 5.6875$

8. Using Bessel's formula and the table given below, find f' (0.04):

x:	0.01	0.02	0.03	0.04	0.05	0.06
$f(x)$:	0.1023	0.1047	0.1071	0.1096	0.1122	0.1148

Ans. f' (0.04) = 0.25625

9. For the values of x and y given below, find f' (4):

x:	1	2	4	8	10
y:	0	1	5	21	27

Ans. 2.833

10. Find the maximum value of $f(x)$ using the table given below:

x:	−1	1	2	3
$f(x)$:	7	5	19	51

Hint: Arguments not equispaced, so use Newton's divided difference formula. Polynomial is $x^3 + 3x^2 - 2x + 3$, maximum value is at $x = 0.291$ and it is 2.6967.

7 Numerical Quadrature

Numerical integration is the process of computing the approximate value of a definite integral using a set of numerical values of the integrand. If the integrand is a function of single variable, the process is called mechanical quadrature. If the integrand is a function of two independent variables, the process of computing double integral is called mechanical cubature.

Like numerical differentiation, the numerical integration is performed by representing the integrand by an interpolation formula and then integrating the interpolation formula between the given limits. Thus, to find $\int_a^b f(x)\,\mathrm{d}x,$ we replace the function f by an interpolation formula involving differences and then integrate this formula between the limits a and b.

7.1 GENERAL QUADRATURE FORMULA

In equidistant interpolation formulae, the relation between x and p is

$$x = x_0 + ph, \tag{7.1}$$

where h is the equidistance between the given nodes. Then

$$\mathrm{d}x = h\mathrm{d}p. \tag{7.2}$$

We integrate Newton's forward difference formula over n equidistant intervals of width h. Let the limit of integration for x be x_0 and $x_0 + nh$. Then equation (7.1) yields the corresponding limits of p as 0 and n. Therefore, integration of Newton's forward difference formula

$$f(x) = f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 f_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 f_0 + \dots$$

yields

$$\int_{x_0}^{x_0+nh} f(x)\mathrm{d}x = h\int_0^n [f_0 + p\Delta f_0 + \binom{p}{2}\Delta^2 f_0 + \binom{p}{3}\Delta^3 f_0 + \binom{p}{4}\Delta^4 f_0 + \dots]\mathrm{d}p$$

$$= h\left[nf_0 + \frac{n^2}{2}\Delta f_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 f_0}{2} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 f_0}{3!} + \dots\right]. \tag{7.3}$$

From this general formula, we obtain the distinct quadrature formulae by putting $n = 1, 2, 3,...$

(A) Trapezoidal Rule: Setting $n = 1$ in the general formula (7.3), we get the differences $\Delta^2, \Delta^3,...$ to be zero and therefore for the interval $[x_0, x_1]$, we have

$$\int_{x_0}^{x_1} f(x)\,\mathrm{d}x = h\left[f_0 + \frac{1}{2}\Delta f_0\right] = h\left[f_0 + \frac{1}{2}(f_1 - f_0)\right] = \frac{h}{2}(f_0 + f_1),$$

which is called trapezoidal rule.

For the next intervals $[x_1, x_2]$, $[x_2, x_3]$, ... $[x_{n-1}, x_n]$, we have

$$\int_{x_1}^{x_2} f(x)\mathrm{d}x = \frac{h}{2}(f_1 + f_2)$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$\int_{x_{n-1}}^{x_n} f(x)\mathrm{d}x = \frac{h}{2}(f_{n-1} + f_n).$$

Adding all these expressions, we get

$$\int_{x_0}^{x_n} f(x)\mathrm{d}x = \frac{h}{2}[f_0 + 2(f_1 + f_2 + \ldots + f_{n-1}) + f_n],$$

which is known as the composite trapezoidal rule.

(B) Simpson's one-third Rule: Setting $n = 2$ in the general formula (7.3), the differences Δ^3, Δ^4,... are all zero. The interval of integration is from x_0 to $x_0 + 2h$ and the functional values available to us are f_0, f_1, and f_2. Thus we have, from general formula (7.3),

$$\int_{x_0}^{x_0+2h} f(x)\mathrm{d}x = h\left[2f_0 + 2\Delta f_0 + \left(\frac{8}{3} - 2\right)\frac{\Delta^2 f_0}{2}\right]$$

$$= h\left[2f_0 + 2(f_1 - f_0) + \frac{1}{3}(f_2 - 2f_1 + f_0)\right]$$

$$= \frac{h}{3}[f_0 + 4f_1 + f_2],$$

which is known as Simpson's one-third rule.
Similarly,

$$\int_{x_2}^{x_4} f(x)dx = \frac{h}{3}[f_2 + 4f_3 + f_4]$$

$$\int_{x_4}^{x_6} f(x)dx = \frac{h}{3}[f_4 + 4f_5 + f_6]$$

$$\ldots \qquad \ldots \quad \ldots \quad \ldots$$

$$\ldots \qquad \ldots \quad \ldots \quad \ldots$$

$$\int_{x_{n-2}}^{x_n} f(x)dx = \frac{h}{3}[f_{n-2} + 4f_{n-1} + f_n].$$

Thus for even n, adding the above expressions gives

$$\int_{x_0}^{x_0+nh} f(x)\mathrm{d}x = \frac{h}{3}[(f_0 + f_n) + 4(f_1 + f_3 + \ldots + f_{n-1}) + 2(f_2 + f_4 + \ldots + f_{n-2})],$$

which is known as composite Simpson's rule or parabolic rule and is probably the most useful formula for mechanical quadrature. Obviously, to use this formula, we divide the interval of integration into an even number of subintervals of width h. The geometric significance of Simpson's rule is that we replace the graph of the given function by $\frac{n}{2}$ arcs of the second degree polynomials or parabolas with vertical axis.

(C) Simpson's three–eight Rule: If we put $n = 3$ in the general formula (7.3), then the values available are f_0, f_1, f_2, f_3 and so the differences $\Delta^4, \Delta^5, \ldots$ are all zero. Then we shall obtain

$$\int_{x_0}^{x_3} f(x)\mathrm{d}x = \int_{x_0}^{x_0+3h} f(x)\mathrm{d}x = \frac{3h}{8}[f_0 + 3f_1 + 3f_2 + f_3]$$

$$\int_{x_3}^{x_6} f(x)\mathrm{d}x = \frac{3h}{8}[f_3 + 3f_4 + 3f_5 + f_6],$$

$$\ldots \qquad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\ldots \qquad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\int_{x_{n-3}}^{x_n} f(x)\mathrm{d}x = \frac{3h}{8}[f_{n-3} + 3f_{n-2} + 3f_{n-1} + f_n].$$

Thus, if n is a multiple of 3, then adding the above expressions, we get

$$\int_{x_0}^{x_0+nh} f(x)\,\mathrm{d}x = \frac{3h}{8}[(f_0 + f_n) + 3(f_1 + f_2 + f_4 + f_5 + \ldots + f_{n-1}) + 2(f_3 + f_6 + \ldots + f_{n-3})],$$

which is called Simpson's three–eight rule. Thus, in this method, we divide the interval of integration into multiple of 3 subintervals.

(D) Boole's Rule: If $n = 4$, the available values of f are f_0, f_1, f_2, f_3, f_4 and therefore $\Delta^5, \Delta^6, \ldots$ are zero. So, putting $n = 4$ in the general quadrature formula, we get

$$\int_{x_0}^{x_0+4h} f(x)\mathrm{d}x = \int_{x_0}^{x_4} f(x)\mathrm{d}x$$

$$= h\left[4f_0 + 8\Delta f_0 + \frac{20}{3}\Delta^2 f_0 + \frac{8}{3}\Delta^3 f_0 + \frac{28}{90}\Delta^4 f_0\right]$$

$$= \frac{2h}{45}[7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4],$$

$$\int_{x_4}^{x_8} f(x)dx = \frac{2h}{45}[7f_4 + 32f_5 + 12f_6 + 32f_7 + 7f_8]$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$\ldots \qquad \ldots \qquad \ldots$$

Adding these integrals, we get

$$\int_{x_0}^{x_0+nh} f(x)\,\mathrm{d}x = \frac{2h}{45}\,[7f_0 + 32f_1 + 12f_2 + 32f_3 + 14f_4 + 32f_5 + 12f_6 + 32f_7 + 14f_8 + \ldots],$$

where n is a multiple of 4. This formula is known as Boole's rule.

(E) Weddle's Rule: If $n = 6$, then $\Delta^7, \Delta^8, \ldots,$ are zero and we have

$$\int_{x_0}^{x_0+6h} f(x)\mathrm{d}x = h\left[6f_0 + 18\Delta f_0 + 27\Delta^2 f_0 + 24\Delta^3 f_0 + \frac{123}{10}\Delta^4 f + \frac{33}{10}\Delta^5 f_0 + \frac{41}{140}\Delta^6 f_0\right].$$

The coefficient of $\Delta^6 f_0$ differs from $\frac{3}{10}$ by a small fraction $\frac{1}{140}$. Therefore, if we replace this coefficient by $\frac{3}{10}$, we commit an error of only $\frac{h}{140}\Delta^6 f_0$. For small values of h, this error is negligible. Making this change, we get

$$\int_{x_0}^{x_0+6h} f(x)\mathrm{d}x = \frac{3h}{10}[f_0 + 5f_1 + f_2 + 6f_3 + f_4 + 5f_5 + f_6].$$

Similarly,

$$\int_{x_0+6h}^{x_0+12h} f(x)dx = \frac{3h}{10}[f_6 + 5f_7 + f_8 + 6f_9 + f_{10} + 5f_{11} + f_{12}]$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$\cdots \qquad \cdots \qquad \cdots$$

So, if n is a multiple of 6, adding all such above expressions, we get

$$\int_{x_0}^{x_0+nh} f(x)\mathrm{d}x = \frac{3h}{10}[f_0 + 5f_1 + f_2 + 6f_3 + f_4 + 5f_5 + 2f_6 + 5f_7 + \ldots + 5f_{n-1} + f_n]$$

$$= \frac{3h}{10}\sum_{i=0}^{n} Kf_i,$$

where

$K = 1, 5, 1, 6, 1, 5, 2, 5, 16, 15, 2$ etc.

This formula is known as Weddle's rule. It is more accurate, in general, than Simpson's rule but requires at least seven consecutive values of the function. The geometric meaning of Weddle's rule is that we replace the graph of the given function by $\frac{n}{6}$ arcs of sixth degree polynomials.

If we integrate Newton's backward difference formula

$$f_p = f_0 + p\nabla f_0 + \frac{p(p+1)}{2!}\nabla^2 f_0 + \frac{p(p+1)(p+2)}{3!}\nabla^3 f_0 + \ldots,$$

then we get

$$\int_{x_0}^{x_1} f(x)\mathrm{d}x = \int_{x_0}^{x_0+h} f(x)\mathrm{d}x$$

$$= h[f_0 + \frac{1}{2}\nabla f_0 + \frac{5}{12}\nabla^2 f_0 + \frac{3}{8}\nabla^3 f_0 + \frac{251}{720}\nabla^4 f_0 + \ldots]. \tag{7.4}$$

If we multiply the right-hand side of equation (7.4) by the identity operator $(I - \nabla)E$, we get

$$\int_{x_0}^{x_1} f(x)\mathrm{d}x = h[f_1 - \frac{1}{2}\nabla f_1 - \frac{1}{12}\nabla^2 f_1 - \frac{1}{24}\nabla^3 f_1 - \frac{19}{720}\nabla^4 f_1 - \ldots \tag{7.5}$$

The above two formulae are used for the numerical solution of differential equations. Formula (7.4) is an extrapolation formula because it uses the ordinates at $x_0, x_{-1}, x_{-2},\ldots$ to find the integral up to x_1. For this reason, it is called a predictor, whereas formula (7.5) is called corrector and is more accurate as its coefficients are smaller, which make it more rapidly convergent than the predictor.

We now integrate Bessel's formula

$$f_p = f_0 + p\delta f_{1/2} + \frac{p(p-1)}{2!}\left(\frac{\delta^2 f_0 + \delta^2 f_1}{2}\right) + \frac{p\left(p-\frac{1}{2}\right)(p-1)}{3!}\delta^3 f_{1/2}$$
$$+\frac{(p+1)p(p-1)(p-2)}{4!}\left(\frac{\delta^4 f_0 + \delta^4 f_1}{2}\right) + \ldots.$$

Integration yields

$$\int_{x_0}^{x_1} f = \int_{x_0}^{x_0+h} f(x)\mathrm{d}x = h\int_0^1 \left[f_0 + p\delta f_{1/2} + \frac{p(p-1)}{2!}\left(\frac{\delta^2 f_0 + \delta^2 f_1}{2}\right) + \frac{p\left(p-\frac{1}{2}\right)(p-1)}{3!}\delta^3 f_{1/2}\right.$$
$$\left. + \frac{(p+1)p(p-1)(p-2)}{4!}\left(\frac{\delta^4 f_0 + \delta^4 f_1}{2}\right) + \ldots \right]\mathrm{d}p$$
$$= h\left[pf_0 + \frac{p^2}{2}\delta f_{1/2} + \frac{\frac{p^3}{3} - \frac{p^2}{2}}{2!}\left(\frac{\delta^2 f_0 + \delta^2 f_1}{2}\right) + \frac{\frac{p^4}{4} - \frac{p^3}{2} + \frac{p^2}{4}}{3!}\delta^3 f_{1/2}\right.$$
$$\left. + \frac{\frac{p^5}{5} - \frac{p^4}{2} - \frac{p^3}{3} + p^2}{4!}\left(\frac{\delta^4 f_0 + \delta^4 f_1}{2}\right) + \ldots \right]_0^1$$
$$= h\left[f_0 + \frac{f_1 - f_0}{2} - \frac{1}{12}\left(\frac{\delta^2 f_0 + \delta^2 f_1}{2}\right) + \frac{11}{720}\left(\frac{\delta^4 f_0 + \delta^4 f_1}{2}\right) + \ldots \right]$$
$$= h\left[\mu f_{1/2} - \frac{1}{2}\mu\delta^2 f_{1/2} + \frac{11}{720}\mu\delta^4 f_{1/2} - \frac{191}{60480}\mu\delta^6 f_{1/2} + \ldots \right].$$

This formula is much more powerful than the integration formula using forward or backward differences, but it cannot be used at the two ends of a table.

7.2 COTE'S FORMULAE

Let the function values of a function f be available at equidistant points $x_0, x_1, x_2,\ldots, x_n$, where $x_n = x_0 + nh$. We replace $f(x)$ by a suitable function

$$P(x) = \sum_{k=0}^{n} L_k(x) f_k,$$

where

$$L_k(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{k-1})(x-x_{k+1})\ldots(x-x_n)}{(x_k-x_0)(x_k-x_1)\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots(x_k-x_n)},$$

and f_k is the function value at x_k. But $x_k = x_0 + kh$. Also, we have $x = x_0 + ph$, and so $\mathrm{d}x = h\mathrm{d}p$. Since

$$x - x_0 = ph, (x - x_1) = x - (x_0 + h) = (x - x_0) - h = ph - h = (p-1)h, \ldots,$$

and

$$(x_k - x_0) = kh, (x_k - x_1) = (k-1)h, \ldots,$$

we get,

$$L_k(x) = \frac{p(p-1)\ldots(p-k+1)(p-k-1)\ldots(p-n)}{k(k-1)\ldots(1)(-1)\ldots(k-n)}.$$

Therefore,

$$\int_{x_0}^{x_n} P(x)\mathrm{d}x = h\int_0^n \left[\sum_{k=0}^{n} L_k f_k\right] \mathrm{d}p$$

$$= nh\left(\frac{1}{n}\sum_{k=0}^{n} f_k \int_0^n L_k \mathrm{d}p\right)$$

$$= nh\sum_{k=0}^{n} C_k^n f_k,$$

where

$$C_k^n = \frac{1}{n}\int_0^n L_k \mathrm{d}p \quad (0 \le k \le n).$$

are called Cote's numbers. It can be seen that

$$C_k^n = C_{n-k}^n \text{ and } \sum_{k=0}^{n} C_k^n = 1.$$

Case I. Let $n = 1$. Then

$$\int_{x_0}^{x_1} P(x)\mathrm{d}x = h\sum_{k=0}^{1} C_k^1 f_k = h[C_0^1 f_0 + C_1^1 f_1]$$

and we have

$$C_0^1 = \frac{1}{1}\int_0^1 \frac{(p-1)}{-1}\mathrm{d}p = \left[\frac{-p^2}{2} + p\right]_0^1 = \frac{1}{2},$$

$$C_1^1 = \frac{1}{1}\int_0^1 p\mathrm{d}p = \left[\frac{p^2}{2}\right]_0^1 = \frac{1}{2}.$$

Therefore,

$$\int_{x_0}^{x_1} f(x)\mathrm{d}x \approx \int_0^1 P(x)\mathrm{d}x = h\left[\frac{f_0+f_1}{2}\right]$$

$$= \frac{h}{2}(f_0+f_1)$$

which is nothing but trapezoidal rule.

Case II. Setting $n = 2$, we get

$$C_0^2 = \frac{1}{2}\int_0^2 \frac{(p-1)(p-2)}{(-1)(-2)}\mathrm{d}p = \frac{1}{6}$$

$$C_1^2 = \frac{1}{2}\int_0^2 \frac{p(p-2)}{1(-1)}\mathrm{d}p = \frac{4}{6}$$

$$C_2^2 = \frac{1}{2}\int_0^2 \frac{p(p-1)}{(2)(1)}\mathrm{d}p = \frac{1}{6}.$$

Hence,

$$\int_{x_0}^{x_2} f(x)\mathrm{d}x \approx nh\left[\frac{1}{6}f_0 + \frac{4}{6}f_1 + \frac{1}{6}f_2\right] = \frac{h}{3}[f_0 + 4f_1 + f_2],$$

which is Simpson's formula.

Case III. Setting $n = 3$, we obtain

$$C_0^3 = \frac{1}{3}\int_0^3 \frac{(p-1)(p-2)(p-3)}{(-1)(-2)(-3)}\mathrm{d}p = -\frac{1}{18}\left[\frac{p^4}{4} - \frac{6p^3}{3} + 11\frac{p^2}{2} - 6p\right]_0^3 = \frac{1}{8}$$

$$C_1^3 = \frac{1}{3}\int_0^3 \frac{p(p-2)(p-3)}{1(-1)(-2)}\mathrm{d}p = \frac{1}{6}\left[\frac{p^4}{4} - \frac{5p^3}{3} + 6\frac{p^2}{2}\right]_0^3 = \frac{3}{8}$$

$$C_2^3 = \frac{1}{3}\int_0^3 \frac{p(p-1)(p-3)}{2(1)(-1)}\mathrm{d}p = \frac{3}{8}$$

$$C_3^3 = \frac{1}{3}\int_0^3 \frac{p(p-1)(p-2)}{3(2)(1)}\mathrm{d}p = \frac{1}{8}.$$

Hence,

$$\int_{x_0}^{x_3} f(x)dx = \frac{3h}{8}[f_0 + 3f_1 + 3f_2 + f_3],$$

which is Simpson's three-eight rule or four point formula.

7.3 ERROR TERM IN QUADRATURE FORMULA

We have seen that in any quadrature formula, the function f is approximated by a polynomial of degree n, say. Thus,

$$f(x) \approx P_n(x)$$

and quadrature formula becomes

$$\int_{x_0}^{x_n} f(x)\mathrm{d}x \approx \int_{x_0}^{x_n} P_n(x)\mathrm{d}x. \tag{7.6}$$

If $R_n(x)$ is the difference between $f(x)$ and $P_n(x)$ at a point belonging to the interval bounded by the extreme points of $(x_0, x_1,..., x_n)$. Then

$$f(x) = P_n(x) + R_n(x).$$

Therefore,

$$\int_{x_0}^{x_n} f(x)\mathrm{d}x = \int_{x_0}^{x_n} P_n(x)\mathrm{d}x + \int_{x_0}^{x_n} R_n(x)\mathrm{d}x.$$

Therefore, the error between the true value of the integral and the value given by the quadrature formula (7.6) is

$$\int_{x_0}^{x_n} R_n(x)\mathrm{d}x.$$

But for a polynomial of degree n, we have

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} F(x),$$

where $F(x) = \prod_{i=0}^{n} (x - x_i)$.

If we consider equispaced ordinates, then $x = x_0 + ph$ and $x_n = x_0 + nh$.
Therefore,

$$\begin{aligned}
x - x_0 &= ph \\
x - x_1 &= (x_0 + ph) - (x_0 + h) = (p-1)h \\
x - x_2 &= (x_0 + ph) - (x_0 + 2h) = (p-2)h \\
\cdots & \qquad\qquad \cdots \qquad\qquad \cdots \\
\cdots & \qquad\qquad \cdots \qquad\qquad \cdots \\
x - xn &= (x_0 + ph) - (x_0 + nh) = (p-n)h
\end{aligned}$$

and so

$$F(x) = h^{n+1}\ [p(p-1)(p-2)\ldots(p-n)].$$

Putting this value of $F(x)$ in $R_n(x)$, we get

$$R_n(x) = h^{n+1} \frac{f^{(n+1)}(\xi)}{(n+1)!} p^{(n+1)}, \quad \xi \in (x_0, x_n),$$

where

$$p^{(n+1)} = p(p-1)\ldots(p-n).$$

Therefore, the error in the quadrature formula is

$$E_n(x) = \int_{x_0}^{x_n} R_n(x)\mathrm{d}x$$

$$= \frac{h}{(n+1)!}\int_0^n h^{n+1} f^{(n+1)}(\xi)p^{(n+1)}\mathrm{d}p$$

$$= \frac{h^{n+2}}{(n+1)!}\int_0^n p^{(n+1)} f^{(n+1)}(\xi)\mathrm{d}p.$$

Since ξ is independent of p, this integral cannot be evaluated directly. It can be shown, however, that this integral takes one of the following two forms:

$$E_n(x) = \frac{h^{n+2}}{(n+1)!} f^{(n+1)}(\xi)\int_0^n p^{(n+1)}\mathrm{d}p \qquad (n \text{ odd}),$$

$$E_n(x) = \frac{h^{n+3}}{(n+1)!} f^{(n+2)}(\xi)\int_0^n \left(p - \frac{n}{2}\right)p^{(n+1)}\mathrm{d}p \qquad (n \text{ even}).$$

For example,

(i) In trapezoidal rule, $n = 1$ (odd). Therefore,

$$E_1(x) = \frac{h^3 f''(\xi)}{2!}\int_0^1 p(p-1)\mathrm{d}p$$

$$= \frac{h^3 f''(\xi)}{2}\left[\frac{p^3}{3} - \frac{p^2}{2}\right]_0^1 = -\frac{h^3 f''(\xi)}{12}, x_0 < \xi < x_1.$$

Summing over n intervals, we get

$$E_n = -\frac{nh^3 f''(\xi)}{12} = -\frac{h^2}{12}(x_n - x_0)f''(\xi),$$

since $nh = x_n - x_0$. Thus, the error in trapezoidal rule is of order 2.

(ii) In the case of Simpson's formula, $n = 2$ (even). Therefore, the error term is given by

$$E_2(x) = \frac{h^5 f^{(\text{iv})}(\xi)}{4!}\int_0^2 (p-1)p(p-1)(p-2)\mathrm{d}p$$

$$= -\frac{h^5}{90} f^{(\text{iv})}(\xi), \qquad x_0 < \xi < x_2.$$

Summing up for $\frac{n}{2}$ intervals, we get

$$E_n = -\frac{n}{2}\frac{h^5}{90} f^{(\text{iv})}(\xi) = -\frac{(x_n - x_0)}{180} h^4 f^{(\text{iv})}(\xi), \quad x_0 < \xi < x_n.$$

Thus, the error is of order 4 in case of Simpson's rule. Therefore, the complete Simpson's formula is

$$\int_{x_0}^{x_n} f(x)\mathrm{d}x = \frac{h}{3}\left[f_0 + 4(f_1 + \ldots + f_{n-1}) + 2(f_2 + f_4 + \ldots + f_{n-2}) + f_n \right] - \frac{(x_n - x_0)}{180} h^4 f^{(\text{iv})}(\xi).$$

The value obtained for E_n shows that E_n is zero when $f^{(\text{iv})}(x) = 0$. Hence, when $f(x)$ is a polynomial of the first, second, or third degree, Simpson's rule yields the exact value of $\int_{x_0}^{x_n} f(x)\,\mathrm{d}x$.

Similarly, we can show that error is of order 8 in case of Weddle's rule.

Taylor's Series Method for Finding Error

Trapezoidal Rule: Let f be a finite continuous function in the interval $x = x_0$ to $x = x_0 + h$ and have continuous first and second derivatives in the said interval. Let

$$F(x) = \int_0^x f(t)\mathrm{d}t.$$

Then, by fundamental theorem of integral calculus,

$$F'(x) = f(x),\ F''(x) = f'(x)\ldots$$

and so

$$\int_{x_0}^{x_0+h} f(x)\,\mathrm{d}x = F(x_0 + h) - F(x_0).$$

On the other hand, by trapezoidal rule,

$$\int_{x_0}^{x_0+h} f(x)\,\mathrm{d}x = \frac{h}{2}[f(x_0) + f(x_0 + h)].$$

Therefore, the error is given by

$$\begin{aligned}
E(x) &= F(x_0+h) - F(x_0) - \frac{h}{2}[f(x_0) + f(x_0 + h)] \\
&= F(x_0) + hf(x_0) + \frac{h^2}{2} f'(x_0) + \frac{h^3}{3!} f''(x_0) + \ldots - F(x_0) - \frac{h}{2}\left[f(x_0) + f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) \right. \\
&\quad \left. + \frac{h^3}{3!} f''(x_0) + \ldots \right] \\
&= \left[\frac{h^3}{3!} f''(x_0) - \frac{h^3}{4} f''(x_0) \right] - \ldots \\
&= -\frac{h^3}{12} f''(x_0) - \ldots.
\end{aligned}$$

Summing over n intervals, we get

$$E_n(x) \le -\frac{nh^3}{12} f''(x_m),$$

where $f''(x_m)$ denotes the greatest value of $f''(x_0), f''(x_1),..., f''(x_{n-1})$. Thus,

$$E_n(x) \le -\frac{(x_n - x_0)}{12} h^2 f''(x_m),$$

which shows that error is of order 2.

Simpson's Rule: Let f be a finite and continuous function in the interval $x = x_0 - h$ to $x = x_0 + h$ and have continuous derivatives of all orders up to and including the fourth in the said interval. Let

$$F(x) = \int_0^x f(t)\mathrm{d}t.$$

Then, by fundamental theorem of integral calculus

$$F'(x) = f(x) \text{ and so } F''(x) = f'(x),\ F'''(x) = f''(x)\ldots.$$

Also, by the same theorem,

$$\int_{x_0-h}^{x_0+h} f(x)\,\mathrm{d}x = F(x_0 + h) - F(x_0 - h).$$

But, by Simpson's rule,

$$\int_{x_0-h}^{x_0+h} f(x)\,\mathrm{d}x = \frac{h}{3}[f(x_0 - h) + 4f(x_0) + f(x_0 + h)].$$

Therefore, the error in Simpson's rule is given by

$$E(x) = F(x_0 + h) - F(x_0 - h) - \frac{h}{3}[f(x_0 - h) + 4f(x_0) + f(x_0 + h)].$$

By Taylor's expansion

$$F(x_0 + h) = F(x_0) + hf(x_0) + \frac{h^2}{2} f'(x_0) + \frac{h^3}{3!} f''(x_0) + \ldots$$

$$F(x_0 - h) = F(x_0) - hf(x_0) + \frac{h^2}{2} f'(x_0) - \frac{h^3}{3!} f''(x_0) + \ldots$$

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \ldots$$

$$f(x_0 - h) = f(x_0) - hf'(x_0) + \frac{h^2}{2} f''(x_0) - \frac{h^3}{3!} f'''(x_0) + \ldots.$$

Substituting these values in the error term, we get

$$\begin{aligned} E(x) &= \left[2hf(x_0) + \frac{2h^3}{3!} f''(x_0) + \frac{2h^5}{5!} f^{(\mathrm{iv})}(x_0) + \ldots \right] \\ &\quad - \frac{h}{3}\left[6f(x_0) + h^2 f''(x_0) + \frac{2h^4}{4!} f^{(\mathrm{iv})}(x_0) + \ldots \right] \\ &= -\frac{h^5}{90}[f^{(\mathrm{iv})}(x_0) + \ldots]. \end{aligned}$$

Summing over $\frac{n}{2}$ intervals, we get

$$E(x) = \frac{h^5}{90}[f^{(iv)}(x_0) + f^{(iv)}(x_2) + \ldots + f^{(iv)}(x_{n-2})].$$

If $f^{(iv)}(x_n)$ denotes the greatest value of any of the $\frac{n}{2}$ values $f^{(iv)}(x_0), f^{(iv)}(x_2), \ldots, f^{(iv)}(x_{n-2})$, then

$$E(x) \leq -\frac{nh^5}{180} f^{(iv)}(x_m) = -\frac{(x_n - x_0)}{180} h^4 f^{(iv)}(x_m),$$

since $x_n - x_0 = nh$. Hence, error in Simpson's rule is of order 4.

7.4 RICHARDSON EXTRAPOLATION (OR DEFERRED APPROACH TO THE LIMIT)

Knowing the order of the error, one can get fairly accurate estimate of the true value Q of the approximate values of the derivative or integral as soon as two approximate values Q_1 and Q_2 of Q have been obtained by means of different spacing, say h_1 and h_2. Thus, if the order of error is n, truncating the error series after its first term, we obtain

$$\left.\begin{aligned} Q - Q_1 &\approx Ch_1^n \\ Q - Q_2 &\approx Ch_2^n \end{aligned}\right\}, \tag{7.7}$$

where in differentiation formulae the constant C depends on the pivotal point at which the derivative is evaluated. From equation (7.7), we get

$$\frac{Q - Q_1}{h_1^n} = \frac{Q - Q_2}{h_2^n} \approx \text{C}$$

or

$$Q\left(\frac{1}{h_2^n} - \frac{1}{h_1^n}\right) = \frac{Q_2}{h_2^n} - \frac{Q_1}{h_1^n}$$

or

$$Q_{12} \approx \frac{\left(\frac{h_1}{h_2}\right)^n Q_2 - Q_1}{\left(\frac{h_1}{h_2}\right)^n - 1}, \tag{7.8}$$

which is called h^n extrapolation formula of Richardson. This formula gives the approximate value Q_{12} of Q.

We, generally, consider the cases where $\frac{h_1}{h_2} = 2$. In trapezoidal rule, order of error is 2. Therefore, the extrapolation formula with $\frac{h_1}{h_2} = 2$ becomes

$$Q_{12} \approx \frac{2^2 Q_2 - Q_1}{2^2 - 1} = \frac{4}{3}Q_2 - \frac{1}{3}Q_1,$$

which is called $\frac{1}{3}h^2$ extrapolation formula.

In Simpson's rule, the order of error is 4. Therefore, the extrapolation formula becomes $\frac{1}{15}h^4$ extrapolation formula given by

$$Q_{12} \approx \frac{16}{15}Q_2 - \frac{1}{15}Q_1 \text{ with error } O(h^4) \text{ and } \frac{h_1}{h_2} = 2.$$

In Boole's rule, the order of error is 6 and so the expression (7.8) yields the following $\frac{1}{63}h^6$ extrapolation formula:

$$Q_{12} \approx \frac{64}{63}Q_2 - \frac{1}{63}Q_1.$$

EXAMPLE 7.1

Dividing the range into 10 equal parts, apply Simpon's one-third rule to evaluate the integral $\int_0^5 \frac{dx}{4x+5}$ correct to four decimal places. Hence, find the approximate value of $\log_e 5$.

Solution. The values of the integrand for $h = \frac{1}{2}$ are.

x:	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2}$	5
$f(0)$:	$\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{9}$	$\frac{1}{11}$	$\frac{1}{13}$	$\frac{1}{15}$	$\frac{1}{17}$	$\frac{1}{19}$	$\frac{1}{21}$	$\frac{1}{23}$	$\frac{1}{25}$.

Therefore, by Simpson's one-third rule,

$$\int_0^5 \frac{dx}{4x+5} = \frac{h}{3}[f_0 + 4(f_1 + f_3 + f_5 + f_7 + f_9) + 2(f_2 + f_4 + f_6 + f_8) + f_{10}]$$

$$= \frac{1}{6}\left[\left(\frac{1}{5} + \frac{1}{25}\right) + 4\left(\frac{1}{7} + \frac{1}{11} + \frac{1}{15} + \frac{1}{19} + \frac{1}{23}\right) + 2\left(\frac{1}{9} + \frac{1}{13} + \frac{1}{17} + \frac{1}{21}\right)\right]$$

$$= \frac{1}{6}\left[\frac{6}{25} + 4(0.142857 + 0.09090 + 0.066666 + 0.05263 + 0.04348) + 2(0.11111 + 0.07692 + 0.05882 + 0.04761)\right]$$

$$= \frac{1}{6}[0.24 + 1.58613 + 0.58892] = 0.4025.$$

Also,

$$\int_0^5 \frac{dx}{4x+5} = \frac{1}{4}[\log(4x+5)]_0^5 = \frac{1}{4}[\log 25 - \log 5]$$

$$= \frac{1}{4}\left[\log\frac{25}{5}\right] = \frac{1}{4}\log_e 5.$$

Hence,

$$\log_e 5 = 4\int_0^5 \frac{dx}{4x+5} = 4(0.4025) = 1.61.$$

The actual value is 1.6094.

EXAMPLE 7.2

Calculate $\int_1^2 \frac{dx}{x}$ by

(i) Simpson's rule with $h = 0.50$,
(ii) Simpson's rule with $h = 0.25$,
(iii) Richardson's extrapolation.

Compare the results with exact value.

Solution. (i) With $h = 0.50$, we have

$$f_0 = 1,\ f_1 = \frac{1}{1.5} = \frac{2}{3},\ f_2 = \frac{1}{2}$$

and so by Simpson's rule

$$\int_1^2 \frac{dx}{x} \approx \frac{h}{3}[f_0 + 4f_1 + f_2] = \frac{0.5}{3}\left[1 + \frac{8}{3} + \frac{1}{2}\right] = \frac{12.5}{18} = 0.69444.$$

(ii) With $h = 0.25$, we have

$$f_0 = 1,\ f_1 = \frac{1}{1.25},\ f_2 = \frac{1}{1.50},\ f_3 = \frac{1}{1.75},\ f_4 = \frac{1}{2}.$$

Then, by Simpson's rule, we have

$$\int_1^2 \frac{dx}{x} \approx \frac{h}{3}[(f_0 + f_4) + 4(f_1 + f_3) + 2f_2]$$

$$= \frac{0.25}{3}[(1 + 0.5) + 4(0.8 + 0.5714) + 1.3333] = 0.69324.$$

(iii) We have

$$\frac{h_1}{h_2} = \frac{0.50}{0.25} = 2.$$

Also Simpson's rule is of order 4. Therefore,

$$Q_{12} \approx \frac{16}{15}(0.69324) - \frac{1}{15}(0.69444) = 0.69316,$$

which is in good agreement with the exact value

$$\log 2 - \log 1 = \log 2 = 0.69315.$$

7.5 SIMPSON'S FORMULA WITH END CORRECTION

We now improve usual Simpson's formula by allowing derivatives in the endpoints. Let

$$\int_{x_0-h}^{x_0+h} f(x)\mathrm{d}x \approx h[af_{-1} + bf_0 + af_1] + h^2[cf'_{-1} - cf'_1]\,. \tag{7.9}$$

Let

$$F(x) = \int f(x)\mathrm{d}x$$

and so

$$\int_{x_0-h}^{x_0+h} f(x)\mathrm{d}x = F(x_0+h) - F(x_0 - h).$$

Therefore formula (7.9) becomes

$$F(x_0 + h) - F(x_0 - h) \approx h[af_{-1} + bf_0 + af_1] + h^2[cf'_{-1} - cf'_1] \tag{7.10}$$

Expanding by Taylor's Theorem, we get

$$\begin{aligned} F(x_0 + h) - F(x_0 - h) &= \left[F(x_0) + hf(x_0) + \frac{h^2}{2} f'(x_0) + \frac{h^3}{3!} f''(x_0) + \ldots \right] \\ &= -\left[F(x_0) - hf(x_0) + \frac{h^2}{2} f'(x_0) - \frac{h^3}{3!} f''(x_0) + \ldots \right] \\ &= 2hf_0 + \frac{2h^3}{3!} f''_0 + \frac{2h^5}{5!} f'' + \ldots. \end{aligned}$$

Also,

$$\begin{aligned} f_{-1} &= f(x_0 - h) = f(x_0) - hf'_0 + \frac{h^2}{2!} f''_0 - \frac{h^3}{3!} f'''_0 + \frac{h^4}{4!} f_0^{(\mathrm{iv})} - \ldots \\ f_1 &= f(x_0 + h) = f(x_0) + hf_0 + \frac{h^2}{2!} f''_0 + \frac{h^3}{3!} f'''_0 + \frac{h^4}{4!} f_0^{(\mathrm{iv})} + \ldots \\ f'_{-1} &= f'_0 - hf''_0 + \frac{h^2}{2!} f'''_0 - \frac{h^3}{3!} f_0^{(\mathrm{iv})} + \ldots \\ f'_1 &= f'_0 + hf''_0 + \frac{h^2}{2!} f''_0 + \frac{h^3}{3!} f_0^{(\mathrm{iv})} + \ldots. \end{aligned}$$

Therefore, the right-hand side of equation (7.10) is

$$bhf_0 + ah\left[2f_0 + \frac{2h^2}{2!} f''_0 + \frac{2h^4}{4!} f_0^{(\mathrm{iv})} + \ldots \right] + ch^2 \left[-2hf''_0 - \frac{2h^3}{3!} f_0^{(\mathrm{iv})} - \ldots \right].$$

Comparing the coefficients of f_0, f''_0, and $f_0^{(\mathrm{iv})}$ and on both sides, we get

$$2a + b = 2,$$

$$a - 2c = \frac{1}{3},$$

$$a - 4c = \frac{1}{5},$$

which yield

$$a = \frac{7}{15},\ b = \frac{16}{15},\ \text{and}\ c = \frac{1}{15}.$$

Hence,

$$\int_{x_0-h}^{x_0+h} f(x)\mathrm{d}x \approx \frac{h}{15}[7f_{-1} + 16f_0 + 7f_1] + \frac{h^2}{15}[f'_{-1} - f'_1]$$

or, we can write

$$\int_{x_0}^{x_0+2h} f(x)\mathrm{d}x \approx \frac{h}{15}[7f_0 + 16f_1 + 7f_2] + \frac{h^2}{15}[f'_0 - f'_2].$$

Similarly,

$$\int_{x_2}^{x_4} f(x)\mathrm{d}x \approx \frac{h}{15}[7f_2 + 16f_3 + 7f_4] + \frac{h^2}{15}[f'_2 - f'_4]$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$\int_{x_{n-2}}^{x_n} f(x)\mathrm{d}x \approx \frac{h}{15}[7f_{n-2} + 16f_{n-1} + 7f_n] + \frac{h^2}{15}[f'_{n-2} - f'_n].$$

Adding, we get

$$\int_{x_0}^{x_n} f(x)\mathrm{d}x \approx \frac{h}{15}[7f_0 + 16f_1 + 14f_2 + 16f_3 + \cdots + 7f_n] + \frac{h^2}{15}[f'_0 - f'_n] + \mathrm{O}(h^6),$$

where $x_n - x_0 = nh$ and $f_0, f_1, ..., f_n$ are function values at the points $x_0, x_1, ..., x_n$.

7.6 ROMBERG'S METHOD

This method makes use of Richardson's extrapolation in a systematic way. We have seen that for trapezoidal rule, the error is $O(h^2)$ and with the spacing ratio $\frac{h_1}{h_2} = 2$ Richardson extrapolation yields

$$R = \frac{4}{3}Q_2 - \frac{1}{3}Q_1 = Q_2 + \frac{Q_2 - Q_1}{3}.$$

We observe that R is the same result as obtained by Simpson's rule.

Since error in R is of order 4 (Simpson's rule), taking again $\frac{h_1}{h_2} = 2$, we get

$$S = \frac{16}{15}R_2 - \frac{1}{15}R_1 = R_2 + \frac{R_2 - R_1}{15}.$$

As a matter of fact, S is the same result as obtained by Cote's formula for $n = 4$. Now, the error is of order 6. So taking $\frac{h_1}{h_2} = 2$, we obtain

$$T = \frac{64}{63}S_2 - \frac{1}{63}S_1 = S_2 + \frac{S_2 - S_1}{63}.$$

The process is repeated till two successive values are sufficiently close to each other.

EXAMPLE 7.3

One wants to construct a quadrature formula of the type

$$\int_0^h f(x)\,\mathrm{d}x = \frac{h}{2}(f_0 + f_1) + ah^2(f_0' - f_1') + R.$$

Calculate the constant a and find the order of the remainder term R.

Solution. Let $F(x) = \int f(x)\mathrm{d}x$. Then

$$\int_0^h f(x)\mathrm{d}x = F(h) - F(0)$$

$$= F(0) + hF'(0) + \frac{h^2}{2}F''(0) + \frac{h^3}{3!}F'''(0) + \frac{h^4}{4!}F^{(\text{iv})}(0)$$

$$+\frac{h^5}{5!}F^{(\text{v})}(0) + \ldots - F(0)$$

$$= hf'(0) + \frac{h^2}{2}f'(0) + \frac{h^3}{3!}f''(0) + \frac{h^4}{4!}f'''(0) + \frac{h^5}{5!}f^{(\text{iv})}(0) + \cdots$$

$$= hf_0 + \frac{h^2}{2}f_0' + \frac{h^3}{3!}f_0'' + \frac{h^4}{4!}f_0''' + \frac{h^5}{5!}f_0^{(\text{iv})} + \cdots. \tag{7.11}$$

Also,

$$f_1 = f_0 + hf_0' + \frac{h^2}{2!}f_0'' + \frac{h^3}{3!}f_0''' + \frac{h^4}{4!}f_0^{(\text{iv})} + \frac{h^5}{5!}f_0^{(\text{v})} + \ldots$$

$$f_1' = f_0' + hf_0'' + \frac{h^2}{2!}f_0'''' + \frac{h^3}{3!}f_0^{iv} + \frac{h^4}{4!}f_0^{(\text{v})} + \frac{h^5}{5!}f_0^{(\text{vi})} + \ldots.$$

Therefore,

$$\frac{h}{2}(f_0 + f_1) + ah^2(f_0' - f_1') = \frac{h}{2}\left[2f_0 + hf_0' + \frac{h^2}{2!}f_0'' + \frac{h^3}{3!}f_0''' + \frac{h^4}{4!}f_0^{(\text{iv})} + \frac{h^5}{5!}f_0^{(\text{v})} + \ldots\right]$$
$$+ah^2\left[-hf_0'' - \frac{h^2}{2}f_0''' - \frac{h^3}{3!}f_0^{(\text{iv})} - \frac{h^4}{4!}f_0^{(\text{v})} - \frac{h^5}{5!}f_0^{(\text{vi})} - \ldots\right] \tag{7.12}$$

Comparing the coefficients of f_0'' in equations (7.11) and (7.12), we obtain

$$\frac{h^3}{3!} = \frac{h^3}{4} - ah^3$$

and so $a = \dfrac{1}{12}$. Hence, the formula is

$$\int_0^h f(x)\mathrm{d}x = hf_0 + \frac{h^2}{2}f_0' + \frac{h^3}{3!}f_0'' + \frac{h^4}{4!}f_0''' + \frac{h^5}{5!}f_0^{(\text{iv})} + \ldots,$$

which clearly shows that R is of order h^5.

EXAMPLE 7.4

If $f(x) = a + bx + cx^2$, find the quadrature formulae of the form

$$\int_0^1 f(x)\mathrm{d}x \approx A_1 f(-1) + B_1 f(1) + C_1 f(2) \tag{7.13}$$

and

$$\int_0^1 f(x)\mathrm{d}x \approx A_2 f(0) + B_2 f(1) + C_2 f(2). \tag{7.14}$$

By finding out the truncation error in both the cases, point out which of the two formulae is more accurate.

Solution. The

$$\text{L.H.S. of equations (7.13) and (7.14)} = \int_0^1 (a + bx + cx^2)\mathrm{d}x = a + \frac{b}{2} + \frac{c}{3}.$$

Now,

$$\text{R.H.S. of equation (7.13)} = A_1(a - b + c) + B_1(a + b + c) + C_1(a + 2b + 4c).$$

Comparing coefficients on both sides, we get

$$A_1 + B_1 + C_1 = 1,$$

$$A_1 + B_1 + 2C_1 = \frac{1}{2},$$

$$3A_1 + 3B_1 + 12C_1 = 1.$$

Solving these equations, we get

$$A_1 = \frac{5}{36}, B_1 = \frac{13}{12}, \text{ and } C_1 = -\frac{2}{9}.$$

Therefore, the first formula is

$$\int_0^1 f(x)\mathrm{d}x = \frac{5}{36} f(-1) + \frac{13}{12} f(1) - \frac{2}{9} f(2) + R_1 \tag{7.15}$$

Further,

$$\text{R.H.S. of equation (7.14)} = A_1(a) + B_2(a + b + c) + C_2(a + 2b + 4c).$$

Comparing coefficient on both sides, we get

$$A_2 + B_2 + C_2 = 1,$$

$$B_2 + 2C_2 = \frac{1}{2},$$

$$B_2 + 4C_2 = \frac{1}{3}.$$

Solving these equations, we get

$$A_2 = \frac{5}{12},\ B_2 = \frac{2}{3},\ \text{and}\ C_2 = -\frac{1}{12}.$$

Therefore, the second formula is

$$\int_0^1 f(x)\,dx = \frac{5}{12} f(0) + \frac{2}{3} f(1) - \frac{1}{12} f(2) + R_2. \tag{7.16}$$

To find the error, we put $F(x) = \int f(x)dx$. Then the left-hand side of equation (7.15) is equal to

$$\begin{aligned} F(1) - F(0) &= F(0) + F'(0) + \frac{1}{2} F''(0) + \frac{1}{3!} F'''(0) + \frac{1}{4!} F^{(iv)}(0) + \frac{1}{5!} F^{(v)}(0) - F(0) \\ &= f(0) + \frac{1}{2} f'(0) + \frac{1}{3!} f''(0) + \frac{1}{4!} f'''(0) + \frac{1}{5!} f^{(v)}(0) + \ldots. \end{aligned} \tag{7.17}$$

Also,

$$f(1) = f(0) + f'(0) + \frac{1}{2!} f''(0) + \frac{1}{3!} f'''(0) + \frac{1}{4!} f^{(iv)}(0) + \frac{1}{5!} f^{(v)}(0) + \ldots$$

$$f(-1) = f(0) - f'(0) + \frac{1}{2!} f''(0) - \frac{1}{3!} f'''(0) + \frac{1}{4!} f^{(iv)}(0) - \frac{1}{5!} f^{(v)}(0) + \ldots$$

$$f(2) = f(0) + 2f'(0) + 2f''(0) + \frac{8}{3!} f'''(0) + \frac{16}{4!} f^{(iv)}(0) + \frac{32}{5!} f^{(v)}(0) + \ldots.$$

Putting these values in equation (7.15), the right-hand side becomes

$$f(0) + \frac{1}{2} f'(0) + \frac{1}{3!} f''(0) - \frac{30}{216} f'''(0) - \ldots$$

Therefore, the order of the remainder term is $Cf'''(0)$, where

$$C = \frac{1}{24} + \frac{30}{216} = \frac{13}{72}.$$

Similarly, the right-hand side of equation (7.16) is equal to

$$f(0) + \frac{1}{2} f'(0) + \frac{1}{6} f''(0) + 0 + \left(-\frac{1}{36}\right) f^{(iv)}(0) + \ldots$$

Therefore, the order of the remainder is $\frac{1}{24} f'''(0)$. Comparing the two errors, we observe that the second formula is better.

EXAMPLE 7.5

Using method of undetermined coefficients, derive Simpson's one-third rule.

Solution. Let

$$\int_{x_0-h}^{x_0+h} f(x)\,dx = af_{-1} + bf_0 + cf_1, \tag{7.18}$$

where the coefficients a, b, c are to be determined. Put $F(x) = \int f(x)\,dx$. Then the left-hand side of equation (7.18) is equal to

$$F(x_0+h)-F(x_0-h)=F(x_0)+hF'(x_0)+\frac{h^2}{2!}F''(x_0)-\frac{h^3}{3!}F'''(x_0)+\frac{h^4}{4!}F^{(\text{iv})}(x_0)+\ldots$$

$$-\{F(x_0)-hF'(x_0)+\frac{h^2}{2!}F''(x_0)-\frac{h^3}{3!}F'''(x_0)+\frac{h^4}{4!}F^{(\text{iv})}(x_0)+\ldots\}$$

$$=2hF'(x_0)+2\frac{h^3}{3!}F'''(x_0)+2\frac{h^5}{5!}F^{(\text{v})}(x_0)+\ldots$$

$$=2hf(x_0)+\frac{2h^3}{6}f''(x_0)+\frac{2h^5}{5!}f^{(\text{iv})}(x_0).$$

Also,

$$f_{-1}=f(x_0-h)=f(x_0)-hf'(x_0)+\frac{h^2}{2!}f''(x_0)-\frac{h^3}{3!}f'''(x_0)+\frac{h^4}{4!}f^{(\text{iv})}(x_0)-\frac{h^5}{5!}f^{(\text{v})}(x_0)+\ldots$$

$$f_1=f(x_0+h)=f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\frac{h^3}{3!}f'''(x_0)+\frac{h^4}{4!}f^{(\text{iv})}(x_0)+\frac{h^5}{5!}f^{(\text{v})}(x_0)+\ldots.$$

Therefore the right-hand side of equation (7.18) is equal to

$$a[f(x_0)-hf'(x_0)+\frac{h^2}{2!}f''(x_0)-\frac{h^3}{3!}f'''(x_0)+\frac{h^4}{4!}f^{(\text{iv})}(x_0)-\frac{h^5}{5!}f^{(\text{v})}(x_0)+\ldots]$$

$$+bf(x_0)+c[f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\frac{h^3}{3!}f'''(x_0)+\ldots]$$

Comparing coefficients of $f(x_0), f'(x), f''(x_0), f^{(\text{iv})}(x_0)$, we get

$$2h=a+b+c,\quad 0=-ah+ch,\quad \frac{2h^3}{6}=\frac{ah^2}{2}+\frac{ch^2}{2},$$

which yield $c=a=\frac{h}{3}$ and $b=\frac{4h}{3}$. Hence,

$$\int_{x_0-h}^{x_0+h} f(x)\,\mathrm{d}x=\frac{h}{3}f_{-1}+\frac{4h}{3}f_0+\frac{h}{3}f_1=\frac{h}{3}[f_{-1}+4f_0+f_1],$$

which is Simpson's one-third rule.

EXAMPLE 7.6

The integral equation

$$y(x)=1+\int_0^x f(t)y(t)\mathrm{d}t,$$

where f is a given function, can be solved by forming a sequence of functions $y_0, y_1, y_2,\ldots$ according to

$$y_{n+1}(x)=1+\int_0^x f(t)y_n(t)\mathrm{d}t.$$

Find the first five functions for $x = 0, 0.25, 0.50$, when $f(x)$ is given in the table below. Start with $y_0 = 1$ and use Bessel's interpolation formula and Simpson's rule.

x	0	0.25	0.50	0.75	1
f	0.5000	0.4794	0.4594	0.4398	0.4207

Solution. Bessel's quadrature formula reads

$$\int_{x_0}^{x_1} f(x)\mathrm{d}x = h\left[\frac{f_0+f_1}{2} - \frac{1}{12}\left(\frac{\delta^2 f_0 + \delta^2 f_1}{2}\right) + \frac{11}{720}\left(\frac{\delta^4 f_0 + \delta^4 f_1}{2}\right) + \ldots\right]$$

and Simpson's formula reads

$$\int_0^{x_2} f(x)\ \mathrm{d}x \approx \frac{h}{3}(f_0 + 4f_1 + f_2).$$

We have

$$y_{n+1}(x) = 1 + \int_0^x f(t)y_n(t)\mathrm{d}t$$

and we start with $y_0 = 1$. For $x = 0$, we have clearly

$$y_1 = 1,\ y_2 = 1,\ y_3 = 1,\ y_4 = 1,\ y_5 = 1.$$

Now for $x = 0.25$, we use Bessel's formula and get

$$y_1 = 1 + \int_0^{0.25} f(t)y_0(t)\ \mathrm{d}t = 1 + h\left[\frac{f_0+f_1}{2}\right],$$

since $y_0(t) = 1$ and higher-order differences contribution is very small. Thus,

$$y_1 = 1 + 0.25\left[\frac{0.5+0.4794}{2}\right] = 0.1224,$$

$$y_2 = 1 + \int_0^{0.25} f(t)y_1(t)\mathrm{d}t = 1 + h\left[\frac{f_0(t)y_1(0) + f_1(t)y_1(0.25)}{2}\right]$$

$$= 1 + 0.25\left[\frac{0.5+1.1224(0.4794)}{2}\right] = 1.1298,$$

$$y_3 = 1 + \int_0^{0.25} f(t)y_2(t)\mathrm{d}t = 1 + h\left[\frac{f_0(t)y_2(0) + f_1(t)y_2(0.25)}{2}\right]$$

$$= 1 + 0.25\left[\frac{0.5+1.1298(0.4794)}{2}\right] = 1.1302,$$

$$y_4 = 1 + \int_0^{0.25} f(t)y_3(t)\mathrm{d}t = 1 + h\left[\frac{f_0(t)y_3(0) + f_1(t)y_3(0.25)}{2}\right]$$

$$= 1 + 0.25\left[\frac{0.5+1.1302(0.4794)}{2}\right] = 1.1302$$

$$y_5 = 1 + \int_0^{0.25} f(t)y_4(t)\mathrm{d}t = 1 + h\left[\frac{f_0(t)y_4(0) + f_1(t)y_4(0.25)}{2}\right],$$

$$= 1 + 0.25\left[\frac{0.5+1.1302(0.4794)}{2}\right] = 1.1302.$$

Now for $x = 0.50$, $h = 0.25$, we use Simpson's rule

$$y_1 = 1 + \int_0^{0.50} f(t)y_0(t)\mathrm{d}t = 1 + \frac{0.25}{3}[0.5 + 4(0.4794) + 0.4594] = 1.2398,$$

$$y_2 = 1 + \int_0^{0.50} f(t)y_1(t)\mathrm{d}t = 1 + \frac{h}{3}[f_0(t)\,y_1(0) + 4f_1(t)\,y_1(0.25) + f_2(t)\,y_1(0.50)]$$

$$= 1 + \frac{0.25}{3}[0.5(1) + 4(0.4794)(1.1224) + 0.4594\,(1.2398)] = 1.2685,$$

$$y_3 = 1 + \int_0^{0.50} f(t)y_2(t)\mathrm{d}t = 1 + \frac{h}{3}[f_0(t)y_2(0) + 4f_1(t)\,y_2(0.25) + f_2(t)\,y_2(0.50)]$$

$$= 1 + \frac{0.25}{3}[\,0.5(1) + 4(0.4794)(1.1298) + 0.4594(1.2685)] = 1.2708,$$

$$y_4 = 1 + \int_0^{0.50} f(t)y_3(t)\mathrm{d}t = 1 + \frac{h}{3}[f_0(t)\,y_3(0) + 4f_1(t)\,y_3(0.25) + f_2(t)\,y_3(0.50)]$$

$$= 1 + \frac{0.25}{3}[0.5(1) + 4(0.4794)(1.1302) + 0.4594(1.2708)] = 1.2709,$$

$$y_5 = 1 + \int_0^{0.50} f(t)y_4(t)\mathrm{d}t = 1 + \frac{h}{3}[f_0(t)\,y_4(0) + 4f_1(t)\,y_4(0.25) + f_2(t)\,y_4(0.50)]$$

$$= 1 + \frac{0.25}{3}[0.5(1) + 4(0.4794)(1.1302) + 0.4594(1.2709)] = 1.2709.$$

EXAMPLE 7.7

The prime number theorem states that the number of primes in the interval $a < x < b$ is approximately $\int_a^b \frac{\mathrm{d}x}{\log x}$. Use this for $a = 100$ and $b = 200$ and compare with the exact value.

Solution. We know that

$$\log_e x = (\log_{10} x)\log_e 10 = (2.302585)\log_{10} x.$$

Therefore,

$$\int_{100}^{200} \frac{\mathrm{d}x}{\log x} = \int_{100}^{200} \frac{\mathrm{d}x}{(2.3025)\log_{10} x}$$

We have the following table:

x	100	150	200
f	$\frac{1}{2(2.302585)}$	$\frac{1}{2.1760(2.302585)}$	$\frac{1}{2.3010(2.302585)}$

Here $h = 50$. We use Simpson's rule and get

$$\int_{100}^{200} \frac{\mathrm{d}x}{\log x} = \frac{h}{3}(f_0 + 4f_1 + f_2)$$

$$= \frac{50}{3}\left(\frac{1}{4.60517} + \frac{4}{5.0104} + \frac{1}{5.282}\right)$$

$$= 16.6667\,(0.2171 + 0.7983 + 0.1887) = 20.068.$$

If $h = 25$, then the table is

x	100	125	150	175	200
f	0.2171	0.2071	0.1996	0.1936	0.1887

and therefore Simpson's formula now yields

$$\int_{100}^{200} \frac{dx}{\log x} = \frac{h}{3}[(f_0 + f_4) + 4(f_1 + f_3) + 2f_2]$$

$$= \frac{25}{3}[0.4058 + 4(0.4007) + 2(0.1996)] = 20.065.$$

The exact number of primes between 100 and 200 is 21.

EXAMPLE 7.8

Use Romberg's method to compute

$$\int_0^1 \frac{1}{1+x}\, dx$$

correct to four decimal places and hence find the value of $\log_e 2$.

Solution. Let $h = 0.5$. Then the values of the integrand $f(x) = \frac{1}{1+x}$ are

x	0	0.5	1.0
$f(x)$	1	0.6667	0.5

Therefore, by trapezoidal rule, we have

$$Q_1 = \int_0^1 \frac{1}{1+x}\, dx = \frac{0.5}{2}[1 + 2(0.6667) + 0.5] = 0.70835 \approx 0.7084.$$

Now, let $h = 0.25$. Then the values of the integrand are

x	0	0.25	0.5	0.75	1.0
$f(x)$	1	0.8	0.6667	0.5714	0.5

Therefore,

$$Q_2 = \frac{0.25}{2}[1 + 2(0.8 + 0.6667 + 0.5714) + 0.5] = 0.6970.$$

Then

$$R_1 = \frac{4}{3}Q_2 - \frac{1}{3}Q_1 = 0.9293 - 0.236 = 0.6932.$$

Now, let $h = 0.125$. Then the values of the integrand are

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1.0
$f(x)$	1	0.8889	0.8	0.7272	0.6667	0.6154	0.5714	0.5333	0.5

Then

$$R_2 = \frac{0.125}{3}[1 + 4(0.8889 + 0.7272 + 0.6154 + 0.5333) + 2(0.8 + 0.6667 + 0.5714) + 0.5]$$
$$= \frac{0.125}{3}[1 + 11.0592 + 4.0762 + 0.5] = 0.6931.$$

Then

$$S = \frac{16}{15}R_2 - \frac{1}{15}R_1 = 0.7393 - 0.0462 = 0.6931.$$

Also,

$$\int_0^1 \frac{dx}{1+x} = \left[\log(1+x)\right]_0^1 = \log_e 2.$$

Hence,

$$\log_e 2 \approx 0.6931.$$

EXAMPLE 7.9

Use Romberg's method to compute

$$\int_4^{5.2} \log x \, dx$$

from the data

x	4	4.2	4.4	4.6	4.8	5.0	5.2
$\log_e x$	1.3863	1.4351	1.4816	1.526	1.5686	1.6094	1.6486

Solution. Let $h = 0.4$. Then by trapezoidal rule,

$$Q_1 = \frac{0.4}{2}[1.3863 + 2(1.4816 + 1.5686) + 1.6486] = 1.8271.$$

Now, let $h = 0.2$. Then, again by trapezoidal rule,

$$Q_2 = \frac{0.2}{2}[1.3863 + 2(1.4356 + 1.4816 + 1.526 + 1.5686 + 1.6094) + 1.6484] = 1.8237.$$

Then

$$R_1 = \frac{4}{3}Q_2 - \frac{1}{3}Q_1 = 2.4316 - 0.6090 = 1.8226.$$

EXAMPLE 7.10

Use Romberg's method to compute $\int_0^1 \frac{dx}{1+x^2}$ correct to four decimal places.

Solution. Let $h = 0.5$. The values of the integrand $f(x) = \frac{1}{1+x^2}$ are

x:	0	0.5	1
$f(x)$:	1	0.8	0.5

Therefore, by trapezoidal rule, we have

$$Q_1 = \int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}[f_0 + 2f_1 + f_2] = \frac{1}{4}[1 + 2(0.8) + 0.5] = 0.775.$$

Now, let $h = 0.25$. Then, the values of the integrand are

x:	0	0.25	0.5	0.75	1.0
$f(x)$:	1	0.9412	0.8	0.64	0.5

Therefore, by trapezoidal rule,

$$Q_2 = \int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}[f_0 + 2(f_1 + f_2 + f_3) + f_4]$$

$$= \frac{1}{8}[1 + 2(0.9412 + 0.8 + 0.64) + 0.5] = 0.7828.$$

Then, by Romberg's method, we get

$$R_1 = \frac{4}{3}Q_2 - \frac{1}{3}Q_1 = \frac{4}{3}(0.7828) - \frac{1}{3}(0.775) = 0.7854.$$

Now, let $h = 0.125$. Then the values of the integrand are

x:	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1.0
$f(x)$:	1	0.9846	0.9412	0.8767	0.8	0.7191	0.64	0.5664	0.5

Then, by Simpson's one-third rule, we have

$$R_2 = \frac{0.125}{3}[1.4(0.9846 + 0.8767 + 0.7191 + 0.5664 + 2(0.9412 + 0.8 + 0.64) + 0.5]$$

$$= \frac{0.125}{3}[1 + 12.5872 + 4.7624 + 0.5] = 0.7854.$$

Therefore, by Romberg's method,

$$S = \frac{16}{15}R_2 - \frac{1}{15}R_1 = \frac{16(0.7854)}{15} - \frac{1}{15}(0.7854)$$

$$= 0.83776 - 0.05236 = 0.7854.$$

EXAMPLE 7.11

Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using

(i) Trapezoidal rule taking $h = \frac{1}{4}$,

(ii) Simpson's one-third rule taking $h = \frac{1}{4}$,

(iii) Simpson's three-eight rule taking $h = \frac{1}{6}$,

(iv) Weddle's rule taking $h = \frac{1}{6}$.

Solution. The value of $f(x) = \frac{1}{1+x^2}$ for first two cases are

x:	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
$f(x)$:	1	0.9412	0.8000	0.6400	0.5000

Case (i): By trapezoidal rule, we have

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}[f_0 + 2(f_1 + f_2 + f_3) + f_4]$$
$$= \frac{1}{8}[1 + 2(0.9412 + 0.8000 + 0.6400) + 0.5000]$$
$$= 0.7828.$$

Case (ii): Using Simpson's one-third rule, we have

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{3}[f_0 + 4(f_1 + f_3) + 2f_2 + f_4]$$
$$= \frac{1}{12}[1 + 4\,(0.9412 + 0.6400) + 2(0.8000) + 0.5000]$$
$$= 0.7854.$$

The values of $f(x)$ for the cases (iii) and (iv) are:

x:	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
$f(x)$:	1	0.9730	0.9000	0.8000	0.6923	0.5902	0.5000

Case (iii): By Simpson's three-eight rule, we have

$$\int_0^1 \frac{dx}{1+x^2} = \frac{3h}{8}[(f_0 + f_6) + 3(f_1 + f_2 + f_4 + f_5) + 2f_3]$$
$$= \frac{1}{16}[(1 + 0.5) + 3(0.9730 + 0.9000 + 0.6923 + 0.5902) + 2(0.8)]$$
$$= 0.78541.$$

Case (iv): By Weddle's rule, we have

$$\int_0^1 \frac{dx}{1+x^2} = \frac{3h}{10}[f_0 + 5f_1 + f_2 + 6f_3 + f_4 + 5f_5 + f_6]$$
$$= \frac{1}{20}[1 + 5(0.9730) + 0.9000 + 6(0.8) + 0.6923 + 5(0.5902) + 0.5000]$$
$$= 0.78542.$$

EXERCISES

1. Using Simpson's rule, find the volume of the solid of revolution formed by rotating about x-axis the area between the x-axis, the lines $x = 0$ and $x = 1$ and a curve through the points (0, 1), (0.25, 0.9896), (0.50, 0.9589), (0.75, 0.9089), and (1, 0.8415).

Hint:
$$\text{Volume} = \int_0^1 \pi y^2 \mathrm{d}x = \pi \int_0^1 y^2 \mathrm{d}x$$
$$= \pi \frac{h}{3}[y_0^2 + 4(y_1^2 + y_3^2) + 2y_2^2 + y_4^2]$$

Ans. 2.8192

2. Find the approximate value of

$$\int_0^{\pi/2} \sqrt{\cos\theta}\,\mathrm{d}\theta$$

by dividing the interval into six parts.

Ans. 1.1873

3. Evaluate

$$\int_1^2 \frac{\mathrm{d}x}{x}$$

by Simpson's rule and compare the approximate value obtained with the exact solution

Ans. 0.6932
Exact value: $\log_2 2 = 0.693147$

4. Evaluate

$$\int_0^{\pi/2} \sin x \mathrm{d}x$$

by Simpson's one-third rule using 11 ordinates.

Ans. 0.9985

5. The velocity v of a particle at distance s from a point on its path is given by the table:

s ft.:	0	10	20	30	40	50	60
v ft/sec.:	47	58	64	65	61	52	38

Using Simpson's one-third rule, determine the time taken by the particle to travel 60 ft.

Hint: $v = \frac{\mathrm{d}s}{\mathrm{d}t}$ and so $\mathrm{d}t = \frac{1}{v}\mathrm{d}s$. So find $\int_0^{60} \frac{1}{v}\mathrm{d}s$.

Ans. 1.063 sec

6. For the case of six known ordinates, show that

$$\int_0^5 f(x)\,\mathrm{d}x = \frac{5}{288}[19(f_0 + f_5) + 75\,(f_1 + f_4) + 50(f_2 + f_3)]$$

7. The velocity v km/min of a moped started from rest is given at fixed intervals of time t(minutes) as follows:

t:	2	4	6	8	10	12	14	16	18	20
v:	10	18	25	29	32	20	11	5	2	0

Using Simpson's rule, find the distance covered in 20 minutes.

Hint: $v = \frac{ds}{dt}$ and so $ds = v dt$. So find $\int_0^{20} v dt$. Take interval length equal to 2 and use Simpson's formula.

Ans. 309.33 km

8. Obtain an estimate of the number of subintervals that should be chosen so as to guarantee that the error committed in evaluating $\int_1^2 \frac{1}{x} dx$ by trapezoidal rule is less than 0.001.

   Hint: $E_n(x) < -\frac{nh^3}{12} f''(\xi)$,

   Ans. $n = 8$

9. Compute the value of

$$\int_0^1 \frac{dx}{1+x^2}$$

   using trapezoidal rule with $h = 0.5$, 0.25, and 0.125. Then use Romberg's method to get better approximation. Compare the result obtained with the true value.

   Ans. 0.77500, 0.78279, 0.78475, 0.7854

10. Use Euler–Maclaurin formula to find the value of log 2 from $\int_0^1 \frac{dx}{1+x}$.

    Hint: $\int_0^1 \frac{dx}{1+x} = \left[\log_e(1+x)\right]_0^1 = \log_e 2$. So find $\int_0^1 \frac{dx}{1+x}$ by Euler–Maclaurin formula.

    Ans. 0.693149

11. Calculate by Simpson's rule an approximate value of $\int_{-3}^{3} x^4 dx$ by taking seven equidistant ordinates. Compare it with exact value and the estimate obtained by using trapezoidal rule.

    Ans. by Simpson's rule: 98
    Exact value: 97.2
    by trapezoidal rule: 115
    So Simpson's rule yields better results

12. Calculate $\int_2^{10} \frac{dx}{1+x}$ by dividing the range into eight equal parts.

    Ans. 1.299

13. If $e^0 = 1$, $e^1 = 2.72$, $e^2 = 7.39$, $e^3 = 20.09$, $e^4 = 54.60$, find $\int_0^4 e^x$ by Simpson's rule.

    Ans. 2.97049

14. A river is 80 feet wide. The depth d(in feet) of the river at a distance x from one bank is given by the following table:

x:	0	10	20	30	40	50	60	70	80
d:	0	4	7	9	12	15	14	8	3

Find approximately the area of the cross-section of the river.
Hint: Since $A = \int y\,dx$ and $h = 10$ we have by Simpson's rule,

$$A = \frac{10}{3}[(0+3) + 4(4+9+15+8) + 2(7+12+14)] = 710 \text{ sq. feet.}$$

15. Show that

$$\int_{-1}^{1} f(x)\,dx = \frac{13}{12}[f(1) + f(-1) - f(3) - f(-3)].$$

8 Ordinary Differential Equations

An ordinary differential equation is an equation containing one independent variable and one dependent variable and at least one of its derivatives with respect to the independent variable. We know that a differential equation of nth order has n independent arbitrary constants in its general solution. Therefore, we need n conditions to compute the numerical solution of an nth order differential equation.

8.1 INITIAL VALUE PROBLEMS AND BOUNDARY VALUE PROBLEMS

Problems in which all the initial conditions are specified at the initial point only are called initial value problems or marching problems. Thus, in an initial value problem, all the auxiliary conditions are specified at a point, for example, value of $y, y', \ldots, y^{(n-1)}$ at the point x_0.

As an illustration, we note that the equation

$$y' = x - y^2,\ y(0) = 1$$

is an initial value problem.

Problems involving second and higher order differential equations in which auxiliary conditions are specified at two or more points are called boundary value problems or jury problems.

As an illustration, we note that the equation

$$y'' = xy, y(0) = 0, y(2) = 1$$

is a boundary value problem.

8.2 CLASSIFICATION OF METHODS OF SOLUTION

Consider first order differential equation $y' = f(x, y)$. Let $x_n = x_0 + nh$ and let y_n be the value of y obtained from a particular method. If the value y_{n+1} appears as a function of just one y-value y_n, then the method is called a single-step method. On the other hand, if the value y_{n+1} appears as a function of several values $y_n, y_{n-1}, \ldots, y_{n-p}$, then the method is called a multistep method. Thus, a single-step method is a method that requires only one preceeding value of y, while a multistep method requires two or more preceeding values of y.

8.3 SINGLE-STEP METHODS

1. Taylor Series Method

Let $f(x, y)$ be a function that is differentiable for sufficient number of times and let

$$\frac{dy}{dx} = y' = f(x, y), \quad y(x_0) = y_0 \tag{8.1}$$

be the initial value problem. We expand $y(x)$ into Taylor series about the point x_0. Thus,

$$y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \cdots + \frac{h^p}{p!} y_0^{(p)} + \frac{h^{p+1}}{(p+1)!} y^{(p+1)}(\xi), \tag{8.2}$$

where ξ is a point in $[x_0,x]$. Since the solution is not known, the derivatives in the expansion are not known. However, they can be obtained by taking total derivative of the differential equation (8.1). Therefore,

$$y' = f(x,y),$$

$$y'' = f_x + f_y y' = f_x + ff_y,$$

$$y''' = f_{xx} + f_{xy}f + f_{yx}f + f_{yy}f^2 + f_y f_x + f_y^2 f$$

$$= f_{xx} + 2f_{xy}f + f^2 f_{yy} + f_y^2 f,$$

and so on.

The number of terms to be included in equation (8.2) is fixed by permissible error. If the permissible error is ε and the series in equation (8.2) is truncated after the term in $y^{(p)}$, then we have

$$\frac{h^{p+1}}{(p+1)!} | y^{(p+1)}(\xi) | < \varepsilon$$

or

$$\frac{h^{p+1}}{(p+1)!} | f^p(\xi) | < \varepsilon.$$

For a given h, we can find p and obtain an upper bound on h. For computational purposes $| f^p(\xi) |$ is replaced by $\max | f^p(\xi_n) |$ in $[x_0, x_n]$.

Advantages:

(i) A large interval can be used by increasing the number of terms.

(ii) No special starting procedure is required.

(iii) The values computed can be checked by applying Taylor's expansion equally on either side of the point x_n. Thus corresponding to y_{n+1}, we may also compute y_{n-1} from the series

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2!} y''_n + \frac{h^3}{3!} y'''_n + \ldots,$$

$$y_{n-1} = y_n - hy'_n + \frac{h^2}{2!} y''_n - \frac{h^3}{3!} y'''_n + \ldots.$$

Disadvantages:

(i) The necessity of calculating the higher derivatives makes this method completely unsuitable on high-speed computers.

(ii) The method is labourious and so is not recommended except for a few equations.

EXAMPLE 8.1

Solve by Taylor series method:

$$y' = y - \frac{2x}{y}, y(0) = 1, \text{for } x = 0.1 \text{ and } x = -0.1.$$

Solution. The given equation is

$$y' = y - \frac{2x}{y},\ y(0) = 1.$$

Therefore,

$$y'' = \frac{y(2yy' - 2) - (y^2 - 2x)y'}{y^2} = \frac{2yy' - 2 - y'^2}{y},$$

$$y''' = \frac{2yy'' - 3y'y'' + 2y'^2}{y},$$

$$\cdots\cdots\cdots\cdots$$

so that

$$y'(0) = y(0) - \frac{2(0)}{y(0)} = y(0) = 1,$$

$$y''(0) = \frac{2y(0)y'(0) - y'^2(0) - 2}{y(0)} = \frac{2 - 1 - 2}{1} = -1,$$

$$y'''(0) = \frac{2y(0)y''(0) - 3y'(0)y''(0) + 2y'^2(0)}{y(0)},$$

$$= \frac{2(1)(-1) - 3(1)(-1) + 2}{1} = 3,$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

Therefore,

$$y(0.1) = y(0) + (0.1)y'(0) + \frac{(0.1)^2}{2!}y''(0) + \frac{(0.1)^3}{3!}y'''(0) + \cdots,$$

$$= 1 + 0.1 + \frac{0.01}{2}(-1) + \frac{0.001}{3!}(3) + \cdots = 1.0955.$$

Similarly,

$$y(-0.1) = y(0) - (0.1)y'(0) + \frac{(0.1)^2}{2!}y''(0) - \frac{(0.1)^3}{3!}y'''(0) + \cdots,$$

$$= 1 - 0.1 + \frac{0.01}{2}(-1) + \frac{0.001}{6}(3) + \cdots = 0.8955.$$

EXAMPLE 8.2

Solve the differential equation $y' = x - y^2$, by series expansion, for $x = 0.2(0.2)1$ under the initial condition $y(0) = 1$.

Solution. We have

$$y' = x - y^2,$$

$$y'' = 1 - 2yy' = 1 - 2y(x - y^2) = 1 - 2xy + 2y^3,$$

$$y''' = -2yy'' - 2y'^2 = -2(y - 4xy^2 + 3y^4 + x^2),$$

$$y^{(iv)} = -2yy''' - 2y'y'' - 4y'y'' = -2yy''' - 6y'y'',$$

$$\cdots\cdots\cdots$$

$$\cdots\cdots\cdots$$

Using the initial condition $y(0) = 1$, we get

$$y'(0) = 0 - (y(0))^2 = -1,$$
$$y''(0) = 1 - 2y(0)y'(0) = 1 - 2(1)(-1) = 3,$$
$$y'''(0) = -2y(0)y''(0) - 2y'^2 = 2(1)(3) - 2(-1)^2 = -8,$$
$$y^{(iv)}(0) = -2y(0)y'''(0) - 6y'(0)y''(0) = -2(1)(-8) - 6(-1)(3) = 34.$$

Therefore,

$$y_1 = y(0.2) \approx y(0) + 0.2y'(0) + \frac{(0.2)^2}{2!}y''(0) + \frac{(0.2)^3}{3!}y'''(0) + \frac{(0.2)^4}{4!}y^{(iv)}(0) + \ldots,$$
$$= 1 - 0.2 + 0.06 - 0.01066 + 0.002266 = 0.8516.$$

Now

$$y_2 = y(0.4) = y_1 + 0.2y_1' + \frac{(0.2)^2}{2!}y_1'' + \frac{(0.2)^3}{3!}y_1''' + \frac{(0.2)^4}{4!}y_1^{(iv)} + \cdots$$

But

$$y_1' = x_1 - y_1^2 = 0.2 - (0.8516)^2 = -0.5252,$$
$$y_1'' = 1 - 2y_1y_1' = 1 - 2(0.8516)(-0.5252) = 1.8945,$$
$$y_1''' = -2y_1y_1'' - 2y_1'^2$$
$$= -2(0.8516)\ (1.8945) - 2(0.5252)^2$$
$$= -3.2267 - 0.5517 = -3.7784,$$
$$y_1^{(iv)} = -2y_1y_1''' - 6y'y_1''$$
$$= -2(0.8516)(-3.7784) - 6(-0.5252)(1.8945)$$
$$= 6.43537 + 5.96995 = 12.40532.$$

Therefore,

$$y_2 \approx 0.8516 + 0.2(-0.5252) + \frac{0.04}{2}(1.8945) + \frac{0.008}{6}(-3.7784) + \frac{0.0016}{24}(12.40532)$$
$$= 0.8516 - 0.10504 + 0.03789 - 0.00504 + 0.000827 = 0.7802.$$

Similarly, we can calculate $y(0.6)$, $y(0.8)$, and $y(1)$.

EXAMPLE 8.3

Solve the differential equation $y'' = xy$ for $x = 0.5$ and $x = 1$ by Taylor series method. Initial values: $x = 0$, $y = 0$, $y' = 1$.

Solution. We have

$$y'' = xy,$$
$$y''' = xy' + y,$$
$$y^{(iv)} = xy'' + y' + y' = xy'' + 2y',$$
$$y^{(v)} = xy''' + y'' + 2y'' = xy''' + 3y''.$$

Initial conditions are $y(0) = 0$, $y'(0) = 1$. Further,

$$y''(0) = 0,$$
$$y'''(0) = 0 + y(0) = 0,$$
$$y^{(iv)}(0) = 0 + 2y'(0) = 2(1) = 2,$$
$$y^{(v)}(0) = 0 + 3y''(0) = 0.$$

Hence,

$$y_1 = y(0.5) = y(0) + 0.5y'(0) + \frac{(0.5)^2}{2!}y''(0) + \frac{(0.5)^3}{3!}y'''(0) + \frac{(0.5)^4}{4!}y^{(iv)}(0) + \frac{(0.5)^5}{5!}y^{(v)}(0) + \cdots$$

$$= 0 + 0.5(1) + \frac{0.0625}{24}(2) = 0.5 + 0.00521 = 0.50521.$$

Now we find $y_2 = y(1)$. We have

$$y_2 = y_1 + 0.5y_1' + \frac{(0.5)^2}{2!}y_1'' + \frac{(0.5)^3}{3!}y_1''' + \frac{(0.5)^4}{4!}y_1^{(iv)} + \cdots$$

But

$$y_1' = y_0' + hy_0'' + \frac{h^2}{2!}y_0''' + \frac{h^2}{3!}y_0^{(iv)} + \cdots$$

$$= 1 + 0.5(0) + \frac{0.25}{2}(0) + \frac{0.125}{6}(2) + \cdots$$

$$= 1.04167.$$

$$y_1'' = x_1y_1 = 0.5(0.50521) = 0.25261.$$

$$y_1''' = x_1y_1' + y_1 = 0.5(1.04167) + 0.50521 = 1.02604.$$

$$y_1^{(iv)} = x_1y_1'' + 2y_1' = 0.5(0.25261) + 2(1.04167) = 2.2096.$$

Hence,

$$y_2 = y(1) = 0.50521 + 0.5(1.04167) + \frac{0.25}{2}(0.25261) + \frac{0.125}{6}(1.02604) + \frac{0.0625}{24}(2.2096) + \cdots$$

$$\approx 0.50521 + 0.52084 + 0.03157 + 0.021376 = 0.00575 = 1.08475.$$

2. Euler's Method

Consider the initial value problem

$$y' = \frac{dy}{dx} = f(x, y), y(x_0) = y_0. \tag{8.3}$$

The Euler method is based on the property that in a small interval, a curve is nearly a straight line. Thus, if $x \in [x_0, x_1]$, a small interval, we approximate the curve by the tangent at the point (x_0, y_0). But the equation of the tangent at (x_0, y_0) is

$$y - y_0 = \left(\frac{dy}{dx}\right)_{(x_0, y_0)}(x - x_0) = f(x_0, y_0)(x - x_0), \qquad \text{using equation (8.3)}$$

or

$$y = y_0 + (x - x_0)f(x_0, y_0).$$

Therefore, the value of y corresponding to x_1 is

$$y_1 = y_0 + (x_1 - x_0)f(x_0, y_0).$$

If $x_n = x_0 + nh$, then we get

$$y_1 = y_0 + hf(x_0, y_0).$$

Similarly, approximating the curve by the tangent in $[x_1, x_2]$ at the point (x_1, y_1) with slope $f(x_1, y_1)$, we have

$$y_2 = y_1 + hf(x_1, y_1),$$

and so, in general

$$y_{n+1} = y_n + hf(x_n, y_n). \tag{8.4}$$

The Euler's method is very slow if h is very small. On the other hand, if h is not small, then this method is too inaccurate. These drawbacks suggest further modifications of Euler's method

Geometric Interpretation: The Euler method has a very simple geometric interpretation. The integral of equation (8.3) yields y as a function of x. Let $y = F(x)$. Then the graph of $y = F(x)$ is a curve in the xy-plane. Since the curve is nearly a straight line in a small interval, we approximate the curve by the tangent at the point (x_0, y_0). Then, as shown in Figure 8.1, the true value Δy (equal to AQ in the figure) is approximated by $\Delta x \tan\theta$ (equal to AB in the figure). Thus,

$$\Delta y \approx \Delta x \tan\theta = \left(\frac{dy}{dx}\right)_{(x_0,y_0)} \Delta x.$$

Therefore

$$y_1 \approx y_0 + \left(\frac{dy}{dx}\right)_{(x_0,y_0)} (x_1 - x_0)$$

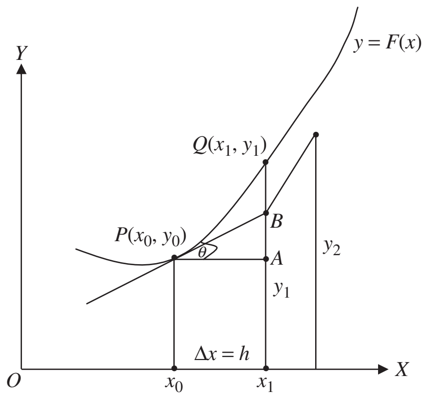


Figure 8.1

In the interval $x_n \le x \le x_{n+1}$, the solution is assumed to follow the line tangent to $y(x)$ at (x_n, y_n). Therefore,

$$y_{n+1} = y_n + hf(x_n, y_n).$$

When this method is applied repeatedly across several intervals in sequence, the numerical solutions traces a polygon segment with sides of slope $f(x_n, y_n)$, $n = 0,1,2, ...$ That is why, this method is also called polygon method.

Error Analysis of Euler's Method

Let $y(x_n)$ be exact value of y at $x = x_n$ and let y_{n+1} be the computed value of y at $x = x_{n+1}$. Then the truncation error after one step, called the local truncation error is given by

$$\begin{aligned} T_{n+1} &= y_{n+1} - y(x_{n+1}) \\ &= y_n + hy'(x_n) - y(x_{n+1}) \quad \text{(by Euler's formula)} \\ &= y_n + hy'(x_n)\left[y_n + hy_n'(x_n) + \frac{h^2}{2} y_n''(\xi)\right], \xi \in [x_n x_{n+1}] \\ &= -\frac{h^2}{2} y_n''(\xi). \end{aligned}$$

Hence, the local truncation error is $O(h^2)$.

The total truncation error is

$$e_n = y_n - y(x_n).$$

We assume that (i) y_0 is exact so that $e_0 = 0$ and y_i are the values of y computed by Euler's method (ii) Lipschitz condition

$$|f(x,y) - f(x,y^*)| \le L|y - y^*|$$

is satisfied, and (iii) $|y''(\xi)| \le M$ in the given interval. By Euler's method, we have

$$y_{n+1} = y_n + hf(x_n, y_n) \tag{8.5}$$

and by Taylor's expansion, we have

$$y(x_{n+1}) = y(x_n) + hf(x_n, y(x_n)) + \frac{h^2}{2!}y''(\xi). \tag{8.6}$$

Subtracting equation (8.6) from equation (8.5), we have

$$e_{n+1} = e_n + h[f(x_n, y_n) - f(x_n, y(x_n)] - \frac{h^2}{2!}y''(\xi).$$

Hence,

$$|e_{n+1}| \le |e_n| + hL|y_n - y(x_n)| + \frac{h^2}{2!}M$$

or

$$|e_{n+1}| \le (1 + hL)|e_n| + \frac{h^2}{2}M.$$

Putting $1 + hL = A$ and $\frac{h^2}{2}M = B$, we get

$$|e_{n+1}| \le A|e_n| + B, n = 0, 1, 2, \ldots, N-1.$$

Thus,

$$|e_1| \le A|e_0| + B$$

$$|e_2| \le A|e_1| + B \le A[A|e_0| + B]$$

$$= A^2|e_0| + (A+1)B = \frac{A^2 - 1}{A - 1}B + A_2|e_0|,$$

$$|e_3| \le A|e_2| + B = A^3|e^0| + \frac{A^3 - 1}{A - 1}B,$$

$$\ldots$$

$$|e_N| \le A^N|e_0| + \frac{A^N - 1}{A - 1}B.$$

But $e_0 = 0$ and

$$A^N = (1 + hL)^N \le e^{NhL} = e^{L(x_N - x_0)}.$$

Hence,

$$|e_N| \le \frac{1}{2}hM\frac{e^{L(x_N - x_0)} - 1}{L} = O(h).$$

The error tends to zero as $h \to 0$ in such a way that $nh = x_n - x_0$ remains constant. From this computation it follows that the Euler method is convergent.

Improved Euler's Method

In this method, the curve in the interval $[x_0, x_1]$ is approximated by a line through (x_0, y_0) whose slope is the average of the slopes at (x_0, y_0) and $(x_1, y_1^{(1)})$ such that

$$y_1^{(1)} = y_0 + hf(x_0, y_0).$$

Thus, the equation of the line becomes

$$y - y_0 = (x - x_0)\left[\frac{1}{2}\left\{f(x_0, y_0) + f(x_1, y_1^{(1)})\right\}\right]$$

and so the line through (x_0, y_0) and (x_1, y_1) is

$$y - y_0 = (x - x_0)\frac{1}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

or

$$\begin{aligned} y_1 &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] \\ &= y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0 + h, y_0 + hf(x_0, y_0))\right]. \end{aligned}$$

Hence, the general formula becomes

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))\right],$$

where $x_n - x_{n-1} = h$.

Geometrical Interpretations: Let Δy computed by Euler's method is represented by AB. If PC is drawn parallel to the tangent at $Q(x_1, y_1)$, then Δy computed by using the slope at Q is represented by AC. On the other hand, if we take the average of the slopes, we have

$$\begin{aligned} \Delta y &= \frac{\left(\frac{dy}{dx}\right)_{(x_0, y_0)} + \left(\frac{dy}{dx}\right)_{(x_1, y_1)}}{2} h \\ &= \frac{1}{2}(AB + AC) = \frac{1}{2}(AB + AB + BC) \\ &= AB + \frac{1}{2}BC, \end{aligned}$$

which is very close to the true value AQ (Figure 8.2).

Therefore,

$$\begin{aligned} y_1 &= y_0 + \frac{h}{2}\left[\left(\frac{dy}{dx}\right)_{(x_0, y_0)} + \left(\frac{dy}{dx}\right)_{(x_1, y_1)}\right] \\ &= y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_0 + hf(x_0, y_0))]. \end{aligned}$$

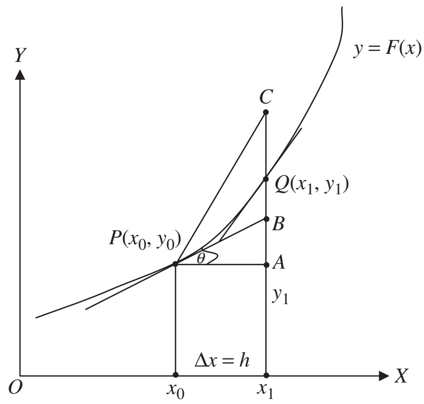


Figure 8.2

Hence, the general formula becomes

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))].$$

Modified Euler's Method

In this method, the curve in the interval $[x_0, x_1]$ is approximated by the line through (x_0, y_0) with slope $f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right)$, that is, the slope at the midpoint whose abscissa is the average of x_0 and x_1, that is, the slope at $x_0 + \frac{h}{2}$. Thus, the equation of the line is

$$y - y_0 = (x - x_0)\left\{f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0))\right\}.$$

Taking $x = x_1$, we have

$$y_1 = y_0 + h\left\{f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right)\right\}.$$

Hence, the general formula becomes

$$y_{n+1} = y_n + h\left\{f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)\right\}.$$

EXAMPLE 8.4

Solve, by Euler's method, the initial value problem

$$\frac{dy}{dx} = \frac{x - y}{2}, \quad y(0) = 1$$

over [0, 3], using step size $\frac{1}{2}$.

Solution. By Euler's method,

$$y_{n+1} = y_n + hf(x_n, y_n).$$

We are given that $h = \frac{1}{2}$ and $f(x, y) = \frac{x - y}{2}$. Therefore,

$$y_{n+1} = y_n + 0.5\left(\frac{x_n - y_n}{2}\right) = 0.25x_n + 0.75y_n.$$

Thus,

$$\begin{aligned}
y_1 &= 0.25x_0 + 0.75y_0 = 0.25(0) + 0.75(1) \\
&= 0.75, \\
y_2 &= 0.25x_1 + 0.75y_1 = 0.25(0.5) + 0.75(0.75) \\
&= 0.125 + 0.5625 = 0.6875, \\
y_3 &= 0.25(1) + 0.75(0.6875) \\
&= 0.25 + 0.515625 = 0.765625, \\
y_4 &= 0.25(1.5) + 0.75(0.765625) = 0.375 + 0.57421875 \\
&= 0.94921875, \\
y_5 &= 0.25(2) + 0.75(0.94921875) = 0.50 + 0.711914062 \\
&= 1.211914063, \\
y_6 &= 0.25(2.5) + 0.75(1.211914063) \\
&= 0.625 + 0.908935546 \\
&= 1.533935547 \approx 1.533936.
\end{aligned}$$

EXAMPLE 8.5

Solve the initial value problem

$$\frac{dy}{dx} = \frac{y - x}{y + x}, \quad y(0) = 1 \text{ for } x = 0.1 \text{ by Euler's method.}$$

Solution. By Euler's method

$$y_{n+1} = y_n + hf(x_n, y_n).$$

We take $h = 0.02$. Therefore,

$$y_{n+1} = y_n + 0.02\left(\frac{y_n - x_n}{y_n + x_n}\right)$$

and so

$$\begin{aligned}
y_1 &= y_0 + 0.02\left(\frac{y_0 - x_0}{y_0 + x_0}\right) \\
&= 1 + 0.02\left(\frac{1 - 0}{1 + 0}\right) = 1.02, \\
y_2 &= y_1 + 0.02\left(\frac{y_1 - x_1}{y_1 + x_1}\right) \\
&= 1.02 + 0.02\left(\frac{1.02 - 0.02}{1.02 + 0.02}\right) = 1.0392,
\end{aligned}$$

$$y_3 = y_2 + 0.02\left(\frac{y_2 - x_2}{y_2 + x_2}\right)$$

$$= 1.0392 + 0.02\left(\frac{1.0392 - 0.04}{1.0392 + 0.04}\right) = 1.05918,$$

$$y_4 = y_3 + 0.02\left(\frac{y_3 - x_3}{y_3 + x_3}\right)$$

$$= 1.05918 + 0.02\left(\frac{1.05918 - 0.06}{1.05918 + 0.06}\right) = 1.07917,$$

$$y_5 = y_4 + 0.02\left(\frac{y_4 - x_4}{y_4 + x_4}\right)$$

$$= 1.07917 + 0.02\left(\frac{1.07917 - 0.08}{1.07917 + 0.08}\right) = 1.09916.$$

Hence, the required solution is 1.09916.

EXAMPLE 8.6

Use Euler's method and its modified form to obtain $y(0.2)$, $y(0.4)$, and $y(0.6)$ correct to three decimal places, given that $y' = y - x^2$ with initial condition $y(0) = 1$.

Solution. By Euler's method,

$$y_{n+1} = y_n + hf(x_n, y_n).$$

Here $f(x, y) = y - x^2$ and $h = 0.2$. Therefore,

$$y_{n+1} = y_n + 0.2(y_n - x_n^2) = 1.2y_n - 0.2x_n^2.$$

Thus,

$$y_1 = 1.2y_0 - 0.2x_0^{\ 2} = 1.2(1) = 1.2,$$

$$y_2 = 1.2y_1 - (0.2)x_1^{\ 2} = (1.2)^2 - (0.2)^3 = 1.44 - 0.008 = 1.4320,$$

$$y_3 = 1.2y_2 - (0.2)x_2^{\ 2} = (1.2)(1.432) - (0.2)(0.4)^2 = 1.6864.$$

Modified Euler's formula is

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}f(x_n, y_n)\right).$$

Taking $h = 0.2$, we have

$$y_1 = y_0 + 0.2\left[y_0 + \frac{0.2}{2}(y_0 - x_0^2) - \left(x_0 + \frac{0.2}{2}\right)^2\right]$$

$$= 1 + 0.2[1 + 0.1(1 - 0) - (0 + 0.1)^2]$$

$$= 1 + 0.2(1 + 0.1 - 0.01) = 1.218,$$

$$y_2 = y_1 + 0.2[y_1 + 0.1(y_1 - x_1^{\ 2}) - (x_1 + 0.1)^2]$$

$$= 1.218 + 0.2[1.218 + 0.1(1.218 + (0.2)^2) - (0.2 + (0.1)^2]$$

$$= 1.218 + 0.2[1.218 + 0.1178 - 0.09] = 1.4672,$$

$$\begin{aligned} y_3 &= y_2 + 0.2[y_2 + 0.1(y_2 - x_2^{\,2}) - (x_2 + 0.1)^2] \\ &= 1.4672 + 0.2[1.4672 + 0.1(1.4672 - (0.4)^2) - (0.4 + 0.1)^2] \\ &= 1.4672 + 0.2[1.4672 + 0.13072 - 0.25] = 1.7368. \end{aligned}$$

EXAMPLE 8.7

Using Euler modified method, obtain a solution of $\frac{\mathrm{d}y}{\mathrm{d}x} = x + \left|\sqrt{y}\right|,\ y(0) = 1$ for the range $0 \le x \le 0.6$ in steps of 0.2.

Solution. The given differential equation is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + \left|\sqrt{y}\right|, y(0) = 1.$$

The modified Euler's formula is

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right).$$

Taking $h = 0.2$, we have

$$\begin{aligned} y_1 &= y_0 + 0.2\left[x_0 + \frac{0.2}{2} + |y_0| + \frac{0.2}{2}\left(x_0 + \left|\sqrt{y_0}\right|\right)\right] \\ &= 1 + 0.2[0.1 + 1 + 0.1(1)] = 1.240. \\ y_2 &= y_1 + 0.2\left[x_1 + \frac{0.2}{2} + |y_1| + \frac{0.2}{2}\left(x_1 + \left|\sqrt{y_1}\right|\right)\right] \\ &= 1.24 + 0.2[0.2 + 0.1 + 1.24 + 0.1(0.2 + 1.114)] \\ &= 1.24 + 0.33428 = 1.574. \\ y_3 &= y_2 + 0.2\left[x_2 + \frac{0.2}{2} + |y_2| + \frac{0.2}{2}\left(x_2 + \left|\sqrt{y_2}\right|\right)\right] \\ &= 1.574 + 0.2[0.4 + 0.1 + 1.547 + 0.1(0.4 + 1.255)] = 2.0219. \end{aligned}$$

3. Picard's Method of Successive Integration

Consider the initial value problem

$$y'(x) = f(x, y(x))\,\text{over}\,[a, b]\,\text{with}\; y(x_0) = y_0. \tag{8.7}$$

Using fundamental theorem of calculus, we have

$$\int_{x_0}^{x_1} f(x, y(x))\mathrm{d}x = \int_{x_0}^{x_1} y'(x)\mathrm{d}x = y(x_1) - y(x_0). \tag{8.8}$$

Thus,

$$y(x_1) = y(x_0) + \int_{x_0}^{x_1} f(x, y(x))\mathrm{d}x. \tag{8.9}$$

Thus, if we start with the approximation $y(x_0)$, then

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x, y_0)\mathrm{d}x;$$

$$y_2 = y_0 + \int_{x_0}^{x_1} f(x, y_1)\mathrm{d}x;$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$y_{n+1} = y_0 + \int_{x_0}^{x_1} f(x, y_n)\mathrm{d}x.$$

We stop the process when $y_{n+1} = y_n$ upto desired decimal places.

The Picard's method of successive integration fails if the function is not easily integrable.

EXAMPLE 8.8

Using Picard's method, solve

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 - y,\ y(0) = 1 \quad \text{for } x = 0.2.$$

Solution. We start with the approximation $y(0) = 1$. Then

$$y_1 = y_0 + \int_0^{0.2} (x^2 - y_0)\mathrm{d}x = 1 + \int_0^{0.2} (x^2 - 1)\mathrm{d}x$$

$$= 1 + \left[\frac{x^3}{3} - x\right]_0^{0.2} = 1 + \left[\frac{(0.2)^3}{3} - 0.2\right] = 0.8027$$

$$y_2 = 1 + \int_0^{0.2} (x^2 - y_1)\mathrm{d}x = 1 + \int_0^{0.2} (x^2 - 0.8027)\mathrm{d}x$$

$$= 1 + \left[\frac{x^3}{3} - 0.8027x\right]_0^{0.2} = 1 + [0.00267 - 0.16054] = 0.8421,$$

$$y_3 = 1 + \int_0^{0.2} (x^2 - y_2)\mathrm{d}x = 1 + \left[\frac{x^3}{3} - y_2 x\right]_0^{0.2}$$

$$= 1 + [0.00267 - (0.8421)(0.2)] = 0.8342,$$

$$y_4 = 1 + [0.00267 - (0.8342)(0.2)] = 0.8358,$$

$$y_5 = 1 + [0.00267 - (0.8358)(0.2)] = 0.8355.$$

Hence $y(0.2) = 0.835$ upto three decimal places.

EXAMPLE 8.9

Solve

$$y' = x^2 + 2xy, \qquad y(0) = 0.$$

Solution. We take first approximation to be $y(0) = 0$. Then,

$$y_1 = y_0 + \int_0^x (x^2 + 2xy(0))\mathrm{d}x = 0 + \int_0^x x^2\mathrm{d}x = \frac{x^3}{3},$$

$$y_2 = 0 + \int_0^x \left(x^2 + 2x\left(\frac{x^3}{3}\right)\right)\mathrm{d}x = \frac{x^3}{3} + \frac{2x^5}{15},$$

$$y_3 = 0 + \int_0^x \left[x^2 + 2x\left(\frac{x^3}{3} + \frac{2x^5}{15}\right)\right]\mathrm{d}x = \frac{x^3}{3} + \frac{2x^5}{3(5)} + \frac{4x^7}{3(5)(7)},$$

$$y_4 = 0 + \int_0^x \left[x^2 + 2x\left(\frac{x^3}{3} + \frac{2x^5}{3(5)} + \frac{4x^7}{3(5)(7)}\right)\right]\mathrm{d}x$$

$$= \frac{x^3}{3} + \frac{2x^5}{3(5)} + \frac{4x^7}{3(5)(7)} + \frac{8x^9}{3(5)(7)(9)}.$$

EXAMPLE 8.10

Solve by Picard's method,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + xy,\ y(0) = 1 \text{ for } x = 0.1.$$

Solution. We take first approximation to be $y(0) = 1$. Then,

$$y_1 = y_0 + \int_0^x f(x, y(0))\mathrm{d}x$$

$$= 1 + \int_0^x (1 + x)\mathrm{d}x = 1 + x + \frac{x^2}{2},$$

$$y_2 = 1 + \int_0^x \left[1 + x\left(1 + x + \frac{x^2}{2}\right)\right]\mathrm{d}x$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8},$$

$$y_3 = 1 + \int_0^x \left[1 + x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}\right)\right]\mathrm{d}x$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}.$$

Thus,

$$y_3(0.1) = 1 + 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{8} + \frac{(0.1)^5}{15} + \frac{(0.1)^6}{48}$$

$$= 1 + 0.1 + \frac{0.01}{2} + \frac{0.001}{3} + \frac{0.0001}{8} + \frac{0.00001}{15} + \frac{0.000001}{48}$$

$$= 1.105346.$$

Further,

$$y_4 = 1 + \int_0^x \left[1 + x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}\right)\right] dx$$
$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48} + \frac{x^7}{105} + \frac{x^8}{384}.$$

Thus,

$$y_4(0.1) = 1 + 0.1 + \frac{0.01}{2} + \frac{0.001}{3} + \frac{0.0001}{8} + \frac{0.00001}{15} + \frac{0.000001}{48} + \frac{0.0000001}{105} + \frac{0.00000001}{384}$$
$$= 1.1053465.$$

Hence,

$$y(0.1) = 1.1053465.$$

4. Heun's Method

Consider the initial value problem

$$y'(x) = f(x, y(x)), y(x_0) = y_0 \tag{8.10}$$

over the interval [a, b]. By fundamental theorem of calculus, we have

$$\int_{x_0}^{x_1} f(x, y(x))dx = \int_{x_0}^{x_1} y'(x)dx = y(x_1) - y(x_0). \tag{8.11}$$

Hence,

$$y(x_1) = y(x_0) + \int_{x_0}^{x_1} f(x, y(x))dx. \tag{8.12}$$

Using trapezoidal rule with $h = x_1 - x_0$, equation (8.12) reduces to

$$y(x_1) \approx y(x_0) + \frac{h}{2}[f(x_0, y(x_0)) + f(x_1, y(x_1))]. \tag{8.13}$$

We observe that $y(x_1)$ appears on both sides of equation (8.13). We replace $y(x_1)$ on right-hand side of equation (8.13) by Euler's method. Thus, $y(x_1)$ on the right-hand side is replaced by

$$y_0 + hf(x_0, y_0).$$

Hence,

$$y(x_1) = y(x_0) + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_0 + hf(x_0, y_0))], \tag{8.14}$$

which is called Heun's method.

The process is repeated to get closer and closer approximation. At each step, Euler's method is used as a predictor and trapezoidal rule is used as corrector. Thus, the general step for Heun's method can be expressed as

$$p_{n+1} = y_n + hf(x_n, y_n)$$
$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, p_{n+1})].$$

EXAMPLE 8.11

Use Heun's method to solve the initial value problem $y'(x)=\frac{x-y}{2}, y(0)=1 \text{ over } [0,2]$ using step size $\frac{1}{2}$.

Solution. By Example 8.4, we have

$$p_1 = 0.75.$$

Therefore,

$$\begin{aligned} y_1 &= y_0 + \frac{h}{2}[f(x_0,y_0)+f(x_1,p_1)] \\ &= 1+\frac{1}{4}\left[-\frac{1}{2}+\frac{(1/2)-0.75}{2}\right] \\ &= 1+\frac{1}{4}\left[-\frac{1}{2}-\frac{0.25}{2}\right]=1-\frac{1.25}{8}=0.84375. \end{aligned}$$

Again, by Example 8.4,

$$p_2 = 0.6875.$$

Therefore,

$$\begin{aligned} y_2 &= y_1 + \frac{h}{2}[f(x_1,y_1)+f(x_2,p_2)] \\ &= 0.84375+\frac{1}{4}\left[\frac{(1/2)-0.84375}{2}+\frac{1-0.6875}{2}\right] \\ &= 0.84375+\frac{1}{8}[-0.34375-0.3125]=0.83984. \end{aligned}$$

Further,

$$p_3 = 0.765625$$

and so

$$\begin{aligned} y_3 &= y_2 + \frac{h}{2}[f(x_2,y_2)+f(x_3,p_3)] \\ &= 0.83984+\frac{1}{4}\left[\frac{1-0.83984}{2}+\frac{1.5-0.765625}{2}\right] \\ &= 0.83984+\frac{1}{8}[0.16016+0.73475]=0.95170375. \end{aligned}$$

Now,

$$p_4 = 0.94921875$$

and so

$$\begin{aligned} y_4 &= y_3 + \frac{h}{2}[f(x_3,y_3)+f(x_4,p_4)] \\ &= 0.95170375+\frac{1}{8}[(1.5-0.9517035)+(2-0.94921875)] \\ &= 0.95170375+\frac{1}{8}[0.5482965+1.05078125]=1.151588. \end{aligned}$$

5. Runge–Kutta Method

Consider the initial value problem

$$y'(x) = f(x, y), y(x_0) = y_0.$$

We note that

$$\begin{aligned} y'(x) &= f(x, y), \\ y''(x) &= f_x + f_y y' = f_x + f_y f, \\ y'''(x) &= f_{xx} + f_{xy} y' + f_y y'' + y'(f_{yx} + f_{yy} y') \\ &= f_{xx} + 2f_{xy} y' + f_y y'' + f_{yy} (y')^2 \\ &= f_{xx} + 2f_{xy} y' + f_y (f_x + f_y f) + f_{yy} f^2 \\ &= f_{xx} + 2f_{xy} y' + f_{yy} f^2 + f_y (f_x + f_y f), \end{aligned}$$

and is general

$$y^{(n)}(x) = d^{(n-1)} f(x, y(x)),$$

where

$$d = \left(\frac{\partial}{\partial x} + f \frac{\partial}{\partial y} \right).$$

If we put

$$\begin{aligned} F_1 &= f_x + ff_y, \\ F_2 &= f_{xx} + 2ff_{xy} + f^2 f_{yy}, \\ F_3 &= f_{xxx} + 3ff_{xxy} + 3f^2 f_{xyy} + f^3 f_{yyy}, \end{aligned}$$

then

$$\begin{aligned} y'(x) &= f(x, y), \\ y''(x) &= F_1, \\ y'''(x) &= F_2 + F_1 f_y, \\ y^{(\text{iv})}(x) &= F_3 + F_2 f_y + 3F_1 f_y + F_1 f_y^2. \end{aligned}$$

Using Taylor's series expansion, we have

$$\begin{aligned} y_{r+1} = y(x_r + h) &= y_r + hy'_r + \frac{h^2}{2!} y''_r + \frac{h^3}{3!} y'''_r + \frac{h^4}{4!} y_r^{\text{iv}} + O(h^5) \\ &= y_r + hf(x_r, y_r) + \frac{h^2}{2} F_1 + \frac{h^3}{3!}(F_2 + F_1 f_y) + \frac{h^4}{4!}(F_3 + F_2 f_y + 3F_1 f_y + F_1 f_y^2) + O(h^5). \end{aligned} \tag{8.15}$$

On the other hand, by Taylor's theorem for function of two variables, we have

$$f(x_r + h, y_r + k) = f(x_r, y_r)\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right) f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^2 f + \cdots \tag{8.16}$$

A Runge–Kutta method of order n is a formula which expresses $y_{r+1} - y_r$ in terms of n values of the function $f(x, y)$ in such a manner that the values obtained coincide with equation (8.15) as far as the terms involving h^n.

Second Order Runge–Kutta Method

Define

$$\left.\begin{aligned} K_1 &= hf(x_r, y_r) \\ K_2 &= hf(x_r + mh, y_r + mK_1). \end{aligned}\right\} \tag{8.17}$$

Our aim is to obtain an expression of the form

$$y_{r+1} = y_r + aK_1 + bK_2. \tag{8.18}$$

If we put

$$\begin{aligned} F_1 &= f_x + ff_y \\ F_2 &= f_{xx} + 2ff_{xy} + f^2 ff_{yy}, \end{aligned}$$

then the left-hand side of equation (8.18) becomes

$$\begin{aligned} y_{r+1} &= y_r + hy_r' + \frac{h^2}{2!} y_r'' + \frac{h^3}{3!} y_r''' + \frac{h^4}{4!} y_r^{(\text{iv})} + O(h^5) \\ &= y_r + hf(x_r, y_r) + \frac{h^2}{2!} F_1 + \frac{h^3}{3!}(F_2 + F_1 f_y + \cdots) \end{aligned} \tag{8.19}$$

Now expanding K_2 by Taylor's series for a function of two variables, we have

$$\begin{aligned} K_2 &= \left[f(x_r, y_r) + \left(mh\frac{\partial}{\partial x} + mK_1 \frac{\partial}{\partial y}\right) f + \frac{1}{2!}\left(mh\frac{\partial}{\partial x} + mK_1 \frac{\partial}{\partial y}\right)^2 f + O(h^3) \right] \\ &= h\left[f(x_r, y_r) + mhf_x + mK_1 f_y + \frac{m^2h^2}{2} f_{xx} + m^2 hK_1 f_{xy} + \frac{m^2 K_1^2}{2} f_{yy} + O(h^3) \right] \end{aligned}$$

Putting the value of K_1 in K_2, we get

$$K_2 = h\left[f(x_r, y_r) + mhf_x + mhf_y f(x_r, y_r) + \frac{m^2h^2}{2} f_{xx} + m^2h^2 f_{xy} f(x_r, y_r) + \frac{m^2h^2}{2} f_{yy} f(x_r, y_r) + O(h^3) \right].$$

Thus, putting the values of K_1 and K_2 from equation (8.17) into equation (8.18), we get

$$\begin{aligned} y_{r+1} &= y_r + ahf(x_r, y_r) \\ &\quad + bh\left[f(x_r, y_r) + mhf_x + mhf_y f(x_r, y_r) + \frac{m^2h^2}{2} f_{xx} + m^2h^2 f_{xy} f(x_r, y_r) + \frac{m^2h^2}{2} f_{yy} f(x_r, y_r) + O(h^3) \right] \\ &= y_r + (a+b)hf + mbh^2(f_x + ff_y) + \frac{m^2bh^3}{2}(f_{xx} + 2ff_{xy} + f^2 f_{yy}) + O(h^4) \\ &= y_r + (a+b)hf + mbh^2 F_1 + \frac{m^2bh^3}{2} F_2 + O(h^4). \end{aligned} \tag{8.20}$$

Comparing coefficients of h and h^2 in equations (8.19) and (8.20), we get

$$\left.\begin{aligned} a + b &= 1 \\ mb &= \frac{1}{2}. \end{aligned}\right\} \tag{8.21}$$

Thus, there are two equations in three unknowns and so there are many solutions to equation (8.21). We choose $a = 0$, $b = 1$, and $m = \frac{1}{2}$ as one of the solution. Then equation (8.18) reduces to

$$\begin{aligned} y_{r+1} &= y_r + K_2 \\ &= y_r + hf\left(x_r + \frac{h}{2}, y_r + \frac{K_1}{2}\right) \\ &= y_r + hf\left(x_r + \frac{h}{2}, y_r + \frac{h}{2} f(x_r, y_r)\right), \end{aligned} \tag{8.22}$$

which is the required second order Runge–Kutta method. We observe that this formula is nothing but modified Euler's method. If we choose $a = b = \frac{1}{2}, m = 1$ as the solution, then we have

$$\begin{aligned} y_{r+1} &= y_r + \frac{1}{2}(K_1 + K_2) \\ &= y_r + \frac{h}{2}[f(x_r, y_r) + f(x_r + h, y_r + hf(x_r, y_r))], \end{aligned}$$

which is nothing but improved Euler's method.

Third Order Runge–Kutta Method

We define

$$\left.\begin{aligned} K_1 &= hf(x_r, y_r), \\ K_2 &= hf(x_r + mh, y_r + mK_1), \\ K_3 &= hf(x_r + nh, y_r + nK_2). \end{aligned}\right\} \tag{8.23}$$

We want to obtain an expression of the form

$$y_{r+1} = y_r + aK_1 + bK_2 + cK_3. \tag{8.24}$$

If we put

$$\begin{aligned} F_1 &= f_x + ff_y, \\ F_2 &= f_{xx} + 2ff_{xy} + f^2 f_{yy}, \end{aligned}$$

then the left-hand side of equation (8.24) is

$$y_{r+1} = y_r + hf + \frac{h^2}{2!}F_1 + \frac{h^3}{3!}(F_2 + F_1 f_y) + O(h^4). \tag{8.25}$$

Also,

$$K_2 = h\left[f(x_r, y_r) + mhf_x + mhf_y f + \frac{m^2h^2}{2} f_{xx} + m^2h^2 f_{xy} f + \frac{m^2h^2}{2} f_{yy} f + O(h^3)\right],$$

$$K_3 = h\left[f(x_r, y_r) + nhf_x + nK_2 f_y + \frac{n^2h^2}{2} f_{xx} + n^2hK_2 f_{xy} + \frac{n^2K_2^2}{2} f_{yy} + O(h^3)\right]$$

$$\begin{aligned} &= h[f(x_r, y_r) + nhf_x + nhf_y\{f(x_r, y_r) + mhf_x + mhf_y f + \frac{m^2h^2}{2} f_{xx} + m^2h^2 f_{xy} f \\ &\quad + \frac{m^2h^2}{2} f_{yy} f + O(h^3)\} + \frac{n^2h^2}{2} f_{xx} + n^2h^2 f_{xy}\{f(x_r, y_r) + mhf_x + mhf_y f + \frac{m^2h^2}{2} f_{xx} \\ &\quad + m^2h^2 f_{xy} f + \frac{m^2h^2}{2} f_{yy} f + O(h^3)\} + \frac{n^2h^2}{2} f_{yy}\{f(x_r, y_r) + mhf_x + mhf_y f + \frac{m^2h^2}{2} f_{xx} \\ &\quad + m^2h^2 f_{xx} f + \frac{m^2h^2}{2} f_{yy} f + O(h^3)\}]. \end{aligned}$$

Therefore, putting the values of K_1, K_2, and K_3 in equation (8.24), we get

$$y_{r+1} = y_r + h(a+b+c)f + h^2(bm+cn)F_1 + \frac{h^3}{2}\left[(bm^2+cn^2)F_2 + 2cmnF_1 f_y\right] + O(h^4). \tag{8.26}$$

Comparing the coefficients of like powers of h in equations (8.25) and (8.26), we get

$$a+b+c=1,$$

$$bm+cn=\frac{1}{2},$$

$$bm^2+cn^2=\frac{1}{3},$$

$$cmn=\frac{1}{6}.$$

Thus, we obtain four equations for five unknowns. Therefore, there exist many solutions. If we take $a=\frac{1}{4}, b=0, c=\frac{3}{4}, n=\frac{2}{3}, m=\frac{1}{3}$ as the solution, then equation (8.24) becomes

$$\begin{aligned} y_{r+1} &= y_r + \frac{1}{4}(K_1+3K_3), \\ K_1 &= hf(x_r, y_r), \\ K_2 &= hf\left(x_r+\frac{h}{3}, y_r+\frac{K_1}{3}\right), \\ K_3 &= hf\left(x_r+\frac{2h}{3}, y_r+\frac{2K_2}{3}\right). \end{aligned} \tag{8.27}$$

Formula (8.27) is called Heun's third order formula.

If we set $a=\frac{2}{9}, b=\frac{1}{3}, c=\frac{4}{9}, m=\frac{1}{2}, n=\frac{3}{4}$, then

$$\begin{aligned} y_{n+1} &= y_r + \frac{1}{9}(2K_1+3K_2+4K_3), \\ K_1 &= hf(x_r, y_r), \\ K_2 &= hf\left(x_r+\frac{h}{2}, y_r+\frac{K_1}{2}\right), \\ K_3 &= hf\left(x_r+\frac{3h}{4}, y_r+\frac{3K_2}{4}\right) \end{aligned}$$

If we set $a=c=\frac{1}{6}, b=\frac{2}{3}, m=\frac{1}{2}, n=1$, then

$$\begin{aligned} y_{r+1} &= y_r + \frac{1}{6}(K_1+4K_2+K_3), \\ K_1 &= hf(x_r, y_r), \\ K_2 &= hf\left(x_r+\frac{h}{2}, y_r+\frac{K_1}{2}\right), \\ K_3 &= hf(x_r+h, y_r+K_2), \end{aligned}$$

which is the most popular third order Runge–Kutta method. It is also known as Kutta's third order rule.

Fourth Order Runge–Kutta Method

Consider the initial value problem

$$y'(x) = f(x, y), y(x_0) = y_0.$$

We define

$$K_1 = hf(x_r, y_r),$$
$$K_2 = hf(x_r + mh, y_r + mK_1),$$
$$K_3 = hf(x_r + nh, y_r + nK_2),$$
$$K_4 = hf(x_r + ph, y_r + pK_3). \tag{8.28}$$

We wish to obtain a formula of the type

$$y_{r+1} = y_r + aK_1 + bK_2 + cK_3 + dK_4. \tag{8.29}$$

Let

$$F_1 = f_x + ff_y,$$
$$F_2 = f_{xx} + 2ff_{xy} + f^2 f_{yy},$$
$$F_3 = f_{xxx} + 3ff_{xxy} + 3f^2 f_{xyy} + f^3 f_{yyy}.$$

Expanding y_{r+1} in the series, we obtain

$$y_{r+1} = y_r + hf + \frac{h^2}{2}F_1 + \frac{h^3}{3!}(F_2 + F_1 f_y) + \frac{h^4}{4!}(F_3 + F_1 f_y^2 + 3F_1(f_{xy} + f_{yy} f)) + O(h^5) \tag{8.30}$$

Further, using Taylor's Theorem for two variables, we have

$$K_1 = hf(x_r, y_r),$$

$$K_2 = h\left[f(x_r, y_r) + mhF_1 + \frac{m^2h^2}{2}F_2 + \frac{m^3h^3}{3!}F_3 + \cdots \right],$$

$$K_3 = h\left[f(x_r, y_r) + nhF_1 + \frac{h^2}{2}(n^2F_2 + 2mnF_1 f_y) + \frac{h^3}{6}(n^3F_3 + 3m^2nF^2 f_y + 6mn^2F_1 f_y') + \cdots \right],$$

$$K_4 = h\left[f(x_r, y_r) + phF_1 + \frac{h^2}{2}(p^2F_2 + 2npF_1 f_y) + \frac{h^3}{6}(p^3F_3 + 3n^2pF_2 f_y + 6np^2F_1 f_y' + 6mnpF_1 f_y^2) + \cdots \right].$$

Putting these values of K_1, K_2, K_3, and K_4 in equation (8.29) and equating the like powers of h in the corresponding expressions for y_{r+1}, we obtain

$$a + b + c + d = 1, \quad cmn + dnp = \frac{1}{6},$$

$$bm + cn + dp = \frac{1}{2}, \quad cmn^2 + dnp^2 = \frac{1}{8},$$

$$bm^2 + cn^2 + dp^2 = \frac{1}{3}, \quad cm^2n + dn^2p = \frac{1}{12},$$

$$bm^3 + cn^3 + dp^3 = \frac{1}{4}, \quad dmnp = \frac{1}{24}. \tag{8.31}$$

Any solution of equation (8.29) will serve our purpose. Let us take $m = n = \frac{1}{2}, p = 1, a = d = \frac{1}{6}, b = c = \frac{1}{3}$. Then

$$K_1 = hf(x_r, y_r),$$

$$K_2 = hf\left(x_r + \frac{h}{2}, y_r + \frac{K_1}{2}\right),$$

$$K_3 = hf\left(x_r + \frac{h}{2}, y_r + \frac{K_2}{2}\right),$$

$$K_4 = hf(x_r + h, y_r + K_3),$$

and

$$y_{r+1} = y_r + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4),$$

which is the required fourth order Runge–Kutta method.

Remark 8.1. Whenever we mention only Runge–Kutta method, we mean the Runge–Kutta method of order 4.

EXAMPLE 8.12

Apply third order Runge–Kutta method to the initial value problem

$$\frac{dy}{dx} = x^2 - y, \quad y(0) = 1,$$

over [0, 0.2] taking $h = 0.1$.

Solution. Taking $h = 0.1$, we have

$$K_1 = hf(x_0, y_0) = 0.1(x_0^2 - y_0) = 0.1(0 - 1) = -0.1,$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.1\left[\left(\frac{0.1}{2}\right)^2 - \left(1 - \frac{0.1}{2}\right)\right]$$

$$= 0.1[0.0025 - 0.95] = -0.09475,$$

$$K_3 = hf(x_0 + h, y_0 + K_2) = 0.1\left[(0.1)^2 - (1 - 0.09475)\right]$$

$$= 0.1[0.01 - 0.90525] = -0.089525.$$

Then, by third order Runge–Kutta method,

$$y_1 = y(0.1) = y_0 + \frac{1}{6}[K_1 + 4K_2 + K_3]$$

$$= 1 + \frac{1}{6}[-0.1 - 4(0.09475) - 0.089525]$$

$$= 0.905245833.$$

To find $y(0.2)$, we have

$$K_1 = hf(x_1, y_1) = 0.1\,(x_1^2 - y_1)$$

$$= 0.1[(0.1)^2 - 0.90524] = -0.089524,$$

$$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right)$$

$$= 0.1\left[\left(0.1 + \frac{0.1}{2}\right)^2\right] - \left(0.905245833 - \frac{0.089524}{2}\right)$$

$$= 0.083798383,$$

$$K_3 = hf(x_1 + h,\ y_1 + K_2)$$

$$= 0.1[(0.1+0.1)^2 - (0.905245833 - 0.083798383)]$$

$$= -0.078144745.$$

Then,

$$y_2 = y(0.2) = y_1 + \frac{1}{6}[K_1 + 4K_2 + K_3]$$

$$= 0.905245833 + \frac{1}{6}[-0.089524 - 4(0.083798383) - 0.078144745]$$

$$= 0.821435453.$$

EXAMPLE 8.13

Use Runge–Kutta method to solve $y' = x + y$, $y(0) = 1$, for $x = 0.1$.

Solution. Taking $h = 0.1$, we obtain

$$K_1 = hf(x_0,\ y_0) = 0.1(x_0 + y_0) = 0.1(0+1) = 0.1,$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.1\left(x_0 + \frac{h}{2} + y_0 + \frac{K_1}{2}\right)$$

$$= 0.1(0 + 0.05 + 1 + 0.05) = 0.11,$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.1\left(0 + 0.05 + 1 + \frac{0.11}{2}\right) = 0.1105,$$

$$K_4 = hf\ (x_0 + h, y_0 + K_3) = 0.1\ (0 + 0.1 + 1 + 0.1105) = 0.12105.$$

Therefore,

$$y_1 = y(0.1) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 1 + \frac{1}{6}(0.1 + 0.22 + 0.2210 + 0.12105) = 0.11034167.$$

EXAMPLE 8.14

Apply fourth order Runge–Kutta method to

$$\frac{dy}{dx} = 3x + \frac{1}{2}y, \quad y(0) = 1$$

to determine $y(0.1)$ and $y(0.2)$ correct to four decimal places.

Solution. Taking $h = 0.1$, we have

$$K_1 = hf(x_0,\ y_0) = 0.1\left(0 + \frac{1}{2}\right) = 0.05,$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.1\left[3(0 + 0.05) + \frac{1}{2}(1 + 0.025)\right] = 0.06625,$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.1\left[3(0+0.05) + \frac{1}{2}\left(1 + \frac{0.06625}{2}\right)\right] = 0.06665625,$$

$$K_4 = hf(x_0 + h,\ y_0 + K_3) = 0.1[3(0+0.1) + \frac{1}{2}(1 + 0.06665625)]$$

$$= 0.1[0.3 + 0.533328125] = 0.0833328125.$$

Hence,

$$y_1 = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$

$$= 1 + \frac{1}{6}[0.05 + 2(0.0625) + 2(0.06665625) + 0.0833328125]$$

$$= 1.06652421875 \approx 1.0665.$$

To find $y(0.2)$, we note that

$$K_1 = hf(x_1,\ y_1) = 0.1[3(0.1) + \frac{1}{2}(1.066524)] = 0.0833262,$$

$$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right)$$

$$= 0.1\left[3(0.1+0.05) + \frac{1}{2}\left(1.066524 + \frac{0.0833262}{2}\right)\right]$$

$$= 0.100409515,$$

$$K_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right)$$

$$= 0.1\left[3(0.1+0.05) + \frac{1}{2}\left(1.066524 + \frac{0.100409515}{2}\right)\right]$$

$$= 0.100836437,$$

$$K_4 = hf(x_1 + h, y_1 + K_3)$$

$$= 0.1\left[3(0.1+0.1) + \frac{1}{2}\left(1.066524 + \frac{0.100836437}{2}\right)\right]$$

$$= 0.11584711.$$

Hence,

$$y(0.2) = y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 1.06652422 + \frac{1}{6}[0.0833262 + 2(0.100409515) + 2(0.100836437) + 0.11584711]$$

$$= 1.166801756 \approx 1.1668.$$

EXAMPLE 8.15

Apply the fourth order Runge–Kutta method to solve

$$\frac{dy}{dx} = x^2 + y^2,\ y(0) = 1.$$

Take step size $h = 0.1$ and determine approximations to $y(0.1)$ and $y(0.2)$ correct to four decimal places.

Solution. Taking $h = 0.1$, we have

$$K_1 = hf(x_0, y_0) = 0.1(0+1) = 0.1,$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.1\left[(0.05)^2 + \left(1 + \frac{0.1}{2}\right)^2\right]$$

$$= 0.1105,$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.1\left[(0.05)^2 + \left(1 + \frac{0.1105}{2}\right)^2\right]$$

$$= 0.111605256,$$

$$K_4 = hf(x_0 + h,\ y_0 + K_3) = 0.1\ [(0.1)^2 + (1 + 0.111605256)^2]$$

$$= 0.124566624.$$

Therefore,

$$y_1 = y(0.1) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 1 + \frac{1}{6}(0.1 + 2(0.1105) + 2(0.111605256) + 0.124566624)$$

$$= 1.111462856 \approx 1.11146.$$

To find $y(0.2)$, we have

$$K_1 = hf(x_1, y_1) = 0.1[x_1^2 + y_1^2]$$

$$= 0.1[(0.1)^2 + (1.1114628)^2] = 0.124534956,$$

$$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right)$$

$$= 0.1\left[\left(0.1 + \frac{0.1}{2}\right)^2 + \left(1.1114628 + \frac{0.124534956}{2}\right)^2\right]$$

$$= 0.1400142,$$

$$K_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right)$$

$$= 0.1\left[0.0225 + \left(1.1114628 + \frac{0.1400142}{2}\right)^2\right]$$

$$= 0.1418371125,$$

$$K_4 = hf(x_1 + h, y_1 + K_3)$$

$$= 0.1[(0.2)^2 + (1.1114628 + 0.141837112)^2]$$

$$= 0.161076063.$$

Hence,

$$y_2 = y(0.2) = y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$
$$= 1.11142856 + \frac{1}{6}[0.124534956 + 2(0.1400142) + 2(0.1418371125) + 0.161076063]$$
$$= 1.2529808 \approx 1.2530.$$

EXAMPLE 8.16

Using Runge–Kutta method of fourth order solve for y at $x = 1.2$, 1.4 from the equation $\frac{dy}{dx} = \frac{2xy + e^x}{x^2 + xe^x}$ with $x_0 = 1$, $y_0 = 0$.

Solution. We have

$$\frac{dy}{dx} = f(x, y) = \frac{2xy + e^x}{x^2 + xe^x},\ y(1) = 0.$$

Thus, $x_0 = 1$, $y_0 = 0$ and we take $h = 0.2$. Then,

$$k_1 = hf(x_0, y_0) = 0.2\left(\frac{e^1}{1 + e^1}\right) = 0.2\left(\frac{2.71828}{3.71828}\right) = 0.1462$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2[f(1.1, 0.0731] = 0.2\left[\frac{0.161 + e^{1.1}}{1.21 + 1.1(e^{1.1})}\right] = 0.2\left[\frac{3.1652}{4.5146}\right] = 0.1402$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = hf(1.1, 0.0701)$$
$$= 0.2\left[\frac{0.15422 + 3.0042}{1.21 + 1.1(e^{1.1})}\right] = 0.2\left[\frac{3.1584}{4.5146}\right] = 0.1399$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = hf(1.2, 0.1399)$$
$$= 0.2\left[\frac{0.3358 + 3.3201}{1.44 + 1.2(e^{1.2})}\right] = 0.2\left[\frac{3.6559}{5.4241}\right] = 0.1348.$$

Therefore,

$$y(1.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= 0 + \frac{1}{6}[0.146 + 2(0.1402) + 2(0.1399) + 0.1348] = 0.1402.$$

Now $x_0 = 1.2$, $y_0 = 0.1402$, $h = 0.2$. Calculate as above k_1, k_2, k_3, and k_4 and then find

$$y(1.4) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= 0.1402 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

It will be approximately 0.264.

6. Runge–Kutta Method for System of First Order Equations

Consider the system of equations

$$y' = F(x, y, z),\ z' = G(x, y, z),$$

with the initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$. Then the Runge–Kutta method for the system becomes

$$y_{r+1} = y_r + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4],$$

$$z_{r+1} = z_r + \frac{1}{6}[L_1 + 2L_2 + 2L_3 + L_4],$$

where

$$K_1 = hF(x_r, y_r, z_r), \qquad L_1 = hG(x_r, y_r, z_r)$$

$$K_2 = hF\left(x_r + \frac{h}{2}, y_r + \frac{K_1}{2}, z_r + \frac{L_1}{2}\right), \quad L_2 = hG\left(x_r + \frac{h}{2}, y_r + \frac{K_1}{2}, z_r + \frac{L_1}{2}\right)$$

$$K_3 = hF\left(x_r + \frac{h}{2}, y_r + \frac{K_2}{2}, z_r + \frac{L_2}{2}\right), \quad L_3 = hG\left(x_r + \frac{h}{2}, y_r + \frac{K_2}{2}, z_r + \frac{L_2}{2}\right)$$

$$K_4 = hF(x_r + h, y_r + K_3, z_r + L_3), \qquad L_4 = hG(x_r + h, y_r + K_3, z_r + L_3).$$

EXAMPLE 8.17

Solve

$$y' = x + z, \qquad y(0) = 0$$

$$z' = x - y, \qquad z(0) = 1$$

for $x = 0.1$ and $x = 0.2$ by Runge–Kutta method.

Solution. We have $h = 0.1$ and

$$F(x, y, z) = x + z, \quad G(x, y, z) = x - y.$$

Then,

$$K_1 = hF(x_0, y_0, z_0) = 0.1(0 + 1) = 0.1, \qquad L_1 = hG(x_0, y_0, z_0) = 0.1(0 - 0) = 0$$

$$K_2 = hF\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{L_1}{2}\right), \qquad L_2 = hG\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{L_1}{2}\right)$$

$$= 0.1\left[\frac{0.1}{2} + 1 + \frac{0}{2}\right] = 0.105, \qquad = 0.1\left[\frac{0.1}{2} - \frac{0.1}{2}\right] = 0$$

$$K_3 = hF\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{L_2}{2}\right), \qquad L_3 = hG\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{L_2}{2}\right)$$

$$= 0.1\left[\frac{0.1}{2} + 1\right] = 0.105, \qquad = 0.1\left[\frac{0.1}{2} - \frac{0.105}{2}\right] = 0.00025$$

$$K_4 = hF(x_0 + h, y_0 + K_3, z_0 + L_3), \qquad L_4 = hG(x_0 + h, y_0 + K_3, z_0 + L_3)$$

$$= 0.1[0.1 + 1 + 0.00025], \qquad = 0.1[0.1 - 0.105]$$

$$= 0.110025., \qquad = 0.0205.$$

Thus,

$$y(0.1) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= \frac{1}{6}[0.1 + 2(0.105) + 2(0.105) + 0.110025] = 0.105004,$$

$$z(0.1) = z_0 + \frac{1}{6}(L_1 + 2L_2 + 2L_3 + L_4)$$
$$= 1 + \frac{1}{6}[2(0.00025) - 0.0205] = 0.99667.$$

Now to find $y(0.2)$ and $z(0.2)$, we have

$$K_1 = hF(x_1, y_1, z_1),$$
$$= 0.1[0.1 + 0.99667] = 0.109667,$$
$$L_1 = hG(x_1, y_1, z_1)$$
$$= 0.1[0.1 - 0.105004] = -0.0005004$$
$$K_2 = hF\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}, z_1 + \frac{L_1}{2}\right),$$
$$= 0.1[0.150 + (0.99667 - 0.00025002)],$$
$$= 0.114642,$$
$$L_2 = hG\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}, z_1 + \frac{L_1}{2}\right)$$
$$= 0.1[0.150 - (0.105004 + 0.054816)]$$
$$= -\ 0.0009837$$
$$K_3 = hF\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}, z_1 + \frac{L_2}{2}\right),$$
$$= 0.1[0.150 + 0.99667 - 0004918],$$
$$= 0.101167,$$
$$L_3 = hG\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}, z_1 + \frac{L_2}{2}\right)$$
$$= 0.1[0.15 - (0.105004 + 0.057321)]$$
$$= -0.0012325$$
$$K_4 = hF(x_1 + h,\ y_1 + K_3, z_1 + L_3),$$
$$= 0.1\ [0.150 + 0.99667 - 0.0012325],$$
$$= 0.11454375.$$
$$L_4 = hG(x_1 + h,\ y_1 + K_3, z_1 + L_3)$$
$$= 0.1[0.15 - (0.105004 + 0.101167)]$$
$$= -0.0056174.$$

Hence,

$$y(0.2) = y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$
$$= 0.105004 + \frac{1}{6}[0.109667 + 2(0.114642) + 2(0.101167) + 0.11454375]$$
$$= 0.2143073$$

and

$$z(0.2) = z_1 + \frac{1}{6}(L_1 + 2L_2 + 2L_3 + L_4)$$
$$= 0.99667 + \frac{1}{6}(-0.0005004 + 2(-0.0009837) + 2(-0.0012325) + (-0.0056174)]$$
$$= 0.99591.$$

7. Runge–Kutta Method for Higher Order Differential Equations

Since the higher order differential equations can be converted into a set of first order differential equations, therefore these equations can be solved by Runge–Kutta method. We illustrate the method in the form of the following example:

EXAMPLE 8.18

Using Runge–Kutta method, solve the differential equation

$$y'' = \frac{x^2 - y^2}{1 + y'^2}, y(0) = 1,\ y'(0) = 0$$

for $x = 0.5$ and $x = 1$.

Solution. We are given that

$$y'' = \frac{x^2 - y^2}{1 + y'^2}, \qquad y(0) = 1,\ y'(0) = 0.$$

Putting $y' = z$, the given equation is equivalent to

$$y' = z = F(x, y, z),$$
$$z' = \frac{x^2 - y^2}{1 + y'^2} = G(x, y, z).$$

Now y and z can be determined by

$$y_{r+1} = y_r + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4),$$
$$z_{r+1} = z_r + \frac{1}{6}(L_1 + 2L_2 + 2L_3 + L_4).$$

Taking $h = 0.5$, we have

$$K_1 = hF(x_0, y_0, z_0) = 0.5(0) = 0, \qquad L_1 = hG(x_0, y_0, z_0) = 0.5\left(\frac{-1}{1}\right) = 0.5$$

$$K_2 = hF\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{L_1}{2}\right), \qquad L_2 = hG\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{L_1}{2}\right)$$

$$= 0.5\left(-\frac{0.5}{2}\right) = -0.125, \qquad = 0.5\left[\frac{(0.25)^2 - 1}{1 + \left(-\frac{0.5}{2}\right)^2}\right] = -0.4412$$

$$K_3 = hF\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{L_2}{2}\right), \qquad L_3 = hG\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{L_2}{2}\right)$$

$$= 0.5\left[\frac{-0.4412}{2}\right] = -0.110294, \qquad = \frac{0.5[0.0625 - (1 - 0.0625)^2]}{1 + (0.2206)^2} = -0.3892$$

$$K_4 = hF(x_0 + h, y_0 + K_3, z_0 + L_3), \qquad L_4 = hG(x_0 + h,\ y_0 + K_3, z_0 + L_3)$$

$$= 0.5(-0.3892) = -0.19460, \qquad = 0.5\left[\frac{0.25 + (1 - 0.11029)^2}{1 + (-0.3892)^2}\right] = 0.2351.$$

Therefore,

$$y(0.5) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$
$$= 1 + \frac{1}{6}(0 - 0.250 - 0.22058 - 0.1946) = -0.88914,$$
$$z(0.5) = z_0 + \frac{1}{6}(L_1 + 2L_2 + 2L_3 + L_4)$$
$$= \frac{1}{6}(0.5 - 0.8826 - 0.7784 - 0.2351) = -0.39935.$$

To find $y(0.2)$, we note that $x_1 = 0.5$, $y_1 = 0.88914$, and $z_1 = 0.39935$. Therefore, proceeding as above, we have

$$K_1 = hF(x_1, y_1, z_1) = -0.199675, \qquad L_1 = hG(x_1, y_1, z_1) = 0.23314$$

$$K_2 = -0.257936, \qquad L_2 = -0.0238965$$

$$K_3 = -0.205632, \qquad L_3 = -0.00656937$$

$$K_4 = -0.2029425, \qquad L_4 = 0.228729.$$

Thus,

$$y_2 = y(1) = y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 0.667517.$$

EXERCISES

1. Solve $\frac{dy}{dx} = 1 - 2xy$, $y(0) = 0$ by Taylor's series method for $x = 0.2$.

   Ans. 0.1947

2. Using Taylor's series method, obtain the values of y at x =0.1, 0.2, 0.3 if y satisfies the equation $\frac{d^2y}{dx^2} + xy = 0$ and $y(0) = 1$, $y'(0) = 0.5$.

   Ans. $y(0.1) = 1.050$, $y(0.2) = 1.099$, $y(0.3) = 1.145$

3. Solve $\frac{dy}{dx} = -xy$, $y(0) = 1$ over [0, 0.1] with $h = 0.05$ using Taylor's series method.

   Ans. $y(0.05) = 0.9987508$, $y(0.1) = 0.9950125$

4. Solve $\frac{dy}{dx} = 1 - y$, $y(0) = 0$ in [0, 0.3] by modified Euler's method taking $h = 0.1$.

   Ans. $y(0.1) = 0.095$, $y(0.2) = 0.180975$, $y(0.3) = 0.2587823$

5. Solve $\frac{dy}{dx} = x + y^2$, $y(0) = 1$ for $x = 0.5$ by modified Euler's method.

   Ans. 2.2352

6. Solve $\frac{dy}{dx} = y - \frac{2x}{y}$, $y(0) = 1$ in [0, 0.2] using Euler's method and taking $h = 0.1$.

   Ans. $y(0.1) = 1.095909$, $y(0.2) = 1.184097$

7. Use Picard's method to solve $\frac{dy}{dx} = x - y^2$, $y(0) = 1$.

   Ans. 0.9138

8. Use Picard's method to solve $y'' + 2xy' + y = 0$, $y(0) = 0.5$, $y'(0) = 0.1$ for $x = 0.1$.

   Ans. 0.5075

9. Use Picard's method to solve $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 0$ for $x = 0.4$.

   Ans. 0.0214

10. Solve for $x = 0.1$, the equation $\frac{dy}{dx} = 3x + y^2$, $y(0) = 1$ by Picard's method.

    Ans. $y(0.1) = 1.127$

11. Use Runge–Kutta method of order four to solve the differential equation $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$, $y(0) = 1$ at $x = 0.2$.

    Ans. $y(0.2) = 1.196$

12. Use fourth order Runge–Kutta method to find $y(0.2)$ for the equation $\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$.

Ans. $y(0.2) = 1.1749$

13. Solve $y' = \frac{y^2 - 2x}{y^2 + x}$, $y(0) = 1$ for $x = 0.1$ and $x = 0.2$ using Runge–Kutta method.

Ans. $y(0.1) = 1.091$, $y(0.2) = 1.168$

14. Using Runge–Kutta method, solve $y' = x + y$, $y(0) = 1$ for $x = 0.2$.

Ans. $y(0.2) = 0.2428$

15. Use Runge–Kutta method to solve $y' = -xy$, $y(0) = 1$ for $x = 0.2$.

Ans. 0.9801987

16. Use Runge–Kutta method to solve $\frac{dy}{dx} = 1 + xz$, $y(0) = 0$; $\frac{dz}{dx} = -xy$, $z(0) = 1$ for $x = 0.3$ and $x = 0.6$.

Ans. $y(0.3) = 0.3448$, $z(0.3) = 0.99$; $y(0.6) = 0.7738$, $z(0.6) = 0.9121$

17. Solve $y' = x + z$, $y(0) = 2$; $z' = x - y^2$, $z(0) = 1$ for $x = 0.1$ by Runge–Kutta method.

Ans. $y(0.1) = 2.0845$, $z(0.1) = 0.586$

18. Use Runge–Kutta method to solve $y'' = y^3$, $y(0) = 10$, $y'(0) = 5$ for $x = 0.1$.

Ans. $y(0.1) = 17.42$

19. Use fourth order Runge–Kutta method to solve $y'' = y + xy'$, $y(0) = 1$, $y'(0) = 0$ for $x = 0.2$.

Ans. $y(0.2) = 0.9802$

20. Solve $y'' = xy'^2 - y^2$, $y(0) = 1$, $y'(0) = 0$ for $x = 0.2$ using Runge–Kutta method.

Ans. $y(0.2) = 0.9801$

21. Use Heun's method to solve the initial value problem $y' = -ty$, $y(0) = 1$, over [0, 0.2] taking $h = 0.1$.

Ans. $y(0.1) = 0.995$, $y(0.2) = 0.980175$

Appendix: Model Question Papers

Model Paper I

1. Attempt any **four** parts of the following: $5 \times 4 = 20$

   (a) Define a harmonic function and conjugate harmonic function. Find the harmonic conjugate function of the function $U(x, y) = 2x(1 - y)$.

   (b) Evaluate $\int_C \left(12z^2 - 4iz\right)$ along the curve C joining the points (1, 1) and (2, 3).

   (c) State the Cauchy's integral formula. Show that $\frac{1}{2\pi i}\int_C \frac{e^{zt}}{z^2+1}dz = \sin t$ if $t > 0$ and C is the circle $|z| = 3$.

   (d) Define the Laurent series expansion of a function. Expand $f(z) = e^{z/(z-2)}$ in a Laurent series about the point $z = 2$.

   (e) Using Residue theorem, evaluate

   $$\frac{1}{2\pi i}\int_C \frac{e^{zt}\,dz}{z^2(z^2+2z+2)}.$$

   where C is the circle $|z| = 3$.

   (f) Show that $\int_\theta^{2\pi} \frac{\cos 3\theta \, d\theta}{5 - 4\cos\theta} = \frac{\pi}{12}$.

2. Attempt any **two** parts of the following: $10 \times 2 = 20$

   (a) The first four moments of a distribution about the value '0' are -0.20, 1.76, -2.36 and 10.88. Find the moments about the mean and measure the kurtosis.

   (b) Fit a second degree parabola to the following data

x	1	2	3	4	5	6	7	8	9	10
y	124	129	140	159	228	289	315	302	263	210

   (c) In a partially destroyed laboratory record of an analysis of correlation data, the following results only are legible. Variance of $x = 9$.
   Regression lines

   $$8x - 10y + 66 = 0$$
   $$40x - 18y - 214 = 0$$

   What were

   (i) The mean value of x and y

   (ii) The standard deviation of y and

   (iii) The coefficient of correlation between x and y.

3. Attempt any **two** parts of the following: $10 \times 2 = 20$

(a) To test the effectiveness of inoculation against chotera, the following table was obtained.

	Attacked	Not attacked	Total
Inoculated	30	160	190
Not inoculated	140	460	600
Total	**170**	**620**	**790**

(The figures represent the number of persons).

Use x^2 –test to Defend or refute the statement. The inoculation prevents attack from cholera.

(b) If there are 3 misprints in a book of 1000 pages, find the probability that a given page will contain (i) no misprint (ii) more than 2 misprints.

(c) Show that the mean deviation from the mean of the normal distribution is about $\frac{4}{5}$ of its standard deviation.

4. Attempt any **two** parts of the following: $10 \times 2 = 20$

(a) Derive the Newton-Raphson formula for finding a root of a non-linear equation. Find a root of

$$f(x) = x^3 + 2x^2 + 10x - 20 = 0$$

up to 10 iterations.

(b) Define the shift operator, forward and backward difference operators, the central difference operator and the average operators. Establish:

(i) $\left(E^{\frac{1}{2}} + E^{\frac{1}{2}}\right) = 2\left(1 + \frac{1}{2}\Delta\right)(1+\Delta)^{-\frac{1}{2}}$

(ii) $\Delta(1+\Delta)^{-\frac{1}{2}} = \nabla(1-\Delta)^{-\frac{1}{2}}$

(iii) $\mu = \sqrt{\left(1 + \frac{1}{4}\delta^2\right)}$

where all the above notations have usual meanings.

(c) Develop the divided-difference table from the data given below and obtain the interpolation polynomial $f(x)$:

x	1	3	5	7	11
$f(x)$	5	11	17	23	29

also, find the value of $f(19.5)$.

5. Attempt any **two** parts of the following: $10 \times 2 = 20$

(a) Describe a method for solving a system of linear equations. Solve the following system of linear equations using Gauss-Seidel method:

$$23x_1 + 13x_2 + 3x_3 = 29$$
$$5x_1 + 23x_2 + 7x_3 = 37$$
$$11x_1 + x_2 + 23x_3 = 43$$

(b) Derive Simpson's $\frac{3}{8}$ – formula for numerical integration. Using this rule evaluate $\int_0^1 \frac{dx}{x^3+x+1} dx$. Choose with steplength 0.25.

(c) Solve the following initial value problem

$$\frac{dy}{dx} = -2xy^2, y(0) = 1,$$

with h = 0.2 on interval [0, 0.6] using fourth order Runge-Kutta method. Compare with the exact solution.

SOLUTIONS

1. (a) Definition: Article work.

Further, we have

$$u(x, y) = 2x - 2xy.$$

Therefore

$$u_1(x, y) = \frac{\partial u}{\partial x} = 2 - 2y,$$

$$u_2(x, y) = \frac{\partial u}{\partial y} = -2x$$

and so

$$f'(z) = u_1(z,0) - i\, u_2(z,0)$$

$$= 2 - i(-2z) = 2 + 2i\, z$$

Integrating, we get

$$f(z) = 2z + 2\, i \frac{z^2}{2} + C\, i$$

$$= 2z + i\, z^2 + C\, i$$

$$= 2(x + i\, y) + i(x + i\, y)^2 + C\, i$$

$$= 2x + 2\, i\, y + i(x^2 - y^2 + 2ixy) + C\, i$$

$$= (2x - 2xy) + i(x^2 - y^2 + 2y + C)$$

$$= u(x, y) + i\, v(x, y), \text{ say.}$$

Comparing real and imaginary parts, we have

$$u(x, y) = 2x - 2xy$$

$$v(x, y) = x^2 - y^2 + 2y + c$$

(b) Equation of the line passing through (1,1) and (2,3) is

$$\frac{y-1}{x-1} = \frac{3-1}{2-1} = \frac{2}{1}$$

or

$$y-1=2(x-1)$$

or

$$y=2x-1$$

Therefore

$$z=x+iy=x+i(2x-1)=(2i+1)x-i$$

and

$$dz=(2i+1)dx.$$

Hence

$$\int_c (12z^2-4iz)dz$$

$$=\int_c [12(x+iy)^2-4i(x+iy)]dz$$

$$=\int_c \{12[x+i(2x-1)]^2-4i[x+i(2x-1)]dz$$

$$=\int_c \{12[x^2+i^2(4x^2+1-4x)+2ix(2x-1)-4i[x+i(2x-1)]\}dz$$

$$=\int \{12[x^2-4x^2-1+4x+4ix^2-2ix]-4ix+8x-4\}dz$$

$$=\int \{12[-3x^2-1+4ix^2+4x-2ix]-4ix+8x-4\}dz$$

$$=\int_1^2 [-36x^2-16+48ix^2+56x-28ix](2i+1)dx$$

$$=(2i+1)\left[-36\frac{x^3}{3}-16+48i\frac{x^3}{3}+56\frac{x^2}{2}-28i\frac{x^2}{2}\right]_1^2$$

$$=(2i+1)\{[-12(8)-32+16i(8)+28(4)-14i(4)]$$
$$-[-12(1)-16+16i+28-14i]\}$$
$$=(2i+1)[-16+112i]=80i-240=80(i-3).$$

(c) For statement of Cauchy's integral formula, see Theorem 1.10.

Let $f(z)=e^{tz}$. Then f is analytic within the circle $|z|=3$. Also $z=\pm i$ lie within $|z|=3$. Hence, by Cauchy's Integral Formula, we have

$$\int_c \frac{e^{tz}}{z^2+1}dz=\int_c \frac{e^{tz}}{(z-i)(z+i)}dz$$

$$=\frac{1}{2i}\left[\int_c \frac{e^{tz}}{z-i}dz-\int_c \frac{e^{tz}}{z+i}dz\right]$$

$$=\frac{1}{2i}[2\pi i\,f(i)-2\pi i\,f(-i)]$$

$$= \frac{1}{2i}[2\pi i e^{it} - 2\pi i e^{-it}]$$

$$= \pi[e^{it} - e^{-it}] = 2\pi i \sin t.$$

(d) For definition, see Theorem 1.20.
We have

$$f(z) = e^{z/(z-2)}.$$

Putting $z - 2 = u$, we have

$$e^{z/(z-2)} = e^{\frac{u+2}{u}} = e^{1+\frac{2}{u}}$$

$$= e\, e^{\frac{2}{u}} = e\left[1 + \frac{2}{u} + \frac{1}{2}\left(\frac{2}{u}\right)^2 + \frac{1}{3!}\left(\frac{2}{u}\right)^3 + \ldots\right]$$

$$= e\left[1 + \frac{2}{u} + \frac{2^2}{2u^2} + \frac{1}{3!}\frac{2^3}{u^3} + \ldots\right]$$

$$= e\left[1 + \frac{2}{z-2} + \frac{2^2}{2!(z-2)^2} + \frac{2^3}{3!(z-2)^3} + \ldots\right]$$

(e) The integrand $\dfrac{e^{zt}}{z^2(z^2+2z+2)}$ has a double pole at $z = 0$ and two simple poles at $z = -1+i$ and $z = -1-i$. All these poles lie within $|z| = 3$. Now

$$\operatorname{Re}s(0) = \lim_{z\to 0}\frac{1}{1!}\frac{d}{dz}\left[z^2\frac{e^{zt}}{z^2(z^2+2z+2)}\right]$$

$$= \lim_{z\to 0}\frac{(z^2+2z+2)(te^{zt}) - e^{zt}(2z+2)}{(z^2+2z+2)^2} = \frac{t-1}{2},$$

$$\operatorname{Re}s(-1+i) = \lim_{z\to -1+i}\left\{[z-(-1+i)]\frac{e^{zt}}{z^2(z^2+2z+2)}\right\} = \frac{e^{(-1+i)t}}{4}$$

$$\operatorname{Re}s(-1-i) = \lim_{z\to -1-i}\left\{[z-(-1-i)]\frac{e^{zt}}{z^2(z^2+2z+2)}\right\} = \frac{e^{(-1-i)t}}{4}.$$

Hence

$$\int_c \frac{e^{zt}}{z^2(z^2+2z+2)}dz = 2\pi i[\Sigma R_i]$$

or

$$\frac{1}{2\pi i}\int_c \frac{e^{zt}}{z^2(z^2+2z+2)}dz = \Sigma R_i = \frac{t-1}{2} + \frac{e^{(-1+i)t}}{4} + \frac{e^{(-1-i)t}}{4}$$

$$= \frac{t-1}{2} + \frac{1}{2}e^{-t}\cos t.$$

(f) Example 1.93.

2. (a) We are given that

$$\mu_1' = -0.20$$
$$\mu_2' = 1.76$$
$$\mu_3' = -2.36$$
$$\mu_4' = 10.88$$

Then the moments about the mean are

$$\mu_1 = 0,$$

$$\mu_2 = \mu_2' - (\mu_1')^2 = 1.76 - (-0.20)^2$$
$$= 1.76 - 0.04 = 1.72,$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$
$$= -2.36 - 3(1.76)(-0.20) + 2(-0.20)^3$$
$$= -2.36 + 1.056 - 2(0.008) = -1.32,$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$
$$= 10.88 - 4(-2.36)(-0.20) - 6(1.76)(-0.20)^2 - 3(-0.20)^4$$
$$= 10.88 + 1.888 + 0.4224 - 0.0048 = 13.1856$$

Further, measure of kurtosis is given by

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{13.1856}{(1.72)^2} = \frac{13.1856}{2.984} = 4.457.$$

(b) Changing the variables by

$$u = x - 5, \quad v = y - 220,$$

we get the following table

u	v	u^2	u^3	u^4	u^v	u^2v
−4	−96	16	−64	256	384	−1536
−3	−91	9	−27	81	273	−819
−2	−80	4	−8	16	160	−320
−1	−61	1	−1	1	61	−61
0	08	0	0	0	0	0
1	69	1	1	1	69	69
2	95	4	8	16	190	380
3	82	9	27	81	246	738
4	43	16	64	256	172	688
5	−10	25	125	625	−50	−250
5	**−41**	**85**	**125**	**833**	**1505**	**−1111**

The normal equations are

$$10a + 5b + 85c = -41$$

$$5a + 85b + 125c = 1505$$

$$85a + 125b + 822c = -1111$$

Solving these equation, we get

$$a = 64.033, \quad b = 100.201, \quad c = -81.71.$$

Hence

$$v = 64.033 + 100.201\,u - 81.71\ u^2$$

or

$$(y - 220) = 64.033 + 100.201(x - 5) - 81.71(x - 5)^2$$

or

$$y = -1758.72 + 917.30x - 81.71x^2$$

(c) Similar to Example 2.27.
The regression lines are

$$8x - 10y + 66 = 0 \tag{1}$$

$$40x - 18y - 214 = 0. \tag{2}$$

Solving these, we get their point of intersection as (13, 17). But point of intersection of regression lines is nothing but $(\bar{x}, \bar{y})$. Hence

$$\bar{x} = 13 \text{ and } \bar{y} = 17.$$

As in Example 2.26, we observe that the line (1) is line of regression of Y on X and (2) is line of regression of X on Y. Also from (1) and (2)

$$y = \frac{4}{5}x + 6.6$$

$$x = \frac{9}{20}y + \frac{214}{40}.$$

Thus the regression coefficients are

$$b_{yx} = \frac{4}{5} \quad \text{and} \quad b_{xy} = \frac{9}{20}.$$

Therefore the coefficient of correlation is given by

$$[\rho(X,Y)]^2 = b_{yx}b_{xy} = \frac{9}{25}$$

or

$$\rho(X,Y) = \frac{3}{5}.$$

Also

$$\sigma_x^2 = 9 \text{ and } b_{yx} = \frac{\text{cov}(X,Y)}{\sigma_x^2}.$$

Therefore

$$Cov(X,Y) = b_{yx}\left(\sigma_x^2\right) = \frac{4}{5}.9 = \frac{36}{5}.$$

Now

$$b_{xy} = \frac{Cov(X,Y)}{\sigma_y^2} \quad \text{so that} \quad \sigma_y^2 = \frac{Cov(X,Y)}{bxy} = 16.$$

Hence $\sigma_y = 4$.

3. (a) Similar to Example 2.120.

Let the null hypothesis be

H_o : The inoculation does not prevent attack from cholera.

The alternative hypothesis is

H_1 = The inoculation prevents attack from cholera.

Using the formula $\frac{Row\,total \times column\,total}{grand\,total}$, the expected frequencies are

			Total
	41	149	190
	129	471	600
Total	**170**	**620**	**790**

Therefore

$$\chi^2 = \frac{(30-41)^2}{41} + \frac{(160-149)}{149} + \frac{(140-129)^2}{129} + \frac{(460-471)^2}{471}$$

$$= \frac{121}{41} + \frac{121}{149} + \frac{121}{129} + \frac{121}{471}$$

$$= 2.951 + 0.812 + 0.938 + 0.257 = 4.958.$$

Also, the number of degree of freedom = (2 – 1) (2 – 1) = 1.

From χ^2 – table, we note that $\chi^2_{0.05}$ for v = 1 is 3.840. Thus the calculated value of χ^2 is greater than the tabulated value of χ^2. Hence H_0 is rejected and H_1 is accepted.

(b) Here, number of pages in the book is 1000 (very large). Then the average number of typographical error per page is given by

$$\lambda = \frac{3}{1000} = 0.003.$$

Therefore, probability of zero error per page is

$$P(X=0) = e^{-\lambda} = e^{-0.003} \text{ (by Poisson's distribution)}$$

$$= 0.997.$$

Further,

$$P(x > 2) = 1 - [P(X=0) + P(X=1) + P(X=2)]$$

$$= 1 - \left[e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2}{2} e^{-\lambda} \right].$$

$$= 1 - e^{-0.003} - (0.003)e^{-0.003} - \frac{1}{2}(0.003)^2 e^{-0.003}$$

$$= 1 - 0.997 - .002991 - .0.000045 \;\; = -ve$$

and therefore the event is impossible.

(c) See Article 2.20 (5)

4. (a) Article 3.6

Further, the given equation is

$$f(x) = x^3 + 2x^2 + 10x - 20 = 0.$$

Proceed as in Example 3.7

(b) For definitions, please see Article 5.1

(i) Since $E = 1 + \Delta$, we have

$$E^{\frac{1}{2}} + E^{-\frac{1}{2}} = (1+\Delta)^{\frac{1}{2}} + \frac{1}{(1+\Delta)^{\frac{1}{2}}}$$

$$= \frac{1+\Delta+1}{(1+\Delta)^{\frac{1}{2}}} = \frac{2+\Delta}{(1+\Delta)^{\frac{1}{2}}}$$

$$= 2\left(1+\frac{\Delta}{2}\right)(1+\Delta)^{-\frac{1}{2}}.$$

(ii) We know that

$$\delta = \frac{\Delta}{\sqrt{1+\Delta}} = \frac{\nabla}{\sqrt{1-\nabla}} \quad \text{(See relations 5.10 and 5.11).}$$

Hence

$$\frac{\Delta}{\sqrt{1+\Delta}} = \frac{\nabla}{\sqrt{1-\nabla}}$$

or

$$\Delta(1+\Delta)^{-\frac{1}{2}} = \nabla(1-\nabla)^{-\frac{1}{2}}.$$

(iii) See relation 5.16

(c) The divided difference table is

	x	$f(x)$				
x_0	1	5				
			3			
x_1	3	11		0		
			3		0	
x_2	5	17		0		$-\frac{1}{320}$
			3		$-\frac{1}{32}$	
x_3	7	23		$-\frac{1}{4}$		
			$\frac{3}{2}$			
x_4	11	29				

Putting these values in the Newton's Divided Difference formula, we have

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0 . x_1, x_2, x_3)$$
$$+ (x - x_0)(x - x_1)(x - x_2)(x - x_3) f(x_0, x_2, x_3, x_4)$$
$$= 5 + 3(x-1) - \frac{1}{320}(x-1)(x-3)(x-5)(x-7)$$
$$= 2 + 3x - \frac{1}{320}[x^4 - 16x^3 + 86x^2 - 176x + 105]$$
$$= \frac{-1}{320}x^4 + \frac{1}{20}x^3 - \frac{86}{320}x^2 + \frac{784}{320}x - \frac{745}{320}.$$

Putting $x = 19.5,$ we shall get $f(19.5).$

5. (a) Article work of Chapter 4

Numerical is similar to Example 4.16.

$$x_1 = \frac{29 - 13x_2 - 3x_3}{23}$$
$$x_2 = \frac{37 - 5x_1 - 7x_3}{23}$$
$$x_3 = \frac{43 - 11x_1 - x_2}{23}.$$

Taking initial approximation as (0,0,0), we have

$$x_1^{(1)} = \frac{29}{23} = 1.26$$
$$x_2^{(1)} = \frac{37 - 5(1.26) - 0}{23} = 1.3$$
$$x_3^{(1)} = \frac{43 - 11(1.26) - 1.3}{23} = 1.2$$

Proceed to get further refinements using Gauss – Seidel Method.

(b) For Derivation see Article 7.1(B)

The table for the values of the integrand for step length $h = 0.25$ is

x	0	0.25	0.50	0.75	1.0
$f(x)$	1	1.08	0.6154	0.4604	0.3333

By Simpson's $\frac{3}{8}$ rule,

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{33h}{8}[f_0 + f_n) + 3(f_1 + f_2 + f_4 + f_5 + ... + f_{n-1})$$
$$+2(f_3 + f_6 + ... f_{n-3})]$$
$$= 0.937[1.3333 + 3(1.08 + 0.6154) + 2(0.4604)] = 0.6878.$$

(c) The given equation is

$$\frac{dy}{dx} = -2xy^2, \quad y(0) = 1.$$

For h = 0.2, we have

$$k_1 = hf(x_0, y_0) = h(-2x_0 y_0^2)$$
$$= 0.2[-2(0)(1^2)] = 0$$
$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2f(0.1, 1)$$
$$= 0.2[-2(0.1)(1^2)] = -0.04$$
$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2f(0.1, 0.98)$$
$$= 0.2[-2(0.1)(0.98)^2] = -0.0384$$
$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2f(0.2, 0.9616)$$
$$= 0.2[-2(0.2)(0.9616)^2] = -0.07397.$$

Therefore, by Runge – Kutta method of order 4, we have

$$y(0.2) = y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= 1 + \frac{1}{6}[0 + 2(-0.04) + 2(-0.0384) + (-0.07397)]$$
$$= 1 + \frac{1}{6}[-0.08 - 0.0768 - 0.07397] = 0.76923.$$

Repeating the process, $y(0.4)$ and $y(0.6)$ may be calculated by the candidate.

Model Paper II

1. Attempt any **two** of the following: 10 × 2 = 20

(a) Find the Fourier transform of

$$f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$$

Using this result evaluate

$$\int_{-\infty}^{\infty} \frac{\sin at \cos at}{t} dt.$$

(b) State and prove the convolution theorem for the Fourier transform. Verify this theorem for the functions

$$f(t) = e^{-t} \text{ and } g(t) = \sin t.$$

(c) Define the Z-transform. Solve the difference equation

$$y_{n+2} + 6y_{n+1} + 9y_n = 2^n \text{ with } y_0 = y_1 = 0.$$

2. Attempt any **two** of the following: $10 \times 2 = 20$

(a) If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u - v = e^{-x}[(x-y)\sin y - (x+y)\cos y]$ then find u, v and the analytic function $f(z)$.

(b) State and prove the Cauchy's integral theorem for the derivative of analytic function.

(c) State and prove Liouville's theorem and using this theorem prove that every polynomial equation of degree n has n roots.

3. Attempt any **four** parts of the following: $5 \times 4 = 20$

(a) Expand the function $f(z) = \tan^{-1} z$ in powers of z.

(b) Define the singularity of a function. Find the singularity (ties) of the functions.

(i) $f(z) = \sin \dfrac{1}{z}$

(ii) $g(z) = \dfrac{e^{\frac{1}{z}}}{z^2}$

(c) Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+2x+2)}$.

(d) Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta}$ if $a > |b|$.

(e) Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2 + 2x + 5}\,dx$.

(f) Define the conformal mapping. Prove that analytic function $f(z)$ ceases to be conformal at the points z_0, where $f'(z_0) = 0$.

4. Attempt any **two** of the following: $10 \times 2 = 20$

(a) Define the coefficients of skewness and Kurtosis. Compute the coefficient of skewness from the following data:

x:	6	7	8	9	10	11	12
f:	3	6	9	13	8	5	4

(b) Define the coefficients of regression and correlation. Calculate the coefficient of correlation between the marks obtained by 8 students in Mathematics and Statistics:

Students:	A	B	C	D	E	F	G	H
Mathematics:	25	30	32	35	37	40	42	45
Statistics:	08	10	15	17	20	23	24	25

(c) Define the binomial distribution and obtain the expression for the Poisson distribution as a limiting case of binomial distribution.

5. Attempt any **two** of the following: $10 \times 2 = 20$

(a) Find the roots of the cubic equation

$$ax^3 + 3bx^2 + 3cx + d = 0$$

using Cardan's method.

(b) Fit a parabola $y = ax^2 + bx + c$ to the following data taking x as independent variable:

x:	1	2	3	5	7	11	13	17	19	23
f:	2	3	5	7	11	13	17	19	23	29

(c) The regression lines of y on x and of x on y are respectively $y = ax + b$ and $x = cy + d$. Show that the means are $\overline{x} = \dfrac{bc+d}{1-ac}$ and $\overline{y} = \dfrac{ad+b}{1-ac}$ and correlation coefficient between x and y is $\sqrt{ac}$. Also, show that the ratio of the standard deviations of y and x is $\sqrt{a/c}$.

SOLUTIONS

1. Not related to New syllabus .

2. (a) Let

$$u + iv = f(z) \tag{1}$$

so that

$$iu - v = i\,f(z). \tag{2}$$

Adding (1) and (2), we get

$$u - v + i(u+v) = (1+i)f(z) = F(z) = U + iV, \text{ say.}$$

Then $F(z) = U + iV$ is an analytic function. We have

$$U = u - v = e^{-x}[(x-y)\sin y - (x+y)\cos y] \text{ (Given).}$$

Therefore

$$\frac{\partial U}{\partial x} = e^{-x}(\sin y - \cos y) - e^{-x}[(x-y)\sin y - (x+y)\cos y]$$

$$= \phi_1(x, y),$$

$$\frac{\partial U}{\partial y} = e^{-x}[x\cos y - (y\cos y + \sin y) + x\sin y - (\cos y - y\sin y)]$$

$$= e^{-x}[(x-y-1)\cos y - (1-x-y)\sin y] = \phi_2(x, y).$$

Therefore, by Milne's Method,

$$F(z) = \int [\phi_1(z,0) - i\,\phi_2(z,0)]dz$$

$$= \int [e^{-z}(z-1) - i(z-1)]dz$$

$$= -e^{-z}(z - 1 - iz + i) - (1-i)e^{-z} + c_1$$

$$= z e^{-z}(i-1) + c_1.$$

Thus

$$(1+i)f(z) = ze^{-z}(i-1)+c_1.$$

or

$$\begin{aligned} f(z) &= \frac{i-1}{i+1} z\, e^{-z} + c_2 \\ &= i\, z\, e^{-z} + c_2 = i(x+iy)e^{-(x+iy)} + c_2 \\ &= e^{-x}(\cos y - i \sin y)(ix - y) + c_2 \\ &= e^{-x}[i\, x\cos y + x \sin y - y\cos y + iy \sin y] + c_2 \\ &= e^{-x}(x \sin y - y \cos y) + i\, e^{-x}(x \cos y + y \sin y). \end{aligned}$$

Comparing real and imaginary parts, we have

$$u = e^{-x}(x \sin y - y \cos y),$$

$$v = e^{-x}(x \cos y + y \sin y).$$

(b) Please see Theorem … 1.11.

(c) Please see Theorem … 1.15. To prove fundamental theorem of algebra, let the polynomial equation $p(z)=0$ has no root. Then $f(z) = \frac{1}{p(z)}$ is analytic for all z. Also $|f(z)| = \frac{1}{|p(z)|}$ is bounded. Hence, by Liouville's Theorem, it follows that $f(z)$ and thus $p(z)$ must be a constant. Thus we arrive at a contradiction. Hence $p(z)=0$ has at least one root. This proves the theorem.

3. (a) Let

$$f(z) = \tan^{-1} z \tag{1}$$

Then

$$f'(z) = \frac{1}{1+z^2} \quad \text{or} \quad (1+z^2)f'(z) - 1 = 0. \tag{2}$$

Differentiating once again, we get

$$(1+z^2)f''(z) + 2zf'(z) = 0 \tag{3}$$

Differentiating n times the relation (3), we have

$$(1+z^2)f^{(n+2)}(z) + 2(n+1)xf^{(n+1)}(z) + n(n+1)f^{(n)}(z) = 0 \tag{4}$$

Substituting $z = 0$ in (1), (2), (3) and (4), we get

$$f(0) = 0\ ,\ f'(0) = 1\ ,\ f''(0) = 0 \text{ and}$$

$$f^{(n+2)}(0) = -n(n+1)f^{(n)}(0).$$

Taking n = 1, 2, 3, we get

$$f^{(3)}(0) = -2,\ f^{(4)}(0) = 0,\ f^{(5)}(0) = 4\,! \text{ and so on}$$

Hence, by Taylor's theorem,

$$\tan^{-1} z = f(0) + zf'(0) + \frac{z^2}{2\,!} f''(0) + \frac{z^3}{3\,!} f'''(0) + \ldots$$

$$= z - \frac{z^3}{3} + \frac{z^5}{5} - \dots .$$

(b) A singularity of a function $f(z)$ is a point at which the function ceases to be regular (analytic).

(i) We have

$$f(z) = \sin\frac{1}{z} = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \dots$$

Since the series does not terminate, there are infinite number of terms in the principal part of the expansion of $\sin\frac{1}{z}$. Hence z = 0 is an isolated essential singularity of $\sin\frac{1}{z}$

(ii) The given function is

$$g(z) = \frac{e^{\frac{1}{z}}}{z^2} = \frac{1}{z^2}\left[1 + \frac{1}{z} + \frac{1}{2!}.\frac{1}{z^2} + \frac{1}{3!}.\frac{1}{z^3} + \dots\right]$$

$$= \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{2!z^4} + \frac{1}{3!z^5} + \dots$$

Since the series does not terminate, z = 0 is an isolated essential singularity.

(c) We have

$$I = \int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} \qquad a > |b|.$$

Substituting $z = e^{i\theta}$, we get $d\theta = \frac{dz}{iz}$ and so

$$I = \frac{1}{i}\int_{|z|=1} \frac{dz}{z\left[a + \frac{b}{2i}\left(z - \frac{1}{z}\right)\right]} = 2\int_{|z|=1} \frac{dz}{bz^2 + 2iaz - b}$$

The poles of the integrand are

$$z = \frac{-a\ i \pm i\sqrt{a^2 - b^2}}{b}.$$

Thus the poles are

$$\alpha = \frac{-ai \pm i\sqrt{a^2 - b^2}}{b} \quad \text{and} \quad \beta = \frac{-ai - i\sqrt{a^2 - b^2}}{b}.$$

Out of these only α lines inside $|z| = 1$. Now

$$Res(\alpha) = \lim_{z\to\alpha}(z - \alpha),\ \frac{2}{bz^2 + 2aiz - b}$$

$$= \lim_{z\to\alpha}\frac{2}{2bz + 2ai} = \frac{1}{b\alpha + ai} = \frac{1}{i\sqrt{a^2 - b^2}}.$$

Hence

$$I = 2\pi i\left(\frac{1}{i\sqrt{a^2-b^2}}\right) = \frac{2\pi}{\sqrt{a^2-b^2}}.$$

(d) The given integral is

$$I = \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)}.$$

Consider the integral

$$\int_c \frac{z^2}{(z^2+1)^2(z^2+2z+2)} dz.$$

The poles of the integrand enclosed by the contour are z = i of order 2 and $z = -1 + i -1 + i$.
Further,

$$\text{Re}\,s(i) = \lim_{z \to i} \frac{d}{dz}\left[(z-i)^2 \frac{z^2}{(z+i)^2(z-i)^2(z^2+2z+2)}\right] = \frac{9i-12}{100}$$

$$\text{Re}\,s(-1+i) = \lim_{z \to -1+i} (z+1-i)\frac{z^2}{(z^2+1)^2(z+1-i)(z+1+i)} = \frac{3-4i}{25}$$

Hence

$$\int_\Gamma \frac{z^2 dz}{(z^2+1)^2(z^2+2z+2)} + \int_{-R}^{R} \frac{x^2}{(x^2+1)^2(x^2+2x+2)} dx = 2\pi i \quad \text{(sum of residues)}$$

The first integral tends to zero as $R \to \infty$. Hence is the limit as $R \to \infty$, we get

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2(x^2+2x+2)} dx = 2\pi i\left[\frac{9i-12}{100} + \frac{3-4i}{25}\right] = \frac{7\pi}{50}.$$

(e) Similar to Example 1.104.
Consider

$$\int_C f(z)dz = \int_C \frac{ze^{\pi i z} dz}{z^2+2z+5} = \int_{-R}^{R} f(x)dx + \int_\Gamma f(z)dz$$

$$= 2\pi i \Sigma(\text{Re}\,s).$$

Since $\frac{1}{z^2+2z+5} \to 0$ as $z \to \infty$, by Jordan's Lemma $\int_\Gamma f(z)dz \to 0$ as $R \to \infty$.
Further, the poles of $f(z)$ are $-1 \pm 2i$. The pole $-1+2i$ lies in the semi-circular contour.

Then

$$\text{Re}\,s(-1+2i) = \lim_{z \to -1+2i} (z+1-2i)\frac{ze^{\pi i z}}{(z+1-2i)(z+1+2i)}$$

$$= \lim_{z \to -1+2i} \frac{ze^{\pi i z}}{z+1+2i} = \frac{(-1+2i)}{4i} e^{(-2-i)\pi}.$$

Therefore

$$\int_C f(z)dz = 2\pi i\left[\frac{-1+2i}{4i}e^{(-2-i)\pi}\right]$$

$$= \frac{\pi}{2}(-1+2i)e^{(-2-i)\pi} = \frac{\pi}{2e^{2\pi}}(-1+2i).e^{-i\pi}$$

$$= \frac{\pi}{2e^{2\pi}}(-1+2i)(\cos\pi - i\sin\pi) = \frac{\pi}{2}(-1+2i)\cos\pi.$$

Equating real and imaginary part, we have

$$\int_{-\infty}^{\infty}\frac{x\sin x}{x^2+2x+5}dx = -\pi.$$

(f) Theorem 1.25

If $f'(z_0) = 0,$ then arg $f'(z_0)$ is indeterminate and so the mapping is not conformal.

4. (a) For definitions see Article 2.3 and 2.4.
The given data is

x:	6	7	8	9	10	11	12
f:	3	6	9	13	8	5	4

We form the table given below:

x	f	fx	fx^2
6	3	18	108
7	6	42	294
8	9	72	576
9	13	117	1053
10	8	80	800
11	5	55	605
12	4	48	576
	48	**432**	**4012**

Then

$$Mean(\overline{x}) = \frac{432}{48} = 9.$$

The value of x corresponding to the maximum frequency is 9. Hence mode is 9.
Also

$$\sigma^2 = \frac{1}{\Sigma f_i}\Sigma f_i x_i^2 - (\overline{x})^2$$

$$= \frac{1}{48}(4012) - 81$$

$$= 83.583 - 81 = 2.583.$$

Pearson's coefficient of skewness $= \dfrac{mean - \text{mod}\, e}{\sigma} = 0$, and so there is no departure from symmetry.

(b) For definition, please see articles 2.7 and 2.8. Setting $u = x - 35$, $v = y - 17$, we have the following table

Math (x)	Stat (y)	u	v	u^2	v^2	uv
25	8	–10	–9	100	81	90
30	10	–5	–7	25	49	35
32	15	–3	–2	9	4	6
35	17	0	0	0	0	0
37	20	2	3	4	9	6
40	23	5	6	25	36	30
42	24	7	7	49	49	49
45	25	10	8	100	64	80
		6	**6**	**312**	**292**	**296**

Now

Karl Pearson coefficient of correlation is given by

$$\rho(X,Y) = \frac{\text{cov}(X,Y)}{\sigma_x \sigma_y}$$

$$= \frac{n\Sigma u_i v_i - \Sigma u_i \Sigma v_i}{\sqrt{n\Sigma u_i^2 - (\Sigma u_i)^2}\sqrt{n(\Sigma v_i^2 - (\Sigma v_i)^2}}$$

$$= \frac{8(296) - 6(6)}{\sqrt{2496 - 36}\sqrt{2336 - 36}} = \frac{2332}{\sqrt{2460}\sqrt{2300}} = 0.938.$$

Since $\rho(X,Y)$ is close to 1, there is high degree of positive correlation.

(c) Please see articles 2.15 and 2.17

5. (a) If is related to theory to equations, which is not included in the new syllabus

(b) Using transformations $X = x - 13$ and $Y = y - 17$ we construct the following table

X	Y	X^2	X^3	X^4	XY	X^2Y
–12	15	144	–1728	20736	180	–2160
–11	–14	121	–1331	14641	154	–1694
–10	–12	100	–1000	10000	120	–1200
–8	–10	64	–512	4096	80	–640
–6	–6	36	–216	1296	36	–216
–2	–4	4	–8	16	8	–16
0	0	0	0	0	0	0
4	2	16	64	256	8	32
6	6	36	216	1296	36	216
10	12	100	1000	10000	120	1200
–29	**–41**	**621**	**3515**	**62337**	**862**	**4478**

The normal equation are

$$10a - 29b + 621c = -41$$
$$-29a + 621b + 3515c = 862$$
$$621a + 35154b + 62337c = 4478.$$

Solve these equations for a, b, c and put in

$$Y = a + bX + cX^2$$

and then substitute $X = x - 13$ and $Y = y - 17$ to get the required parabola.

(c) The regression lines of Y on X and of X on Y are respectively.

$$y = ax + b \quad (1)$$

and

$$x = cy + d \quad (2)$$

Since the point of intersection of the regression lines is $(\bar{x}, \bar{y})$, the means $\bar{x}$ and $\bar{y}$ lie on the two regression lines. Thus, we have

$$a\bar{x} - \bar{y} + b = 0 \text{ and } \bar{x} - c\bar{y} - d = 0.$$

Solving these equation, we get the means

$$\bar{x} = (bc + d)/(1 - ac) \quad \text{and} \quad \bar{y} = (ad + b)/(1 - ac)$$

Further, the equation (1) and (2) imply that the regression coefficients are

$$b_{yx} = a \quad \text{and} \quad b_{xy} = c$$

We know that

$$b_{yx} = \frac{Cov(X,Y)}{\sigma_x^2} \quad \text{and} \quad b_{xy} = \frac{Cov(X,Y)}{\sigma_y^2}$$

or

$$a = \frac{\text{cov}(X,Y)}{\sigma_x^2} \quad \text{and} \quad c = \frac{\text{cov}(X,Y)}{\sigma_y^2}.$$

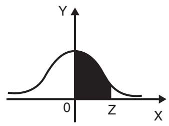
Hence

$$\frac{a}{c} = \frac{\sigma_y^2}{\sigma_x^2}$$

or

$$\frac{\sigma_y}{\sigma_x} = \sqrt{\frac{a}{c}}.$$

Area Under Normal Curve from 0 to z

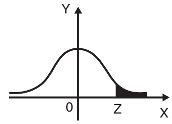


z	00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.0000	0.004	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0479	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0754
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2258	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2518	0.2549
0.7	0.2580	0.2612	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2996	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4611	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993

Examples: (i) $P(0 \le z \le 0.27) = 0.1064$ (ii) $P(z \ge 0.81) = 0.5 - P(0 \le z \le 0.81) = 0.5 - 0.2910$ $= 0.2090$ (iii) $P(-3 \le z \le 3) = P(-3 \le z \le 0) + P(0 \le z \le 3) = P(0 \le z \le 3) + P(0 \le z \le 3)$

$$= 2P(0 \le z \le 3) = 2(0.4987) = 0.9974.$$

Area under the Normal Curve from z to ∞



z										
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.1	0.4602	0.4562	0.4521	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4246
0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
0.6	0.2742	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2482	0.2451
0.7	0.2420	0.2388	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
0.8	0.2119	0.2090	0.2061	0.2033	0.2004	0.1977	0.1949	0.1922	0.1894	0.1867
0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007

Examples: (i) $P(z \geq 0.81) = 0.2090$ (ii) $P(0 \leq z \leq 0.27) = 0.5 - P(z \geq 0.27) = 0.5 - 0.3936 = 0.1064$.

Values of | t | with probability P and degree of freedom v

v	$P = 0.50$	$P = 0.10$	$P = 0.05$	$P = 0.02$	$P = 0.01$
1	1.000	6.340	12.710	31.820	63.660
2	0.816	2.920	4.300	6.960	9.920
3	0.765	2.350	3.180	4.540	5.840
4	0.741	2.130	2.780	3.750	4.600
5	0.727	2.020	2.570	3.360	4.030
6	0.718	1.940	2.450	3.140	3.710
7	0.711	1.900	2.360	3.000	3.500
8	0.706	1.860	2.310	2.900	3.360
9	0.703	1.830	2.260	2.820	3.250
10	0.700	1.810	2.230	2.760	3.170
11	0.697	1.800	2.200	2.720	3.110
12	0.695	1.780	2.180	2.680	3.060
13	0.694	1.770	2.160	2.650	3.010
14	0.692	1.760	2.140	2.620	2.980
15	0.691	1.750	2.130	2.600	2.950
16	0.690	1.750	2.120	2.580	2.920
17	0.689	1.740	2.110	2.570	2.900
18	0.688	1.730	2.100	2.550	2.880
19	0.688	1.730	2.090	2.540	2.860
20	0.687	1.720	2.090	2.530	2.840
21	0.686	1.720	2.080	2.520	2.830
22	0.686	1.720	2.070	2.510	2.820
23	0.685	1.710	2.070	2.500	2.810
24	0.685	1.710	2.060	2.490	2.800
25	0.684	1.710	2.060	2.480	2.790
26	0.684	1.710	2.060	2.480	2.780
27	0.684	1.700	2.050	2.470	2.770
28	0.683	1.700	2.050	2.470	2.760
29	0.683	1.700	2.040	2.460	2.760
30	0.683	1.700	2.04	2.460	2.750

Examples: (i) $t_{0.05} = 2.26$ for $v = 9$ (ii) $t_{0.05} = 2.14$ for $v = 14$ (ii) $t_{0.01} = 3.50$ for $v = 7$.

Values of χ^2 with probability *P* and degree of freedom *v*

v	$P = 0.99$	$P = 0.95$	$P = 0.50$	$P = 0.30$	$P = 0.20$	$P = 0.10$	$P = 0.05$	$P = 0.01$
1	0.0002	0.004	0.460	1.070	1.640	2.710	3.840	6.640
2	0.020	0.103	1.390	2.410	3.220	4.600	5.990	9.210
3	0.115	0.350	2.370	3.660	4.640	6.250	7.820	11.340
4	0.300	0.710	3.360	4.880	5.990	7.780	9.490	13.280
5	0.550	1.140	4.350	6.060	7.290	9.240	11.070	15.090
6	0.870	1.640	5.350	7.230	8.560	10.64	12.590	16.810
7	1.240	2.170	6.350	8.380	9.800	12.020	14.070	18.480
8	1.650	2.730	7.340	9.520	11.030	13.360	15.510	20.090
9	2.090	3.320	8.340	10.660	12.240	14.680	16.920	21.670
10	2.560	3.940	9.340	11.780	13.440	15.990	18.310	23.210
11	3.050	4.580	10.340	12.900	14.630	17.280	19.680	24.720
12	3.570	5.230	11.340	14.010	15.810	18.550	21.030	26.220
13	4.110	5.890	12.340	15.120	16.980	19.810	22.360	27.690
14	4.660	6.570	13.340	16.220	18.150	21.060	23.680	29.140
15	5.230	7.260	14.340	17.320	19.310	22.310	25.000	30.580
16	5.810	7.960	15.340	18.420	20.460	23.540	26.300	32.000
17	6.410	8.670	16.340	19.510	21.620	24.770	27.590	33.410
18	7.020	9.390	17.340	20.600	22.760	25.990	28.870	34.800
19	7.630	10.120	18.340	21.690	23.900	27.200	30.140	36.190
20	8.260	10.850	19.340	22.780	25.040	28.410	31.410	37.570
21	8.900	11.590	20.340	23.860	26.170	29.620	32.670	38.930
22	9.540	12.340	21.340	24.940	27.300	30.810	33.920	40.290
23	10.200	13.090	22.340	26.020	28.430	32.010	35.170	41.640
24	10.860	13.850	23.340	27.100	29.550	33.200	36.420	42.980
25	11.520	14.610	24.340	28.170	30.680	34.680	37.650	44.310
26	12.200	15.380	25.340	29.250	31.800	35.560	38.880	45.640
27	12.880	16.150	26.340	30.320	32.910	36.740	40.110	46.960
28	13.560	16.930	27.340	31.390	34.030	37.920	41.340	48.280
29	14.260	17.710	28.340	32.460	35.140	39.090	42.560	49.590
30	14.950	18.490	29.340	33.530	36.250	40.260	43.770	50.890

Examples: (i) $\chi^2_{0.05} = 11.07$ for $v = 5$.

***F* – Distribution $F_{0.05}(n_1, n_2)$**

$n_1\backslash n_2$	1	2	3	4	5	6	7	8	9	10	12
1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54	241.88	243.91
2	18.513	19.000	19.164	19.247	19.296	19.330	19.353	19.371	19.385	19.396	19.413
3	10.128	9.5521	9.2766	9.1172	9.0135	8.9406	8.8868	8.8452	8.8123	8.7855	8.7446
4	7.7086	6.9443	6.5914	6.3883	6.2560	6.1631	6.0942	6.0410	5.9988	5.9644	5.9117
5	6.6079	5.7861	5.4095	5.1922	5.0503	4.9503	4.8753	4.8183	4.7725	4.7351	4.6777
6	5.9874	5.1433	4.7571	4.5337	4.3874	4.2839	4.2066	4.1468	4.0990	4.0600	3.9999
7	5.5914	4.7374	4.3468	4.1203	3.9715	3.8660	3.7870	3.7257	3.6767	3.6365	3.5747
8	5.3177	4.4590	4.0662	3.8378	3.6875	3.5806	3.5005	3.4381	3.3881	3.3472	3.2840
9	5.1174	4.2565	3.8626	3.6331	3.4817	3.3738	3.2927	3.2296	3.1789	3.1373	3.0729
10	4.9646	4.1028	3.7083	3.4780	3.3258	3.2172	3.1355	3.0717	3.0204	2.9782	2.9130
11	4.8443	3.9823	3.5874	3.3567	3.2039	3.0946	3.0123	2.9480	2.8962	2.8536	2.7876
12	4.7272	3.8853	3.4903	3.2502	3.1059	2.9961	2.9134	2.8486	2.7964	2.7534	2.6866
13	4.6672	3.8056	3.4105	3.1791	3.0254	2.9153	2.8321	2.7669	2.7144	2.6710	2.6037
14	4.6001	3.7389	3.3439	3.1122	2.9582	2.8477	2.7642	2.6987	2.6458	2.6021	2.5342
15	4.5431	3.6823	3.2874	3.0556	2.9013	2.7905	2.7066	2.6408	2.5876	2.5437	2.4753
16	4.4940	3.6337	3.2389	3.0069	2.8524	2.7413	2.6572	2.5911	2.5377	2.4935	2.4247
17	4.4513	3.5915	3.1968	2.9647	2.8100	2.6987	2.6143	2.5480	2.4943	2.4499	2.3807
18	4.4139	3.5546	3.1599	2.9277	2.7729	2.6613	2.5767	2.5102	2.4563	2.4117	2.3421
19	4.3808	3.5219	3.1274	2.8951	2.7401	2.6283	2.5435	2.4768	2.4227	2.3779	2.3080
20	4.3513	3.4928	3.0984	2.8661	2.7100	2.5990	2.5140	2.4471	2.3928	2.3479	2.2776
21	4.3248	3.4668	3.0725	2.8401	2.6848	2.5727	2.4876	2.4205	2.3661	2.3210	2.2504
22	4.3009	3.4434	3.0491	2.8167	2.6613	2.5491	2.4638	2.3965	2.3419	2.2967	2.2258
23	4.2793	3.4221	3.0280	2.7955	2.6500	2.5277	2.4422	2.3748	2.3201	2.2747	2.2036
24	4.2597	3.4028	3.0088	2.7763	2.6207	2.5082	2.4226	2.3551	2.3002	2.2547	2.1834
25	4.2417	3.3852	2.9912	2.7587	2.6030	2.4904	2.4047	2.3371	2.2821	2.2365	2.1649
26	4.2252	3.3690	2.9751	2.7426	2.5868	2.4741	2.3883	2.3205	2.2655	2.2197	2.1479
27	4.2100	3.3541	2.9604	2.7278	2.5719	2.4591	2.3732	2.3053	2.2501	2.2043	2.1323
28	4.1960	3.3404	2.9467	2.7141	2.5581	2.4453	2.3593	2.2913	2.2360	2.1900	2.1179
29	4.1830	3.3277	2.9340	2.7014	2.5454	2.4324	2.3463	2.2782	2.2229	2.1768	2.1045
30	4.1709	3.3158	2.9223	2.6896	2.5336	2.4205	2.3343	2.2662	2.2107	2.1646	2.0921

Examples: (i) $F_{0.05}(3.8) = 4.0662$ (ii) $F_{0.05}(2.9) = 4.2565$ (iii) $F_{0.05}(5.24) = 2.6207$.

Index