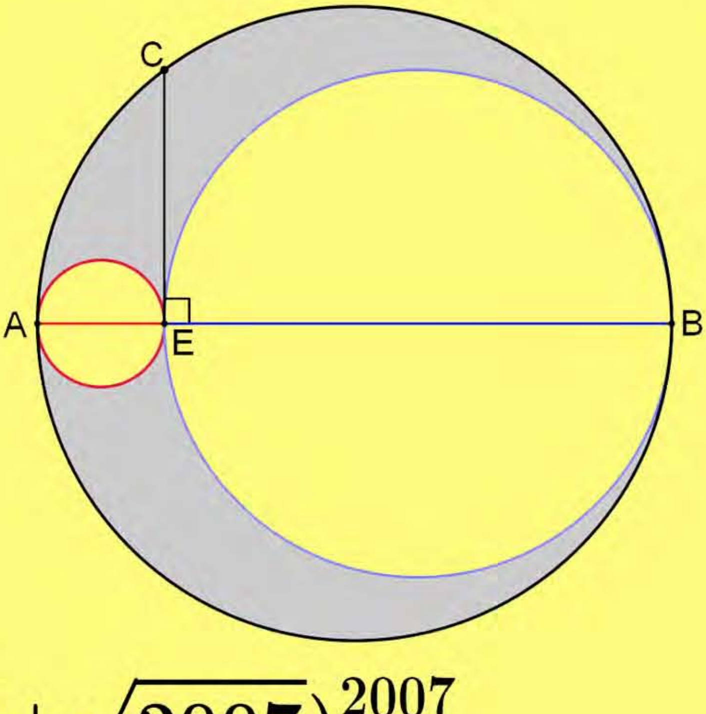
កម្រងវិញ្ញាសាគណិតវិទ្យា

ពីឆ្នាំ1982 ដល់ 2018

Final Event

ប្រឡងជ្រើសរើសសិស្សពូកែគណិតវិទ្យាប្រចាំ ហុង កុង

Hong Kong Mathematics Olympiad



$$\begin{array}{r} \text{GOLDEN} \\ \times \quad \text{J} \\ \hline \text{DENGOL} \end{array}$$

$$\frac{\left(\sqrt{2008}+\sqrt{2007}\right)^{2007}}{\left(\sqrt{2007}-\sqrt{2008}\right)^{-2007}}$$

សម្រាប់សិស្សកម្រិតអនុវិទ្យាល័យ

ឧសភា ២០១៩

ចងក្រងដោយលោកគ្រូ សារឿន មិនា

May 2019

HONG KONG MATHEMATICS OLYMPIAD - HKMO

Final Event

Past Test Papers 1982-2018

Compiled by Mr. SAROEUN Minea

Math Teacher, Kurmul International School

អារម្ភកថា

Hong Kong Mathematics Olympiad គឺជាប្រភេទនៃការប្រឡងប្រជែងជ្រើសរើសសិស្សពូកែ ឬក្រុមសិស្សពូកែផ្នែកគណិតវិទ្យាសម្រាប់សិស្សកម្រិតអនុវិទ្យាល័យ។ ប្អូនៗ និងអ្នកសិក្សាស្រាវជ្រាវផ្នែកគណិតវិទ្យា ឬស្រឡាញ់វិស័យគណិតវិទ្យា អាចសិក្សាស្វែងយល់បន្ថែមអំពីរចនាសម្ពន្ធនៃការរៀបចំការប្រឡងនេះនាទំព័របន្ទាប់ស្តីអំពីប្រវត្តិនៃបង្កើត និងដំណើរការរបស់ HKMO ។

ខ្ញុំបានរៀបចំចងក្រងវិញ្ញាសាតាមច្បាប់ដើមនេះឡើង ដើម្បីជាឯកសារស្រាវជ្រាវបន្ថែមដល់លោកគ្រូអ្នកគ្រូដែលកំពុងបង្ហាត់បង្រៀនសិស្សពូកែកម្រិតអនុវិទ្យាល័យ និងសម្រាប់ប្អូនៗដែលចង់ក្លាយខ្លួនជាសិស្សពូកែផ្នែកគណិតវិទ្យា។ មិនតែប៉ុណ្ណោះ កម្រងវិញ្ញាសានេះនឹងជាផ្នែកមួយដ៏សំខាន់សម្រាប់ប្អូនៗ ដែលមានបំណងចូលរួមការប្រឡងប្រជែងគណិតវិទ្យាកម្រិតអន្តរជាតិ ឬប្រឡងយកអាហារូបករណ៍ទៅសិក្សានៅបរទេស។ កម្រងវិញ្ញាសានេះក៏ជាផ្នែកដ៏សំខាន់សម្រាប់សិស្សានុសិស្សដែលមានផែនការចូលរួមប្រឡងប្រជែងជ្រើសរើសសិស្សពូកែគណិតកម្រិតសាលា កម្រិតថ្នាក់ខេត្តរាជធានី និងទូទាំងប្រទេសសម្រាប់ថ្នាក់ទី៩។

សូមកត់សម្គាល់ថា វិញ្ញាសាទាំងអស់នេះគឺធ្វើឡើងជាភាសាចិន និងភាសាអង់គ្លេស។ ដូចនេះ យ៉ាងហោចណាស់ ប្អូនៗត្រូវមានចំណេះដឹងភាសាអង់គ្លេស ដែលប្រើប្រាស់ក្នុងគណិតវិទ្យា។ វាអាចពិបាកខ្លាំង តែមិនដល់ថ្នាក់យើងធ្វើមិនកើតនោះទេ។ វាពិបាកក្នុងដំណោះស្រាយ តែមិនដល់ថ្នាក់ថាគិតមិនចេញសោះនោះទេ វាគ្រាន់តែត្រូវការពេលវេលាច្រើនបន្តិក្នុងការសិក្សាស្វែងយល់អមពីវា។

បួនសំណួរដែលគួរហាត់សួរខ្លួនឯងពេលដោះស្រាយលំហាត់ណាមួយ។ ១. តើខ្ញុំសង្កេតឃើញអ្វីខ្លះ តាមរយៈទម្រង់លំហាត់ដែលផ្តល់ឱ្យ? ២. តើរូបមន្ត លក្ខណៈ ឬទ្រឹស្តីណាខ្លះត្រូវយកមកប្រើ ដើម្បីដោះស្រាយលំហាត់នេះ? ៣. តើខ្ញុំត្រូវសរសេរប្រមាណវិធីដោយរបៀបណា? តាមគំរូពីមុន ឬគំរូថ្មីរបស់ខ្ញុំ? ៤. តើមានវិធីដោះស្រាយណាផ្សេងទៀតដែរឬទេ សម្រាប់ដោះស្រាយលំហាត់នេះ? ជាជំនួយដល់អ្នកស្វ័យសិក្សា គឺនៅផ្នែកទីពីរនៃកម្រងសៀវភៅវិញ្ញសានេះបានបង្ហាញគន្លឹះសម្រាប់ដំណោះស្រាយតាមឆ្នាំនីមួយៗ។

ភ្នំពេញ, ថ្ងៃចន្ទ ទី២០ ខែឧសភា ឆ្នាំ២០១៩

សារឿន មិនា

Hong Kong Mathematics Olympiad

Hong Kong Mathematics Olympiad (HKMO, Chinese: 香港數學競賽) is a Mathematics Competition held in Hong Kong every year, jointly organized by the The Education University of Hong Kong and Education Bureau. At present, more than 250 secondary schools send teams of 4-6 students of or below Form 5 to enter the competition. It is made up of a Heat Event and a Final Event, which both forbid the usage of calculators and calculation assisting equipments (e.g. printed mathematical table). Though it bears the term Mathematics Olympiad, it has no relationship with the International Mathematical Olympiad.

History

The predecessor of HKMO is the Inter-school Mathematics Olympiad initiated by the Mathematics Society of Northcote College of Education in 1974, which had attracted 20 secondary schools to participate. Since 1983, the competition is jointly conducted by the Mathematics Department of Northcote College of Education and the Mathematics Section of the Advisory Inspectorate Division of the Education Department. Also in 1983, the competition is formally renamed as Hong Kong Mathematics Olympiad.

Format and Scoring in the Heat Event

The Heat Event is usually held in four venues, for contestants from schools on Hong Kong Island, and in Kowloon, New Territories East and New Territories West respectively. It comprises an individual event and a group event. Each team sends 4 contestants among 4-6 team members for each event.

For the individual event, 1 mark and 2 marks will be given to each correct answer in Part A and Part B respectively. The maximum score for a team should be 80. For the group event, 2 marks will be given to each correct answer. The maximum score for a team should be 20. For the geometric construction event, the maximum score for a team should be 20 (all working, including construction work, must be clearly shown). In other words, a contesting school may earn 120 marks at most in the Heat Event. The top 50 may enter the Final Event.

Format and Scoring in the Final Event

The Final Event is usually held at the Education University of Hong Kong in Tai Po. It comprises 4 individual events and 4 group events. Before the real events begin, there is a mock event which carries no marks. Each team may send any 4 students for the individual events, and any 4 students for the group events. For every events, only answers are required.

There are 4 questions in each Final Individual Event. The questions have to be solved by alternate contestants independently, and no discussions are allowed. For each event, the questions are interrelated, i.e. to solve the second question, the answer of the first question is needed, and to solve the third, the answer from the second is needed, etc..

There are also 4 questions in each Final Group Event, which may be interrelated or not. The four contestants shall complete each event together, and discussion is allowed.For each event, 5 minutes is given. There are timekeepers to report the time taken used for each team in each event. The detailed scoring method is:

(A) Score for Accuracy

Individual Events	Score	Group Events	Score
correct in (i)	1	Any 1 answer correct	2
correct in (ii)	2	Any 2 answers correct	4
correct in (iii)	3	Any 3 answers correct	7
correct in (iv)	4	All answers correct	10

(B) Multiplying Factor for Speed

Time Taken	Multiplying Factor
Within 1 minute	4
Within 2 minutes	3
Within 3 minutes	2
More than 3 minutes	1

(C) Bonus Score

If all answers from a team in an event are correct, 20 marks are given as a bonus. The score for an event is equal to (A)×(B)+(C). The honour of Champion, 1st Runner-up and 2nd Runner-up are given according to the total score earned in eight events.

Past Champion (1984-2019)

1984: Hong Kong Sze Yap Commercial &
Industrial Association Wong Tai Shan Memorial School
1985: Methodist College
1986: Ying Wa College
1987: King's College
1988: Ying Wa College
1989: King's College
1990: Clementi Secondary School
1991: Queen's College
1992: New Territories Heung Yee Kuk Yuen Long District Secondary School
1993: Clementi Secondary School
1994: King's College
1995: Tsuen Wan Public Ho Chuen Yiu Memorial School
1996: Mongkok Workers' Children School (Secondary Section)
1997: Queen's College
1998: Diocesan Boys' School
1999: SKH Bishop Baker Secondary School
2000: La Salle College
2001: Yuen Long Merchants Association Secondary School
2002: King's College
2003: La Salle College
2004: Bishop Jubilee School
2005: La Salle College
2006: Cheung Chuk Shan College
2007: La Salle College
2008: La Salle College
2009: La Salle College
2010: St. Paul Co-educational College
2011: St. Paul Co-educational College
2012: La Salle College
2013: La Salle College
2014: La Salle College
2015: La Salle College
2016: La Salle College
2017: Pui Ching Middle School
2018: Pui Ching Middle School
2019: La Salle College

<u>決賽規則</u>

1. 競賽共分八項，個人及團體各佔*四*項。
2. 每隊由已報名參加初賽的同學組成。其中任何四位可參加「個人項目」；又其中任何四位可參加「團體項目」。不足四位同學的隊伍將不獲准出賽。
3. 每隊隊員必須**<u>穿著整齊校服，並由負責老師帶領</u>**，於**<u>上午 9 時正或以前</u>**向會場接待處註冊，同時**<u>必須出示身分證/學生證明文件</u>**，否則將被**<u>撤銷參賽資格</u>**。
4. 粵語將會被採用為指示語言。若參賽者不諳粵語，則可獲發給一份中、英文指示。比賽題目則中英並列。
5. 每一「個人項目」包括四部份。每一隊員回答其中一部份，其他隊友不得從旁協助，否則此項目所得分數會被取消。
6. 「個人項目」中，四部份互有關連。解答第二部份之隊員需利用第一部份之答案，如此類推。
7. 每一「團體項目」亦包括四部份。但各部份不一定相關，且可由全隊共同作答。各隊員可進行討論，但必須將聲浪降至最低。
8. 比賽時，參賽者不可使用計算工具，違例者將被取消資格或扣分。
9. 參賽者如有攜帶電子通訊器材（包括平板電腦、手提電話、多媒體播放器、電子字典、具文字顯示功能的手錶、智能手錶或其他穿戴式附有通訊或資料貯存功能之科技用品）或其他響鬧裝置，應把它關掉，並放入手提包內或座位的椅下，否則大會有權**<u>取消該隊參賽資格</u>**。
10. 除另有聲明外，所有答案須為數字，並應化簡，但無需呈交證明及算草。
11. 每一項目限時五分鐘。
12. 計分辦法如下：

(甲) 準確分:

個人項目	積分	團體項目	積分
答對第一部分	1	答對任何一部分	2
答對第二部分	2	答對任何兩部分	4
答對第三部分	3	答對任何三部分	7
答對第四部分	4	答對所有四部分	10
合共	10		

(乙) 快捷分

	積分所乘倍數
參賽隊伍完成並交出答案的時間 < 1 分鐘	**4**
1 分鐘 ≤ 參賽隊伍完成並交出答案的時間 < 2 分鐘	**3**
2 分鐘 ≤ 參賽隊伍完成並交出答案的時間 < 3 分鐘	**2**
參賽隊伍完成並交出答案的時間 ≥3 分鐘	**1**

(丙) 獎勵分

任何一隊在某一個人/團體項目競賽中，若全部答對時，可額外獲得 20 分。

(丁) 每項目之總分

準確分×倍數 + 獎勵分

13. 如有任何疑問，參賽者須於最後一項個人/團體賽完畢後 10 分鐘內向評判團提出。所有提出之疑問，將由評判團作最後裁決。
14. 得分最高之三隊將獲得獎盃及獎品。冠軍學校可保存總冠軍盾牌至下一屆香港數學競賽。
15. 總成績將由評判團作最後裁決。

Regulations (Final Events)

1. The competition consists of ***8*** events, which are divided into ***4*** individual events and ***4*** group events.
2. Each participating should consist of students who have enrolled in the heat event. Any 4 of them may take part in the individual event and any 4 of them may take part in the group event. Teams of less than 4 members will not be allowed to participate.
3. Members of each team, **accompanied by the teacher-in-charge, should wear proper school uniform** and present **ID Card or student identification document** when registering at the venue reception **not later than 9:00 a.m.** Failing to do so, the team **will be disqualified.**
4. Verbal instruction will be given in Cantonese. However, for competitors who do not understand Cantonese, instructions written in both Chinese and English will be provided. Question papers are printed in both English and Chinese.
5. Each individual event consists of 4 parts. Each part must be completed by one member of the team. Help from other team members would result in disqualification for that particular event.
6. In an individual event, the four parts are interrelated. When solving Part 2, one has to make use of the answer obtained in Part 1, and so on.
7. In a group event, the four parts are to be done by the whole team and the parts may or may not be interrelated. Discussions are allowed provided that voice level is kept to a minimum.
8. The use of calculating devices will not be allowed; otherwise the team will risk disqualification or deduction of marks.
9. Participants having electronic communication devices (include tablets, mobile phones, multimedia players, electronic dictionaries, databank watches, smart watches or other wearable technologies with communication or data storage functions) or any alarm device(s), should turned them off (including the alarm function) and be put inside the bags or under the chairs. Failing to do so, the team **will risk disqualification.**
10. All answers should be numerical and reduced to the simplest form unless stated otherwise. No proof or working is required.
11. The time limit for each event is 5 minutes.
12. The Marking System is as follows:
    (*a*) Scores for accuracy:

Individual Events	Scores	Group Events	Scores
Part 1 correct ···	1	Any 1 part correct	···2
Part 2 correct ···	2	Any 2 parts correct	···4
Part 3 correct ···	3	Any 3 parts correct	···7
Part 4 correct ···	4	All 4 parts correct	···10
Total ············	10		

    (*b*) Multiplying factors for speed:

Time taken for the teams to hand in their answer < 1 min.	***4***
1 min. ≤Time taken for the teams to hand in their answer < 2 min.	***3***
2 min. ≤Time taken for the teams to hand in their answer < 3 min.	***2***
Time taken for the teams to hand in their answer ≥3 min.	***1***

    (*c*) Bonus Score:
    Teams, which hand in their answers of anyone individual/group event have all the answers in that event correct, will be awarded a bonus score of 20 marks.
    (*d*) Total score for each event:
    (Score for accuracy) × (Multiplying factor) + (Bonus score)
13. Any queries should reach the Judging Panel within 10 minutes after the end of the last individual group event. The decision of the Judging Panel on the queries is final.
14. Trophies and prizes will be given to the three schools achieving the highest scores. The champion school may keep the Champion shield until the next Hong Kong Mathematics Olympiad.
15. The decision of the Judging Panel on the overall results is final.

Part 1: Test Papers

Hong Kong Mathematics Olympiad (1981 – 1982)

Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 求 a 的值，若 $a = 5 + 8 + 11 + \ldots + 38$。
Find the value of a if $a = 5 + 8 + 11 + \ldots + 38$.

$a =$

(ii) 設 $b = a$ 的所有位值之和，求 b 的值。
Let b = the sum of the digits of the number a. Find the value of b .

$b =$

(iii) 若 $c = b^2$，求 c 的值。
If $c = b^2$, find the value of c .

$c =$

(iv) 已知 $3d = c$，求 d 的值。
Given that $3d = c$, find the value of d .

$d =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]　　Team No. [ ]

\+ Bonus score [ ]　　Time [ ] [ ]

Total score [ ]　　Min.　Sec.

Compiled by Mr. SAROEUN Minea

Hong Kong Mathematics Olympiad (1981 – 1982)

Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 從一副撲克牌中抽出兩張，而不放回原位。若抽得兩張都是紅心的機會率為 $\frac{1}{a}$，求 a 的值。
Two cards are drawn at random from a pack and not replaced.
If the probability that both cards are hearts is $\frac{1}{a}$, find the value of a.

a =

(ii) 在 17 人之中揀選 a 人，共有 b 種方法，求 b 的值。
If there are b ways of choosing 15 people from ‘a’ people, find the value of b.

b =

(iii) 一共有 $\frac{b}{2a}$ 幅不同顏色的旗，每次升起最少一幅。
如果不考慮顏色的次序，求一共有多少種不同的訊號 c？
If c signals can be made with $\frac{b}{2a}$ flags of different colours by raising at least one of the flags, without considering the arrangement of colours, find the value of c.

c =

(iv) 一個袋有 c 個球，其中 3 個是紅色。從中抽取一個，問抽到紅球的概率為何？
There are c balls in a bag, of which 3 are red .
What is the probability of drawing a red ball ?

Probability =

FOR OFFICIAL USE

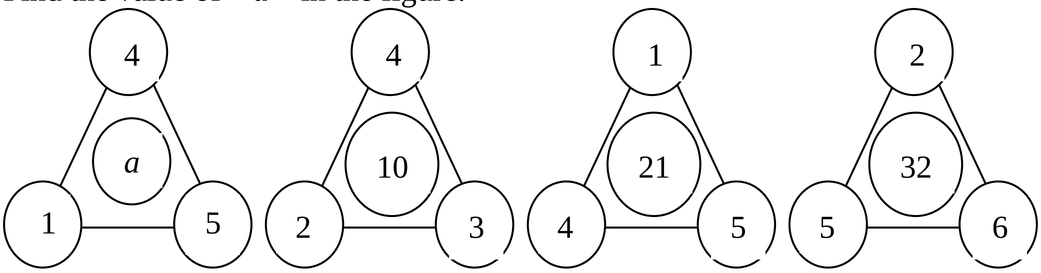
Score for accuracy		×	Mult. factor for speed		=		Team No.		
			+ Bonus score				Time		
			Total score					Min.	Sec.

Hong Kong Mathematics Olympiad (1981 – 1982)

Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 下圖中，求 a 的值。
Find the value of a in the figure.



a =

(ii) 求 b 的值，若 $\frac{\sin(4b)^\circ}{\cos(4b)^\circ}=\sqrt{\sqrt{a}}$ $(0 < 4b < 90)$。

Find the value of b if $\frac{\sin(4b)^\circ}{\cos(4b)^\circ}=\sqrt{\sqrt{a}}$ $(0 < 4b < 90)$.

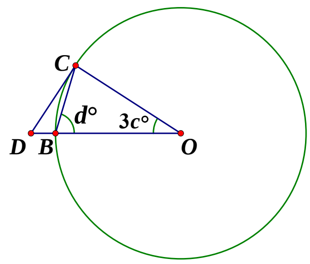
b =

(iii) 在以下數列中求 c 的值。

Find the value of c from the sequence: $\frac{3}{12}, \frac{7}{34}, \frac{c}{56}, \frac{b}{78}$.

c =

(iv) 圖中，O 為圓心，B 和 C 為圓周上的點，
使得 $\angle BOC = 3c°$, $\angle OBC = d°$。求 d 的值。
In the figure, O is the centre, B and C are points on the circumference. $\angle BOC = 3c°$, $\angle OBC = d°$.
Find the value of d.



d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (1981 – 1982)

Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $x=\dfrac{\log a^3}{\log a^2}$，其中 $a>0$ 及 $a\neq 1$，求 x 的值。

Find the value of x if $x=\dfrac{\log a^3}{\log a^2}$ where $a>0$ and $a\neq 1$.

$x =$

(ii) 若 $y-1=\log x+\log 2-\log 3$，求 y 的值。
If $y-1=\log x+\log 2-\log 3$, find the value of y.

$y =$

(iii) 若 $\log_2 Z^y=3$ 則 Z 的值為何？
What is the value of Z if $\log_2 Z^y=3$?

$Z =$

(iv) 求 $\log_z y$ 的值。
Find the value of $\log_z y$.

$\log_z y =$

FOR OFFICIAL USE

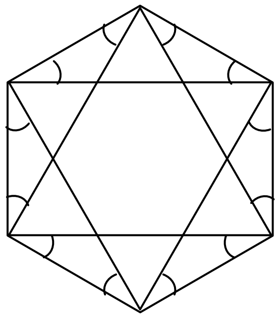
Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1981 – 1982)

Final Event 5 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

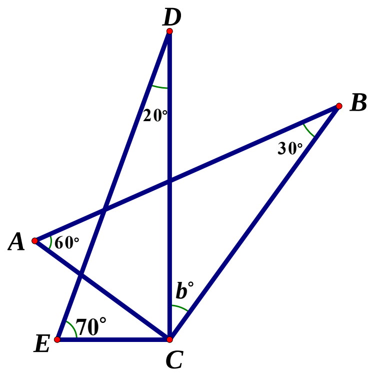
(i) 如圖，所有有記號的角的總和是 $a°$，求 a 的值。
Let the sum of the marked angles be $a°$.
Find the value of a.



$a =$

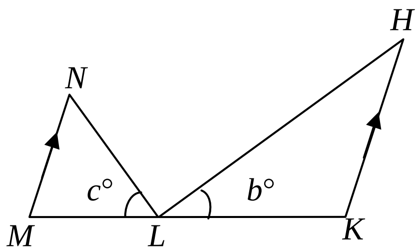
(ii) 若 $\angle ACE = \left(\frac{a}{10}\right)^{\circ}$。求 b 的值。

$\angle ACE = \left(\frac{a}{10}\right)^{\circ}$. Find the value of b.



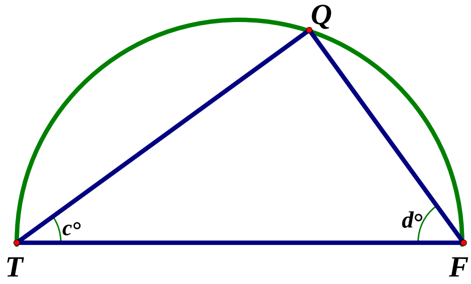
$b =$

(iii) 若 $HK = KL$，$LM = MN$，$HK // MN$，
求 c 的值。
If $HK = KL$, $LM = MN$, $HK // MN$,
find the value of c.



$c =$

(iv) TQF 為一半圓形，求 d 的值。
TQF is a semi-circle. Find the value of d.



$d =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

\+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (1981 – 1982)

Final Event 6 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

Let $\log 2 = a$ 設 $\log 2 = a$
$\log 3 = b$ $\log 3 = b$
$\log 5 = c$ $\log 5 = c$

(i) 以 a、b 及 c 表示 $\log 6$。
Express $\log 6$ in terms of a, b and c.

$\log 6 =$

(ii) 計算 $3.5a + 3.5c$。
Evaluate $3.5a + 3.5c$.

$3.5a+3.5c =$

(iii) 以 a、b 及 c 表示 $\frac{\log 30}{\log 15}$。
Express $\frac{\log 30}{\log 15}$ in terms of a, b and c.

$\frac{\log 30}{\log 15} =$

(iv) 以 a、b 及 c 表示 $(\log 15)^2 - \log 15$。
Express $(\log 15)^2 - \log 15$ in terms of a, b and c.

$(\log 15)^2 - \log 15 =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (1981 – 1982)

Final Event 7 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 右圖顯示一圓錐體及一半球體。OB = 12 cm，r = 10 cm，以 π 表示該立體的表面面積。
Figure 1 shows a cone and a hemisphere.
OB = 12 cm, r = 10 cm. Express the surface area of the solid in terms of π.

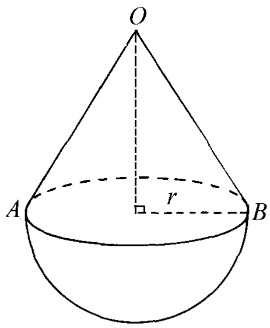


Figure 1

surface area =

(ii) 以 π 表示上圖立體的體積。
What is the volume of the hemisphere shown in figure 1?
Give your answer in terms of π.

volume =

(iii) 圖二顯示一圓錐體放置在一個半徑相等(r)、高度相同(h)的圓柱體內，以 r 及 h 表示兩者之間的空間的體積。
In figure 2, a right circular cone stands inside a right cylinder of same base radius r and height h. Express the volume of the space between them in terms of r and h.

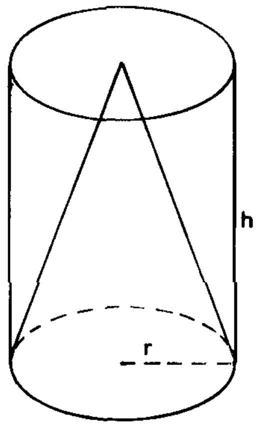


Figure 2

Volume =

(iv) 求圓柱體與圓錐體體積之比。
Find the ratio of the volume of the cylinder to that of the cone.

ratio =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1981 – 1982)

Final Event 8 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

依下圖
Given that:

1	2	3
4	5	6
7	8	9

1 stands for A 1 表示 A
2 stands for B 2 表示 B
……………………………
25 stands for Y 25 表示 Y
26 stands for Z 26 表示 Z

(i) 以下符號 ⊔ ∟ ⊐ ⊓ □ 表示甚麼數字？
What number does the code ⊔ ∟ ⊐ ⊓ □ stand for?

⊔ ∟ ⊐ ⊓ □ =

(ii) 以 Δ 表示 零。計算以下公式並以符號表示答案。
Put Δ stands for zero. Calculate the following and give the answer in code.
(⊔Δ)(⊐ Δ) + ┘┌ – ∟Δ

answer =

(iii) "3 8 18 9 19 20 13 1 19" 表示一個英文字。它是甚麼？
"3 8 18 9 19 20 13 1 19" stands for a word. What is it?

word =

(iv) 將以下密碼翻譯成英文字。一共有兩個英文字。
Decode the following message:
(┘Δ ┘□ ┌ ┘⊐) (⊔┘ ┘┌)
There are two words in the message.

message =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min. Sec.

Hong Kong Mathematics Olympiad (1981 – 1982)

Final Event 9 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 在以下數列中求 A 的值。
Find the value of A from the sequence:
0, 3, 8, A, 24, 35, ⋯

A =

(ii) 方程 $x^2 - 15x + B = 0$ 的根為 7 及 C。求 B 和 C 的值。
The roots of the equation $x^2 - Ax + B = 0$ are 7 and C.
Find the values of B and C.

B =

C =

(iii) 若 $\log_7 B = \log_7 C + 7^X$， 求 X 的值。
If $\log_7 B = \log_7 C + 7^X$; find the value of X.

X =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1981 – 1982)

Final Event 10 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $N = 2^{12} \times 5^{8}$，N 是一個多少位的數字？
How many digits are there in the number N if $N = 2^{12} \times 5^{8}$?

Number of digits

(ii) $(2^{48} - 1)$ 可被兩個介乎於 60 至 70 之間的整數，求該兩數。
If $(2^{48} - 1)$ is divisible by two whole numbers between 60 and 70, find them.

smaller number =

larger number =

(iii) 以下兩個數，哪一個較大？$2^{\frac{1}{2}} \times 9^{\frac{1}{9}}$，$3^{\frac{1}{3}} \times 8^{\frac{1}{8}}$。
Given $2^{\frac{1}{2}} \times 9^{\frac{1}{9}}$, $3^{\frac{1}{3}} \times 8^{\frac{1}{8}}$. What is the greatest number?

greatest number =

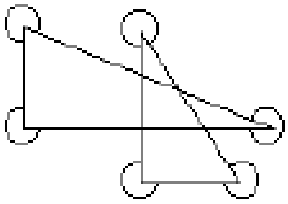
FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1982-83)
Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 如圖，所有有記號的角的總和是 $a°$，求 a 的值。
In the following figure, the sum of the marked angles is $a°$, find the value of a.



$a =$

(ii) 一正 b-邊形的內角和是 $a°$。求 b 的值。
The sum of the interior angles of a regular b-sided polygon is $a°$. Find the value of b.

$b =$

(iii) 求 c 的值，若 $2^b = c^4$ 及 $c > 0$。
Find the value of c, if $2^b = c^4$ and $c > 0$.

$c =$

(iv) 若 $\frac{b}{c} = k$ 及 $c : d = k : 100$，求 d 的值。
Find the value of d, if $\frac{b}{c} = k$ and $c : d = k : 100$.

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

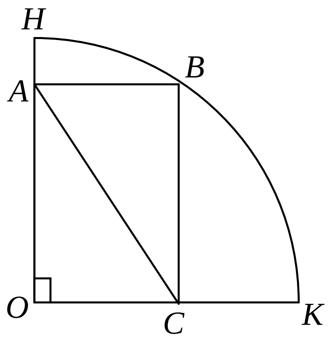
Hong Kong Mathematics Olympiad (1982-83)
Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $a = 1.8 \times 5.0865 + 1 - 0.0865 \times 1.8$，求 a 的值。
If $a = 1.8 \times 5.0865 + 1 - 0.0865 \times 1.8$, find the value of a .

$a =$

(ii) 如圖，$OH = OK = 10$ 及 $OABC$ 為一個長方形。$AC = b$，問 b 為何值？
In the diagram shown, $OH = OK = a$ units and $OABC$ is a rectangle. $AC = b$ units.
What is the value of b ?



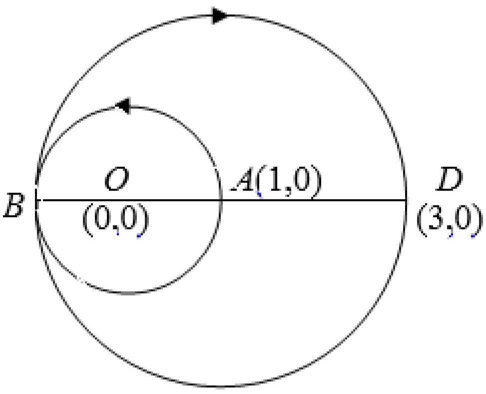
$b =$

(iii) 依下圖之分數，當計算至分子是 x^8 時，c 為何值？
In the expression shown, what is c when it is expanded to the term with $x^{(b-2)}$ as the numerator?

$$\cfrac{x^0}{2+\cfrac{x^1}{6+\cfrac{x^2}{10+\cfrac{x^3}{14+\cfrac{\cdots\cdots\cdots}{\cfrac{\cdots}{c+\cfrac{x^{(b-2)}}{\ddots}}}}}}}$$

$c =$

(iv) 如圖，一兔子花了 c 分鐘經半圓跑道由 A 去到 B。以相同速度，牠花了 d 分鐘經半圓跑道由 $A \to B \to D$。問 d 為何值？
As shown a rabbit spends c minutes in travelling from A to B along half circle. With the same speed, it spends d minutes in travelling from $A \to B \to D$ along half circles. What is the value of d ?



$d =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

\+ Bonus score [ ]

Total score [ ]

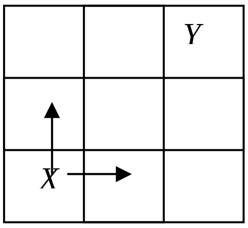
Team No. [ ]

Time [ ] Min. [ ] Sec.

Hong Kong Mathematics Olympiad (1982-83)
Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 右方棋盤為一 3×3 九宮格。一隻棋子放置在 X 的位置上，每次只可向上行一格，或向右行一格。
問：由 X 行到 Y，共有多少種不同的路徑？
The figure shows a board consisting of nine squares. A counter originally on square X can be moved either upwards or to the right one square at a time. By how many different routes may the counter be moved from X to Y?



(ii) 已知 $\sqrt{2a} = -b\tan\frac{\pi}{3}$。求 b 的值。
Given $\sqrt{2a} = -b\tan\frac{\pi}{3}$. Find the value of b.

$b =$

(iii) 已知 $p * q = \frac{p-q}{p}$，求 c 的值，若 $c = (a+b) * (b-a)$。
Given that $p * q = \frac{p-q}{p}$, find the value of c if $c = (a+b) * (b-a)$.

$c =$

(iv) 把一 c cm 的鐵綫屈曲成一半徑為 1 cm 的扇形。問扇形的圓心角為何？
A wire of c cm is bent to form a sector of radius 1 cm. What is the angle of the sector in degrees (correct to the nearest degree)?

angle =

FOR OFFICIAL USE

Score for accuracy × Mult. factor for speed =

+ Bonus score

Total score

Team No.

Time Min. Sec.

Hong Kong Mathematics Olympiad (1982-83)
Event 5 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $a(x+1) \equiv x^3 + 3x^2 + 3x + 1$，以 x 表示 a。
If $a(x+1) \equiv x^3 + 3x^2 + 3x + 1$, express a in terms of x.

$a =$

(ii) 若 $a - 1 = 0$，則 x 的解為 0 或 b，求 b 的值。
If $a - 1 = 0$, then the value of x is 0 or b, what is the value of b?

$b =$

(iii) 若 $pc^4 = 32$，$pc = b^2$ 及 c 為正數，c 的值為何？
If $pc^4 = 32$, $pc = b^2$ and c is positive, what is the value of c?

$c =$

(iv) P 為一運算子使得 $P(A \cdot B) = P(A) + P(B)$。
$P(A) = y$ 的意思是 $A = 10^y$。若 $d = A \cdot B$，$P(A) = 1$ 及 $P(B) = c$，求 d 的值。
P is an operation such that $P(A \cdot B) = P(A) + P(B)$.
$P(A) = y$ means $A = 10^y$.
If $d = A \cdot B$, $P(A) = 1$ and $P(B) = c$, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1982-83)
Event 6 (Group)

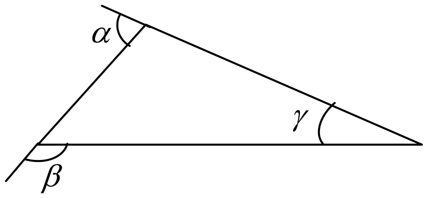
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 右表顯示二元運算子*定義於 P、Q、R、S 時的結果。假設 a 為 P 的反元素。求 a 的值。
The table shows the results of the operation * on P, Q, R, S taken two at a time.
Let a be the inverse of P. Find the value of a.

*	P	Q	R	S
P	Q	R	S	P
Q	R	S	P	Q
R	S	P	Q	R
S	P	Q	R	S

a =

(ii) α 與 β 的平均值是 105°，α、β 與 γ 的平均值是 b°。求 b 的值。
The average of α and β is 105°, the average of α, β and γ is b°. Find the value of b.



b =

(iii) 兩數之和為 10，其乘積為 20。若該兩倒數之和為 c，求 c 的值。
The sum of two numbers is 10, their product is 20. The sum of their reciprocal is c. What is the value of c?

c =

(iv) 已知 $\sqrt{90} = 9.49$ (準至兩位小數)
若 $d < 7\sqrt{0.9} < d + 1$，其中 d 為整數，求 d 的值。
It is given that $\sqrt{90} = 9.49$, to 2 decimal places.
If $d < 7\sqrt{0.9} < d + 1$, where d is an integer, what is the value of d?

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

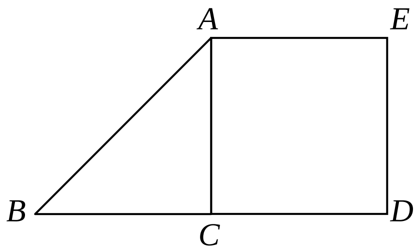
Hong Kong Mathematics Olympiad (1982-83)
Event 7 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 求 $3 + 6 + 9 + \ldots + 45$ 的值。
Find the value of $3 + 6 + 9 + \ldots + 45$.

sum =

(ii) 圖中，$ACDE$ 為一正方形，$AC = BC$ 及 $\angle ACB = 90°$。若 $ACDE$ 的面積為 10 cm^2，求 ΔABC 的面積。
In the figure shown, $ACDE$ is a square and $AC = BC$, $\angle ACB = 90°$. Find the area of ΔABC if the area of $ACDE$ is 10 cm^2.



Area =

(iii) 若 $a+\frac{1}{a}=3$，求 $a^3+\frac{1}{a^3}$ 的值。
Given that $a+\frac{1}{a}=3$. Evaluate $a^3+\frac{1}{a^3}$.

(iv) 已知 $\sum_{y=1}^{n}\frac{1}{y}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$。
求 $\sum_{y=3}^{10}\frac{1}{y-2}-\sum_{y=3}^{10}\frac{1}{y-1}$ 的值。(答案以份數表示。)
Given that $\sum_{y=1}^{n}\frac{1}{y}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$.
Find the value of $\sum_{y=3}^{10}\frac{1}{y-2}-\sum_{y=3}^{10}\frac{1}{y-1}$. (Express your answer in fraction.)

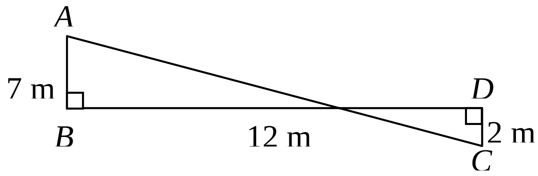
FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

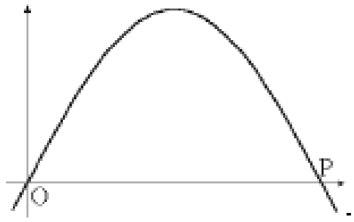
Hong Kong Mathematics Olympiad (1982-83)
Event 8 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 如圖，彼得站 A 點而約翰站在 C 點，BD 的距離 12 m。問彼得和約翰之間的最短距離為何？
Peter is standing at A and John is at C. The distance between B and D is 12 m. What is the shortest distance between John and Peter?



(ii) 右圖顯示 $y = \sin 3x°$ 的圖像，求 P 點的 x 座標。
The following figure shows a part of the graph $y = \sin 3x°$. What is the x-coordinate of P ?



$x =$

(iii) 若 $f(x) = x^2$，以 x 表示 $f(x) – f(x – 1)$。
If $f(x) = x^2$, then express $f(x) – f(x – 1)$ in terms of x.

(iv) 若果 mnp、nmp、mmp 及 nnp 為十進制數字，其位值是由 m、n 及 p 組成，且 $mnp – nmp = 180$ 及 $mmp – nnp = d$。求 d 的值。
If mnp, nmp, mmp and nnp are numbers in base 10 composed of the digits m, n and p, such that: $mnp – nmp = 180$ and $mmp – nnp = d$. Find the value of d .

$d =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

\+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (1982-83)
Event 9 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $\sin\theta = \frac{3}{5}$，$a = \sqrt{\tan^2\theta + 1}$，求 a 的值。

If $\sin\theta = \frac{3}{5}$, $a = \sqrt{\tan^2\theta + 1}$, find the value of a.

$a =$

(ii) 考慮以下步驟，用以證明 $\frac{1}{8} > \frac{1}{4}$。

Examine the following proof carefully: To prove $\frac{1}{8} > \frac{1}{4}$.

step ______

步驟 Steps

1	$3 > 2$	$3 > 2$
2	兩邊乘以 $\log\left(\frac{1}{2}\right)$， 使得 $3\log\left(\frac{1}{2}\right) > 2\log\left(\frac{1}{2}\right)$	Multiply both sides by $\log\left(\frac{1}{2}\right)$, then $3\log\left(\frac{1}{2}\right) > 2\log\left(\frac{1}{2}\right)$
3	$\log\left(\frac{1}{2}\right)^3 > \log\left(\frac{1}{2}\right)^2$	$\log\left(\frac{1}{2}\right)^3 > \log\left(\frac{1}{2}\right)^2$
4	$\left(\frac{1}{2}\right)^3 > \left(\frac{1}{2}\right)^2$	$\left(\frac{1}{2}\right)^3 > \left(\frac{1}{2}\right)^2$

$\therefore \frac{1}{8} > \frac{1}{4}$

Which step is incorrect? 以上哪一步是錯的？

(iii) 若兩直綫 $2y + x + 3 = 0$ 及 $3y + cx + 2 = 0$ 互相垂直，求 c 的值。
If the lines $2y + x + 3 = 0$ and $3y + cx + 2 = 0$ are perpendicular, find the value of c.

$c =$

(iv) 在箱子內有 4 個紅球和 3 個黑球。若從中一個接一個抽出 3 個球，每次抽完之後皆將抽到的球放回原位。求抽到 2 個紅球和 1 個黑球的概率。
There are 4 red balls and 3 black balls in a box. If 3 balls are chosen one by one with replacement, what is the probability of choosing 2 red balls and 1 black ball?

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (1982-83)
Event 10 (Group)

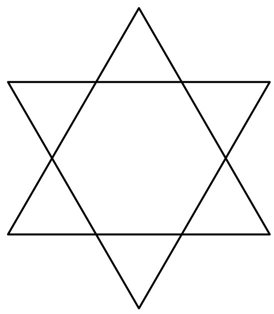
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) $1^2 - 1 = 0 \times 2$
$2^2 - 1 = 1 \times 3$
$3^2 - 1 = 2 \times 4$
$4^2 - 1 = 3 \times 5$
………………
$A^2 - 1 = 3577 \times 3579$
若 $A > 0$，求 A 的值。

$1^2 - 1 = 0 \times 2$
$2^2 - 1 = 1 \times 3$
$3^2 - 1 = 2 \times 4$
$4^2 - 1 = 3 \times 5$
………………
$A^2 - 1 = 3577 \times 3579$
If $A > 0$, find the value of A.

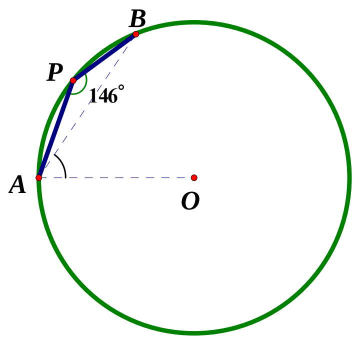
$A =$

(ii) 一正 N-邊形的邊向外延長形成一個“星形”。如果該星形的每一隻角均為 108°，求 N 的值。(例如，由正 6 邊形形成的 6 角星如右圖所示。)
The sides of an N-sided regular polygon are produced to form a “star”. If the angle at each point of that “star” is 108°, find the value of N.
(For example, the “star” of a six-sided polygon is given as shown in the diagram.)



6-sided regular polygon.

(iii) A、P 及 B 三點均在圓周上，圓心為 O。
若 $\angle APB = 146°$，求 $\angle OAB$ 的值。
A, P, B are three points on a circle with centre O.
If $\angle APB = 146°$, find the value of $\angle OAB$.



$\angle OAB =$

(iv) 一兩位數 X 的個位與十位相乘等於 24，若將個位與十位對掉，新的兩位數比原來的兩位數大了 18，求 X 的值。
A number X consists of 2 digits whose product is 24. By reversing the digits, the new number formed is 18 greater than the original one. What is the value of X?

$X =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]　　Team No. [ ]

\+ Bonus score [ ]　　Time [ ] [ ]

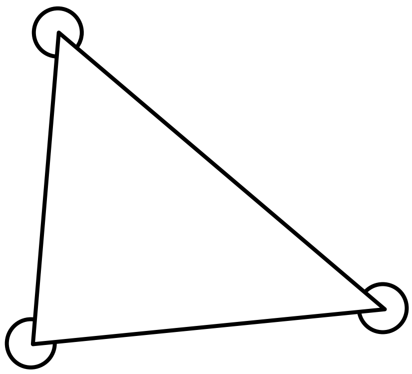
Total score [ ]　　Min.　Sec.

Hong Kong Mathematics Olympiad (1983 – 1984)
Sample Event (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 附圖所示三角之和為 $a°$，求 a 的值。
In the given diagram, the sum of the three marked angles is $a°$.
Find the value of a .

a =



(ii) 一正 b 邊形之內角和為 $a°$，求 b 的值。
The sum of the interior angles of a regular b-sided polygon is $a°$.
Find the value of b .

b =

(iii) 若 $8^b = c^{21}$，求 c 的值。
If $8^b = c^{21}$, find the value of c .

c =

(iv) 若 $c = \log_d 81$，求 d 的值。
If $c = \log_d 81$, find the value of d .

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (1983 – 1984)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $100a = 35^2 - 15^2$，求 a 的值。
If $100a = 35^2 - 15^2$, find the value of a.

a =

(ii) 若 $(a - 1)^2 = 3^{4b}$，求 b 的值。
If $(a - 1)^2 = 3^{4b}$, find the value of b.

b =

(iii) 若 b 為 $x^2 + cx - 5 = 0$ 之一根，求 c 的值。
If b is a root of $x^2 + cx - 5 = 0$, find the value of c.

c =

(iv) 若 $x + c$ 為 $2x^2 + 3x + 4d$ 之因式，求 d 的值。
If $x + c$ is a factor of $2x^2 + 3x + 4d$, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

+ Bonus score [ ]

Total score [ ]

Team No. [ ]

Time [ ] Min. [ ] Sec.

Hong Kong Mathematics Olympiad (1983 – 1984)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 α、β為 $x^2 - 10x + 20 = 0$ 之根，且 $a = \frac{1}{\alpha} + \frac{1}{\beta}$，求 a 的值。

If α, β are roots of $x^2 - 10x + 20 = 0$, find the value of a, where $a = \frac{1}{\alpha} + \frac{1}{\beta}$.

$a =$

(ii) 若 $\sin\theta = a$ $(0° < \theta < 90°)$，且 $10\cos 2\theta = b$，求 b 的值。

If $\sin\theta = a$ $(0° < \theta < 90°)$, and $10\cos 2\theta = b$, find the value of b.

$b =$

(iii) 點 $A(b, c)$ 在直線 $2y = x + 15$ 上，求 c 的值。

The point $A(b, c)$ lies on the line $2y = x + 15$. Find the value of c.

$c =$

(iv) 若 $x^2 - cx + 40 \equiv (x + k)^2 + d$，求 d 的值。

If $x^2 - cx + 40 \equiv (x + k)^2 + d$, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.		
			+ Bonus score				Time		
			Total score					Min.	Sec.

Hong Kong Mathematics Olympiad (1983 – 1984)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 a 為 $2x^3 - 3x^2 + x - 1$ 被 $x + 1$ 除所得之餘數，求 a 的值。
If a is the remainder when $2x^3 - 3x^2 + x - 1$ is divided by $x + 1$, find the value of a.

a =

(ii) If b cm^2 is the total surface area of a cube of side $(8 + a)$ cm, find the value of b.
若 b cm^2 為一邊長$(8 + a)$ cm 的立方體之總表面積，求 b 的值。

b =

(iii) 一袋內有紅球 $b + 4$ 個，白球 $2b - 2$ 個。若隨意於袋內取球一個，而該球為白色之機會為 x，求 x 的值。
One ball is taken at random from a bag containing $b + 4$ red balls and $2b - 2$ white balls. If x is the probability that the ball is white, find the value of x.

x =

(iv) 若 $\sin \theta = x$ $(90° < \theta < 180°)$及 $\tan(\theta - 15°) = y$，求 y 的值。
If $\sin \theta = x$ $(90° < \theta < 180°)$ and $\tan(\theta - 15°) = y$, find the value of y.

y =

FOR OFFICIAL USE

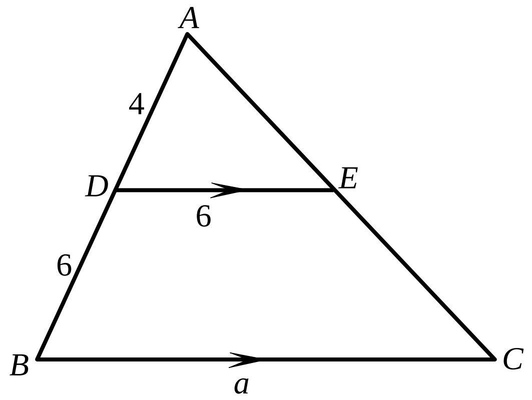
Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1983 – 1984)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 在圖一中，$DE//BC$，若 $AD = 4$，$DB = 6$，$DE = 6$，且 $BC = a$，求 a 的值。
In figure 1, DE // BC. If $AD = 4$, $DB = 6$, $DE = 6$ and $BC = a$, find the value of a.

$a =$



圖一 Figure 1

(ii) θ為銳角，$\cos\theta = \frac{a}{17}$。若 $\tan\theta = \frac{b}{15}$，求 b 的值。
θ is an acute angle such that $\cos\theta = \frac{a}{17}$. If $\tan\theta = \frac{b}{15}$, find the value of b.

$b =$

(iii) 若 $c^3 = b^2$，求 c 的值。
If $c^3 = b^2$, find the value of c.

$c =$

(iv) 一等邊三角形之面積為 $c\sqrt{3}$ cm^2。若其周界長 d cm，求 d 的值。
The area of an equilateral triangle is $c\sqrt{3}$ cm^2.
If its perimeter is d cm, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy ☐ × Mult. factor for speed ☐ = ☐ Team No. ☐

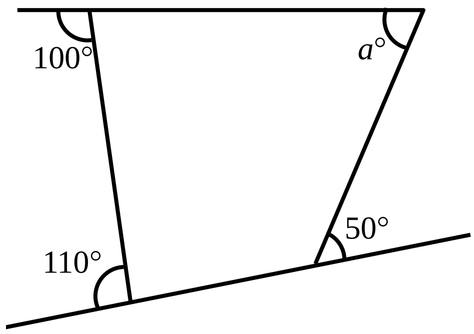
+ Bonus score ☐ Time ☐ ☐

Total score ☐ Min. Sec.

Hong Kong Mathematics Olympiad (1983 – 1984)
Final Event 5 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 在圖二，求 a 的值。
In Figure 2, find the value of a.



圖二 Figure 2

a =

(ii) 若 $b=\log_2\left(\frac{a}{5}\right)$，求 b 的值。
If $b=\log_2\left(\frac{a}{5}\right)$, find the value of b.

b =

(iii) 一繩長 20 m，依 $b-2:b:b+2$ 之比例分成三段。
若最長一段為 N m，求 N 的值。
A piece of string, 20 m long, is divided into 3 parts in the ratio of $b-2:b:b+2$.
If N m is the length of the longest portion, find the value of N.

N =

(iv) 正 N 邊形之每一內角為 $x°$。求 x 的值。
Each interior angle of an N-sided regular polygon is $x°$. Find the value of x.

x =

FOR OFFICIAL USE

Score for accuracy ☐ × Mult. factor for speed ☐ = ☐ Team No. ☐

\+ Bonus score ☐ Time ☐ ☐

Total score ☐ Min. Sec.

Hong Kong Mathematics Olympiad (1983 – 1984)
Sample Event (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 某兩數之和為 20，其積為 10，若該兩數倒數之和為 a，求 a 的值。
The sum of 2 numbers is 20 , their product is 10 .
If the sum of their reciprocals is a , find the value of a .

a =

(ii) $1^2 - 1 = 0\times2$，$2^2 - 1 = 1\times3$，$3^2 - 1 = 2\times4$，…，$b^2 - 1 = 135\times137$。
若 $b > 0$，求 b 的值。
$1^2 - 1 = 0\times2$, $2^2 - 1 = 1\times3$, $3^2 - 1 = 2\times4$, … , $b^2 - 1 = 135\times137$.
If $b > 0$, find the value of b .

b =

(iii) 若兩直綫 $x + 2y + 1 = 0$ 及 $cx + 3y + 1 = 0$ 互相垂直，求 c 的值。
If the lines $x + 2y + 1 = 0$ and $cx + 3y + 1 = 0$ are perpendicular, find the value of c .

c =

(iv) (2, −1)、(0, 1)、(c, d)三點共線。求 d 的值。
The points (2, −1), (0, 1), (c, d) are collinear. Find the value of d .

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

+ Bonus score [ ]

Total score [ ]

Team No. [ ]

Time [ ] Min. [ ] Sec.

Hong Kong Mathematics Olympiad (1983 – 1984)
Final Event 6 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $p=\dfrac{21^3-11^3}{21^2+21\times 11+11^2}$，求 p 的值。
If $p=\dfrac{21^3-11^3}{21^2+21\times 11+11^2}$, find the value of p.

p =

(ii) 若 p 人可在 6 日完成某一工程，且 4 人可在 q 日完成同一工程，求 q 的值。
If p men can do a job in 6 days and 4 men can do the same job in q days, find the value of q.

b =

(iii) 某年三月第 q 日為星期三，而同年三月第 r 日為星期五，且 $18 < r < 26$，求 r 的值。
If the q^{th} day of March in a year is Wednesday and the r^{th} day of March in the same year is Friday, where $18 < r < 26$, find the value of r.

r =

(iv) 若 $a*b = ab + 1$，且 $s = (3*4)*2$，求 s 的值。
If $a*b = ab + 1$, and $s = (3*4)*2$, find the value of s.

s =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.
			+ Bonus score				Time: Min. / Sec.
			Total score				

Hong Kong Mathematics Olympiad (1983 – 1984)
Final Event 7 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 凌晨三點卅分，時鐘兩針間之銳角為 $p°$，求 p 的值。
The acute angle between the 2 hands of a clock at 3:30 a.m. is $p°$.
Find the value of p.

$p =$

(ii) 在ΔABC 中，$\angle B = \angle C = p°$。若 $q = \sin A$，求 q 的值。
In ΔABC, $\angle B = \angle C = p°$. If $q = \sin A$, find the value of q.

$q =$

(iii) 三點(1, 3)、(a, 5)、(4, 9)共綫，求 a 的值。
The 3 points (1, 3), (a, 5), (4, 9) are collinear. Find the value of a.

$a =$

(iv) 7、9、x、y、17 之平均數為 10。若 m 為 $x + 3$、$x + 5$、$y + 2$、8、$y + 18$ 之平均數，求 m 的值。
The average of 7, 9, x, y, 17 is 10.
If m is the average of $x +3$, $x +5$, $y +2$,8, $y +18$, find the value of m.

$m =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1983 – 1984)
Final Event 8 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

如圖所示加法中，每字母代表由零至九之不同整數。
已知 $S = 9$，O =零，$E = 5$。
求下列字母所代表之數字：

$$\begin{array}{cccccc} & & S & E & N & D \\ + & & M & O & R & E \\ \hline & M & O & N & E & Y \end{array}$$

(i) M
(ii) N
(iii) R
(iv) Y

In the addition shown, each letter represents a different digit ranging from zero to nine. It is already known that $S = 9$, O = zero, $E = 5$.
Find the numbers represented by

$$\begin{array}{cccccc} & & S & E & N & D \\ + & & M & O & R & E \\ \hline & M & O & N & E & Y \end{array}$$

(i) M
(ii) N
(iii) R
(iv) Y

M =

N =

R =

Y =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (1983 – 1984)
Final Event 9 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $x=\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{100}\right)$，試以最簡單的分數表 x。

If $x=\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{100}\right)$, find x in the simplest fractional form.

$x =$

(ii) 一長方體之長、闊、高依次為 2、3 及 4。若其總面積為 A，求 A 的值。
The length, width and height of a rectangular block are 2, 3 and 4 respectively.
Its total surface area is A, find the value of A.

$A =$

(iii) 若 m 為 1、2、3、...、1001 之平均數，求 m 的值。
The average of the integers 1, 2, 3, ... , 1001 is m. Find the value of m.

$m =$

(iv) 一面積為 12π 之圓，內接於一周界為 P 之等邊三角形，求 P 的值。
The area of a circle inscribed in an equilateral triangle is 12π.
If P is the perimeter of this triangle, find the value of P.

$P =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1983 – 1984)
Final Event 10 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 一正方形內接於一直徑為 10 之圓。若 A 為正方形的面積，求 A 的值。
If A is the area of a square inscribed in a circle of diameter 10, find the value of A.

A =

(ii) 若 $a+\frac{1}{a}=2$，及 $S=a^3+\frac{1}{a^3}$，求 S 的值。
If $a+\frac{1}{a}=2$, and $S=a^3+\frac{1}{a^3}$, find the value of S.

S =

(iii) 一凸 n 邊形有 14 條對角線，求 n 的值。
An n-sided convex polygon has 14 diagonals. Find the value of n.

n =

(iv) 若 d 為兩點(2, 3)及(−1, 7)間之距離，求 d 的值。
If d is the distance between the 2 points (2, 3) and (−1, 7), find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1984 – 1985)
Sample Event (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 某兩數之和為 40，其積為 20。若該兩數倒數之和為 $\boldsymbol{a}$，求 $\boldsymbol{a}$ 的值。
The sum of two numbers is 40, and their product is 20.
If the sum of their reciprocals is $\boldsymbol{a}$, find the value of $\boldsymbol{a}$.

$\boldsymbol{a}$ =

(ii) 若一邊長 $(\boldsymbol{a}+1)$ cm 之正方體之總表面積為 $\boldsymbol{b}$ cm^2，求 $\boldsymbol{b}$ 的值。
If $\boldsymbol{b}$ cm^2 is the total surface area of a cube of side $(\boldsymbol{a}+1)$ cm, find the value of $\boldsymbol{b}$.

$\boldsymbol{b}$ =

(iii) 一袋內有 $\boldsymbol{b}-4$ 個白球，$\boldsymbol{b}+46$ 個紅球。若隨意於袋內取一球，而該球為白色之概率為 $\frac{\boldsymbol{c}}{6}$，求 $\boldsymbol{c}$ 的值。
One ball is taken at random from a bag containing $\boldsymbol{b}-4$ white balls and $\boldsymbol{b}+46$ red balls. If $\frac{\boldsymbol{c}}{6}$ is the probability that the ball is white, find the value of $\boldsymbol{c}$.

$\boldsymbol{c}$ =

(iv) 若一邊長 $\boldsymbol{c}$ cm 之正三角形之面積為 $\boldsymbol{d}\sqrt{3}$ cm^2，求 $\boldsymbol{d}$ 的值。
The length of a side of an equilateral triangle is $\boldsymbol{c}$ cm. If its area is $\boldsymbol{d}\sqrt{3}$ cm^2, find the value of $\boldsymbol{d}$.

$\boldsymbol{d}$ =

FOR OFFICIAL USE

Score for accuracy ☐ × Mult. factor for speed ☐ = ☐ Team No. ☐

+ Bonus score ☐ Time ☐ ☐

Total score ☐ Min. Sec.

Hong Kong Mathematics Olympiad (1984 – 1985)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $a = \log_5 \frac{(125)(625)}{25}$，求 a 的值。

Find the value of a if $a = \log_5 \frac{(125)(625)}{25}$.

a =

(ii) 若 $\left(r+\frac{1}{r}\right)^2 = a-2$ 且 $r^3+\frac{1}{r^3} = b$，求 b 的值。

If $\left(r+\frac{1}{r}\right)^2 = a-2$ and $r^3+\frac{1}{r^3} = b$, find the value of b.

b =

(iii) 若 2 為方程 $x^3 + cx + 10 = b$ 之一根，求 c 的值。

If one root of the equation $x^3 + cx + 10 = b$ is 2, find the value of c.

c =

(iv) 若 $9^{d+2} = (6489 + c) + 9^d$，求 d 的值。

Find the value of d if $9^{d+2} = (6489 + c) + 9^d$.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1984 – 1985)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 在以下數列中，求 a 的值：

1, 8, 27, 64, a, 216, ……

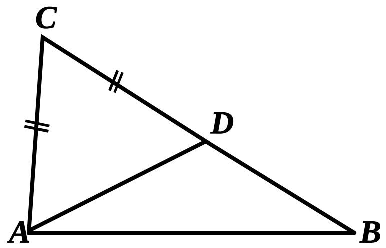
Find a in the following sequence:

1, 8, 27, 64, a, 216, ……

a =

(ii) 在圖一中，$AC = CD$，$\angle CAB - \angle ABC = (a - 95)°$。若$\angle BAD = b°$，求 b 的值。
In Figure 1, $AC = CD$ and $\angle CAB - \angle ABC = (a - 95)°$.
If $\angle BAD = b°$, find the value of b.

b =



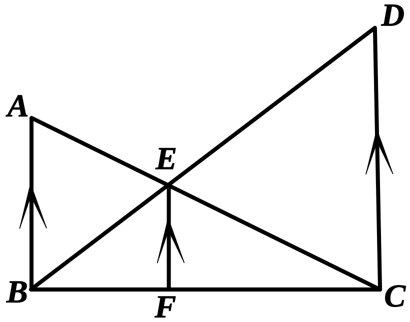
圖一 Figure 1

(iii) 一直綫過(–1, 1)及(3, b – 6)。若其 y 截距為 c，求 c 的值。
A line passes through the points (–1, 1) and (3, b – 6). If the y-intercept of the line is c, find the value of c.

c =

(iv) 在圖二中，$AB = c + 17$，$BC = 100$，$CD = 80$。若 $EF = d$，求 d 的值。
In Figure 2, $AB = c + 17$, $BC = 100$, $CD = 80$. If $EF = d$, find the value of d.

d =



圖二 Figure 2

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1984 – 1985)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 在二時十五分，時鐘兩針所構成之銳角為 $\left(18\frac{1}{2}+\boldsymbol{a}\right)^{\circ}$，求 $\boldsymbol{a}$ 的值。

The acute angle formed by the hands of a clock at 2:15 is $\left(18\frac{1}{2}+\boldsymbol{a}\right)^{\circ}$.

Find the value of $\boldsymbol{a}$.

$\boldsymbol{a}$ =

(ii) 若 $(\boldsymbol{x}+\boldsymbol{y})^{\boldsymbol{a}}$ 的展開式之係數總和是 $\boldsymbol{b}$，求 $\boldsymbol{b}$ 的值。

If the sum of the coefficients in the expansion of $(\boldsymbol{x}+\boldsymbol{y})^{\boldsymbol{a}}$ is $\boldsymbol{b}$, find the value of $\boldsymbol{b}$.

$\boldsymbol{b}$ =

(iii) 若 $\mathrm{f}(\boldsymbol{x}) = \boldsymbol{x} - 2$，$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{y}^2 + \boldsymbol{x}$，且 $\boldsymbol{c} = \boldsymbol{F}(3, \mathrm{f}(\boldsymbol{b}))$，求 $\boldsymbol{c}$ 的值。

If $\mathrm{f}(\boldsymbol{x}) = \boldsymbol{x} - 2$, $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{y}^2 + \boldsymbol{x}$ and $\boldsymbol{c} = \boldsymbol{F}(3, \mathrm{f}(\boldsymbol{b}))$, find the value of $\boldsymbol{c}$.

$\boldsymbol{c}$ =

(iv) $\boldsymbol{x}$，$\boldsymbol{y}$ 為實數。若 $\boldsymbol{x}+\boldsymbol{y} = \boldsymbol{c} - 195$ 及 $\boldsymbol{d}$ 為 $\boldsymbol{xy}$ 之最大值，求 $\boldsymbol{d}$ 的值。

$\boldsymbol{x}$, $\boldsymbol{y}$ are real numbers. If $\boldsymbol{x}+\boldsymbol{y} = \boldsymbol{c} - 195$ and $\boldsymbol{d}$ is the maximum value of $\boldsymbol{xy}$, find the value of $\boldsymbol{d}$.

$\boldsymbol{d}$ =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1984 – 1985)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若兩綫 $x + 2y + 3 = 0$ 及 $4x - ay + 5 = 0$ 互相垂直，求 a 的值。
If the lines $x + 2y + 3 = 0$ and $4x - ay + 5 = 0$ are perpendicular to each other, find the value of a.

a =

(ii) 在圖一中，$ABCD$ 為一梯形，AB 與 DC 平行且$\angle ABC = \angle DCB = 90°$。若 $AB = a$，$BC = CD = 8$ 及 $AD = b$，求 b 的值。
In Figure 1, $ABCD$ is a trapezium with AB parallel to DC and $\angle ABC = \angle DCB = 90°$. If $AB = a$, $BC = CD = 8$ and $AD = b$, find the value of b.

b =

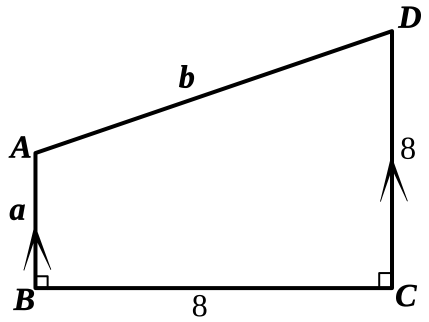


Figure 1
圖一

(iii) 在圖二中，$BD = \frac{b}{2}$，$DE = 4$，$EC = 3$。若ΔAEC 之面積為 24 及ΔABC 之面積為 c，求 c 的值。
In Figure 2, $BD = \frac{b}{2}$, $DE = 4$, $EC = 3$.
If the area of ΔAEC is 24 and the area of ΔABC is c, find the value of c.

c =

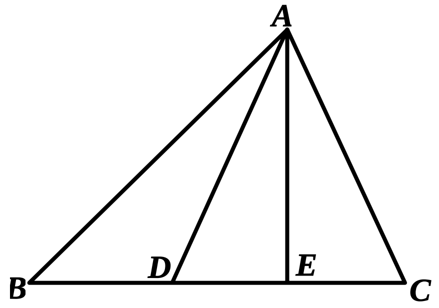


Figure 2
圖二

(iv) 若 $3x^3 - 2x^2 + dx - c$ 可被 $x - 1$ 整除，求 d 的值。
If $3x^3 - 2x^2 + dx - c$ is divisible by $x - 1$, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1984 – 1985)
Final Event 5 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $1 + 2 + 3 + 4 + \ldots + t = 36$，求 t 的值。
If $1 + 2 + 3 + 4 + \ldots + t = 36$, find the value of t.

t =

(ii) 若 $\sin u° = \frac{2}{\sqrt{t}}$ 且 $90 < u < 180$，求 u 的值。
If $\sin u° = \frac{2}{\sqrt{t}}$ and $90 < u < 180$, find the value of u.

u =

(iii) 在圖一中，$\angle ABC = 30°$，且 $AC = (u - 90)$ cm。若ΔABC之外接圓半徑為 v cm，求 v 的值。
In Figure 1, $\angle ABC = 30°$ and $AC = (u - 90)$ cm.
If the radius of the circumcircle of ΔABC is v cm, find the value of v.

v =

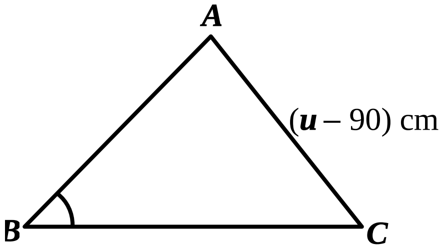


Figure 1
圖一

(iv) 在圖二中，ΔPAB 由切於圓之三切綫形成，且 O 為圓心，
若 $\angle APB = (v - 5)°$，且 $\angle AOB = w°$,，求 w 的值。
In Figure 2, ΔPAB is formed by the 3 tangents of the circle with centre O.
If $\angle APB = (v - 5)°$ and $\angle AOB = w°$, find the value of w.

w =

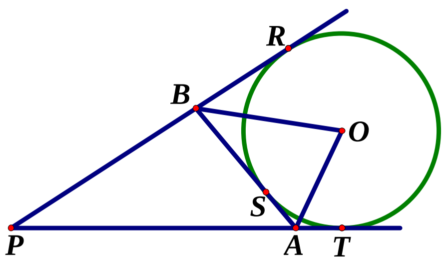


Figure 2
圖二

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1984 – 1985)
Sample Event (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $\boldsymbol{a} * \boldsymbol{b} = \boldsymbol{ab} + 1$，且 $\boldsymbol{s} = (2*4)*2$，求 $\boldsymbol{s}$ 的值。
If $\boldsymbol{a} * \boldsymbol{b} = \boldsymbol{ab} + 1$ and $\boldsymbol{s} = (2*4)*2$, find the value of $\boldsymbol{s}$.

$\boldsymbol{s}$ =

(ii) 若第 $\boldsymbol{n}$ 個質數為 $\boldsymbol{s}$，求 $\boldsymbol{n}$ 的值。
If the $\boldsymbol{n}^{\text{th}}$ prime number is $\boldsymbol{s}$, find the value of $\boldsymbol{n}$.

$\boldsymbol{n}$ =

(iii) 若 $\boldsymbol{K} = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{50}\right)$，試以最簡單之分數表 $\boldsymbol{K}$。
If $\boldsymbol{K} = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{50}\right)$,
find the value of $\boldsymbol{K}$ in the simplest fractional form.

$\boldsymbol{K}$ =

(iv) 一正方形內接於一個半徑為 10 之圓。若正方形之面積為 $\boldsymbol{A}$，求 $\boldsymbol{A}$ 的值。
If $\boldsymbol{A}$ is the area of a square inscribed in a circle of radius 10, find the value of $\boldsymbol{A}$.

$\boldsymbol{A}$ =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1984 – 1985)
Final Event 6 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) p、q、r 之平均數為 4。 p、q、r、x 之平均數為 5。求 x 的值。
The average of p, q, r is 4 . The average of p, q, r, x is 5. Find the value of x .

x =

(ii) 一行車速率為 60 km/h 的貨車之一輪每秒轉動 4 周，
若其直徑為 $\frac{y}{6\pi}$ m，求 y 的值。
A wheel of a truck travelling at 60 km/h makes 4 revolutions per second.
If its diameter is $\frac{y}{6\pi}$ m, find the value of y .

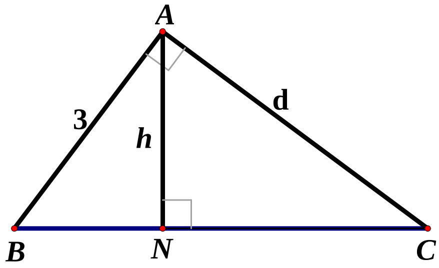
y =

(iii) If $\sin(55-y)° = \frac{d}{x}$, find the value of d .
若 $\sin(55-y)° = \frac{d}{x}$，求 d 的值。

d =

(iv) 如附圖所示，$BA \perp AC$ 及 $AN \perp BC$。若 $AB = 3$，$AC = d$，$AN = h$，求 h 的值。
In the figure, $BA \perp AC$ and $AN \perp BC$. If $AB = 3$, $AC = d$, $AN = h$,
find the value of h .

h =



FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1984 – 1985)
Final Event 7 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 設 $\boldsymbol{M}=\dfrac{78^3+22^3}{78^2-78\times 22+22^2}$。求 $\boldsymbol{M}$ 的值。

Let $\boldsymbol{M}=\dfrac{78^3+22^3}{78^2-78\times 22+22^2}$. Find the value of $\boldsymbol{M}$.

$\boldsymbol{M}$ =

(ii) 正整數 $\boldsymbol{N}$ 分別被 6、5、4、3 及 2 除時，其餘數依次為 5、4、3、2 及 1。求 $\boldsymbol{N}$ 之最小值。

When the positive integer $\boldsymbol{N}$ is divided by 6, 5, 4, 3 and 2, the remainders are 5, 4, 3, 2 and 1 respectively. Find the least value of $\boldsymbol{N}$.

$\boldsymbol{N}$ =

(iii) 一人以 4 km/h 之速率步行 10 km，再以 6 km/h 之速率步行另 10 km。若全程之平均速率為 $\boldsymbol{x}$ km/h，求 $\boldsymbol{x}$ 的值。

A man travels 10 km at a speed of 4 km/h and another 10 km at a speed of 6 km/h. If the average speed of the whole journey is $\boldsymbol{x}$ km/h, find the value of $\boldsymbol{x}$.

$\boldsymbol{x}$ =

(iv) 若 $\boldsymbol{S}$ = 1 + 2 – 3 – 4 + 5 + 6 – 7 – 8 + … + 1985，求 $\boldsymbol{S}$ 的值。

If $\boldsymbol{S}$ = 1 + 2 – 3 – 4 + 5 + 6 – 7 – 8 + … + 1985, find the value of $\boldsymbol{S}$.

$\boldsymbol{S}$ =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1984 – 1985)
Final Event 8 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

M、***N*** 均為小於 10 之正整數，且 $258024M8 \times 9 = 2111110N \times 11$。

M, ***N*** are positive integers less than 10 and $258024M8 \times 9 = 2111110N \times 11$.

(i) 求 ***M*** 的值。
Find the value of ***M***.

M =

(ii) 求 ***N*** 的值。
Find the value of ***N***.

N =

(iii) 一凸 20 邊形有 ***x*** 條對角線。求 ***x*** 的值。
A convex 20-sided polygon has ***x*** diagonals. Find the value of ***x***.

x =

(iv) 若 $y = ab + a + b + 1$ 且 $a = 99$，$b = 49$，求 ***y*** 的值。
If $y = ab + a + b + 1$ and $a = 99$, $b = 49$, find the value of ***y***.

y =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1984 – 1985)
Final Event 9 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) $\Delta \boldsymbol{LMN}$ 之三邊長分別為 8、15 及 17。若 $\Delta \boldsymbol{LMN}$ 之面積為 $\boldsymbol{A}$，求 $\boldsymbol{A}$ 的值。
The lengths of the 3 sides of $\Delta \boldsymbol{LMN}$ are 8, 15 and 17 respectively.
If the area of $\Delta \boldsymbol{LMN}$ is $\boldsymbol{A}$, find the value of $\boldsymbol{A}$.

$\boldsymbol{A}$ =

(ii) 若 $\Delta \boldsymbol{LMN}$ 之內接圓之半徑為 $\boldsymbol{r}$，求 $\boldsymbol{r}$ 的值。
If $\boldsymbol{r}$ is the length of the radius of the circle inscribed in $\Delta \boldsymbol{LMN}$, find the value of $\boldsymbol{r}$.

$\boldsymbol{r}$ =

(iii) 若某年五月第 $\boldsymbol{r}$ 日為星期五，且同年五月第 $\boldsymbol{n}$ 日為星期一，其中 $15 < \boldsymbol{n} < 25$，求 $\boldsymbol{n}$ 的值。
If the $\boldsymbol{r}^{\text{th}}$ day of May in a year is Friday and the $\boldsymbol{n}^{\text{th}}$ day of May in the same year is Monday, where $15 < \boldsymbol{n} < 25$, find the value of $\boldsymbol{n}$.

$\boldsymbol{n}$ =

(iv) 若一凸 $\boldsymbol{n}$ 邊形之內角和為 $\boldsymbol{x}°$，求 $\boldsymbol{x}$ 的值。
If the sum of the interior angles of an $\boldsymbol{n}$-sided convex polygon is $\boldsymbol{x}°$, find the value of $\boldsymbol{x}$.

$\boldsymbol{x}$ =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1984 – 1985)
Final Event 10 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 三連續奇數(最小者為 ***k***)之和為 51。求 ***k*** 的值。
The sum of 3 consecutive odd integers (the smallest being ***k***) is 51.
Find the value of ***k***.

k =

(ii) 若 $x^2 + 6x + k \equiv (x + a)^2 + C$，且 a、C 為常數，求 C 的值。
If $x^2 + 6x + k \equiv (x + a)^2 + C$, where a, C are constants, find the value of C.

C =

(iii) 若 $\frac{p}{q} = \frac{q}{r} = \frac{r}{s} = 2$ 且 $R = \frac{p}{s}$，求 R 的值。
If $\frac{p}{q} = \frac{q}{r} = \frac{r}{s} = 2$ and $R = \frac{p}{s}$, find the value of R.

R =

(iv) 若 $A = \frac{3^n \cdot 9^{n+1}}{27^{n-1}}$，求 A 的值。
If $A = \frac{3^n \cdot 9^{n+1}}{27^{n-1}}$, find the value of A.

A =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ] Team No. [ ]

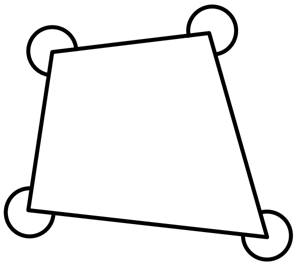
\+ Bonus score [ ] Time [ ] [ ]

Total score [ ] Min. Sec.

Hong Kong Mathematics Olympiad (1985 – 1986)
Sample Event (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 附圖所示四角之和為 $a°$，求 a 的值。
In the given figure, the sum of the four marked angles is $a°$. Find the value of a .



$a =$

(ii) 一正 b 邊形之內角和為 $a°$，求 b 的值。
The sum of the interior angles of a regular b-sided polygon is $a°$.
Find the value of b .

$b =$

(iii) 若 $b^5 = 32^c$，求 c 的值。
If $b^5 = 32^c$, find the value of c .

$c =$

(iv) 若 $c = \log_4 d$，求 d 的值。
If $c = \log_4 d$, find the value of d .

$d =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] [ ]

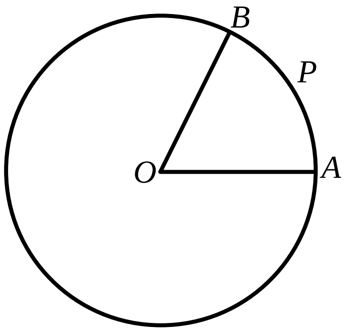
Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (1985 – 1986)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 附圖所示的圓之半徑為 18 cm，圓心為 O。
若 $\angle AOB = \frac{\pi}{3}$，且弧 APB 之長為 $a\pi$ cm，求 a 的值。
The given figure shows a circle of radius 18 cm, centre O.
If $\angle AOB = \frac{\pi}{3}$ and the length of arc APB is $a\pi$ cm, find the value of a.

a =



(ii) 若不等式 $2x^2 - ax + 4 < 0$ 之解為 $1 < x < b$，求 b 的值。
If the solution of the inequality $2x^2 - ax + 4 < 0$ is $1 < x < b$, find the value of b.

b =

(iii) 若 $b(2x - 5) + x + 3 \equiv 5x - c$，求 c 的值。
If $b(2x - 5) + x + 3 \equiv 5x - c$, find the value of c.

c =

(iv) 過 (2, 6) 及 (5, c) 之直綫與 x-軸相交於(d, 0)。求 d 的值。
The line through (2, 6) and (5, c) cuts the x-axis at (d, 0). Find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ] Team No. [ ]

\+ Bonus score [ ] Time [ ] [ ]

Total score [ ] Min. Sec.

Hong Kong Mathematics Olympiad (1985 – 1986)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若方程 $3x^2 - 4x + \frac{h}{3} = 0$ 有等根，求 h 的值。

If the equation $3x^2 - 4x + \frac{h}{3} = 0$ has equal roots, find the value of h.

h =

(ii) 若一圓柱體之高增加一倍，且新半徑為原來之 h 倍，則新體積為原來之 k 倍，求 k 的值。

If the height of a cylinder is doubled and the new radius is h times the original, then the new volume is k times the original. Find the value of k.

k =

(iii) 若 $\log_{10}210 + \log_{10}k - \log_{10}56 + \log_{10}40 - \log_{10}120 + \log_{10}25 = p$，求 p 的值。

If $\log_{10}210 + \log_{10}k - \log_{10}56 + \log_{10}40 - \log_{10}120 + \log_{10}25 = p$, find the value of p.

p =

(iv) 若 $\sin A = \frac{p}{5}$ 且 $\frac{\cos A}{\tan A} = \frac{q}{15}$，求 q 的值。

If $\sin A = \frac{p}{5}$ and $\frac{\cos A}{\tan A} = \frac{q}{15}$, find the value of q.

q =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

\+ Bonus score [ ]

Total score [ ]

Team No. [ ]

Time [ ] Min. [ ] Sec.

Hong Kong Mathematics Olympiad (1985 – 1986)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 某公司的一百個員工之月薪如附表所示。若平均月薪為$m，求 m 的值。

月薪($)	6000	4000	2500
員工人數	5	15	80

The monthly salaries of 100 employees in a company are as shown:

Salaries ($)	6000	4000	2500
No. of employees	5	15	80

If the mean salary is $m, find the value of m.

m =

(ii) 若 $8\sin^2(m+10)^\circ + 12\cos^2(m+25)^\circ = x$，求 x 的值。

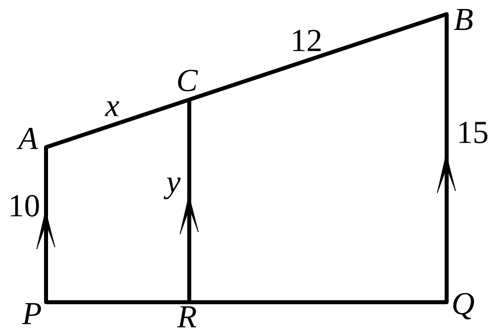
If $8\sin^2(m+10)^\circ + 12\cos^2(m+25)^\circ = x$, find the value of x.

x =

(iii) 如圖所示，$AP // CR // BQ$，$AC = x$，$CB = 12$，$AP = 10$，$BQ = 15$ 及 $CR = y$。求 y 的值。

In the figure, $AP // CR // BQ$, $AC = x$, $CB = 12$, $AP = 10$, $BQ = 15$ and $CR = y$. Find the value of y.

`



(iv) 定義$(a, b, c)\cdot(p, q, r) = ap + bq + cr$，其中 a、b、c、p、q、r 為實數。若$(3, 4, 5)\cdot(y, -2, 1) = n$，求 n 的值。

Define $(a, b, c)\cdot(p, q, r) = ap + bq + cr$, where a, b, c, p, q, r are real numbers. If $(3, 4, 5)\cdot(y, -2, 1) = n$, find the value of n.

n =

FOR OFFICIAL USE

Score for accuracy × Mult. factor for speed =

Team No.

+ Bonus score

Time

Total score

Min. Sec.

Hong Kong Mathematics Olympiad (1985 – 1986)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 已知 $\begin{cases}1=1^2\\1+3=2^2\\1+3+5=3^2\\1+3+5+7=4^2\end{cases}$ It is known that $\begin{cases}1=1^2\\1+3=2^2\\1+3+5=3^2\\1+3+5+7=4^2\end{cases}$

若 $1+3+5+\cdots+n=20^2$，求 n 的值。 If $1+3+5+\cdots+n=20^2$, find the value of n.

n =

(ii) 若直綫 $x+2y=3$ 及 $nx+my=4$ 平行，求 m 的值。
If the lines $x+2y=3$ and $nx+my=4$ are parallel, find the value of m.

m =

(iii) 若由整數 1 至 m 抽出一個數字，而每一數字被抽出之機會均等，被抽出數字為 m 之因數的或然率為 $\frac{p}{39}$，求 p 的值。
If a number is selected from the whole numbers 1 to m, and if each number has an equal chance of being selected, the probability that the number is a factor of m is $\frac{p}{39}$, find the value of p.

p =

(iv) 某小童以速率 p km/h 由家步行上學，並依照原來路線以速率 3 km/h 步行回家。若來回兩程之平均速率為 $\frac{24}{q}$ km/h，求 q 的值。
A boy walks from home to school at a speed of p km/h and returns home along the same route at a speed of 3 km/h.
If the average speed for the double journey is $\frac{24}{q}$ km/h, find the value of q.

q =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.		
			+ Bonus score				Time		
			Total score					Min.	Sec.

Hong Kong Mathematics Olympiad (1985 – 1986)
Final Event 5 (Individual)

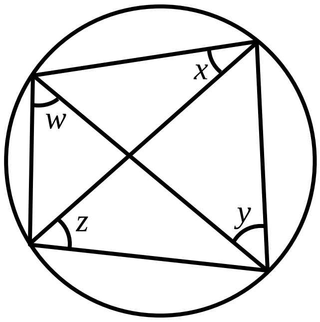
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 投擲一骰子，若擲出質數之或然率為 $\frac{a}{72}$，求 a 的值。

A die is rolled. If the probability of getting a prime number is $\frac{a}{72}$, find the value of a.

a =

(ii) 如圖所示，$x = a°$，$y = 44°$，$z = 52°$及 $w = b°$。求 b 的值。
In the figure, $x = a°$, $y = 44°$, $z = 52°$ and $w = b°$. Find the value of b.



b =

(iii) A，B 兩城相距 b km。彼得從 A 城以速率 7 km/h 踏單車往 B 城，與此同時，約翰從 B 城以速率 5 km/h 踏單車往 A 城。若兩人於 p 小時後相遇，求 p 的值。
A, B are two towns b km apart. Peter cycles at a speed of 7 km/h from A to B and at the same time John cycles from B to A at a speed of 5 km/h.
If they meet after p hours, find the value of p.

p =

(iv) 一角錐體之底為三角形，其邊長分別為 3 cm，p cm 及 5 cm。若該角錐體之高及體積依次為 q cm 及 12 cm^3，求 q 的值。
The base of a pyramid is a triangle with sides 3 cm, p cm and 5 cm. If the height and volume of the pyramid are q cm and 12 cm^3 respectively, find the value of q.

q =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1985 – 1986)
Sample Event (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 某兩數之和為 50，其積為 25。若該兩數倒數之和為 a，求 a 的值。
The sum of two numbers is 50, and their product is 25.
If the sum of their reciprocals is a, find the value of a.

a =

(ii) 若直綫 $ax + 2y + 1 = 0$ 及 $3x + by + 5 = 0$ 互相垂直，求 b 的值。
If the lines $ax + 2y + 1 = 0$ and $3x + by + 5 = 0$ are perpendicular, find the value of b.

b =

(iii) 一正三角形之面積為 $100\sqrt{3}$ cm^2。若其周界為 p cm，求 p 的值。
The area of an equilateral triangle is $100\sqrt{3}$ cm^2. If its perimeter is p cm, find the value of p.

p =

(iv) 若 $x^3 - 2x^2 + px + q$ 可被 $x + 2$ 整除，求 q 的值。
If $x^3 - 2x^2 + px + q$ is divisible by $x + 2$, find the value of q.

q =

FOR OFFICIAL USE

Score for accuracy × Mult. factor for speed =

\+ Bonus score

Total score

Team No.

Time

Min. Sec.

Hong Kong Mathematics Olympiad (1985 – 1986)
Final Event 6 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $12345\times6789 = a\times10^p$，其中 p 為正整數，且 $1 \le a < 10$，求 p 的值。
If $12345\times6789 = a\times10^p$ where p is a positive integer and $1 \le a < 10$, find the value of p.

$p =$

(ii) 若 (p, q)、$(5, 3)$ 及 $(1, -1)$ 共綫，求 q 的值。
If (p, q), $(5, 3)$ and $(1, -1)$ are collinear, find the value of q.

$q =$

(iii) 若 $\tan\theta = \frac{-7}{24}$，$90° < \theta < 180°$ 及 $100\cos\theta = r$，求 r 的值。
If $\tan\theta = \frac{-7}{24}$, $90° < \theta < 180°$ and $100\cos\theta = r$, find the value of r.

$r =$

(iv) x、y、z 之平均數為 10。x、y、z、t 之平均數為 12。求 t 的值。
The average of x, y, z is 10. The average of x, y, z, t is 12. Find the value of t.

$t =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] Min. [ ] Sec.

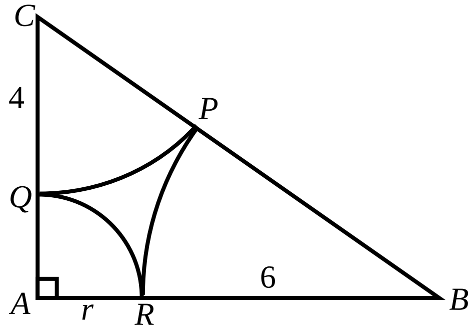
Total score [ ]

Hong Kong Mathematics Olympiad (1985 – 1986)
Final Event 7 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 如圖所示，依次以 A、B、C 為圓心之弧 QR、RP、PQ 相切於 R、P、Q。若 $AR = r$，$RB = 6$，$QC = 4$，$\angle A = 90°$，求 r 的值。
In the figure, QR, RP, PQ are 3 arcs, centres at A, B, C respectively, touching one another at R, P, Q. If $AR = r$, $RB = 6$, $QC = 4$, $\angle A = 90°$, find the value of r.

$r =$



(ii) M、N 依次為 $(3, 2)$ 及 $(9, 5)$。若 $P(s, t)$ 為 MN 上一點使 $MP : PN = 4 : r$，求 s 的值。
M, N are the points $(3, 2)$ and $(9, 5)$ respectively. If $P(s, t)$ is a point on MN such that $MP : PN = 4 : r$, find the value of s.

$s =$

(iii) $x^2 + 10x + t \equiv (x + a)^2 + k$，其中 t、a、k 為常數，求 a 的值。
$x^2 + 10x + t \equiv (x + a)^2 + k$, where t, a, k are constants. Find the value of a.

$a =$

(iv) 若 $9^{p+2} = 240 + 9^p$，求 p 的值。
If $9^{p+2} = 240 + 9^p$, find the value of p.

$p =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1985 – 1986)
Final Event 8 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

在所示乘法中，不同字母代表可能為 2、4、5、6、7、8、9 之不同整數。

In the given multiplication, different letters represent different integers whose possible values are 2, 4, 5, 6, 7, 8, 9.

	1	A	B	C	D	E
×						3
	A	B	C	D	E	1

(i) 求 *A* 的值。
Find the value of *A* .

A =

(ii) 求 *B* 的值。
Find the value of *B* .

B =

(iii) 求 *C* 的值。
Find the value of *C* .

C =

(iv) 求 *D* 的值。
Find the value of *D* .

D =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1985 – 1986)
Final Event 9 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 7 個橙和 5 個蘋果值\$13。3 個橙和 4 個蘋果值\$8。37 個橙和 45 個蘋果值\$$C$。求 C 的值。
7 oranges and 5 apples cost \$13. 3 oranges and 4 apples cost \$8. 37 oranges and 45 apples cost \$$C$. Find the value of C.

C =

(ii) 方程$(\sin^2\theta - 1)(2\sin^2\theta - 1) = 0$，其中 $0° \le \theta \le 360°$，共有 n 個根。求 n 的值。
There are exactly n values of θ satisfying the equation $(\sin^2\theta - 1)(2\sin^2\theta - 1) = 0$, where $0° \le \theta \le 360°$. Find the value of n.

n =

(iii) 若 $S = ab + a - b - 1$ 及 $a = 101$，$b = 49$，求 S 的值。
If $S = ab + a - b - 1$ and $a = 101$, $b = 49$, find the value of S.

S =

(iv) 若 (13, 5) 與 (5, −10) 兩點之距離為 d，求 d 的值。
If d is the distance between the points (13, 5) and (5, −10), find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy		× Mult. factor for speed		=		Team No.	
		+ Bonus score				Time	
		Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1985 – 1986)
Final Event 10 (Group)

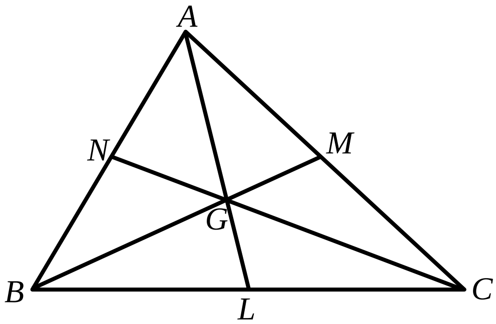
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $b+c=3$，$c+a=6$，$a+b=7$，且 $P=abc$，求 P 的值。
If $b+c=3$, $c+a=6$, $a+b=7$ and $P=abc$, find the value of P.

$P=$

(ii) ΔABC 之中綫 AL、BM、CN 相交於 G。
若 ΔABC 之面積為 54 cm^2，ΔANG 之面積為 x cm^2，求 x 的值。
The medians AL, BM, CN of ΔABC meet at G.
If the area of ΔABC is 54 cm^2 and the area of ΔANG is x cm^2. Find the value of x.

$x=$



(iii) 若 $k=\dfrac{3\sin\theta+5\cos\theta}{2\sin\theta+\cos\theta}$ 及 $\tan\theta=3$，求 k 的值。
If $k=\dfrac{3\sin\theta+5\cos\theta}{2\sin\theta+\cos\theta}$ and $\tan\theta=3$, find the value of k.

$k=$

(iv) 若 $S=\left(1-\dfrac{1}{2^2}\right)\left(1-\dfrac{1}{3^2}\right)\left(1-\dfrac{1}{4^2}\right)\cdots\left(1-\dfrac{1}{10^2}\right)$，求 S 的值。
If $S=\left(1-\dfrac{1}{2^2}\right)\left(1-\dfrac{1}{3^2}\right)\left(1-\dfrac{1}{4^2}\right)\cdots\left(1-\dfrac{1}{10^2}\right)$, find the value of S.

$S=$

FOR OFFICIAL USE

Score for accuracy × Mult. factor for speed =

Team No.

+ Bonus score

Time

Total score

Min. Sec.

Hong Kong Mathematics Olympiad (1986 – 1987)
Sample Event (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $x^2 - 8x + 26 \equiv (x + k)^2 + a$，求 a 的值。
If $x^2 - 8x + 26 \equiv (x + k)^2 + a$, find the value of a.

$a =$

(ii) 若 $\sin a° = \cos b°$，其中 $270 < b < 360$，求 b 的值。
If $\sin a° = \cos b°$, where $270 < b < 360$, find the value of b.

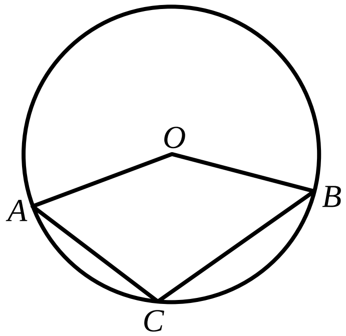
$b =$

(iii) X 以\$$b$ 出售一貨品與 Y 而虧蝕 30%。若 X 購入該貨品之成本為\$$c$，求 c 的值。
X sold an article to Y for \$$b$ at a loss of 30%.
If the cost price of the article for X is \$$c$, find the value of c.

$c =$

(iv) 附圖中，O 為圓心。若$\angle ACB = \frac{3c°}{10}$ 及$\angle AOB = d°$，求 d 的值。
In the figure, O is the centre of the circle.
If $\angle ACB = \frac{3c°}{10}$ and $\angle AOB = d°$, find the value of d.

$d =$



FOR OFFICIAL USE

Score for accuracy □ × Mult. factor for speed □ = □ Team No. □

+ Bonus score □ Time □ □

Total score □ Min. Sec.

Hong Kong Mathematics Olympiad (1986 – 1987)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

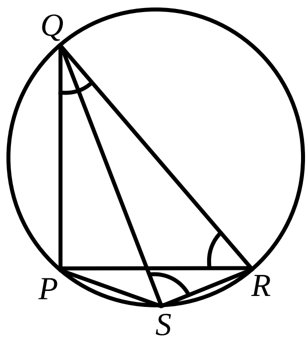
(i) 若 $A = 11 + 12 + 13 + \ldots + 29$，求 A 的值。
If $A = 11 + 12 + 13 + \ldots + 29$, find the value of A.

$A =$

(ii) 若 $\sin A° = \cos B°$，其中 $0 < B < 90$，求 B 的值。
If $\sin A° = \cos B°$, where $0 < B < 90$, find the value of B.

$B =$

(iii) 附圖中，$\angle PQR = B°$，$\angle PRQ = 50°$。若$\angle QSR = n°$，求 n 的值。
In the given figure, $\angle PQR = B°$, $\angle PRQ = 50°$. If $\angle QSR = n°$, find the value of n.



$n =$

(iv) 由 1 至 n 號卡片中隨意抽出一張。若得到 5 之倍數之概率為 $\frac{1}{m}$，求 m 的值。
n cards are marked from 1 to n and one is drawn at random. If the chance of it being a multiple of 5 is $\frac{1}{m}$, find the value of m.

$m =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1986 – 1987)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 某球體之半徑為 r，體積為 36π，求 r 的值。
The volume of a sphere with radius r is 36π, find the value of r.

$r =$

(ii) 若 $r^x + r^{1-x} = 4$，且 $x > 0$，求 x 的值。
If $r^x + r^{1-x} = 4$ and $x > 0$, find the value of x.

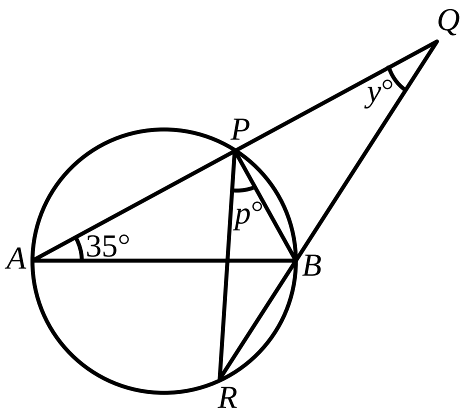
$x =$

(iii) 若 $a : b = 5 : 4$，$b : c = 3 : x$ 且 $a : c = y : 4$，求 y 的值。
In $a : b = 5 : 4$, $b : c = 3 : x$ and $a : c = y : 4$, find the value of y.

$y =$

(iv) 附圖中，AB 為該圓之直徑。APQ 及 RBQ 為直綫。若$\angle PAB = 35°$，$\angle PQB = y°$ 及 $\angle RPB = p°$，求 p 的值。
In the figure, AB is a diameter of the circle. APQ and RBQ are straight lines.
If $\angle PAB = 35°$, $\angle PQB = y°$ and $\angle RPB = p°$, find the value of p.

$p =$



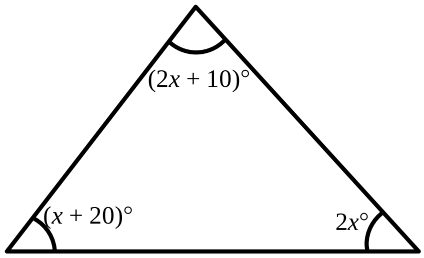
FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (1986 – 1987)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 如圖所示，求 x 的值。
In the figure, find the value of x .



x =

(ii) P，Q 之坐標依次為$(a, 2)$及$(x, -6)$。若 PQ 的中點之坐標為$(18, b)$，求 a 的值。
The coordinates of the points P and Q are $(a, 2)$ and $(x, -6)$ respectively.
If the coordinates of the mid-point of PQ is $(18, b)$, find the value of a .

a =

(iii) 某人以均勻速度 a km/h 由 X 往 Y，並以均勻速度 $2a$ km/h 由 Y 返 X。
若其平均速度為 c km/h，求 c 的值。
A man travels from X to Y at a uniform speed of a km/h and returns at a uniform speed of $2a$ km/h. If his average speed is c km/h, find the value of c .

c =

(iv) 若 $f(y) = 2y^2 + cy - 1$，求 $f(4)$ 的值。
If $f(y) = 2y^2 + cy - 1$, find the value of $f(4)$.

$f(4)$=

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

Team No. [ ]

\+ Bonus score [ ]

Time [ ] [ ]

Total score [ ]

Min. Sec.

Hong Kong Mathematics Olympiad (1986 – 1987)
Final Event 4 (Individual)

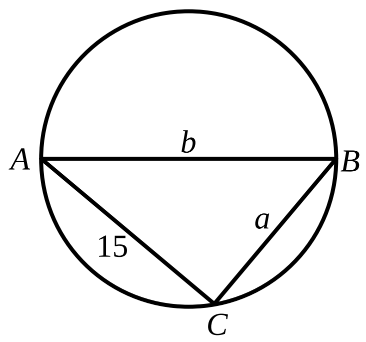
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若曲線 $y = 2x^2 - 8x + a$ 與 x-軸相切，求 a 的值。
If the curve $y = 2x^2 - 8x + a$ touches the x-axis, find the value of a.

a =

(ii) 附圖中，AB 為該圓之直徑。若 $AC = 15$，$BC = a$ 及 $AB = b$，求 b 的值。
In the figure, AB is a diameter of the circle. If $AC = 15$, $BC = a$ and $AB = b$, find the value of b.

b =



(iii) 直綫 $5x + by + 2 = d$ 過點(40, 5)。求 d 的值。
The line $5x + by + 2 = d$ passes through (40, 5). Find the value of d.

d =

(iv) X 以\$$d$ 出售一貨品與 Y，得利潤 2.5%。若 X 購入該貨品之成本為\$$K$，求 K 的值。
X sold an article to Y for \$$d$ at a profit of 2.5%. If the cost price of the article for X is \$$K$, find the value of K.

K =

FOR OFFICIAL USE

Score for accuracy × Mult. factor for speed =

+ Bonus score

Total score

Team No.

Time Min. Sec.

Hong Kong Mathematics Olympiad (1986 – 1987)
Final Event 5 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 設 $x = 19.\dot{8}\dot{7}$ 。若 $19.\dot{8}\dot{7} = \frac{a}{99}$ ，求 a 的值。
(提示：$99x = 100x - x$)
Let $x = 19.\dot{8}\dot{7}$. If $19.\dot{8}\dot{7} = \frac{a}{99}$, find the value of a .
(Hint: $99x = 100x - x$)

a =

(ii) 若 $f(y) = 4\sin y°$，且 $f(a - 18) = b$，求 b 的值。
If $f(y) = 4\sin y°$ and $f(a - 18) = b$, find the value of b .

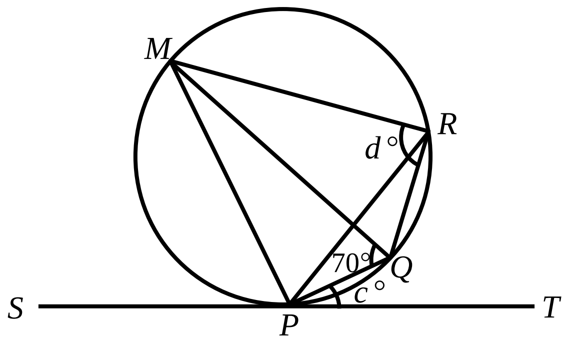
b =

(iii) 若 $\frac{\sqrt{3}}{b\sqrt{7} - \sqrt{3}} = \frac{2\sqrt{21} + 3}{c}$ ，求 c 的值。
If $\frac{\sqrt{3}}{b\sqrt{7} - \sqrt{3}} = \frac{2\sqrt{21} + 3}{c}$, find the value of c.

c =

(iv) 附圖中，ST 與圓相切於 P。若$\angle MQP = 70°$，$\angle QPT = c°$及$\angle MRQ = d°$，求 d 的值。
In the figure, ST is a tangent to the circle at P.
If $\angle MQP = 70°$, $\angle QPT = c°$ and $\angle MRQ = d°$, find the value of d .

d =



FOR OFFICIAL USE

Score for accuracy × Mult. factor for speed =

\+ Bonus score

Total score

Team No.

Time

Min. Sec.

Hong Kong Mathematics Olympiad (1986 – 1987)
Sample Event (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

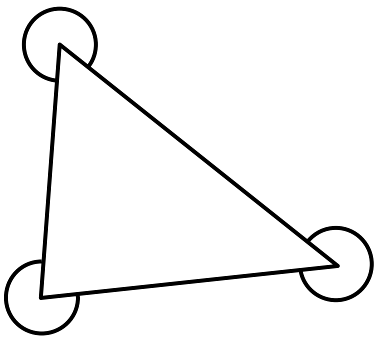
(i) 若 $100A = 35^2 - 15^2$，求 A 的值。
If $100A = 35^2 - 15^2$, find the value of A.

A =

(ii) 若 $(A - 1)^6 = 27^B$，求 B 的值。
If $(A - 1)^6 = 27^B$, find the value of B.

B =

(iii) 附圖所示三角之和是 $C°$。求 C 的值。
In the given diagram, the sum of the three marked angles is $C°$. Find the value of C.



C =

(iv) 若直綫 $x + 2y + 1 = 0$ 及 $9x + Dy + 1 = 0$ 互相平行，求 D 的值。
If the lines $x + 2y + 1 = 0$ and $9x + Dy + 1 = 0$ are parallel, find D.

D =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

\+ Bonus score [ ]

Total score [ ]

Team No. [ ]

Time [ ] Min. [ ] Sec.

Hong Kong Mathematics Olympiad (1986 – 1987)
Final Event 6 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 α、β 為 $x^2 - 10x + 20 = 0$ 之根，且 $p = \alpha^2 + \beta^2$，求 p 的值。
If α, β are the roots of $x^2 - 10x + 20 = 0$, and $p = \alpha^2 + \beta^2$, find the value of p.

$p =$

(ii) 一正三角形之周界為 p。若其面積為 $k\sqrt{3}$，求 k 的值。
The perimeter of an equilateral triangle is p. If its area is $k\sqrt{3}$, find the value of k.

$k =$

(iii) 一正 N 邊形之每一內角為 140°。求 N 的值。
Each interior angle of an N-sided regular polygon is 140°. Find the value of N.

$N =$

(iv) 若 $M = (10^2 + 10\times1 + 1^2)(10^2 - 1^2)(10^2 - 10\times1 + 1^2)$，求 M 的值。
If $M = (10^2 + 10\times1 + 1^2)(10^2 - 1^2)(10^2 - 10\times1 + 1^2)$, find the value of M.

$M =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1986 – 1987)
Final Event 7 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 在下午三點三十分時，時鐘兩針所構成之銳角為 $A°$。求 A 的值。
The acute angle formed by the hands of a clock at 3:30 p.m. is $A°$.
Find the value of A.

$A =$

(ii) 若 $\tan(3A + 15)° = \sqrt{B}$，求 B 的值。
If $\tan(3A + 15)° = \sqrt{B}$, find the value of B.

$B =$

(iii) 若 $\log_{10}AB = C \log_{10}15$，求 C 的值。
If $\log_{10}AB = C \log_{10}15$, find the value of C.

$C =$

(iv) 點 (1, 3)、(4, 9) 及 (2, D) 共线。求 D 的值。
The points (1, 3), (4, 9) and (2, D) are collinear. Find the value of D.

$D =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	Min. / Sec.
			Total score					

Hong Kong Mathematics Olympiad (1986 – 1987)
Final Event 8 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $A=\frac{5\sin\theta+4\cos\theta}{3\sin\theta+\cos\theta}$，且 $\tan\theta=2$，求 A 的值。
If $A=\frac{5\sin\theta+4\cos\theta}{3\sin\theta+\cos\theta}$ and $\tan\theta=2$, find the value of A.

$A=$

(ii) 若 $x+\frac{1}{x}=2A$，且 $x^3+\frac{1}{x^3}=B$，求 B 的值。
If $x+\frac{1}{x}=2A$, and $x^3+\frac{1}{x^3}=B$, find the value of B.

$B=$

(iii) 共有 N 個 α 值可滿足方程 $\cos^3\alpha-\cos\alpha=0$，其中 $0°\le\alpha\le360°$。求 N 的值。
There are exactly N values of α satisfying the equation $\cos^3\alpha-\cos\alpha=0$, where $0°\le\alpha\le360°$. Find the value of N.

$N=$

(iv) 若某年五月第 N 日為星期四，且同年五月第 K 日為星期一，其中 $10<K<20$，求 K 的值。
If the N^{th} day of May in a year is Thursday and the K^{th} day of May in the same year is Monday, where $10<K<20$, find the value of K.

$K=$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (1986 – 1987)
Final Event 9 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

在所示乘法中，不同字母代表由 0 至 9 之不同整數。

In the given multiplication, different letters represent different integers ranging from 0 to 9.

$$\begin{array}{r} A\ B\ C\ D \\ \times \qquad\quad 9 \\ \hline D\ C\ B\ A \end{array}$$

(i) 求 A 的值。
Find the value of A . | A =

(ii) 求 B 的值。
Find the value of B . | B =

(iii) 求 C 的值。
Find the value of C . | C =

(iv) 求 D 的值。
Find the value of D . | D =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1986 – 1987)
Final Event 10 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) p、q、r 及 s 之平均數為 5。
p、q、r、s 及 A 之平均數為 8。求 A 的值。
The average of p, q, r and s is 5.
The average of p, q, r, s and A is 8. Find the value of A.

A =

(ii) 若直綫 $3x - 2y + 1 = 0$ 及 $Ax + By + 1 = 0$ 互相垂直，求 B 的值。
If the lines $3x - 2y + 1 = 0$ and $Ax + By + 1 = 0$ are perpendicular, find the value of B.

B =

(iii) 若 $Cx^3 - 3x^2 + x - 1$ 除以 $x + 1$ 得之餘數為 -7。求 C 的值。
When $Cx^3 - 3x^2 + x - 1$ is divided by $x + 1$, the remainder is -7. Find the value of C.

C =

(iv) 若 P、Q 為正整數使 $P + Q + PQ = 90$，且 $D = P + Q$，求 D 的值。
(提示：因式分解 $1 + P + Q + PQ$)
If P, Q are positive integers such that $P + Q + PQ = 90$ and $D = P + Q$,
find the value of D. (Hint: Factorise $1 + P + Q + PQ$)

D =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

+ Bonus score [ ]

Total score [ ]

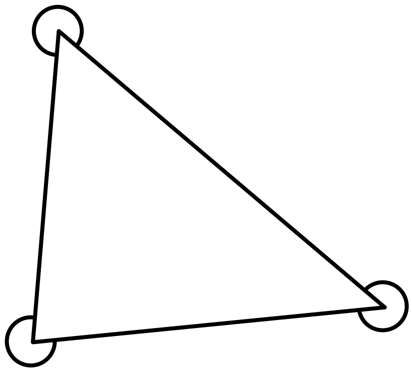
Team No. [ ]

Time [ ] Min. [ ] Sec.

Hong Kong Mathematics Olympiad (1987 – 1988)
Sample Event (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 附圖所示三角的和是 $a°$，求 a 的值。
In the given diagram, the sum of the three marked angles is $a°$. Find the value of a.



a =

(ii) 一正 b 邊形的內角和為 $a°$，求 b 的值。
The sum of the interior angles of a regular b-sided polygon is $a°$. Find the value of b.

b =

(iii) 若 $8^b = p^{21}$，求 p 的值。
If $8^b = p^{21}$, find the value of p.

p =

(iv) 若 $p = \log_q 81$，求 q 的值。
If $p = \log_q 81$, find the value of q.

q =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

\+ Bonus score [ ]

Total score [ ]

Team No. [ ]

Time [ ] Min. [ ] Sec.

Hong Kong Mathematics Olympiad (1987 – 1988)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $N(t) = 100\times18^t$，且 $P = N(0)$，求 P 的值。
If $N(t) = 100\times18^t$ and $P = N(0)$, find the value of P.

$P =$

(ii) A fox ate P grapes in 5 days, each day eating 6 more than on the previous day. If he ate Q grapes on the first day, find the value of Q.
一狐狸在 5 天內吃提子 P 粒，而每天較前一天多吃 6 粒。假如牠在第一天吃了 Q 粒提子，求 Q 的值。

$Q =$

(iii) 若 $\frac{25}{32}$ 的 Q%是 R 的 $\frac{1}{Q}$%，求 R 的值。
If Q% of $\frac{25}{32}$ is $\frac{1}{Q}$% of R, find the value of R.

$R =$

(iv) 若 $3x^2 - ax + R = 0$ 的其中一根是 $\frac{50}{9}$，而另一根是 S，求 S 的值。
If one root of the equation $3x^2 - ax + R = 0$ is $\frac{50}{9}$ and the other root is S, find the value of S.

$S =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1987 – 1988)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$，且 $\begin{vmatrix} 3 & 4 \\ 2 & k \end{vmatrix} = k$，求 k 的值。

If $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ and $\begin{vmatrix} 3 & 4 \\ 2 & k \end{vmatrix} = k$, find the value of k.

$k =$

(ii) 若 $50m = 54^2 - k^2$，求 m 的值。

If $50m = 54^2 - k^2$, find the value of m.

$m =$

(iii) 若 $(m + 6)^a = 2^{12}$，求 a 的值。

If $(m + 6)^a = 2^{12}$, find the value of a.

$a =$

(iv) A、B 及 C 依次為 $(a, 5)$、$(2, 3)$ 及 $(4, b)$。若 $AB \perp BC$，求 b 的值。

A, B and C are the points $(a, 5)$, $(2, 3)$ and $(4, b)$ respectively.
If $AB \perp BC$, find the value of b.

$b =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time		
				Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1987 – 1988)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $\frac{\sqrt{3}}{2\sqrt{7}-\sqrt{3}}=\frac{2\sqrt{21}+h}{25}$，求 h 的值。

If $\frac{\sqrt{3}}{2\sqrt{7}-\sqrt{3}}=\frac{2\sqrt{21}+h}{25}$, find the value of h.

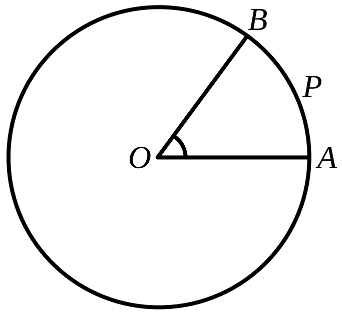
$h =$

(ii) 附圖所示圓形的半徑是 $2h$ cm，圓心是 O。若 $\angle AOB=\frac{\pi}{3}$，且扇形 $AOBP$ 的面積是 $k\pi$ cm^2，求 k 的值。

The given figure shows a circle of radius $2h$ cm, centre O.

If $\angle AOB=\frac{\pi}{3}$, and the area of sector $AOBP$ is $k\pi$ cm^2, find the value of k.

$k =$



(iii) 甲可在 k 日完成某一工程，乙可在$(k+6)$日完成同一工程。
假如甲、乙合作，可在 m 日完成該工程。求 m 的值。

A can do a job in k days, B can do the same job in $(k + 6)$ days.

If they work together, they can finish the job in m days. Find the value of m.

$m =$

(iv) 同時擲 m 個硬幣。若其中至少有一個正面出現的概率是 p，求 p 的值。

m coins are tossed. If the probability of obtaining at least one head is p, find the value of p.

$p =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1987 – 1988)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $f(t) = 2 - \frac{t}{3}$，且 $f(a) = -4$，求 a 的值。

If $f(t) = 2 - \frac{t}{3}$, and $f(a) = -4$, find the value of a.

$a =$

(ii) 若 $a + 9 = 12Q + r$，其中 Q、r 是整數，且 $0 < r < 12$，求 r 的值。
If $a + 9 = 12Q + r$, where Q, r are integers and $0 < r < 12$, find the value of r.

$r =$

(iii) x、y 是實數。若 $x + y = r$，且 M 是 xy 的最大值，求 M 的值。
x, y are real numbers. If $x + y = r$ and M is the maximum value of xy, find the value of M.

$M =$

(iv) 若 w 是實數，且 $2^{2w} - 2^{w} - \frac{8}{9}M = 0$，求 w 的值。

If w is a real number and $2^{2w} - 2^{w} - \frac{8}{9}M = 0$, find the value of w.

$w =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1987 – 1988)
Final Event 5 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $0.3\dot{5}\dot{7} = \frac{177}{a}$，求 a 的值。

If $0.3\dot{5}\dot{7} = \frac{177}{a}$, find the value of a.

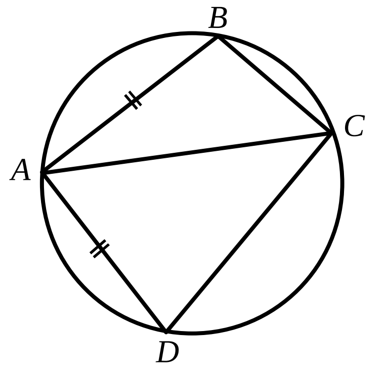
$a =$

(ii) 若 $\tan^2 a° + 1 = b$，求 b 的值。
If $\tan^2 a° + 1 = b$, find the value of b.

$b =$

(iii) 附圖中，$AB = AD$，$\angle BAC = 26° + b°$，$\angle BCD = 106°$。
若$\angle ABC = x°$，求 x 的值。
In the figure, $AB = AD$, $\angle BAC = 26° + b°$, $\angle BCD = 106°$.
If $\angle ABC = x°$, find the value of x.

$x =$



(iv) 若 $\begin{pmatrix} h & k \end{pmatrix}\begin{pmatrix} m & p \\ n & q \end{pmatrix} = \begin{pmatrix} hm+kn & hp+kq \end{pmatrix}$，且 $\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & x \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 11 & Y \end{pmatrix}$，求 Y 的值。

If $\begin{pmatrix} h & k \end{pmatrix}\begin{pmatrix} m & p \\ n & q \end{pmatrix} = \begin{pmatrix} hm+kn & hp+kq \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & x \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 11 & Y \end{pmatrix}$,
find the value of Y.

$Y =$

FOR OFFICIAL USE

Score for accuracy ☐ × Mult. factor for speed ☐ = ☐ Team No. ☐

\+ Bonus score ☐ Time ☐ ☐

Total score ☐ Min. Sec.

Hong Kong Mathematics Olympiad (1987 – 1988)
Sample Event (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 在下午三點卅分，時鐘兩針間之銳角為 $p°$，求 p 的值。
The acute angle between the 2 hands of a clock at 3:30 p.m. is $p°$. Find the value of p.

p =

(ii) 在 ΔABC 中，$\angle B = \angle C = p°$。若 $q = \sin A$，求 q 的值。
In ΔABC, $\angle B = \angle C = p°$. If $q = \sin A$, find the value of q.

q =

(iii) 三點(1, 3)、(2, 5)、(4, a)共綫，求 a 的值。
The 3 points (1, 3), (2, 5), (4, a) are collinear. Find the value of a.

a =

(iv) 7、9、x、y 及 17 之平均數為 10。
若 $x + 3$、$x + 5$、$y + 2$、8 及 $y + 18$ 的平均數是 m，求 m 的值。
The average of 7 , 9 , x , y and 17 is 10 .
If the average of $x + 3$, $x + 5$, $y + 2$, 8 and $y + 18$ is m, find the value of m.

m =

FOR OFFICIAL USE

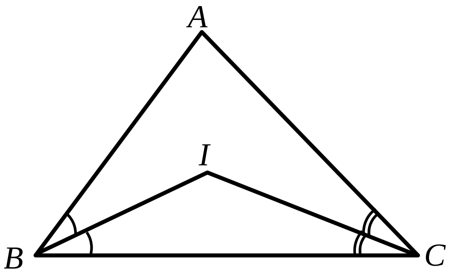
Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1987 – 1988)
Final Event 6 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 附圖中∠B 及∠C 的平分線相交於 I。若∠$A = 70°$，∠$BIC = x°$，求 x 的值。
In the figure, the bisectors of ∠B and ∠C meet at I. If ∠$A = 70°$ and ∠$BIC = x°$, find the value of x.

x =



(ii) 一凸 n 邊形有 35 條對角綫。求 n 的值。
A convex n-sided polygon has 35 diagonals. Find the value of n.

n =

(iii) 若 $y = ab - a + b - 1$，且 $a = 49$，$b = 21$，求 y 的值。
If $y = ab - a + b - 1$ and $a = 49$, $b = 21$, find the value of y.

y =

(iv) 若 $K = 1 + 2 - 3 - 4 + 5 + 6 - 7 - 8 + \ldots + 1001 + 1002$，求 K 的值。
If $K = 1 + 2 - 3 - 4 + 5 + 6 - 7 - 8 + \ldots + 1001 + 1002$, find the value of K.

K =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ] Team No. [ ]

\+ Bonus score [ ] Time [ ] Min. [ ] Sec.

Total score [ ]

Hong Kong Mathematics Olympiad (1987 – 1988)
Final Event 7 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

M、N 是小於 10 的正整數，且 $8M420852 \times 9 = N9889788 \times 11$。

M, N are positive integers less than 10 and $8M420852 \times 9 = N9889788 \times 11$.

(i) 求 M 的值。
Find the value of M.

$M =$

(ii) 求 N 的值。
Find the value of N.

$N =$

(iii) 經過 (4, 3) 及 (12, −3) 的直綫方程是 $\frac{x}{a}+\frac{y}{b}=1$。求 a 的值。

The equation of the line through (4, 3) and (12, −3) is $\frac{x}{a}+\frac{y}{b}=1$.

Find the value of a.

$a =$

(iv) 若 $x + k$ 是 $3x^2 + 14x + a$ 的因式，求 k 的值。(k 是整數)
If $x + k$ is a factor of $3x^2 + 14x + a$, find the value of k. (k is an integer.)

$k =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

+ Bonus score [ ]

Total score [ ]

Team No. [ ]

Time [ ] Min. [ ] Sec.

Hong Kong Mathematics Olympiad (1987 – 1988)
Final Event 8 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $\log_9 S = \frac{3}{2}$，求 S 的值。

If $\log_9 S = \frac{3}{2}$, find the value of S.

$S =$

(ii) 若直綫 $x + 5y = 0$ 及 $Tx - Sy = 0$ 互相垂直，求 T 的值。
If the lines $x + 5y = 0$ and $Tx - Sy = 0$ are perpendicular to each other, find the value of T.

$T =$

三位數 AAA(其中 $A \neq 0$)及六位數 $AAABBB$ 滿足下列等式：$AAA \times AAA + AAA = AAABBB$。
The 3-digit number AAA, where $A \neq 0$, and the 6-digit number $AAABBB$ satisfy the following equality: $AAA \times AAA + AAA = AAABBB$.

(iii) 求 A 的值。
Find the value of A.

$A =$

(iv) 求 B 的值。
Find the value of B.

$B =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ] Team No. [ ]

+ Bonus score [ ] Time [ ] [ ]

Total score [ ] Min. Sec.

Hong Kong Mathematics Olympiad (1987 – 1988)
Final Event 9 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 一正三角形的面積是 $50\sqrt{12}$。若它的周界是 p，求 p 的值。
The area of an equilateral triangle is $50\sqrt{12}$.
If its perimeter is p, find the value of p.

p =

(ii) q、y、z 的平均數是 14。q、y、z、t 的平均數是 13。求 t 的值。
The average of q, y, z is 14 . The average of q, y, z, t is 13 . Find the value of t .

t =

(iii) 若 $7 - 24x - 4x^2 \equiv K + A(x + B)^2$，且 K、A、B 是常數，求 K 的值。
If $7 - 24x - 4x^2 \equiv K + A(x + B)^2$, where K, A, B are constants, find the value of K.

K =

(iv) 若 $C = \frac{3^{4n} \cdot 9^{n+4}}{27^{2n+2}}$，求 C 的值。
If $C = \frac{3^{4n} \cdot 9^{n+4}}{27^{2n+2}}$, find the value of C .

C =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1987 – 1988)
Final Event 10 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

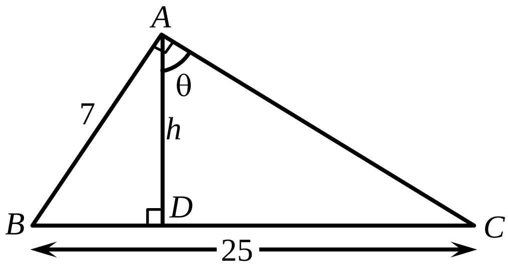
(i) 一正 n 邊形每一內角是 160°。求 n 的值。
Each interior angle of an n-sided regular polygon is 160°. Find the value of n.

n =

(ii) 某年五月第 n 日是星期五。同年五月第 k 日是星期二，且 $20 < k < 26$。求 k 的值。
The n^{th} day of May in a year is Friday. The k^{th} day of May in the same year is Tuesday, where $20 < k < 26$. Find the value of k.

k =

在圖中，$AD \perp BC$，$BA \perp CA$，$AB = 7$，$BC = 25$，$AD = h$ 及 $\angle CAD = \theta$。
In the figure, $AD \perp BC$, $BA \perp CA$, $AB = 7$, $BC = 25$, $AD = h$ and $\angle CAD = \theta$.



(iii) 若 $100 \sin \theta = t$，求 t 的值。
If $100 \sin \theta = t$, find the value of t.

t =

(iv) 求 h 的值。
Find the value of h.

h =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]　　Team No. [ ]

\+ Bonus score [ ]　　Time [ ] [ ]

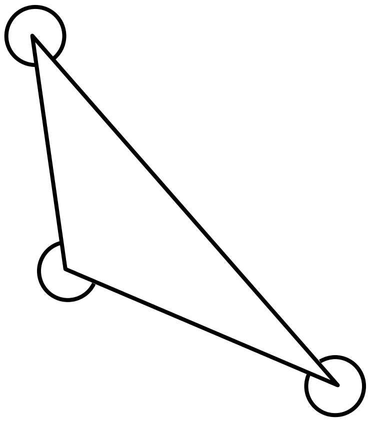
Total score [ ]　　Min.　Sec.

Hong Kong Mathematics Olympiad (1988 – 1989)
Sample Event (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 附圖所示三角的和是 $a°$，求 a 的值。
In the given diagram, the sum of the three marked angles is $a°$. Find the value of a .

a =



(ii) 一凸 b 邊形的內角和為 $a°$，求 b 的值。
The sum of the interior angles of a convex b-sided polygon is $a°$.
Find the value of b .

b =

(iii) 若 $27^{b-1} = c^{18}$，求 c 的值。
If $27^{b-1} = c^{18}$, find the value of c .

c =

(iv) 若 $c = \log_d 125$，求 d 的值。
If $c = \log_d 125$, find the value of d .

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1988 – 1989)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 在十時三十分，時鐘兩針構成的鈍角是$(100 + a)°$，求 a 的值。
The obtuse angle formed by the hands of a clock at 10:30 is $(100 + a)°$.
Find the value of a.

$a =$

(ii) 兩直綫 $ax + by = 0$ 及 $x - 5y + 1 = 0$ 互相垂直。求 b 的值。
The lines $ax + by = 0$ and $x - 5y + 1 = 0$ are perpendicular to each other.
Find the value of b.

$b =$

(iii) 已知 $(b + 1)^4 = 2^{c+2}$，求 c 的值。
If $(b + 1)^4 = 2^{c+2}$, find the value of c.

$c =$

(iv) 已知 $c - 9 = \log_c (6d - 2)$，求 d 的值。
If $c - 9 = \log_c (6d - 2)$, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ] Team No. [ ]

\+ Bonus score [ ] Time [ ] Min. [ ] Sec.

Total score [ ]

Hong Kong Mathematics Olympiad (1988 – 1989)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 已知 $1000a = 85^2 - 15^2$，求 a 的值。
If $1000a = 85^2 - 15^2$, find the value of a.

$a =$

(ii) 假設點(a, b)在直綫 $5x + 2y = 41$ 上。求 b 的值。
The point (a, b) lies on the line $5x + 2y = 41$. Find the value of b.

$b =$

(iii) $x + b$ 是 $x^2 + 6x + c$ 的因式。求 c 的值。
$x + b$ is a factor of $x^2 + 6x + c$. Find the value of c.

$c =$

(iv) 設 d 是兩點 $(c, 1)$ 及 $(5, 4)$ 間的距離，求 d 的值。
If d is the distance between the points $(c, 1)$ and $(5, 4)$, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.		
			+	Bonus score			Time		
				Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1988 – 1989)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 已知 $\alpha+\beta=11$，$\alpha\beta=24$，且 $\alpha>\beta$，求 α 的值。
If $\alpha+\beta=11$, $\alpha\beta=24$ and $\alpha>\beta$, find the value of α.

$\alpha=$

(ii) 已知 $\tan\theta=\frac{-\alpha}{15}$，$90°<\theta<180°$，且 $\sin\theta=\frac{b}{34}$，求 b 的值。
If $\tan\theta=\frac{-\alpha}{15}$, $90°<\theta<180°$ and $\sin\theta=\frac{b}{34}$, find the value of b.

$b=$

(iii) 一正方形內接一個直徑為 b 的圓。設正方形的面積為 A，求 A 的值。
If A is the area of a square inscribed in a circle of diameter b, find the value of A.

$A=$

(iv) 已知 $x^2+22x+A\equiv(x+k)^2+d$，其中 k，d 是常數，求 d 的值。
If $x^2+22x+A\equiv(x+k)^2+d$, where k, d are constants, find the value of d.

$d=$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

Team No. [ ]

\+ Bonus score [ ]

Time [ ] Min. [ ] Sec.

Total score [ ]

Hong Kong Mathematics Olympiad (1988 – 1989)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 已知 p、q、r 的平均數是 12，且 p、q、r、t、$2t$ 的平均數是 15。求 t 的值。
The average of p, q, r is 12. The average of p, q, r, t, $2t$ is 15. Find the value of t.

$t =$

(ii) k 是實數，且 $k^4 + \frac{1}{k^4} = t + 1$，設 $s = k^2 + \frac{1}{k^2}$。求 s 的值。
k is a real number such that $k^4 + \frac{1}{k^4} = t + 1$, and $s = k^2 + \frac{1}{k^2}$. Find the value of s.

$s =$

(iii) M 及 N 依次是(1, 2)，(11, 7)兩點。$P(a, b)$是 MN 上一點使 $MP : PN = 1 : s$。求 a 的值。
M and N are the points (1, 2) and (11, 7) respectively. $P(a, b)$ is a point on MN such that $MP : PN = 1 : s$. Find the value of a.

$a =$

(iv) 已知曲線 $y = ax^2 + 12x + c$ 與 x-軸相切，求 c 的值。
If the curve $y = ax^2 + 12x + c$ touches the x-axis, find the value of c.

$c =$

FOR OFFICIAL USE

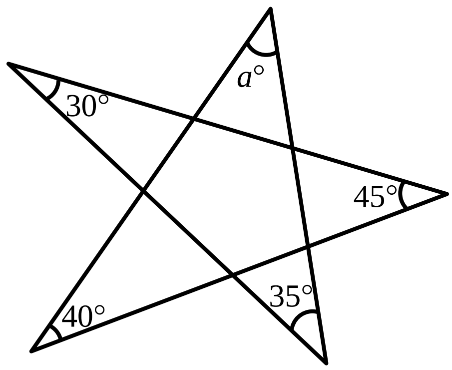
Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (1988 – 1989)
Final Event 5 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 如圖所示，求 a 的值。
In the figure, find the value of a.

a =



(ii) 已知 $\sin(a° + 210°) = \cos b°$，且 $90° < b < 180°$，求 b 的值。
If $\sin(a° + 210°) = \cos b°$, and $90° < b < 180°$, find the value of b.

b =

(iii) 一正 n 邊形的每一內角是 $b°$。求 n 的值。
Each interior angle of an n-sided regular polygon is $b°$. Find the value of n.

n =

(iv) 某年三月第 n 日是星期五，同年三月第 k 日是星期三，且 $20 < k < 25$。求 k 的值。
The n^{th} day of March in a year is Friday. The k^{th} day of March in the same year is Wednesday, where $20 < k < 25$. Find te value of k.

k =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1988 – 1989)
Sample Event (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 已知 $2at^2 + 12t + 9 = 0$ 有等根，求 a 的值。
If $2at^2 + 12t + 9 = 0$ has equal roots, find the value of a.

$a =$

(ii) 已知 $ax + by = 1$ 及 $4x + 18y = 3$ 平行，求 b 的值。
If $ax + by = 1$ and $4x + 18y = 3$ are parallel, find the value of b.

$b =$

(iii) 第 b 個質數是 p。求 p 的值。
The b^{th} prime number is p. Find the value of p.

$p =$

(iv) 已知 $k = \dfrac{4\sin\theta + 3\cos\theta}{2\sin\theta - \cos\theta}$，且 $\tan\theta = 3$，求 k 的值。
If $k = \dfrac{4\sin\theta + 3\cos\theta}{2\sin\theta - \cos\theta}$ and $\tan\theta = 3$, find the value of k.

$k =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1988 – 1989)
Final Event 6 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 一凸 n 邊形有 20 條對角線。求 n 的值。
An n-sided convex polygon has 20 diagonals. Find the value of n.

n =

(ii) 兩骰同擲，所得點數之和是 n 的概率是 $\frac{k}{36}$。求 k 的值。
Two dice are thrown. The probability of getting a total of n is $\frac{k}{36}$. Find the value of k.

k =

(iii) 某人以 25 km/h 的速率行車 3 小時，再以 50 km/h 的速率行車 2 小時。若全程的平均速率是 u km/h，求 u 的值。
A man drives at 25 km/h for 3 hours and then at 50 km/h for 2 hours.
His average speed for the whole journey is u km/h. Find the value of u.

u =

(iv) 已知 $a\Delta b = ab + 1$，且$(2\Delta a)\Delta 3 = 10$，求 a 的值。
If $a\Delta b = ab + 1$ and $(2\Delta a)\Delta 3 = 10$, find the value of a.

a =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (1988 – 1989)
Final Event 7 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

在下圖所示乘法中，不同字母代表由 1 至 9 的不同整數。設字母 O 及 J 依次代表 4 及 6。求
In the attached calculation, different letters represent different integers ranging from 1 to 9.
If the letters O and J represent 4 and 6 respectively, find

$$\begin{array}{rcccccc} & G & O & L & D & E & N \\ \times & & & & & & J \\ \hline & D & E & N & G & O & L \end{array}$$

(i) G 的值。
the value of G. 　　G =

(ii) D 的值。
the value of D. 　　D =

(iii) L 的值。
the value of L. 　　L =

(iv) E 的值。
the value of E. 　　E =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1988 – 1989)
Final Event 8 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

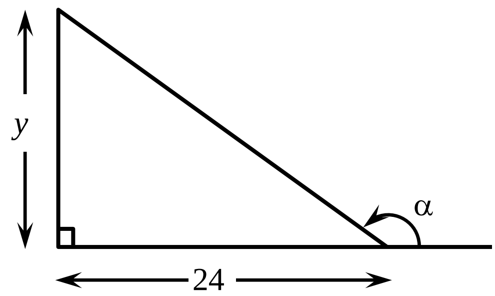
(i) 設 y 是 $\frac{14}{5+3\sin\theta}$ 的最大值。求 y 的值。

If y is the greatest value of $\frac{14}{5+3\sin\theta}$, find the value of y.

$y =$

(ii) 如圖所示，$100\cos\alpha = k$。求 k 的值。
In the figure, $100\cos\alpha = k$. Find the value of k.

$k =$



(iii) $3x^2 + 4x + a$ 被 $x + 2$ 除所得的餘數是 5。求 a 的值。
When $3x^2 + 4x + a$ is divided by $x + 2$, the remainder is 5. Find the value of a.

$a =$

(iv) $3t^2 - 5t - 2 < 0$ 的解是 $-\frac{1}{3} < t < m$。求 m 的值。

The solution for $3t^2 - 5t - 2 < 0$ is $-\frac{1}{3} < t < m$. Find the value of m.

$m =$

FOR OFFICIAL USE

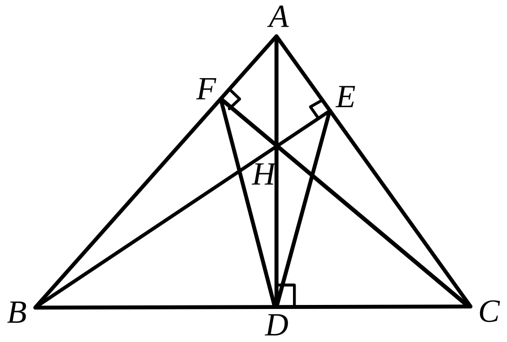
Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time		
				Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1988 – 1989)
Final Event 9 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 圖中，$\angle BAC = 70°$，且$\angle FDE = x°$，求 x 的值。

In the figure, $\angle BAC = 70°$ and $\angle FDE = x°$. Find the value of x.



$x =$

(ii) 一長方體闊 y cm，長 6 cm，高 5 cm。它的表面積是 126 cm^2，求 y 的值。
A cuboid is y cm wide, 6 cm long and 5 cm high. Its surface area is 126 cm^2. Find the value of y.

$y =$

(iii) 已知 $\log_9(\log_2 k) = \frac{1}{2}$，求 k 的值。

If $\log_9(\log_2 k) = \frac{1}{2}$, find the value of k.

$k =$

(iv) 已知 $a : b = 3 : 8$，$b : c = 5 : 6$，且 $a : c = r : 16$，求 r 的值。
If $a : b = 3 : 8$, $b : c = 5 : 6$ and $a : c = r : 16$, find the value of r.

$r =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1988 – 1989)
Final Event 10 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

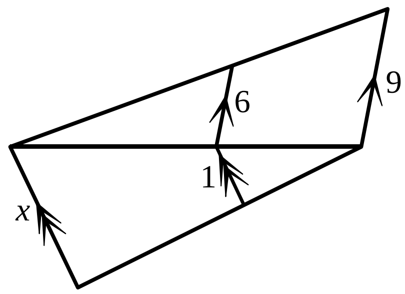
(i) 已知 $\dfrac{6\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}=3\sqrt{a}+6$，求 a 的值。

If $\dfrac{6\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}=3\sqrt{a}+6$, find the value of a.

$a =$

(ii) 如圖所示，求 x 的值。

In the figure, find the value of x.



$x =$

(iii) 已知 $k=\dfrac{6\cos^2\theta+2\sin\theta\cos\theta+\sin^2\theta}{\cos^2\theta+\sin\theta\cos\theta+\sin^2\theta}$，且 $\tan\theta = 2$，求 k 的值。

If $k=\dfrac{6\cos^2\theta+2\sin\theta\cos\theta+\sin^2\theta}{\cos^2\theta+\sin\theta\cos\theta+\sin^2\theta}$ and $\tan\theta = 2$, find the value of k.

$k =$

(iv) 已知 $y=\dfrac{3(2^k)-4(2^{k-2})}{2^k-2^{k-1}}$，求 y 的值。

If $y=\dfrac{3(2^k)-4(2^{k-2})}{2^k-2^{k-1}}$, find the value of y.

$y =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1989 – 1990)
Sample Event (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若方程 $3x^2 - 4x + \frac{h}{3} = 0$ 有等根，求 h 的值。

If the equation $3x^2 - 4x + \frac{h}{3} = 0$ has equal roots, find the value of h.

$h =$

(ii) 若一圓柱體之高增加一倍，且新半徑為原來之 h 倍，則新體積為原來之 k 倍，求 k 的值。

If the height of a cylinder is doubled and the new radius is h times the original, then the new volume is k times the original. Find the value of k.

$k =$

(iii) 若 $\log_{10} 210 + \log_{10}k - \log_{10}56 + \log_{10}40 - \log_{10}120 + \log_{10}25 = p$,求 p 的值。

If $\log_{10} 210 + \log_{10}k - \log_{10}56 + \log_{10}40 - \log_{10}120 + \log_{10}25 = p$, find the value of p.

$p =$

(iv) 若 $\sin A = \frac{p}{5}$ 且 $\frac{\cos A}{\tan A} = \frac{q}{15}$，求 q 的值。

If $\sin A = \frac{p}{5}$ and $\frac{\cos A}{\tan A} = \frac{q}{15}$, find the value of q.

$q =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.
		+	Bonus score			Time	
			Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (1989 – 1990)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $2t+1$ 是 $4t^2+12t+a$ 的因式，求 a 的值。
Find the value of a if $2t+1$ is a factor of $4t^2+12t+a$.

$a =$

(ii) 對 $K \geq 0$，$\sqrt{K}$ 表 K 的非負平方根。若 b 是方程 $\sqrt{a-x} = x-3$ 的根，求 b 的值。
$\sqrt{K}$ denotes the nonnegative square root of K, where $K \geq 0$.
If b is the root of the equation $\sqrt{a-x} = x-3$, find the value of b.

$b =$

(iii) 若 c 是 $\dfrac{20}{4+2\cos\theta}$ 的最大值，求 c 的值。
If c is the greatest value of $\dfrac{20}{4+2\cos\theta}$, find the value of c.

$c =$

(iv) 某人以 $3c$ km/h 的速率行車 3 小時，再以 $4c$ km/h 的速率行車 2 小時。
若全程的平均速率是 d km/h，求 d 的值。
A man drives a car at $3c$ km/h for 3 hours and then $4c$ km/h for 2 hours.
If his average speed for the whole journey is d km/h, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (1989 – 1990)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $0° \le \theta < 360°$，θ 的方程 $3\cos\theta + \frac{1}{\cos\theta} = 4$ 有 p 個根，求 p 的值。

If $0° \le \theta < 360°$, the equation in θ: $3\cos\theta + \frac{1}{\cos\theta} = 4$ has p roots.

Find the value of p.

$p =$

(ii) 若 $x - \frac{1}{x} = p$，且 $x^3 - \frac{1}{x^3} = q$，求 q 的值。

If $x - \frac{1}{x} = p$ and $x^3 - \frac{1}{x^3} = q$, find the value of q.

$q =$

(iii) 一圓內接於一周界長 q cm 的正三角形。若圓的面積是 $k\pi$ cm^2，求 k 的值。

A circle is inscribed in an equilateral triangle of perimeter q cm.
If the area of the circle is $k\pi$ cm^2, find the value of k.

$k =$

(iv) 正 k 邊形的每一內角為 $m°$。求 m 的值。

Each interior angle of a regular polygon of k sides is $m°$. Find the value of m.

$m =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.		
			+ Bonus score				Time		
			Total score					Min.	Sec.

Hong Kong Mathematics Olympiad (1989 – 1990)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $998a + 1 = 999^2$，求 a 的值。
If $998a + 1 = 999^2$, find the value of a.

a =

(ii) 若 $\log_{10}a = \log_2 b$，求 b 的值。
If $\log_{10}a = \log_2 b$, find the value of b.

b =

(iii) 以 x 軸，y 軸及直綫 $2x + y = b$ 所圍成的三角形的面積是 c 平方單位，求 c 的值。
The area of the triangle formed by the x-axis, the y-axis and the line $2x + y = b$ is c sq. units. Find the value of c.

c =

(iv) 若 $64t^2 + ct + d$ 是完全平方，求 d 的值。
If $64t^2 + ct + d$ is a perfect square, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min. Sec.

Hong Kong Mathematics Olympiad (1989 – 1990)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特别聲明，答案須用數字表達，並化至最簡。

(i) 解下列 a 的方程 $2^{a+1}+2^{a}+2^{a-1}=112$。

Solve for a in the equation $2^{a+1}+2^{a}+2^{a-1}=112$.

a =

(ii) 若 a 是方程 $x^2-bx+35=0$ 的一個根，求 b 的值。

If a is one root of the equation $x^2-bx+35=0$, find the value of b.

b =

(iii) 若 $\sin\theta=\frac{-b}{15}$，其中 $180°<\theta<270°$，且 $\tan\theta=\frac{c}{3}$，求 c 的值。

If $\sin\theta=\frac{-b}{15}$, where $180°<\theta<270°$, and $\tan\theta=\frac{c}{3}$, find the value of c.

c =

(iv) 兩骰同擲，所得點數之和為 c 的概率是 $\frac{1}{d}$。求 d 的值。

The probability of getting a sum of c in throwing two dice is $\frac{1}{d}$. Find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

Team No. [ ]

\+ Bonus score [ ]

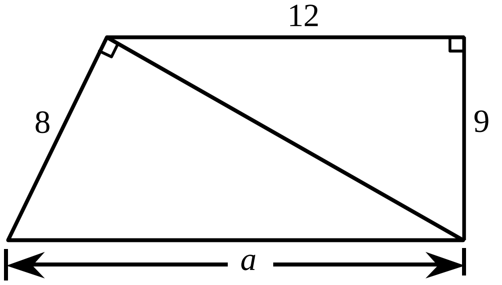
Time [ ] Min. [ ] Sec.

Total score [ ]

Hong Kong Mathematics Olympiad (1989 – 1990)
Final Event 5 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 如圖所示，求 a 的值。
In the figure, find the value of a.



a =

(ii) 若直綫 $ax + by = 1$ 及 $10x - 34y = 3$ 互相垂直，求 b 的值。
If the lines $ax + by = 1$ and $10x - 34y = 3$ are perpendicular to each other, find the value of b.

b =

(iii) 某年五月第 b 日為星期五，而同年五月第 c 日為星期二，且 $16 < c < 24$，求 c 的值。
If the b^{th} day of May in a year is Friday and the c^{th} day of May in the same year is Tuesday, where $16 < c < 24$, find the value of c.

c =

(iv) c 是第 d 個質數。求 d 的值。
c is the d^{th} prime number. Find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy ☐ × Mult. factor for speed ☐ = ☐ Team No. ☐

+ Bonus score ☐ Time ☐ ☐

Total score ☐ Min. Sec.

Hong Kong Mathematics Olympiad (1989 – 1990)
Sample Event (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 某兩數之和為 50，其積為 25。若該兩數倒數之和為 a，求 a 的值。
The sum of two numbers is 50, and their product is 25.
If the sum of their reciprocals is a, find the value of a.

a =

(ii) 若直綫 $ax + 2y + 1 = 0$ 及 $3x + by + 5 = 0$ 互相垂直，求 b 的值。
If the lines $ax + 2y + 1 = 0$ and $3x + by + 5 = 0$ are perpendicular, find the value of b.

b =

(iii) 一正三角形之面積為 $100\sqrt{3}\text{ cm}^2$。若其周界為 p cm，求 p 的值。
The area of an equilateral triangle is $100\sqrt{3}\text{ cm}^2$.
If its perimeter is p cm, find the value of p.

p =

(iv) 若 $x^3 - 2x^2 + px + q$ 可被 $x + 2$ 整除，求 q 的值。
If $x^3 - 2x^2 + px + q$ is divisible by $x + 2$, find the value of q.

q =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] Min. [ ] Sec.

Total score [ ]

Hong Kong Mathematics Olympiad (1989 – 1990)
Final Event 6 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $a=\frac{(68^3-65^3)\cdot(32^3+18^3)}{(32^2-32\times18+18^2)\cdot(68^2+68\times65+65^2)}$，求 a 的值。

If $a=\frac{(68^3-65^3)\cdot(32^3+18^3)}{(32^2-32\times18+18^2)\cdot(68^2+68\times65+65^2)}$, find the value of a.

a =

(ii) 若三點(a, b)，(10, –4)及(20, –3)共綫，求 b 的值。

If the 3 points (a, b), (10, –4) and (20, –3) are collinear, find the value of b.

b =

(iii) 若在四時十五分，時鐘兩針之間的銳角是 $k°$，求 k 的值。

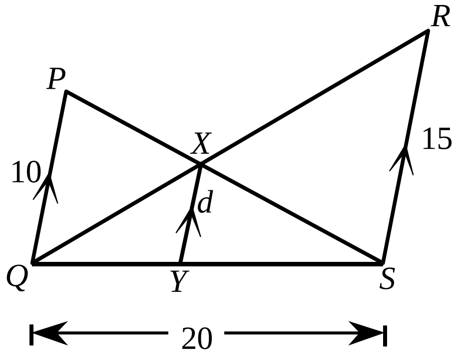
If the acute angle formed by the hands of a clock at 4:15 is $k°$, find the value of k.

k =

(iv) 在圖中，PQ = 10, RS = 15, QS = 20。若 $XY = d$，求 d 的值。

In the figure, PQ = 10, RS = 15, QS = 20. If $XY = d$, find the value of d.

d =



FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.
			+	Bonus score			Time
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (1989 – 1990)
Final Event 7 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 2 個蘋果和 3 個橙共值 6 元。
4 個蘋果和 7 個橙共值 13 元。
16 個蘋果和 23 個橙共值 C 元，求 C 的值。
2 apples and 3 oranges cost 6 dollars.
4 apples and 7 oranges cost 13 dollars.
16 apples and 23 oranges cost C dollars. Find the value of C.

C =

(ii) 若 $K=\dfrac{6\cos\theta+5\sin\theta}{2\cos\theta+3\sin\theta}$，且 $\tan\theta=2$，求 K 的值。
If $K=\dfrac{6\cos\theta+5\sin\theta}{2\cos\theta+3\sin\theta}$ and $\tan\theta=2$, find the value of K.

K =

A、B 均為小於 10 的正整數，且 $21A104\times 11=2B8016\times 9$。
A, B are positive integers less than 10 such that $21A104\times 11=2B8016\times 9$.

(iii) 求 A 的值。
Find the value of A.

A =

(iv) 求 B 的值。
Find the value of B.

B =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.		
			+ Bonus score				Time		
			Total score					Min.	Sec.

Hong Kong Mathematics Olympiad (1989 – 1990)
Final Event 8 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

在所示乘法中，字母 A、B、C 及 K (其中 $A < B$)代表由 1 至 9 的不同整數。
In the multiplication shown, the letters A, B, C and K ($A<B$) represent different integers from 1 to 9.

$$\begin{array}{rccc} & & A & C \\ \times) & & B & C \\ \hline & K & K & K \end{array}$$

(i) 求 A 的值。
Find the value of A. | A =

(ii) 求 B 的值。
Find the value of B. | B =

(iii) 求 C 的值。
Find the value of C. | C =

(iv) 求 K 的值。
Find the value of K. | K =

(提示：$KKK = K\times 111$。)
(Hint: $KKK = K\times 111$.)

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]
+ Bonus score [ ]
Total score [ ]

Team No. [ ]
Time [ ] Min. [ ] Sec.

Hong Kong Mathematics Olympiad (1989 – 1990)
Final Event 9 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $S = ab - 1 + a - b$，且 $a = 101$，$b = 9$，求 S 的值。
If $S = ab - 1 + a - b$ and $a = 101$, $b = 9$, find the value of S.

$S =$

(ii) 若 $x = 1.9\dot{8}\dot{9}$，且 $x - 1 = \frac{K}{99}$，求 K 的值。
If $x = 1.9\dot{8}\dot{9}$ and $x - 1 = \frac{K}{99}$, find the value of K.

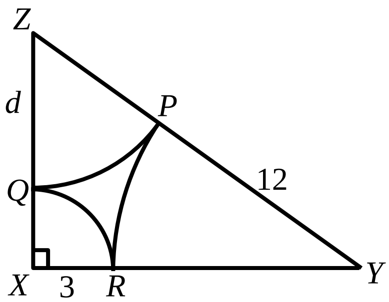
$K =$

(iii) p、q 及 r 的平均值是 18。$p + 1$、$q - 2$、$r + 3$ 及 t 的平均值是 19。求 t 的值。
The average of p, q and r is 18.
The average of $p + 1$, $q - 2$, $r + 3$ and t is 19. Find the value of t.

$t =$

(iv) 如圖所示，依次以 X，Y，Z 為圓心之三弧 $\overset{\frown}{QR}$ 、 $\overset{\frown}{RP}$ 、 $\overset{\frown}{PQ}$ 互相切於 P、Q、R.。
若 $ZQ = d$，$XR = 3$，$YP = 12$，$\angle X = 90°$，求 d 的值。
In the figure, $\overset{\frown}{QR}$, $\overset{\frown}{RP}$, $\overset{\frown}{PQ}$ are 3 arcs, centres at X, Y and Z respectively, touching one another at P, Q and R. If $ZQ = d$, $XR = 3$, $YP = 12$, $\angle X = 90°$, find the value of d.

$d =$



FOR OFFICIAL USE

Score for accuracy × Mult. factor for speed =

Team No.

\+ Bonus score

Time

Total score

Min. Sec.

Hong Kong Mathematics Olympiad (1989 – 1990)
Final Event 10 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $A = 1 + 2 - 3 + 4 + 5 - 6 + 7 + 8 - 9 + \ldots + 97 + 98 - 99$，求 A 的值。
If $A = 1 + 2 - 3 + 4 + 5 - 6 + 7 + 8 - 9 + \ldots + 97 + 98 - 99$, find the value of A.

$A =$

(ii) 若 $\log_{10}(k-1) - \log_{10}(k^2 - 5k + 4) + 1 = 0$，求 k 的值。
If $\log_{10}(k-1) - \log_{10}(k^2 - 5k + 4) + 1 = 0$, find the value of k.

$k =$

一凸 n 邊形其中一內角為 $x°$，而其餘內角之和為 2180°。
One interior angle of a convex n-sided polygon is $x°$.
The sum of the remaining interior angles is 2180°.

(iii) 求 x 的值。
Find the value of x.

$x =$

(iv) 求 n 的值。
Find the value of n.

$n =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1990 – 1991)
Sample Event (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $a = -1 + 2 - 3 + 4 - 5 + 6 - \ldots + 100$，求 a 的值。
If $a = -1 + 2 - 3 + 4 - 5 + 6 - \ldots + 100$, find the value of a.

$a =$

(ii) 首 b 個正奇數之和是 $2a$。求 b 的值。
The sum of the first b positive odd numbers is $2a$. Find the value of b.

$b =$

(iii) 袋中有白球 b 個，黑球 3 個。現任意取出二球。
若得到兩個不同顏色的球的概率為 $\frac{c}{13}$，求 c 的值。
A bag contains b white balls and 3 black balls. Two balls are drawn from the bag at random. If the probability of getting 2 balls of different colours is $\frac{c}{13}$, find the value of c.

$c =$

(iv) 若直綫 $cx + 10y = 4$ 及 $dx - y = 5$ 互相垂直，求 d 的值。
If the lines $cx + 10y = 4$ and $dx - y = 5$ are perpendicular to each other, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

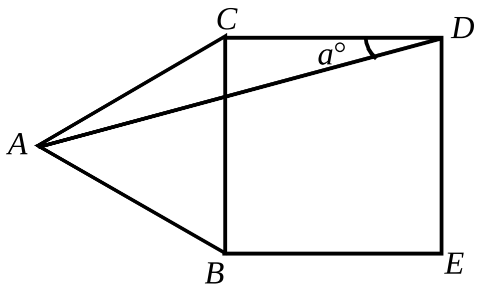
+ Bonus score [ ]   Time [ ] Min. [ ] Sec.

Total score [ ]

Hong Kong Mathematics Olympiad (1990 – 1991)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 如圖所示，ABC 是等邊三角形，$BCDE$ 是正方形。若$\angle ADC = a^\circ$，求 a 的值。
In the figure, ABC is an equilateral triangle and $BCDE$ is a square.
If $\angle ADC = a^\circ$, find the value of a.



a =

(ii) 若 $rb = 15$，且 $br^4 = 125a$，其中 r 是整數，求 b 的值。
If $rb = 15$ and $br^4 = 125a$, where r is an integer, find the value of b.

b =

(iii) 若方程 $bx^2 - 252x - 13431 = 0$ 之正根是 c，求 c 的值。
If the positive root of the equation $bx^2 - 252x - 13431 = 0$ is c, find the value of c.

c =

(iv) 已知 $x \# y = \frac{y-1}{x} - x + y$。若 $d = 10 \# c$，求 d 的值。
Given $x \# y = \frac{y-1}{x} - x + y$. If $d = 10 \# c$, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1990 – 1991)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $a^2 - 1 = 123\times125$，且 $a > 0$，求 a 的值。
If $a^2 - 1 = 123\times125$ and $a > 0$, find the value of a.

$a =$

(ii) 若 $x^3 - 16x^2 - 9x + a$ 除以 $x - 2$ 之餘數為 b，求 b 的值。
If the remainder of $x^3 - 16x^2 - 9x + a$ when divided by $x - 2$ is b, find the value of b.

$b =$

(iii) 若一凸 n 邊形有$(b + 4)$條對角綫，求 n 的值。
If an n-sided polygon has $(b + 4)$ diagonals, find the value of n.

$n =$

(iv) 若點$(3, n)$、$(5, 1)$、$(7, d)$共綫，求 d 的值。
If the points $(3, n)$, $(5, 1)$ and $(7, d)$ are collinear, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

\+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (1990 – 1991)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 6 位數 168a26 可被 3 整除，求 a 之最大可能值。
If the 6-digit number 168a26 is divisible by 3, find the greatest possible value of a.

a =

(ii) 一個邊長 a cm 之正方體在全部面上都塗上紅色後，再被分割為邊長 1 cm 之正方體。若所有面都未有被塗上顏色之正方體數目為 b，求 b 的值。
A cube with edge a cm long is painted red on all faces.
It is then cut into cubes with edge 1 cm long.
If the number of cubes with all the faces not painted is b, find the value of b.

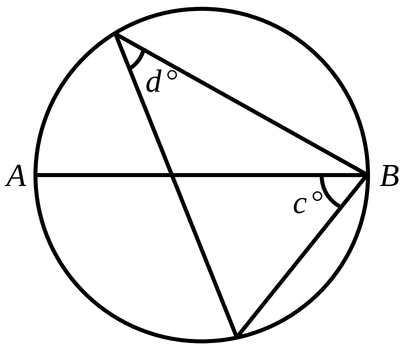
b =

(iii) 若 $(x-85)(x-c) \equiv x^2 - bx + 85c$，求 c 的值。
If $(x-85)(x-c) \equiv x^2 - bx + 85c$, find the value of c.

c =

(iv) 在圖中，AB 是該圓形的直徑。求 d 的值。
In the figure, AB is a diameter of the circle. Find the value of d.

d =



FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1990 – 1991)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) Given $x-\frac{1}{x}=3$. If $a=x^2+\frac{1}{x^2}$, find the value of a.

已知 $x-\frac{1}{x}=3$。若 $a=x^2+\frac{1}{x^2}$，求 a 的值。

$a =$

(ii) 若 $f(x)=\log_2 x$，且 $f(a+21)=b$，求 b。

If $f(x)=\log_2 x$ and $f(a+21)=b$, find b.

$b =$

(iii) 若 $\cos\theta=\frac{8b}{41}$，其中 θ 為銳角，且 $c=\frac{1}{\sin\theta}+\frac{1}{\tan\theta}$，求 c 的值。

If $\cos\theta=\frac{8b}{41}$, where θ is an acute angle, and $c=\frac{1}{\sin\theta}+\frac{1}{\tan\theta}$, find the value of c.

$c =$

(iv) 兩骰同擲，得和為 7 或 c 之概率為 $\frac{d}{18}$，求 d 的值。

Two dice are tossed. If the probability of getting a sum of 7 or c is $\frac{d}{18}$, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

Team No. [ ]

+ Bonus score [ ]

Time [ ] Min. [ ] Sec.

Total score [ ]

Hong Kong Mathematics Olympiad (1990 – 1991)
Final Event 5 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 在圖一中，若多邊形之內角和是 $a°$，求 a 的值。
In Figure 1, if the sum of the interior angles is $a°$, find the value of a.

$a =$

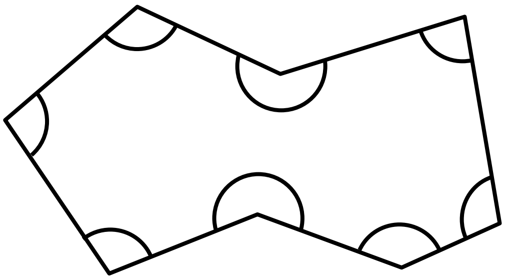


Figure 1 (圖一)

(ii) 若算術級數 80, 130, 180, 230, 280, … 之第 n 項是 a，求 n 的值。
If the n^{th} term of the arithmetic progression 80, 130, 180, 230, 280, $\cdots$ is a, find the value of n.

$n =$

(iii) 在圖二中，$AP : PB = 2 : 1$。若 $AC = 33$ cm，$BD = n$ cm，$PQ = x$ cm，求 x 的值。
In Figure 2, $AP : PB = 2 : 1$. If $AC = 33$ cm, $BD = n$ cm, $PQ = x$ cm, find the value of x.

$x =$

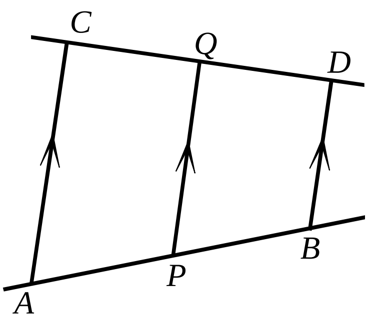


Figure 2 (圖二)

(iv) 若 $K = \dfrac{\sin 65^\circ \tan^2 60^\circ}{\tan 30^\circ \cos 30^\circ \cos x^\circ}$，求 K 的值。

If $K = \dfrac{\sin 65^\circ \tan^2 60^\circ}{\tan 30^\circ \cos 30^\circ \cos x^\circ}$, find the value of K.

$K =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (1990 – 1991)
Sample Event (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 一等邊三角形的高是 $8\sqrt{3}$ cm，面積是 $a\sqrt{3}$ cm^2。求 a 的值。
The height of an equilateral triangle is $8\sqrt{3}$ cm and the area of the triangle is $a\sqrt{3}$ cm^2. Find the value of a.

$a =$

(ii) 已知 $\sum_{x=1}^{n}\frac{1}{x}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$，及 $\sum_{x=4}^{10}\frac{1}{x-2}-\sum_{x=4}^{10}\frac{1}{x-1}=\frac{b}{18}$。求 b 的值。

$b =$

Given that $\sum_{x=1}^{n}\frac{1}{x}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$, and $\sum_{x=4}^{10}\frac{1}{x-2}-\sum_{x=4}^{10}\frac{1}{x-1}=\frac{b}{18}$. Find the value of b.

某童把一平行四邊形兩鄰邊相乘當作該圖形之面積，其結果為正確答案之兩倍。
若該圖形之銳角及鈍角依次為 $h°$ 及 $k°$。
A boy tries to find the area of a parallelogram by multiplying together the lengths of two adjacent sides. His answer is double the correct answer.
If the acute angle and the obtuse angle of the figure are $h°$ and $k°$ respectively,

(iii) 求 h 的值。
find the value of h.

$h =$

(iv) 求 k 的值。
find the value of k.

$k =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1990 – 1991)
Final Event 6 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

某兩位數 x 之個位數字是 M，十位數字是 N。另一兩位數 y 之個位數字是 N，十位數字是 M。若 $x > y$，且他們的和是他們的差的十一倍，

A 2-digit number x has M as the units digit and N as the tens digit. Another 2-digit number y has N as the units digit and M as the tens digit.

If $x > y$ and their sum is equal to eleven times their differences,

(i) 求 M 的值。
find the value of M.

M =

(ii) 求 N 的值。
find the value of N.

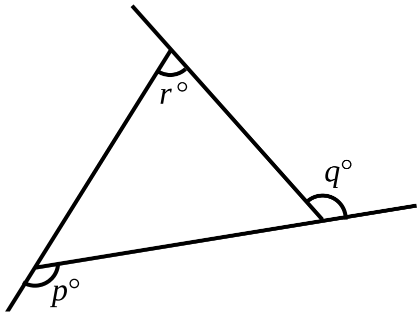
N =

(iii) 兩數之和是 20，積是 5。若該兩數倒數之和是 z，求 z 的值。
The sum of two numbers is 20 and their product is 5.
If the sum of their reciprocals is z, find the value of z.

z =

(iv) 圖中，p 與 q 的平均值是 $121 + z$。求 r 的值。
In the figure, the average of p and q is $121 + z$. Find the value of r.

r =



FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1990 – 1991)
Final Event 7 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 5 部印刷機可在 5 天內印 5 本書。
若要在 100 天內印 100 本書，則需要 n 部印刷機，求 n 的值。
5 printing machines can print 5 books in 5 days. If n printing machines are required in order to have 100 books printed in 100 days, find the value of n.

n =

(ii) 某方程 $x^2 + 2x + c = 0$ 無實根，且 c 為小於 3 之整數，求 c 的值。
If the equation $x^2+2x+c = 0$ has no real root and c is an integer less than 3, find the value of c.

c =

雞蛋每只\$0.50，鴨蛋每只\$0.60，鵝蛋每只\$0.90。某人賣出 x 只雞蛋，y 只鴨蛋，z 只鵝蛋，共得\$60。
若 x、y、z 皆為正數，且 $x + y + z = 100$，及在 x、y、z 中有兩數相同，
Chicken eggs cost \$0.50 each, duck eggs cost \$0.60 each and goose eggs cost \$0.90 each.
A man sold x chicken eggs, y duck eggs, z goose eggs and received \$60.
If x, y, z are all positive numbers with $x + y + z = 100$ and two of the values x, y, z are equal,

(iii) 求 x 的值。
find the value of x.

x =

(iv) 求 y 的值。
find the value of y.

y =

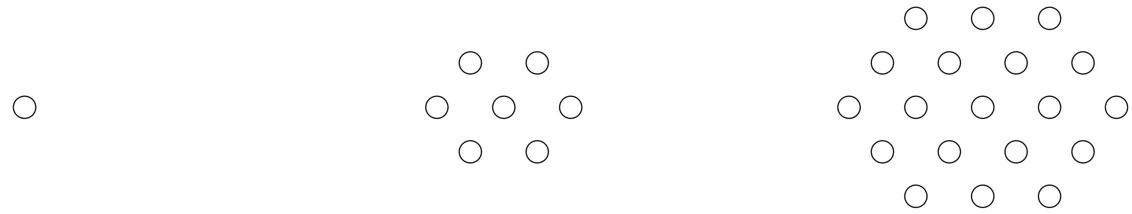
FOR OFFICIAL USE

Score for accuracy		× Mult. factor for speed		=		Team No.	
		+ Bonus score				Time	
		Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1990 – 1991)
Final Event 8 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

細看以下之六邊形數：
Consider the following hexagonal numbers :



$H_1 = 1$ $H_2 = 7$ $H_3 = 19$

(i) 求 H_5 的值。
Find the value of H_5 .

$H_5 =$

(ii) 若 $H_n = an^2 + bn + c$，其中 n 為正整數，求 a 的值。
If $H_n = an^2 + bn + c$, where n is any positive integer, find the value of a .

$a =$

(iii) 若 $p : q = 2 : 3$，$q : r = 4 : 5$，且 $p : q : r = 8 : t : 15$，求 t 的值。
If $p : q = 2 : 3$, $q : r = 4 : 5$ and $p : q : r = 8 : t : 15$, find the value of t.

$t =$

(iv) 若 $\frac{1}{x} : \frac{1}{y} = 4 : 3$，且 $\frac{1}{x+y} : \frac{1}{x} = 3 : m$，求 m 的值。
If $\frac{1}{x} : \frac{1}{y} = 4 : 3$ and $\frac{1}{x+y} : \frac{1}{x} = 3 : m$, find the value of m .

$m =$

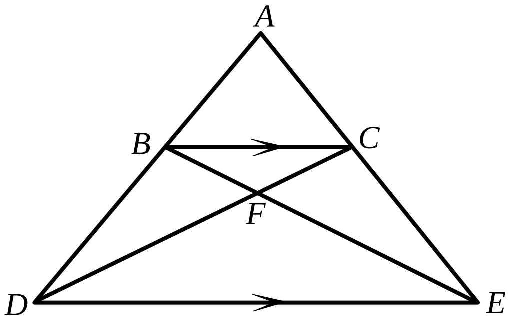
FOR OFFICIAL USE

Score for accuracy × Mult. factor for speed =
+ Bonus score
Total score

Team No.
Time
Min. Sec.

Hong Kong Mathematics Olympiad (1990 – 1991)
Final Event 9 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。



圖中，BC 與 DE 平行。
若 $AB : BC : BF : CF : FE = 5 : 4 : 2 : 3 : 5$，
且ΔBCF 之面積為 12，求
In the figure, BC is parallel to DE.
If $AB : BC : BF : CF : FE = 5 : 4 : 2 : 3 : 5$
and the area of ΔBCF is 12, find

(i) ΔBDF 之面積，
the area of ΔBDF,

Area of ΔBDF =

(ii) ΔFDE 之面積，
the area of ΔFDE,

Area of ΔFDE =

(iii) ΔABC 之面積。
the area of ΔABC.

Area of ΔABC =

(iv) 若一球體之體積增加 72.8%，則其表面面積增加 x%。求 x 的值。
If the volume of a sphere is increased by 72.8%, then the surface area of the sphere is increased by x%. Find the value of x.

x =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1990 – 1991)
Final Event 10 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

在所附除法算式中
In the attached division

			1	D	E
2 1 5)	A	7	B	9	C
	F	G	H		
	J	5	K	9	
	L	5	M	5	
			N	4	P
			Q	R	S

(i) 求 A 的值。
find the value of A.

A =

(ii) 求 B 的值。
find the value of B.

B =

(iii) 求 C 的值。
find the value of C.

C =

(iv) 求 D 的值。
find the value of D.

D =

FOR OFFICIAL USE

Score for accuracy ☐ × Mult. factor for speed ☐ = ☐ Team No. ☐

+ Bonus score ☐ Time ☐ ☐

Total score ☐ Min. Sec.

Hong Kong Mathematics Olympiad (1991 – 1992)
Sample Event (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 已知 $A = (b^m)^n + b^{m+n}$。當 $b = 4$，$m = n = 1$ 時，求 A 的值。
Given $A = (b^m)^n + b^{m+n}$. Find the value of A when $b = 4$, $m = n = 1$.

$A =$

(ii) 若 $2^A = B^{10}$ 且 $B > 0$，求 B 的值。
If $2^A = B^{10}$ and $B > 0$, find the value fo B.

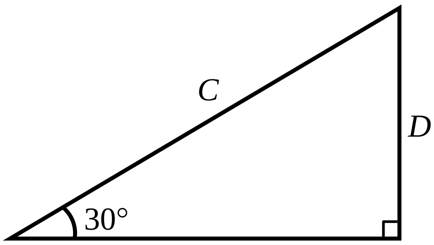
$B =$

(iii) 從下列方程求 C：$\sqrt{\frac{20B+45}{C}} = C$。
Solve for C in the following equation: $\sqrt{\frac{20B+45}{C}} = C$.

$C =$

(iv) 如圖所示，求 D 的值。
Find the value of D in the figure.

$D =$



FOR OFFICIAL USE

Score for accuracy × Mult. factor for speed =

\+ Bonus score

Total score

Team No.

Time

Min. Sec.

Hong Kong Mathematics Olympiad (1991 – 1992)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若一凸 n 邊形之內角和為 1440°，求 n 的值。
If the sum of the interior angles of an n-sided polygon is 1440°, find the value of n .

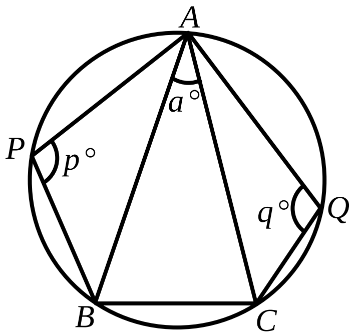
n =

(ii) 若 $x^2 - nx + a = 0$ 有兩等根，求 a 的值。
If $x^2 - nx + a = 0$ has 2 equal roots, find the value of a .

a =

(iii) 如圖所示，若 $z = p + q$，求 z 的值。
In the figure, if $z = p + q$, find the value of z .

z =



(iv) 若 $S = 1 + 2 - 3 - 4 + 5 + 6 - 7 - 8 + \ldots + z$，求 S 的值。
If $S = 1 + 2 - 3 - 4 + 5 + 6 - 7 - 8 + \ldots + z$, find the value of S .

S =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] Min. [ ] Sec.

Total score [ ]

Hong Kong Mathematics Olympiad (1991 – 1992)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $ar = 24$ 及 $ar^4 = 3$，求 a 的值。
If $ar = 24$ and $ar^4 = 3$, find the value of a.

$a =$

(ii) 若 $\left(x+\frac{a}{4}\right)^2 = x^2 + \frac{a}{2}\cdot x + b$，求 b 的值。
If $\left(x+\frac{a}{4}\right)^2 = x^2 + \frac{a}{2}\cdot x + b$, find the value of b.

$b =$

(iii) 若 $c = \log_2 \frac{b}{9}$，求 c 的值。
If $c = \log_2 \frac{b}{9}$, find the value of c.

$c =$

(iv) If $d = 12^c - 142^2$, find the value of d.
若 $d = 12^c - 142^2$，求 d 的值。

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (1991 – 1992)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $a=\frac{\sin 15^\circ}{\cos 75^\circ}+\frac{1}{\sin^2 75^\circ}-\tan^2 15^\circ$，求 a 的值。

If $a=\frac{\sin 15^\circ}{\cos 75^\circ}+\frac{1}{\sin^2 75^\circ}-\tan^2 15^\circ$, find the value of a.

a =

(ii) 若直綫 $ax + 2y + 1 = 0$ 與 $3x + by + 5 = 0$ 互相垂直，求 b 的值。

If the lines $ax + 2y + 1 = 0$ and $3x + by + 5 = 0$ are perpendicular to each other, find the value of b.

b =

(iii) 三點 $(2, b)$、$(4, -b)$ 及 $(5, \frac{c}{2})$ 共綫，求 c 的值。

The three points $(2, b)$, $(4, -b)$ and $(5, \frac{c}{2})$ are collinear. Find the value of c.

c =

(iv) 若 $\frac{1}{x}:\frac{1}{y}:\frac{1}{z}=3:4:5$ 且 $\frac{1}{x+y}:\frac{1}{y+z}=9c:d$，求 d 的值。

If $\frac{1}{x}:\frac{1}{y}:\frac{1}{z}=3:4:5$ and $\frac{1}{x+y}:\frac{1}{y+z}=9c:d$, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy ☐ × Mult. factor for speed ☐ = ☐ Team No. ☐

\+ Bonus score ☐ Time ☐ ☐

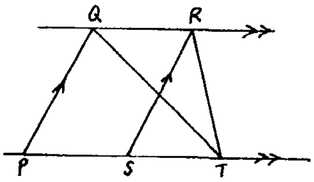
Total score ☐ Min. Sec.

Hong Kong Mathematics Olympiad (1991 – 1992)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 在圖中，*PQRS* 之面積為 80 cm^2。若ΔQRT 之面積為 A cm^2，求 A 的值。
In the figure, the area of *PQRS* is 80 cm^2. If the area of ΔQRT is A cm^2, find the value of A.

A =



(ii) 若 $B=\log_2\left(\frac{8A}{5}\right)$，求 B 的值。
If $B=\log_2\left(\frac{8A}{5}\right)$, find the value of B.

B =

(iii) 已知 $x+\frac{1}{x}=B$。若 $C=x^3+\frac{1}{x^3}$ ，求 C 的值。
Given $x+\frac{1}{x}=B$. If $C=x^3+\frac{1}{x^3}$, find the value of C.

C =

(iv) 設 $(p, q)=qD+p$。若$(C, 2)=212$，求 D 的值。
Let $(p, q)=qD+p$. If $(C, 2)=212$, find the value of D.

D =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time		
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (1991 – 1992)
Final Event 5 (Individual)

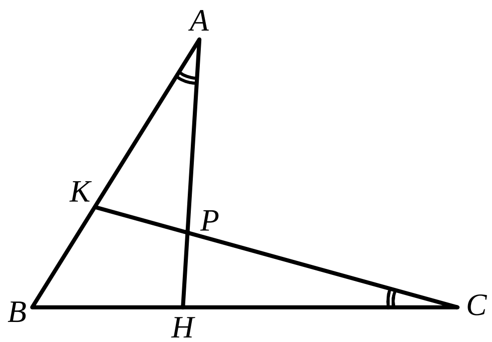
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 設 p、q 為二次方程 $x^2 - 3x - 2 = 0$ 的兩根，且 $a = p^3 + q^3$，求 a 的值。
Let p, q be the roots of the quadratic equation $x^2 - 3x - 2 = 0$ and $a = p^3 + q^3$. Find the value of a.

a =

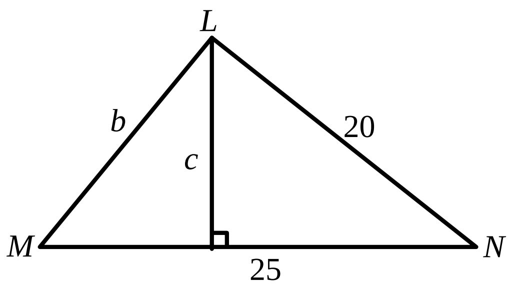
(ii) 若 $AH = a$，$CK = 36$，$BK = 12$，$BH = b$，求 b 的值。
If $AH = a$, $CK = 36$, $BK = 12$ and $BH = b$, find the value of b.

b =



(iii) 求 c 的值。
Find the value of c.

c =



(iv) 設 $\sqrt{2x+23}+\sqrt{2x-1}=c$ 及 $d=\sqrt{2x+23}-\sqrt{2x-1}$。求 d 的值。
Let $\sqrt{2x+23}+\sqrt{2x-1}=c$ and $d=\sqrt{2x+23}-\sqrt{2x-1}$. Find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1991 – 1992)
Sample Event (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

細看下列各組數字：
Consider the following groups of numbers:

(2)
(4, 6)
(8, 10, 12)
(14, 16, 18, 20)
(22, 24, 26, 28, 30)
……………………………

(i) 求第 50 組的最後一個數字。
Find the last number of the 50th group.

(ii) 求第 50 組的第一個數字。
Find the first number of the 50th group.

(iii) 若第 50 組的數字之和為 $50P$，求 P 的值。
Find the value of P if the sum of the numbers in the 50th group is $50P$.

P =

(iv) 若第 100 組的數字之和為 $100Q$，求 Q 的值。
Find the value of Q if the sum of the numbers in the 100th group is $100Q$.

Q =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ] Team No. [ ]

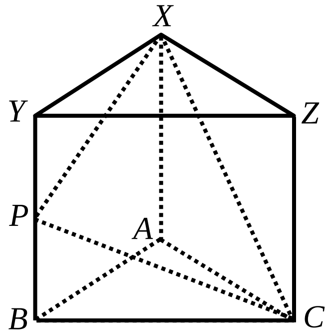
\+ Bonus score [ ] Time [ ] Min. [ ] Sec.

Total score [ ]

Hong Kong Mathematics Olympiad (1991 – 1992)
Final Event 6 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

如圖所示，ΔABC 及ΔXYZ 為等邊三角形，同時亦為一柱體的底和面。
P 為 BY 的中點，且 $BP = 3$ cm，$XY = 4$ cm。
As shown in the figure, ΔABC and ΔXYZ are equilateral triangles and are ends of a right prism. P is the mid-point of BY and $BP = 3$ cm, $XY = 4$ cm.



(i) If $a = \dfrac{CP}{PX}$, find the value of a.
若 $a = \dfrac{CP}{PX}$，求 a 的值。

$a =$

(ii) If $CX = \sqrt{b}$ cm, find the value of b.
若 $CX = \sqrt{b}$ cm，求 b 的值。

$b =$

(iii) If $\cos\angle PCX = \dfrac{\sqrt{c}}{5}$, find the value of c.
若 $\cos\angle PCX = \dfrac{\sqrt{c}}{5}$，求 c 的值。

$c =$

(iv) If $\sin\angle PCX = \dfrac{2\sqrt{d}}{5}$, find the value of d.
若 $\sin\angle PCX = \dfrac{2\sqrt{d}}{5}$，求 d 的值。

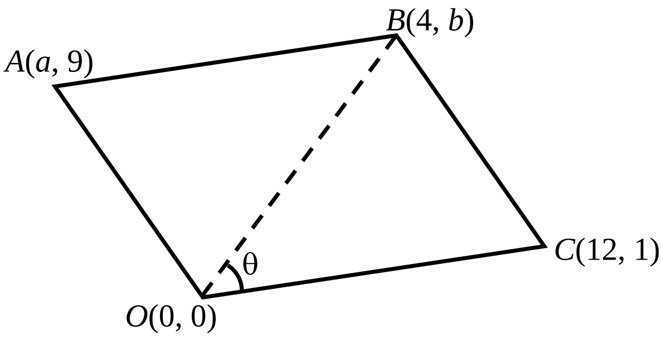
$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	Min.	Sec.
				Total score					

Hong Kong Mathematics Olympiad (1991 – 1992)
Final Event 7 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。



已知 *OABC* 為一平行四邊形。

Given that *OABC* is a parallelogram.

(i) 求 a 的值。
Find the value of a.

a =

(ii) 求 b 的值。
Find the value of b.

b =

(iii) 求 *OABC* 的面積。
Find the area of *OABC*.

Area =

(iv) 求 $\tan\theta$ 的值。
Find the value of $\tan\theta$.

$\tan\theta$ =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1991 – 1992)
Final Event 8 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 一邊長 A cm 的等邊三角形之面積為 $\sqrt{3}$ cm^2。求 A 的值。
The area of an equilateral triangle of side A cm is $\sqrt{3}$ cm^2. Find the value of A.

A =

(ii) 若 $19\times 243^{\frac{A}{5}}=b$，求 b 的值。
If $19\times 243^{\frac{A}{5}}=b$, find the value of b.

b =

(iii) 方程 $x^3 - 173x^2 + 339x + 513 = 0$ 之根為 -1、b 及 c。求 c 的值。
The roots of the equation $x^3 - 173x^2 + 339x + 513 = 0$ are -1, b and c.
Find the value of c.

c =

(iv) 某三角錐體之底為一邊長 $2c$ cm 之等邊三角形。
若該三角錐體之高為 $\sqrt{27}$ cm，且其體積為 d cm^3，求 d 的值。
The base of a triangular pyramid is an equilateral triangle of side $2c$ cm.
If the height of the pyramid is $\sqrt{27}$ cm, and its volume is d cm^3,
find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1991 – 1992)
Final Event 9 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

若一正六邊形 $ABCDEF$ 之面積為 $54\sqrt{3}\text{ cm}^2$，且 $AB = x$ cm，$AC = y\sqrt{3}$ cm，
If the area of a regular hexagon $ABCDEF$ is $54\sqrt{3}\text{ cm}^2$ and $AB = x$ cm, $AC = y\sqrt{3}$ cm,

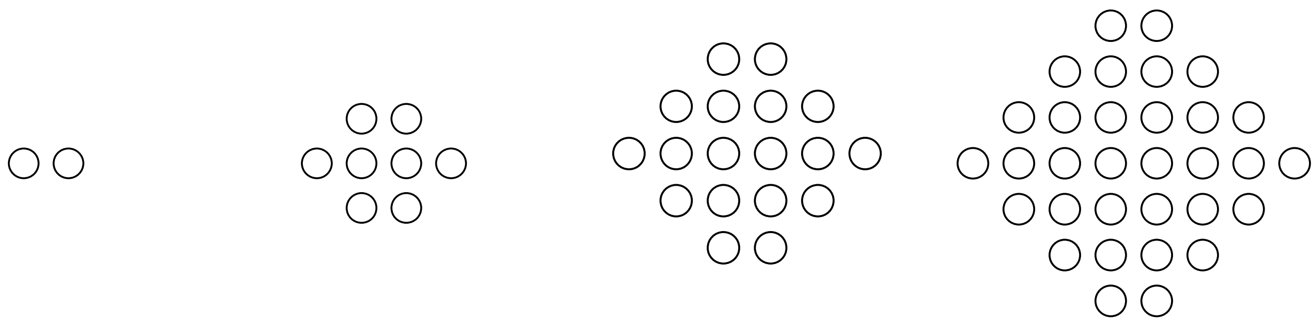
(i) 求 x 的值。
find the value of x.

$x =$

(ii) 求 y 的值。
find the value of y.

$y =$

細看以下之數形：
Consider the following number pattern:



$T_1 = 2$ $T_2 = 8$ $T_3 = 18$ $T_4 = 32$

(iii) 求 T_{10} 的值。
Find the value of T_{10}.

$T_{10} =$

(iv) 若 $T_n = 722$，求 n 的值。
If $T_n = 722$, find the value of n.

$n =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

\+ Bonus score [ ]   Time [ ] [ ]

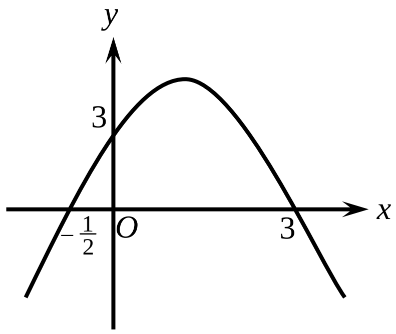
Total score [ ]   Min. Sec.

Hong Kong Mathematics Olympiad (1991 – 1992)
Final Event 10 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

下圖為 $y = ax^2 + bx + c$ 的圖形。

The following shows the graph of $y = ax^2 + bx + c$.



(i) 求 c 的值。
Find the value of c .

c =

(ii) 求 a 的值。
Find the value of a .

a =

(iii) 求 b 的值。
Find the value of b .

b =

(iv) 若 $y = x + d$ 為 $y = ax^2 + bx + c$ 的切線，求 d 的值。
If $y = x + d$ is tangent to $y = ax^2 + bx + c$, find the value of d .

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1992 – 93)

Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 已知 $7^{2x} = 36$ 及 $7^{-x} = (6)^{-\frac{a}{2}}$，求 a 的值。
Given that $7^{2x} = 36$ and $7^{-x} = (6)^{-\frac{a}{2}}$, find the value of a.

$a =$

(ii) 若 $\log_2\{\log_2[\log_2(2b) + a] + a\} = a$，求 b 的值。
Find the value of b if $\log_2\{\log_2[\log_2(2b) + a] + a\} = a$.

$b =$

(iii) 若方程 $(x - b)(x - 2)(x + 1) = 3(x - b)(x + 1)$ 正根的總數為 c，求 c 的值。
If c is the total number of positive roots of the equation $(x - b)(x - 2)(x + 1) = 3(x - b)(x + 1)$, find the value of c.

$c =$

(iv) 若 $\sqrt{3-2\sqrt{2}} = \sqrt{c} - \sqrt{d}$，求 d 的值。
If $\sqrt{3-2\sqrt{2}} = \sqrt{c} - \sqrt{d}$, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

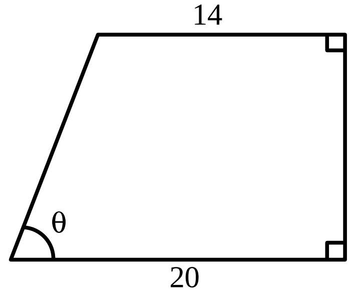
Hong Kong Mathematics Olympiad (1992 – 93)

Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $\sin\theta = \frac{4}{5}$，求四邊形面積 a。

If $\sin\theta = \frac{4}{5}$, find a, the area of the quadrilateral.



$a =$

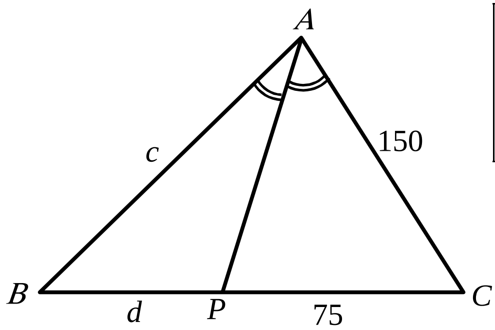
(ii) 若 $b = 126^2 - a^2$，求 b 的值。
If $b = 126^2 - a^2$, find the value of b.

$b =$

(iii) 將\$(3000 + b)按 5 : 6 : 8 分成 3 份，最小的一份為\$$c$。求 c 的值。
Dividing \$(3000 + b) in a ratio 5 : 6 : 8, the smallest part is \$ c.
Find the value of c.

$c =$

(iv) 圖中 AP 等分$\angle BAC$。已知 $AB = c$，$BP = d$，$PC = 75$ 及 $AC = 150$，求 d 的值。
In the figure, AP bisects $\angle BAC$.
Given that $AB = c$, $BP = d$, $PC = 75$ and $AC = 150$, find the value of d.



$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.		
			+ Bonus score				Time		
			Total score					Min.	Sec.

Hong Kong Mathematics Olympiad (1992 – 93)

Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 a 為以 13 除 2614303940317 的餘數，求 a 的值。
If a is the remainder when 2614303940317 is divided by 13, find the value of a.

a =

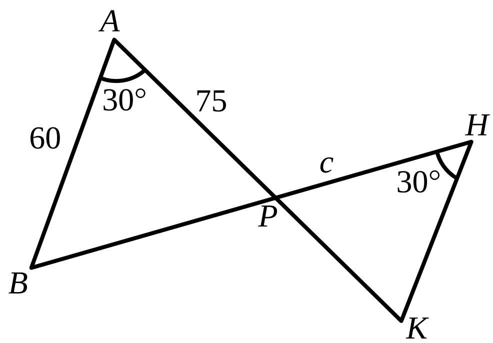
(ii) 設 $P(x, b)$為直綫 $x + y = 30$ 上的點且滿足 OP 斜率為 a (O 乃原點)。求 b 的值。
Let $P(x, b)$ be a point on the straight line $x + y = 30$ such that slope of $OP = a$ (O is the origin). Determine the value of b.

b =

(iii) 兩人踏單車，起始時相距(b + 26) km，以時速 40 km/h 及 60 km/h 相向而行。一蒼蠅以時速 100 km/h 往返兩人鼻尖，若牠在兩人碰上前共飛 c km，求 c 的值。
Two cyclists, initially (b + 26) km apart travelling towards each other with speeds 40 km/h and 60 km/h respectively. A fly flies back and forth between their noses at 100 km/h. If the fly flied c km before crushed between the cyclists, find the value of c.

c =

(iv) 圖中 APK 及 BPH 為直綫。若 $d = \Delta HPK$ 的面積，求 d 的值。
In the figure, APK and BPH are straight lines.
If d = area of triangle HPK, find the value of d.



d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1992 – 93)

Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

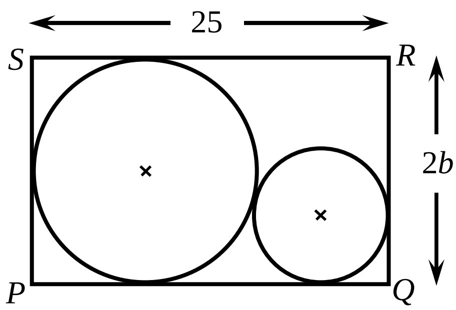
(i) 已知 x 和 y、y 和 z、z 和 x 的平均值分別為 5、9、10。
若 x、y、z 的平均值是 a，求 a 的值。
Given that the means of x and y, y and z, z and x are respectively 5, 9, 10.
If a is the mean of x, y, z, find the value of a .

a =

(ii) 某兩數的比例為 $5 : a$。當每邊加 12 時，兩數的比例變為 $3 : 4$。
若 b 為原本兩數之差及 $b > 0$，求 b 的值。
The ratio of two numbers is $5 : a$. If 12 is added to each of them, the ratio becomes $3 : 4$. If b is the difference of the original numbers and $b > 0$, find the value of b .

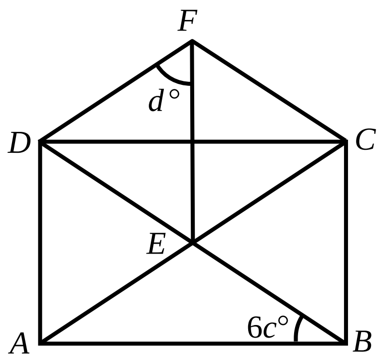
b =

(iii) $PQRS$ 為一長方形，
若細圓的半徑為 c，求 c 的值。
$PQRS$ is a rectangle. If c is the radius of the smaller circle, find the value of c.



c =

(iv) $ABCD$ 為一長方形及 CEF 為一等邊三角形，
$\angle ABD = 6c°$，求 d 的值。
$ABCD$ is a rectangle and CEF is an equilateral triangle, $\angle ABD = 6c°$, find the value of d .



d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1992 – 93)

Event 5 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 長方形兩對邊同時加長 50%，而其餘兩對邊則縮短 20%。
若長方形的面積增加 a%，求 a 的值。
Two opposite sides of a rectangle are increased by 50% while the other two are decreased by 20%. If the area of the rectangle is increased by a%, find the value of a.

a =

(ii) 設 $f(x) = x^3 - 20x^2 + x - a$ 及 $g(x) = x^4 + 3x^2 + 2$。若 $h(x)$為 $f(x)$和 $g(x)$的最大公因子，求 $b = h(1)$的值。
Let $f(x) = x^3 - 20x^2 + x - a$ and $g(x) = x^4 + 3x^2 + 2$.
If $h(x)$ is the highest common factor of $f(x)$ and $g(x)$, find the value of $b = h(1)$.

b =

(iii) It is known that $b^{16} - 1$ has four distinct prime factors, determine the largest one, denoted by c.
已知 $b^{16} - 1$ 共有四質因子，求其中最大的一個，以 c 表它。

c =

(iv) When c is represented in binary scale, there are d '0's. Find the value of d.
當以二進制表示 c，則其中有 d 個'0'。求 d 的值。

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

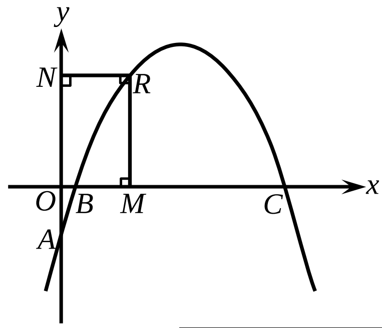
Hong Kong Mathematics Olympiad (1992 – 93)

Event 6 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

右圖所示為 $y = px^2 + 5x + p$ 的圖像。$A = (0, -2)$、$B = \left(\frac{1}{2}, 0\right)$、$C = (2, 0)$、$O = (0, 0)$。

The following shows the graph of $y = px^2 + 5x + p$. $A = (0, -2)$, $B = \left(\frac{1}{2}, 0\right)$, $C = (2, 0)$, $O = (0, 0)$.



(i) 求 p 的值。
Find the value of p .

p =

(ii) 若 y 的最大值為 $\frac{9}{m}$，求 m 的值。
If $\frac{9}{m}$ is the maximum value of y, find the value of m .

m =

(iii) 設 R 為曲綫上一點且 $OMRN$ 為一正方形。若 R 的 x 坐標為 r，求 r 的值。
Let R be a point on the curve such that $OMRN$ is a square.
If r is the x-coordinate of R , find the value of r .

r =

(iv) 一斜率為–2 及通過原點的直綫與上述曲綫相交於兩點 E 及 F。若 EF 中點的 y 坐標為 $\frac{7}{s}$，求 s 的值。
A straight line with slope = –2 passes through the origin cutting the curve at two points E and F.
If $\frac{7}{s}$ is the y-coordinate of the midpoint of EF, find the value of s .

s =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

\+ Bonus score [ ]

Total score [ ]

Team No. [ ]

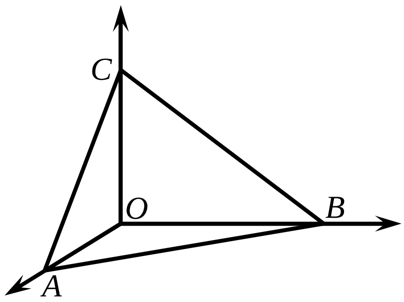
Time [ ] Min. [ ] Sec.

Hong Kong Mathematics Olympiad (1992 – 93)

Event 7 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

OABC 為一四面體，其中 *OA*、*OB* 及 *OC* 互相垂直。
已知 $OA = OB = OC = 6x$。
OABC is a tetrahedron with *OA*, *OB* and *OC* being mutually perpendicular. Given that $OA = OB = OC = 6x$.



(i) 若 *OABC* 的體積為 ax^3，求 a 的值。
If the volume of *OABC* is ax^3, find the value of a.

$a =$

(ii) 若 ΔABC 的面積為 $b\sqrt{3}x^2$，求 b 的值。
If the area of ΔABC is $b\sqrt{3}x^2$, find the value of b.

$b =$

(iii) 若由 O 至 ΔABC 的距離為 $c\sqrt{3}x$，求 c 的值。
If the distance from O to ΔABC is $c\sqrt{3}x$, find the value of c.

$c =$

(iv) 若由 C 至 AB 中點的俯角為θ，且 $\sin\theta = \frac{\sqrt{d}}{3}$，求 d 的值。
If θ is the angle of depression from C to the midpoint of AB and $\sin\theta = \frac{\sqrt{d}}{3}$, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1992 – 93)

Event 8 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

已知方程 $x^2 + (m + 1)x - 2 = 0$ 有兩整數根$(\alpha + 1)$及$(\beta + 1)$，且$\alpha < \beta$及 $m \neq 0$。設 $d = \beta - \alpha$。
Given that the equation $x^2 + (m + 1)x - 2 = 0$ has 2 integral roots $(\alpha + 1)$ and $(\beta + 1)$ with $\alpha < \beta$ and $m \neq 0$.
Let $d = \beta - \alpha$.

(i) 求 m 的值。
Find the value of m.

m =

(ii) 求 d 的值。
Find the value of d.

d =

設 n 為由 1 至 2000 內被 3 或 7 除時，餘數都為 1 的整數的總數。
Let n be the total number of integers between 1 and 2000 such that each of them gives a remainder of 1 when it is divided by 3 or 7.

(iii) 求 n 的值。
Find the value of n.

n =

(iv) 若 s 為上述 n 個整數的總和，求 s 的值。
If s is the sum of all these n integers, find the value of s.

s =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ] Team No. [ ]

\+ Bonus score [ ] Time [ ] [ ]

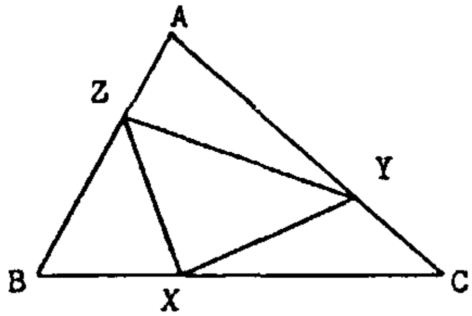
Total score [ ] Min. Sec.

Hong Kong Mathematics Olympiad (1992 – 93)

Event 9 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

點 X、Y、Z 依次將 BC、CA、AB 分成 $1:2$。
設 ΔAZY 的面積: ΔABC 的面積 $=2:a$ 及
ΔAZY 的面積: ΔXYZ 的面積 $=2:b$。
BC, CA, AB are divided respectively by the points X, Y, Z in the ratio $1:2$.
Let area of ΔAZY : area of $\Delta ABC = 2:a$ and
area of ΔAZY : area of $\Delta XYZ = 2:b$.



(i) 求 a 的值。
Find the value of a .

$a =$

(ii) 求 b 的值。
Find the value of b .

$b =$

擲一枚骰子兩次。設 $\frac{x}{36}$ 為擲得點數總和為 7 或 8 的概率，$\frac{y}{36}$ 為擲得兩數之差為 1 的概率。

A die is thrown 2 times. Let $\frac{x}{36}$ be the probability that the sum of numbers obtained is 7 or 8 and $\frac{y}{36}$ be the probability that the difference of numbers obtained is 1.

(iii) 求 x 的值。
Find the value of x .

$x =$

(iv) 求 y 的值。
Find the value of y .

$y =$

FOR OFFICIAL USE

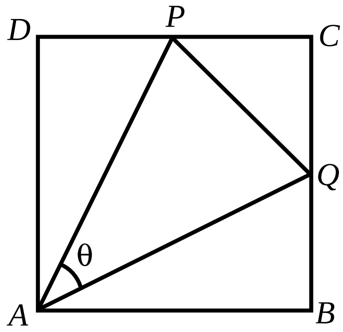
Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1992 – 93)

Event 10 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

ABCD 乃一邊長為 $20\sqrt{5}x$ 的正方形。*P*、*Q* 分別為 *DC* 及 *BC* 的中點。
ABCD is a square of side length $20\sqrt{5}x$. *P*, *Q* are midpoints of *DC* and *BC* respectively.



(i) 若 $AP = ax$，求 a 的值。
If $AP = ax$, find the value of a.

$a =$

(ii) 若 $PQ = b\sqrt{10}x$，求 b 的值。
If $PQ = b\sqrt{10}x$, find the value of b.

$b =$

(iii) 若由 *A* 至 *PQ* 的距離為 $c\sqrt{10}x$，求 c 的值。
If the distance from *A* to *PQ* is $c\sqrt{10}x$, find the value of c.

$c =$

(iv) 若 $\sin\theta = \frac{d}{100}$，求 d 的值。
If $\sin\theta = \frac{d}{100}$, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1993 – 94)

Sample Event (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 某兩數之和為 40，其積為 20。若該兩數倒數之和為 a，求 a 的值。
The sum of two numbers is 40, their product is 20.
If the sum of their reciprocals is a, find the value of a.

a =

(ii) 若一邊長 $(a+1)$ 厘米之正方體之總表面積為 b 平方厘米，求 b 的值。
If b cm^2 is the total surface area of a cube of side $(a+1)$ cm, find the value of b.

b =

(iii) 一袋內有$(b-4)$個白球，$(b+46)$個紅球。若隨意於袋內取一球，而該球為白色之概率為 $\frac{c}{6}$，求 c 的值。
One ball is taken at random from a bag containing $(b-4)$ white balls and $(b+46)$ red balls. If $\frac{c}{6}$ is the probability that the ball is white, find the value of c.

c =

(iv) 若一邊長 c 厘米之正三角形之面積 $d\sqrt{3}$ 平方厘米，求 d 的值。
The length of a side of an equilateral triangle is c cm. If its area is $d\sqrt{3}$ cm^2, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

\+ Bonus score [ ]

Total score [ ]

Team No. [ ]

Time [ ] Min. [ ] Sec.

Hong Kong Mathematics Olympiad (1993 – 94)

Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 方程式 $x^2 - ax + (a + 3) = 0$ 有等根。若 a 為一正整數，求 a 的值。
The equation $x^2 - ax + (a + 3) = 0$ has equal roots. Find the value of a, if a is a positive integer.

a =

(ii) 在一次測驗中，共 20 題。做對一題給 a 分，做錯一題要倒扣 3 分。一學生做了全部的 20 題，而得到 48 分。他答對了的題目數目是 b。求 b 的值。
In a test, there are 20 questions. a marks will be given to a correct answer and 3 marks will be deducted for each wrong answer. A student has done all the 20 questions and scored 48 marks. Find b, the number of questions that he has answered correctly.

b =

(iii) 若

$x : y = 2 : 3$
$x : z = 4 : 5$
$y : z = b : c$,

求 c。

If

$x : y = 2 : 3$
$x : z = 4 : 5$
$y : z = b : c$,

find c.

c =

(iv) 設 $P(x, d)$為直綫 $x + y = 22$ 上的點，且 OP 的斜率為 c (O 為原點)。求 d 的值。
Let $P(x, d)$ be a point on the straight line $x + y = 22$ such that the slope of OP equals to c (O is the origin). Determine the value of d.

d =

<u>FOR OFFICIAL USE</u>

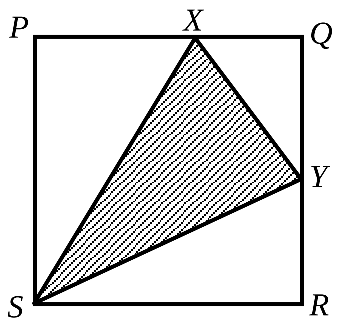
Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1993 – 94)

Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 在正方形 $PQRS$ 中，Y 為 QR 之中點，且 $PX=\frac{3}{4}PQ$。若 A 為陰影部分三角形面積與正方形面積的比，求 A 的值。
In square $PQRS$, Y is the mid-point of the side QR and $PX=\frac{3}{4}PQ$. If A is the ratio of the area of the shaded triangle to the area of the square, find the value of A.

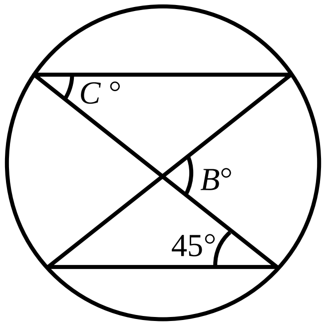


A =

(ii) 某甲買了一些乒乓球，需多付出銷售稅 16A%。若他毋須付稅，則可用同等金錢多買 3 個乒乓球。假設 B 是他所買乒乓球的個數，求 B 的值。
A man bought a number of ping-pong balls where a 16A% sales tax is added. If he did not have to pay tax he could have bought 3 more balls for the same amount of money. If B is the total number of balls that he bought, find the value of B.

B =

(iii) 如圖，求 C 的值。
Refer to the diagram, find the value of C.



C =

(iv) 2C 個連續偶數之和為 1170。若 D 為其中最大之偶數，求 D 的值。
The sum of 2C consecutive even numbers is 1170. If D is the largest of them, find the value of D.

D =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1993 – 94)

Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 183*a*8 為 287 的倍數，求 *a* 的值。
If 183*a*8 is a mu1tiple of 287, find the value of a.

a =

(ii) a^2 這個數共有 b 個正因數，求 b 的值。
The number of positive factors of a^2 is b, find the value of b.

b =

(iii) 瓶中有球 c 個，其中 b 個是黑色或紅色的，$(b + 2)$個是紅色或白色的，而黑色或白色的有 12 個。求 c 的值。
In an urn, there are c balls, b of them are either black or red, $(b + 2)$ of them are either red or white and 12 of them are either black or white. Find the value of c.

c =

(iv) 已知對所有 x，$f(3 + x) = f(3 – x)$，且方程式 $f(x) = 0$ 有 c 個不等根，求所有根的總和 d。
Given $f(3 + x) = f(3 – x)$ for all values of x, and the equation $f(x) = 0$ has exactly c distinct roots. Find d, the sum of these roots.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1993 – 94)

Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

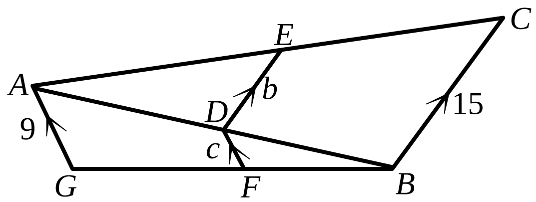
(i) $x^6 - 8x^3 + 6$ 除以$(x - 1)(x - 2)$，其餘數為 $7x - a$，求 a 的值。
The remainder when $x^6 - 8x^3 + 6$ is divided by $(x - 1)(x - 2)$ is $7x - a$, find the value of a .

a =

(ii) 若 $x^2 - x + 1 = 0$ 及 $b = x^3 - 3x^2 + 3x + a$，求 b 的值。
If $x^2 - x + 1 = 0$ and $b = x^3 - 3x^2 + 3x + a$, find the value of b .

b =

(iii) 如圖，求 c 的值。
Refer to the diagram, find the value of c.



c =

(iv) 有 c 個兒童，他們均生於一九九零年六月，若果他們生於不同日子的概率是 $\frac{d}{225}$，求 d 的值。
If c boys were all born in June 1990 and the probability that their birthdays are all different is $\frac{d}{225}$, find the value of d .

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1993 – 94)

Event 5 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 已知$1-\frac{4}{x}+\frac{4}{x^2}=0$。若$A=\frac{2}{x}$，求$A$的值。

Given $1-\frac{4}{x}+\frac{4}{x^2}=0$. If $A=\frac{2}{x}$, find the value of A.

A =

(ii) 若B條內直徑為A厘米的圓形水管的輸水量與一內直徑為 6 厘米的圓形水管相等，求B的值。

If B circular pipes each with an internal diameter of A cm carry the same amount of water as a pipe with an internal diameter 6 cm, find the value of B.

B =

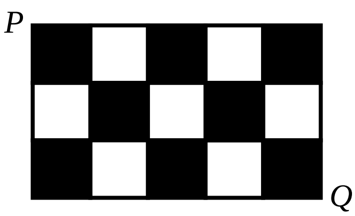
(iii) 若一個由x軸、y軸及直綫$Bx + 9y = 18$所圍成之三角形之面積為C，求C的值。

If C is the area of the triangle formed by x-axis, y-axis and the line $Bx + 9y = 18$, find the value of C.

C =

(iv) 十五塊邊長為 $10C$ 單位的正方形磚如圖排列。一蟻沿磚之邊緣爬行，而其左邊必為一黑磚。求D，此蟻由P爬至Q之最短距離。

Fifteen square tiles with side $10C$ units long are arranged as shown. An ant walks along the edges of the tiles, always keeping a black tile on its left.
Find the shortest distance D that the ant would walk in going from P to Q.



D =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1993 – 94)

Sample Event (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $x*y = xy + 1$，且 $a = (2*4)*2$，求 a 的值。
If $x*y = xy + 1$ and $a = (2*4)*2$, find the value of a.

a =

(ii) 若第 b 個質數為 a，求 b 的值。
If the b^{th} prime number is a, find the value of b.

b =

(iii) 若 $c = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{50}\right)$，試以最簡單之分數表 c。
If $c = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{50}\right)$, find c in the simplest fractional form.

c =

(iv) 一正方形內接於一個半徑為 10 之圓。若正方形之面積為 d，求 d 的值。
If d is the area of a square inscribed in a circle of radius 10, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1993 – 94)

Event 6 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $\log_2 a - 2\log_a 2 = 1$，求 a 的值。
If $\log_2 a - 2\log_a 2 = 1$, find the value of a.

$a =$

(ii) 若 $b = \log_3[2(3 + 1)(3^2 + 1)(3^4 + 1)(3^8 + 1) + 1]$，求 b 的值。
If $b = \log_3[2(3 + 1)(3^2 + 1)(3^4 + 1)(3^8 + 1) + 1]$, find the value of b.

$b =$

(iii) 若任意選擇一個有三十一日的月份，求該月有五個星期天的機率 c。
If a 31-day month is taken at random,
find c, the probability that there are 5 Sundays in the month.

$c =$

(iv) 從六名男士及四名女士中選出五人，組成一組。若其間共有 d 種選法，使男士必多於女士，求 d 的值。
A group of 5 people is to be selected from 6 men and 4 women.
Find d, the number of ways that there are always more men than women.

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1993 – 94)

Event 7 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 在 $1\times2\times3\times\ldots\times100$ 的積數中，最末的 a 個位都是 0。求 a 的值。
There are a zeros at the end of the product $1\times2\times3\times\ldots\times100$. Find the value of a.

a =

(ii) 1998^{10} 除以 10^4，所得餘數為 b，求 b 的值。
Find the value of b, if b is the remainder when 1998^{10} is divided by 10^4.

b =

(iii) 若 $c = 2 - x + 2\sqrt{x-1}$ 且 $x > 1$，求 c 之最大值。
Find the largest value of c, if $c = 2 - x + 2\sqrt{x-1}$ and $x > 1$.

c =

(iv) 若 $\left|\frac{3-2d}{5}+2\right| \le 3$，求 d 的最小值。
Find the least value of d, if $\left|\frac{3-2d}{5}+2\right| \le 3$.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1993 – 94)

Event 8 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 由 1 至 121，有 a 個數是 3 或是 5 的倍數。求 a 的值。
From 1 to 121, there are a numbers which are multiples of 3 or 5.
Find the value of a.

a =

(ii) 由 1 至 121，有 b 個數不能被 5 或 7 整除。求 b 的值。
From 1 to 121, there are b numbers which are not divisible by 5 nor 7.
Find the value of b.

b =

用 1、2、3、4 這四個數字，而每個數字均可重複使用，則可組成一些 4 位數。求
From the digits 1, 2, 3, 4, when each digit can be used repeatedly, 4-digit numbers are formed. Find

(iii) 共可組成的 4 位數的個數 c。
c, the number of 4-digit numbers that can be formed.

c =

(iv) 所組成的 4 位數的總和 d。
d, the sum of all these 4-digit numbers.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1993 – 94)

Event 9 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

A、*B*、*C*、D 為由 0 至 9 間的不同整數，而

$$\begin{array}{r} A\ B\ A \\ \times \quad A\ B\ A \\ \hline C\ C\ D\ C\ C \end{array}$$

求 *A*、*B*、*C* 及 *D* 的值。

A, *B*, *C*, *D* are different integers ranging from 0 to 9 and

$$\begin{array}{r} A\ B\ A \\ \times \quad A\ B\ A \\ \hline C\ C\ D\ C\ C \end{array}$$

Find the values of *A*, *B*, *C* and *D*.

A =	*B* =	*C* =	*D* =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1993 – 94)

Event 10 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

在長方形 $ABCD$ 中，$AD = 10$，$CD = 15$，P 為長方形內一點，使 $PB = 9$，$PA = 12$。求
In rectangle $ABCD$, $AD = 10$, $CD = 15$, P is a point inside the rectangle such that $PB = 9$, $PA = 12$. Find

(i) PD 之長 a，及
a, the length of PD and

$a =$

(ii) PC 之長 b。
b, the length of PC.

$b =$

(iii) 已知 $\sin 2\theta = 2\sin\theta\cos\theta$。求 c，若 $c = \frac{\sin 20^\circ \cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ}{\sin 160^\circ}$ 的值。
It is given that $\sin 2\theta = 2\sin\theta\cos\theta$. Find the value of c, if

$$c = \frac{\sin 20^\circ \cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ}{\sin 160^\circ}.$$

$c =$

(iv) 已知 $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$，求 d 的值，若
$d = (1 + \tan 21^\circ)(1 + \tan 22^\circ)(1 + \tan 23^\circ)(1 + \tan 24^\circ)$。
It is given that $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$. Find the value of d, if

$$d = (1 + \tan 21^\circ)(1 + \tan 22^\circ)(1 + \tan 23^\circ)(1 + \tan 24^\circ).$$

$d =$

FOR OFFICIAL USE

Score for accuracy		× Mult. factor for speed		=		Team No.	
		+ Bonus score				Time		
		Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1994-95)

Event 1 (Individual)

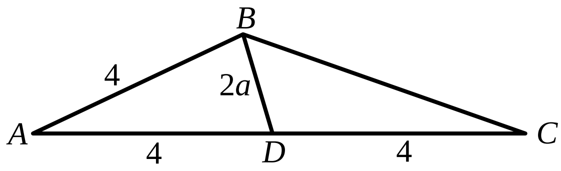
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $a=\log_{\frac{1}{4}}\frac{1}{2}$，求 a 的值。

Find the value of a, if $a=\log_{\frac{1}{4}}\frac{1}{2}$.

a =

(ii) 如圖示，$AB = AD = DC = 4$, $BD = 2a$。
若 BC 之長為 b，求 b 的值。
In the figure, $AB = AD = DC = 4$,
$BD = 2a$. Find b, the length of BC.



b =

(iii) 已知 $f(x) = px^3 + qx + 5$ 且 $f(-7) = \sqrt{2}\,b + 1$。若 $c = f(7)$，求 c 的值。
It is given that $f(x) = px^3 + qx + 5$ and $f(-7) = \sqrt{2}\,b + 1$.
Find the value of c, if $c = f(7)$.

c =

(iv) 若 $d^c + 1000$ 可被 $10 + c$ 所整除，求 d 的最小正整數值。
Find the least positive integer d, such that $d^c + 1000$ is divisible by $10 + c$.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (1994-95)

Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $\dfrac{x}{(x-1)(x-4)}=\dfrac{x}{(x-2)(x-3)}$，求 x 的值。
If $\dfrac{x}{(x-1)(x-4)}=\dfrac{x}{(x-2)(x-3)}$, find the value of x.

$x =$

(ii) 若 $f(t) = 3\times52^t$ 且 $y = f(x)$。求 y 的值。
If $f(t) = 3\times52^t$ and $y = f(x)$, find the value of y.

$y =$

(iii) 甲可在 y 日完成某一項工程，乙可在$(y + 3)$日完成同一工程。
假如甲乙二人合作，可在 z 日完成，求 z 的值。
A can finish a job in y days, B can finish a job in $(y + 3)$ days.
If they worked together, they can finish the job in z days, find the value of z.

$z =$

(iv) 用 z 粒骰子擲得 7 點的概率是 w，求 w 的值。
The probability of throwing z dice to score 7 is w, find the value of w.

$w =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1994-95)

Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $a = \sin 30° + \sin 300° + \sin 3000°$，求 a 的值。
If $a = \sin 30° + \sin 300° + \sin 3000°$, find the value of a.

$a =$

(ii) 已知 $\frac{x+y}{2} = \frac{z+x}{3} = \frac{y+z}{4}$ 且 $x + y + z = 36a$。求 b 之值，若 $b = x + y$。
It is given that $\frac{x+y}{2} = \frac{z+x}{3} = \frac{y+z}{4}$ and $x + y + z = 36a$.
Find the value of b, if $b = x + y$.

$b =$

(iii) 已知方程 $x + 6 + 8k = k(x + b)$有正整數解。求 k 的最小值 c。
It is given that the equation $x + 6 + 8k = k(x + b)$ has positive integral solution. Find c, the least value of k.

$c =$

(iv) 一輛汽車以平均時速 $40c$ km/h 完成了旅程的 40 %。為著使全程的平均速度為 100 km/h，車速被調至 d km/h 行畢全程。求 d 的值。
A car has already travelled 40% of its journey at an average speed of $40c$ km/h. In order to make the average speed of the whole journey become 100 km/h, the speed of the car is adjusted to d km/h to complete the rest of the journey.
Find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

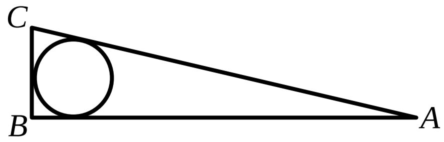
Hong Kong Mathematics Olympiad (1994-95)

Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 在三角形 ABC 中，$\angle B = 90°$，$BC = 7$ 且 $AB = 24$。若 r 為內切圓之半徑，求 r 的值。
In triangle ABC, $\angle B = 90°$, $BC = 7$ and $AB = 24$. If r is the radius of the inscribed circle, find the value of r.

r =



(ii) 若 $x^2 + x - 1 = 0$ 且 $s = x^3 + 2x^2 + r$，求 s 的值。
If $x^2 + x - 1 = 0$ and $s = x^3 + 2x^2 + r$, find the value of s.

s =

(iii) 已知 $F_1 = F_2 = 1$ 且 $F_n = F_{n-1} + F_{n-2}$，其中 $n \geq 3$。若 $F_t = s + 1$，求 t 的值。
It is given that $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$, where $n \geq 3$. If $F_t = s + 1$, find the value of t.

t =

(iv) 若 $u = \sqrt{t(t+1)(t+2)(t+3)+1}$，求 u 的值。
If $u = \sqrt{t(t+1)(t+2)(t+3)+1}$, find the value of u.

u =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1994-95)

Event 5 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) It is given that $\log_7(\log_3(\log_2 x)) = 0$. Find the value of a, if $a = x^{\frac{1}{3}}$.

已知 $\log_7(\log_3(\log_2 x)) = 0$。若 $a = x^{\frac{1}{3}}$，求 a 的值。

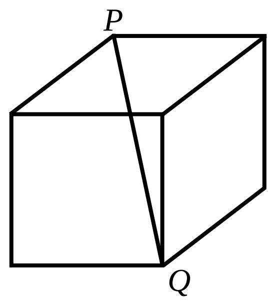
a =

(ii) 如圖示，PQ 是正方體的一條對角綫，且 $PQ = \frac{a}{2}$。

若 b 為此正方體的總表面積，求 b 的值。

In the figure, PQ is a diagonal of the cube and $PQ = \frac{a}{2}$.

Find the value of b, if b is the total surface area of the cube.

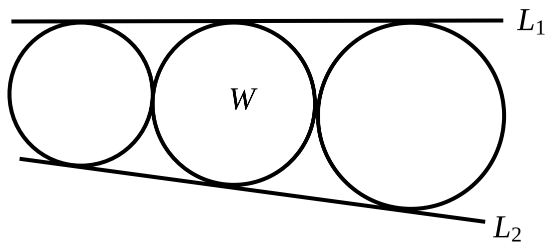


b =

(iii) 如圖示，L_1、L_2 為三個圓的切綫。如果最大圓的半徑是 18，最小圓半徑是 $4b$，求 c，若 c 為圓 W 的半徑。

In the figure, L_1 and L_2 are tangents to the three circles. If the radius of the largest circle is 18 and the radius of the smallest circle is $4b$, find c, where c is the radius of the circle W.

c =



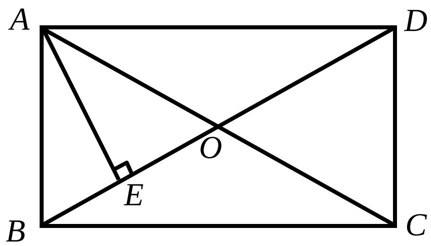
(iv) 如圖，$ABCD$ 為一長方形。$AE \perp BD$

且 $BE = EO = \frac{c}{6}$。求長方形 $ABCD$ 之面積 d。

Refer to the figure, $ABCD$ is a rectangle. $AE \perp BD$

and $BE = EO = \frac{c}{6}$.

Find d, the area of the rectangle $ABCD$.



d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.
			+ Bonus score				Time
			Total score				Min. / Sec.

Hong Kong Mathematics Olympiad (1994-95)

Event 6 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) $2^a \cdot 9^b$ 為一四位數，其千位數是 2，百位數是 a，十位數是 9，個位數是 b，求 a 及 b 的值。
$2^a \cdot 9^b$ is a four digit number and its thousands digit is 2, its hundreds digit is a, its tens digit is 9 and its units digit is b, find the values of a and b.

a =

(ii)

b =

(iii) 若 $c=\left(1+\frac{1}{2}+\frac{1}{3}\right)\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\left(\frac{1}{2}+\frac{1}{3}\right)$，求 c 的值。

Find the value of c, if $c=\left(1+\frac{1}{2}+\frac{1}{3}\right)\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\left(\frac{1}{2}+\frac{1}{3}\right)$.

c =

(iv) 求 d 的值，若

$$d=\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1994}\right)\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1995}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1995}\right)\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1994}\right)。$$

Find the value of d, if

$$d=\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1994}\right)\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1995}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1995}\right)\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1994}\right).$$

d =

FOR OFFICIAL USE

Score for accuracy ☐ × Mult. factor for speed ☐ = ☐

\+ Bonus score ☐

Total score ☐

Team No. ☐

Time ☐ Min. ☐ Sec.

Hong Kong Mathematics Olympiad (1994-95)

Event 7 (Group)

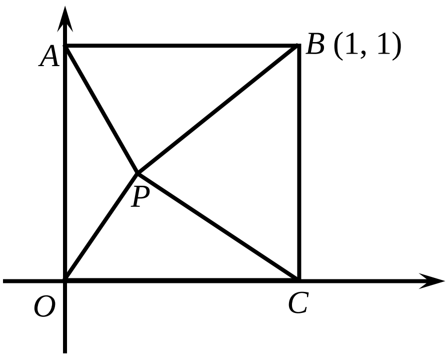
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 設 p、q、r 為三角形 PQR 的三邊。若 $p^4 + q^4 + r^4 = 2r^2(p^2 + q^2)$，且 $a = \cos^2 R$，其中 R 的對邊為 r，求 a 的值。
Let p, q, r be the three sides of triangle PQR. If $p^4 + q^4 + r^4 = 2r^2(p^2 + q^2)$, find the value of a, where $a = \cos^2 R$ and R denotes the angle opposite r.

a =

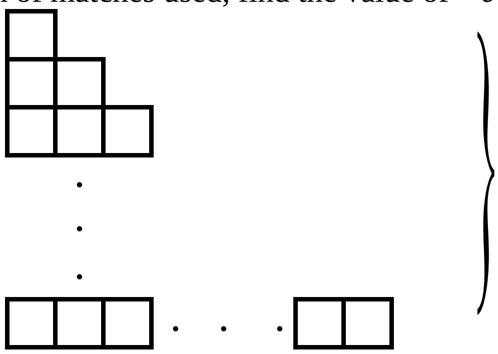
(ii) 如圖，P 為正方形 $OABC$ 內的任意點，且 b 為 $PO + PA + PB + PC$ 之最小值，求 b 的值。
Refer to the diagram, P is any point inside the square $OABC$ and b is the minimum value of $PO + PA + PB + PC$, find the value of b.

b =



(iii) 長度同為 1 的火柴被排成下列圖案。若以 c 表示用去火柴枝的總長，求 c 的值。
Identical matches of length 1 are used to arrange the following pattern, if c denotes the total length of matches used, find the value of c.

c =



(iv) 求 d 的值，若 $d = \sqrt{111111 - 222}$。
Find the value of d, where $d = \sqrt{111111 - 222}$.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ] Team No. [ ]

\+ Bonus score [ ] Time [ ] Min. [ ] Sec.

Total score [ ]

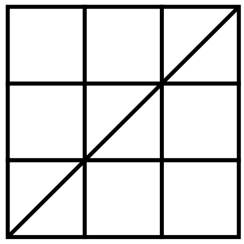
Hong Kong Mathematics Olympiad (1994-95)

Event 8 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

在方格紙上繪畫尺寸為 $\ell \times b$ 的長方形，其中 ℓ、b 為正整數並添上對角綫一條。以 V 代表相交的端點總數(不包括首尾兩點在內)。(如右圖示)

Rectangles of length ℓ and breadth b where ℓ, b are positive integers, are drawn on square grid paper. For each of these rectangles, a diagonal is drawn and the number of vertices V intersected (excluding the two end points) is counted (see the figure).



$\ell = b = 3$

$V = 2$

(i) 當 $\ell = 6$，$b = 4$ 時，求 V 的值。

Find the value of V, when $\ell = 6$, $b = 4$.

$V =$

(ii) 當 $\ell = 5$，$b = 3$ 時，求 V 的值。

Find the value of V, when $\ell = 5$, $b = 3$.

$V =$

(iii) 當 $\ell = 12$ 且 $1 < b < 12$ 時，求使 $V = 0$ 時，b 的不同個數 r。

When $\ell = 12$ and $1 < b < 12$, find r, the number of different values of b that makes $V = 0$?

$r =$

(iv) 當 $\ell = 108$，$b = 72$ 時，求 V 的值。

Find the value of V, when $\ell = 108$, $b = 72$.

$V =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1994-95)

Event 9 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

A、B、C、D 為自 0 至 9 間的不同整數，且

$$\begin{array}{rcccc} & A & A & B & C \\ - & B & A & C & B \\ \hline & D & A & C & D \end{array}$$

求 A、B、C 及 D 之值。

A, B, C, D are different integers ranging from 0 to 9 and

$$\begin{array}{rcccc} & A & A & B & C \\ - & B & A & C & B \\ \hline & D & A & C & D \end{array}$$

Find the values of A, B, C and D .

A =	B =	C =	D =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.		
			+ Bonus score				Time		
			Total score					Min.	Sec.

Hong Kong Mathematics Olympiad (1994-95)
Event 10 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

在直角坐標平面上，x- 和 y- 坐標同為整數的點稱為格點。P 是起始時位於(0, 0)的移動點，它每一步必須沿坐標綫的其中一過個方向走 1 個單位的距離。

Lattice points are points on a rectangular coordinate plane having both x- and y-coordinates being integers. A moving point P is initially located at (0, 0). It moves 1 unit along the coordinate lines (in either directions) in a single step.

(i) 若 P 走 1 步，它可到達 a 個格點，求 a 的值。
If P moves 1 step then P can reach a different lattice points, find the value of a.

a =

(ii) 若 P 可走不超過 2 步，它可到達 b 個格點，求 b 的值。
If P moves not more than 2 steps then P can reach b different lattice points, find the value of b.

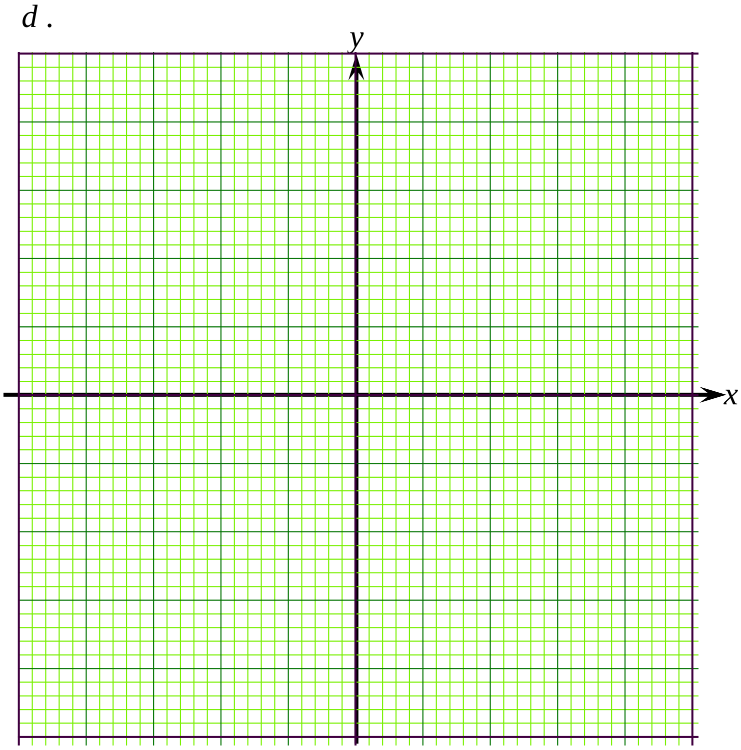
b =

(iii) 若 P 走 3 步，它可到達 c 個格點，求 c 的值。
If P moves 3 steps then P can reach c different lattice points, find the value of c.

c =

(iv) 若 P 走 9 步，它停在直綫 $x + y = 9$ 上的概率是 d，求 d 的值。
If d is the probability that P lies on the straight line $x + y = 9$ when P advances 9 steps, find the value of d.

d =



FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1995-96)
Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

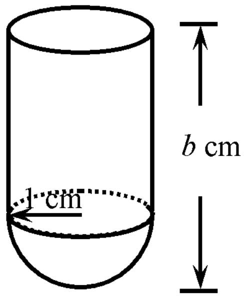
(i) 若一個等邊三角形與一個正六邊形周長相等，而其面積比為 $2 : a$，求 a 的值。
The perimeter of an equilateral triangle is exactly the same in length as the perimeter of a regular hexagon. The ratio of the areas of the triangle and the hexagon is $2 : a$, find the value of a.

$a =$

(ii) 若 $5^x + 5^{-x} = a$ 和 $5^{3x} + 5^{-3x} = b$ 求 b 的值。
If $5^x + 5^{-x} = a$ and $5^{3x} + 5^{-3x} = b$, find the value of b.

$b =$

(iii) 圖中為一圓柱體和半球體組成的無蓋空心物體。半球體和圓柱體的半徑均為 1 cm。若這物體的長度為 b cm，且表面面積為 $c\pi$ cm^2，求 c 的值。
The figure shows an open cylindrical tube (radius = 1 cm) with a hemispherical bottom of radius 1 cm. The height of the tube is b cm and the external surface area of the tube is $c\pi$ cm^2.
Find the value of c.



$c =$

(iv) 拋擲兩粒正常骰子，設取得點數總和是 $\frac{c}{6}$ 的概率為 d，求 d 的值。
Two fair dice are thrown. Let d be the probability of getting the sum of scores to be $\frac{c}{6}$. Find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1995-96)
Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 已知 $m, n > 0$ 和 $m + n = 1$。若 $\left(1+\frac{1}{m}\right)\left(1+\frac{1}{n}\right)$ 之最小值為 a，求 a 的值。

It is given that $m, n > 0$ and $m + n = 1$.

If the minimum value of $\left(1+\frac{1}{m}\right)\left(1+\frac{1}{n}\right)$ is a, find the value of a.

$a =$

(ii) 方程 $x^2 - (10 + a)x + 25 = 0$ 的根是 $x^2 + bx = 5$ 的根的平方，求 b 的正數值。

If the roots of the equation $x^2 - (10 + a)x + 25 = 0$ are the square of the roots of the equation $x^2 + bx = 5$, find the positive value of b.

$b =$

(iii) 若 $(xy - 2)^{b-1} + (x - 2y)^{b-1} = 0$ 及 $c = x^2 + y^2 - 1$，求 c 的值。

If $(xy - 2)^{b-1} + (x - 2y)^{b-1} = 0$ and $c = x^2 + y^2 - 1$, find the value of c.

$c =$

(iv) 若 $f(x)$ 是一二次多項式，$f(f(x)) = x^4 - 2x^2$ 及 $d = f(c)$，求 d 的值。

If $f(x)$ is a polynomial of degree two, $f(f(x)) = x^4 - 2x^2$ and $d = f(c)$, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1995-96)
Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 a 為實數及 $2a^3 + a^2 - 275 = 0$，求 a 的值。
If a is a real number and $2a^3 + a^2 - 275 = 0$, find the value of a.

a =

(ii) 若 $3^2 \cdot 3^5 \cdot 3^8 \cdots 3^{3b-1} = 27^a$，求 b 的值。
Find the value of b if $3^2 \cdot 3^5 \cdot 3^8 \cdots 3^{3b-1} = 27^a$.

b =

(iii) 若 $\log_b(b^c - 8) = 2 - c$，求 c 的值。
Find the value of c if $\log_b(b^c - 8) = 2 - c$.

c =

(iv) 若 $[(4^c)^c]^c = 2^d$，求 d 的值。
If $[(4^c)^c]^c = 2^d$, find the value of d.

d =

FOR OFFICIAL USE

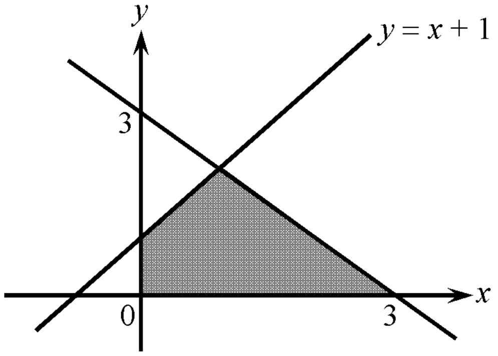
Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1995-96)
Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 圖中陰影部分面積是 a，求 a 的值。
In the figure, the area of the shaded region is a. Find the value of a.

a =



(ii) 若 $8^b = 4^a - 4^3$，求 b 的值。
If $8^b = 4^a - 4^3$, find the value of b.

b =

(iii) 已知 c 是方程式 $x^2 - 100b + \frac{10000}{x^2} = 0$ 之正根，求 c 的值。

Given that c is the positive root of the equation $x^2 - 100b + \frac{10000}{x^2} = 0$, find the value of c.

c =

(iv) 若 $d = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{(c-1)\times c}$，求 d 的值。
If $d = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{(c-1)\times c}$, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (1995-96)
Event 5 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 同時投擲四顆骰子。設取得最小一半骰子的結果為偶數的概率為 a，求 a 的值。
Four fair dice are thrown. Let a be the probability of getting at least half of the outcome of the dice to be even. Find the value of a.

a =

(ii) 已知 $f(x)=\frac{3}{8}x^2(81)^{-\frac{1}{x}}$ 和 $g(x) = 4\log_{10}(14x) - 2\log_{10}49$。
求 $b = f\{g[16(1-a)]\}$ 的值。
It is given that $f(x)=\frac{3}{8}x^2(81)^{-\frac{1}{x}}$ and $g(x) = 4\log_{10}(14x) - 2\log_{10}49$.
Find the value of $b = f\{g[16(1-a)]\}$.

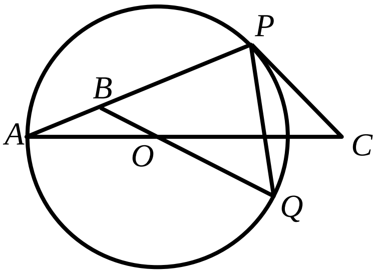
b =

(iii) 設 $c=\frac{1}{b^2-1}+\frac{1}{(2b)^2-1}+\frac{1}{(3b)^2-1}+\cdots+\frac{1}{(10b)^2-1}$，求 c 的值。
提示：$\frac{1}{x^2-1}=\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)$
Let $c=\frac{1}{b^2-1}+\frac{1}{(2b)^2-1}+\frac{1}{(3b)^2-1}+\cdots+\frac{1}{(10b)^2-1}$, find the value of c.
Hint: $\frac{1}{x^2-1}=\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)$

c =

(iv) 在下圖中，PC 是圓(圓心為 O)的切綫，切點在 P。ΔABO 是等腰三角形，$AB = OB$，$\angle PCO = c$ 及 $d = \angle QPC$，其中 c、d 為弧度。求 d 的值。(取$\pi=\frac{22}{7}$)
In the following diagram, PC is a tangent to the circle (centre O) at the point P, and ΔABO is an isosceles triangle, $AB = OB$, $\angle PCO = c$ and $d = \angle QPC$, where c and d are radian measures. Find the value of d. (Take $\pi=\frac{22}{7}$)

d =



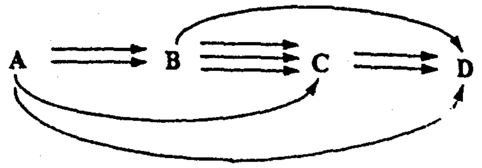
FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1995-96)
Spare Event (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 下圖中，由 A 到 D 共有 a 條路徑，求 a 的值。
From the following figure, determine the number of routes a from A to D.



a =

(ii) 若 $\sin(2b° + 2a°) = \cos(6b° - 16°)$，其中 $0 < b < 90$，求 b 的值。
If $\sin(2b° + 2a°) = \cos(6b° - 16°)$, where $0 < b < 90$, find the value of b.

b =

(iii) 直綫 $(bx - 6y + 3) + k(x - y + 1) = 0$ 經過 $P(c, m)$，其中 k 是任何實數，求 c 的值。
The lines $(bx - 6y + 3) + k(x - y + 1) = 0$, where k is any real constant, pass through a fixed point $P(c, m)$, find the value of c.

c =

(iv) 已知 $d^2 - c = 257\times259$。求 d 的正值。
It is known that $d^2 - c = 257\times259$. Find the positive value of d.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (1995-96)
Event 6 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 一籃子雞蛋的數目為 a，分三輪派發。第一輪派出一半另半枚，第二輪派出剩下的一半另半枚，第三輪又派出剩下的一半另半枚。籃子中的雞蛋便全部派光，求 a 的值。
The number of eggs in a basket was a. Eggs were given out in three rounds. In the first round half of egg plus half an egg were given out. In the second round, half of the remaining eggs plus half an egg were given out. In the third round, again, half of the remaining eggs plus half an egg were given out. The basket then became empty.
Find the value of a.

a =

(ii) 若 $p - q = 2; p - r = 1$ 及 $b = (r - q)[(p - q)^2 + (p - q)(p - r) + (p - r)^2]$，求 b 的值。
If $p - q = 2; p - r = 1$ and $b = (r - q)[(p - q)^2 + (p - q)(p - r) + (p - r)^2]$.
Find the value of b.

b =

(iii) 若 n 是一正整數，$m^{2n} = 2$ 及 $c = 2m^{6n} - 4$，求 c 的值。
If n is a positive integer, $m^{2n} = 2$ and $c = 2m^{6n} - 4$, find the value of c.

c =

(iv) 若 r, s, t, u 是正整數及 $r^5 = s^4, t^3 = u^2, t - r = 19$ 及 $d = u - s$，求 d 的值。
If r, s, t, u are positive integers and $r^5 = s^4, t^3 = u^2, t - r = 19$ and $d = u - s$, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Compiled by Mr. SAROEUN Minea

Hong Kong Mathematics Olympiad (1995-96)
Event 7 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若方程 $ax^2 - mx + 1996 = 0$ 的兩個不等根是質數，求 a 的值。
If the two distinct roots of the equation $ax^2 - mx + 1996 = 0$ are primes, find the value of a.

$a =$

(ii) 六位數 $111aaa$ 是兩個連續正整數 b 和 $b + 1$ 之積，求 b 的值。
A six-digit figure $111aaa$ is the product of two consecutive positive integers b and $b + 1$, find the value of b.

$b =$

(iii) 若 p, q, r 是非零實數，$p^2 + q^2 + r^2 = 1$，$p\left(\frac{1}{q}+\frac{1}{r}\right)+q\left(\frac{1}{r}+\frac{1}{p}\right)+r\left(\frac{1}{p}+\frac{1}{q}\right)+3=0$，及 $c = p + q + r$，求 c 的最大值。
If p, q, r are non-zero real numbers;
$p^2 + q^2 + r^2 = 1$, $p\left(\frac{1}{q}+\frac{1}{r}\right)+q\left(\frac{1}{r}+\frac{1}{p}\right)+r\left(\frac{1}{p}+\frac{1}{q}\right)+3=0$ and $c = p + q + r$,
find the largest value of c.

$c =$

(iv) 若 7^{14} 之個位是 d，求 d 的值。
If the units digit of 7^{14} is d, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

\+ Bonus score [ ]   Time [ ] [ ]

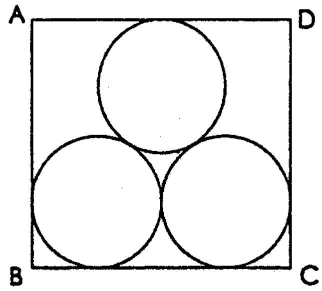
Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (1995-96)
Event 8 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

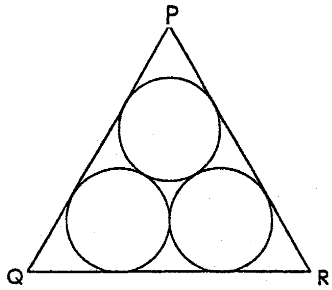
In this question, all unnamed circles are unit circles.
在本題內，所有不命名的圓皆是單位圓。

(i) 若矩形 *ABCD* 的面積是 $a+4\sqrt{3}$，求 a 的值。
If the area of the rectangle *ABCD* is $a+4\sqrt{3}$, find the value of a.



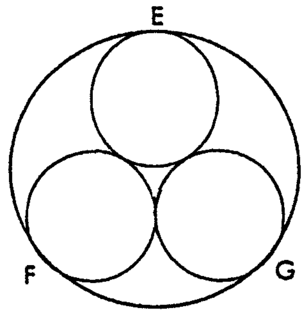
$a =$

(ii) 若等邊三角形 *PQR* 的面積是 $6+b\sqrt{3}$，求 b 的值。
If the area of the equilateral triangle *PQR* is $6+b\sqrt{3}$, find the value of b.



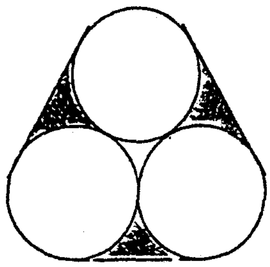
$b =$

(iii) 若圓 *EFG* 的面積是 $\frac{(c+4\sqrt{3})\pi}{3}$，求 c 的值。
If the area of the circle *EFG* is $\frac{(c+4\sqrt{3})\pi}{3}$, find the value of c.



$c =$

(iv) 若下圖所有直綫皆是兩個圓的公切綫，且陰影部份的面積是 $6+d\pi$，求 d 的值。
If all the straight lines in the diagram below are common tangents to the two circles, and the area of the shaded part is $6+d\pi$, find the value of d.



$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1995-96)
Event 9 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $(1995)^a + (1996)^a + (1997)^a$ 能被 10 整除，求 a 的最小可能整數值。
If $(1995)^a + (1996)^a + (1997)^a$ is divisible by 10,
find the least possible integral value of a .

$a =$

(ii) 若 $(x^2 + y^2)^2 \le b(x^4 + y^4)$ 對任意實數 x 和 y 都成立，求 b 的最小可能整數值。
If the expression $(x^2 + y^2)^2 \le b(x^4 + y^4)$ holds for all real values of x and y,
find the least possible integral value of b .

$b =$

(iii) 若 $c = 1996\times19971997 - 1995\times19961996$，求 c 的值。
If $c = 1996\times19971997 - 1995\times19961996$, find the value of c .

$c =$

(iv) 若
$$d=\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{60}\right)+\left(\frac{2}{3}+\frac{2}{4}+\frac{2}{5}+\cdots+\frac{2}{60}\right)+\left(\frac{3}{4}+\frac{3}{5}+\cdots+\frac{3}{60}\right)+\cdots+\left(\frac{58}{59}+\frac{58}{60}\right)+\frac{59}{60}$$，
求 d 的值。
Find the sum d where
$$d=\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{60}\right)+\left(\frac{2}{3}+\frac{2}{4}+\frac{2}{5}+\cdots+\frac{2}{60}\right)+\left(\frac{3}{4}+\frac{3}{5}+\cdots+\frac{3}{60}\right)+\cdots+\left(\frac{58}{59}+\frac{58}{60}\right)+\frac{59}{60}.$$

$d =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (1995-96)
Event 10 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 已知 $3\times4\times5\times6 = 19^2 - 1$
$4\times5\times6\times7 = 29^2 - 1$
$5\times6\times7\times8 = 41^2 - 1$
$6\times7\times8\times9 = 55^2 - 1$
若 $a^2 = 1000\times1001\times1002\times1003 + 1$，求 a 的值。

It is given that $3\times4\times5\times6 = 19^2 - 1$
$4\times5\times6\times7 = 29^2 - 1$
$5\times6\times7\times8 = 41^2 - 1$
$6\times7\times8\times9 = 55^2 - 1$
If $a^2 = 1000\times1001\times1002\times1003 + 1$, find the value of a.

$a =$

(ii) 設 $f(x) = x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$。
當 $f(x^{10})$ 除以 $f(x)$，餘數是 b。求 b 的值。
Let $f(x) = x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$.
When $f(x^{10})$ is divided by $f(x)$, the remainder is b. Find the value of b.

$b =$

(iii) 分數 $\frac{p}{q}$ 已化成最簡形式。若 $\frac{7}{10} < \frac{p}{q} < \frac{11}{15}$，當中 q 是最小可能正整數，且 $c = pq$，求 c 的值。
The fraction $\frac{p}{q}$ is in its simplest form. If $\frac{7}{10} < \frac{p}{q} < \frac{11}{15}$ where q is the smallest possible positive integer and $c = pq$. Find the value of c.

$c =$

(iv) 若正整數 d 除以 7，餘數是 1；除以 5 餘數是 2；除以 3 餘數是 2。求 d 的最小可能值。
A positive integer d when divided by 7 will have 1 as its remainder; when divided by 5 will have 2 as its remainder and when divided by 3 will have 2 as its remainder. Find the least possible value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	Min. / Sec.
			Total score					

Hong Kong Mathematics Olympiad (1996-97)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 已知$\frac{3}{a}+\frac{1}{u}=\frac{7}{2}$及$\frac{2}{a}-\frac{3}{u}=6$為 a 與 u 的聯立方程。求 a 的解。

Given that $\frac{3}{a}+\frac{1}{u}=\frac{7}{2}$ and $\frac{2}{a}-\frac{3}{u}=6$ are simultaneous equations in a and u. Solve for a.

a =

(ii) 方程 $px + qy + bz = 1$ 的根分別為 $(0, 3a, 1)$、$(9a, -1, 2)$ 和 $(0, 3a, 0)$。求係數 b 的值。

Three solutions of the equation $px + qy + bz = 1$ are $(0, 3a, 1)$, $(9a, -1, 2)$ and $(0, 3a, 0)$. Find the value of the coefficient b.

b =

(iii) 若 $y = mx + c$ 的圖像經過 $(b + 4, 5)$ 及 $(-2, 2)$ 兩點。求 c 的值。

Find the value of c so that the graph of $y = mx + c$ passes through the two points $(b + 4, 5)$ and $(-2, 2)$.

c =

(iv) 不等式 $x^2 + 5x - 2c \le 0$ 的解為 $d \le x \le 1$。求 d 的值。

The solution of the inequality $x^2 + 5x - 2c \le 0$ is $d \le x \le 1$. Find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

\+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min. Sec.

Hong Kong Mathematics Olympiad (1996-97)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 考慮：$\frac{1^2}{1}=1$，$\frac{1^2+2^2}{1+2}=\frac{5}{3}$，$\frac{1^2+2^2+3^2}{1+2+3}=\frac{7}{3}$，$\frac{1^2+2^2+3^2+4^2}{1+2+3+4}=3$，

求 a 的值使得 $\frac{1^2+2^2+\cdots+a^2}{1+2+\cdots+a}=\frac{25}{3}$。

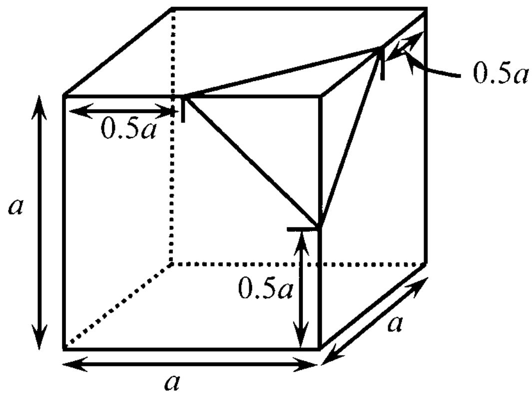
By considering: $\frac{1^2}{1}=1, \frac{1^2+2^2}{1+2}=\frac{5}{3}, \frac{1^2+2^2+3^2}{1+2+3}=\frac{7}{3}, \frac{1^2+2^2+3^2+4^2}{1+2+3+4}=3$,

find the value of a such that $\frac{1^2+2^2+\cdots+a^2}{1+2+\cdots+a}=\frac{25}{3}$.

a =

(ii) 如圖所示，從邊長為 a cm 的正立方體的一角割出一個三角錐體。
若三角錐體的體積為 b cm^3，求 b 的值。
A triangular pyramid is cut from a corner of a cube with side length a cm as the figure shown. If the volume of the pyramid is b cm^3, find the value of b.

b =



(iii) 若對於所有實數 x，x^2+cx+b 不小於 0，求 c 的最大值。
If the value of x^2+cx+b is not less than 0 for all real number x, find the maximum value of c.

c =

(iv) 若 1997^{1997} 的個位數為 $c-d$，求 d 的值。
If the units digit of 1997^{1997} is $c-d$, find d.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

\+ Bonus score [ ]   Time [ ] Min. [ ] Sec.

Total score [ ]

Hong Kong Mathematics Olympiad (1996-97)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) a、b、c 和 d 的平均值為 8。若 a、b、c、d 和 P 的平均值為 P，求 P 的值。
The average of a, b c and d is 8. If the average of a, b, c, d and P is P, find the value of P.

P =

(ii) 若直綫 $2x + 3y + 2 = 0$ 和 $Px + Qy + 3 = 0$ 互相平行，求 Q 的值。
If the lines $2x + 3y + 2 = 0$ and $Px + Qy + 3 = 0$ are parallel, find the value of Q.

Q =

(iii) 若等邊三角形的周界和面積分別為 Q cm 和 $\sqrt{3}R\,\text{cm}^2$。求 R 的值。
The perimeter and the area of an equilateral triangle are Q cm and $\sqrt{3}R\,\text{cm}^2$ respectively. Find the value of R.

R =

(iv) 若 $(1 + 2 + \ldots + R)^2 = 1^2 + 2^2 + \ldots + R^2 + S$，求 S 的值。
If $(1 + 2 + \ldots + R)^2 = 1^2 + 2^2 + \ldots + R^2 + S$, find the value of S.

S =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

\+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (1996-97)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若正 n 邊形的內角為 140°，求 n 的值。
If each interior angle of a n-sided regular polygon is 140°, find the value of n.

n =

(ii) 若不等式 $2x^2 - nx + 9 < 0$ 的解為 $k < x < b$，求 b 的值。
If the solution of the inequality $2x^2 - nx + 9 < 0$ is $k < x < b$, find the value of b.

b =

(iii) 若 $cx^3 - bx + x - 1$ 除以 $x + 1$，餘數為 –7，求 c 的值。
If $cx^3 - bx + x - 1$ is divided by $x + 1$, the remainder is –7, find the value of c.

c =

(iv) 若 $x+\frac{1}{x}=c$ 和 $x^2+\frac{1}{x^2}=d$，求 d 的值。
If $x+\frac{1}{x}=c$ and $x^2+\frac{1}{x^2}=d$, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1996-97)
Final Event 5 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 一直徑為 a 的半球體的體積為 18π cm^3，求 a 的值。
The volume of a hemisphere with diameter a cm is 18π cm^3, find the value of a.

a =

(ii) 若 $\sin 10a° = \cos(360° - b°)$ 和 $0 < b < 90$，求 b 的值。
If $\sin 10a° = \cos(360° - b°)$ and $0 < b < 90$, find the value of b.

b =

(iii) 一三角形是由 x-軸、y-軸和直綫 $bx + 2by = 120$ 所組成。
若所包圍之三角形的面積為 c，求 c 的值。
The triangle is formed by the x-axis and y-axis and the line $bx + 2by = 120$.
If the bounded area of the triangle is c, find the value of c.

c =

(iv) 若方程式 $x^2 - (c + 2)x + (c + 1) = 0$ 兩根之差為 d，求 d 的值。
If the difference of the two roots of the equation $x^2 - (c + 2)x + (c + 1) = 0$ is d, find the value of d.

d =

FOR OFFICIAL USE

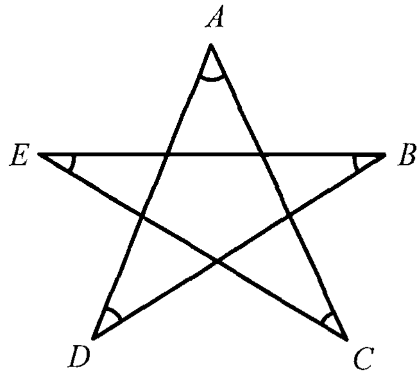
Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1996-97)
Final Event 1 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 圖中，$\angle A + \angle B + \angle C + \angle D + \angle E = a°$。求 a 的值。
In the diagram, $\angle A + \angle B + \angle C + \angle D + \angle E = a°$. Find the value of a.

a =



(ii) 代數式 $x^6 + x^6 + x^6 + \ldots + x^6$ 有 x 項及其總和為 x^b。求 b 的值。
There are x terms in the algebraic expression $x^6 + x^6 + x^6 + \ldots + x^6$ and its sum is x^b. Find the value of b.

b =

(iii) 若 $1 + 3 + 3^2 + 3^3 + \ldots + 3^8 = \frac{3^c - 1}{2}$，求 c 的值。

If $1 + 3 + 3^2 + 3^3 + \ldots + 3^8 = \frac{3^c - 1}{2}$, find the value of c.

c =

(iv) 從 16 張寫上 1 至 16 的咭紙中隨意抽出一張，若果抽出的號碼是一個完全平方數的概率為 $\frac{1}{d}$，求 d 之值。
16 cards are marked from 1 to 16 and one is drawn at random.
If the chance of it being a perfect square number is $\frac{1}{d}$, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

\+ Bonus score [ ]   Time [ ] Min. [ ] Sec.

Total score [ ]

Hong Kong Mathematics Olympiad (1996-97)
Final Event 2 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若數列 1、6 + 2*a*、10 + 5*a*、⋯ 是一算術級數，求 *a* 的值。

If the sequence 1, 6 + 2*a*, 10 + 5*a*, ⋯ forms an A.P., find the value of a.

$a =$

(ii) 若 $(0.0025\times40)^b=\frac{1}{100}$，求 *b* 的值。

If $(0.0025\times40)^b=\frac{1}{100}$, find the vale of b.

$b =$

(iii) 若 *c* 為正整數及 $c^3+3c+\frac{3}{c}+\frac{1}{c^3}=8$，求 *c* 的值。

If *c* is an integer and $c^3+3c+\frac{3}{c}+\frac{1}{c^3}=8$, find the value of c.

$c =$

(iv) 若將 5 個女孩排成一列，共有 *d* 個不同方法。求 *d* 的值。
There are *d* different ways for arranging 5 girls in a row. Find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ] Team No. [ ]

+ Bonus score [ ] Time [ ] [ ]

Total score [ ] Min. Sec.

Hong Kong Mathematics Olympiad (1996-97)
Final Event 3 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 設 m 為滿足不等式 $14x - 7(3x - 8) < 4(25 + x)$的整數。求 m 的最小值。
Let m be an integer satisfying the inequality: $14x - 7(3x - 8) < 4(25 + x)$.
Find the least value of m.

m =

(ii) 已知 $f(x) = \frac{1}{3}x^3 - 2x^2 + \frac{2}{3}x^3 + 3x^2 + 5x + 7 - 4x$。若 f(–2) = b，求 b 的值。
It is given that $f(x) = \frac{1}{3}x^3 - 2x^2 + \frac{2}{3}x^3 + 3x^2 + 5x + 7 - 4x$.
If f(–2) = b, find the value of b.

b =

(iii) 已知 $\log\frac{x}{2} = 0.5$ 及 $\log\frac{y}{5} = 0.1$。若 $\log xy = c$，求 c 的值。
It is given that $\log\frac{x}{2} = 0.5$ and $\log\frac{y}{5} = 0.1$. If $\log xy = c$, find the value of c.

c =

(iv) d、e 及 f 為三個小於 10 之質數且滿足兩個條件 $d + e = f$ 及 $d < e$。求 d 的值。
Three prime numbers d, e and f which are all less than 10, satisfy the two conditions $d + e = f$ and $d < e$. Find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

\+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (1996-97)
Final Event 4 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 已知 $a = 103\times97\times10009$，求 a 的值。
It is given that $a = 103\times97\times10009$, find the value of a.

$a =$

(ii) 已知 $1 + x + x^2 + x^3 + x^4 = 0$。若 $b = 2 + x + x^2 + x^3 + x^4 + \ldots + x^{1989}$，求 b 的值。
It is given that $1 + x + x^2 + x^3 + x^4 = 0$.
If $b = 2 + x + x^2 + x^3 + x^4 + \ldots + x^{1989}$, find the value of b.

$b =$

(iii) 已知 m 及 n 為兩個不大於 10 的自然數。
若 c 為 m 及 n 滿足方程 $mx = n$ 之組數，其中 $\frac{1}{4} < x < \frac{1}{3}$。求 c 的值。
It is given that m and n are two natural numbers and both are not greater than 10. If c is the number of pairs of m and n satisfying the equation $mx = n$, where $\frac{1}{4} < x < \frac{1}{3}$, find the value of c.

$c =$

(iv) 設 x 及 y 為實數且定義運算*為 $x*y = px^y + q + 1$。已知 $1*2 = 869$ 及 $2*3 = 883$。若 $2*9 = d$，求 d 的值。
Let x and y be real numbers and define the operation * as $x*y = px^y + q + 1$.
It is given that $1*2 = 869$ and $2*3 = 883$. If $2*9 = d$, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score			Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1996-97)
Final Event 5 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 a 是 5 的正倍數，且被 3 除時餘 1，求 a 之最小可能數值。
If a is a positive multiple of 5, which gives remainder 1 when divided by 3, find the smallest possible value of a.

a =

(ii) 若 $x^3 + 6x^2 + 12x + 17 \equiv (x + 2)^3 + b$，求 b 的值。
If $x^3 + 6x^2 + 12x + 17 \equiv (x + 2)^3 + b$, find the value of b.

b =

(iii) 若 c 是一兩位正整數，其兩位之和是 10 而兩位之積是 25。求 c 的值。
If c is a 2 digit positive integer such that sum of its digits is 10 and product of its digit is 25, find the value of c.

c =

(iv) 設 S_1、S_2、…、S_{10} 是一個由正整數組成的 A.P.之首 10 項。
若 $S_1 + S_2 + \ldots + S_{10} = 55$ 及 $(S_{10} - S_8) + (S_9 - S_7) + \ldots + (S_3 - S_1) = d$。求 d 的值。
Let $S_1, S_2, \ldots, S_{10}$ be the first ten terms of an A.P., which consists of positive integers.
If $S_1 + S_2 + \ldots + S_{10} = 55$ and $(S_{10} - S_8) + (S_9 - S_7) + \ldots + (S_3 - S_1) = d$, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]　　Team No. [ ]

\+ Bonus score [ ]　　Time [ ] [ ]

Total score [ ]　　Min.　Sec.

Hong Kong Mathematics Olympiad (1996-97)
Final Event (Spare Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) E 是平行四邊形 $ABCD$ 其中一條邊 CD 的中點。若三角形 ADE 與平行四邊形 $ABCD$ 面積的比等於 $1 : a$，求 a 的值。
$ABCD$ is a parallelogram and E is the midpoint of CD. If the ratio of the area of the triangle ADE to the area of the parallelogram $ABCD$ is $1 : a$, find the value of a.

a =

(ii) E 是平行四邊形 $ABCD$ 其中一條邊 CD 的中點，且 AE 和 BD 相交於 M；若 $DM : MB = 1 : k$，求 k 的值。
$ABCD$ is a parallelogram and E is the midpoint of CD. AE and BD meet at M. If $DM : MB = 1 : k$, find the value of k.

k =

(iii) 若 5 的平方根是 2.236，以同一準確度，80 的平方根是 d。求 d 的值。
If the square root of 5 is approximately 2.236, the square root of 80 with the same precision is d. Find the value of d

d =

(iv) 將一個正方形的長增加 20%，同時又將它的闊減少 20%，則我們可得一個長方形。若長方形與正方形面積的比為 $1 : r$，求 r 的值。
A square is changed into a rectangle by increasing its length by 20% and decreasing its width by 20%. If the ratio of the area of the rectangle to the area of the square is $1 : r$, find the value of r .

r =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1997-98)
Sample Event (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 已知 $\frac{3}{a}+\frac{1}{u}=\frac{7}{2}$ 及 $\frac{2}{a}-\frac{3}{u}=6$ 為 a 與 u 的聯立方程。求 a 的解。

Given that $\frac{3}{a}+\frac{1}{u}=\frac{7}{2}$ and $\frac{2}{a}-\frac{3}{u}=6$ are simultaneous equations in a and u. Solve for a.

a =

(ii) 方程 $px + qy + bz = 1$ 的根分別為(0, 3a, 1)、(9a, –1, 2)和(0, 3a, 0)。求係數 b 的值。

Three solutions of the equation $px + qy + bz = 1$ are (0, 3a, 1), (9a, –1, 2) and (0, 3a, 0). Find the value of the coefficient b.

b =

(iii) 若 $y = mx + c$ 的圖像經過 (b + 4, 5) 及 (–2, 2) 兩點。求 c 的值。

Find the value of c so that the graph of $y = mx + c$ passes through the two points (b + 4, 5) and (–2, 2).

c =

(iv) 不等式 $x^2 + 5x - 2c \le 0$ 的解為 $d \le x \le 1$。求 d 的值。

The solution of the inequality $x^2 + 5x - 2c \le 0$ is $d \le x \le 1$. Find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

\+ Bonus score [ ]

Total score [ ]

Team No. [ ]

Time [ ] Min. [ ] Sec.

Hong Kong Mathematics Olympiad (1997-98)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 a 是 $\frac{1}{2}\sin^2 3\theta - \frac{1}{2}\cos 2\theta$ 的最大值，求 a 的值。

If a is the maximum value of $\frac{1}{2}\sin^2 3\theta - \frac{1}{2}\cos 2\theta$, find the value of a.

a =

(ii) 若 $\begin{cases} x+y=2 \\ xy-z^2=a \\ b=x+y+z \end{cases}$，求 b 的值。

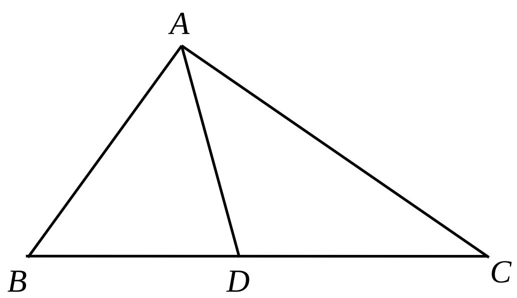
If $\begin{cases} x+y=2 \\ xy-z^2=a \\ b=x+y+z \end{cases}$, find the value of b.

b =

(iii) 在圖中，$BD = b$ cm，$DC = c$ cm，且 ΔABD 的面積 $= \frac{1}{3} \times \Delta ABC$ 的面積，求 c 的值。

In the figure, $BD = b$ cm, $DC = c$ cm and area of $\Delta ABD = \frac{1}{3} \times$ area of ΔABC, find the value of c.

c =



(iv) 設 d 為 $500 + c$ 的正因數的數目，求 d 的值。
Suppose d is the number of positive factors of $500 + c$, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (1997-98)
Final Event 2 (Individual)

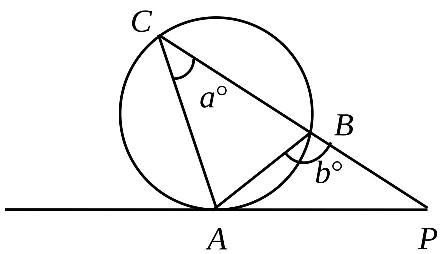
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $A(1, 3)$、$B(5, 8)$及 $C(29, a)$共綫，求 a 的值。
If $A(1, 3)$, $B(5, 8)$ and $C(29, a)$ are collinear, find the value of a.

$a =$

(ii) 在圖中，PA 切圓 ABC 於 A。PBC 為一直綫、$AB = BP$、$\angle ACB = a°$。若 $\angle ABP = b°$，求 b 的值。
In the figure, PA touches the circle ABC at A, PBC is a straight line, $AB = PB$, $\angle ACB = a°$. If $\angle ABP = b°$, find the value of b .

$b =$



(iii) 若 c 為二次函數 $y = x^2 + 4x + b$ 之最小值，求 c 的值。
If c is the minimum value of the quadratic function $y = x^2 + 4x + b$, find the value of c.

$c =$

(iv) 若 $d = 1 - 2 + 3 - 4 + \ldots - c$，求 d 的值。
If $d = 1 - 2 + 3 - 4 + \ldots - c$, find the value of d .

$d =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ] Team No. [ ]

\+ Bonus score [ ] Time [ ] [ ]

Total score [ ] Min. Sec.

Hong Kong Mathematics Olympiad (1997-98)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $\{p, q\} = q \times a + p$ 且 $\{2, 5\} = 52$，求 a 的值。
If $\{p, q\} = q \times a + p$ and $\{2, 5\} = 52$, find the value of a.

$a =$

(ii) 若數列 a，$\frac{37}{2}$，b 為一等差數列，求 b 的值。
If $a, \frac{37}{2}, b$ is an arithmetic progression, find the value of b.

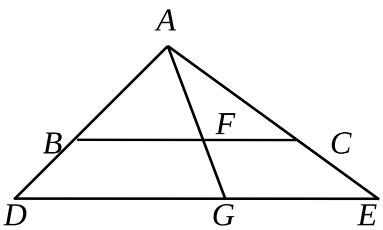
$b =$

(iii) 若 $b^2 - c^2 = 200$ 及 $c > 0$，求 c 的值。
If $b^2 - c^2 = 200$ and $c > 0$, find the value of c.

$c =$

(iv) 在圖中，已知 $BC \parallel DE$、$BC : DE = 10 : c$ 及 $AF : FG = 20 : d$，求 d 的值。
Given that in the figure, $BC \parallel DE$, $BC : DE = 10 : c$ and $AF : FG = 20 : d$, find the value of d.

$d =$



FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1997-98)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 已知 $\frac{10x-3y}{x+2y}=2$ 且 $p=\frac{y+x}{y-x}$，求 p 的值。
Given that $\frac{10x-3y}{x+2y}=2$ and $p=\frac{y+x}{y-x}$, find the value of p.

$p =$

(ii) 已知 $a \neq b$ 且 $ax = bx$。若 $p + q = 19(a - b)^x$，求 q 的值。
Given that $a \neq b$ and $ax = bx$. If $p + q = 19(a - b)^x$, find the value of q.

$q =$

(iii) 已知 q 個連續數之和為 222，其中最大的是 r，求 r 的數值。
Given that the sum of q consecutive numbers is 222, and the largest of these consecutive numbers is r, find the value of r.

$r =$

(iv) 若 $\tan^2(r + s)° = 3$ 且 $0 \leq r + s \leq 90$，求 s 的值。
If $\tan^2(r + s)° = 3$ and $0 \leq r + s \leq 90$, find the value of s.

$s =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

\+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (1997-98)
Final Event 5 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若方程 $5x^2 + ax - 2 = 0$ 的根的和為它的根的積的兩倍，求 a 的值。
If the sum of roots of $5x^2 + ax - 2 = 0$ is twice the product of its roots, find the value of a.

a =

(ii) 已知 $y = ax^2 - bx - 13$ 穿過(3, 8)，求 b 的值。
Given that $y = ax^2 - bx - 13$ passes through (3, 8), find the value of b.

b =

(iii) 若有 c 種排法把 b 位女孩排成一圓，求 c 的值。
If there are c ways of arranging b girls in a circle, find the value of c.

c =

(iv) 若 $\frac{c}{4}$ 條直綫和 3 個圓畫於一白紙上，且它們的最多交點數量為 d，求 d 的值。
If $\frac{c}{4}$ straight lines and 3 circles are drawn on a paper, and d is the largest numbers of points of intersection, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1997-98)
Sample Event (Group)

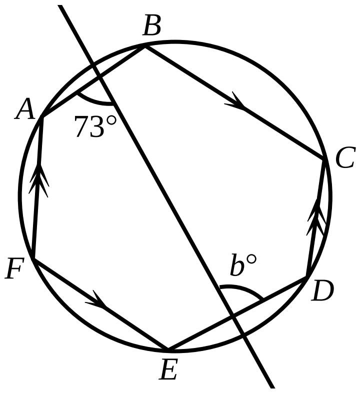
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 a 是最小的正整數被 3 除時餘 1 而能被 5 整除，求 a 的值。
If a is the smallest positive integer which gives remainder 1 when divided by 3 and is a multiple of 5, find the value of a.

a =

(ii) 下圖中，$FA//DC$ 及 $FE//BC$。求 b 的值。
In the following diagram, $FA//DC$ and $FE//BC$. Find the value of b.

b =



(iii) 若 c 是一兩位正整數，其兩位之和是 10 而兩位之積是 25，求 c 的值。
If c is a 2 digit positive integer such that sum of its digits is 10 and product of its digit is 25, find the value of c.

c =

(iv) 若 $S_1, S_2, \ldots, S_{10}$ 是一由正整數組成的 A.P.的頭十項使得
$S_1 + S_2 + \ldots + S_{10} = 55$ 及$(S_{10} - S_8) + (S_9 - S_7) + \ldots + (S_3 - S_1) = d$。求 d 的值。
If $S_1, S_2, \ldots, S_{10}$ are the first ten terms of an A.P. consisting of positive integers such that $S_1 + S_2 + \ldots + S_{10} = 55$ and $(S_{10} - S_8) + (S_9 - S_7) + \ldots + (S_3 - S_1) = d$, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

Team No. [ ]

\+ Bonus score [ ]

Time [ ] Min. [ ] Sec.

Total score [ ]

Hong Kong Mathematics Olympiad (1997-98)
Final Event 1 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若扇形面積 $s = 4\text{ cm}^2$、扇形半徑 $r = 2$ cm 及扇形的弧長 $A = p$ cm，求 p 的值。
If the area of a given sector $s = 4\text{ cm}^2$, the radius of this sector $r = 2$ cm and the arc length of this sector $A = p$ cm, find the value of p.

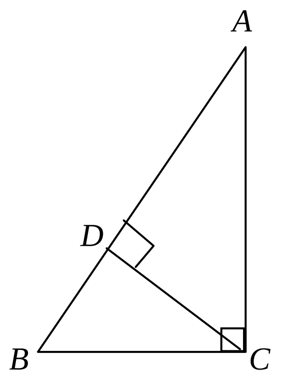
$p =$

(ii) 已知 $\frac{a}{2b+c} = \frac{b}{2c+a} = \frac{c}{2a+b}$ 且 $a + b + c \neq 0$。若 $q = \frac{2b+c}{a}$，求 q 的值。
Given that $\frac{a}{2b+c} = \frac{b}{2c+a} = \frac{c}{2a+b}$ and $a + b + c \neq 0$.
If $q = \frac{2b+c}{a}$, find the value of q.

$q =$

(iii) 設直角三角形 ABC 中，CD 是斜邊 AB 上的高，$AC = 3$, $DB = \frac{5}{2}$, $AD = r$，求 r 的值。
Let ABC be a right-angled triangle, CD is the altitude on AB, $AC = 3$, $DB = \frac{5}{2}$, $AD = r$, find the value of r.

$r =$



(iv) 若 $x^3 + px^2 + qx + 17 \equiv (x + 2)^3 + a$，求 a 的值。
If $x^3 + px^2 + qx + 17 \equiv (x + 2)^3 + a$, find the value of a.

$a =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1997-98)
Final Event 2 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $\frac{137}{a}=0.1\dot{2}3\dot{4}$，求 a 的值。
If $\frac{137}{a}=0.1\dot{2}3\dot{4}$, find the value of a.

a =

(ii) 若 b = 1999×19981998 – 1998×19991999 + 1，求 b 的數值。
If b = 1999×19981998 – 1998×19991999 + 1, find the value of b.

b =

(iii) 若參數方程 $\begin{cases} x=\sqrt{3-t^2} \\ y=t-3 \end{cases}$ 可轉換為 $x^2+y^2+cx+dy+6=0$，求 c 及 d 的值。
If the parametric equation $\begin{cases} x=\sqrt{3-t^2} \\ y=t-3 \end{cases}$ can be transformed into $x^2+y^2+cx+dy+6=0$, find the values of c and d.

c =

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] Min. [ ] Sec.

Total score [ ]

Hong Kong Mathematics Olympiad (1997-98)
Final Event 3 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 在ΔABC 中，$\angle ABC = 2\angle ACB$，$BC = 2AB$。若$\angle BAC = a^\circ$，求 a 的值。
In ΔABC, $\angle ABC = 2\angle ACB$, $BC = 2AB$. If $\angle BAC = a^\circ$, find the value of a.

$a =$

(ii) 已知 $x+\frac{1}{x}=\sqrt{2}$，$\frac{x^2}{x^4+x^2+1}=b$，求 b 的值。
Given that $x+\frac{1}{x}=\sqrt{2}$, $\frac{x^2}{x^4+x^2+1}=b$, find the value of b.

$b =$

(iii) 若方程 $x + y + 2xy = 141$ 有 c 個正整數解，求 c 的值。
If the number of positive integral root(s) of the equation $x + y + 2xy = 141$ is c, find the value of c.

$c =$

(iv) 已知 $x + y + z = 0$、$x^2 + y^2 + z^2 = 1$ 及 $d = 2(x^4 + y^4 + z^4)$，求 d 的值。
Given that $x + y + z = 0$, $x^2 + y^2 + z^2 = 1$ and $d = 2(x^4 + y^4 + z^4)$, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

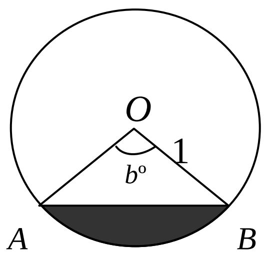
Hong Kong Mathematics Olympiad (1997-98)
Final Event 4 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $0.\dot{1}+0.0\dot{2}+0.00\dot{3}+\ldots+0.00000000\dot{9}=a$，求 a 的值。(答案以小數表示。)
If $0.\dot{1}+0.0\dot{2}+0.00\dot{3}+\ldots+0.00000000\dot{9}=a$, find the value of a.
(Give your answer in decimal)

a =

(ii) 圖中的圓之圓心為 O，半徑為 1，A 和 B 是圓形上的點。
已知 $\dfrac{\text{陰影部分}}{\text{沒有陰影部分}}=\dfrac{\pi-2}{3\pi+2}$ 且 $\angle AOB=b^{\circ}$，求 b 的值。
The circle in the figure has centre O and radius 1, A and B are points on the circle.
Given that $\dfrac{\text{Area of shaded part}}{\text{Area of unshaded part}}=\dfrac{\pi-2}{3\pi+2}$ and $\angle AOB=b^{\circ}$, find the value of b.



b =

(iii) 圖形 S_0，S_1，S_2，…用以下方法構成：把綫段[0, 1]的中間三分之一取去，得到 S_0，把 S_0 的兩條組成綫段，每段的中間三分之一取去，得到 S_1，把 S_1 的四條組成綫段，每段的中間三分之一取去，得到 S_2，S_3、S_4 …等用類似方法獲得。求在構成 S_5 的過程中取去的綫段的總長度 c (答案以分數表示)。
A sequence of figures S_0, S_1, S_2, … are constructed as follows. S_0 is obtained by removing the middle third of [0,1] interval; S_1 by removing the middle third of each of the two intervals in S_0; S_2 by removing the middle third of each of the four intervals in S_1; S_3, S_4, … are obtained similarly. Find the total length c of the intervals removed in the construction of S_5 (Give your answer in fraction).

|————————(   )————————| S_0
0   $\frac{1}{3}$ $\frac{2}{3}$ 1

|——(   )——(   )——(   )——| S_1
0   $\frac{1}{9}$ $\frac{2}{9}$ $\frac{1}{3}$ $\frac{2}{3}$ $\frac{7}{9}$ $\frac{8}{9}$ 1

|–( )–(   )–( )–(   )–( )–(   )–( )–| S_2

c =

(iv) 把所有整數用下表的方法編碼。若編碼 101 至 200 的所有整數之和為 d，求 d 的值。
All integers are coded as shown in the following table. If the sum of all integers coded from 101 to 200 is d, find the value of d.

整數 Integer	…	…	–3	–2	–1	0	1	2	3	…	…
編碼 Code	…	…	7	5	3	1	2	4	6	…	…

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (1997-98)
Final Event 5 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $1\times2\times3 + 2\times3\times4 + 3\times4\times5 + \ldots + 10\times11\times12 = a$，求 a 的值。
If $1\times2\times3 + 2\times3\times4 + 3\times4\times5 + \ldots + 10\times11\times12 = a$, find the value of a.

a =

(ii) 已知 $5^x + 5^{-x} = 3$。若 $5^{3x} + 5^{-3x} = b$，求 b 的值。
Given that $5^x + 5^{-x} = 3$. If $5^{3x} + 5^{-3x} = b$, find the value of b.

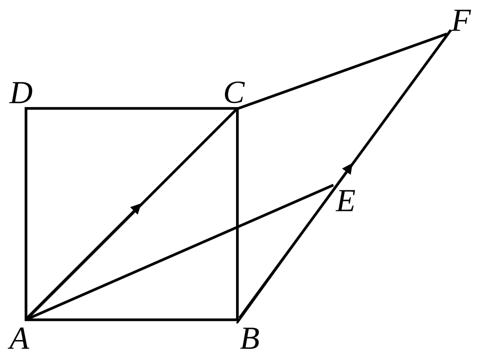
b =

(iii) 已知二次方程 $x^2 + mx + n = 0$ 的根為 98 和 99，且 $y = x^2 + mx + n$。
若 x 取 0、1、2、…、100，則有 c 個 y 的數值能被 6 整除。求 c 的值。
Given that the roots of equation $x^2 + mx + n = 0$ are 98 and 99 and $y = x^2 + mx + n$.
If x takes on the values of 0, 1, 2, …, 100, then there are c values of y that can be divisible by 6. Find the value of c.

c =

(iv) 在圖中，$ABCD$ 為一正方形，$BF \parallel AC$，且 $AEFC$ 為一菱形。
若 $\angle EAC = d°$，求 d 的值。
In the figure, $ABCD$ is a square, $BF \parallel AC$, and $AEFC$ is a rhombus.
If $\angle EAC = d°$, find the value of d.

d =



FOR OFFICIAL USE

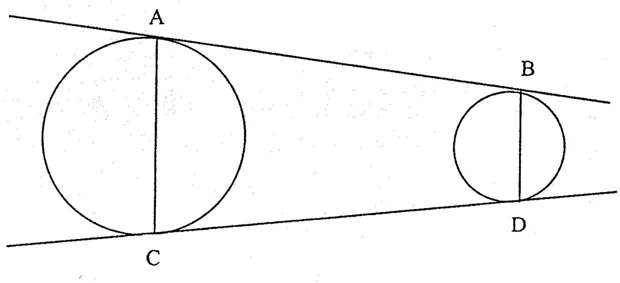
Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1997-98)
Final Event Spare (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

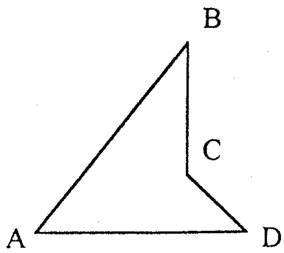
(i) 在圖中，有兩外公切綫，此外公切綫與圓相交於點 A、B、C 及 D。
若 $AC = 9$ cm，$BD = 3$ cm，$\angle BAC = 60°$及 $AB = s$ cm，求 s 的值。
In the figure, there are two common tangents. These common tangents meet the circles at points A, B, C and D. If $AC = 9$ cm, $BD = 3$ cm, $\angle BAC = 60°$ and $AB = s$ cm, find the value of s.

$s =$



(ii) 在圖中，$ABCD$ 為一四邊形，其中內角$\angle A$、$\angle B$ 及$\angle D$ 均為 45°。BC 的延綫與 AD 互相垂直。若 $AC = 10$，$BD = b$，求 b 的值。
In the figure, $ABCD$ is a quadrilateral, where the interior angles $\angle A$, $\angle B$ and $\angle D$ are all equal to 45°. When produced, BC is perpendicular to AD.
If $AC = 10$ and $BD = b$, find the value of b.

$b =$



(iii) 若 $\log_c 27 = 0.75$，求 c 的值。
If $\log_c 27 = 0.75$, find the value of c .

$c =$

(iv) 若數據 30，80，50，40，d 的平均數、眾數和中位數都相等，求 d 的值。
If the mean, mode and median of the data 30, 80, 50, 40, d are all equal, find the value of d .

$d =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (1998-99)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若一個 P-邊的多邊形的內角形成一算術級數，且最小和最大的角分別為 $20°$ 及 $160°$，求 P 之值。

If the interior angles of a P-sided polygon form an Arithmetic Progression and the smallest and the largest angles are $20°$ and $160°$ respectively. Find the value of P .

P =

(ii) 在ΔABC 中，$AB = 5$, $AC = 6$ 及 $BC = P$，若 $\frac{1}{Q} = \cos 2A$，求 Q 之值。

(提示: $\cos 2A = 2\cos^2 A - 1$)

In ΔABC, $AB = 5$, $AC = 6$ and $BC = P$. If $\frac{1}{Q} = \cos 2A$, find the value of Q.

(Hint: $\cos 2A = 2\cos^2 A - 1$)

Q =

(iii) 若 $\log_2 Q + \log_4 Q + \log_8 Q = \frac{R}{2}$，求 R 之值。

If $\log_2 Q + \log_4 Q + \log_8 Q = \frac{R}{2}$, find the value of R .

R =

(iv) 若兩數 R 和 $\frac{11}{S}$ 的積等於它們的和，求 S 之值。

If the product of the numbers R and $\frac{11}{S}$ is the same as their sum, find the value of S.

S =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1998-99)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 x、y 及 z 為正實數使得 $\frac{x+y-z}{z}=\frac{x-y+z}{y}=\frac{-x+y+z}{x}$，且 $a=\frac{(x+y)\cdot(y+z)\cdot(z+x)}{xyz}$，求 a 之值。

If x, y and z are positive real numbers such that $\frac{x+y-z}{z}=\frac{x-y+z}{y}=\frac{-x+y+z}{x}$ and $a=\frac{(x+y)\cdot(y+z)\cdot(z+x)}{xyz}$, find the value of a.

$a =$

(ii) 設 u 和 t 為正整數使得 $u + t + ut = 4a + 2$，若 $b = u + t$，求 b 之值。
Let u and t be positive integers such that $u + t + ut = 4a + 2$.
If $b = u + t$, find the value of b.

$b =$

(iii) 在圖一，OAB 為四分之一圓，且以 OA、OB 為直徑繪出兩個半圓，
若 p、q 代表陰影部分之面積，其中 $p = (b - 9)$ cm^2 及 $q = c$ cm^2，求 c 之值。
In Figure 1, OAB is a quadrant of a circle and semi-circles are drawn on OA and OB.
If p, q denotes the areas of the shaded regions, where $p = (b - 9)$ cm^2 and $q = c$ cm^2, find the value of c.

$c =$

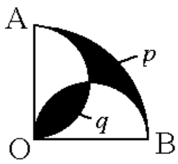


Figure 1 圖一

(iv) 設 $f_0(x)=\frac{1}{c-x}$，且 $f_n(x) = f_0(f_{n-1}(x))$，$n = 1, 2, 3, \cdots\cdots$
若 $f_{2000}(2000) = d$，求 d 之值。
Let $f_0(x)=\frac{1}{c-x}$ and $f_n(x) = f_0(f_{n-1}(x))$, $n = 1, 2, 3, \cdots\cdots$
If $f_{2000}(2000) = d$, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy		× Mult. factor for speed		=		Team No.	
		+ Bonus score				Time	Min. / Sec.
		Total score					

Hong Kong Mathematics Olympiad (1998-99)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 對任意整數 m 及 n，$m \otimes n$ 之定義如下：$m \otimes n = m^n + n^m$。
若 $2 \otimes a = 100$，求 a 之值。
For all integers m and n, $m \otimes n$ is defined as: $m \otimes n = m^n + n^m$.
If $2 \otimes a = 100$, find the value of a.

$a =$

(ii) 若 $\sqrt[3]{13b+6a+1}-\sqrt[3]{13b-6a-1}=\sqrt[3]{2}$，其中 $b>0$，求 b 之值。
If $\sqrt[3]{13b+6a+1}-\sqrt[3]{13b-6a-1}=\sqrt[3]{2}$, where $b > 0$, find the value of b.

$b =$

(iii) 在圖二，$AB = AC$ 和 $KL = LM$。若 $LC = b - 6$ cm 及 $KB = c$ cm，求 c 之值。
In figure 2, $AB = AC$ and $KL = LM$. If $LC = b - 6$ cm and $KB = c$ cm, find the value of c.

$c =$

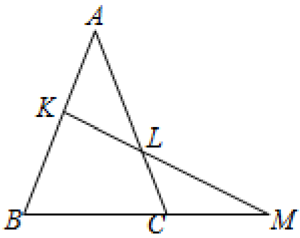


Figure 2 圖二

(iv) 數列$\{a_n\}$的定義如下：$a_1 = c$，$a_{n+1} = a_n + 2n$ $(n \ge 1)$。若 $a_{100} = d$，求 d 之值。
The sequence $\{a_n\}$ is defined as $a_1 = c$, $a_{n+1} = a_n + 2n$ $(n \ge 1)$.
If $a_{100} = d$, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1998-99)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 李先生今年 a 歲，$a < 100$。若把李先生的出生月份與 a 相乘，其結果是 253。求 a 的值。

Mr. Lee is a years old, $a < 100$.

If the product of a and his month of birth is 253, find the value of a.

$a =$

(ii) 李先生有糖 $a + b$ 粒，若平均分給 10 人，則餘下 5 粒。
若平均分給 7 人，則欠 3 粒。求 b 之最小值。
Mr. Lee has $a + b$ sweets. If he divides them equally among 10 persons, 5 sweets will be remained. If he divides them equally among 7 persons, 3 more sweets are needed.
Find the minimum value of b.

$b =$

(iii) 設 c 為一正實數，若 $x^2 + 2\sqrt{c}\,x + b = 0$ 僅有一實數解，求 c 之值。
Let c be a positive real number.
If $x^2 + 2\sqrt{c}\,x + b = 0$ has one real root only, find the value of c.

$c =$

(iv) 在圖三，正方形 $ABCD$ 之面積為 d。若 E，F，G，H 分別是 AB，BC，CD，DA 之中心點，及 $EF = c$，求 d 之值。

In figure 3, the area of the square $ABCD$ is equal to d. If E, F, G, H are the mid-points of AB, BC, CD and DA respectively and $EF = c$, find the value of d.

$d =$

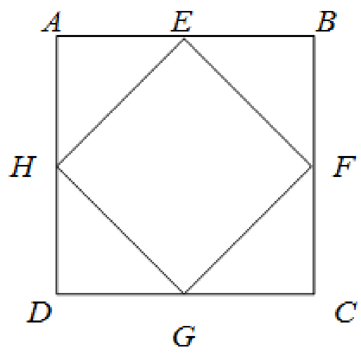


Figure 3 圖三

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1998-99)
Final Event 5 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $144^p = 10$，$1728^q = 5$ 及 $a = 12^{2p-3q}$，求 a 之值。

If $144^p = 10$, $1728^q = 5$ and $a = 12^{2p-3q}$, find the value of a.

a =

(ii) 若 $1 - \frac{4}{x} + \frac{4}{x^2} = 0$，及 $b = \frac{a}{x}$，求 b 之值。

If $1 - \frac{4}{x} + \frac{4}{x^2} = 0$, $b = \frac{a}{x}$, find the value of b.

b =

(iii) 若方程 $x^2 - bx + 1 = 0$ 有 c 個實數解，求 c 之值。

If the number of real roots of the equation $x^2 - bx + 1 = 0$ is c, find the value of c.

c =

(iv) 設 $f(1) = c + 1$ 及 $f(n) = (n - 1) f(n - 1)$，其中 $n > 1$。若 $d = f(4)$，求 d 之值。

Let $f(1) = c + 1$ and $f(n) = (n - 1) f(n - 1)$, where $n > 1$.

If $d = f(4)$, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1998-99)
Spare Event (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 a 為能整除 $3^{11}+5^{13}$ 的最小質數，求 a 之值。
If a is the smallest prime number which can divide the sum $3^{11}+5^{13}$, find the value of a.

a =

(ii) 對任意實數 x 及 y, $x\oplus y$ 之定義如下：$x\oplus y=\frac{1}{xy}$。
若 $b=4\oplus(a\oplus 1540)$，求 b 之值。
For all real number x and y, $x\oplus y$ is defined as: $x\oplus y=\frac{1}{xy}$.
If $b=4\oplus(a\oplus 1540)$, find the value of b.

b =

(iii) W 和 F 為兩大於 20 的整數。
若 W 與 F 之積為 b，W 與 F 之和為 c，求 c 之值。
W and F are two integers which are greater than 20.
If the product of W and F is b and the sum of W and F is c, find the value of c.

c =

(iv) 若 $\frac{d}{114}=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{c^2}\right)$，求 d 之值。
If $\frac{d}{114}=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{c^2}\right)$, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ] Team No. [ ]

\+ Bonus score [ ] Time [ ] [ ]

Total score [ ] Min. Sec.

Hong Kong Mathematics Olympiad (1998-99)
Final Event 1 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 設 $x * y = x + y - xy$，其中 x, y 為實數，若 $a = 1 * (0 * 1)$，求 a 之值。
Let $x * y = x + y - xy$, where x, y are real numbers.
If $a = 1 * (0 * 1)$, find the value of a.

a =

(ii) 在圖一，AB 平行於 DC，$\angle ACB$ 為一直角，$AC = CB$ 及 $AB = BD$，
若 $\angle CBD = b°$，求 b 之值。
In figure 1, AB is parallel to DC, $\angle ACB$ is a right angle, $AC = CB$ and $AB = BD$.
If $\angle CBD = b°$, find the value of b.

b =

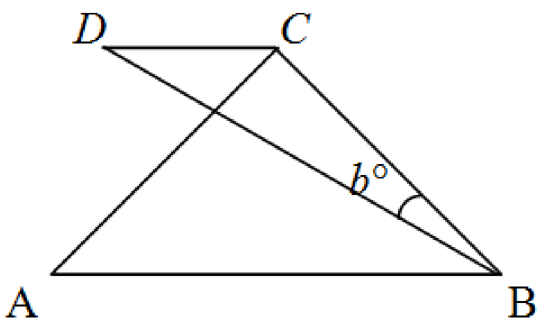


Figure 1 圖一

(iii) 設 x, y 為非零實數，若 x 是 y 的 250%，而 $2y$ 是 x 的 c %，求 c 之值。
Let x, y be non-zero real numbers.
If x is 250% of y and $2y$ is c % of x, find the value of c.

c =

(iv) 若 $\log_p x = 2$，$\log_q x = 3$，$\log_r x = 6$ 及 $\log_{pqr} x = d$，求 d 之值。
If $\log_p x = 2$, $\log_q x = 3$, $\log_r x = 6$ and $\log_{pqr} x = d$, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1998-99)
Final Event 2 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $a = x^4 + x^{-4}$ 及 $x^2 + x + 1 = 0$ ，求 a 之值。

If $a = x^4 + x^{-4}$ and $x^2 + x + 1 = 0$, find the value of a.

$a =$

(ii) 若 $6^b + 6^{b+1} = 2^b + 2^{b+1} + 2^{b+2}$，求 b 之值。

If $6^b + 6^{b+1} = 2^b + 2^{b+1} + 2^{b+2}$, find the value of b.

$b =$

(iii) 設 c 為質數，若 $11c + 1$ 是一正整數之平方，求 c 之值。

Let c be a prime number.

If $11c + 1$ is the square of a positive integer, find the value of c.

$c =$

(iv) 設 d 為奇質數，若 $89 - (d + 3)^2$ 是一整數之平方，求 d 之值。

Let d be an odd prime number.

If $89 - (d + 3)^2$ is the square of an integer, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1998-99)
Final Event 3 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 設小於 100 的正整數，同時又是完全平方及完全立方的數目共有 a 個，求 a 之值。

Let a be the number of positive integers less than 100 such that they are both square and cubic numbers, find the value of a.

a =

(ii) 數列$\{a_k\}$定義如下：$a_1 = 1$、$a_2 = 1$ 及 $a_k = a_{k-1} + a_{k-2}$ $(k > 2)$。
若 $a_1 + a_2 + \cdots + a_{10} = 11\, a_b$，求 b 之值。
The sequence $\{a_k\}$ is defined as:
$a_1 = 1$, $a_2 = 1$ and $a_k = a_{k-1} + a_{k-2}$ $(k > 2)$.
If $a_1 + a_2 + \cdots + a_{10} = 11\, a_b$, find the value of b.

b =

(iii) 若 c 是 $\log(\sin x)$的最大值，其中 $0 < x < \pi$，求 c 之值。
If c is the maximum value of $\log(\sin x)$, where $0 < x < \pi$, find the value of c.

c =

(iv) 設 $x \ge 0$ and $y \ge 0$。已知 $x + y = 18$。若 $\sqrt{x} + \sqrt{y}$ 之最大值是 d，求 d 之值。
Let $x \ge 0$ and $y \ge 0$. Given that $x + y = 18$.
If the maximum value of $\sqrt{x} + \sqrt{y}$ is d, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

\+ Bonus score [ ]   Time [ ] [ ]

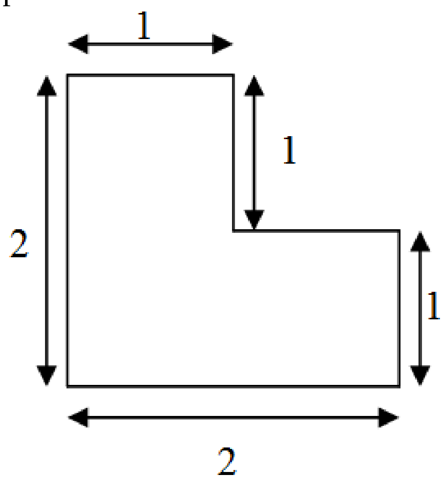
Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (1998-99)
Final Event 4 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若以 a 塊 L 形的瓷磚（圖二），不重疊地拼出一幅與之相似，但面積較大的圖形，求 a 的最小可能值。
If a tiles of L-shape are used to form a larger similar figure (figure 2) without overlapping, find the least possible value of a.

a =



圖二 Figure 2

(ii) 設α、β 是 $x^2 + bx - 2 = 0$ 的根。若 $\alpha > 1$ 及 $\beta < -1$，且 b 為一整數，求 b 之值。
Let α, β be the roots of $x^2 + bx - 2 = 0$.
If $\alpha > 1$ and $\beta < -1$, and b is an integer, find the value of b.

b =

(iii) 已知 m，c 是小於 10 的正整數。若 $m = 2c$，且 $0.\dot{m}\dot{c} = \frac{c+4}{m+5}$，求 c 之值。
Given that m, c are positive integers less than 10. If $m = 2c$ and $0.\dot{m}\dot{c} = \frac{c+4}{m+5}$,
find the value of c.

c =

(iv) 一個袋子裏有 d 個球，其中 x 個是黑球，$x + 1$ 個是紅球，$x + 2$ 個是白球。
若從袋裏隨機抽出一個黑球之概率小於 $\frac{1}{6}$，求 d 之值。
A bag contains d balls of which x are black, $x + 1$ are red and $x + 2$ are white.
If the probability of drawing a black ball randomly from the bag is less than $\frac{1}{6}$,
find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

\+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (1998-99)
Final Event 5 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若 $x^2 - 2x - P = 0$ 的根相差 12，求 P 之值。
If the roots of $x^2 - 2x - P = 0$ differ by 12, find the value of P.

P =

(ii) 已知方程式 $x^2 + ax + 2b = 0$ 及 $x^2 + 2bx + a = 0$ 的根為實數，且 $a, b > 0$。
若 $a + b$ 的最小值為 Q，求 Q 之值。
Given that the roots of $x^2 + ax + 2b = 0$ and $x^2 + 2bx + a = 0$ are both real and $a, b > 0$. If the minimum value of $a + b$ is Q, find the value of Q.

Q =

(iii) If $R^{2000} < 5^{3000}$, where R is a positive integer, find the largest value of R.
若 $R^{2000} < 5^{3000}$，其中 R 為正整數，求 R 之最大值。

R =

(iv) 在圖三，直角三角形 ABC 中，$BH \perp AC$。
若 $AB = 15$，$HC = 16$ 及 ΔABC 的面積是 S，求 S 之值。
In figure 3, ΔABC is a right-angled triangle and $BH \perp AC$.
If $AB = 15$, $HC = 16$ and the area of ΔABC is S, find the value of S.

S =

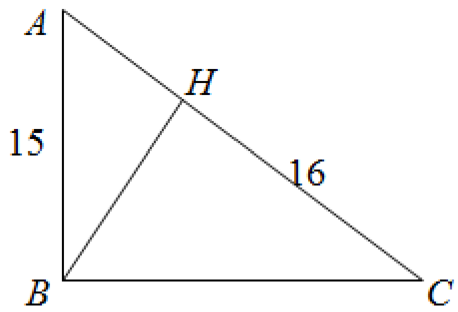


Figure 3 圖三

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time		
				Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1998-99)

Spare Event (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 若從正整數集中任意抽取一數 N，N^4 的個位數字為 1 的概率是 $\frac{P}{10}$，求 P 之值。
If a number N is chosen randomly from the set of positive integers, the probability of the unit digit of N^4 being unity is $\frac{P}{10}$, find the value of P.

P =

(ii) 設 $x \geq 0$ and $y \geq 0$。已知 $x + y = 18$。若 $\sqrt{x} + \sqrt{y}$ 的最大值為 Q，求 Q 之值。
Let $x \geq 0$ and $y \geq 0$. Given that $x + y = 18$.
If the maximum value of $\sqrt{x} + \sqrt{y}$ is Q, find the value of Q.

Q =

(iii) 若 $x^2 - 2x - R = 0$ 的兩根之差為 12，求 R 之值。
If the roots of $x^2 - 2x - R = 0$ differs by 12, find the value of R.

R =

(iv) 若一四位數 ab$S$d 與 9 的積恰為四位數 dSba，求 S 之值。
If the product of a 4-digit number ab$S$d and 9 is equal to another 4-digit number dSba , find the value of S .

S =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1999-2000)

Final Event (Individual) Example

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 對任意整數 m 及 n，$m \otimes n$ 之定義如下：$m \otimes n = m^n + n^m$。
若 $2 \otimes P = 100$，求 P 之值。
For all integers m and n, $m \otimes n$ is defined as $m \otimes n = m^n + n^m$.
If $2 \otimes P = 100$, find the value of P.

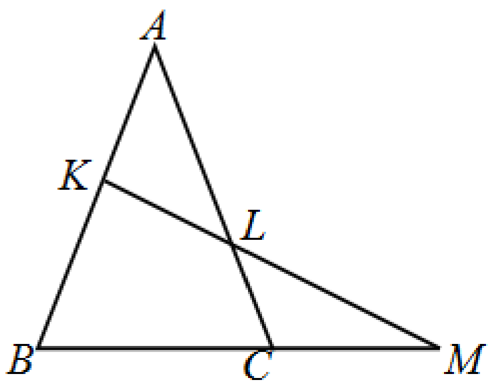
$P =$

(ii) 若 $\sqrt[3]{13Q+6P+1} - \sqrt[3]{13Q-6P-1} = \sqrt[3]{2}$，其中 $Q > 0$，求 Q 之值。
If $\sqrt[3]{13Q+6P+1} - \sqrt[3]{13Q-6P-1} = \sqrt[3]{2}$, where $Q > 0$, find the value of Q.

$Q =$

(iii) 在圖一，$AB = AC$ 和 $KL = LM$。若 $LC = Q - 6$ cm 及 $KB = R$ cm，求 R 之值。
In figure 1, $AB = AC$ and $KL = LM$. If $LC = Q - 6$ cm and $KB = R$ cm, find the value of R.

$R =$



圖一 Figure 1

(iv) 數列$\{a_n\}$的定義如下：$a_1 = R$，$a_{n+1} = a_n + 2n$ $(n \geq 1)$。若 $a_{100} = S$，求 S 之值。
The sequence $\{a_n\}$ is defined as $a_1 = R$, $a_{n+1} = a_n + 2n$ $(n \geq 1)$.
If $a_{100} = S$, find the value of S.

$S =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] Min. [ ] Sec.

Total score [ ]

Hong Kong Mathematics Olympiad (1999 – 2000)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 設 $[x]$ 表示小數 x 的整數部份。

已知 $[3.126]+[3.126+\frac{1}{8}]+[3.126+\frac{2}{8}]+\ldots+[3.126+\frac{7}{8}]=P$，求 P 的值。

Let $[x]$ represents the integral part of the decimal number x. Given that

$[3.126]+[3.126+\frac{1}{8}]+[3.126+\frac{2}{8}]+\ldots+[3.126+\frac{7}{8}]=$ P, find the value of P.

P =

(ii) 設 $a+b+c=0$。已知 $\frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ac}+\frac{c^2}{2c^2+ab}=P-3Q$，求 Q 的值。

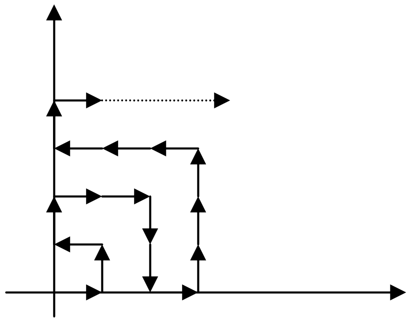
Let $a+b+c=0$. Given that $\frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ac}+\frac{c^2}{2c^2+ab}=P-3Q$,

find the value of Q.

Q =

(iii) 在直角座標平面的第一象限中，把座標為整數的點按以下方法編號：

點 (0, 0) 為第 1 號，
點 (1, 0) 為第 2 號，
點 (1, 1) 為第 3 號，
點 (0, 1) 為第 4 號，
點 (0, 2) 為第 5 號，
點 (1, 2) 為第 6 號，
點 (2, 2) 為第 7 號，
點 (2, 1) 為第 8 號，
…………………………

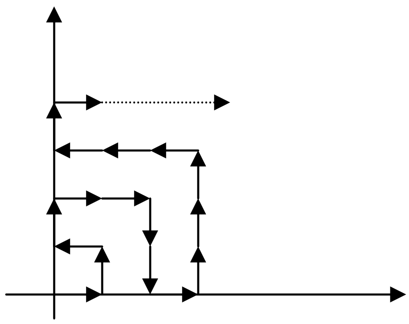


已知 $(Q-1, Q)$ 點為第 R 號，求 R 的值。

R =

In the first quadrant of the rectangular co-ordinate plane, all integral points are numbered as follows,

point (0, 0) is numbered as 1,
point (1, 0) is numbered as 2,
point (1, 1) is numbered as 3,
point (0, 1) is numbered as 4,
point (0, 2) is numbered as 5,
point (1, 2) is numbered as 6,
point (2, 2) is numbered as 7,
point (2, 1) is numbered as 8,
………………………………



Given that point $(Q-1, Q)$ is numbered as R, find the value of R.

(iv) 當 $x+y=4$ 時，$3x^2+y^2$ 的最小值為 $\frac{R}{S}$，求 S 的值。

When $x+y=4$, the minimum value of $3x^2+y^2$ is $\frac{R}{S}$, find the value of S.

S =

FOR OFFICIAL USE

Score for accuracy × Mult. factor for speed =

\+ Bonus score

Total score

Team No.

Time Min. Sec.

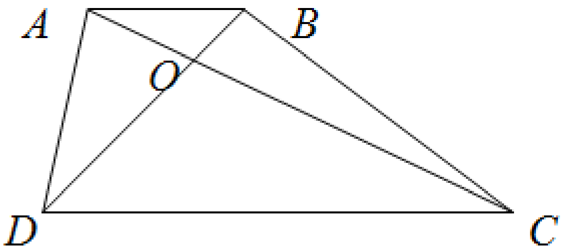
Hong Kong Mathematics Olympiad (1999 – 2000)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 如果 $\log_2(\log_4 P) = \log_4(\log_2 P)$ 及 $P \neq 1$，求 P 的值。
If $\log_2(\log_4 P) = \log_4(\log_2 P)$ and $P \neq 1$, find the value of P.

$P =$

(ii) 在梯形 $ABCD$ 中，$AB // DC$。AC 和 BD 相交於 O。三角形 AOB 和 COD 的面積分別為 P 和 25。已知梯形的面積為 Q，求 Q 的值。



In the trapezium $ABCD$, $AB // DC$. AC and BD intersect at O. The areas of triangles AOB and COD are P and 25 respectively. Given that the area of the trapezium is Q, find the value of Q.

$Q =$

(iii) 當 1999^Q 被 7 除時，餘數為 R。求 R 的值。
When 1999^Q is divided by 7, the remainder is R. Find the value of R.

$R =$

(iv) 如果 $111111111111 - 222222 = (R + S)^2$，求正數 S 的值。
If $111111111111 - 222222 = (R + S)^2$, find the positive value of S.

$S =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1999 – 2000)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 已知 1 +2 +3 + ⋯ +1997 +1998+ 1999+ 1998+ 1997 + ⋯ + 3 + 2 + 1 的個位數是 P，求 P 的值。
Given that the units digit of 1+2+3+...+1997+1998+1999+1998+1997+...+3+2+1 is P, find the value of P.

P =

(ii) 已知 $x+\frac{1}{x}=P$。如果 $x^6+\frac{1}{x^6}=Q$，求 Q 的值。
Given that $x+\frac{1}{x}=P$. If $x^6+\frac{1}{x^6}=Q$, find the value of Q.

Q =

(iii) 已知 $\frac{Q}{\sqrt{Q}+\sqrt{2Q}}+\frac{Q}{\sqrt{2Q}+\sqrt{3Q}}+\cdots+\frac{Q}{\sqrt{1998Q}+\sqrt{1999Q}}=\frac{R}{\sqrt{Q}+\sqrt{1999Q}}$，求 R 的值。
Given that
$$\frac{Q}{\sqrt{Q}+\sqrt{2Q}}+\frac{Q}{\sqrt{2Q}+\sqrt{3Q}}+\cdots+\frac{Q}{\sqrt{1998Q}+\sqrt{1999Q}}=\frac{R}{\sqrt{Q}+\sqrt{1999Q}},$$
find the value of R.

R =

(iv) 設 f(0) = 0；f(n) = f(n – 1) + 3 當 n = 1、2、3、4、⋯。
如果 2 f(S) = R，求 S 的值。
Let f(0) = 0 ; f(n) = f(n – 1) +3 when n = 1, 2, 3, 4, ⋯ .
If 2 f(S) = R, find the value of S.

S =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]　Team No. [ ]

\+ Bonus score [ ]　Time [ ] [ ]

Total score [ ]　Min.　Sec.

Hong Kong Mathematics Olympiad (1999 – 2000)
Final Event 4 (Individual)

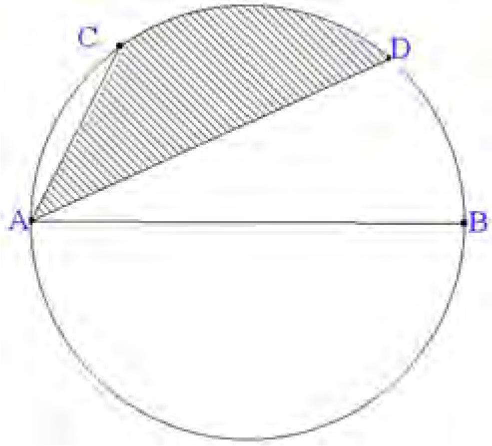
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 假設 $a+\frac{1}{a+1}=b+\frac{1}{b-1}-2$，其中 $a\neq -1$，$b\neq 1$ 和 $a-b+2\neq 0$。
已知 $ab-a+b=P$，求 P 的值。
Suppose $a+\frac{1}{a+1}=b+\frac{1}{b-1}-2$, where $a\neq -1$, $b\neq 1$, and $a-b+2\neq 0$.
Given that $ab-a+b=P$, find the value of P.

$P =$

(ii) 在下圖中，AB 為圓的直徑。C 和 D 把弧 AB 分為三等份。斜綫面積為 P。
若圓的面積為 Q，求 Q 的值。
In the following figure, AB is a diameter of the circle. C and D divide the arc AB into three equal parts. The shaded area is P.
If the area of the circle is Q, find the value of Q.

$Q =$



(iii) 已知兩個 Q 位數 $1111\cdots11$ 和 $9999\cdots99$ 的乘積中有 R 個數字是奇數，
求 R 的值。
Given that there are R odd numbers in the digits of the product of the two Q-digit numbers $1111\cdots11$ and $9999\cdots99$, find the value of R.

$R =$

(iv) 設 a_1、a_2、$\cdots$、a_R 為正整數，其中 $a_1<a_2<a_3<\cdots<a_{R-1}<a_R$。
已知這 R 個正整數的和為 90 及 a_1 的最大值為 S，求 S 的值。
Let $a_1, a_2, \cdots, a_R$ be positive integers such that $a_1<a_2<a_3<\cdots<a_{R-1}<a_R$.
Given that the sum of these R integers is 90 and the maximum value of a_1 is S, find the value of S.

$S =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	Min. / Sec.
				Total score				

Hong Kong Mathematics Olympiad (1999 – 2000)
Final Event 5 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 如果 $\left(\dfrac{1\times2\times4+2\times4\times8+3\times6\times12+\cdots+1999\times3998\times7996}{1^3+2^3+3^3+\cdots+1999^3}\right)^{\frac{1}{3}}=P$，求 P 的值。

If $\left(\dfrac{1\times2\times4+2\times4\times8+3\times6\times12+\cdots+1999\times3998\times7996}{1^3+2^3+3^3+\cdots+1999^3}\right)^{\frac{1}{3}}=P$,
find the value of P.

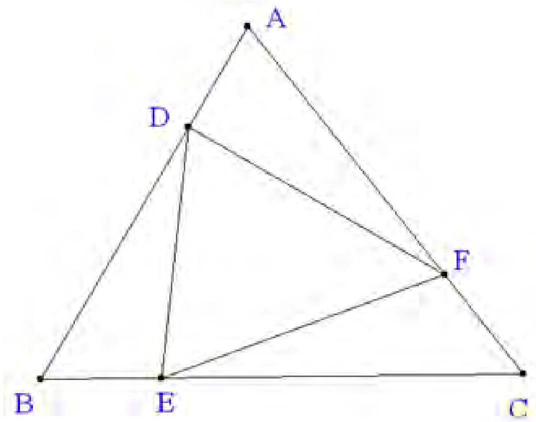
$P =$

(ii) 如果 $(x-P)(x-2Q)-1=0$ 有兩個整數根，求 Q 的值。
If $(x-P)(x-2Q)-1=0$ has two integral roots, find the value of Q.

$Q =$

(iii) 已知 ΔABC 的面積為 $3Q$；D、E 和 F 分別為 AB、BC 和 CA 上的點使得 $AD=\frac{1}{3}AB$，$BE=\frac{1}{3}BC$，$CF=\frac{1}{3}CA$。如果 ΔDEF 的面積為 R，求 R 的值。
Given that the area of the ΔABC is $3Q$; D, E and F are the points on AB, BC and CA respectively such that $AD=\frac{1}{3}AB$, $BE=\frac{1}{3}BC$, $CF=\frac{1}{3}CA$.
If the area of ΔDEF is R, find the value of R.

$R =$



(iv) 已知 $(Rx^2-x+1)^{1999}\equiv a_0+a_1x+a_2x^2+\cdots+a_{3998}x^{3998}$。
設 $S=a_0+a_1+a_2+\cdots+a_{3997}$，求 S 的值。
Given that $(Rx^2-x+1)^{1999}\equiv a_0+a_1x+a_2x^2+\cdots+a_{3998}x^{3998}$.
If $S=a_0+a_1+a_2+\cdots+a_{3997}$, find the value of S.

$S =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1999 – 2000)
Final Event (Group) Example

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 設 $x * y = x + y - xy$，其中 x, y 為實數，若 $a = 1 * (0 * 1)$，求 a 之值。
Let $x * y = x + y - xy$, where x, y are real numbers.
If $a = 1 * (0 * 1)$, find the value of a.

$a =$

(ii) 在圖一，AB 平行於 DC，$\angle ACB$ 為一直角，$AC = CB$ 及 $AB = BD$.，若 $\angle CBD = b°$，求 b 之值。
In figure 1, AB is parallel to DC, $\angle ACB$ is a right angle, $AC = CB$ and $AB = BD$.
If $\angle CBD = b°$, find the value of b.

$b =$

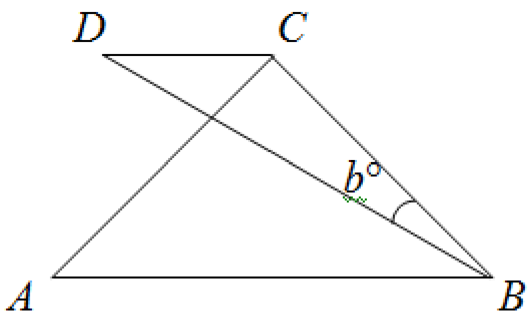


Figure 1 圖一

(iii) 設 x、y 為非零實數，若 x 是 y 的 250%，而 $2y$ 是 x 的 c%，求 c 之值。
Let x, y be non-zero real numbers.
If x is 250% of y and $2y$ is c% of x, find the value of c.

$c =$

(iv) 若 $\log_p x = 2$，$\log_q x = 3$，$\log_r x = 6$ 及 $\log_{pqr} x = d$，求 d 之值。
If $\log_p x = 2$, $\log_q x = 3$, $\log_r x = 6$ and $\log_{pqr} x = d$, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1999 – 2000)

Final Event 1 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 已知整數 n 除 81849、106392 及 124374 得出的餘數相等，求 n 的最大值 a。
Given that when 81849, 106392 and 124374 are divided by an integer n, the remainders are equal. If a is the maximum value of n, find a.

$a =$

(ii) 設 $x=\frac{1-\sqrt{3}}{1+\sqrt{3}}$ 及 $y=\frac{1+\sqrt{3}}{1-\sqrt{3}}$。如果 $b = 2x^2 - 3xy + 2y^2$，求 b 的值。
Let $x=\frac{1-\sqrt{3}}{1+\sqrt{3}}$ and $y=\frac{1+\sqrt{3}}{1-\sqrt{3}}$. If $b = 2x^2 - 3xy + 2y^2$, find the value of b.

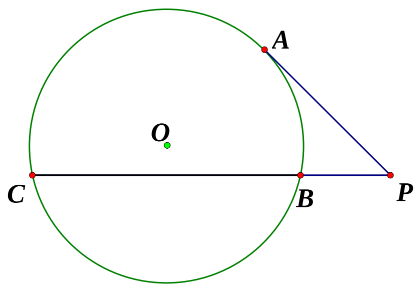
$b =$

(iii) 已知 c 為正數，如果只有一條直綫穿過點 A(1, c)且與曲綫 C: $x^2 + y^2 - 2x - 2y - 7 = 0$ 相交於一點，求 c 的值。
Given that c is a positive number. If there is only one straight line which passes through point A(1, c) and meets the curve C: $x^2 + y^2 - 2x - 2y - 7 = 0$ at only one point, find the value of c.

$c =$

(iv) 在圖一，PA 切圓於 A，O 為圓心。如果 $PA = 6$，$BC = 9$，$PB = d$，求 d 的值。
In Figure 1, PA touches the circle with centre O at A. If $PA = 6$, $BC = 9$, $PB = d$, find the value of d.

$d =$



圖一
Figure 1

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.
			+ Bonus score				Time
			Total score				Min. / Sec.

Hong Kong Mathematics Olympiad (1999 – 2000)

Final Event 2 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

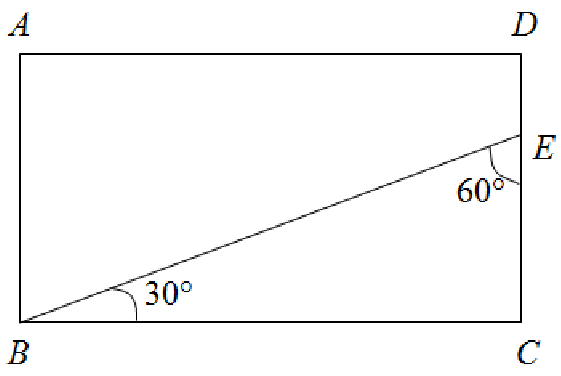
(i) 如果 191 為兩個連續平方數之差，而 a 為其中最小的平方數，求 a 的值。
If 191 is the difference of two consecutive perfect squares, find the value of the smallest square number, a.

a =

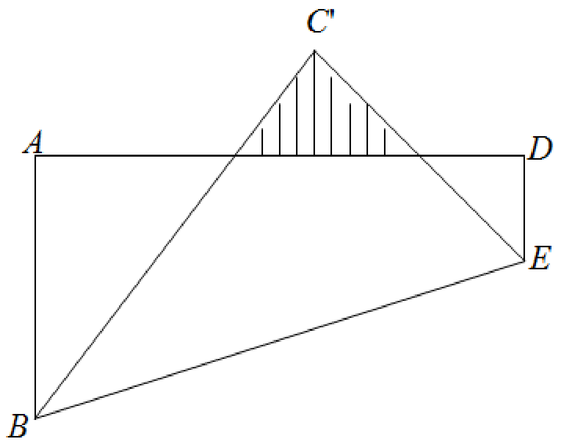
(ii) 在圖二(a)，$ABCD$ 是一長方形。$DE : EC = 1 : 5$，且 $DE = 12^{\frac{1}{4}}$。
ΔBCE 沿 BE 摺去另一方。設 b 為圖二(b)中陰影部份的面積，求 b 的值。

In Figure 2(a), $ABCD$ is a rectangle. $DE:EC = 1: 5$, and $DE = 12^{\frac{1}{4}}$.
ΔBCE is folded along the side BE.
If b is the area of the shaded part as shown in Figure 2(b), find the value of b.

b =



圖二(a) Figure 2(a)



圖二(b) Figure 2(b)

(iii) 設曲綫 $y = x^2 - 7x + 12$ 與 x 軸的交點為 A 及 B，而與 y 軸的交點為 C。如果 c 是 ΔABC 的面積，求 c 的值。
Let the curve $y = x^2 - 7x + 12$ intersect the x-axis at points A and B, and intersect the y-axis at C. If c is the area of ΔABC, find the value of c.

c =

(iv) 設 $f(x) = 41x^2 - 4x + 4$，$g(x) = -2x^2 + x$。如果 $f(x) + kg(x) = 0$ 只有一個根，求 k 的最小值 d。
Let $f(x) = 41x^2 - 4x + 4$ and $g(x) = -2x^2 + x$. If d is the smallest value of k such that $f(x) + kg(x) = 0$ has a single root, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]　　Team No. [ ]

\+ Bonus score [ ]　　Time [ ] [ ]

Total score [ ]　　Min.　Sec.

Hong Kong Mathematics Olympiad (1999 – 2000)

Final Event 3 (Group)

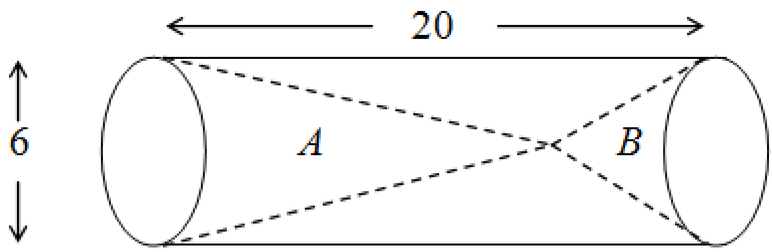
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 設 $a=\sqrt{1997\times1998\times1999\times2000+1}$，求 a 的值。
Let $a=\sqrt{1997\times1998\times1999\times2000+1}$, find the value of a.

a =

(ii) 在圖三，圓管的長為 20 及直徑為 6，內有兩個圓錐體 A 和 B。A 及 B 的體積比例為 3 : 1。如果 b 是 B 的高度，求 b 的值。
In Figure 3, A and B are two cones inside a cylindrical tube with length of 20 and diameter of 6. If the volumes of A and B are in the ratio 3:1 and b is the height of the cone B, find the value of b.

b =



圖三 Figure 3

(iii) 現有點 $A\left(\frac{\sqrt{10}}{2},\frac{\sqrt{10}}{2}\right)$ 和圓 $C: x^2+y^2=1$。
如果 c 是通過點 A 與圓相切直綫的最大斜率，求 c 的值。
If c is the largest slope of the tangents from the point $A\left(\frac{\sqrt{10}}{2},\frac{\sqrt{10}}{2}\right)$ to the circle C: $x^2+y^2=1$, find the value of c.

c =

(iv) 在座標平面的原點上有一點 P。假如擲出骰子的點數 n 是偶數，P 在 x 方向右前進 n；如果 n 是奇數，P 在 y 方向上前進 n。
如果有 d 種不同擲法使得 P 到達點 (4, 4)，求 d 的值。
P is a point located at the origin of the coordinate plane. When a dice is thrown and the number n shown is even, P moves to the right by n. If n is odd, P moves upward by n. Find the value of d, the total number of tossing sequences for P to move to the point (4, 4) .

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (1999 – 2000)

Final Event 4 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 如果 a 是一個三位數，駁在 504 之後，新組成的六位數可被 7、9、11 整除，求 a 的值。
Let a be a 3-digit number. If the 6-digit number formed by putting a at the end of the number 504 is divisible by 7, 9, and 11, find the value of a.

a =

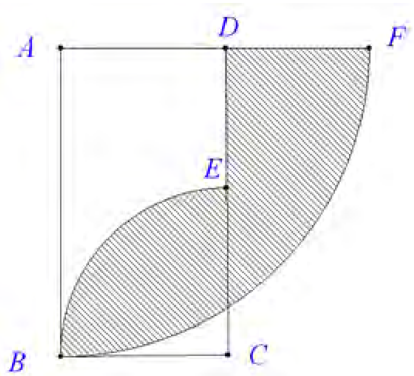
(ii) 在圖四，$ABCD$ 為長方形，

$$AB=\sqrt{\frac{8+\sqrt{64-\pi^2}}{\pi}}，BC=\sqrt{\frac{8-\sqrt{64-\pi^2}}{\pi}}。$$

BE、BF 分別是以 C、A 為圓心的弧。
若 b 是陰影部份之面積，求 b 的值。
In Figure 4, $ABCD$ is a rectangle with

$$AB=\sqrt{\frac{8+\sqrt{64-\pi^2}}{\pi}} \text{ and } BC=\sqrt{\frac{8-\sqrt{64-\pi^2}}{\pi}}.$$

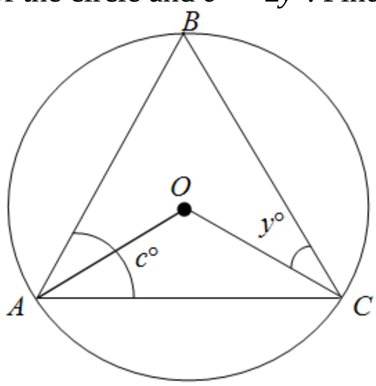
BE and BF are the arcs of circles with centres at C and A respectively. If b is the total area of the shaded parts, find the value of b.



圖四 Figure 4

b =

(iii) 在圖五，O 為圓心，$c° = 2y°$，求 c 的值。
In Figure 5, O is the centre of the circle and $c° = 2y°$. Find the value of c.



圖五 Figure 5

c =

(iv) A、B、C、D、E、F、G 七個人圍圓桌而坐。
如果 B 及 G 都與 C 相鄰而坐的坐法總數為 d，求 d 的值。
A, B, C, D, E, F, G are seven people sitting around a circular table.
If d is the total number of ways that B and G must sit next to C, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (1999 – 2000)

Final Event 5 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

(i) 如果 a 是可被 810 整除的最小立方數，求 a 的值。
If a is the smallest cubic number divisible by 810, find the value of a.

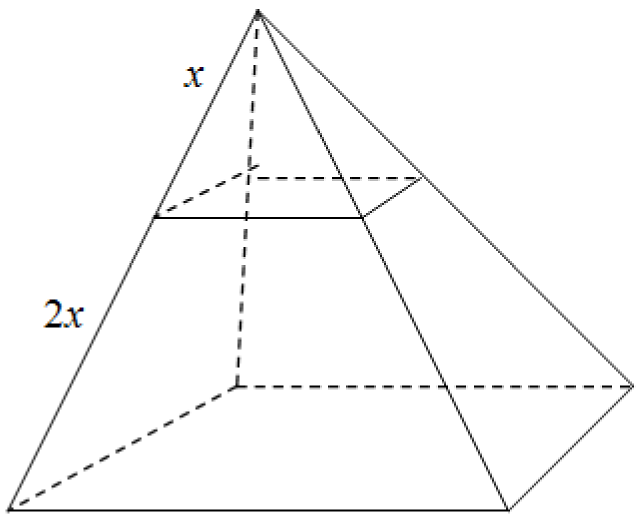
a =

(ii) 設 b 是函數 $y = |x^2 - 4| - 6x$（其中 $-2 \le x \le 5$）的最大值，求 b 的值。
Let b be the maximum of the function $y = |x^2 - 4| - 6x$ (where $-2 \le x \le 5$), find the value of b.

b =

(iii) 圖六為一個正方形底的錐體。若從底部向上並在 $\frac{2}{3}$ 之高度平行橫切，並設 $1 : c$ 為上面細錐與餘下底部體積的比，求 c 的值。
In Figure 6, a square-based pyramid is cut into two shapes by a cut running parallel to the base and made $\frac{2}{3}$ of the way up. Let $1 : c$ be the ratio of the volume of the small pyramid to that of the truncated base, find the value of c.

c =



圖六 Figure 6

(iv) 如果 $\cos^6\theta + \sin^6\theta = 0.4$，及 $d = 2 + 5\cos^2\theta\sin^2\theta$，求 d 的值。
If $\cos^6\theta + \sin^6\theta = 0.4$ and $d = 2 + 5\cos^2\theta\sin^2\theta$, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2000 – 2001)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. a、b 和 c 分別為 ΔABC 的 $\angle A$、$\angle B$ 和 $\angle C$ 的相對邊的長度。
若 $\angle C = 60°$ 及 $\frac{a}{b+c}+\frac{b}{a+c}=P$，求 P 的值。
a, b and c are the lengths of the opposite sides $\angle A$, $\angle B$ and $\angle C$ of the ΔABC respectively. If $\angle C = 60°$ and $\frac{a}{b+c}+\frac{b}{a+c}=P$, find the value of P .

$P =$

2. 已知 $f(x) = x^2 + ax + b$ 是 $x^3 + 4x^2 + 5x + 6$ 和 $2x^3 + 7x^2 + 9x + 10$ 的公因式。
若 $f(P) = Q$，求 Q 的值。
Given that $f(x) = x^2 + ax + b$ is the common factor of $x^3 + 4x^2 + 5x + 6$ and $2x^3 + 7x^2 + 9x + 10$. If $f(P) = Q$, find the value of Q .

$Q =$

3. 已知 $\frac{1}{a}+\frac{1}{b}=\frac{Q}{a+b}$ 及 $\frac{a}{b}+\frac{b}{a}=R$，求 R 的值。
Given that $\frac{1}{a}+\frac{1}{b}=\frac{Q}{a+b}$ and $\frac{a}{b}+\frac{b}{a}=R$, find the value of R .

$R =$

4. 已知 $\begin{cases} a+b=R \\ a^2+b^2=12 \end{cases}$ 及 $a^3 + b^3 = S$，求 S 的值。
Given that $\begin{cases} a+b=R \\ a^2+b^2=12 \end{cases}$ and $a^3 + b^3 = S$, find the value of S .

$S =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

\+ Bonus score [ ]

Total score [ ]

Team No. [ ]

Time [ ] Min. [ ] Sec.

Hong Kong Mathematics Olympiad (2000 – 2001)
Final Event 2 (Individual)

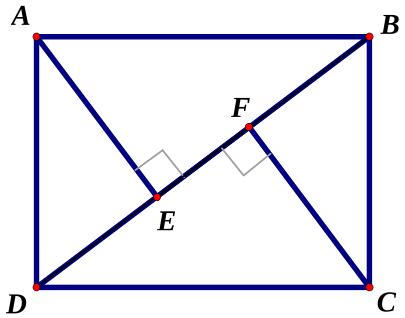
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 P 為整數，及 $5 < P < 20$。
若方程 $x^2 - 2(2P - 3)x + 4P^2 - 14P + 8 = 0$ 的兩個根皆為整數，求 P 的值。
Suppose P is an integer and $5 < P < 20$. If the roots of the equation $x^2 - 2(2P - 3)x + 4P^2 - 14P + 8 = 0$ are integers, find the value of P.

$P =$

2. $ABCD$ 是一長方形。若 $AB = 3P + 4$，$AD = 2P + 6$，AE 和 CF 分別垂直於對角綫 BD，及 $EF = Q$，求 Q 的值。
$ABCD$ is a rectangle. $AB = 3P + 4$, $AD = 2P + 6$.
AE and CF are perpendiculars to the diagonal BD. If $EF = Q$, find the value of Q.

$Q =$



3. 某班學生的人數少於 $4Q$ 人。在一次數學測驗中有 $\frac{1}{3}$ 學生得甲等，$\frac{1}{7}$ 學生得乙等，一半學生得丙等，餘下的學生都不及格。
已知不及格的學生人數是 R，求 R 的值。
There are less than $4Q$ students in a class. In a mathematics test, $\frac{1}{3}$ of the students got grade A, $\frac{1}{7}$ of the students got grade B, half of the students got grade C, and the rest failed. Given that R students failed in the mathematics test, find the value of R.

$R =$

4. $[a]$ 表示不大於 a 的最大整數。例如 $\left[2\frac{1}{3}\right] = 2$。已知方程 $[3x + R] = 2x + \frac{3}{2}$ 的所有根的和為 S，求 S 的值。
$[a]$ represents the largest integer not greater than a. For example, $\left[2\frac{1}{3}\right] = 2$. Given that the sum of the roots of the equation $[3x + R] = 2x + \frac{3}{2}$ is S, find the value of S.

$S =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time		
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2000 – 2001)
Final Event 3 (Individual)

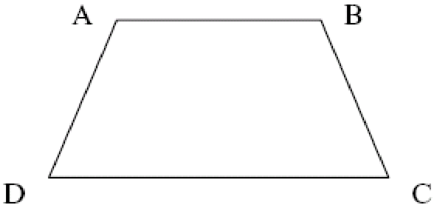
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. $ABCD$ 是一個梯形，其中 $\angle ADC = \angle BCD = 60°$ 及 $AB = BC = AD = \frac{1}{2}CD$。
若把這梯形分割為 P 等份 $(P > 1)$，使其分割所得的每份與梯形 $ABCD$ 相似。
求 P 的最小值。

$ABCD$ is a trapezium such that $\angle ADC = \angle BCD = 60°$ and $AB = BC = AD = \frac{1}{2}CD$.

If this trapezium is divided into P equal portions $(P > 1)$ and each portion is similar to trapezium $ABCD$ itself, find the minimum value of P.

$P =$



2. $(P + 1)^{2001}$ 的個位數字與十位數字的和是 Q，求 Q 的值。
The sum of tens and units digits of $(P + 1)^{2001}$ is Q. Find the value of Q.

$Q =$

3. 若 $\sin 30° + \sin^2 30° + \ldots + \sin^Q 30° = 1 - \cos^R 45°$，求 R 的值。
If $\sin 30° + \sin^2 30° + \ldots + \sin^Q 30° = 1 - \cos^R 45°$, find the value of R.

$R =$

4. 設方程 $x^2 - 8x + (R + 1) = 0$ 的根為 α 和 β。
若 $\frac{1}{\alpha^2}$ 和 $\frac{1}{\beta^2}$ 是方程 $225x^2 - Sx + 1 = 0$ 的根，求 S 的值。
Let α and β be the roots of the equation $x^2 - 8x + (R + 1) = 0$.
If $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$ are the roots of the equation $225x^2 - Sx + 1 = 0$, find the value of S.

$S =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2000 – 2001)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知 $a^{\frac{2}{3}}+b^{\frac{2}{3}}=17\frac{1}{2}$，$x=a+3a^{\frac{1}{3}}b^{\frac{2}{3}}$，$y=b+3a^{\frac{2}{3}}b^{\frac{1}{3}}$。
若 $P=(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}$，求 P 的值。
Let $a^{\frac{2}{3}}+b^{\frac{2}{3}}=17\frac{1}{2}$, $x=a+3a^{\frac{1}{3}}b^{\frac{2}{3}}$ and $y=b+3a^{\frac{2}{3}}b^{\frac{1}{3}}$.
If $P=(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}$, find the value of P.

P =

2. 若一正 Q 邊形有 P 條對角綫，求 Q 的值。
If a regular Q-sided polygon has P diagonals, find the value of Q.

Q =

3. 已知 $x=\sqrt{\frac{Q}{2}+\sqrt{\frac{Q}{2}}}$，$y=\sqrt{\frac{Q}{2}-\sqrt{\frac{Q}{2}}}$。若 $R=\frac{x^6+y^6}{40}$，求 R 的值。
Let $x=\sqrt{\frac{Q}{2}+\sqrt{\frac{Q}{2}}}$ and $y=\sqrt{\frac{Q}{2}-\sqrt{\frac{Q}{2}}}$. If $R=\frac{x^6+y^6}{40}$, find the value of R.

R =

4. 已知 $[a]$ 表示不大於 a 的最大整數。例如 $[2.5]=2$。
若 $S=\left[\frac{2001}{R}\right]+\left[\frac{2001}{R^2}\right]+\left[\frac{2001}{R^3}\right]+\cdots$，求 S 的值。
$[a]$ represents the largest integer not greater than a. For example, $[2.5]=2$.
If $S=\left[\frac{2001}{R}\right]+\left[\frac{2001}{R^2}\right]+\left[\frac{2001}{R^3}\right]+\cdots$, find the value of S.

S =

FOR OFFICIAL USE

Score for accuracy × Mult. factor for speed =
\+ Bonus score
Total score

Team No.
Time Min. Sec.

Hong Kong Mathematics Olympiad (2000 – 2001)
Final Event 1 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知 $(a+b+c)^2 = 3(a^2+b^2+c^2)$ 及 $a+b+c=12$。求 a 的值。
Given that $(a+b+c)^2 = 3(a^2+b^2+c^2)$ and $a+b+c=12$, find the value of a.

$a =$

2. 已知 $b\left[\frac{1}{1\times 3}+\frac{1}{3\times 5}+\cdots+\frac{1}{1999\times 2001}\right]=2\times\left[\frac{1^2}{1\times 3}+\frac{2^2}{3\times 5}+\cdots+\frac{1000^2}{1999\times 2001}\right]$，求 b 的值。

Given that $b\left[\frac{1}{1\times 3}+\frac{1}{3\times 5}+\cdots+\frac{1}{1999\times 2001}\right]=2\times\left[\frac{1^2}{1\times 3}+\frac{2^2}{3\times 5}+\cdots+\frac{1000^2}{1999\times 2001}\right]$,

find the value of b.

$b =$

3. 一六位數 $1234xy$ 能同時被 8 和 9 整除。已知 $x+y=c$，求 c 的值。
A six-digit number $1234xy$ is divisible by both 8 and 9. Given that $x+y=c$, find the value of c.

$c =$

4. 已知 $\log_x t = 6$，$\log_y t = 10$，$\log_z t = 15$。若 $\log_{xyz} t = d$，求 d 的值。
Suppose $\log_x t = 6$, $\log_y t = 10$ and $\log_z t = 15$. If $\log_{xyz} t = d$, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2000 – 2001)
Final Event 2 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知 $x=\sqrt{7-4\sqrt{3}}$ 及 $\frac{x^2-4x+5}{x^2-4x+3}=a$，求 a 的值。

   Given that $x=\sqrt{7-4\sqrt{3}}$ and $\frac{x^2-4x+5}{x^2-4x+3}=a$, find the value of a.

   $a =$

2. E 是長方形 $ABCD$ 內一點。已知 EA、EB、EC 和 ED 的長度分別為 2、$\sqrt{11}$、4 和 b，求 b 的值。

   E is an interior point of the rectangle $ABCD$. Given that the lengths of EA, EB, EC and ED are 2, $\sqrt{11}$, 4 and b respectively, find the value of b.

   $b =$

3. 已知 $111111222222 = c\times(c+1)$，求 c 的值。

   Given that $111111222222 = c\times(c+1)$, find the value of c.

   $c =$

4. 已知 $\cos 16° = \sin 14° + \sin d°$ 及 $0 < d < 90$，求 d 的值。

   Given that $\cos 16° = \sin 14° + \sin d°$ and $0 < d < 90$, find the value of d.

   $d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2000 – 2001)
Final Event 3 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知方程 $\sqrt{3x+1}+\sqrt{3x+6}=\sqrt{4x-2}+\sqrt{4x+3}$ 的解為 a，求 a 的值。
Given that the solution of the equation $\sqrt{3x+1}+\sqrt{3x+6}=\sqrt{4x-2}+\sqrt{4x+3}$ is a, find the value of a.

a =

2. 已知方程 $x^2y - x^2 - 3y - 14 = 0$ 只得一組正整數解 (x_0, y_0)。若 $x_0 + y_0 = b$，求 b 的值。
Suppose the equation $x^2y - x^2 - 3y - 14 = 0$ has only one positive integral solution (x_0, y_0). If $x_0 + y_0 = b$, find the value of b.

b =

3. $ABCD$ 是一圓內接四邊形。AC 和 BD 相交於 G。
已知 AC =16 cm，$BC = CD$ = 8 cm，$BG = x$ cm 和 $GD = y$ cm。
若 x 和 y 皆為整數且 $x + y = c$，求 c 的值。
$ABCD$ is a cyclic quadrilateral. AC and BD intersect at G.
Suppose AC =16 cm, $BC = CD$ = 8 cm, $BG = x$ cm and $GD = y$ cm.
If x and y are integers and $x + y = c$, find the value of c.

c =

4. 已知 $5^{\log 30}\times\left(\frac{1}{3}\right)^{\log 0.5}=d$。求 d 的值。
Given that $5^{\log 30}\times\left(\frac{1}{3}\right)^{\log 0.5}=d$, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2000 – 2001)
Final Event 4 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. $x_1 = 2001$。當 $n > 1$，$x_n = \dfrac{n}{x_{n-1}}$。已知 $x_1x_2x_3\ldots x_{10} = a$，求 a 的值。

   $x_1 = 2001$. When $n > 1$, $x_n = \dfrac{n}{x_{n-1}}$. Given that $x_1x_2x_3\ldots x_{10} = a$, find the value of a.

   $a =$

2. 已知 $1^3 + 2^3 + 3^3 + \ldots + 2001^3$ 的個位數字為 b，求 b 的值。
   Given that the units digit of $1^3 + 2^3 + 3^3 + \ldots + 2001^3$ is b, find the value of b.

   $b =$

3. 甲乙兩人在一圓形跑道上同時同地相背以均速開跑。他們第一次相遇後，乙再跑 1 分鐘到達原起步點。已知甲和乙分別需要 6 分鐘和 c 分鐘繞跑道一周，求 c 的值。

   A and B ran around a circular path with constant speeds. They started from the same place and at the same time in opposite directions. After their first meeting, B took 1 minute to go back to the starting place. If A and B need 6 minutes and c minutes respectively to complete one round of the path, find the value of c.

   $c =$

4. 方程 $x^2 - 45x + m = 0$ 的兩個根皆為質數。已知兩根的平方和為 d，求 d 的值。
   The roots of the equation $x^2 - 45x + m = 0$ are prime numbers.
   Given that the sum of the squares of the roots is d, find the value of d.

   $d =$

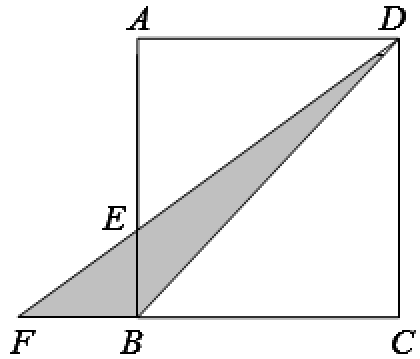
FOR OFFICIAL USE

Score for accuracy		× Mult. factor for speed		=		Team No.	
		+ Bonus score				Time	
		Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2001 - 2002)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 在右圖中，$ABCD$ 是一邊長為 10 cm 的正方形，AEB、FED 及 FBC 為直綫，ΔAED 的面積比 ΔFEB 的面積大 10 cm^2。若 ΔDFB 的面積為 P cm^2，求 P 的值。
In the following figure, $ABCD$ is a square of length 10 cm. AEB, FED and FBC are straight lines. The area of ΔAED is larger than that of ΔFEB by 10 cm^2. If the area of ΔDFB is P cm^2, find the value of P .

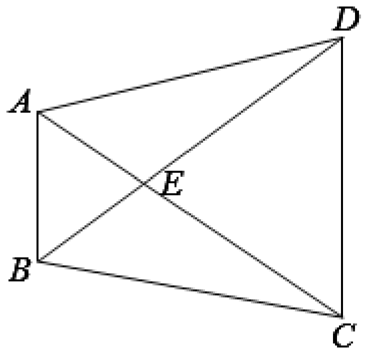


P =

2. 一件工程，甲單獨需時 90 天完成，而乙則需時 Q 天。若甲、乙二人合做只需 P 天完成，求 Q 的值。
Workman A needs 90 days to finish a task independently while workman B needs Q days for the same task. If they only need P days to finish the task when working together, find the value of Q .

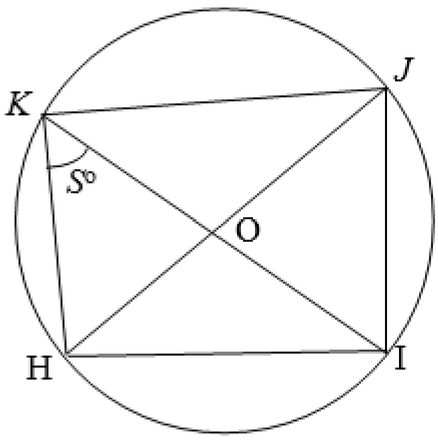
Q =

3. 在右圖中，$AB//CD$，梯形 $ABCD$ 的面積為 R cm^2。已知 ΔABE 和 ΔCDE 的面積分別為 Q cm^2 和 $4Q$ cm^2，求 R 的值。
In the following figure, $AB // CD$, the area of trapezium $ABCD$ is R cm^2. Given that the areas of ΔABE and ΔCDE are Q cm^2 and $4Q$ cm^2 respectively, find the value of R .



R =

4. 在右圖中，O 為圓心，HJ 和 IK 為圓的直徑以及 $\angle HKI = S°$。
已知 $\angle HKI + \angle HOI + \angle HJI = \frac{1}{4}R°$，求 S 的值。
In the following figure, O is the centre of the circle, HJ and IK are diameters and $\angle HKI = S°$.
Given that $\angle HKI + \angle HOI + \angle HJI = \frac{1}{4}R°$,
find the value of S .



S =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2001 - 2002)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知 $P=\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{99\times 100}$，求 P 的值。
Given that $P=\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{99\times 100}$, find the value of P.

$P =$

2. 已知 $99Q=P\times(1+\frac{99}{100}+\frac{99^2}{100^2}+\frac{99^3}{100^3}+\ldots)$，求 Q 的值。
Given that $99Q=P\times(1+\frac{99}{100}+\frac{99^2}{100^2}+\frac{99^3}{100^3}+\ldots)$, find the value of Q.

$Q =$

3. 已知 x 及 R 為實數。若對所有 x，$\frac{2x^2+2Rx+R}{4x^2+6x+3}\le Q$，求 R 的最大值。
Given that x and R are real numbers and $\frac{2x^2+2Rx+R}{4x^2+6x+3}\le Q$ for all x, find the maximum value of R.

$R =$

4. 已知 $S=\log_{144}\sqrt[R]{2}+\log_{144}\sqrt[2R]{R}$，求 S 的值。
Given that $S=\log_{144}\sqrt[R]{2}+\log_{144}\sqrt[2R]{R}$, find the value of S.

$S =$

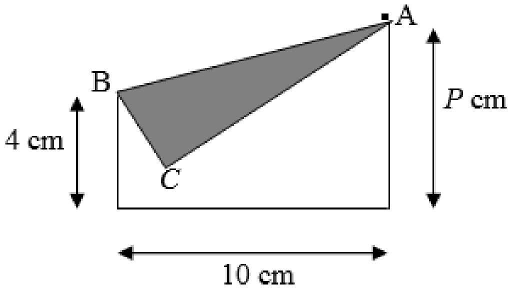
FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2001 - 2002)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 將一長方形紙摺出以下的圖形。若△ABC 的面積是原長方形紙面積的 $\frac{1}{3}$，求 P 的值。
A rectangular piece of paper is folded into the following figure. If the area of △ABC is $\frac{1}{3}$ of the area of the original rectangular piece of paper, find the value of P .



P =

2. 已知 $\frac{P}{2}(4^x + 4^{-x}) - 35(2^x + 2^{-x}) + 62 = 0$。若 Q 是此方程的正整數解，求 Q 的值。
If Q is the positive integral solution of the equation $\frac{P}{2}(4^x + 4^{-x}) - 35(2^x + 2^{-x}) + 62 = 0$, find the value of Q .

Q =

3. 設 $[a]$ 表示不大於 a 的最大整數，例如 $[2.5] = 2$。
若 $R = [\sqrt{1}] + [\sqrt{2}] + \ldots + [\sqrt{99Q}]$，求 R 的值。
Let $[a]$ be the largest integer not greater than a. For example, $[2.5] = 2$.
If $R = [\sqrt{1}] + [\sqrt{2}] + \ldots + [\sqrt{99Q}]$, find the value of R .

R =

4. 一個凸多邊形，除了內角 A 以外，其他內角的和是 $4R°$。若 $\angle A = S°$，求 S 的值。
In a convex polygon, other than the interior angle A, the sum of all the remaining interior angles is equal to $4R°$. If $\angle A = S°$, find the value of S .

S =

FOR OFFICIAL USE

Score for accuracy ☐ × Mult. factor for speed ☐ = ☐   Team No. ☐

+ Bonus score ☐   Time ☐ ☐

Total score ☐   Min. Sec.

Hong Kong Mathematics Olympiad (2001 - 2002)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

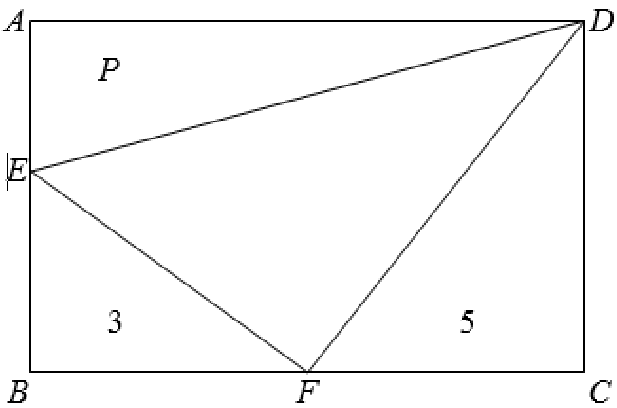
1. 已知 $f(x) = (x^2 + x - 2)^{2002} + 3$ 及 $f\left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right) = P$，求 P 的值。

Given that $f(x) = (x^2 + x - 2)^{2002} + 3$ and $f\left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right) = P$, find the value of P.

P =

2. 在下圖中，$ABCD$ 為一長方形。E 和 F 分別是 AB 和 BC 上的點。三角形 AED、EBF 和 FCD 的面積分別為 P、3 和 5。若ΔEFD 的面積為 Q，求 Q 的值。

In the following figure, $ABCD$ is a rectangle. E and F are points on AB and BC respectively. The areas of triangles AED, EBF and FCD are P, 3 and 5 respectively. If the area of ΔEFD is Q, find the value of Q.



Q =

3. 已知 x 和 y 為兩正整數。若不等式 $x^2 + y^2 \le Q$ 的解(x, y)的數目為 R，求 R 的值。

It is given that x and y are positive integers.

If the number of solutions (x, y) of the inequality $x^2 + y^2 \le Q$ is R, find the value of R.

R =

4. 已知 α 和 β 是方程 $x^2 - ax + a - R = 0$ 的兩個根，其中 a 為實數。
若 $(\alpha+1)^2 + (\beta+1)^2$ 的最小值為 S，求 S 的值。

It is given that α and β are roots of the equation $x^2 - ax + a - R = 0$, where a is real.
If the minimum value of $(\alpha+1)^2 + (\beta+1)^2$ is S, find the value of S.

S =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2001 - 2002)
Final Event 1 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

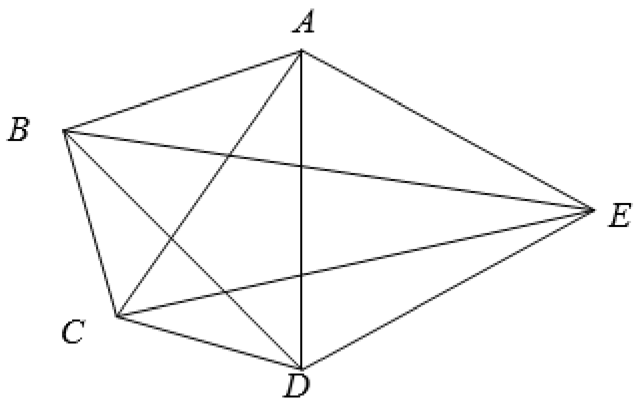
1. 假設曲綫 $x^2 + 3y^2 = 12$ 及直綫 $mx + y = 16$ 只相交於一點。若 $a = m^2$，求 a 的值。
Assume that the curve $x^2 + 3y^2 = 12$ and the straight line $mx + y = 16$ intersect at only one point. If $a = m^2$, find the value of a .

$a =$

2. 已知 $x + y = 1$ 及 $x^2 + y^2 = 2$。若 $x^3 + y^3 = b$，求 b 的值。
It is given that $x + y = 1$ and $x^2 + y^2 = 2$. If $x^3 + y^3 = b$, find the value of b .

$b =$

3. 在右圖中，$AC = AD = AE = ED = DB$ 及 $\angle BEC = c°$。已知 $\angle BDC = 26°$及 $\angle ADB = 46°$，求 c 的值。
In the following figure, $AC = AD = AE = ED = DB$ and $\angle BEC = c°$. Given that $\angle BDC = 26°$ and $\angle ADB = 46°$, find the value of c .



$c =$

4. 已知 $4 \cos^4 \theta + 5 \sin^2 \theta - 4 = 0$，其中 $0° < \theta < 360°$。若 θ 的最大值為 d，求 d 的值。
It is given that $4 \cos^4 \theta + 5 \sin^2 \theta - 4 = 0$, where $0° < \theta < 360°$.
If the maximum value of θ is d, find the value of d .

$d =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]　　Team No. [ ]

+ Bonus score [ ]　　Time [ ] [ ]

Total score [ ]　　Min.　Sec.

Hong Kong Mathematics Olympiad (2001 - 2002)
Final Event 2 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知三角形三邊的長分別為 6、8 和 10。若這三角形的面積為 a，求 a 的值。
It is given that the lengths of the sides of a triangle are 6, 8, and 10.
If the area of the triangle is a, find the value of a .

a =

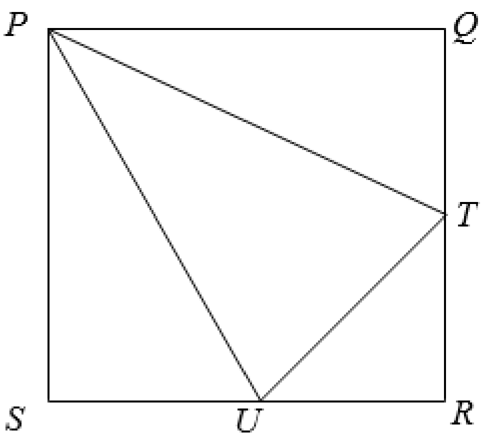
2. 已知 $f\left(x+\frac{1}{x}\right)=x^3+\frac{1}{x^3}$ 。若 f(4) = b，求 b 的值。
Given that $f\left(x+\frac{1}{x}\right)=x^3+\frac{1}{x^3}$ and f(4) = b, find the value of b .

b =

3. 已知 $2002^2 - 2001^2 + 2000^2 - 1999^2 + \ldots + 4^2 - 3^2 + 2^2 - 1^2 = c$，求 c 的值。
Given that $2002^2 - 2001^2 + 2000^2 - 1999^2 + \ldots + 4^2 - 3^2 + 2^2 - 1^2 = c$,
find the value of c .

c =

4. $PQRS$ 為一正方形，PTU 為一等腰三角形及 $\angle TPU = 30°$。T 及 U 分別為 QR 及 RS 上的點。ΔPTU 之面積為 1。
若正方形 $PQRS$ 之面積為 d，求 d 的值。
$PQRS$ is a square, PTU is an isosceles triangle, and $\angle TPU = 30°$. Points T and U lie on QR and RS respectively. The area of ΔPTU is 1.
If the area of $PQRS$ is d, find the value of d .



d =

FOR OFFICIAL USE

Score for accuracy × Mult. factor for speed =

Team No.

+ Bonus score

Time

Total score

Min. Sec.

Hong Kong Mathematics Olympiad (2001 - 2002)
Final Event 3 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 $\dfrac{2002^3+4\times 2002^2+6006}{2002^2+2002}=a$，求 a 的值。

   If $\dfrac{2002^3+4\times 2002^2+6006}{2002^2+2002}=a$, find the value of a.

   a =

2. 已知 x 和 y 為兩實數且滿足關係 $y=\dfrac{x}{2x-1}$。若 $\dfrac{1}{x^2}+\dfrac{1}{y^2}$ 的最小值為 b，求 b 的值。

   It is given that the real numbers x and y satisfy the relation $y=\dfrac{x}{2x-1}$.

   If the minimum value of $\dfrac{1}{x^2}+\dfrac{1}{y^2}$ is b, find the value of b.

   b =

3. 從 50 個正整數 1，2，3，…，50 中任意抽兩個不同的數。
   已知兩數之和不少於 50。若抽取這兩數共有 c 種取法，求 c 的值。
   Suppose two different numbers are chosen randomly from the 50 positive integers 1, 2, 3, … , 50, and the sum of these two numbers is not less than 50.
   If the number of ways of choosing these two numbers is c, find the value of c.

   c =

4. 已知 $x-y=1+\sqrt{5}$，$y-z=1-\sqrt{5}$。若 $x^2+y^2+z^2-xy-yz-zx=d$，求 d 的值。
   Given that $x-y=1+\sqrt{5}$, $y-z=1-\sqrt{5}$.
   If $x^2+y^2+z^2-xy-yz-zx=d$, find the value of d.

   d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2001 - 2002)
Final Event 4 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 a 是 2002 的所有正因數之和，求 a 的值。
If a is the sum of all the positive factors of 2002, find the value of a .

a =

2. 設 $x > 0$，$y > 0$ 且 $\sqrt{x}(\sqrt{x}+\sqrt{y}) = 3\sqrt{y}(\sqrt{x}+5\sqrt{y})$。
若 $b = \dfrac{2x+\sqrt{xy}+3y}{x+\sqrt{xy}-y}$，求 b 的值。
It is given that $x > 0$, $y > 0$ and $\sqrt{x}(\sqrt{x}+\sqrt{y}) = 3\sqrt{y}(\sqrt{x}+5\sqrt{y})$.
If $b = \dfrac{2x+\sqrt{xy}+3y}{x+\sqrt{xy}-y}$, find the value of b .

b =

3. 若方程 $\left||x-2|-1\right| = c$ 只有 3 個整數解，求 c 的值。
Given that the equation $\left||x-2|-1\right| = c$ = c has only 3 integral solutions, find the value of c .

c =

4. 若 d 是方程 $\frac{1}{2}\left\{\frac{1}{2}\left[\frac{1}{2}\left(\frac{1}{2}x^2+2\right)+2\right]+2\right\}=2$ 的正實數解，求 d 的值。
If d is the positive real root of the equation $\frac{1}{2}\left\{\frac{1}{2}\left[\frac{1}{2}\left(\frac{1}{2}x^2+2\right)+2\right]+2\right\}=2$, find the value of d .

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	(Min.)	(Sec.)
			Total score					

Hong Kong Mathematics Olympiad (2002 – 2003)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設 P 是 $3^{2003}\times 5^{2002}\times 7^{2001}$ 的個位數。求 P 的值。
Let P be the units digit of $3^{2003}\times 5^{2002}\times 7^{2001}$. Find the value of P .

P =

2. 若方程 $(x^2-x-1)^{x+P-1}=1$ 有 Q 個整數解，求 Q 的值。
If the equation $(x^2-x-1)^{x+P-1}=1$ has Q integral solutions, find the value of Q .

Q =

3. 設 x、y 為實數且 $xy=1$。若 $\frac{1}{x^4}+\frac{1}{Qy^4}$ 的最小值是 R，求 R 的值。
Let x, y be real numbers and $xy = 1$. If the minimum value of $\frac{1}{x^4}+\frac{1}{Qy^4}$ is R, find the value of R .

R =

4. 設 x_R、x_{R+1}、…、x_K $(K>R)$ 為 $K-R+1$ 個不相同的正整數且 $x_R+x_{R+1}+\ldots+x_K=2003$。若 S 是 K 的最大可能的值，求 S 的值。
Let $x_R, x_{R+1}, \ldots, x_K$ $(K>R)$ be $K-R+1$ distinct positive integers and $x_R+x_{R+1}+\ldots+x_K=2003$.
If S is the maximum possible value of K, find the value of S .

S =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2002 – 2003)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若一個兩位數 P 的 50 次方是一個 69 位數，求 P 的值。
(已知 $\log 2 = 0.3010$，$\log 3 = 0.4771$，$\log 11 = 1.0414$)
If the 50^{th} power of a two-digit number P is a 69-digit number, find the value of P .
(Given that $\log 2 = 0.3010$, $\log 3 = 0.4771$, $\log 11 = 1.0414$.)

P =

2. 方程式 $x^2 + ax - P + 7 = 0$ 的根是 α 和 β；而方程式 $x^2 + bx - r = 0$ 的根是 $-\alpha$ 和 $-\beta$。若方程式 $(x^2 + ax - P + 7) + (x^2 + bx - r) = 0$ 的正根是 Q，求 Q 的值。
The roots of the equation $x^2 + ax - P + 7 = 0$ are α and β, whereas the roots of the equation $x^2 + bx - r = 0$ are $-\alpha$ and $-\beta$. If the positive root of the equation $(x^2 + ax - P + 7) + (x^2 + bx - r) = 0$ is Q , find the value of Q .

Q =

3. 已知ΔABC 為一等腰三角形，$AB = AC = \sqrt{2}$ 及 BC 上有 Q 個點 D_1、D_2、 …、D_Q。設 $m_i = AD_i^2 + BD_i\times D_iC$。若 $m_1 + m_2 + m_3 + \ldots + m_Q = R$，求 R 的值。
Given that ΔABC is an isosceles triangle, $AB = AC = \sqrt{2}$, and D_1, D_2, … , D_Q are Q points on BC. Let $m_i = AD_i^2 + BD_i\times D_iC$.
If $m_1 + m_2 + m_3 + \ldots + m_Q = R$, find the value of R .

R =

4. 有 2003 個袋從左至右排列。已知最左面的袋裝有 R 個球，而且每 7 個相鄰的袋共裝有 19 個球。若最右面的袋有 S 個球，求 S 的值。
There are 2003 bags arranged from left to right. It is given that the leftmost bag contains R balls, and every 7 consecutive bags contains 19 balls altogether.
If the rightmost bag contains S balls, find the value of S .

S =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2002 – 2003)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知 $\begin{cases} wxyz = 4 \\ w - xyz = 3 \end{cases}$ 且 $w > 0$。若 w 的解是 P，求 P 的值。

Given that $\begin{cases} wxyz = 4 \\ w - xyz = 3 \end{cases}$ and $w > 0$. If the solution of w is P, find the value of P.

$P =$

2. 設 $[y]$ 表示小數 y 的整數部分，如 $[3.14] = 3$。若 $\left[\left(\sqrt{2}+1\right)^P\right] = Q$，求 Q 的值。

Let $[y]$ represents the integral part of the decimal number y.

For example, $[3.14] = 3$. If $\left[\left(\sqrt{2}+1\right)^P\right] = Q$, find the value of Q.

$Q =$

3. 已知 $x_0y_0 \neq 0$ 及 $Qx_0^2 - 22\sqrt{3}\,x_0y_0 + 11y_0^2 = 0$。若 $\frac{6x_0^2 + y_0^2}{6x_0^2 - y_0^2} = R$,，求 R 的值。

Given that $x_0y_0 \neq 0$ and $Qx_0^2 - 22\sqrt{3}\,x_0y_0 + 11y_0^2 = 0$.

If $\frac{6x_0^2 + y_0^2}{6x_0^2 - y_0^2} = R$, find the value of R.

$R =$

4. 四邊形 $ABCD$ 兩對角綫 AC 和 BD 互相垂直。$AB = 5$，$BC = 4$，$CD = R$。若 $DA = S$，求 S 的值。
The diagonals AC and BD of a quadrilateral $ABCD$ are perpendicular to each other. Given that $AB = 5$, $BC = 4$, $CD = R$. If $DA = S$, find the value of S.

$S =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.		
			+	Bonus score			Time		
				Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2002 – 2003)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 如果 9 位數 $\overline{32x35717y}$ 是 72 的倍數，$P = xy$，求 P 的值。
Suppose the 9-digit number $\overline{32x35717y}$ is a multiple of 72, and $P = xy$, find the value of P.

$P =$

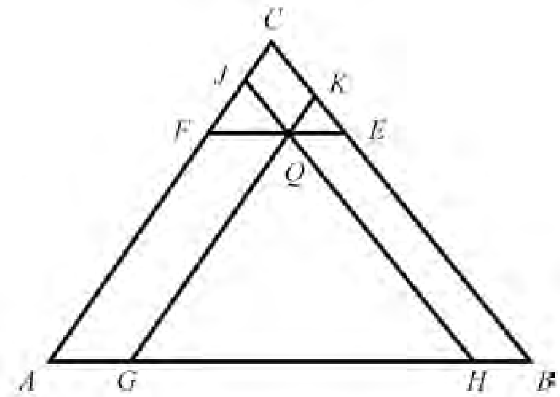
2. 已知三條直綫 $4x + y = \frac{P}{3}$，$mx + y = 0$ 和 $2x - 3my = 4$ 不能構成一個三角形。若 $m > 0$ 及 Q 是 m 的最小可能的值，求 Q 的值。
Given that the lines $4x + y = \frac{P}{3}$, $mx + y = 0$ and $2x - 3my = 4$ cannot form a triangle. Suppose that $m > 0$ and Q is the minimum possible value of m, find Q.

$Q =$

3. 已知 R，x，y 及 z 是整數且 $R > x > y > z$。若 R，x，y 及 z 滿足方程 $2^R + 2^x + 2^y + 2^z = \frac{495Q}{16}$，求 R 的值。
Given that R, x, y, z are integers and $R > x > y > z$.
If R, x, y, z satisfy the equation $2^R + 2^x + 2^y + 2^z = \frac{495Q}{16}$, find the value of R.

$R =$

4.



圖一 Figure 1

如圖一，ΔABC 內任選一點 Q，通過 Q 作三條分別平行於各邊的直綫，其中 $FE // AB$，$GK // AC$ 及 $HJ // BC$。ΔKQE，ΔJFQ 及 ΔQGH 的面積分別是 R，9 及 49。若 ΔABC 的面積是 S，求 S 的值。
In Figure 1, Q is the interior point of ΔABC. Three straight lines passing through Q are parallel to the sides of the triangle such that $FE // AB$, $GK // AC$ and $HJ // BC$.
Given that the areas of ΔKQE, ΔJFQ and ΔQGH are R, 9 and 49 respectively.
If the area of ΔABC is S, find the value of S.

$S =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

\+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min. Sec.

Hong Kong Mathematics Olympiad (2002 – 2003)
Final Event 1 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

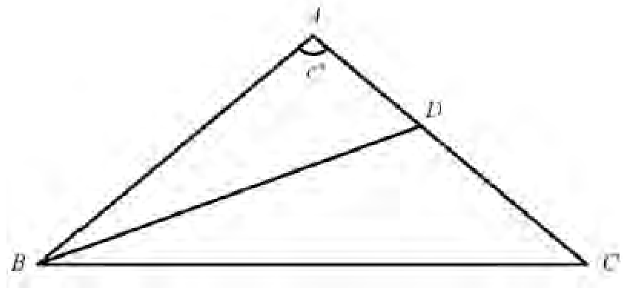
1. 已知 n、k 皆為自然數，且 $1<k<n$。
若 $\dfrac{(1+2+3+\cdots+n)-k}{n-1}=10$ 及 $n+k=a$，求 a 的值。
Given that n and k are natural numbers and $1<k<n$.
If $\dfrac{(1+2+3+\cdots+n)-k}{n-1}=10$ and $n+k=a$, find the value of a.

a =

2. 已知$(x-1)^2+y^2=4$，其中 x 和 y 是實數。若 $2x+y^2$ 的極大值是 b，求 b 的值。
Given that $(x-1)^2+y^2=4$, where x and y are real numbers.
If the maximum value of $2x+y^2$ is b, find the value of b.

b =

3.



圖一
Figure 1

如圖一，ΔABC 是一個等腰三角形，其中 $AB=AC$。
若 $\angle B$ 的角平分綫交 AC 於 D 且 $BC=BD+AD$。設 $\angle A=c°$，求 c 的值。
In Figure 1, ΔABC is an isosceles triangle and $AB=AC$. Suppose the angle bisector of $\angle B$ meets AC at D and $BC=BD+AD$. Let $\angle A=c°$, find the value of c.

c =

4. 兩質數之和為 105。若這兩質數之積為 d，求 d 的值。
Given that the sum of two prime numbers is 105.
If the product of these prime numbers is d, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

\+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (2002 – 2003)
Final Event 2 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

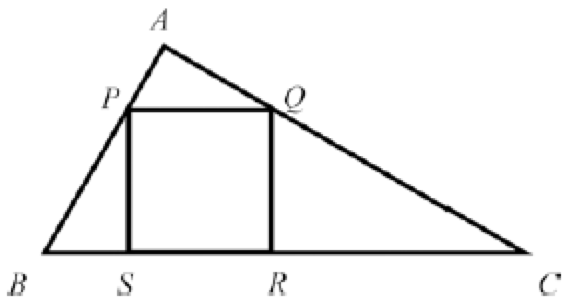
1. 設方程 $ax(x+1) + bx(x+2) + c(x+1)(x+2) = 0$ 有根 1 和 2。若 $a + b + c = 2$，求 a 的值。
Given that the equation $ax(x + 1) + bx(x + 2) + c(x + 1)(x + 2) = 0$ has roots 1 and 2. If $a + b + c = 2$, find the value of a.

$a =$

2. 設 $48^x = 2$，$48^y = 3$。若 $8^{\frac{x+y}{1-x-y}} = b$，求 b 的值。
Given that $48^x = 2$ and $48^y = 3$. If $8^{\frac{x+y}{1-x-y}} = b$, find the value of b.

$b =$

3.



圖一
Figure 1

如圖一，正方形 $PQRS$ 內接於 ΔABC。ΔAPQ、ΔPBS 和 ΔQRC 的面積分別為 4、4 和 12。若正方形 $PQRS$ 的面積為 c，求 c 的值。
In Figure 1, the square $PQRS$ is inscribed in ΔABC. The areas of ΔAPQ, ΔPBS and ΔQRC are 4, 4 and 12 respectively. If the area of the square is c, find the value of c.

$c =$

4. 在 ΔABC 中，$\cos A = \frac{4}{5}$ 和 $\cos B = \frac{7}{25}$。若 $\cos C = d$，求 d 的值。
In ΔABC, $\cos A = \frac{4}{5}$ and $\cos B = \frac{7}{25}$. If $\cos C = d$, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

\+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (2002 – 2003)
Final Event 3 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設 f 為一函數，f(1) = 1，並對任意整數 m 及 n，$f(m + n) = f(m) + f(n) + mn$。若 $a = \frac{f(2003)}{6}$，求 a 的值。
Let f be a function such that f(1) = 1 and for any integers m and n, $f(m + n) = f(m) + f(n) + mn$. If $a = \frac{f(2003)}{6}$, find the value of a.

a =

2. 若 $x^{\frac{1}{2}} + x^{-\frac{1}{2}} = 3$，$b = \frac{x^{\frac{3}{2}} + x^{-\frac{3}{2}} - 3}{x^2 + x^{-2} - 2}$，求 b 的值。
Suppose $x^{\frac{1}{2}} + x^{-\frac{1}{2}} = 3$, $b = \frac{x^{\frac{3}{2}} + x^{-\frac{3}{2}} - 3}{x^2 + x^{-2} - 2}$, find the value of b.

b =

3. 已知 $f(n) = \sin\frac{n\pi}{4}$，其中 n 是整數。若 $c = f(1) + f(2) + \cdots + f(2003)$，求 c 的值。
Given that $f(n) = \sin\frac{n\pi}{4}$, where n is an integer.
If $c = f(1) + f(2) + \cdots + f(2003)$, find the value of c.

c =

4. 已知函數 $f(x) = \begin{cases} -2x+1, & \text{when } x<1 \\ x^2 - 2x, & \text{when } x \ge 1 \end{cases}$。若 d 是 $f(x) = 3$ 的最大整數解，求 d 的值。
Given that $f(x) = \begin{cases} -2x+1, & \text{when } x<1 \\ x^2 - 2x, & \text{when } x \ge 1 \end{cases}$.
If d is the maximum integral solution of $f(x) = 3$, find the value of d.

d =

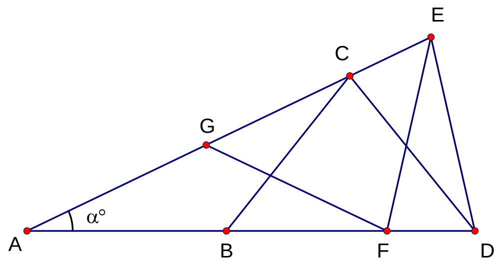
FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time		
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2002 – 2003)
Final Event 4 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 如圖一，AE 、AD 是直綫且 $AB = BC = CD = DE = EF = FG = GA$。若 $\angle DAE = \alpha°$，求 α 的值。
In Figure 1, AE and AD are two straight lines and $AB = BC = CD = DE = EF = FG = GA$. If $\angle DAE = \alpha°$, find the value of α.



圖一
Figure 1

α =

2. 設 $P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_8x^8$ 為八次多項式，其中 a_0、a_1、…、a_8 為實數。若 $P(k) = \frac{1}{k}$ 當 $k = 1, 2, \ldots, 9$, 及 $b = P(10)$，求 b 的值。

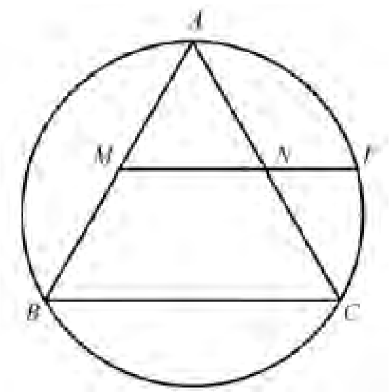
Suppose $P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_8x^8$ is a polynomial of degree 8 with real coefficients $a_0, a_1, \ldots, a_8$. If $P(k) = \frac{1}{k}$ when $k = 1, 2, \ldots, 9$, and $b = P(10)$, find the value of b.

b =

3. 已知 x，y 為兩正整數使 $xy - (x + y) = \text{HCF}(x, y) + \text{LCM}(x, y)$，其中 HCF($x$, y) 和 LCM(x, y) 分別是 x 和 y 的最大公因數和最小公倍數。若 c 是 $x + y$ 的最大可能的值，求 c。
Given two positive integers x and y, $xy - (x + y) = \text{HCF}(x, y) + \text{LCM}(x, y)$, where HCF($x$, y) and LCM(x, y) are respectively the greatest common divisor and the least common multiple of x and y. If c is the maximum possible value of $x + y$, find c.

c =

4. 如圖二，ΔABC 是等邊三角形，M 及 N 分別是 AB 及 AC 的中點，F 是直綫 MN 與圓 ABC 的交點。若 $d = \frac{MF}{MN}$，求 d 的值。
In Figure 2, ΔABC is an equilateral triangle, points M and N are the midpoints of sides AB and AC respectively, and F is the intersection of the line MN with the circle ABC.
If $d = \frac{MF}{MN}$, find the value of d.



圖二 Figure 2

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time		
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2003-04)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知有 a 個少於 200 的正整數，它們每個都只有三個正因數，求 a 的值。
Given that there are a positive integers less than 200 and each of them has exactly three positive factors, find the value of a .

a =

2. 若 a 個斜邊是 $\sqrt{2}$ cm 的等腰直角三角形能拼成一個周界是 b cm 的梯形，求 b 的最小可能的值。(答案用根號表示)
If a copies of a right-angled isosceles triangle with hypotenuse $\sqrt{2}$ cm can be assembled to form a trapezium with perimeter equal to b cm,
find the least possible value of b. (give the answer in surd form).

b =

3. 若 $\sin(c^2 - 3c + 17)° = \frac{4}{b-2}$，其中 $0 < c^2 - 3c + 17 < 90$ 及 $c > 0$，求 c 的值。
If $\sin(c^2 - 3c + 17)° = \frac{4}{b-2}$, where $0 < c^2 - 3c + 17 < 90$ and $c > 0$,
find the value of c.

c =

4. 已知兩個三位數 $\overline{xyz}$ 和 $\overline{zyx}$ 的差等於 $700 - c$，其中 $x > z$。
若 d 是 $x + z$ 的最大值，求 d 的值。
Given that the difference between two 3-digit numbers $\overline{xyz}$ and $\overline{zyx}$ is $700 - c$,
where $x > z$. If d is the greatest value of $x + z$, find the value of d .

d =

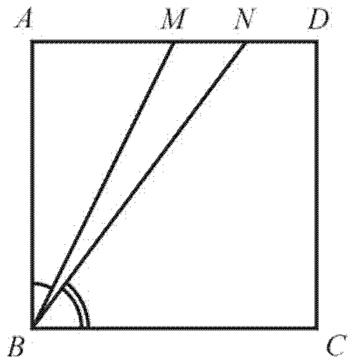
FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2003-04)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 如圖一，$ABCD$ 為一正方形，M 是 AD 的中點及 N 是 MD 的中點及 N 是 MD 的中點。
若 $\angle CBN : \angle MBA = P : 1$，求 P 的值。
In Figure 1, $ABCD$ is a square, M is the mid-point of AD and N is the mid-point of MD.
If $\angle CBN : \angle MBA = P : 1$, find the value of P.



圖一
Figure 1

P =

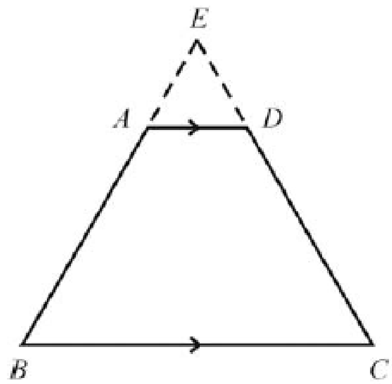
2. 已知 $ABCD$ 為一坐標平面上的菱形，其頂點的座標分別為 $A(0, 0)$，$B(P, 1)$，$C(u, v)$ 及 $D(1, P)$。若 $u + v = Q$，求 Q 的值。
Given that $ABCD$ is a rhombus on a Cartesian plane, and the co-ordinates of its vertices are $A(0, 0)$, $B(P, 1)$, $C(u, v)$ and $D(1, P)$ respectively.
If $u + v = Q$, find the value of Q.

Q =

3. 若 $1 + (1 + 2) + (1 + 2 + 3) + \ldots + (1 + 2 + 3 + \ldots + Q) = R$，求 R 的值。
If $1 + (1 + 2) + (1 + 2 + 3) + \ldots + (1 + 2 + 3 + \ldots + Q) = R$, find the value of R.

R =

4. 如圖二，EBC 是一等邊三角形，A 和 D 分別在 EB 和 EC 上。已知 $AD // BC$，$AB = CD = R$，且 $AC \perp BD$。
若梯形 $ABCD$ 的面積是 S，求 S 的值。
In figure 2, EBC is an equilateral triangle, and A, D lie on EB and EC respectively. Given that $AD // BC$, $AB = CD = R$ and $AC \perp BD$. If the area of the trapezium $ABCD$ is S, find the value of S.



圖二
Figure 2

S =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.		
			+ Bonus score				Time		
			Total score					Min.	Sec.

Hong Kong Mathematics Olympiad (2003-04)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設 $x \neq \pm 1$ 及 $x \neq -3$。若 a 是方程 $\frac{1}{x-1}+\frac{1}{x+3}=\frac{2}{x^2-1}$ 的實根，求 a 的值。

Let $x \neq \pm 1$ and $x \neq -3$. If a is the real root of the equation $\frac{1}{x-1}+\frac{1}{x+3}=\frac{2}{x^2-1}$, find the value of a.

$a =$

2. 設 $b > 1$，$f(b)=\frac{-a}{\log_2 b}$ 及 $g(b)=1+\frac{1}{\log_3 b}$。

若 b 滿足方程 $|f(b) - g(b)| + f(b) + g(b) = 3$，求 b 的值。

If $b > 1$，$f(b)=\frac{-a}{\log_2 b}$ and $g(b)=1+\frac{1}{\log_3 b}$.

If b satisfies the equation $|f(b) - g(b)| + f(b) + g(b) = 3$, find the value of b.

$b =$

3. 已知實數 x_0 滿足方程 $x^2 - 5x + (b - 8) = 0$。若 $c=\frac{x_0^2}{x_0^4+x_0^2+1}$，求 c 的值。

Given that x_0 satisfies the equation $x^2 - 5x + (b - 8) = 0$.

If $c=\frac{x_0^2}{x_0^4+x_0^2+1}$, find the value of c.

$c =$

4. 若 -2 和 $216c$ 是方程 $px^2 + dx = 1$ 的根，求 d 的值。

If -2 and $216c$ are the roots of the equation $px^2 + dx = 1$, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	Min. / Sec.
			Total score					

Hong Kong Mathematics Olympiad (2003-04)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

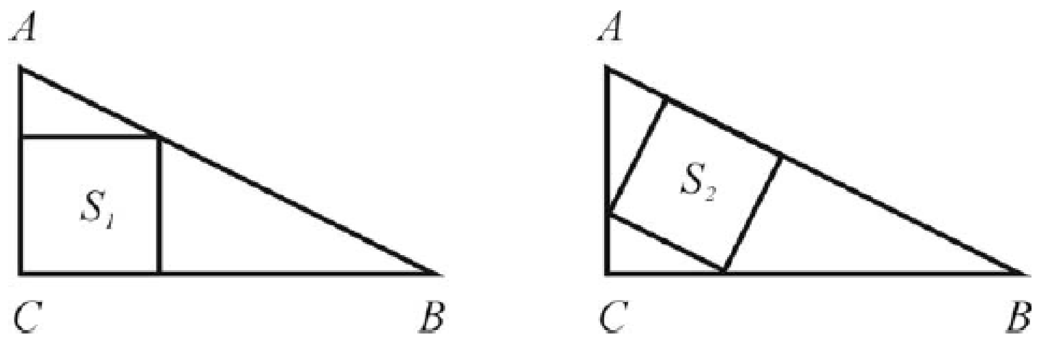
1. 設 a 為實數。若 a 滿足方程 $\log_2(4^x + 4) = x + \log_2(2^{x+1} - 3)$，求 a 的數值。
Let a be a real number.
If a satisfies the equation $\log_2(4^x + 4) = x + \log_2(2^{x+1} - 3)$, find the value of a .

a =

2. 已知 n 是自然數。若 $b = n^3 - 4an^2 - 12n + 144$ 是質數，求 b 的數值。
Given that n is a natural number. If $b = n^3 - 4an^2 - 12n + 144$ is a prime number, find the value of b .

b =

3.



圖一
Figure 1

如圖一，S_1 和 S_2 都是直角三角形 ABC 的兩個不同的正方形。
若 S_1 的面積是 $40b + 1$， S_2 的面積是 $40b$，及 $AC + CB = c$，求 c 的值。
In Figure 1, S_1 and S_2 are two different inscribed squares of the right-angled triangle ABC.
If the area of S_1 is $40b + 1$, the area of S_2 is $40b$ and $AC + CB = c$, find the value of c .

c =

4. 已知 $241c + 214 = d^2$，求 d 的正數值。
Given that $241c + 214 = d^2$, find the positive value of d .

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

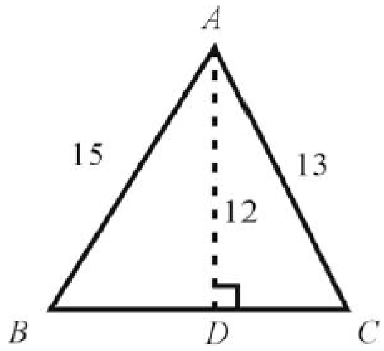
+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (2003-04)
Final Event Spare (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 如圖一，ΔABC 為一銳角三角形，$AB = 15$，$AC = 13$，而高 $AD = 12$。若ΔABC 的面積為 P，求 P 的值。
In figure 1, ΔABC is an acute triangle, $AB = 15$, $AC = 13$, and its altitude $AD = 12$. If the area of the ΔABC is P, find the value of P .



圖一
Figure 1

$P =$

2. 已知 x 和 y 是正整數。若 $x^4 + y^4$ 除以 $x + y$，所得的商是 $P + 13$，餘數是 Q，求 Q 的值。
Given that x and y are positive integers. If $x^4 + y^4$ is divided by $x + y$, the quotient is $P + 13$ and the remainder is Q, find the value of Q .

$Q =$

3. 已知一等邊三角形的周界與一個半徑是 $\frac{12}{Q}$ cm 的圓的周界相等。若這三角形的面積是 $R\pi^2$ cm^2，求 R 的值。(答案以根式表示)。
Given that the perimeter of an equilateral triangle equals to that of a circle with radius $\frac{12}{Q}$ cm. If the area of the triangle is $R\pi^2$ cm^2, find the value of R .

$R =$

4. 設 $W = \frac{\sqrt{3}}{2R}$，$S = W + \cfrac{1}{W + \cfrac{1}{W + \cfrac{1}{W + \cdots}}}$，求 S 的值。
Let $W = \frac{\sqrt{3}}{2R}$, $S = W + \cfrac{1}{W + \cfrac{1}{W + \cfrac{1}{W + \cdots}}}$, find the value of S .

$S =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

\+ Bonus score [ ]

Total score [ ]

Team No. [ ]

Time [ ] Min. [ ] Sec.

Hong Kong Mathematics Olympiad (2003-04)
Final Event 1 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知 a 為整數。若 50! 能被 2^a 整除，求 a 的最大可能的值。
Given that a is an integer.
If 50! is divisible by 2^a, find the largest possible value of a.

a =

2. 設 $[x]$ 表示不大於 x 的最大整數，例如 $[2.5] = 2$。
若 $b=\left[100\times\frac{11\times77+12\times78+13\times79+14\times80}{11\times76+12\times77+13\times78+14\times79}\right]$，求 b 的值。
Let $[x]$ be the largest integer not greater than x. For example, $[2.5] = 2$.
If $b=\left[100\times\frac{11\times77+12\times78+13\times79+14\times80}{11\times76+12\times77+13\times78+14\times79}\right]$, find the value of b.

b =

3. 若在 200 至 500 之間有 c 個數是 7 的倍數，求 c 的值。
If there are c multiples of 7 between 200 and 500, find the value of c.

c =

4. 已知 $0\le x_0\le\frac{\pi}{2}$ 且 x_0 滿足方程 $\sqrt{\sin x+1}-\sqrt{1-\sin x}=\sin\frac{x}{2}$。
若 $d=\tan x_0$，求 d 的值。
Given that $0\le x_0\le\frac{\pi}{2}$ and x_0 satisfies the equation $\sqrt{\sin x+1}-\sqrt{1-\sin x}=\sin\frac{x}{2}$.
If $d=\tan x_0$, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	Min.	Sec.
			Total score					

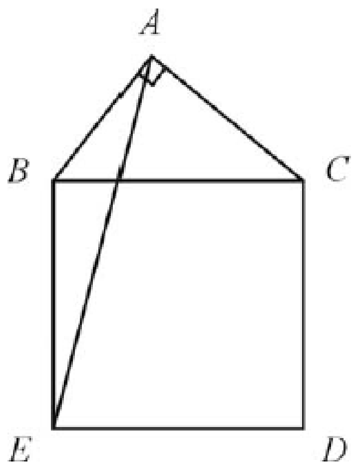
Hong Kong Mathematics Olympiad (2003-04)
Final Event 2 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 5^{5^5} 的十位數是 a，求 a 的值。
If the tens digit of 5^{5^5} is a, find the value of a.

a =

2. 如圖一，ΔABC 是一直角三角形，$AB = 3$ cm，$AC = 4$ cm 及 $BC = 5$ cm。若 $BCDE$ 是一正方形且 ΔABE 的面積是 b cm^2，求 b 的值。
In Figure 1, ΔABC is a right-angled triangle, $AB = 3$ cm, $AC = 4$ cm and $BC = 5$ cm. If $BCDE$ is a square and the area of ΔABE is b cm^2, find the value of b.



圖一 Figure 1

b =

3. 已知在 100 以內的質數中，其個位並非平方數的數目有 c 個，求 c 的值。
Given that there are c prime numbers less than 100 such that their unit digits are not square numbers, find the values of c.

c =

4. 若直綫 $y = x + d$ 與 $x = -y + d$ 相交於點 $(d - 1, d)$，求 d 的值。
If the lines $y = x + d$ and $x = -y + d$ intersect at the point $(d - 1, d)$, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2003-04)
Final Event 3 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 a 是方程 $\sqrt{x(x+1)(x+2)(x+3)+1}=71$ 的最小實數解，求 a 的值。
If a is the smallest real root of the equation $\sqrt{x(x+1)(x+2)(x+3)+1}=71$, find the value of a.

$a =$

2. 已知質數 p 和 q 滿足方程 $18p + 30q = 186$。若 $\log_8 \frac{p}{3q+1} = b \ge 0$，求 b 的值。
Given that p and q are prime numbers satisfying the equation $18p + 30q = 186$.
If $\log_8 \frac{p}{3q+1} = b \ge 0$, find the value of b.

$b =$

3. 已知對任意實數 x、y 及 z，運算 $\oplus$ 滿足
(i) $x \oplus 0 = 1$；及
(ii) $(x \oplus y) \oplus z = (z \oplus xy) + z$。
若 $1 \oplus 2004 = c$，求 c 的值。
Given that for any real numbers x, y and z, $\oplus$ is an operation satisfying
(i) $x \oplus 0 = 1$, and
(ii) $(x \oplus y) \oplus z = (z \oplus xy) + z$.
If $1 \oplus 2004 = c$, find the value of c.

$c =$

4. 已知 $f(x) = (x^4 + 2x^3 + 4x - 5)^{2004} + 2004$，若 $f\left(\sqrt{3}-1\right) = d$，求 d 的值。
Given that $f(x) = (x^4 + 2x^3 + 4x - 5)^{2004} + 2004$. If $f\left(\sqrt{3}-1\right) = d$, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

\+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (2003-04)
Final Event 4 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 $f(x)=\dfrac{4^x}{4^x+2}$ 及 $P=f\left(\dfrac{1}{1001}\right)+f\left(\dfrac{2}{1001}\right)+\cdots+f\left(\dfrac{1000}{1001}\right)$，求 P 的數值。

If $f(x)=\dfrac{4^x}{4^x+2}$ and $P=f\left(\dfrac{1}{1001}\right)+f\left(\dfrac{2}{1001}\right)+\cdots+f\left(\dfrac{1000}{1001}\right)$, find the value of P.

$P =$

2. 設 $f(x)=|x-a|+|x-15|+|x-a-15|$，其中 $a\le x\le 15$ 及 $0<a<15$。
若 Q 是 $f(x)$ 的最小值，求 Q 的值。
Let $f(x)=|x-a|+|x-15|+|x-a-15|$, where $a\le x\le 15$ and $0<a<15$.
If Q is the smallest value of $f(x)$, find the value of Q.

$Q =$

3. 若 $2^m=3^n=36$ 及 $R=\dfrac{1}{m}+\dfrac{1}{n}$，求 R 的值。

If $2^m=3^n=36$ and $R=\dfrac{1}{m}+\dfrac{1}{n}$, find the value of R.

$R =$

4. 設 $[x]$ 表示不大於 x 的最大整數，例如 $[2.5]=2$。
若 $a=1+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\cdots+\dfrac{1}{2004^2}$ 及 $S=[a]$，求 S 的值。
Let $[x]$ be the largest integer not greater than x, for example, $[2.5]=2$.
If $a=1+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\cdots+\dfrac{1}{2004^2}$ and $S=[a]$, find the value of S.

$S =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2003-04)
Final Event Spare (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

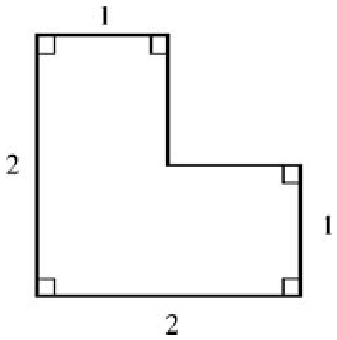
1. 對任意整數 n，F_n 的定義如下：$F_n = F_{n-1} + F_{n-2}$，$F_0 = 0$ 及 $F_1 = 1$。
若 $a = F_{-5} + F_{-4} + \ldots + F_4 + F_5$，求 a 的值。
For all integers n, F_n is defined by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$ and $F_1 = 1$.
If $a = F_{-5} + F_{-4} + \ldots + F_4 + F_5$, find the value of a.

$a =$

2. 已知 x_0 滿足方程 $x^2 + x + 2 = 0$。若 $b = x_0^4 + 2x_0^3 + 3x_0^2 + 2x_0 + 1$，求 b 的值。
Given that x_0 satisfies the equation $x^2 + x + 2 = 0$.
If $b = x_0^4 + 2x_0^3 + 3x_0^2 + 2x_0 + 1$, find the value of b.

$b =$

3. 圖一所示為一瓷磚圖形。若最少可用 C 塊該類瓷磚便能鋪滿一正方形，求 C 的值。
Figure 1 shows a tile. If C is the minimum number of tiles required to tile a square, find the value of C.



圖一 Figure 1

$C =$

4. 若直綫 $5x + 2y - 100 = 0$ 上有 d 個點，其 x 及 y 坐標的值都是正整數，求 d 的值。
If the line $5x + 2y - 100 = 0$ has d points whose x and y coordinates are both positive integers, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

Team No. [ ]

+ Bonus score [ ]

Time [ ] Min. [ ] Sec.

Total score [ ]

Hong Kong Mathematics Olympiad (2004 – 2005)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 一個動物園內有 a 頭駱駝，單峯的比雙峯的多 10 頭。若牠們共有 55 個峯，求 a 的值。
There are a camels in a zoo.
The number of one-hump camels exceeds that of two-hump camels by 10.
If there have 55 humps altogether, find the value of a .

a =

2. 若 LCM(a, b) = 280 及 HCF(a, b) = 10，求 b 的值。
If LCM(a, b) = 280 and HCF(a, b) = 10, find the value of b .

b =

3. 設 C 是一正整數且小於 $\sqrt{b}$ 。若 b 除以 C，餘數是 2。除以(C + 2)，餘數是 C，求 C 的值。
Let C be a positive integer less than $\sqrt{b}$. If b is divided by C, the remainder is 2; when divided by C + 2, the remainder is C, find the value of C .

C =

4. 一個正 2C 邊形共有 d 條對角綫，求 d 的值。
A regular 2C-sided polygon has d diagonals, find the value of d .

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2004 – 2005)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

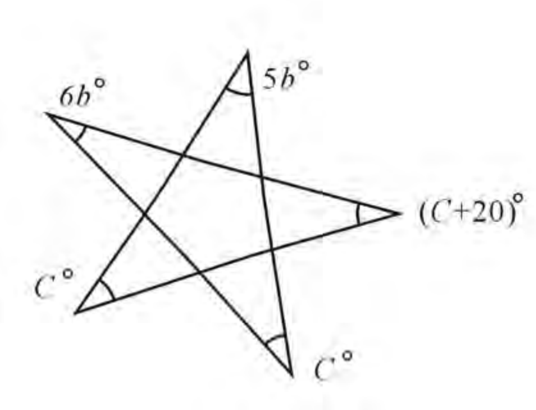
1. 陳先生有 8 個兒子和 a 個女兒，他的每個兒子都有 8 個兒子和 a 個女兒。他的每個女兒都有 a 個兒子和 8 個女兒。已知陳先生的男孫比女孫多 1 個及 a 是個質數，求 a 的值。
Mr. Chan has 8 sons and a daughters. Each of his sons has 8 sons and a daughters. Each of his daughters has a sons and 8 daughters. It is known that the number of his grand sons is one more than the number of his grand daughters and a is a prime number, find the value of a .

a =

2. 設 $\frac{a}{7}=\sqrt[3]{2+\sqrt{b}}+\sqrt[3]{2-\sqrt{b}}$ ，求 b 的值。
Let $\frac{a}{7}=\sqrt[3]{2+\sqrt{b}}+\sqrt[3]{2-\sqrt{b}}$. Find the value of b .

b =

3. 如圖一，求 C 的值。
In Figure 1, find the value of C .



圖一 Figure 1

C =

4. 已知 P_1、P_2、⋯、P_d 是 d 個連續質數。
若 $P_1+P_2+\cdots+P_{d-2}=P_{d-1}+P_d=C+1$，求 d 的值。
Given that $P_1, P_2, \cdots, P_d$ are d consecutive prime numbers.
If $P_1+P_2+\cdots+P_{d-2}=P_{d-1}+P_d=C+1$, find the value of d .

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time		
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2004 – 2005)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知 a 是方程 $2^{x+1}=8^{\frac{1}{x}-\frac{1}{3}}$ 的正實數解，求 a 的值。

Given that a is a positive real root of the equation $2^{x+1}=8^{\frac{1}{x}-\frac{1}{3}}$. Find the value of a.

$a =$

2. 在周界為 a 米的長方形中，最大面積的一個長方形的面積是 b 平方米，求 b 的值。

The largest area of the rectangle with perimeter a meter is b square meter, find the value of b.

$b =$

3. 若 $c = [1234^3 - 1232\times(1234^2 + 2472)]\times b$，求 c 的值。

If $c = [1234^3 - 1232\times(1234^2 + 2472)]\times b$, find the value of c.

$c =$

4. 若 $\frac{1}{(c+1)(c+2)}+\frac{1}{(c+2)(c+3)}+\cdots+\frac{1}{(c+d)(c+d+1)}=\frac{8}{15}$，求 d 的值。

If $\frac{1}{(c+1)(c+2)}+\frac{1}{(c+2)(c+3)}+\cdots+\frac{1}{(c+d)(c+d+1)}=\frac{8}{15}$, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

\+ Bonus score [ ]

Total score [ ]

Team No. [ ]

Time [ ] Min. [ ] Sec.

Hong Kong Mathematics Olympiad (2004 – 2005)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 $A^2 + B^2 + C^2 = AB + BC + CA = 3$ 及 $a = A^2$，求 a 的值。
If $A^2 + B^2 + C^2 = AB + BC + CA = 3$ and $a = A^2$, find the value of a.

$a =$

2. 已知 n 及 b 是整數，並滿足方程 $29n + 42b = a$，若 $5 < b < 10$，求 b 的值。
Given that n and b are integers satisfying the equation $29n + 42b = a$.
If $5 < b < 10$, find the value b.

$b =$

3. 若 $\dfrac{\sqrt{3}-\sqrt{5}+\sqrt{7}}{\sqrt{3}+\sqrt{5}+\sqrt{7}} = \dfrac{c\sqrt{21}-18\sqrt{15}-2\sqrt{35}+b}{59}$，求 c 的值。
If $\dfrac{\sqrt{3}-\sqrt{5}+\sqrt{7}}{\sqrt{3}+\sqrt{5}+\sqrt{7}} = \dfrac{c\sqrt{21}-18\sqrt{15}-2\sqrt{35}+b}{59}$, find the value of c.

$c =$

4. 若 c 有 d 個正因數，求 d 的值。
If c has d positive factors, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	Min. / Sec.
			Total score					

Hong Kong Mathematics Olympiad (2004 – 2005)
Final Event 1 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若在 1 至 200 內能同時被 3 和 7 整除的數有 a 個，求 a 的值。
Suppose there are a numbers between 1 and 200 that can be divisible by 3 and 7, find the value of a .

a =

2. 設質數 p 和 q 是方程 $x^2 - 13x + R = 0$ 的兩個不同的根，其中 R 是實數。
若 $b = p^2 + q^2$，求 b 的值。
Let p and q be prime numbers that are the two distinct roots of the equation $x^2 - 13x + R = 0$, where R is a real number. If $b = p^2 + q^2$, find the value of b .

b =

3. 已知 $\tan\alpha = -\frac{1}{2}$。若 $c = \frac{2\cos\alpha - \sin\alpha}{\sin\alpha + \cos\alpha}$，求 c 的值。
Given that $\tan\alpha = -\frac{1}{2}$. If $c = \frac{2\cos\alpha - \sin\alpha}{\sin\alpha + \cos\alpha}$, find the value of c .

c =

4. 設 r 和 s 是方程 $2\left(x^2 + \frac{1}{x^2}\right) - 3\left(x + \frac{1}{x}\right) = 1$ 的兩個不同的實數根。
若 $d = r + s$，求 d 的值。
Let r and s be the two distinct real roots of the equation $2\left(x^2 + \frac{1}{x^2}\right) - 3\left(x + \frac{1}{x}\right) = 1$. If $d = r + s$, find the value of d .

d =

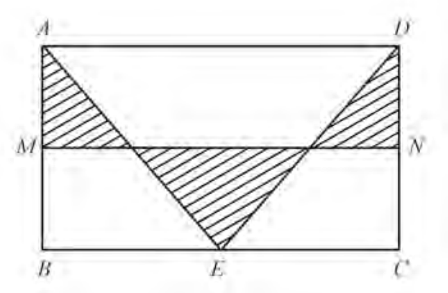
FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.
			+ Bonus score				Time
			Total score				Min. / Sec.

Hong Kong Mathematics Olympiad (2004 – 2005)
Final Event 2 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 如圖一，在長方形 $ABCD$ 中，$AB = 6$ cm，$BC = 10$ cm。M 和 N 分別是 AB 和 DC 的中點。若陰影部分的面積是 a cm^2，求 a 的值。
In Figure 1, $ABCD$ is a rectangle, $AB = 6$ cm and $BC = 10$ cm. M and N are the midpoints of AB and DC respectively. If the area of the shaded region is a cm^2, find the value of a.



圖一 Figure 1

a =

2. 設 $b = 89 + 899 + 8999 + 89999 + 899999$，求 b 的值。
Let $b = 89 + 899 + 8999 + 89999 + 899999$, find the value of b.

b =

3. 已知 $2x + 5y = 3$。若 $c = \sqrt{4^{x+\frac{1}{2}} \times 32^y}$，求 c 的值。
Given that $2x + 5y = 3$. If $c = \sqrt{4^{x+\frac{1}{2}} \times 32^y}$, find the value of c.

c =

4. 設 $d = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \ldots + \frac{10}{2^{10}}$，求 d 的值。
Let $d = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \ldots + \frac{10}{2^{10}}$, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ] Team No. [ ]

+ Bonus score [ ] Time [ ] [ ]

Total score [ ] Min. Sec.

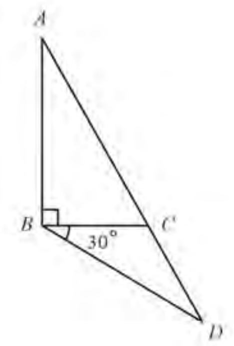
Hong Kong Mathematics Olympiad (2004 – 2005)
Final Event 3 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設 $0° < \alpha < 45°$。若 $\sin\alpha\cos\alpha = \frac{3\sqrt{7}}{16}$ 及 $A = \sin\alpha$，求 A 的值。
Let $0° < \alpha < 45°$. If $\sin\alpha\cos\alpha = \frac{3\sqrt{7}}{16}$ and $A = \sin\alpha$, find the value of A.

$A =$

2. 如圖一，C 在 AD 上且 $AB = BD = 1$ cm，$\angle ABC = 90°$，$\angle CBD = 30°$。若 $CD = b$ cm，求 b 的值。
In figure 1, C lies on AD, $AB = BD = 1$ cm, $\angle ABC = 90°$ and $\angle CBD = 30°$. If $CD = b$ cm, find the value of b.

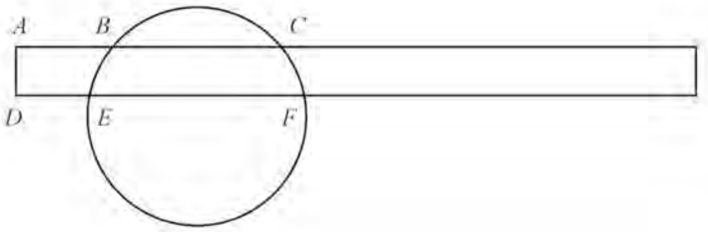


圖一 Figure 1

$b =$

3. 如圖二，一長方形與圓相交於點 B、C、E 及 F。已知 $AB = 4$ cm，$BC = 5$ cm 及 $DE = 3$ cm。若 $EF = c$ cm，求 c 的值。
In Figure 2, a rectangle intersects a circle at points B, C, E and F.
Given that AB =4 cm, $BC = 5$ cm and $DE = 3$ cm. If $EF = c$ cm, find the value of c.

$c =$



圖二 Figure 2

4. 假設 x 和 y 都是正數並且成反比。若 x 增加了 10%，則 y 減少了 d%，求 d 的值。
Let x and y be two positive numbers that are inversely proportional to each other.
If x is increased by 10 %, y will be decreased by d %, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy × Mult. factor for speed =

\+ Bonus score

Total score

Team No.

Time Min. Sec.

Hong Kong Mathematics Olympiad (2004 – 2005)
Final Event 4 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 $a=\log_{\frac{1}{2}}0.125$，求 a 的值。
If $a=\log_{\frac{1}{2}}0.125$, find the value of a.

a =

2. 若方程 $\left|x-|2x+1|\right|=3$ 有 b 個不同的解，求 b 的值。
Suppose there are b distinct solutions of the equation $\left|x-|2x+1|\right|=3$, find the value of b.

b =

3. 若 $c=2\sqrt{3}\times\sqrt[3]{1.5}\times\sqrt[6]{12}$，求 c 的值。
If $c=2\sqrt{3}\times\sqrt[3]{1.5}\times\sqrt[6]{12}$, find the value of c.

c =

4. 已知 $f_1=0$，$f_2=1$ 及對正整數 $n\ge 3$，$f_n=f_{n-1}+2f_{n-2}$。若 $d=f_{10}$，求 d 的值。
Given that $f_1=0$, $f_2=1$ and for any positive integer $n\ge 3$, $f_n=f_{n-1}+2f_{n-2}$.
If $d=f_{10}$, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
				Total score			Min.	Sec.

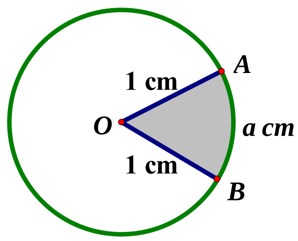
Hong Kong Mathematics Olympiad (2005 – 2006)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 a 為實數且滿足方程 $\log_2(x+3) - \log_2(x+1) = 1$，求 a 的值。
If a is a real number satisfying $\log_2(x+3) - \log_2(x+1) = 1$, find the value of a .

a =

2. 如圖一，O 是半徑 1 cm 的圓的圓心。若弧 AB 的長度是 a cm 及著色部份扇形 OAB 的面積是 b cm^2，求 b 的值。(取 $\pi = 3$)
In Figure 1, O is the centre of the circle with radius 1 cm. If the length of the arc AB is equal to a cm and the area of the shaded sector OAB is equal to b cm^2, find the value of b . (Take $\pi = 3$)



圖一 Figure 1

b =

3. 一個正 C 邊形的一隻內角是 $288b°$，求 C 的值。
An interior angle of a regular C-sided polygon is $288b°$, find the value of C .

C =

4. 已知 10 是方程 $kx^2 + 2x + 5 = 0$ 的一個根，其中 k 為常數。
若 D 是另一個根，求 D 的值。
Given that C is a root of the equation $kx^2 + 2x + 5 = 0$, where k is a constant.
If D is another root, find the value of D .

D =

FOR OFFICIAL USE

Score for accuracy × Mult. factor for speed =

\+ Bonus score

Total score

Team No.

Time

Min. Sec.

Hong Kong Mathematics Olympiad (2005 – 2006)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知 $a:b:c=6:3:1$。若 $R=\dfrac{3b^2}{2a^2+bc}$，求 R 的值。

   Given that $a:b:c=6:3:1$. If $R=\dfrac{3b^2}{2a^2+bc}$, find the value of R.

   $R =$

2. 已知 $\dfrac{|k+R|}{|R|}=0$，若 $S=\dfrac{|k+2R|}{|2k+R|}$，求 S 的值。

   Given that $\dfrac{|k+R|}{|R|}=0$. If $S=\dfrac{|k+2R|}{|2k+R|}$, find the value of S.

   $S =$

3. 已知 $T=\sin 50°\times(S+\sqrt{3}\times\tan 10°)$，求 T 的值。
   Given that $T=\sin 50°\times(S+\sqrt{3}\times\tan 10°)$, find the value of T.

   $T =$

4. 已知 x_0 和 y_0 是實數且滿足方程組 $\begin{cases} y=\dfrac{T}{x} \\ y=|x|+T \end{cases}$，
   若 $W=x_0+y_0$，求 W 的值。

   Given that x_0 and y_0 are real numbers satisfying the system of equations $\begin{cases} y=\dfrac{T}{x} \\ y=|x|+T \end{cases}$.

   If $W=x_0+y_0$, find the value of W.

   $W =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time		
				Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2005 – 2006)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知 $\frac{2x-3}{x^2-x}=\frac{A}{x-1}+\frac{B}{x}$，其中 A 和 B 是常數。若 $S=A^2+B^2$，求 S 的值。

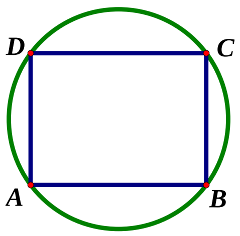
Given that $\frac{2x-3}{x^2-x}=\frac{A}{x-1}+\frac{B}{x}$, where A and B are constants.

If $S=A^2+B^2$, find the value of S.

$S=$

2. 如圖一，$ABCD$ 是圓內長方形，$AB=(S-2)$ cm 及 $AD=(S-4)$ cm。若圓形的圓周是 R cm，求 R 的值。(取 $\pi=3$)

In Figure 1, $ABCD$ is an inscribed rectangle, $AB=(S-2)$ cm and $AD=(S-4)$ cm. If the circumference of the circle is R cm, find the value of R. (Take $\pi=3$)



圖一 Figure 1

$R=$

3. 已知整數 x 和 y 滿足 $\frac{R}{2}xy=21x+20y-13$。若 $T=xy$，求 T 的值。

Given that x and y are integers satisfying the equation $\frac{R}{2}xy=21x+20y-13$.

If $T=xy$, find the value of T.

$T=$

4. 設 a 是方程 $x^2-2x-T=0$ 的一個正根。若 $P=3+\cfrac{T}{2+\cfrac{T}{2+\cfrac{T}{2+\cfrac{T}{a}}}}$，求 P 的值。

Let a be the positive root of the equation $x^2-2x-T=0$.

If $P=3+\cfrac{T}{2+\cfrac{T}{2+\cfrac{T}{2+\cfrac{T}{a}}}}$, find the value of P.

$P=$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	Min.	Sec.
			Total score					

Hong Kong Mathematics Olympiad (2005 – 2006)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設 $\frac{k}{4}=\left(1+\frac{1}{2}+\frac{1}{3}\right)\times\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\times\left(\frac{1}{2}+\frac{1}{3}\right)$，求 k 的值。
Let $\frac{k}{4}=\left(1+\frac{1}{2}+\frac{1}{3}\right)\times\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\times\left(\frac{1}{2}+\frac{1}{3}\right)$, find the value of k .

$k =$

2. 設 x 和 y 是實數且滿足方程 $y^2 + 4y + 4 + \sqrt{x+y+k} = 0$。若 $r = |xy|$，求 r 的值。
Let x and y be real numbers satisfying the equation $y^2 + 4y + 4 + \sqrt{x+y+k} = 0$.
If $r = |xy|$, find the value of r .

$r =$

3. 如圖一，八個正數排成一列，從第三個數開始，每個數都等於前面兩個數的乘積。已知第五個是 $\frac{1}{r}$，而第八個數是 $\frac{1}{r^4}$。若第一個是 s，求 s 的值。
In Figure 1, there are eight positive numbers in series. Starting from the 3rd number, each number is the product of the previous two numbers. Given that the 5th number is $\frac{1}{r}$ and the 8th number is $\frac{1}{r^4}$.
If the first number is s, find the value of s .

$s =$

s				$\frac{1}{r}$			$\frac{1}{r^4}$

圖一 Figure 1

4. 設 $[x]$ 表示不大於 x 的最大整數，例如 $[2.5] = 2$。
若 $w = 1 + [10\times s^2] + [10\times s^4] + [10\times s^6] + \ldots + [10\times s^{2n}] + \cdots$，求 w 的值。
Let $[x]$ be the largest integer not greater than x. For example, $[2.5] = 2$.
Let $w = 1 + [10\times s^2] + [10\times s^4] + [10\times s^6] + \ldots + [10\times s^{2n}] + \cdots$, find the value of w.

$w =$

FOR OFFICIAL USE

Score for accuracy × Mult. factor for speed =

\+ Bonus score

Total score

Team No.

Time

Min. Sec.

Hong Kong Mathematics Olympiad (2005 – 2006)
Final Event 1 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知 k 為實數。若 $x^2 + 2kx - 3k^2$ 能被 $x - 1$ 整除，求 k 最大可能的值。
Given that k is a real number. If $x^2 + 2kx - 3k^2$ can be divisible by $x - 1$, find the greatest value of k.

$k =$

2. 已知 $x = x_0$ 及 $y = y_0$ 滿足方程組 $\begin{cases} \frac{x}{3}+\frac{y}{5}=1 \\ \frac{x}{5}+\frac{y}{3}=1 \end{cases}$。若 $B=\frac{1}{x_0}+\frac{1}{y_0}$，求 B 的值。

$B =$

Given that $x = x_0$ and $y = y_0$ satisfy the system of equations $\begin{cases} \frac{x}{3}+\frac{y}{5}=1 \\ \frac{x}{5}+\frac{y}{3}=1 \end{cases}$.

If $B=\frac{1}{x_0}+\frac{1}{y_0}$, find the value of B.

3. 已知 $x = 2+\sqrt{3}$ 是方程 $x^2 - (\tan\alpha+\cot\alpha)x + 1 = 0$ 的一個根。
若 $C = \sin\alpha \times \cos\alpha$，求 C 的值。
Given that $x = 2+\sqrt{3}$ is a root of the equation $x^2 - (\tan\alpha + \cot\alpha)x + 1 = 0$.
If $C = \sin\alpha \times \cos\alpha$, find the value of C.

$C =$

4. 設 a 為整數。若不等式 $|x + 1| < a - 1.5$ 沒有整數解，求 a 最大可能的值。
Let a be an integer. If the inequality $|x + 1| < a - 1.5$ has no integral solution, find the greatest value of a.

$a =$

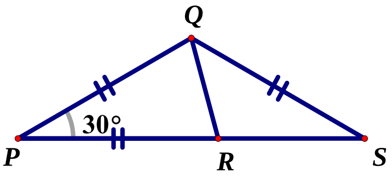
FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2005 – 2006)
Final Event 2 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 如圖一， PRS 是一直綫， $PQ = PR = QS$ 及 $\angle QPR = 30°$。若 $\angle RQS = w°$，求 w 的值。
In Figure 1, PRS is a straight line, $PQ = PR = QS$ and $\angle QPR = 30°$. If $\angle RQS = w°$, find the value of w.



圖一 Figure 1

$w =$

2. 設 $f(x) = px^7 + qx^3 + rx - 5$，其中 p、q 及 r 是實數。若 $f(-6) = 3$ 及 $z = f(6)$，求 z 的值。
Let $f(x) = px^7 + qx^3 + rx - 5$, where p, q and r are real numbers.
If $f(-6) = 3$ and $z = f(6)$, find the value of z.

$z =$

3. 若 $n \neq 0$ 及 $s = \left(\frac{20}{2^{2n+4} + 2^{2n+2}}\right)^{\frac{1}{n}}$，求 s 的值。

If $n \neq 0$ and $s = \left(\frac{20}{2^{2n+4} + 2^{2n+2}}\right)^{\frac{1}{n}}$, find the value of s.

$s =$

4. 已知 x 和 y 是正整數及 $x + y + xy = 54$。若 $t = x + y$，求 t 的值。
Given that x and y are positive integers and $x + y + xy = 54$.
If $t = x + y$, find the value of t.

$t =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

+ Bonus score [ ]

Total score [ ]

Team No. [ ]

Time [ ] Min. [ ] Sec.

Hong Kong Mathematics Olympiad (2005 – 2006)
Final Event 3 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知 $r = 2006 \times \frac{\sqrt{8} - \sqrt{2}}{\sqrt{2}}$，求 r 的值。
Given that $r = 2006 \times \frac{\sqrt{8} - \sqrt{2}}{\sqrt{2}}$, find the value of r.

$r =$

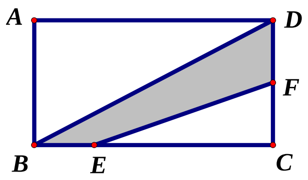
2. 已知 $6^{x+y} = 36$ 及 $6^{x+5y} = 216$，求 x 的值。
Given that $6^{x+y} = 36$ and $6^{x+5y} = 216$, find the value of x.

$x =$

3. 已知 $\tan x + \tan y + 1 = \cot x + \cot y = 6$。若 $z = \tan(x + y)$，求 z 的值。
Given that $\tan x + \tan y + 1 = \cot x + \cot y = 6$.
If $z = \tan(x + y)$, find the value of z.

$z =$

4. 如圖一，$ABCD$ 是一長方形，F 是 CD 的中點及 $BE : EC = 1 : 3$。若長方形 $ABCD$ 的面積是 12 cm^2 及陰影部份 $BEFD$ 的面積是 $R\text{ cm}^2$，求 R 的值。
In Figure 1, $ABCD$ is a rectangle, F is the midpoint of CD and $BE:EC = 1:3$. If the area of the rectangle $ABCD$ is 12 cm^2 and the area of $BEFD$ is $R\text{ cm}^2$, find the value of R.



圖一 Figure 1

$R =$

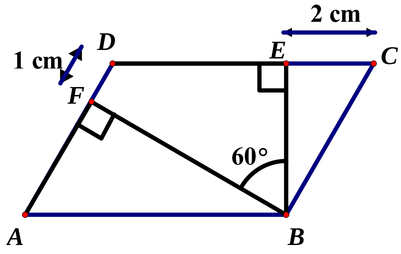
FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2005 – 2006)
Final Event 4 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 如圖一，平行四邊形 $ABCD$ 中，$BE \perp CD$，$BF \perp AD$，$CE = 2$ cm，$DF = 1$ cm 及 $\angle EBF = 60°$。若平行四邊形 $ABCD$ 的面積是 R cm^2，求 R 的值。
In Figure 1, $ABCD$ is a parallelogram, $BE \perp CD$, $BF \perp AD$, $CE = 2$ cm, $DF = 1$ cm and $\angle EBF = 60°$. If the area of the parallelogram $ABCD$ is R cm^2, find the value of R.



圖一 Figure 1

$R =$

2. 已知 a 和 b 是正數且 $a + b = 2$。若 $S = \left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2$，求 S 的最小值。
Given that a and b are positive numbers and $a + b = 2$.
If $S = \left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2$, find the minimum value S.

$S =$

3. 設 $2^x = 7^y = 196$。若 $T = \frac{1}{x} + \frac{1}{y}$，求 T 的值。
Let $2^x = 7^y = 196$. If $T = \frac{1}{x} + \frac{1}{y}$, find the value of T.

$T =$

4. 若 $W = 2006^2 - 2005^2 + 2004^2 - 2003^2 + \ldots + 4^2 - 3^2 + 2^2 - 1^2$，求 W 的值。
If $W = 2006^2 - 2005^2 + 2004^2 - 2003^2 + \ldots + 4^2 - 3^2 + 2^2 - 1^2$, find the value of W.

$W =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

\+ Bonus score [ ]

Total score [ ]

Team No. [ ]

Time [ ] Min. [ ] Sec.

Hong Kong Mathematics Olympiad (2006 – 2007)
Final Event 1 (Individual)

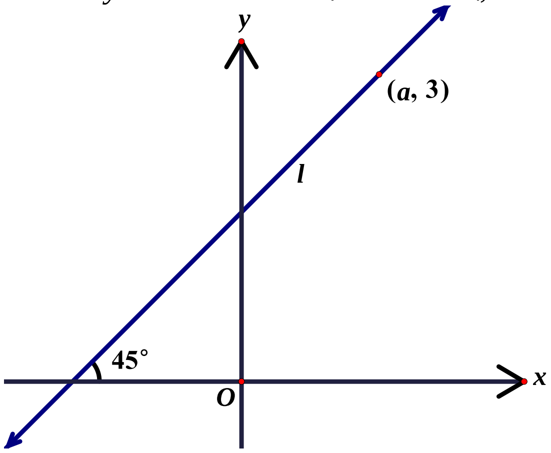
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設 a 為實數，且 $\sqrt{a}=\sqrt{7+\sqrt{13}}-\sqrt{7-\sqrt{13}}$，求 a 的值。
Let a be a real number and $\sqrt{a}=\sqrt{7+\sqrt{13}}-\sqrt{7-\sqrt{13}}$. Find the value of a.

a =

2. 如圖一，直綫 ℓ 經過點 $(a, 3)$並與 x 軸成 45° 夾角。
若 ℓ 的方程是 $x+my+n=0$ 及 $b=|1+m+n|$，求 b 的值。
In Figure 1, the straight line ℓ passes though the point $(a, 3)$, and makes an angle 45° with the x-axis.
If the equation of ℓ is $x+my+n=0$ and $b=|1+m+n|$, find the value of b.

b =



圖一
Figure 1

3. 若 $x-b$ 為 $x^3-6x^2+11x+c$ 的因式，求 c 的值。
If $x-b$ is a factor of $x^3-6x^2+11x+c$, find the value of c.

c =

4. 若 $\cos x+\sin x=-\frac{c}{5}$ 及 $d=\tan x+\cot x$，求 d 的值。
If $\cos x+\sin x=-\frac{c}{5}$ and $d=\tan x+\cot x$, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

\+ Bonus score [ ]

Total score [ ]

Team No. [ ]

Time [ ] Min. [ ] Sec.

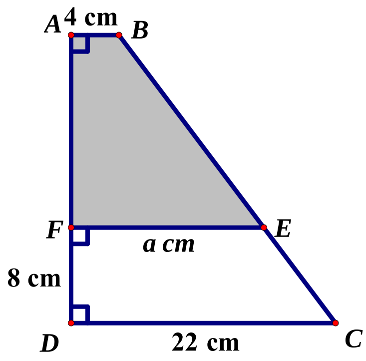
Hong Kong Mathematics Olympiad (2006 – 2007)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設 $n = 1 + 3 + 5 + \ldots + 31$ 及 $m = 2 + 4 + 6 \ldots + 32$。若 $a = m - n$，求 a 的值。
Let $n = 1 + 3 + 5 + \ldots + 31$ and $m = 2 + 4 + 6 \ldots + 32$.
If $a = m - n$, find the value of a.

$a =$

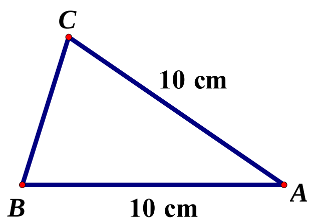
2. 如圖一，$ABCD$ 是一梯形，$AB = 4$ cm，$EF = a$ cm，$CD = 22$ cm 及 $FD = 8$ cm。
若 $ABEF$ 的面積是 b cm^2，求 b 的值。
If Figure 1, $ABCD$ is a trapezium, $AB = 4$ cm, $EF = a$ cm, $CD = 22$ cm and $FD = 8$ cm, if the area of $ABEF$ is b cm^2, find the value of b.



圖一 Figure 1

$b =$

3. 如圖二，ΔABC 是一個三角形，$AB = AC = 10$ cm 及 $\angle ABC = b° - 100°$。若 ΔABC 有 c 條對稱軸，求 c 的值。
In Figure 2, ΔABC is a triangle, $AB = AC = 10$ cm and $\angle ABC = b° - 100°$.
If ΔABC has c axis of symmetry, find the value of c.



圖二 Figure 2

$c =$

4. 設 d 為方程 $cx^{\frac{2}{3}} - 8x^{\frac{1}{3}} + 4 = 0$ 的最小實根，求 d 的值。
Let d be the least real root of the $cx^{\frac{2}{3}} - 8x^{\frac{1}{3}} + 4 = 0$, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time		
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2006 – 2007)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設 $a = \cos^4\theta - \sin^4\theta - 2\cos^2\theta$，求 a 的值。
Suppose that $a = \cos^4\theta - \sin^4\theta - 2\cos^2\theta$, find the value of a.

$a =$

2. 若 $x^y = 3$ 及 $b = x^{3y} + 10a$，求 b 的值。
If $x^y = 3$ and $b = x^{3y} + 10a$, find the value of b.

$b =$

3. 若有 c 個正整數 n 使得 $\frac{n+b}{n-7}$ 也是正整數，求 c 的值。
If there is (are) c positive integer(s) n such that $\frac{n+b}{n-7}$ is also a positive integer, find the value of c.

$c =$

4. 設 $d = \log_4 2 + \log_4 4 + \log_4 8 + \cdots + \log_4 2^c$，求 d 的值。
Suppose that $d = \log_4 2 + \log_4 4 + \log_4 8 + \cdots + \log_4 2^c$, find the value of d.

$d =$

FOR OFFICIAL USE

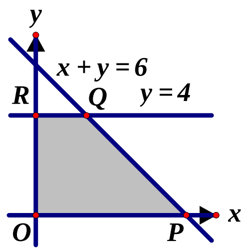
Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2006 – 2007)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 如圖一，設直綫 $x + y = 6$，$y = 4$，$x = 0$ 及 $y = 0$ 所圍成的封閉區域的面積是 A 平方單位，求 A 的值。
In Figure 1, let the area of the closed region bounded by the straight line $x + y = 6$ and $y = 4$, $x = 0$ and $y = 0$ be A square units, find the value of A.

A =



圖一
Figure 1

2. 設[x]表示不大於 x 的最大整數，例如 $[2.5] = 2$。
若 b 滿足方程組 $\begin{cases} Ax^2 - 4 = 0 \\ 3 + 2(x + [x]) = 0 \end{cases}$，求 b 的值。
Let [x] be the largest integer not greater than x. For example, $[2.5] = 2$.
If b satisfies the system of equation $\begin{cases} Ax^2 - 4 = 0 \\ 3 + 2(x + [x]) = 0 \end{cases}$, find the value of b.

b =

3. 設 c 為數 $(2x + \frac{b}{\sqrt{x}})^3$ 展開式中的常數項，求 c 的值。
Let c be the constant term in the expansion of $(2x + \frac{b}{\sqrt{x}})^3$. Find the value of c.

c =

4. 若滿足不等式 $\left|\frac{x}{2} - \sqrt{2}\right| < c$ 的整數有 d 個，求 d 的值。
If the number of integral solutions of the inequality $\left|\frac{x}{2} - \sqrt{2}\right| < c$ is d, find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

\+ Bonus score [ ]

Total score [ ]

Team No. [ ]

Time [ ] Min. [ ] Sec.

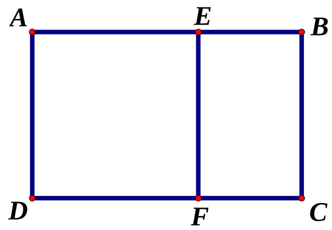
Hong Kong Mathematics Olympiad (2006 – 2007)
Final Event 1 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 如圖一，*AEFD* 是邊長為一單位的正方形。長方形 *ABCD* 的長闊的比例與長方形 *BCFE* 的長闊的比例相同。若 *AB* 的長度是 *W* 單位，求 *W* 的值。

   In Figure 1, *AEFD* is a unit square. The ratio of the length of the rectangle *ABCD* to its width is equal to the ratio of the length of the rectangle *BCFE* to its width.
   If the length of *AB* is *W* units, find the value of *W*.

   $W =$



圖一 Figure 1

2. 在座標平面上滿足 $x^2 + y^2 < 10$，其中 x 及 y 為整數的點 (x, y) 共有 T 個，求 T 的值。
   On the coordinate plane, there are T points (x, y) , where x, y are integers, satisfying $x^2 + y^2 < 10$, find the value of T.

   $T =$

3. 設 P 及 $P + 2$ 均為質數並滿足 $P(P + 2) \le 2007$。
   若 S 是符合上述要求的質數 P 的總和，求 S 的值。
   Let P and $P + 2$ be both prime numbers satisfying $P(P + 2) \le 2007$.
   If S represents the sum of such possible values of P , find the value of S .

   $S =$

4. 已知 $\log_{10}(2007^{2006}\times 2006^{2007}) = a\times 10^k$，其中 $1 \le a < 10$ 及 k 是整數，求 k 的值。
   It is known that $\log_{10}(2007^{2006}\times 2006^{2007}) = a\times 10^k$, where $1 \le a < 10$ and k is an integer.
   Find the value of k.

   $k =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2006 – 2007)
Final Event 2 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 $R = 1\times 2 + 2\times 2^2 + 3\times 2^3 + \ldots + 10\times 2^{10}$，求 R 的值。
If $R = 1\times 2 + 2\times 2^2 + 3\times 2^3 + \ldots + 10\times 2^{10}$, find the value of R.

$R =$

2. 若整數 x 滿足 $x \geq 3+\sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3}}}}}$，求 x 的最小值。
If integer x satisfies $x \geq 3+\sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3}}}}}$, find the minimum value of x.

$x =$

3. 設 $y = \dfrac{146410000-12100}{12099}$，求 y 的值。
Let $y = \dfrac{146410000-12100}{12099}$, find the value of y.

$y =$

4. 在座標平面上，某圓以 $T(3, 3)$ 為中心及經過原點 $O(0, 0)$。若 A 為該圓上的一點使得 $\angle AOT = 45°$ 及 ΔAOT 的面積是 Q 個平方單位，求 Q 的值。
On the coordinate plane, a circle with centre $T(3, 3)$ passes through the origin $O(0, 0)$. If A is a point on the circle such that $\angle AOT = 45°$ and the area of ΔAOT is Q square units, find the value of Q.

$Q =$

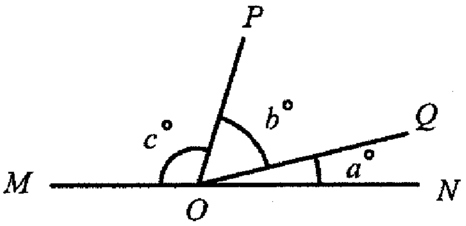
FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score					Min.	Sec.

Hong Kong Mathematics Olympiad (2006 – 2007)
Final Event 3 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 如圖一，MN 是一直綫，$\angle QON = a°$，$\angle POQ = b°$及$\angle POM = c°$。若 $b : a = 2 : 1$ 及 $c : b = 3 : 1$，求 b 的值。
In figure 1, MN is a straight line, $\angle QON = a°$, $\angle POQ = b°$ and $\angle POM = c°$. If $b : a = 2 : 1$ and $c : b = 3 : 1$, find the value of b.



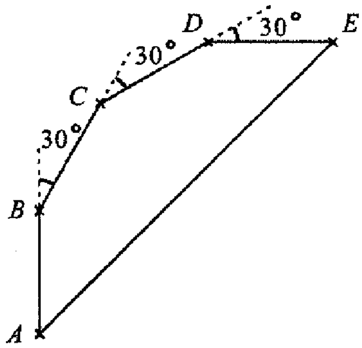
圖一 Figure 1

$b =$

2. 已知 $\sqrt{\frac{50+120+130}{2}\times(150-50)\times(150-120)\times(150-130)} = \frac{50\times130\times k}{2}$。若 $t = \frac{k}{\sqrt{1-k^2}}$，求 t 的值。
It is known that $\sqrt{\frac{50+120+130}{2}\times(150-50)\times(150-120)\times(150-130)} = \frac{50\times130\times k}{2}$.
If $t = \frac{k}{\sqrt{1-k^2}}$, find the value of t.

$t =$

3. 如圖二，一螞蟻由 A 點出發，往前直走 $5 \sec 15°$ 厘米至 B 點；接著右轉 $30°$，往前直走 $5 \sec 15°$厘米至 C 點。螞蟻再重覆右轉 $30°$及往前直走 $5 \sec 15°$厘米兩次，分別到達 D 點及 E 點。若 AE 的距離是 x 厘米，求 x 的值。
In Figure 2, an ant runs ahead straightly for $5 \sec 15°$ cm from point A to point B. It then turns $30°$ to the right and run $5 \sec 15°$ cm to point C. Again it repeatedly turns $30°$ to the right and run $5 \sec 15°$ cm twice to reach the points D and E respectively. If the distance of AE is x cm, find the value of x.



圖二 Figure 2

$x =$

4. 某數學比賽共有 4 條題目。以下述方式為每個題目評分：答對得 2 分、答錯扣一分、不作答得零分。若至少有 S 名參賽者才可保證比賽中有三人同分，求 S 的值。
There are 4 problems in a mathematics competition. The scores of each problem are allocated in the following ways: 2 marks will be given for a correct answer, 1 mark will be deducted from a wrong answer and 0 mark will be given for a blank answer. To ensure that 3 candidates will have the same scores, there should be at least S candidates in the competition. Find the value of S.

$S =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]　　Team No. [ ]

\+ Bonus score [ ]　　Time [ ] Min. [ ] Sec.

Total score [ ]

Hong Kong Mathematics Olympiad (2006 – 2007)
Final Event 4 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 有糖果 x 粒及 $120 \le x \le 150$。將糖果分成小堆，若每堆 5 粒，則餘 2 粒；若每堆 6 粒，則餘 5 粒。求 x 的值。

Let x be the number of candies satisfies the inequalities $120 \le x \le 150$. 2 candies will be remained if they are divided into groups of 5 candies each; 5 candies will be remained if they are divided into groups of 6 candies each. Find the value of x .

$x =$

2. 在座標平面上，點 $A(3, 7)$及 $B(8, 14)$沿直綫 $y = kx + c$ 反射，當中 k 和 c 是常數，其像分別是點 $C(5, 5)$及 $D(12, 10)$。若 $R = \frac{k}{c}$，求 R 的值。

On the coordinate plane, the points $A(3, 7)$ and $B(8, 14)$ are reflected about the line $y = kx + c$, where k and c are constants, their images are $C(5, 5)$ and $D(12,10)$ respectively. If $R = \frac{k}{c}$, find the value of R .

$R =$

3. 已知 $z = \sqrt[3]{456533}$ 是一整數，求 z 的值。

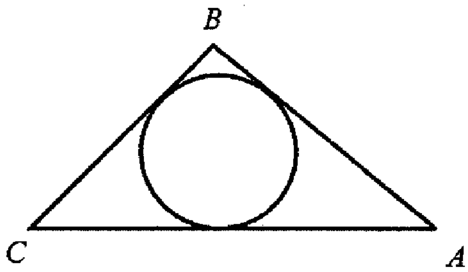
Given that $z = \sqrt[3]{456533}$ is an integer, find the value of z .

$z =$

4. 如圖一，ΔABC 是一等腰三角形，$AB = BC = 20$ cm 及 $\tan\angle BAC = \frac{4}{3}$。若$\Delta ABC$ 的內切圓的半徑為 r cm，求 r 的值。

In Figure 1, ΔABC is an isosceles triangle, $AB = BC = 20$ cm and $\tan \angle BAC = \frac{4}{3}$. If the length of radius of the inscribed circle of ΔABC is r cm, find the value of r .

$r =$



圖一
Figure 1

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

\+ Bonus score [ ]

Total score [ ]

Team No. [ ]

Time [ ] Min. [ ] Sec.

Hong Kong Mathematics Olympiad (2007 – 2008)
Final Event Sample (Individual)

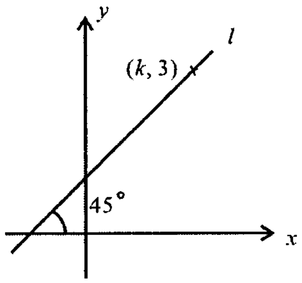
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設 $\sqrt{k}=\sqrt{7+\sqrt{13}}-\sqrt{7-\sqrt{13}}$，求 k 的值。
Let $\sqrt{k}=\sqrt{7+\sqrt{13}}-\sqrt{7-\sqrt{13}}$, find the value of k.

k =

2. 如圖一，直綫 ℓ 經過點 $(k, 3)$ 並與 x 軸成 $45°$ 夾角。若 ℓ 的方程是 $x + by + c = 0$ 及 $b = |1 + b + c|$，求 d 的值。
In Figure 1, the straight line ℓ passes though the point $(k, 3)$ and makes an angle $45°$ with the x-axis.
If the equation of ℓ is $x + by + c = 0$ and $d = |1 + b + c|$, find the value of d.

d =



圖一 Figure 1

3. 若 $x - d$ 為 $x^3 - 6x^2 + 11x + a$ 的因式,，求 a 的值。
If $x - d$ is a factor of $x^3 - 6x^2 + 11x + a$, find the value of a.

a =

4. 若 $\cos x + \sin x = -\frac{a}{5}$ 及 $t = \tan x + \cot x$，求 t 的值。
If $\cos x + \sin x = -\frac{a}{5}$ and $t = \tan x + \cot x$, find the value of t.

t =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2007 – 2008)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設 $A = 15\times\tan 44°\times\tan 45°\times\tan 46°$，求 A 的值。
Let $A = 15\times\tan 44°\times\tan 45°\times\tan 46°$, find the value of A.

$A =$

2. 設 n 為正整數及 $\overbrace{20082008\cdots2008}^{n\text{個}2008}15$ 能被 A 整除。
若 n 的最小可能值是 B，求 B 的值。
Let n be a positive integer and $\overbrace{20082008\cdots2008}^{n\ 2008's}15$ is divisible by A.
If the least possible value of n is B, find the value of B.

$B =$

3. 已知有 C 個整數滿足方程 $|x-2|+|x+1|=B$，求 C 的值。
Given that there are C integers that satisfy the equation $|x-2|+|x+1|=B$, find the value of C.

$C =$

4. 在座標平面上，點 $(-C, 0)$ 與直綫 $y = x$ 的距離是 $\sqrt{D}$，求 D 的值。
In the coordinate plane, the distance from the point $(-C, 0)$ to the straight line $y = x$ is $\sqrt{D}$, find the value of D.

$D =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2007 – 2008)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知 $P=\left[\sqrt[3]{6}\times\left(\sqrt[3]{\frac{1}{162}}\right)\right]^{-1}$，求 P 的值。

Given that $P=\left[\sqrt[3]{6}\times\left(\sqrt[3]{\frac{1}{162}}\right)\right]^{-1}$, find the value of P.

$P =$

2. 設 a、b 和 c 是實數且 $b:(a+c)=1:2$ 及 $a:(b+c)=1:P$。
若 $Q=\frac{a+b+c}{a}$，求 Q 的值。
Let a, b and c be real numbers with ratios $b:(a+c)=1:2$ and $a:(b+c)=1:P$.
If $Q=\frac{a+b+c}{a}$, find the value of Q.

$Q =$

3. 設 $R=\left(\sqrt{\sqrt{3}+\sqrt{2}}\right)^Q+\left(\sqrt{\sqrt{3}-\sqrt{2}}\right)^Q$。求 R 的值。
Let $R=\left(\sqrt{\sqrt{3}+\sqrt{2}}\right)^Q+\left(\sqrt{\sqrt{3}-\sqrt{2}}\right)^Q$. Find the value of R.

$R =$

4. 設 $S=(x-R)^2+(x+5)^2$，其中 x 為實數。求 S 的最小值。
Let $S=(x-R)^2+(x+5)^2$, where x is a real number. Find the minimum value of S.

$S =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] Min. [ ] Sec.

Total score [ ]

Hong Kong Mathematics Olympiad (2007 – 2008)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

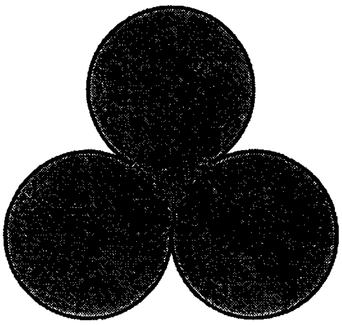
1. 已知 $\frac{1-\sqrt{3}}{2}$ 滿足方程 $x^2+px+q=0$，其中 p 和 q 是有理數。
若 $A=|p|+2|q|$，求 A 的值。
Given that $\frac{1-\sqrt{3}}{2}$ satisfies the equation $x^2+px+q=0$, where p and q are rational numbers. If $A=|p|+2|q|$, find the value of A.

A =

2. U_1 及 U_2 兩袋有大小相同的紅球和白球。U_1 裝有 A 個紅球，2 個白球。U_2 裝有 2 個紅球，B 個白球。從每袋中各取出 2 個球。
若取到四個紅球的概率是 $\frac{1}{60}$，求 B 的值。
Two bags U_1 and U_2 contain identical red and white balls. U_1 contains A red balls and 2 white balls. U_2 contains 2 red balls and B white balls. Take two balls out of each bag. If the probability of all four balls are red is $\frac{1}{60}$, find the value of B.

B =

3. 圖一由三個大小相同互切的圓所組成，三個圓的半徑均是 B cm。
若陰影部分的周界是 C cm，求 C 的值。(取 $\pi=3$)
Figure 1 is formed by three identical circles touching one another, the radius of each circle is B cm. If the perimeter of the shaded region is C cm, find the value of C. (Take $\pi=3$)


圖一 Figure 1

C =

4. 設與 $\sqrt{C}$ 最接近的整數是 D，求 D 的值。
Let D be the integer closest to $\sqrt{C}$, find the value of D.

D =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	Min.	Sec.
			Total score						

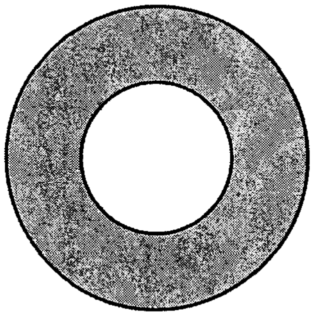
Hong Kong Mathematics Olympiad (2007 – 2008)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知 x 及 y 為實數，且滿足$|x| + x + y = 10$ 及 $|y| + x - y = 10$。
若 $P = x + y$，求 P 的值。
Given that x and y are real numbers such that $|x| + x + y = 10$ and $|y| + x - y = 10$.
If $P = x + y$, find the value of P.

$P =$

2. 如圖一，陰影部分由兩同心圓所組成，其面積為 96π cm^2。若該兩圓的半徑相差 $2P$ cm 及大圓的面積為 Q cm^2，求 Q 的值。(取 $\pi = 3$)
In Figure 1, the shaded area is formed by two concentric circles and has area 96π cm^2. If the two radii differ by $2P$ cm and the large circle has area Q cm^2, find the value of Q. (Take $\pi = 3$)



圖一 Figure 1

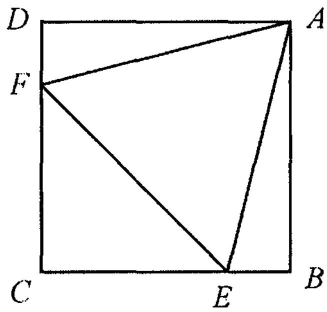
$Q =$

3. 設 R 為最大的整數使得 $R^Q < 5^{200}$ 成立，求 R 的值。
Let R be the largest integer such that $R^Q < 5^{200}$, find the value of R.

$R =$

4. 圖二顯示一個邊長為 $(R - 1)$ cm 的正方形 $ABCD$ 及一個等邊三角形 AEF (E 及 F 分別是直綫 BC 及 CD 上的點)。若 ΔAEF 的面積是$(S - 3)$ cm^2，求 S 的值。
In Figure 2, there are a square $ABCD$ with side length $(R - 1)$ cm and an equilateral triangle AEF. (E and F are points on BC and CD respectively).
If the area of ΔAEF is $(S - 3)$ cm^2, find the value of S.

$S =$



圖二 Figure 2

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2007 – 2008)
Final Event Spare (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 28 的所有正因子是 $d_1, d_2, \ldots, d_n$ 及 $a = \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_n}$，求 a 的值。

   If all the positive factors of 28 are $d_1, d_2, \ldots, d_n$ and $a = \frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_n}$,

   find the value of a.

   $a =$

2. 已知 x 為負實數且 $\dfrac{1}{x + \frac{1}{x+2}} = a$。若 $b = x + \frac{7}{2}$，求 b 的值。

   Given that x is a negative real number that satisfy $\dfrac{1}{x + \frac{1}{x+2}} = a$.

   If $b = x + \frac{7}{2}$, find the value of b.

   $b =$

3. 設 α 和 β 是方程 $x^2 + cx + b = 0$ 的兩個根，其中 $c < 0$ 及 $\alpha - \beta = 1$。求 c 的值。

   Let α and β be the two roots of the equation $x^2 + cx + b = 0$, where $c < 0$ and $\alpha - \beta = 1$. Find the value of c.

   $c =$

4. 設 d 為$(196c)^{2008}$ 除以 97 所得的餘數。求 d 的值。

   Let d be the remainder of $(196c)^{2008}$ divided by 97. Find the value of d.

   $d =$

FOR OFFICIAL USE

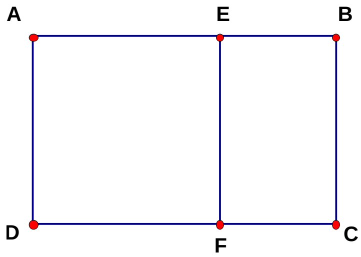
Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2007 – 2008)
Final Event Sample (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 如圖一，*AEFD* 是邊長為一單位的正方形。長方形 *ABCD* 的長闊的比例與長方形 *BCFE* 的長闊比例相同。若 *AB* 的長度是 *W* 單位，求 *W* 的值。
In Figure 1, *AEFD* is a unit square. The ratio of the length of the rectangle *ABCD* to its width is equal to the ratio of the length of the rectangle *BCFE* to its width.
If the length of *AB* is *W* units, find the value of *W* .

$W =$



圖一
Figure 1

2. 在座標平面上滿足 $x^2 + y^2 < 10$，其中 x 及 y 為整數的點(x, y)共有 T 個，求 T 的值。
On the coordinate plane, there are T points (x, y), where x, y are integers, satisfying $x^2 + y^2 < 10$, find the value of T .

$T =$

3. 設 P 及 $P + 2$ 均為質數並滿足 $P(P + 2) \leq 2007$。
若 S 是符合上述要求的質數 P 的總和，求 S 的值。
Let P and $P + 2$ be both prime numbers satisfying $P(P + 2) \leq 2007$.
If S represents the sum of such possible values of P, find the value of S .

$S =$

4. 已知 $\log_{10}(2007^{2006} \times 2006^{2007}) = a\times 10^k$，其中 $1 \leq a < 10$ 及 k 是整數，求 k 的值。
It is known that $\log_{10}(2007^{2006} \times 2006^{2007}) = a\times 10^k$, where $1 \leq a < 10$ and k is an integer. Find the value of k.

$k =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

Team No. [ ]

+ Bonus score [ ]

Time [ ] Min. [ ] Sec.

Total score [ ]

Hong Kong Mathematics Olympiad (2007 – 2008)
Final Event 1 (Group)

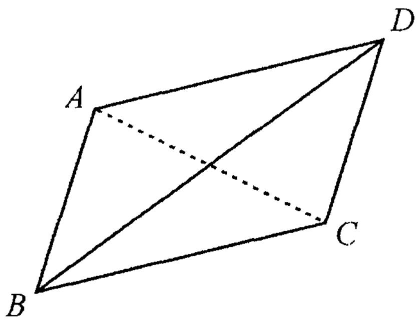
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知座標平面上三點：$O(0, 0)$、$A(12, 2)$ 及 $B(0, 8)$。ΔOAB 經直綫 $y = 6$ 作反射後得ΔPQR。若ΔOAB 及ΔPQR 重疊部分的面積是 m 平方單位，求 m 的值。
Given that there are three points on the coordinate plane: $O(0, 0)$, $A(12, 2)$ and $B(0, 8)$. A reflection of ΔOAB along the straight line $y = 6$ creates ΔPQR. If the overlapped area of ΔOAB and ΔPQR is m square units, find the value of m.

m =

2. 如圖一，$ABCD$ 是平行四邊形，$BA = 3$ cm、$BC = 4$ cm 及 $BD = \sqrt{37}$ cm。若 $AC = h$ cm，求 h 的值。
In Figure 1, $ABCD$ is a parallelogram with $BA = 3$ cm, $BC = 4$ cm and $BD = \sqrt{37}$ cm. If $AC = h$ cm, find the value of h.

h =



圖一
Figure 1

3. 已知 x、y 及 z 為正整數及分數$\frac{151}{44}$可寫成 $3+\cfrac{1}{x+\cfrac{1}{y+\cfrac{1}{z}}}$ 的形式。求 $x+y+z$ 的值。
Given that x, y and z are positive integers and the fraction $\frac{151}{44}$ can be written in the form of $3+\cfrac{1}{x+\cfrac{1}{y+\cfrac{1}{z}}}$. Find the value of $x+y+z$.

z =

4. 當 491 除以一個兩位數，餘數是 59。求這兩位數。
When 491 is divided by a two-digit integer, the remainder is 59.
Find this two-digit integer.

FOR OFFICIAL USE

Score for accuracy × Mult. factor for speed =

+ Bonus score

Total score

Team No.

Time

Min. Sec.

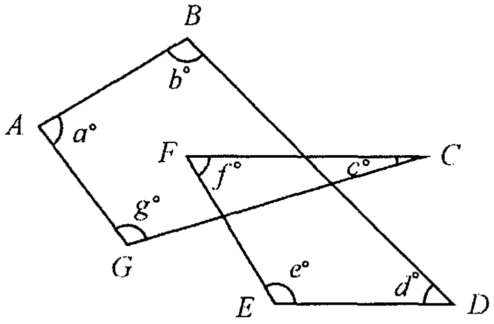
Hong Kong Mathematics Olympiad (2007 – 2008)

Final Event 2 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 如圖一，BD、FC、GC 及 FE 為直綫。若 $z = a + b + c + d + e + f + g$，求 z 的值。
In Figure 1, BD, FC, GC and FE are straight lines.
If $z = a + b + c + d + e + f + g$, find the value of z .

$z =$



圖一

Figure 1

2. 若 $1^6 + 2^6 + 3^6 + 4^6 + 5^6 + 6^6$ 被 7 除後的餘數是 R，求 R 的值。
If R is the remainder of $1^6 + 2^6 + 3^6 + 4^6 + 5^6 + 6^6$ divided by 7, find the value of R .

$R =$

3. 若 14! 能被 6^k 整除，其中 k 為整數，求 k 的最大可能值。
If 14! is divisible by 6^k, where k is an integer, find the largest possible value of k .

$k =$

4. 設實數 x、y 及 z 滿足 $x + \frac{1}{y} = 4$，$y + \frac{1}{z} = 1$ 及 $z + \frac{1}{x} = \frac{7}{3}$。求 xyz 的值。
Let x, y and z be real numbers that satisfy $x + \frac{1}{y} = 4$, $y + \frac{1}{z} = 1$ and $z + \frac{1}{x} = \frac{7}{3}$.
Find the value of xyz .

$xyz =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ] Team No. [ ]

+ Bonus score [ ] Time [ ] [ ]

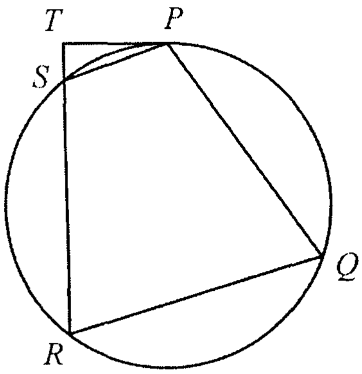
Total score [ ] Min. Sec.

Hong Kong Mathematics Olympiad (2007 – 2008)
Final Event 3 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 如圖一，*PQRS* 是一個圓內接四邊形，其中 *S* 在直綫 *RT* 上且 *TP* 為該圓的切綫。若 *RS* = 8 cm，*RT* = 11 cm 及 *TP* = *k* cm，求 *k* 的值。
In Figure 1, *PQRS* is a cyclic quadrilateral, where *S* is on the straight line *RT* and *TP* is tangent to the circle. If *RS* = 8 cm, *RT* = 11 cm and *TP* = *k* cm, find the value of *k* .

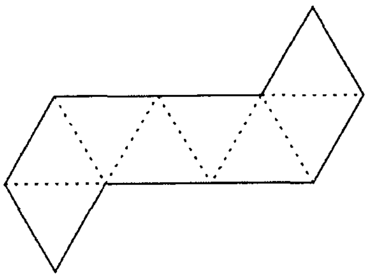
k =



圖一
Figure 1

2. 圖二中的摺紙圖樣能摺出一多面體。若該多面體有 *v* 個頂點，求 *v* 的值。
The layout in Figure 2 can be used to fold a polyhedron.
If this polyhedron has *v* vertices, find the value of *v* .

v =



圖二
Figure 2

3. 對任意實數 *x*，定義[*x*]是小於或等於 *x* 的最大整數。例如，[2] = 2，[3.4] = 3。求 $[1.008^8 \times 100]$ 的值。
For arbitrary real number *x*, define [*x*] to be the largest integer less than or equal to *x*. For instance, [2] = 2 and [3.4] = 3. Find the value of $[1.008^8 \times 100]$.

4. 當從標明了 1 至 30 的 30 個號碼球中選出 4 個，而選出的球均不放回重選時，能得 *r* 個組合，求 *r* 的值。
When choosing, without replacement, 4 out of 30 labelled balls that are marked from 1 to 30, there are *r* combinations. Find the value of *r* .

r =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2007 – 2008)
Final Event 4 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 利用相同的正 m 邊形能密鋪平面，求所有可能 m 值的總和。
Regular tessellation is formed by identical regular m-polygons for some fixed m.
Find the sum of all possible values of m .

sum of m =

2. 在 3624、36024、360924、3609924、36099924、360999924 及 3609999924 這七個數中，能被 38 整除的有 n 個，求 n 的值。
Amongst the seven numbers 3624, 36024, 360924, 3609924, 36099924, 360999924 and 3609999924, there are n of them that are divisible by 38.
Find the value of n .

n =

3. 若 $208208 = 8^5a + 8^4b + 8^3c + 8^2d + 8e + f$，其中 a、b、c、d、e 及 f 為整數且 $0 \le a, b, c, d, e, f \le 7$，求 $a\times b\times c + d\times e\times f$ 的值。
If $208208 = 8^5a + 8^4b + 8^3c + 8^2d + 8e + f$, where a, b, c, d, e, and f are integers and $0 \le a, b, c, d, e, f \le 7$, find the value of $a\times b\times c + d\times e\times f$.

4. 在座標平面上，點 $A(6, 8)$ 繞原點 $O(0, 0)$逆時針轉 20070° 至點 $B(p, q)$。求 $p + q$ 的值。
In the coordinate plane, rotate point $A(6, 8)$ about the origin $O(0, 0)$ counter-clockwise for 20070° to point $B(p, q)$. Find the value of $p + q$.

$p + q$ =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

+ Bonus score [ ]

Total score [ ]

Team No. [ ]

Time [ ] Min. [ ] Sec.

Hong Kong Mathematics Olympiad (2007 – 2008)
Final Event Spare (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 計算 $\left(\sqrt{2008}+\sqrt{2007}\right)^{2007}\times\left(\sqrt{2007}-\sqrt{2008}\right)^{2007}$ 的值。

   Calculate the value of $\left(\sqrt{2008}+\sqrt{2007}\right)^{2007}\times\left(\sqrt{2007}-\sqrt{2008}\right)^{2007}$.

2. 若 $x-\frac{1}{x}=\sqrt{2007}$，求 $x^4+\frac{1}{x^4}$ 的值。

   If $x-\frac{1}{x}=\sqrt{2007}$, find the value of $x^4+\frac{1}{x^4}$.

3. 已知 $\cos\alpha=-\frac{99}{101}$ 及 $180°<\alpha<270°$。求 $\cot\alpha$ 的值。

   Given that $\cos\alpha=-\frac{99}{101}$ and $180°<\alpha<270°$. Find the value of $\cot\alpha$.

4. 求 $\frac{2008^3+4015^3}{2007^3+4015^3}$ 的值。

   Calculate the value of $\frac{2008^3+4015^3}{2007^3+4015^3}$.

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2008 – 2009)
Final Event Sample (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設 $A = 15\times\tan 44°\times\tan 45°\times\tan 46°$，求 A 的值。
Let $A = 15\times\tan 44°\times\tan 45°\times\tan 46°$, find the value of A .

A =

2. 設 n 為正整數及 $\overbrace{20082008\cdots2008}^{n\text{個}2008}15$ 能被 A 整除。
若 n 的最小可能值是 B，求 B 的值。

Let n be a positive integer and $\overbrace{20082008\cdots2008}^{n\ 2008's}15$ is divisible by A .
If the least possible value of n is B, find the value of B .

B =

3. 已知有 C 個整數滿足方程 $|x-2|+|x+1|=B$，求 C 的值。
Given that there are C integers that satisfy the equation $|x-2|+|x+1|=B$,
find the value of C .

C =

4. 在座標平面上，點 $(-C, 0)$ 與直綫 $y=x$ 的距離是 $\sqrt{D}$ ，求 D 的值。
In the coordinate plane, the distance from the point $(-C, 0)$ to the straight line $y=x$ is $\sqrt{D}$, find the value of D .

D =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time		
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2008 – 2009)
Final Event 1 (Individual)

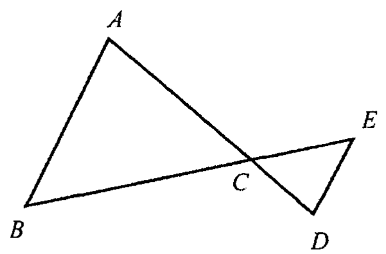
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設 a、b、c 及 d 為方程 $x^4 - 15x^2 + 56 = 0$ 相異的根。
若 $R = a^2 + b^2 + c^2 + d^2$，求 R 的值。
Let a, b, c and d be the distinct roots of the equation $x^4 - 15x^2 + 56 = 0$.
If $R = a^2 + b^2 + c^2 + d^2$, find the value of R.

$R =$

2. 如圖一，AD 及 BE 為直綫且 $AB = AC$ 及 $AB \parallel ED$。
若 $\angle ABC = R°$ 及 $\angle ADE = S°$，求 S 的值。
In Figure 1, AD and BE are straight lines with $AB = AC$ and $AB \parallel ED$.
If $\angle ABC = R°$ and $\angle ADE = S°$, find the value of S.

$S =$



圖一
Figure 1

3. 設 $F = 1 + 2 + 2^2 + 2^3 + \ldots + 2^S$ 及 $T = \sqrt{\dfrac{\log(1+F)}{\log 2}}$，求 T 的值。

Let $F = 1 + 2 + 2^2 + 2^3 + \ldots + 2^S$ and $T = \sqrt{\dfrac{\log(1+F)}{\log 2}}$, find the value of T.

$T =$

4. 設 f(x)是一個函數使得對所有整數 $n \geq 6$ 時，$f(n) = (n-1)\, f(n-1)$及 $f(n) \neq 0$。
若 $U = \dfrac{f(T)}{(T-1)\, f(T-3)}$，求 U 的值。
Let f(x) be a function such that $f(n) = (n-1)\, f(n-1)$
and $f(n) \neq 0$ hold for all integers $n \geq 6$. If $U = \dfrac{f(T)}{(T-1)\, f(T-3)}$, find the value of U.

$U =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2008 – 2009)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設 $[x]$ 是不超過 x 的最大整數。若 $a=\left[\left(\sqrt{3}-\sqrt{2}\right)^{2009}\right]+16$，求 a 的值。

Let $[x]$ be the largest integer not greater than x.
If $a=\left[\left(\sqrt{3}-\sqrt{2}\right)^{2009}\right]+16$, find the value of a.

$a =$

2. 在坐標平面上，若 x-軸、y-軸與直綫 $3x+ay=12$ 所圍成三角形的面積是 b 平方單位，求 b 的值。
In the coordinate plane, if the area of the triangle formed by the x-axis, y-axis and the line $3x+ay=12$ is b square units, find the value of b.

$b =$

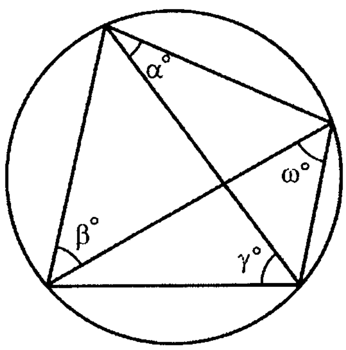
3. 已知 $x-\frac{1}{x}=2b$ 及 $x^3-\frac{1}{x^3}=c$，求 c 的值。

Given that $x-\frac{1}{x}=2b$ and $x^3-\frac{1}{x^3}=c$, find the value of c.

$c =$

4. 如圖一，$\alpha=c$, $\beta=43$, $\gamma=59$ 及 $\omega=d$，求 d 的值。
In Figure 1, $\alpha=c$, $\beta=43$, $\gamma=59$ and $\omega=d$, find the value of d.

$d =$



圖一
Figure 1

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2008 – 2009)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

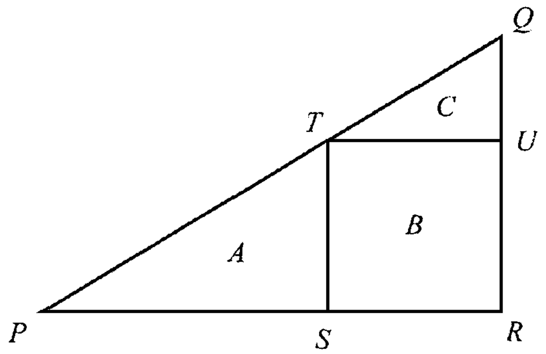
1. 已知 $\frac{4}{\sqrt{6}+\sqrt{2}}-\frac{1}{\sqrt{3}+\sqrt{2}}=\sqrt{a}-\sqrt{b}$。若 $m=a-b$，求 m 的值。

Given that $\frac{4}{\sqrt{6}+\sqrt{2}}-\frac{1}{\sqrt{3}+\sqrt{2}}=\sqrt{a}-\sqrt{b}$. If $m = a - b$, find the value of m.

$m =$

2. 如圖一，PQR 為直角三角形及 $RSTU$ 為矩形。設 A，B 及 C 是相對圖形的面積。若 $A:B=m:2$ 及 $A:C=n:1$，求 n 的值。
In figure 1, PQR is a right-angled triangle and $RSTU$ is a rectangle.
Let A, B and C be the areas of the corresponding regions.
If $A : B = m : 2$ and $A : C = n : 1$, find the value of n.

$n =$



圖一
Figure 1

3. 設 x_1、x_2、x_3、x_4 為實數及 $x_1 \neq x_2$。若 $(x_1+x_3)(x_1+x_4)=(x_2+x_3)(x_2+x_4)=n-10$ 及 $p=(x_1+x_3)(x_2+x_3)+(x_1+x_4)(x_2+x_4)$，求 p 的值。
Let x_1, x_2, x_3, x_4 be real numbers and $x_1 \neq x_2$.
If $(x_1+x_3)(x_1+x_4)=(x_2+x_3)(x_2+x_4)=n-10$ and
$p=(x_1+x_3)(x_2+x_3)+(x_1+x_4)(x_2+x_4)$, find the value of p.

$p =$

4. 已知某校學生人數是 7 的倍數且不少於 1000。若學生人數被 $p+1$、$p+2$ 及 $p+3$ 除後的餘數均是 1。設學生人數的最小可能值為 q，求 q 的值。
The total number of students in a school is a multiple of 7 and not less than 1000. Given that the same remainder 1 will be obtained when the number of students is divided by $p + 1$, $p + 2$ and $p + 3$. Let q be the least of the possible numbers of students in the school, find the value of q.

$q =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score					Min.	Sec.

Hong Kong Mathematics Olympiad (2008 – 2009)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知 $x_0^2+x_0-1=0$。若 $m=x_0^3+2x_0^2+2$，求 m 的值。

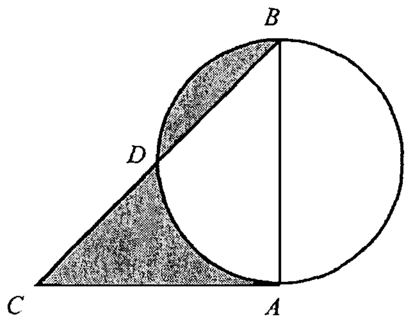
Given that $x_0^2+x_0-1=0$. If $m=x_0^3+2x_0^2+2$, find the value of m.

$m =$

2. 如圖一，ΔBAC 是一直角三角形，$AB = AC = m$ cm。已知直徑為 AB 的圓與 BC 相交於 D 且陰影部分的面積是 n cm^2，求 n 的值。

In Figure 1, ΔBAC is a right-angled triangle, $AB = AC = m$ cm. Suppose that the circle with diameter AB intersects the line BC at D, and the total area of the shaded region is n cm^2. Find the value of n.

$n =$



圖一
Figure 1

3. 已知 $p=4n\left(\frac{1}{2^{2009}}\right)^{\log(1)}$，求 p 的值。

Given that $p=4n\left(\frac{1}{2^{2009}}\right)^{\log(1)}$, find the value of p.

$p =$

4. 設 x 及 y 為實數並滿足方程 $\left(x-\sqrt{p}\right)^2+\left(y-\sqrt{p}\right)^2=1$。
若 $k=\frac{y}{x-3}$ 及 q 是 k^2 的最小可能值，求 q 的值。

Let x and y be real numbers satisfying the equation $\left(x-\sqrt{p}\right)^2+\left(y-\sqrt{p}\right)^2=1$.
If $k=\frac{y}{x-3}$ and q is the least possible values of k^2, find the value of q.

$q =$

FOR OFFICIAL USE

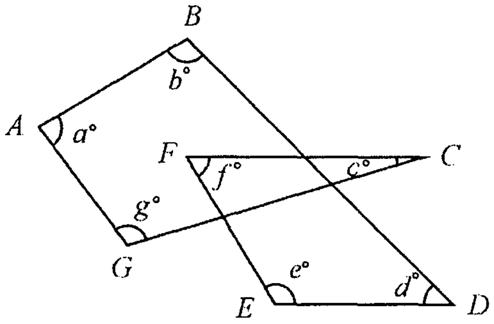
Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2008 – 2009)
Final Event Sample (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 如圖一，BD、FC、GC 及 FE 為直綫。
若 $z = a + b + c + d + e + f + g$，求 z 的值。
In Figure 1, BD, FC, GC and FE are straight lines.
If $z = a + b + c + d + e + f + g$, find the value of z.

$z =$



圖一

Figure 1

2. 若 $1^6 + 2^6 + 3^6 + 4^6 + 5^6 + 6^6$ 被 7 除後的餘數是 R，求 R 的值。
If R is the remainder of $1^6 + 2^6 + 3^6 + 4^6 + 5^6 + 6^6$ divided by 7, find the value of R.

$R =$

3. 若 14!能被 6^k 整除，其中 k 為整數，求 k 的最大可能值。
If 14! is divisible by 6^k, where k is an integer, find the largest possible value of k.

$k =$

4. 設實數 x、y 及 z 滿足 $x+\frac{1}{y}=4$，$y+\frac{1}{z}=1$ 及 $z+\frac{1}{x}=\frac{7}{3}$。求 xyz 的值。
Let x, y and z be real numbers that satisfy $x+\frac{1}{y}=4$, $y+\frac{1}{z}=1$ and $z+\frac{1}{x}=\frac{7}{3}$.
Find the value of xyz.

$xyz =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2008 – 2009)
Final Event 1 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知三角形三邊的長度分別是 a cm、2 cm 及 b cm，其中 a 和 b 是整數且 $a \le 2 \le b$。若有 q 種不全等的三角形滿足上述條件，求 q 的值。
Given some triangles with side lengths a cm, 2 cm and b cm, where a and b are integers and $a \le 2 \le b$. If there are q non-congruent classes of triangles satisfying the above conditions, find the value of q.

$q =$

2. 已知方程 $|x| - \frac{4}{x} = \frac{3|x|}{x}$ 有 k 個相異實根，求 k 的值。
Given that the equation $|x| - \frac{4}{x} = \frac{3|x|}{x}$ has k distinct real root(s), find the value of k.

$k =$

3. 已知 x 及 y 為非零實數且滿足方程 $\frac{\sqrt{x}}{\sqrt{y}} - \frac{\sqrt{y}}{\sqrt{x}} = \frac{7}{12}$ 及 $x - y = 7$。
若 $w = x + y$，求 w 的值。
Given that x and y are non-zero real numbers satisfying the equations $\frac{\sqrt{x}}{\sqrt{y}} - \frac{\sqrt{y}}{\sqrt{x}} = \frac{7}{12}$ and $x - y = 7$. If $w = x + y$, find the value of w.

$w =$

4. 已知 x 及 y 為實數且 $\left|x - \frac{1}{2}\right| + \sqrt{y^2 - 1} = 0$。設 $p = |x| + |y|$，求 p 的值。
Given that x and y are real numbers and $\left|x - \frac{1}{2}\right| + \sqrt{y^2 - 1} = 0$.
Let $p = |x| + |y|$, find the value of p.

$p =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score					Min.	Sec.

Hong Kong Mathematics Olympiad (2008 – 2009)
Final Event 2 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知 $\tan\theta=\frac{5}{12}$，其中 $180°\le\theta\le270°$。若 $A=\cos\theta+\sin\theta$，求 A 的值。

   Given $\tan\theta=\frac{5}{12}$, where $180°\le\theta\le270°$. If $A=\cos\theta+\sin\theta$, find the value of A.

   A =

2. 設 $[x]$ 是不超過 x 的最大整數。
   若 $B=\left[10+\sqrt{10+\sqrt{10+\sqrt{10+\cdots}}}\right]$，求 B 的值。

   Let $[x]$ be the largest integer not greater than x.
   If $B=\left[10+\sqrt{10+\sqrt{10+\sqrt{10+\cdots}}}\right]$, find the value of B.

   B =

3. 設 $a\oplus b=ab+10$。若 $C=(1\oplus2)\oplus3$，求 C 的值。

   Let $a\oplus b=ab+10$. If $C=(1\oplus2)\oplus3$, find the value of C.

   C =

4. 在座標平面上，用以下直綫所圍成圖形的面積為 D 平方單位，求 D 的值。

   L_1: $y-2=0$
   L_2: $y+2=0$
   L_3: $4x+7y-10=0$
   L_4: $4x+7y+20=0$

   In the coordinate plane, the area of the region bounded by the following lines is D square units, find the value of D.

   L_1: $y-2=0$
   L_2: $y+2=0$
   L_3: $4x+7y-10=0$
   L_4: $4x+7y+20=0$

   D =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2008 – 2009)
Final Event 3 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設[x]是不超過 x 的最大整數。
若 $A=\left[\dfrac{2008\times80+2009\times130+2010\times180}{2008\times15+2009\times25+2010\times35}\right]$，求 A 的值。
Let [x] be the largest integer not greater than x.
If $A=\left[\dfrac{2008\times80+2009\times130+2010\times180}{2008\times15+2009\times25+2010\times35}\right]$, find the value of A.

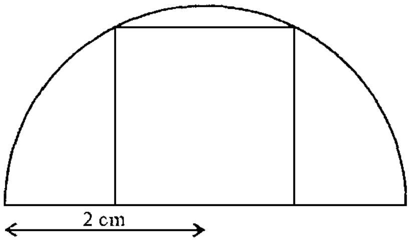
A =

2. 在 $\underbrace{99\ldots9}_{2009個9}\times\underbrace{99\ldots9}_{2009個9}+1\underbrace{99\ldots9}_{2009個9}$ 中，末位的 0 共有 R 個，求 R 的值。
There are R zeros at the end of $\underbrace{99\ldots9}_{2009\text{ of }9's}\times\underbrace{99\ldots9}_{2009\text{ of }9's}+1\underbrace{99\ldots9}_{2009\text{ of }9's}$, find the value of R.

R =

3. 如圖一，邊長為 Q cm 的正方形內接於半徑為 2 cm 的半圓中，求 Q 的值。
In Figure 1, a square of side length Q cm is inscribed in a semi-circle of radius 2 cm. Find the value of Q.

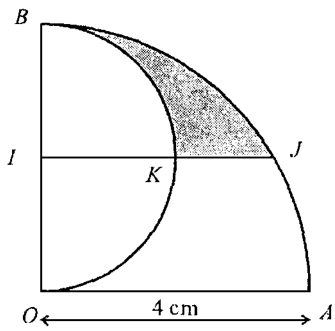
Q =



圖一
Figure 1

4. 如圖二，扇形 OAB 的半徑為 4 cm 及 $\angle AOB$ 為直角。設以 OB 為直徑的半圓，其圓心為 I 且 IJ // OA 及 IJ 與該半圓相交於 K。若陰影部分的面積為 T cm^2，求 T 的值。(取 $\pi=3$)
In Figure 2, the sector OAB has radius 4 cm and $\angle AOB$ is a right angle.
Let the semi-circle with diameter OB be centred at I with IJ // OA, and IJ intersects the semi-circle at K.
If the area of the shaded region is T cm^2, find the value of T. (Take $\pi=3$)

T =



圖二
Figure 2

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2008 – 2009)
Final Event 4 (Group)

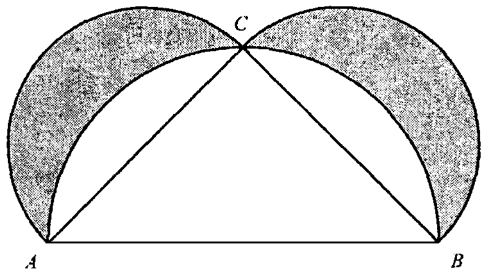
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設 P 為實數。若 $\sqrt{3-2P}+\sqrt{1-2P}=2$，求 P 的值。
Let P be a real number. If $\sqrt{3-2P}+\sqrt{1-2P}=2$, find the value of P.

P =

2. 如圖一，設 AB、AC 及 BC 為相應半圓的直徑。
若 $AC = BC = 1$ cm 及陰影部分的面積是 R cm^2，求 R 的值。
In Figure 1, let AB, AC and BC be the diameters of the corresponding three semi-circles. If $AC = BC = 1$ cm and the area of the shaded region is R cm^2.
Find the value of R.

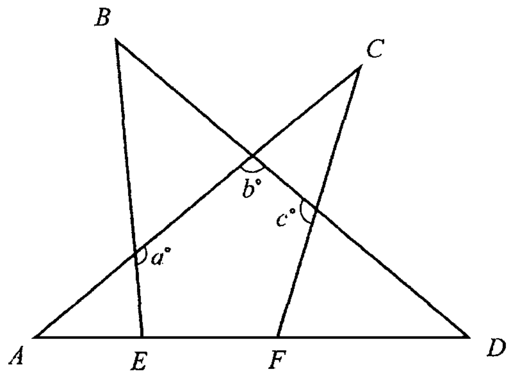
R =



圖一
Figure 1

3. 如圖二，AC、AD、BD、BE 及 CF 為直綫。
若 $\angle A + \angle B + \angle C + \angle D = 140°$ 及 $a + b + c = S$，求 S 的值。
In Figure 2, AC, AD, BD, BE and CF are straight lines.
If $\angle A + \angle B + \angle C + \angle D = 140°$ and $a + b + c = S$, find the value of S.

S =



圖二
Figure 2

4. 設 $Q=\log_{2+\sqrt{2^2-1}}\left(2-\sqrt{2^2-1}\right)$，求 Q 的值。
Let $Q=\log_{2+\sqrt{2^2-1}}\left(2-\sqrt{2^2-1}\right)$, find the value of Q.

Q =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]　　Team No. [ ]

\+ Bonus score [ ]　　Time [ ] [ ]

Total score [ ]　　Min.　Sec.

Hong Kong Mathematics Olympiad (2009 – 2010)
Final Event Sample (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設$[x]$是不超過 x 的最大整數。若 $a=\left[\left(\sqrt{3}-\sqrt{2}\right)^{2009}\right]+16$，求 a 的值。

   Let $[x]$ be the largest integer not greater than x.

   If $a=\left[\left(\sqrt{3}-\sqrt{2}\right)^{2009}\right]+16$, find the value of a.

   $a=$

2. 在座標平面上，若 x-軸、y-軸與直綫 $3x+ay=12$ 所圍成三角形的面積是 b 平方單位，求 b 的值。

   In the coordinate plane, if the area of the triangle formed by the x-axis, y-axis and the line $3x+ay=12$ is b square units, find the value of b.

   $b=$

3. 已知 $x-\frac{1}{x}=2b$ 及 $x^3-\frac{1}{x^3}=c$，求 c 的值。

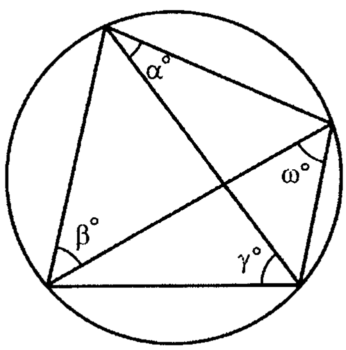
   Given that $x-\frac{1}{x}=2b$ and $x^3-\frac{1}{x^3}=c$, find the value of c.

   $c=$

4. 如圖一，$\alpha=c$、$\beta=43$、$\gamma=59$ 及 $\omega=d$，求 d 的值。

   In Figure 1, $\alpha=c$, $\beta=43$, $\gamma=59$ and $\omega=d$, find the value of d.

   $d=$



圖一
Figure 1

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.
			+ Bonus score				Time: Min. / Sec.
			Total score				

Hong Kong Mathematics Olympiad (2009 – 2010)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 把三個體積分別為 1、8、27 的正立方體，以面同貼面的方法黏合起來。
若 a 為所得的多面體的最小總表面積，求 a 的值。
Three cubes with volumes 1, 8, 27 are glued together at their faces.
If a is the smallest possible surface area of the resulting polyhedron, find the value of a.

a =

2. 已知 $f(x) = -x^2 + 10x + 9$，且 $2 \le x \le \frac{a}{9}$。若 b 是 f 的最大及最小值之差，求 b 的值。
Given that $f(x) = -x^2 + 10x + 9$, and $2 \le x \le \frac{a}{9}$.
If b is the difference of the maximum and minimum values of f, find the value of b .

b =

3. 已知 p 及 q 是實數，且 $pq = b$ 及 $p^2q + q^2p + p + q = 70$。若 $c = p^2 + q^2$，求 c 的值。
Given that p and q are real numbers with $pq = b$ and $p^2q + q^2p + p + q = 70$.
If $c = p^2 + q^2$, find the value of c .

c =

4. 在一個有 c 行的演奏廳中，每一行都比前一行多兩個座位。
若中間的行有 64 個座位，這演奏廳共有多少個座位 (d)？
There are c rows in a concert hall and each succeeding row has two more seats than the previous row.
If the middle row has 64 seats, how many seats (d) does the concert have ?

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.		
			+	Bonus score			Time		
				Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2009 – 2010)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 a，p，q 是質數，且滿足 $a < p$ 及 $a + p = q$，求 a 的值。
If a, p, q are primes with $a < p$ and $a + p = q$, find the value of a .

a =

2. 若 b 及 h 為正整數，且滿足 $b < h$ 及 $b^2 + h^2 = b(a + h) + ah$，求 b 的值。
If b and h are positive integers with $b < h$ and $b^2 + h^2 = b(a + h) + ah$, find the value of b .

b =

3. 在一個$(2b + 1)\times(2b + 1)$的棋盤上任意選取兩個不在同一橫行上方格。若 c 為選取的兩個不同方格的組合數目，求 c 的值。
In a $(2b + 1)\times(2b + 1)$ checkerboard, two squares not lying in the same row are randomly chosen. If c is the number of combinations of different pairs of squares chosen, find the value of c .

c =

4. 已知 $f(x) = c\left|\frac{1}{x} - \left\lfloor \frac{1}{x} + \frac{1}{2} \right\rfloor\right|$，其中$\lfloor x \rfloor$是小於或等於實數 x 的最大整數。若 d 為 $f(x)$的最大值，求 d 的值。
Given that $f(x) = c\left|\frac{1}{x} - \left\lfloor \frac{1}{x} + \frac{1}{2} \right\rfloor\right|$，where $\lfloor x \rfloor$ is the greatest integer less than or equal to the real number x. If d is the maximum value of $f(x)$, find the value of d .

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time		
				Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2009 – 2010)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 a 為 15147 的相異質因數的數目。求 a 的值。
If a is the number of distinct prime factors of 15147, find the value of a .

a =

2. 若 $x+\frac{1}{x}=a$ 及 $x^3+\frac{1}{x^3}=b$ ，求 b 的值。
If $x+\frac{1}{x}=a$ and $x^3+\frac{1}{x^3}=b$, find the value of b .

b =

3. 設 $f(x)=\begin{cases} x+5 & \text{當 } x \text{ 是一奇數} \\ \frac{x}{2} & \text{當 } x \text{ 是一偶數} \end{cases}$。
若 c 是一奇數及 $f(f(f(c)))=b$，求 c 的最小值。
Let $f(x)=\begin{cases} x+5 & \text{if } x \text{ is an odd integer} \\ \frac{x}{2} & \text{if } x \text{ is an even integer} \end{cases}$.
If c is an odd integer and $f(f(f(c)))=b$, find the least value of c.

c =

4. 設 $f\left(\frac{x}{3}\right)=x^2+x+1$。若 d 為所有滿足 $f(3x)=c$ 的 x 之和，求 d 的值。
Let $f\left(\frac{x}{3}\right)=x^2+x+1$. If d is the sum of all x for which $f(3x)=c$,
find the value of d .

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

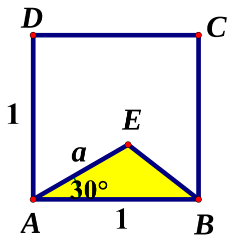
\+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min. Sec.

Hong Kong Mathematics Olympiad (2009 – 2010)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 在圖一中，$ABCD$ 為一正方形，E 為一點及 $\angle EAB = 30°$。若 $ABCD$ 的面積是 ΔABE 的面積的六倍，則 $AE : AB = a : 1$。求 a 的值。
In Figure 1, $ABCD$ is a square, E is a point and $\angle EAB = 30°$. If the area of $ABCD$ is six times that of ΔABE, then the ratio of $AE : AB = a : 1$. Find the value of a.



圖一
Figure 1

$a =$

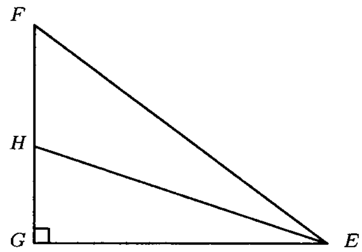
2. 已知 $b = \dfrac{\log 8^a + \log 27^a + \log 125^a}{\log 9 + \log 25 + \log 2 - \log 15}$，求 b 的值。
Given that $b = \dfrac{\log 8^a + \log 27^a + \log 125^a}{\log 9 + \log 25 + \log 2 - \log 15}$, find the value of b.

$b =$

3. 設 c 為 $1^3 + 2^3 + \ldots + 2009^3 + 2010^3$ 除以 b^2 的餘數，求 c 的值。
Let c be the remainder of $1^3 + 2^3 + \ldots + 2009^3 + 2010^3$ divided by b^2, find the value of c.

$c =$

4. 在圖二中，EFG 為一直角三角形。已知 H 為 FG 上的一點，使得 $GH : HF = 4 : 5$ 及 $\angle GEH = \angle FEH$。若 $EG = c$ 及 $FG = d$，求 d 的值。
In Figure 2, EFG is a right-angled triangle.
Given that H is a point on FG, such that $GH : HF = 4 : 5$ and $\angle GEH = \angle FEH$.
If $EG = c$ and $FG = d$, find the value of d.



圖二
Figure 2

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time		
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2009 – 2010)
Final Event Spare (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知 $a=\sqrt{(19.19)^2+(39.19)^2-(38.38)(39.19)}$。求 a 的值。

Given that $a=\sqrt{(19.19)^2+(39.19)^2-(38.38)(39.19)}$. Find the value of m.

$a =$

2. 給定四點 $R(0, 0)$、$S(a, 0)$、$T(a, 6)$及 $U(0, 6)$。
若直綫 $y = b(x - 7) + 4$ 把四邊形 $RSTU$ 分成兩份，其面積相等，求 b 的值。
Given four points $R(0, 0)$, $S(a, 0)$, $T(a, 6)$ and $U(0, 6)$. If the line $y = b(x - 7) + 4$ cuts the quadrilateral $RSTU$ into two halves of equal area, find the value of b.

$b =$

3. 已知 c 為 $\mathrm{f}(x)=\dfrac{x^2-2x+\frac{1}{b}}{2x^2+2x+1}$ 的最小值。求 c 的值。

Given that c is the minimum value of $\mathrm{f}(x)=\dfrac{x^2-2x+\frac{1}{b}}{2x^2+2x+1}$. Find the value of c.

$c =$

4. 已知 $\mathrm{f}(x) = px^6 + qx^4 + 3x - \sqrt{2}$，且 p、q 為非零實數。
若 $d = \mathrm{f}(c) - \mathrm{f}(-c)$，求 d 的值。
Given that $\mathrm{f}(x) = px^6 + qx^4 + 3x - \sqrt{2}$, and p, q are non-zero real numbers.
If $d = \mathrm{f}(c) - \mathrm{f}(-c)$, find the value of d.

$d =$

FOR OFFICIAL USE

Score for accuracy		× Mult. factor for speed		=	Team No.	
		+ Bonus score			Time	
		Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2009 – 2010)
Final Event Sample (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知 $\tan\theta = \frac{5}{12}$，其中 $180° \le \theta \le 270°$。若 $A = \cos\theta + \sin\theta$，求 A 的值。
Given $\tan\theta = \frac{5}{12}$, where $180° \le \theta \le 270°$.
If $A = \cos\theta + \sin\theta$, find the value of A.

A =

2. 設 $[x]$是不超過 x 的最大整數。若 $B = \left[10+\sqrt{10+\sqrt{10+\sqrt{10+\cdots}}}\right]$，求 B 的值。
Let $[x]$ be the largest integer not greater than x.
If $B = \left[10+\sqrt{10+\sqrt{10+\sqrt{10+\cdots}}}\right]$, find the value of B.

B =

3. 設 $a \oplus b = ab + 10$。若 $C = (1\oplus 2) \oplus 3$，求 C 的值。
Let $a \oplus b = ab + 10$. If $C = (1\oplus 2) \oplus 3$, find the value of C.

C =

4. 在座標平面上，用以下直綫所圍成圖形的面積為 D 平方單位，求 D 的值。

L_1: $y - 2 = 0$
L_2: $y + 2 = 0$
L_3: $4x + 7y - 10 = 0$
L_4: $4x + 7y + 20 = 0$

In the coordinate plane, the area of the region bounded by the following lines is D square units, find the value of D.

L_1: $y - 2 = 0$
L_2: $y + 2 = 0$
L_3: $4x + 7y - 10 = 0$
L_4: $4x + 7y + 20 = 0$

D =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score					Min.	Sec.

Hong Kong Mathematics Olympiad (2009 – 2010)
Final Event 1 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 求 $\sin^2 1° + \sin^2 2° + \ldots + \sin^2 89°$的值。
Find the value of $\sin^2 1° + \sin^2 2° + \ldots + \sin^2 89°$.

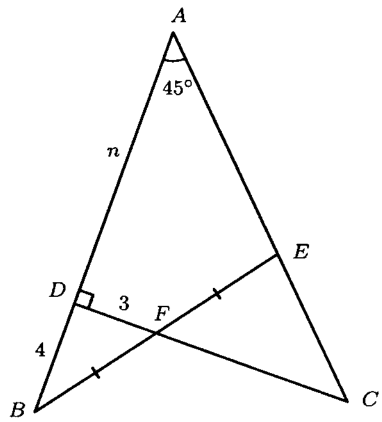
2. 已知 $\frac{x+z}{2z-x} = \frac{z+2y}{2x-z} = \frac{x}{y}$，其中 x、y、z 為正數。求 $\frac{x}{y}$ 的值。
Given that $\frac{x+z}{2z-x} = \frac{z+2y}{2x-z} = \frac{x}{y}$, where x, y and z are positive numbers.
Find the value of $\frac{x}{y}$.

$\frac{x}{y} =$

3. 求方程 $(2^x - 4)^3 + (4^x - 2)^3 = (4^x + 2^x - 6)^3$ 的所有實根 x 的總和。
Find the sum of all real roots x of the equation $(2^x - 4)^3 + (4^x - 2)^3 = (4^x + 2^x - 6)^3$.

4. 在圖一，若 $AB \perp CD$，F 是 BE 的中點，$\angle A = 45°$，$DF = 3$, $BD = 4$ 及 $AD = n$，求 n 的值。
In Figure 1, if $AB \perp CD$, F is the midpoint of BE, $\angle A = 45°$, $DF = 3$, $BD = 4$ and $AD = n$, find the value of n .

$n =$



圖一
Figure 1

FOR OFFICIAL USE

Score for accuracy × Mult. factor for speed =

Team No.

+ Bonus score

Time

Total score

Min. Sec.

Hong Kong Mathematics Olympiad (2009 – 2010)
Final Event 2 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 $p = 2 - 2^2 - 2^3 - 2^4 - \ldots - 2^9 - 2^{10} + 2^{11}$，求 p 的值。
If $p = 2 - 2^2 - 2^3 - 2^4 - \ldots - 2^9 - 2^{10} + 2^{11}$, find the value of p .

$p =$

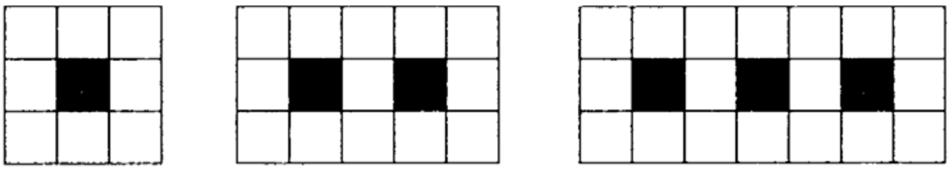
2. 已知 x、y、z 為 3 個相異實數。若 $x+\frac{1}{y}=y+\frac{1}{z}=z+\frac{1}{x}$ 及 $m = x^2y^2z^2$。
求 m 的值。
Given that x, y, z are three distinct real numbers.
If $x+\frac{1}{y}=y+\frac{1}{z}=z+\frac{1}{x}$ and $m = x^2y^2 z^2$, find the value of m .

$m =$

3. 已知 x 為一正實數，且滿足 $x\cdot 3^x = 3^{18}$。若 k 是一正整數且 $k < x < k + 1$，求 k 的值。
Given that x is a positive real number and $x\cdot 3^x = 3^{18}$.
If k is a positive integer and $k < x < k + 1$, find the value of k .

$k =$

4. 圖一所示為利用黑白兩種顏色湊成有規律的圖形。
求第 95 個圖形的白色格子的數目。
Figure 1 shows the sequence of figures that are made of squares of white and black.
Find the number of white squares in the 95th figure.



第一個 1st figure　　第二個 2nd figure　　第三個 3rd figure

圖一

Figure 1

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]　　Team No. [ ]

\+ Bonus score [ ]　　Time [ ] [ ]

Total score [ ]　　Min.　Sec.

Hong Kong Mathematics Olympiad (2009 – 2010)
Final Event 3 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

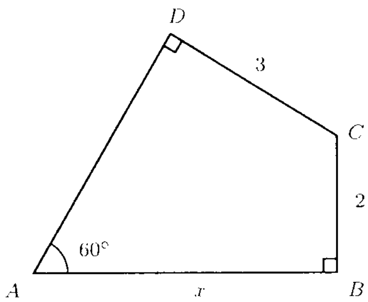
1. 求 $101^{303} + 301^{101}$ 的最小質因子。
Find the smallest prime factor of $101^{303} + 301^{101}$.

2. 設 n 為 $\dfrac{1}{\frac{1}{1980}+\frac{1}{1981}+\cdots+\frac{1}{2009}}$ 的整數部分，求 n 的值。
Let n be the integral part of $\dfrac{1}{\frac{1}{1980}+\frac{1}{1981}+\cdots+\frac{1}{2009}}$. Find the value of n.

$n =$

3. 在圖一中，若 $\angle A = 60^\circ$，$\angle B = \angle D = 90^\circ$。$BC = 2$，$CD = 3$ 及 $AB = x$，求 x 的值。
In Figure 1, $\angle A = 60^\circ$, $\angle B = \angle D = 90^\circ$. $BC = 2$, $CD = 3$ and $AB = x$, find the value of x.

$x =$



圖一
Figure 1

4. 已知函數 f 對所有實數 x 皆滿足 $f(2 + x) = f(2 - x)$，且 $f(x) = 0$ 恰好有四個相異實根。求這四個相異實根之和。
Given that the function f satisfies $f(2 + x) = f(2 - x)$ for every real number x and that $f(x) = 0$ has exactly four distinct real roots.
Find the sum of these four distinct real roots.

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2009 – 2010)
Final Event 4 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設 a 為整數及 $a \neq 1$。已知方程 $(a-1)x^2 - mx + a = 0$ 的兩根均為正整數。求 m 的值。
Let a be an integer and $a \neq 1$. Given that the equation $(a-1)x^2 - mx + a = 0$ has two roots which are positive integers. Find the value of m .

m =

2. 已知 x 為一實數及 $y=\sqrt{x^2-2x+2}+\sqrt{x^2-10x+34}$ 。求 y 的最小值。

Given that x is a real number and $y=\sqrt{x^2-2x+2}+\sqrt{x^2-10x+34}$.

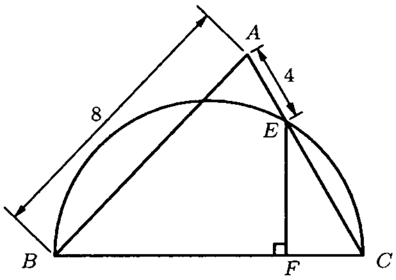
Find the minimum value of y .

y =

3. 已知 A、B、C 為正整數，且 A、B 和 C 的最大公因數等於 1。若 A、B、C 滿足 $A\log_{500}5 + B\log_{500}2 = C$，求 $A+B+C$ 的值。
Given that A, B, C are positive integers with their greatest common divisor equal to 1 . If A, B, C satisfy $A\log_{500}5 + B\log_{500}2 = C$, find the value of $A+B+C$.

$A+B+C$=

4. 在圖一中，BEC 是一半圓形及 F 是直徑 BC 上的一點。已知 $BF : FC = 3 : 1$，$AB = 8$ 及 $AE = 4$。求 EC 的長度。
In figure 1, BEC is a semicircle and F is a point on the diameter BC. Given that $BF : FC = 3 : 1$, $AB = 8$ and $AE = 4$. Find the length of EC .



圖一

Figure 1

EC =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2009 – 2010)
Final Event Spare (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知 n 為一正整數。若 $n^2 + 5n + 13$ 為一完全平方數，求 n 的值。
Given that n is a positive integer.
If $n^2 + 5n + 13$ is a perfect square, find the value of n.

$n =$

2. 已知 $1^3 + 2^3 + \ldots + k^3 = \left(\frac{k(k+1)}{2}\right)^2$，求 $11^3 + 12^3 + \ldots + 24^3$ 的值。
Given that $1^3 + 2^3 + \ldots + k^3 = \left(\frac{k(k+1)}{2}\right)^2$. Find the value of $11^3 + 12^3 + \ldots + 24^3$.

3. 若 P 是等邊三角形 ABC 內部的隨意一點，求ΔABP 的面積同時大於ΔACP 及ΔBCP 的面積的概率。
If P is an arbitrary point in the interior of the equilateral triangle ABC, find the probability that the area of ΔABP is greater than each of the areas of ΔACP and ΔBCP.

4. 共有多少個正整數 m 使得通過點 $A(-m, 0)$ 及點 $B(0, 2)$的直綫亦通過 $P(7, k)$，其中 k 為一正整數？
How many positive integers m are there for which the straight line passing through points $A(-m, 0)$ and $B(0, 2)$ and also passes through the point $P(7, k)$, where k is a positive integer?

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.
			+	Bonus score			Time
				Total score			Min. / Sec.

Hong Kong Mathematics Olympiad (2010 – 2011)
Final Event Sample (Individual)

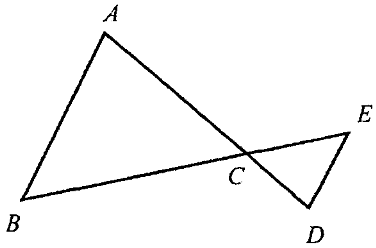
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設 a、b、c 及 d 為方程 $x^4 - 15x^2 + 56 = 0$ 相異的根。
若 $R = a^2 + b^2 + c^2 + d^2$ ，求 R 的值。
Let a, b, c and d be the distinct roots of the equation $x^4 - 15x^2 + 56 = 0$.
If $R = a^2 + b^2 + c^2 + d^2$, find the value of R .

R =

2. 如圖一，AD 及 BE 為直綫且 $AB = AC$ 及 $AB // ED$。
若 $\angle ABC = R°$及$\angle ADE = S°$，求 S 的值。
In Figure 1, AD and BE are straight lines with $AB = AC$ and $AB // ED$.
If $\angle ABC = R°$ and $\angle ADE = S°$, find the value of S .

S =



圖一
Figure 1

3. 設 $F = 1 + 2 + 2^2 + 2^3 + \ldots + 2^S$ 及 $T = \sqrt{\frac{\log(1+F)}{\log 2}}$ ，求 T 的值。

Let $F = 1 + 2 + 2^2 + 2^3 + \ldots + 2^S$ and $T = \sqrt{\frac{\log(1+F)}{\log 2}}$, find the value of T .

T =

4. 設 f(x)是一個函數使得對所有整數 $n \geq 6$ 時，$f(n) = (n-1)\, f(n-1)$及 $f(n) \neq 0$。
若 $U = \frac{f(T)}{(T-1) f(T-3)}$ ，求 U 的值。
Let f(x) be a function such that $f(n) = (n-1)\, f(n-1)$
and $f(n) \neq 0$ hold for all integers $n \geq 6$. If $U = \frac{f(T)}{(T-1) f(T-3)}$, find the value of U .

U =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=	Team No.	
		+	Bonus score			Time	
			Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2010 – 2011)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 a、b 及 c 的平均值為 12，和 $2a + 1$、$2b + 2$、$2c + 3$ 及 2 的平均值為 P，求 P 的值。
If the average of a, b and c is 12, and the average of $2a + 1$, $2b + 2$, $2c + 3$ and 2 is P, find the value of P .

$P =$

2. 設 $20112011 = aP^5 + bP^4 + cP^3 + dP^2 + eP + f$，其中 a、b、c、d、e 及 f 為整數及 $0 \le a, b, c, d, e, f < P$。若 $Q = a + b + c + d + e + f$，求 Q 的值。
Let $20112011 = aP^5 + bP^4 + cP^3 + dP^2 + eP + f$, where a, b, c, d, e and f are integers and $0 \le a, b, c, d, e, f < P$.If $Q = a + b + c + d + e + f$, find the value of Q .

$Q =$

3. 若 R 為 $8^Q + 7^{10Q} + 6^{100Q} + 5^{1000Q}$ 的個位數，求 R 的值。
If R is the units digit of the value of $8^Q + 7^{10Q} + 6^{100Q} + 5^{1000Q}$, find the value of R .

$R =$

4. 若 S 為安排 R 個人圍成圓形的數目，求 S 的值。
If S is the number of ways to arrange R persons in a circle, find the value of S .

$S =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2010 – 2011)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若方程組 $\begin{cases} x+y=P \\ 3x+5y=13 \end{cases}$ 的解為正整數，求 P 的值。

If the solution of the system of equations $\begin{cases} x+y=P \\ 3x+5y=13 \end{cases}$ are positive integers, find the value of P.

$P =$

2. 若 $x+y=P$，$x^2+y^2=Q$ 及 $x^3+y^3=P^2$，求 Q 的值。
If $x+y=P$, $x^2+y^2=Q$ and $x^3+y^3=P^2$, find the value of Q.

$Q =$

3. 若 a 及 b 為相異質數且 $a^2-aQ+R=0$ 及 $b^2-bQ+R=0$，求 R 的值。
If a and b are distinct prime numbers and $a^2-aQ+R=0$ and $b^2-bQ+R=0$, find the value of R.

$R =$

4. 若 $S>0$ 及 $\frac{1}{S(S-1)}+\frac{1}{(S+1)S}+\cdots+\frac{1}{(S+20)(S+19)}=1-\frac{1}{R}$，求 S 的值。
If $S>0$ and $\frac{1}{S(S-1)}+\frac{1}{(S+1)S}+\cdots+\frac{1}{(S+20)(S+19)}=1-\frac{1}{R}$, find the value of S.

$S =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.		
			+ Bonus score				Time		
			Total score					Min.	Sec.

Hong Kong Mathematics Olympiad (2010 – 2011)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 P 為一質數，而且方程 $x^2 + 2(P + 1)x + P^2 – P – 14 = 0$ 的根為整數，求 P 的最小值。
If P is a prime number and the roots of the equation $x^2 + 2(P+1)x + P^2 –P –14 = 0$ are integers, find the least value of P .

$P =$

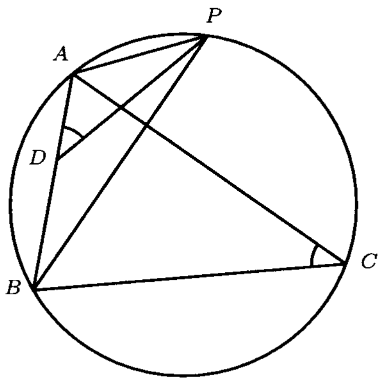
2. 已知 $x^2 + ax + b$ 為 $2x^3 + 5x^2 + 24x + 11$ 及 $x^3 + Px – 22$ 的公因式。若 $Q = a + b$，求 Q 的值。
Given that $x^2 + ax + b$ is a common factor of $2x^3 + 5x^2 + 24x + 11$ and $x^3 + Px – 22$. If $Q = a + b$, find the value of Q .

$Q =$

3. 若 R 為一正整數及 $R^3 + 4R^2 + (Q – 93)R + 14Q + 10$ 為一質數，求 R 的值。
If R is a positive integer and $R^3 + 4R^2 + (Q – 93)R + 14Q + 10$ is a prime number, find the value of R .

$R =$

4. 在圖一中，AP、AB、PB、PD、AC 及 BC 為綫段及 D 為 AB 上的一點。若 AB 的長度為 AD 的長度的 R 倍，$\angle ADP = \angle ACB$ 及 $S = \frac{PB}{PD}$，求 S 的值。
In Figure 1, AP, AB, PB, PD, AC and BC are line segments and D is a point on AB. If the length of AB is R times that of AD, $\angle ADP = \angle ACB$ and $S = \frac{PB}{PD}$, find the value of S .



圖一
Figure 1

$S =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]
\+ Bonus score [ ]
Total score [ ]

Team No. [ ]
Time [ ] Min. [ ] Sec.

Hong Kong Mathematics Olympiad (2010 – 2011)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 考慮函數 $y = \sin x + \sqrt{3}\cos x$。設 a 為 y 的最大值。求 a 的值。
Consider the function $y = \sin x + \sqrt{3}\cos x$. Let a be the maximum value of y. Find the value of a.

$a =$

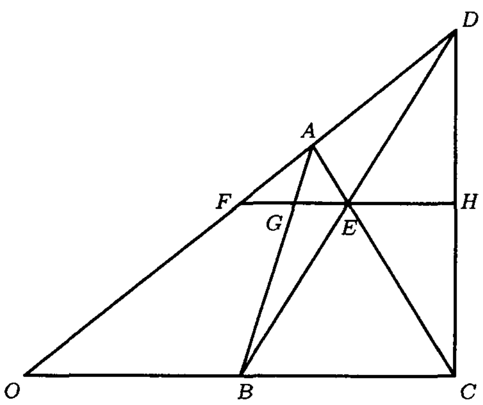
2. 若 b 及 y 滿足 $|b - y| = b + y - a$ 及 $|b + y| = b + a$。求 b 的值。
Find the value of b if b and y satisfy $|b - y| = b + y - a$ and $|b + y| = b + a$.

$b =$

3. 設 x、y 及 z 為正整數。若 $|x - y|^{2010} + |z - x|^{2011} = b$，而且 $c = |x - y| + |y - z| + |z - x|$，求 c 的值。
Let x, y and z be positive integers. If $|x - y|^{2010} + |z - x|^{2011} = b$ and $c = |x - y| + |y - z| + |z - x|$, find the value of c.

$c =$

4. 在圖一中，ODC 為一三角形。已知 FH、AB、AC 及 BD 為綫段使得 AB 及 FH 相交於 G，綫段 AC、BD 及 FH 相交於 E，$GE = 1$，$EH = c$ 及 $FH \parallel OC$。若 $d = EF$，求 d 的值。
In Figure 1, let ODC be a triangle. Given that FH, AB, AC and BD are line segments such that AB intersects FH at G, AC, BD and FH intersect at E, $GE = 1$, $EH = c$ and $FH \parallel OC$. If $d = EF$, find the value of d.



圖一
Figure 1

$d =$

FOR OFFICIAL USE

Score for accuracy × Mult. factor for speed =

\+ Bonus score

Total score

Team No.

Time

Min. Sec.

Hong Kong Mathematics Olympiad (2010 – 2011)
Final Event Spare (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設 P 為邊長為整數小於或等於 9 的三角形的數目。求 P 的值。
Let P be the number of triangles whose side lengths are integers less than or equal to 9. Find the value of P.

P =

2. 設 $Q = \log_{128} 2^3 + \log_{128} 2^5 + \log_{128} 2^7 + \ldots + \log_{128} 2^P$。求 Q 的值。
Let $Q = \log_{128} 2^3 + \log_{128} 2^5 + \log_{128} 2^7 + \ldots + \log_{128} 2^P$. Find the value of Q.

Q =

3. 考慮直綫 $12x - 4y + (Q - 305) = 0$。
若 x-軸、y-軸及此直綫所形成的三角形的面積為 R 平方單位，求 R 的值。
Consider the line $12x - 4y + (Q - 305) = 0$. If the area of the triangle formed by the x-axis, the y-axis and this line is R square units, what is the value of R?

R =

4. 若 $x + \frac{1}{x} = R$ 及 $x^3 + \frac{1}{x^3} = S$，求 S 的值。
If $x + \frac{1}{x} = R$ and $x^3 + \frac{1}{x^3} = S$, find the value of S.

S =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2010 – 2011)
Final Event Sample (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知三角形三邊的長度分別是 a cm、2 cm 及 b cm，其中 a 和 b 是整數且 $a \le 2 \le b$。若有 q 種不全等的三角形滿足上述條件，求 q 的值。
Given some triangles with side lengths a cm, 2 cm and b cm, where a and b are integers and $a \le 2 \le b$. If there are q non-congruent classes of triangles satisfying the above conditions, find the value of q.

$q =$

2. 已知方程 $|x|-\frac{4}{x}=\frac{3|x|}{x}$ 有 k 個相異實根，求 k 的值。
Given that the equation $|x|-\frac{4}{x}=\frac{3|x|}{x}$ has k distinct real root(s), find the value of k.

$k =$

3. 已知 x 及 y 為非零實數且滿足方程 $\frac{\sqrt{x}}{\sqrt{y}}-\frac{\sqrt{y}}{\sqrt{x}}=\frac{7}{12}$ 及 $x-y=7$。若 $w=x+y$，求 w 的值。
Given that x and y are non-zero real numbers satisfying the equations $\frac{\sqrt{x}}{\sqrt{y}}-\frac{\sqrt{y}}{\sqrt{x}}=\frac{7}{12}$ and $x-y=7$. If $w=x+y$, find the value of w.

$w =$

4. 已知 x 及 y 為實數且 $\left|x-\frac{1}{2}\right|+\sqrt{y^2-1}=0$。設 $p=|x|+|y|$，求 p 的值。
Given that x and y are real numbers and $\left|x-\frac{1}{2}\right|+\sqrt{y^2-1}=0$.
Let $p=|x|+|y|$, find the value of p.

$p =$

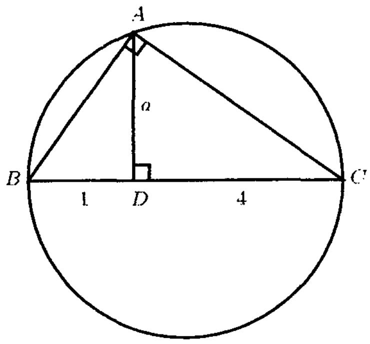
FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.
			+ Bonus score				Time
			Total score				Min. Sec.

Hong Kong Mathematics Olympiad (2010 – 2011)
Final Event 1 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 在圖一中，BC 為圓的直徑，A 為圓上的一點，AB、AC 及 AD 為綫段，而且 AD 垂直 BC。
若 $BD = 1$，$DC = 4$ 及 $AD = a$，求 a 的值。
In Figure 1, BC is the diameter of the circle. A is a point on the circle, AB and AC are line segments and AD is a line segment perpendicular to BC.
If $BD = 1$, $DC = 4$ and $AD = a$, find the value of a.



圖一
Figure 1

$a =$

2. 若 $b = 1 - \cfrac{1}{1 - \cfrac{1}{1 - \cfrac{1}{1 - \cfrac{1}{-\frac{1}{2}}}}}$，求 b 的值。

If $b = 1 - \cfrac{1}{1 - \cfrac{1}{1 - \cfrac{1}{1 - \cfrac{1}{-\frac{1}{2}}}}}$, find the value of b.

$b =$

3. 若 x、y 及 z 為實數，$xyz \neq 0$，$2xy = 3yz = 5xz$ 及 $c = \frac{x+3y-3z}{x+3y-6z}$。求 c 的值。

If x, y and z are real numbers, $xyz \neq 0$, $2xy = 3yz = 5xz$ and $c = \frac{x+3y-3z}{x+3y-6z}$,

find the value of c.

$c =$

4. 若 x 為一整數滿足 $\log_{1/4}(2x+1) < \log_{1/2}(x-1)$，求 x 的最大值。
If x is an integer satisfying $\log_{1/4}(2x+1) < \log_{1/2}(x-1)$, find the maximum value of x.

$x =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

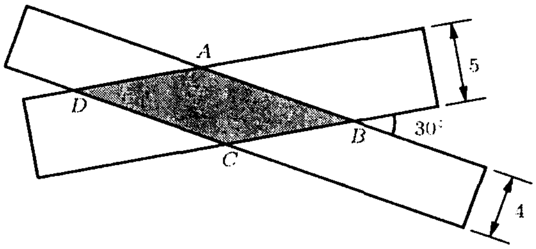
+ Bonus score [ ]   Time [ ] Min. [ ] Sec.

Total score [ ]

Hong Kong Mathematics Olympiad (2010 – 2011)
Final Event 2 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

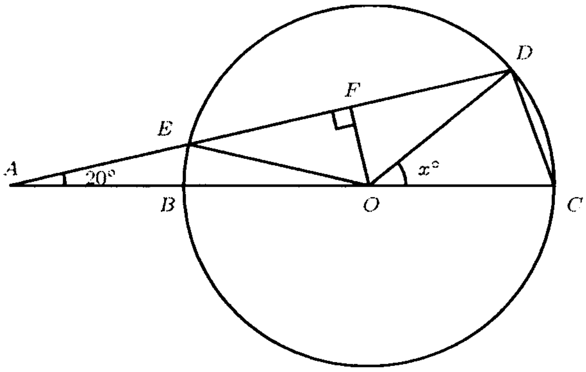
1. 在圖一中，兩闊度為 4 及 5 單位的長方形間的夾角為 30°。求重疊部份的面積。
In Figure 1, two rectangles with widths 4 and 5 units cross each other at 30°. Find the area of the overlapped region.



圖一
Figure 1

2. 從 1 到 100 選取兩整數(容許重覆)其和大於 100。問可選得多少對？
From 1 to 100, take a pair of integers (repetitions allowed) so that their sum is greater than 100. How many ways are there to pick such pairs?

3. 在圖二中的圓，其圓心為 *O* 及半徑為 *r*，三角形 *ACD* 與圓相交於 *B*、*C*、*D* 及 *E* 點。綫段 *AE* 的長度與圓的半徑相同。
若 $\angle DAC = 20°$ 及 $\angle DOC = x°$，求 *x* 的值。
In Figure 2, there is a circle with centre *O* and radius *r*. Triangle *ACD* intersects the circle at *B*, *C*, *D* and *E*. Line segment *AE* has the same length as the radius. If $\angle DAC = 20°$ and $\angle DOC = x°$, find the value of *x*.



圖二
Figure 2

$x =$

4. 已知 $\frac{1}{x}+\frac{2}{y}+\frac{3}{z}=0$ 及 $\frac{1}{x}-\frac{6}{y}-\frac{5}{z}=0$。若 $P=\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$，求 *P* 的值。
Given that $\frac{1}{x}+\frac{2}{y}+\frac{3}{z}=0$ and $\frac{1}{x}-\frac{6}{y}-\frac{5}{z}=0$. If $P=\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$, find the value of *P*.

$P =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.
			+ Bonus score				Time: Min. / Sec.
			Total score				

Hong Kong Mathematics Olympiad (2010 – 2011)
Final Event 3 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 a 為一正整數及 $a^2 + 100a$ 為一質數，求 a 的最大值。
If a is a positive integer and $a^2 + 100a$ is a prime number, find the maximum value of a.

$a =$

2. 設 a、b 及 c 為實數。若 1 為 $x^2 + ax + 2 = 0$ 的根及 a 和 b 為 $x^2 + 5x + c = 0$ 的根，求 $a + b + c$ 的值。
Let a, b and c be real numbers. If 1 is a root of $x^2 + ax + 2 = 0$ and a and b be roots of $x^2 + 5x + c = 0$, find the value of $a + b + c$.

$a+b+c =$

3. 設 x 及 y 為正實數且 $x < y$。若 $\sqrt{x}+\sqrt{y}=1$、$\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{10}{3}$ 及 $x < y$，求 $y - x$ 的值。
Let x and y be positive real numbers with $x < y$.
If $\sqrt{x}+\sqrt{y}=1$, $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{10}{3}$ and $x < y$, find the value of $y - x$.

$y - x =$

4. 把數字 1, 2, … , 10 分成兩組並設 P_1 及 P_2 分別為該兩組的乘積。若 P_1 為 P_2 的倍數，求 $\frac{P_1}{P_2}$ 的最小值。
Spilt the numbers 1, 2, … , 10 into two groups and let P_1 be the product of the first group and P_2 the product of the second group.
If P_1 is a multiple of P_2, find the minimum value of $\frac{P_1}{P_2}$.

$\frac{P_1}{P_2} =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] Min. [ ] Sec.

Total score [ ]

Hong Kong Mathematics Olympiad (2010 – 2011)
Final Event 4 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 $P = 2\sqrt[4]{2007 \cdot 2009 \cdot 2011 \cdot 2013 + 10 \cdot 2010 \cdot 2010 - 9} - 4000$，求 P 的值。
If $P = 2\sqrt[4]{2007 \cdot 2009 \cdot 2011 \cdot 2013 + 10 \cdot 2010 \cdot 2010 - 9} - 4000$, find the value of P.

$P =$

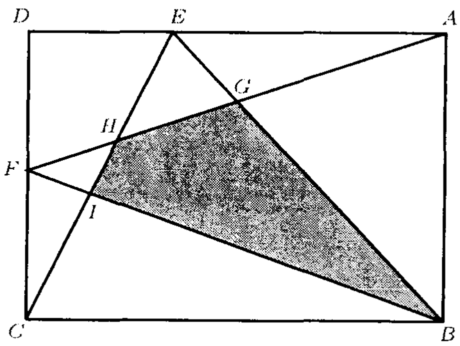
2. 若 $9x^2 + nx + 1$ 及 $4y^2 + 12y + m$ 為平方數及 $n > 0$，求 $\frac{n}{m}$ 的值。
If $9x^2 + nx + 1$ and $4y^2 + 12y + m$ are squares with $n > 0$, find the value of $\frac{n}{m}$.

$\frac{n}{m} =$

3. 設 n 及 $\frac{47}{5}\left(\frac{4}{47} + \frac{n}{141}\right)$ 為正整數。若 r 為 n 被 15 除的餘數，求 r 的最值。
Let n and $\frac{47}{5}\left(\frac{4}{47} + \frac{n}{141}\right)$ be positive integers. If r is the remainder of n divided by 15, find the value of r.

$r =$

4. 在圖一中，$ABCD$ 為一長方形，及 E 及 F 分別為綫段 AD 及 DC 上的點。點 G 為綫段 AF 及 BE 的交點，點 H 為綫段 AF 及 CE 的交點，點 I 為綫段 BF 及 CE 的交點。若 AGE，$DEHF$ 及 CIF 的面積分別為 2、3 及 1，求灰色部份 $BGHI$ 的面積。



圖一 Figure 1

In figure 1, $ABCD$ is a rectangle, and E and F are points on AD and DC, respectively. Also, G is the intersection of AF and BE, H is the intersection of AF and CE, and I is the intersection of BF and CE. If the areas of AGE, $DEHF$ and CIF are 2, 3 and 1, respectively, find the area of the grey region $BGHI$.

Shaded area =

FOR OFFICIAL USE

Score for accuracy × Mult. factor for speed =

\+ Bonus score

Total score

Team No.

Time Min. Sec.

Hong Kong Mathematics Olympiad (2010 – 2011)
Final Event Spare (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設 α 及 β 為方程 $y^2 - 6y + 5 = 0$ 的實根。設 m 為 $|x - \alpha| + |x - \beta|$ 對任何實數 x 的最小值。求 m 的值。
Let α and β be the real roots of $y^2 - 6y + 5 = 0$.
Let m be the minimum value of $|x - \alpha| + |x - \beta|$ over all real values of x.
Find the value of m.

$m =$

2. 設 α、β、γ 為實數且滿足 $\alpha + \beta + \gamma = 2$ 及$\alpha\beta\gamma = 4$。設 v 為 $|\alpha| + |\beta| + |\gamma|$的最小值，求 v 的值。
Let α, β, γ be real numbers satisfying $\alpha + \beta + \gamma = 2$ and $\alpha\beta\gamma = 4$.
Let v be the minimum value of $|\alpha| + |\beta| + |\gamma|$. Find the value of v.

$v =$

3. 設 $y = |x + 1| - 2|x| + |x - 2|$及$-1 \le x \le 2$。設 α 為 y 的最大值，求 α 的值。
Let $y = |x + 1| - 2|x| + |x - 2|$ and $-1 \le x \le 2$.
Let α be the maximum value of y. Find the value of α.

$\alpha =$

4. 設 F 為方程 $x^2 + y^2 + z^2 + w^2 = 3(x + y + z + w)$的整數解的數目。求 F 的值。
Let F be the number of integral solutions of $x^2 + y^2 + z^2 + w^2 = 3(x + y + z + w)$.
Find the value of F.

$F =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2011 – 2012)
Final Event Sample (Individual)

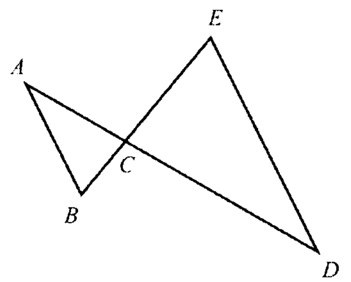
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設 a、b、c 及 d 為方程 $x^4 - 15x^2 + 56 = 0$ 的根。
若 $P = a^2 + b^2 + c^2 + d^2$，求 P 的值。
Let a, b, c and d be the roots of the equation $x^4 - 15x^2 + 56 = 0$.
If $P = a^2 + b^2 + c^2 + d^2$, find the value of P.

$P =$

2. 如圖一，$AB = AC$ 及 $AB \parallel ED$。若$\angle ABC = P°$及$\angle ADE = Q°$，求 Q 的值。
In Figure 1, $AB = AC$ and $AB \parallel ED$. If $\angle ABC = P°$ and $\angle ADE = Q°$,
find the value of Q.

$Q =$



圖一
Figure 1

3. 設 $F = 1 + 2 + 2^2 + 2^3 + \dots + 2^Q$ 及 $R = \sqrt{\frac{\log(1+F)}{\log 2}}$，求 R 的值。

Let $F = 1 + 2 + 2^2 + 2^3 + \dots + 2^Q$ and $R = \sqrt{\frac{\log(1+F)}{\log 2}}$, find the value of R.

$R =$

4. 設 f(x)是一個函數使得對所有正整數 n，$f(n) = (n-1)\,f(n-1)$及 $f(1) \neq 0$。
若 $S = \frac{f(R)}{(R-1)\,f(R-3)}$，求 S 的值。
Let f(x) be a function such that $f(n) = (n-1)\,f(n-1)$ and $f(1) \neq 0$ for all positive integers n. If $S = \frac{f(R)}{(R-1)\,f(R-3)}$, find the value of S.

$S =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2011 – 2012)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 A 是多項式 $x^4 + 6x^3 + 12x^2 + 9x + 2$ 的所有根的平方之和，求 A 的值。
If A is the sum of the squares of the roots of $x^4 + 6x^3 + 12x^2 + 9x + 2$,
find the value of A .

A =

2. 設 x、y、z、w 為正 A 邊形的四個相連端點。若綫段 xy 的長度為 2
及四邊形 $xyzw$ 的面積是 $a + \sqrt{b}$ ，求 $B = 2^a \cdot 3^b$ 的值。
Let x, y, z, w be four consecutive vertices of a regular A-gon. If the length of the line segment xy is 2 and the area of the quadrilateral $xyzw$ is $a + \sqrt{b}$,
find the value of $B = 2^a \cdot 3^b$.

B =

3. 若 C 是 B 的所有正因子之和，其中 B 的因子包括 1 和 B，求 C 的值。
If C is the sum of all positive factors of B, including 1 and B itself,
find the value of C .

C =

4. 若 $C! = 10^D \cdot k$，其中 D 及 k 皆為整數且 k 不是 10 的倍數，求 D 的值。
If $C! = 10^D \cdot k$, where D and k are integers such that k is not divisible by 10 ,
find the value of D .

D =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time		
				Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2011 – 2012)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 P 是方程 $x^2 + 9x + 13 = 2\sqrt{x^2+9x+21}$ 的所有實根之乘積，求 P 的值。
If the product of the real roots of the equation $x^2 + 9x + 13 = 2\sqrt{x^2+9x+21}$ is P, find the value of P.

$P =$

2. 若 $\mathrm{f}(x) = \dfrac{25^x}{25^x + P}$ 及 $Q = \mathrm{f}\left(\dfrac{1}{25}\right) + \mathrm{f}\left(\dfrac{2}{25}\right) + \cdots + \mathrm{f}\left(\dfrac{24}{25}\right)$，求 Q 的值。
If $\mathrm{f}(x) = \dfrac{25^x}{25^x + P}$ and $Q = \mathrm{f}\left(\dfrac{1}{25}\right) + \mathrm{f}\left(\dfrac{2}{25}\right) + \cdots + \mathrm{f}\left(\dfrac{24}{25}\right)$, find the value of Q.

$Q =$

3. 若 $X = \sqrt{(100)(102)(103)(105) + (Q-3)}$ 是整數及 R 是 X 的個位數，求 R 的值。
If $X = \sqrt{(100)(102)(103)(105) + (Q-3)}$ is an integer and R is the units digit of X, find the value of R.

$R =$

4. 若 S 是方程 $\sqrt{2012} \cdot x^{\log_{2012} x} = x^R$ 的所有正根之乘積的最後 3 位數字(個位數，十位數，百位數)之和，求 S 的值。
If S is the sum of the last 3 digits (hundreds, tens, units) of the product of the positive roots of $\sqrt{2012} \cdot x^{\log_{2012} x} = x^R$, find the value of S.

$S =$

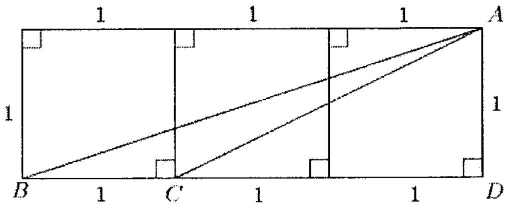
FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2011 – 2012)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 在圖一中，長方形由三個邊長為 1 之正方形組成。
若 $\alpha° = \angle ABD + \angle ACD$，求 α 的值。
In Figure 1, a rectangle is sub-divided into 3 identical squares of side length 1 .
If $\alpha° = \angle ABD + \angle ACD$,
find the value of α.



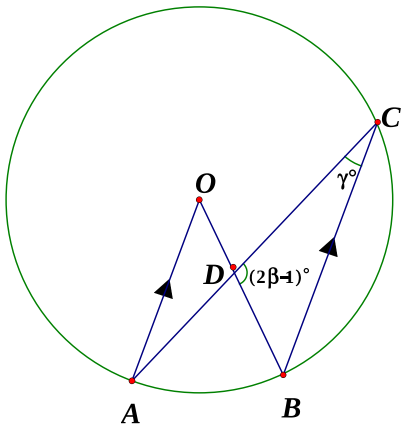
圖一 Figure 1

α =

2. 設 ABC 為一銳角三角形。若 $\sin A = \dfrac{36}{\alpha}$，$\sin B = \dfrac{12}{13}$ 及 $\sin C = \dfrac{\beta}{y}$，
求 β 的值，其中 β 及 y 是最簡化之代表形式。
Let ABC be an acute-angled triangle. If $\sin A = \dfrac{36}{\alpha}$, $\sin B = \dfrac{12}{13}$ and $\sin C = \dfrac{\beta}{y}$,
find the value of β, where β and y are in the lowest terms.

β =

3. 在圖二中，有一個圓心在 O 的圓，其圓周上有點 A、B 及 C，四條綫段：OA、OB、AC 與 BC，且 OA 與 BC 平行。
若 D 是 OB 及 AC 之交點且$\angle BDC = (2\beta - 1)°$ 及 $\angle ACB = \gamma°$，求 γ 的值。
In Figure 2, a circle at centre O has three points on its circumference, A, B and C. There are line segments OA, OB, AC and BC, where OA is parallel to BC. If D is the intersection of OB and AC with $\angle BDC = (2\beta - 1)°$ and $\angle ACB = \gamma°$,
find the value of γ.



圖二 Figure 2

γ =

4. 在 $(ax + b)^{2012}$ 的展開式中，a 與 b 為互質之正整數，
若 x^{γ} 與 $x^{\gamma+1}$ 的系數相同，求 $\delta = a + b$ 的值。
In the expansion of $(ax + b)^{2012}$, where a and b are relatively prime positive integers. If the coefficients of x^{γ} and $x^{\gamma+1}$ are equal, find the value of $\delta = a + b$.

δ =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score			Time		
			Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2011 – 2012)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 A 為一正整數且 $\frac{1}{1\times 3}+\frac{1}{3\times 5}+\cdots+\frac{1}{(A+1)(A+3)}=\frac{12}{25}$，求 A 的值。

   If A is a positive integer such that $\frac{1}{1\times 3}+\frac{1}{3\times 5}+\cdots+\frac{1}{(A+1)(A+3)}$,

   find the value of A .

   A =

2. 若 x 與 y 為正整數且 $x > y > 1$ 及 $xy = x + y + A$。設 $B=\frac{x}{y}$，求 B 的值。

   If x and y be positive integers such that $x > y > 1$ and $xy = x + y + A$.

   Let $B=\frac{x}{y}$, find the value of B.

   B =

3. 設 f 為為一函數並滿足以下條件：
   (i) 對所有正整數 n，f(n) 必為整數；
   (ii) f(2) = 2;
   (iii) 對所有正整數 m 及 n，$f(mn) = f(m)\cdot f(n)$及
   (iv) 當 $m > n$，$f(m) > f(n)$。
   若 C = f(B)，求 C 的值。
   Let f be a function satisfying the following conditions:
   (i) f(n) is an integer for every positive integer n;
   (ii) f(2) = 2;
   (iii) $f(mn) = f(m)\cdot f(n)$ for all positive integers m and n and
   (iv) $f(m) > f(n)$ if $m > n$.
   If C = f(B), find the value of C .

   C =

4. 設 D 為 2401×7^{C} (以十進制表示)的最後三位數字之和。求 D 的值。
   Let D be the sum of the last three digits of 2401×7^{C} (in the denary system).
   Find the value of D.

   D =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.		
			+ Bonus score				Time		
			Total score					Min.	Sec.

Hong Kong Mathematics Olympiad (2011 – 2012)
Final Event Spare (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設 P 為邊長為整數小於或等於 9 的三角形的數目。求 P 的值。
Let P be the number of triangles whose side lengths are integers less than or equal to 9. Find the value of P.

P =

2. 設 $Q = \log_{128} 2^3 + \log_{128} 2^5 + \log_{128} 2^7 + \ldots + \log_{128} 2^P$。求 Q 的值。
Let $Q = \log_{128} 2^3 + \log_{128} 2^5 + \log_{128} 2^7 + \ldots + \log_{128} 2^P$. Find the value of Q.

Q =

3. 考慮直綫 $12x - 4y + (Q - 305) = 0$。
若 x-軸、y-軸及此直綫所形成的三角形的面積為 R 平方單位，求 R 的值。
Consider the line $12x - 4y + (Q - 305) = 0$. If the area of the triangle formed by the x-axis, the y-axis and this line is R square units, what is the value of R?

R =

4. 若 $x + \frac{1}{x} = R$ 及 $x^3 + \frac{1}{x^3} = S$，求 S 的值。
If $x + \frac{1}{x} = R$ and $x^3 + \frac{1}{x^3} = S$, find the value of S.

S =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score			Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2011 – 2012)
Final Event Sample (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知三角形三邊的長度分別是 a cm、2 cm 及 b cm，其中 a 和 b 是整數且 $a \le 2 \le b$。若有 q 種不全等的三角形滿足上述條件，求 q 的值。
Given some triangles with side lengths a cm, 2 cm and b cm, where a and b are integers and $a \le 2 \le b$. If there are q non-congruent classes of triangles satisfying the above conditions, find the value of q.

$q =$

2. 已知方程 $|x| - \frac{4}{x} = \frac{3|x|}{x}$ 有 k 個相異實根，求 k 的值。
Given that the equation $|x| - \frac{4}{x} = \frac{3|x|}{x}$ has k distinct real root(s), find the value of k.

$k =$

3. 已知 x 及 y 為非零實數且滿足方程 $\frac{\sqrt{x}}{\sqrt{y}} - \frac{\sqrt{y}}{\sqrt{x}} = \frac{7}{12}$ 及 $x - y = 7$。
若 $w = x + y$，求 w 的值。
Given that x and y are non-zero real numbers satisfying the equations $\frac{\sqrt{x}}{\sqrt{y}} - \frac{\sqrt{y}}{\sqrt{x}} = \frac{7}{12}$ and $x - y = 7$. If $w = x + y$, find the value of w.

$w =$

4. 已知 x 及 y 為實數且 $\left|x - \frac{1}{2}\right| + \sqrt{y^2 - 1} = 0$。設 $p = |x| + |y|$，求 p 的值。
Given that x and y are real numbers and $\left|x - \frac{1}{2}\right| + \sqrt{y^2 - 1} = 0$.
Let $p = |x| + |y|$, find the value of p.

$p =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Compiled by Mr. SAROEUN Minea

Hong Kong Mathematics Olympiad (2011 – 2012)
Final Event 1 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 求 2011^{2011} 的十位數。
Calculate the tens digit of 2011^{2011} .

tens digit =

2. 設 a_1、a_2、a_3、… 為一等差數列，公差是 1 及 $a_1 + a_2 + a_3 + \ldots + a_{100} = 2012$。如果 $P = a_2 + a_4 + a_6 + \ldots + a_{100}$，求 P 的值。
Let a_1, a_2, a_3, … be an arithmetic sequence with common difference 1 and $a_1 + a_2 + a_3 + \ldots + a_{100} = 2012$. If $P = a_2 + a_4 + a_6 + \ldots + a_{100}$, find the value of P .

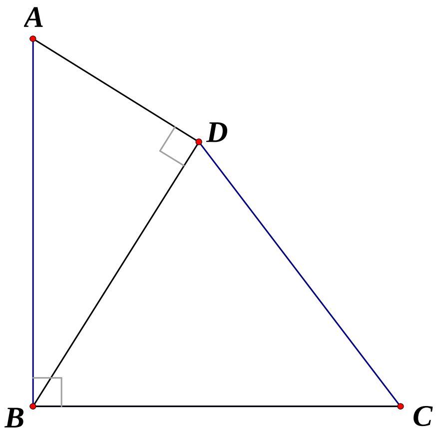
P =

3. 若 90! 可被 10^k 整除，當中 k 是正整數，求 k 的最大可能值。
If 90! is divisible by 10^k, where k is a positive integer, find the greatest possible value of k .

k =

4. 在圖一中，ΔABC 是一直角三形且 $AB \perp BC$。若 $AB = BC$，D 是一點使得 $AD \perp BD$，且 $AD = 5$ 及 $BD = 8$，求 ΔBCD 的面積的值。
In Figure 1, ΔABC is a right-angled triangle with $AB \perp BC$. If $AB = BC$, D is a point such that $AD \perp BD$ with $AD = 5$ and $BD = 8$, find the value of the area of ΔBCD .

$S_{\Delta BCD}$ =



圖一 Figure 1

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2011 – 2012)
Final Event 2 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 求 $2\times\tan 1°\times\tan 2°\times \tan 3°\times\ldots\times \tan 87°\times \tan 88°\times \tan 89°$ 的值。
Find the value of $2\times\tan 1°\times\tan 2°\times \tan 3°\times\ldots\times \tan 87°\times \tan 88°\times \tan 89°$.

2. 若方程 $(x^2 - 3x + 2)^2 - 3(x^2 - 3x) - 4 = 0$ 有 K 個整數解，求 K 的值。
If there are K integers that satisfy the equation $(x^2 - 3x + 2)^2 - 3(x^2 - 3x) - 4 = 0$, find the value of K.

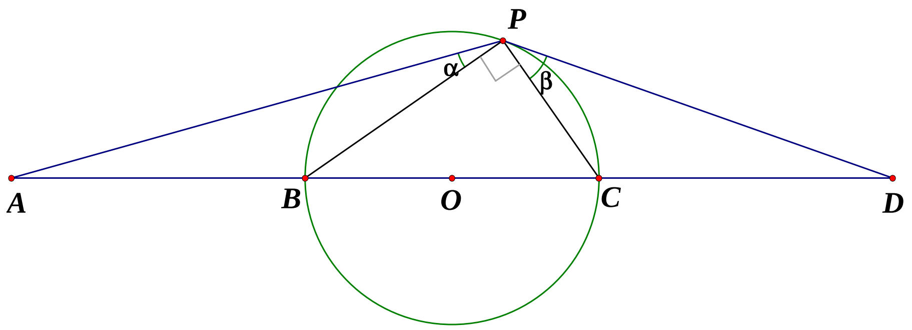
$K =$

3. 若 ℓ 為 $|x - 2| + |x - 47|$ 的最小值，求 ℓ 的值。
If ℓ is the minimum value of $|x - 2| + |x - 47|$, find the value of ℓ.

$\ell =$

4. 在圖一，圓有直徑 BC，圓心在 O，P、B 及 C 皆為圓周上的點。若 $AB = BC = CD$ 及 AD 為一綫段，$\alpha = \angle APB$ 及 $\beta = \angle CPD$，求 $(\tan \alpha)(\tan \beta)$ 的值。
In Figure 1, P, B and C are points on a circle with centre O and diameter BC.
If $AB = BC = CD$ and AD is a line segment, $\alpha = \angle APB$ and $\beta = \angle CPD$, find the value of $(\tan \alpha)(\tan \beta)$.

$(\tan \alpha)(\tan \beta) =$



圖一 Figure 1

FOR OFFICIAL USE

Score for accuracy × Mult. factor for speed =

\+ Bonus score

Total score

Team No.

Time

Min. Sec.

Hong Kong Mathematics Olympiad (2011 – 2012)
Final Event 3 (Group)

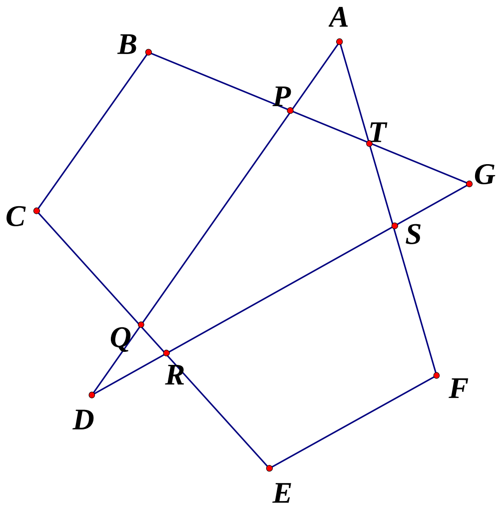
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設 $x=\frac{\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}}$，$y=\frac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}}$ 及 $192z=x^4+y^4+(x+y)^4$，求 z 的值。
Let $x=\frac{\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}}$, $y=\frac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}}$ and $192z=x^4+y^4+(x+y)^4$, find the value of z.

z =

2. 在圖一中，AD、DG、GB、BC、CE、EF 及 FA 都是直綫綫段。
若 $\angle FAD+\angle GBC+\angle BCE+\angle ADG+\angle CEF+\angle AFE+\angle DGB=r°$，求 r 的值。
In Figure 1, AD, DG, GB, BC, CE, EF and FA are line segments.
If $\angle FAD+\angle GBC+\angle BCE+\angle ADG+\angle CEF+\angle AFE+\angle DGB=r°$, find the value of r.

r =



圖一 Figure 1

3. 設 k 為正整數及函數 $f(k)$的定義是若 $\frac{k-1}{k}=0.k_1k_2k_3\ldots\ldots$，則 $f(k)=\overline{k_1k_2k_3}$，
例如 $f(3)=666$ 因為 $\frac{3-1}{3}=0.666\ldots\ldots$，求 $D=f(f(f(f(f(112)))))$的值。
Let k be positive integer and $f(k)$ a function that if $\frac{k-1}{k}=0.k_1k_2k_3\cdots\cdots$,
then $f(k)=\overline{k_1k_2k_3}$, for example, $f(3)=666$ because $\frac{3-1}{3}=0.666\cdots\cdots$,
find the value of $D=f(f(f(f(f(112)))))$.

D =

4. 若 F_n 為一整數值函數，其定義為 $F_n(k)=F_1(F_{n-1}(k))$，$n\geq 2$
且 $F_1(k)$是 k 的所有位數的平方之和，求 $F_{2012}(7)$的值。
If F_n is an integral valued function defined recursively by $F_n(k)=F_1(F_{n-1}(k))$ for $n\geq 2$ where $F_1(k)$ is the sum of squares of the digits of k, find the value of $F_{2012}(7)$.

$F_{2012}(7)$ =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
				Total score			Min.	Sec.

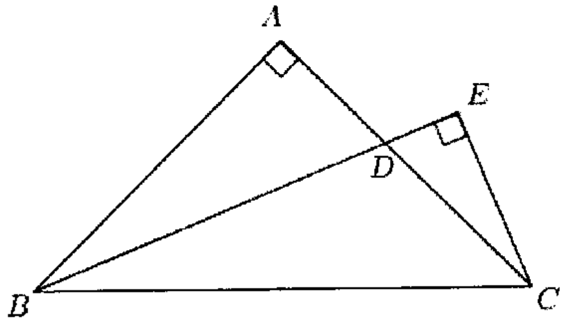
Hong Kong Mathematics Olympiad (2011 – 2012)
Final Event 4 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 在圖一中，ABC 及 EBC 是兩個直角三角形，$\angle BAC = \angle BEC = 90°$，$AB = AC$ 及 EDB 為 $\angle ABC$ 的角平分綫。求 $\frac{BD}{CE}$ 的值。

In figure 1, ABC and EBC are two right-angled triangles, $\angle BAC = \angle BEC = 90°$，$AB = AC$ and EDB is the angle bisector of $\angle ABC$. Find the value of $\frac{BD}{CE}$.

$\frac{BD}{CE} =$



圖一 Figure 1

2. 若 $Q > 0$ 並滿足 $\left|3Q - |1-2Q|\right| = 2$，求 Q 的值。

If $Q > 0$ and satisfies $\left|3Q - |1-2Q|\right| = 2$, find the value of Q.

$Q =$

3. 設 $xyzt = 1$。若 $R = \frac{1}{1+x+xy+xyz} + \frac{1}{1+y+yz+yzt} + \frac{1}{1+z+zt+ztx} + \frac{1}{1+t+tx+txy}$，求 R 的值。

Let $xyzt = 1$. If $R = \frac{1}{1+x+xy+xyz} + \frac{1}{1+y+yz+yzt} + \frac{1}{1+z+zt+ztx} + \frac{1}{1+t+tx+txy}$, find the value of R.

$R =$

4. 若 x_1、x_2、x_3、x_4 與 x_5 為正整數並滿足 $x_1 + x_2 + x_3 + x_4 + x_5 = x_1x_2x_3x_4x_5$，即是，五數之和等於五數之乘積，求 x_5 的最大值。

If x_1, x_2, x_3, x_4 and x_5 are positive integers that satisfy $x_1 + x_2 + x_3 + x_4 + x_5 = x_1x_2x_3x_4x_5$, that is the sum is the product, find the maximum value of x_5.

max x_5 =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
		+	Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2011 – 2012)
Final Event Spare (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設α及β為方程 $y^2 - 6y + 5 = 0$ 的實根。
設 m 為$|x - \alpha| + |x - \beta|$對任何實數 x 的最小值。求 m 的值。
Let α and β be the real roots of $y^2 - 6y + 5 = 0$. Let m be the minimum value of $|x - \alpha| + |x - \beta|$ over all real values of x. Find the value of m.

m =

2. 設α、β、γ 為實數且滿足 $\alpha + \beta + \gamma = 2$ 及 $\alpha\beta\gamma = 4$。
設 v 為 $|\alpha| + |\beta| + |\gamma|$ 的最小值，求 v 的值。
Let α, β, γ be real numbers satisfying $\alpha + \beta + \gamma = 2$ and $\alpha\beta\gamma = 4$.
Let v be the minimum value of $|\alpha| + |\beta| + |\gamma|$. Find the value of v .

v =

3. 設 $y = |x + 1| - 2|x| + |x - 2|$及$-1 \le x \le 2$。設α為 y 的最大值，求 α 的值。
Let $y = |x + 1| - 2|x| + |x - 2|$ and $-1 \le x \le 2$. Let α be the maximum value of y .
Find the value of α .

α =

4. 設 F 為方程 $x^2 + y^2 + z^2 + w^2 = 3(x + y + z + w)$的整數解的數目。求 F 的值。
Let F be the number of integral solutions of $x^2 + y^2 + z^2 + w^2 = 3(x + y + z + w)$.
Find the value of F .

F =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2012 – 2013)
Final Event Sample (Individual)

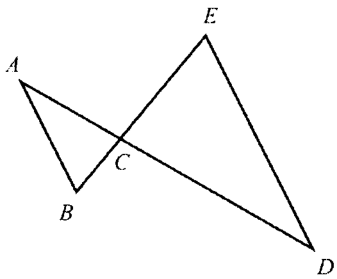
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設 a、b、c 及 d 為方程 $x^4 - 15x^2 + 56 = 0$ 的根。若 $P = a^2 + b^2 + c^2 + d^2$，求 P 的值。
Let a, b, c and d be the roots of the equation $x^4 - 15x^2 + 56 = 0$.
If $P = a^2 + b^2 + c^2 + d^2$, find the value of P.

P =

2. 如圖一，$AB = AC$ 及 $AB // ED$。若 $\angle ABC = P°$ 及 $\angle ADE = Q°$，求 Q 的值。
In Figure 1, $AB = AC$ and $AB // ED$. If $\angle ABC = P°$ and $\angle ADE = Q°$, find the value of Q.

Q =



圖一
Figure 1

3. 設 $F = 1 + 2 + 2^2 + 2^3 + \ldots + 2^Q$ 及 $R = \sqrt{\frac{\log(1+F)}{\log 2}}$，求 R 的值。

Let $F = 1 + 2 + 2^2 + 2^3 + \ldots + 2^Q$ and $R = \sqrt{\frac{\log(1+F)}{\log 2}}$, find the value of R.

R =

4. 設 f(x)是一個函數使得對所有正整數 n，$f(n) = (n - 1)\, f(n - 1)$ 及 $f(1) \neq 0$。
若 $S = \frac{f(R)}{(R-1)\, f(R-3)}$，求 S 的值。
Let f(x) be a function such that $f(n) = (n - 1)\, f(n - 1)$ and $f(1) \neq 0$ for all positive integers n. If $S = \frac{f(R)}{(R-1)\, f(R-3)}$, find the value of S.

S =

FOR OFFICIAL USE

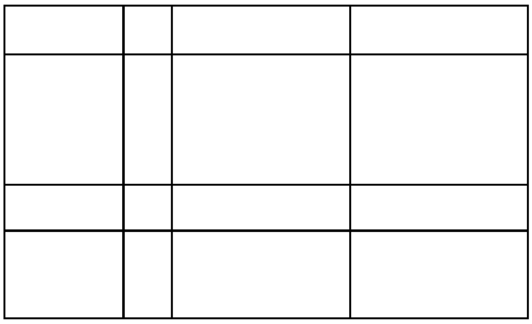
Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2012 – 2013)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 圖一共有 a 個長方形，求 a 的值。
Figure 1 has a rectangles, find the value of a .

a =



圖一 Figure 1

2. 已知 111111 能被 7 整除。若 b 為 $\underbrace{111111\ldots111111}_{a\text{個}}$ 除以 7 的餘數，求 b 的值。
Given that 7 divides 111111.
If b is the remainder when $\underbrace{111111\ldots111111}_{a-times}$ is divided by 7, find the value of b .

b =

3. 若 c 為 $\left[(b-2)^{4b^2}+(b-1)^{2b^2}+b^{b^2}\right]$ 除以 3 的餘數，求 c 的數值。
If c is the remainder of $\left[(b-2)^{4b^2}+(b-1)^{2b^2}+b^{b^2}\right]$ divided by 3 , find the value of c .

c =

4. 若 $|x + 1| + |y - 1| + |z| = c$，求 $d = x^2 + y^2 + z^2$ 的值。
If $|x + 1| + |y - 1| + |z| = c$，find the value of $d = x^2 + y^2 + z^2$.

d =

FOR OFFICIAL USE

Score for accuracy × Mult. factor for speed =

\+ Bonus score

Total score

Team No.

Time

Min. Sec.

Hong Kong Mathematics Olympiad (2012 – 2013)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知函數 $f(x) = x^2 + rx + s$ 和 $g(x) = x^2 - 9x + 6$ 有以下特性：
$f(x)$ 的根之和是 $g(x)$ 的根之積，且 $f(x)$ 的根之積是 $g(x)$ 的根之和。
若 $f(x)$ 的最小值取值於 $x = a$，求 a 的值。
Given that functions $f(x) = x^2 + rx + s$ and $g(x) = x^2 - 9x + 6$ have the properties that the sum of roots of $f(x)$ is the product of the roots of $g(x)$, and the product of roots of $f(x)$ is the sum of roots of $g(x)$. If $f(x)$ attains its minimum at $x = a$, find the value of a.

$a =$

2. 一正方體的表面積是 b cm^2。
若它每一條邊的長度增加 3 cm，它的體積隨之增加 $(2b - a)$ cm^3，求 b 的值。
The surface area of a cube is b cm^2. If the length of each side is increased by 3 cm, its volume is increased by $(2b - a)$ cm^3, find the value of b.

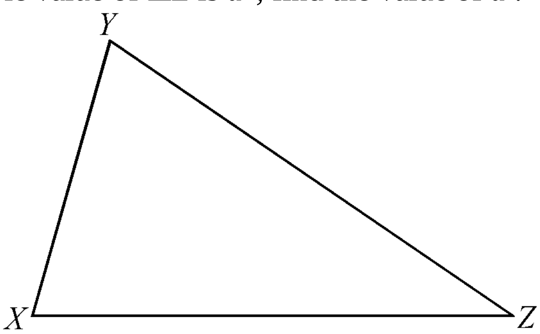
$b =$

3. 設 $f(1) = 3$，$f(2) = 5$ 且對所有正整數 n，$f(n + 2) = f(n + 1) + f(n)$。
當 $f(b)$ 除以 3 的餘數是 c，求 c 的值。
Let $f(1) = 3$, $f(2) = 5$ and $f(n + 2) = f(n + 1) + f(n)$ for positive integers n.
If c is the remainder of $f(b)$ divided by 3, find the value of c.

$c =$

4. 如圖二，三角形 XYZ 的角度滿足 $\angle Z \le \angle Y \le \angle X$ 且 $c\cdot\angle X = 6\cdot\angle Z$。
若$\angle Z$ 的最大可能值是 $d°$，求 d 的值。
In Figure 2, the angles of triangle XYZ satisfy $\angle Z \le \angle Y \le \angle X$ and $c\cdot\angle X = 6\cdot\angle Z$.
If the maximum possible value of $\angle Z$ is $d°$, find the value of d.

$d =$



圖二 Figure 2

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2012 – 2013)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 $a=\dfrac{\left(7+4\sqrt{3}\right)^{\frac{1}{2}}-\left(7-4\sqrt{3}\right)^{\frac{1}{2}}}{\sqrt{3}}$，求 a 的值。

If $a=\dfrac{\left(7+4\sqrt{3}\right)^{\frac{1}{2}}-\left(7-4\sqrt{3}\right)^{\frac{1}{2}}}{\sqrt{3}}$, find the value of a.

α =

2. 設 f(x) = $x - a$ 及 F(x, y) = $y^2 + x$。如果 b = F(3, f(4))，求 b 的值。
Suppose f(x) = $x - a$ and F(x, y) = $y^2 + x$. If b = F(3, f(4)), find the value of b.

b =

3. 已知 392 除以一個兩位正整數的餘數是 b，
符合這個條件的兩位正整數共有 c 個，求 c 的值。
The remainder when 392 is divided by a 2-digit positive integer is b.
If c is the number of such 2-digit positive integers, find the value of c.

c =

4. 若 x 為實數及 d 為函數 $y=\dfrac{3x^2+3x+c}{x^2+x+1}$ 的最大值，求 d 的值。

If x is a real number and d is the maximum value of the function $y=\dfrac{3x^2+3x+c}{x^2+x+1}$,
find the value of d.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min.   Sec.

Hong Kong Mathematics Olympiad (2012 – 2013)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設實函數 f(x)對於所有實數 x 及 y 滿足 f(xy) = f(x)·f(y)，且 f(0) ≠ 0。
求 a = f(1)的值。
Let f(x) be a real value function that satisfies f(xy) = f(x)·f(y) for all real numbers x and y and f(0) ≠ 0. Find the value of a = f(1) .

a =

2. 設函數 $F(n)$滿足 $F(1) = F(2) = F(3) = a$ 及 $F(n+1) = \frac{F(n)\cdot F(n-1)+1}{F(n-2)}$，
其中 $n \ge 3$ 為正整數。求 $b = F(6)$的值。
Let $F(n)$ be a function with $F(1) = F(2) = F(3) = a$ and $F(n+1) = \frac{F(n)\cdot F(n-1)+1}{F(n-2)}$ for positive integer $n \ge 3$, find the value of $b = F(6)$.

b =

3. 若 $b-6$、$b-5$ 及 $b-4$ 為方程 $x^4 + rx^2 + sx + t = 0$ 的根，求 $c = r + t$ 的值。
If $b-6$, $b-5$, $b-4$ are three roots of the equation $x^4 + rx^2 + sx + t = 0$,
find the value of $c = r + t$.

c =

4. 設(x_0, y_0) 是以下方程組的一個解：
$$\begin{cases} xy = 6 \\ x^2y + xy^2 + x + y + c = 2 \end{cases}$$
求 $d = x_0^2 + y_0^2$ 的值。
Suppose that (x_0, y_0) is a solution of the system:
$$\begin{cases} xy = 6 \\ x^2y + xy^2 + x + y + c = 2 \end{cases}$$
Find the value of $d = x_0^2 + y_0^2$.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2012 – 2013)
Final Event Sample (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知三角形三邊的長度分別是 a cm、2 cm 及 b cm，其中 a 和 b 是整數且 $a \le 2 \le b$。若有 q 種不全等的三角形滿足上述條件，求 q 的值。
Given some triangles with side lengths a cm, 2 cm and b cm, where a and b are integers and $a \le 2 \le b$. If there are q non-congruent classes of triangles satisfying the above conditions, find the value of q.

q =

2. 已知方程 $|x|-\frac{4}{x}=\frac{3|x|}{x}$ 有 k 個相異實根，求 k 的值。
Given that the equation $|x|-\frac{4}{x}=\frac{3|x|}{x}$ has k distinct real root(s), find the value of k.

k =

3. 已知 x 及 y 為非零實數且滿足方程 $\frac{\sqrt{x}}{\sqrt{y}}-\frac{\sqrt{y}}{\sqrt{x}}=\frac{7}{12}$ 及 $x-y=7$。
若 $w=x+y$，求 w 的值。
Given that x and y are non-zero real numbers satisfying the equations $\frac{\sqrt{x}}{\sqrt{y}}-\frac{\sqrt{y}}{\sqrt{x}}=\frac{7}{12}$ and $x-y=7$. If $w=x+y$, find the value of w.

w =

4. 已知 x 及 y 為實數且 $\left|x-\frac{1}{2}\right|+\sqrt{y^2-1}=0$。設 $p=|x|+|y|$，求 p 的值。
Given that x and y are real numbers and $\left|x-\frac{1}{2}\right|+\sqrt{y^2-1}=0$.
Let $p=|x|+|y|$, find the value of p.

p =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	Min. / Sec.
			Total score					

Hong Kong Mathematics Olympiad (2012 – 2013)
Final Event 1 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 求 $(2^{13}+1)(2^{14}+1)(2^{15}+1)(2^{16}+1)$ 的個位數字。
Find the units digit of $(2^{13}+1)(2^{14}+1)(2^{15}+1)(2^{16}+1)$.

unit digit =

2. 求 $16 \div (0.40 + 0.41 + 0.42 + \ldots + 0.59)$ 的值的整數部分。
Find the integral part of $16 \div (0.40 + 0.41 + 0.42 + \ldots + 0.59)$.

integral part

3. 從 1、2、4、6、7 中選三個數字組成三位數。
這些三位數有多少個能被 3 整除？
Choose three digits from 1, 2, 4, 6, 7 to construct three-digit numbers.
Of these three-digit numbers, how many of them are divisible by 3?

4. 用 1、2、3、4、5、6 組成一個位數：*ABCDEF*，使得 *A* 能被 1 整除，*AB* 能被 2 整除，*ABC* 能被 3 整除，*ABCD* 能被 4 整除，*ABCDE* 能被 5 整除，及 *ABCDEF* 能被 6 整除。求 *A* 的最大值。
Using numbers: 1, 2, 3, 4, 5, 6 to form a six-digit number: *ABCDEF* such that *A* is divisible by 1, *AB* is divisible by 2, *ABC* is divisible by 3, *ABCD* is divisible by 4, *ABCDE* is divisible by 5, *ABCDEF* is divisible by 6. Find the greatest value of *A* .

Greatest *A*

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2012 – 2013)
Final Event 2 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 $4^3 + 4^r + 4^4$ 是一平方數，其中 r 是正整數，求 r 的最小值。
If $4^3 + 4^r + 4^4$ is a perfect square and r is a positive integer, find the minimum value of r.

minimum r

2. 三男 B_1、B_2、B_3 和三女 G_1、G_2、G_3 就坐一排座位，並滿足以下兩個條件：
1) 一男不會坐在另一男旁邊及一女不會坐在另一女旁邊
2) B_1 必須坐在 G_1 旁邊
若 s 是這樣就坐的排列數量，求 s 的值。
Three boys B_1, B_2, B_3 and three girls G_1, G_2, G_3 are to be seated in a row according to the following rules:
1) A boy will not sit next to another boy and a girl will not sit next to another girl ,
2) Boy B_1 must sit next to girl G_1
If s is the number of different such seating arrangements, find the value of s.

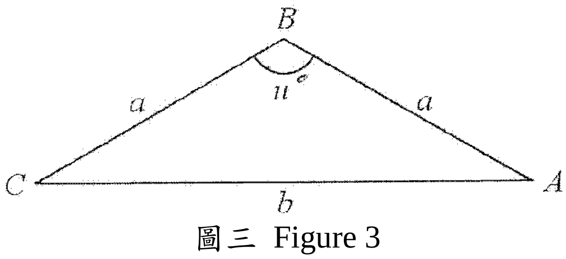
s =

3. 設 $f(x) = \dfrac{x+a}{x^2+\frac{1}{2}}$，$x$ 為實數且 $f(x)$ 的最大值和最小值分別是 $\frac{1}{2}$ 和 -1。
若 $t = f(0)$，求 t 的值。
Let $f(x) = \dfrac{x+a}{x^2+\frac{1}{2}}$, where x is a real number and the maximum value of $f(x)$ is $\frac{1}{2}$ and the minimum value of $f(x)$ is -1. If $t = f(0)$, find the value of t.

t =

4. 在圖三，ABC 是一等腰三角形，其中 $\angle ABC = u°$，$AB = BC = a$ 和 $AC = b$。
若二次方程 $ax^2 - \sqrt{2}\cdot bx + a = 0$ 有兩個實根，它們的絕對差為 $\sqrt{2}$，求 u 的值。
In Figure 3, ABC is an isosceles triangle with $\angle ABC = u°$, $AB = BC = a$ and $AC = b$.
If the quadratic equation $ax^2 - \sqrt{2}\cdot bx + a = 0$ has two real roots, whose absolute difference is $\sqrt{2}$, find the value of u.

u =



圖三 Figure 3

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

Team No. [ ]

\+ Bonus score [ ]

Time [ ] Min. [ ] Sec.

Total score [ ]

Hong Kong Mathematics Olympiad (2012 – 2013)
Final Event 3 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 m 和 n 是正整數且 $m^2 - n^2 = 43$，求 $m^3 - n^3$ 的值。
If m and n are positive integers with $m^2 - n^2 = 43$, find the value of $m^3 - n^3$.

$m^3 - n^3 =$

2. 設 x_1、x_2、…、x_{10} 為非零整數，且滿足 $-1 \le x_i \le 2$，其中 $i = 1$、2、…、10。
若 $x_1 + x_2 + \ldots + x_{10} = 11$，求 $x_1^2 + x_2^2 + \cdots + x_{10}^2$ 的最大可能值。
Let $x_1, x_2, \ldots, x_{10}$ be non-zero integers satisfying $-1 \le x_i \le 2$ for $i = 1, 2, \ldots, 10$.
If $x_1 + x_2 + \cdots + x_{10} = 11$, find the maximum possible value for $x_1^2 + x_2^2 + \cdots + x_{10}^2$.

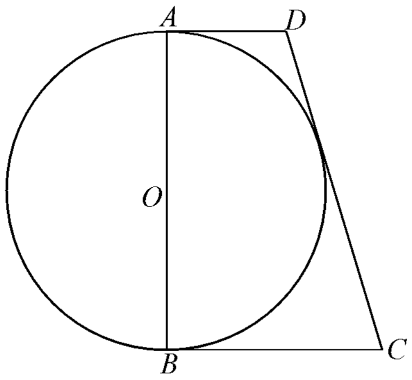
Maximum

3. 若 $f(n) = a^n + b^n$，其中 n 是正整數且 $f(3) = [f(1)]^3 + f(1)$，求 $a \cdot b$ 的值。
If $f(n) = a^n + b^n$, where n is a positive integer and $f(3) = [f(1)]^3 + f(1)$, find the value of $a \cdot b$.

$a \cdot b =$

4. 在圖四，AD、BC 和 CD 是以 O 作圓心且直徑 $AB = 12$ 的圓的切綫。
若 $AD = 4$，求 BC 的值。
In Figure 4, AD, BC and CD are tangents to the circle with centre at O and diameter $AB = 12$. If $AD = 4$, find the value of BC .

$BC =$



圖四 Figure 4

FOR OFFICIAL USE

Score for accuracy × Mult. factor for speed =

+ Bonus score

Total score

Team No.

Time

Min. Sec.

Hong Kong Mathematics Olympiad (2012 – 2013)
Final Event 4 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 P 為整數 3,659,893,456,789,325,678 與 342,973,489,379,256 的乘積，求 P 的位數。
In P be the product of 3,659,893,456,789,325,678 and 342,973,489,379,256 , find the number of digits of P .

no. of digits

2. 若 $\frac{1}{4}+4\left(\frac{1}{2013}+\frac{1}{x}\right)=\frac{7}{4}$，求 $1872+48\times\left(\frac{2013x}{x+2013}\right)$ 的值。
If $\frac{1}{4}+4\left(\frac{1}{2013}+\frac{1}{x}\right)=\frac{7}{4}$, find the value of $1872+48\times\left(\frac{2013x}{x+2013}\right)$.

3. 有一個整數被 10 除，餘數為 9；被 9 除，餘數為 8；被 8 除，餘數為 7；等等直至被 2 除，餘數為 1。求此整數的最小值。
The remainders of an integer when divided by 10, 9, 8, … , 2 are 9, 8, 7, … , 1 respectively. Find the smallest such an integer .

4. 如圖五，A、B、C、D、E 代表不同的個位數字。求 $A+B+C+D+E$ 的值。
In Figure 5, A, B, C, D, E represent different digits.
Find the value of $A+B+C+D+E$.

$$\begin{array}{r} ABCDE \\ \times \quad\quad 9 \\ \hline 1AAA0E \end{array}$$

圖五 Figure 5

FOR OFFICIAL USE

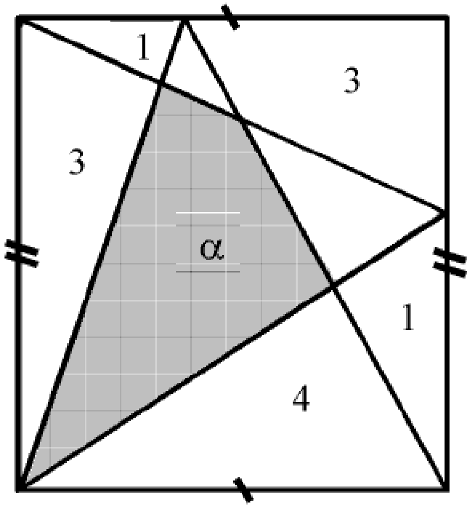
Score for accuracy		× Mult. factor for speed		=	Team No.	
		+ Bonus score			Time	
		Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2013 – 2014)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 求下圖中陰影部分的面積 α。
Determine the area of the shaded region, α , in the figure below.

α =



圖一 Figure 1

2. 如果 10 個不同的正整數的平均值是 2α，
求這 10 個數中，最大的一個數 β 最大可能值。
If the average of 10 distinct positive integers is 2α,
what is the largest possible value of the largest integer, β, of the ten integers?

β =

3. 考慮兩組由正整數組成的有限數列：$1, 3, 5, 7, \cdots, \beta$ 和 $1, 6, 11, 16, \cdots, \beta+1$。
求它們之間相同數字的數目 γ。
Given that $1, 3, 5, 7, \cdots, \beta$ and $1, 6, 11, 16, \cdots, \beta + 1$ are two finite sequences of positive integers.
Determine γ, the numbers of positive integers common to both sequences.

γ =

4. 若 $\log_2 a + \log_2 b \geq \gamma$，求 $a + b$ 的最小值 δ。
If $\log_2 a + \log_2 b \geq \gamma$, determine the smallest positive value δ for $a + b$.

δ =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] [ ]

Total score [ ]   Min. Sec.

Hong Kong Mathematics Olympiad (2013 – 2014)
Final Event 2 (Individual)

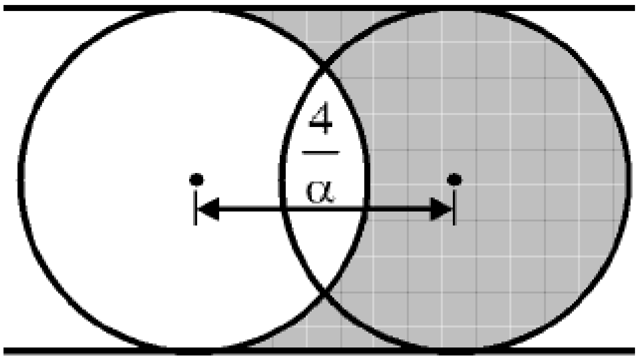
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 求方程 $\sqrt{\left(x+\sqrt{x}\right)}-\sqrt{\left(x-\sqrt{x}\right)}=\sqrt{x}$ 的正實根 α。
Determine the positive real root, α, of $\sqrt{\left(x+\sqrt{x}\right)}-\sqrt{\left(x-\sqrt{x}\right)}=\sqrt{x}$.

α =

2. 下圖為兩個半徑為 4 的圓，其圓心相隔 $\frac{4}{\alpha}$。求陰影部分的面積 β。
In the figure below, two circles of radii 4 with their centres placed apart by $\frac{4}{\alpha}$.
Determine the area β, of the shaded region.

β =



3. 求正整數 γ 的最小值，以使得方程 $\sqrt{x}-\sqrt{\beta\gamma}=4\sqrt{2}$ 對 x 有正整數解。
Determine the smallest positive integer γ such that the equation
$\sqrt{x}-\sqrt{\beta\gamma}=4\sqrt{2}$ has an integer solution in x.

γ =

4. 求 $\left(\left(\gamma^{\gamma}\right)^{\gamma}\right)^{\gamma}$ 的個位數 δ。
Determine the units digit, δ, of $\left(\left(\gamma^{\gamma}\right)^{\gamma}\right)^{\gamma}$.

δ =

FOR OFFICIAL USE

Score for accuracy		× Mult. factor for speed		=		Team No.	
		+ Bonus score				Time	
		Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2013 – 2014)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若數列 $10^{\frac{1}{11}}$、$10^{\frac{2}{11}}$、$10^{\frac{3}{11}}$、…、$10^{\frac{\alpha}{11}}$ 中所有數字的乘積為 1 000 000，求正整數 α 的值。

   If the product of numbers in the sequence $10^{\frac{1}{11}}, 10^{\frac{2}{11}}, 10^{\frac{3}{11}}, \ldots, 10^{\frac{\alpha}{11}}$ is 1 000 000, determine the value of the positive integer α .

   α =

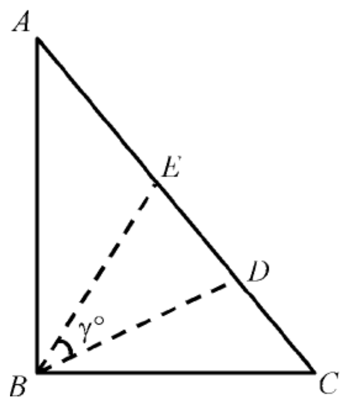
2. 若 $\frac{\beta}{1\times2\times3}+\frac{\beta}{2\times3\times4}+\cdots+\frac{\beta}{8\times9\times10}=\alpha$，求 β 的值。

   Determine the value of β if $\frac{\beta}{1\times2\times3}+\frac{\beta}{2\times3\times4}+\cdots+\frac{\beta}{8\times9\times10}=\alpha$.

   β =

3. 在下圖的三角形 *ABC* 中，$\angle ABC = 2\beta^\circ$，*AB* = *AD* 及 *CB* = *CE*。設 $\gamma^\circ = \angle DBE$，求 γ 的值。
   In the figure below, triangle *ABC* has $\angle ABC = 2\beta^\circ$, *AB* = *AD* and *CB* = *CE*.
   If $\gamma^\circ = \angle DBE$, determine the value of γ .

   γ =



4. 考慮數列 1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 2, 1, 2, …，求首 γ 項的和 δ。
   For the sequence 1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 2, 1, 2, … ,
   determine the sum δ of the first γ terms.

   δ =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2013 – 2014)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 $\frac{6\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}=3\sqrt{\alpha}-6$，求 α 的值。

   If $\frac{6\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}=3\sqrt{\alpha}-6$, determine the value of α.

   α =

2. 考慮形如 $\frac{n}{n+1}$ 的分數，當中 n 是一個正整數。若同時把該分數的分子和分母減去 1，得出的分數是小於 $\frac{\alpha}{7}$，且大於 0，求這樣的分數的數目 β。

   Consider fractions of the form $\frac{n}{n+1}$, where n is a positive integer. If 1 is subtracted from both the numerator and the denominator, and the resultant fraction remains positive and is strictly less than $\frac{\alpha}{7}$, determine, β, the number of these fractions.

   β =

3. 一個等邊三角形和一個正六邊形的周長相同。若該等邊三角形的面積為 β 平方單位，求正六邊形的面積 γ (平方單位)。

   The perimeters of an equilateral triangle and a regular hexagon are equal. If the area of the triangle is β square units, determine the area, γ, of the hexagon in square units.

   γ =

4. 求 $\delta=\frac{3}{2}+\frac{5}{4}+\frac{9}{8}+\frac{17}{16}+\frac{33}{32}+\frac{65}{64}-\gamma$ 的值。

   Determine the value of $\delta=\frac{3}{2}+\frac{5}{4}+\frac{9}{8}+\frac{17}{16}+\frac{33}{32}+\frac{65}{64}-\gamma$.

   δ =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]   Team No. [ ]

+ Bonus score [ ]   Time [ ] Min. [ ] Sec.

Total score [ ]

Hong Kong Mathematics Olympiad (2013– 2014)
Final Event 1 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若一個等腰三角形對應底邊(不是兩條等腰邊)的高是 8，且周長是 32，求該三角形的面積。
If an isosceles triangle has height 8 from the base, not the legs, and perimeters 32, determine the area of the triangle.

area =

2. 若 $f(x)=\dfrac{\left(x+\frac{1}{x}\right)^6-\left(x^6+\frac{1}{x^6}\right)-2}{\left(x+\frac{1}{x}\right)^3+\left(x^3+\frac{1}{x^3}\right)}$ 當中 x 是一個正實數，求 $f(x)$ 的最小值。

If $f(x)=\dfrac{\left(x+\frac{1}{x}\right)^6-\left(x^6+\frac{1}{x^6}\right)-2}{\left(x+\frac{1}{x}\right)^3+\left(x^3+\frac{1}{x^3}\right)}$ where x is a positive real number, determine the minimum value of $f(x)$.

minimum =

3. 求 81 位數 $\overline{111\cdots1}$ 除以 81 的餘數。
Determine the remainder of the 81-digit integer $\overline{111\cdots1}$ divided by 81 .

remainder =

4. 給定一實數數列 $a_1, a_2, a_3, \ldots$ ，它滿足
1) $a_1=\dfrac{1}{2}$ ，及
2) 對 $k\geq 2$，有 $a_1+a_2+\ldots+a_k=k^2a_k$。
求 a_{100} 的值。
Given a sequence of real numbers $a_1, a_2, a_3, \ldots$ that satisfy
1) $a_1=\dfrac{1}{2}$, and
2) $a_1+a_2+\ldots+a_k=k^2a_k$, for $k\geq 2$.
Determine the value of a_{100} .

a_{100} =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.		
			+	Bonus score			Time		
			Total score					Min.	Sec.

Hong Kong Mathematics Olympiad (2013 – 2014)
Final Event 2 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若在 $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}$ 中刪去若干項後剩 1，求刪去各項的乘積。

By removing certain terms from the sum, $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}$, we can get 1 .

What is the product of the removed term(s) ?

Product =

2. 若 $S_n = 1 - 2 + 3 - 4 + \ldots + (-1)^{n-1} n$，當中 n 是正整數，求 $S_{17} + S_{33} + S_{50}$ 的值。
If $S_n = 1 - 2 + 3 - 4 + \ldots + (-1)^{n-1} n$, where n is a positive integer, determine the value of $S_{17} + S_{33} + S_{50}$.

$S_{17}+S_{33}+S_{50}$ =

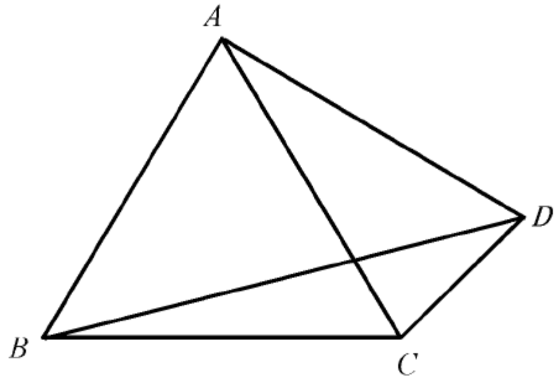
3. *A*, *B*, *C*, *D*, *E* 和 *F* 六人根據英文字母的順序輪班工作。*A* 在第一個星期日當值，然後 *B* 在星期一當值，如此類推。*A* 於第 50 個星期的哪一天當值？(答案以數字 0 代表星期日，數字 1 代表星期一，……，數字 6 代表星期六)。

Six persons *A*, *B*, *C*, *D*, *E* and *F* are to rotate for night shifts in alphabetical order with *A* serving on the first Sunday, *B* on the first Monday and so on. In the fiftieth week, which day does *A* serve on? (Represent Sunday by 0, Monday by 1, … , Saturday by 6 in your answer.)

Day

4. 在下圖中，*D* 以直綫連接著等邊三角形 *ABC* 的頂點，當中 $AB = AD$。
設 $\angle BDC = \alpha°$，求 α 的值。
In the figure below, vertices of equilateral triangle *ABC* are connected to *D* in straight line segments with $AB = AD$. If $\angle BDC = \alpha°$, determine the value of α .

α =



FOR OFFICIAL USE

Score for accuracy × Mult. factor for speed =

+ Bonus score

Total score

Team No.

Time Min. Sec.

Hong Kong Mathematics Olympiad (2013 – 2014)
Final Event 3 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 求乘積 $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{10^2}\right)$ 的值。

   Determine the value of the product $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{10^2}\right)$.

   Product =

2. 求和 $\frac{1}{\log_2 100!}+\frac{1}{\log_3 100!}+\frac{1}{\log_4 100!}+\cdots+\frac{1}{\log_{100} 100!}$ 的值，當中 $100! = 100\times99\times98\times\ldots\times3\times2\times1$。

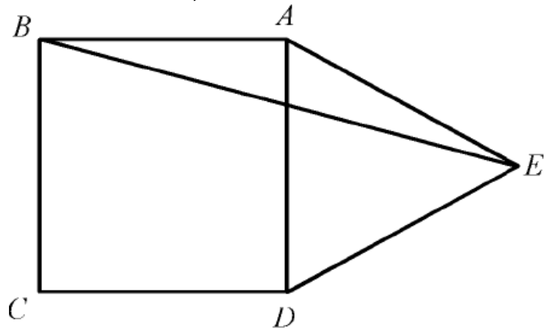
   Determine the value of the sum $\frac{1}{\log_2 100!}+\frac{1}{\log_3 100!}+\frac{1}{\log_4 100!}+\cdots+\frac{1}{\log_{100} 100!}$ where $100! = 100\times99\times98\times\ldots\times3\times2\times1$.

   Sum =

3. 在下圖中，*ABCD* 是一個正方形，*ADE* 是一個等邊三角形，且 *E* 是正方形 *ABCD* 外的一點。設 $\angle AEB = \alpha°$，求 α 的值。

   In the figure below, *ABCD* is a square, *ADE* is an equilateral triangle and *E* is a point outside of the square *ABCD*. If $\angle AEB = \alpha°$, determine the value of α.

   α =



4. 把不同的非零個位數填進下表白色的正方格內，使所有橫、直的等式均成立。求 α 的值。

   Fill the white squares in the figure below with distinct non-zero digits so that the arithmetical expressions, read both horizontally and vertically, are correct.
   What is the value of α?

   α =

	÷		=	
+	■	×	■	■
	+		=	α
=	■	=	■	■
	■		■	■

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]　　Team No. [ ]

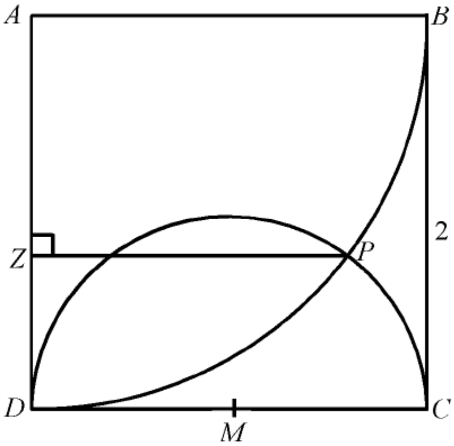
+ Bonus score [ ]　　Time [ ] [ ]

Total score [ ]　　Min.　Sec.

Hong Kong Mathematics Olympiad (2013 – 2014)
Final Event 4 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 在下圖，$ABCD$ 是一個邊長為 2 的正方形。先以 A 為圓心畫出弧 BD，再以 CD 的中點 M 為圓心從 C 到 D 畫出一個半圓。弧 BD 和弧 DC 相交於 P。求 P 與 AD 的最短距離，即 PZ 的長度。
In the figure below, $ABCD$ is a square of side length 2. A circular arc with centre at A is drawn from B to D. A semicircle with centre at M, the midpoint of CD, is drawn from C to D and sits inside the square. Determine the shortest distance from P, the intersection of the two arcs, to side AD, that is, the length of PZ.



$PZ =$

2. 若 $x=\frac{\sqrt{5}+1}{2}$ 及 $y=\frac{\sqrt{5}-1}{2}$，求 $x^3y+2x^2y^2+xy^3$ 的值。
If $x=\frac{\sqrt{5}+1}{2}$ and $y=\frac{\sqrt{5}-1}{2}$, determine the value of $x^3y+2x^2y^2+xy^3$.

$x^3y+2x^2y^2+xy^3=$

3. 若 a, b, c 及 d 是不同的個位數，且
$$\begin{array}{r} a\,a\,b\,c\,d \\ -\,d\,a\,a\,b\,c \\ \hline 2\,0\,1\,4\,d \end{array}$$
求 d 的值。
If a, b, c and d are distinct digits and
$$\begin{array}{r} a\,a\,b\,c\,d \\ -\,d\,a\,a\,b\,c \\ \hline 2\,0\,1\,4\,d \end{array}$$
determine the value of d.

$d =$

4. 求方程 $x^4+(x-4)^4=32$ 所有實根的乘積。
Determine the product of all real roots of the equation $x^4+(x-4)^4=32$.

Product =

FOR OFFICIAL USE

Score for accuracy ☐ × Mult. factor for speed ☐ = ☐ Team No. ☐

\+ Bonus score ☐ Time ☐ ☐

Total score ☐ Min. Sec.

Hong Kong Mathematics Olympiad (2014 – 2015)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 $\left|x+\sqrt{5}\right|+\left|y-\sqrt{5}\right|+\left|z\right|=0$，求 $\alpha = x^2 + y^2 + z^2$。

If $\left|x+\sqrt{5}\right|+\left|y-\sqrt{5}\right|+\left|z\right|=0$, determine $\alpha = x^2 + y^2 + z^2$.

α =

2. 若 β 為乘積 $\underbrace{11111\cdots11}_{\alpha個1}\times\underbrace{99999\cdots99}_{\alpha個9}$ 所有數位的數字之和，求 β 的值。

If β is the sum of all digits of the product $\underbrace{11111\cdots11}_{\alpha\ 1's}\times\underbrace{99999\cdots99}_{\alpha\ 9's}$,

determine the value of β .

β =

3. 設實函數 f(x) 對於所有實數 x 及 y 滿足 f(xy) = f(x) f(y)，且 f(1) < 1。
求 γ = f(β) + 100 – β 的值。

Suppose that the real function f(x) satisfies f(xy) = f(x) f(y) for all real numbers x and y, and f(1) < 1. Determine the value of γ = f(β) + 100 – β .

γ =

4. 若 n 為正整數及 $f(n) = 2^n + 2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2^1 + 1$，求 δ = f(γ) 的最值。

If n is a positive integer and $f(n) = 2^n + 2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2^1 + 1$,

determine the value of δ = f(γ) .

δ =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2014 – 2015)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 x_0, y_0, z_0 為以下方程組的解，求 $\alpha = x_0 + y_0 + z_0$ 的值。
If x_0, y_0, z_0 is a solution to the simultaneous equations below, determine the value of $\alpha = x_0 + y_0 + z_0$.

$$\begin{cases} x - y - z = -1 \\ y - x - z = -2 \\ z - x - y = -4 \end{cases}$$

α =

2. 若 β 為 $\underbrace{111\cdots111}_{100\text{個}1} \div \alpha$ 的餘數。求 β 的值。
If β is the reminder of $\underbrace{111\cdots111}_{100\ 1's} \div \alpha$, determine the value of β .

β =

3. 若 γ 為 $[(\beta - 2)^{100} + \beta^{50} + (\beta + 2)^{25}] \div 3$ 的餘數，求 γ 的值。
If γ is the remainder of $[(\beta - 2)^{100} + \beta^{50} + (\beta + 2)^{25}] \div 3$, determine the value of γ .

γ =

4. 若方程 $x^4 + ax^2 + bx + \delta = 0$ 有四實根，且已知其中三個為 1、γ 及 γ^2，求 δ 的值。
If the equation $x^4 + ax^2 + bx + \delta = 0$ has four real roots with three of them being 1, γ and γ^2 , determine the value of δ .

δ =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.		
			+	Bonus score			Time		
				Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2014 – 2015)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 由 1 至 1000 的正整數中包括 1 及 1000，有 α 個不能被 5 或 7 整除。求 α 的值。
Of the positive integers from 1 to 1000, including 1 and 1000, there are α of them that are not divisible by 5 or 7 . Determine the value of α .

α =

2. 求 $\beta = 1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^\alpha (\alpha + 1)^2$ 的值。
Determine the value of $\beta = 1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^\alpha (\alpha + 1)^2$.

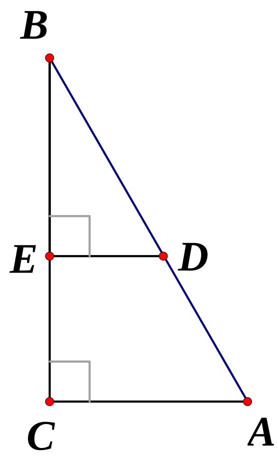
β =

3. 若 γ 為當 β 除以以下數列中的第 1993 項時的餘數：
1，2，2，3，3，3，4，4，4，4，5，5，5，5，5，…
求 γ 的值。
If γ is the remainder of β divided by the 1993rd term of the following sequence:
1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, ⋯.
Determine the value of γ .

γ =

4. 在下圖中，$BE = AC$，$BD = \frac{1}{2}$ 及 $DE + BC = 1$。若 δ 是 ED 的長度的 γ 倍，求 δ 的值。
In the figure below, $BE = AC$, $BD = \frac{1}{2}$ and $DE + BC = 1$.
If δ is γ times the length of ED, determine the value of δ .

δ =



FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2014 – 2015)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設 α 為 2^{1000} 除以 13 的餘數，求 α 的值。
Let α be the remainder of 2^{1000} divided by 13, determine the value of α.

$\alpha =$

2. 求 $\beta = \dfrac{\left(7+4\sqrt{\alpha}\right)^{\frac{1}{2}} - \left(7-4\sqrt{\alpha}\right)^{\frac{1}{2}}}{\sqrt{\alpha}}$ 的值。
Determine the value of $\beta = \dfrac{\left(7+4\sqrt{\alpha}\right)^{\frac{1}{2}} - \left(7-4\sqrt{\alpha}\right)^{\frac{1}{2}}}{\sqrt{\alpha}}$.

$\beta =$

3. 若 $f(a) = a - \beta$ 且 $F(a, b) = b^2 + a$，求 $\gamma = F(3, f(4))$ 的值。
If $f(a) = a - \beta$ and $F(a, b) = b^2 + a$, determine the value of $\gamma = F(3, f(4))$.

$\gamma =$

4. 若 δ 是方程 $x^{\log_\gamma x} = 10$ 所有實根的積，求 δ 的值。
If δ is the product of all real roots of $x^{\log_\gamma x} = 10$, determine the value of δ.

$\delta =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

\+ Bonus score [ ]

Total score [ ]

Team No. [ ]

Time [ ] Min. [ ] Sec.

Hong Kong Mathematics Olympiad (2014– 2015)
Final Event 1 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 化簡 $\left(\dfrac{1\times 3\times 9+2\times 6\times 18+\cdots+n\times 3n\times 9n}{1\times 5\times 25+2\times 10\times 50+\cdots+n\times 5n\times 25n}\right)^{\frac{1}{3}}$ 。

   Simplify $\left(\dfrac{1\times 3\times 9+2\times 6\times 18+\cdots+n\times 3n\times 9n}{1\times 5\times 25+2\times 10\times 50+\cdots+n\times 5n\times 25n}\right)^{\frac{1}{3}}$.

2. 在 50 隊香港數學競賽的參賽隊伍中，沒有一隊能答對一團體項目中的全部共四個題目。若該項目中的第一題有 45 隊答中，第二題有 40 隊答中，第三題有 35 隊答中，及第四題有 30 隊答中。請計算有多少隊伍同時答中第三及第四題。
   Among 50 school teams joining the HKMO, no team answered all four questions correctly in the paper of a group event. If the first question was solved by 45 teams, the second by 40 teams, the third by 35 teams and the fourth by 30 teams.
   How many teams solved both the third and the fourth questions?

3. 設 n 為 3659893456789325678 和 342973489379256 的乘積。
   求 n 中數字的位數。
   Let n be the product 3659893456789325678 and 342973489379256.
   Determine the number of digits of n.

4. 三個半徑分別為 2、3 及 10 單位的圓同時放於另一大圓內，
   使得四個圓剛好彼此接觸。求大圓的半徑的值。
   Three circles of radii 2, 3 and 10 units are placed inside another big circle in such a way that all circles are touching one another.
   Determine the value of the radius of the big circle.

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
				+ Bonus score			Time	Min.	Sec.
			Total score					

Hong Kong Mathematics Olympiad (2014 – 2015)
Final Event 2 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 在一個 3×3 的方格內的九個正方形上，分別填上紅色或藍色。
若α為不同着色方法的數量而使得所有 2×2 方格中所包含的正方形都不是全為紅色，求 α 的值。
On a 3×3 grid of 9 squares, each squares is to be painted with either Red or Blue.
If α is the total number of possible colouring in which no 2×2 grid consists of only Red squares, determine the value of α.

α =

2. 若 25 個連續正整數之和剛好等於三個質數的積，這三個質數之和最小是多少？
If the sum of 25 consecutive positive integers is the product of 3 prime numbers, what is the minimum sum of these 3 prime numbers?

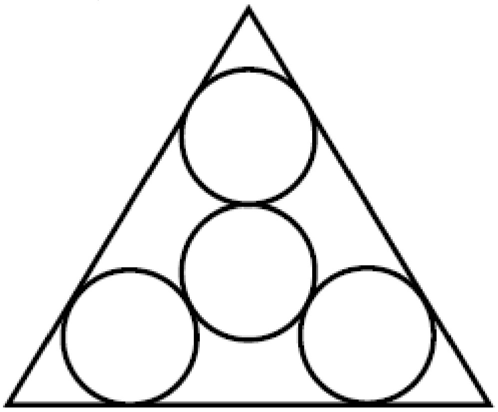
3. 求以下方程的所有實根之和

$$|x + 3| - |x - 1| = x + 1。$$

Determine the sum of all real roots of the following equation

$$|x + 3| - |x - 1| = x + 1.$$

4. 在下圖中，四個大小相同的圓形剛好放入一等邊三角形內。
若圓的半徑為 1 單位，求三角形的面積的值。
In the figure below, there are 4 identical circles placed inside an equilateral triangle. If the radii of the circles are 1 unit, what is the value of the area of the triangle?



FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2014 – 2015)
Final Event 3 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 化簡 $\sqrt{3+\sqrt{5}}+\sqrt{3-\sqrt{5}}$ 。
Simplify $\sqrt{3+\sqrt{5}}+\sqrt{3-\sqrt{5}}$.

2. 設 p 為質數及 m 為整數。若 $p(p+m)+2p=(m+2)^3$，找出 m 的最大可能值。
Let p be a prime and m be an integer.
If $p(p+m)+2p=(m+2)^3$, find the greatest possible value of m.

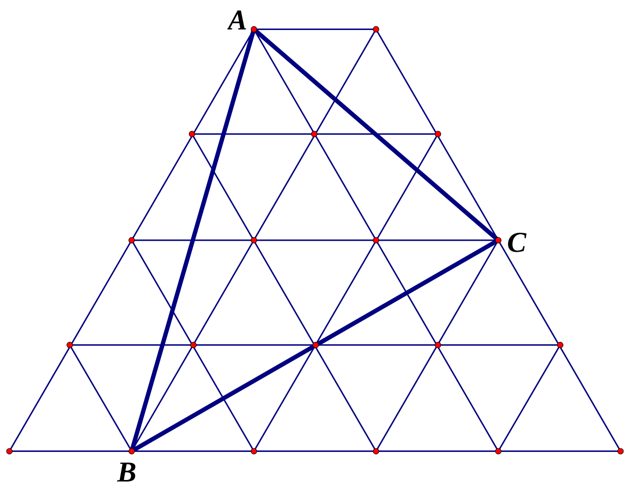
m =

3. 求以下方程的根
$$x=\left(x-\frac{1}{x}\right)^{\frac{1}{2}}+\left(1-\frac{1}{x}\right)^{\frac{1}{2}}$$ 。
Determine a root to $x=\left(x-\frac{1}{x}\right)^{\frac{1}{2}}+\left(1-\frac{1}{x}\right)^{\frac{1}{2}}$.

x =

4. 下圖中，每個小三角形的面積皆為 1，求三角形 ABC 的面積的值。
In the figure below, the area of each small triangle is 1. Determine the value of the area of the triangle ABC.

area =



FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2014 – 2015)
Final Event 4 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

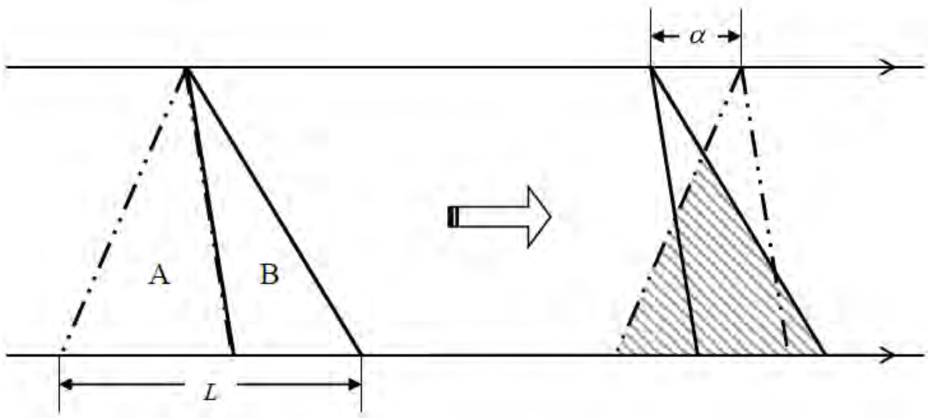
1. 設 $b = 1^2 - 2^2 + 3^2 - 4^2 + 5^2 - \ldots - 2012^2 + 2013^2$，求 b 除以 2015 的餘數。
Let $b = 1^2 - 2^2 + 3^2 - 4^2 + 5^2 - \ldots - 2012^2 + 2013^2$.
Determine the remainder of b divided by 2015 .

2. 考慮所有最大數位為 6 的數，及當把這個最大數位移除後，餘下數值剛為原來數值的 $\frac{1}{25}$ 的正整數。找出在這些正整數中，數值最小的一個。
There are positive integers with leading digits being 6 and upon removing this leading digit, the resulting integer is $\frac{1}{25}$ of the original value.
Determine the least of such positive integers.

3. 若 $x + \frac{1}{x} = 1$，求 $x^5 + \frac{1}{x^5}$ 的值。
If $x + \frac{1}{x} = 1$, determine the value of $x^5 + \frac{1}{x^5}$.

4. 在下圖中，若三角形 A 向右移動 α 單位後，所形成的陰影部分的面積為三角形 A 及 B 面積總和的 $\frac{\alpha}{L}$ 倍，求 $\frac{\alpha}{L}$ 的值。
In the figure below, when triangle A shifts α units to the right, the area of shaded region is $\frac{\alpha}{L}$ times of the total area of the triangles A and B. Determine the value of $\frac{\alpha}{L}$.

$\frac{\alpha}{L} =$



FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.
			+	Bonus score			Time
				Total score			Min. / Sec.

Hong Kong Mathematics Olympiad (2015 – 2016)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 解方程 $\log_5 a + \log_3 a = \log_5 a \cdot \log_3 a$，其中 $a > 1$ 為實數。
Solve the equation $\log_5 a + \log_3 a = \log_5 a \cdot \log_3 a$ for real number $a > 1$.

a =

2. 若 $\sqrt{b} = \sqrt{8+\sqrt{a}} + \sqrt{8-\sqrt{a}}$ ，求 b 的實數值。
If $\sqrt{b} = \sqrt{8+\sqrt{a}} + \sqrt{8-\sqrt{a}}$, determine the real value of b .

b =

3. 若方程 $x^2 - cx + b = 0$ 有兩個實數根及兩根之差為 1，求兩根之和的最大可能值 c。
If the equation $x^2 - cx + b = 0$ has two distinct real roots and their difference is 1, determine the greatest possible value of the sum of the roots, c .

c =

4. 設 $d = \overline{xyz}$ 為一**不能**被 10 整除的三位數。若 $\overline{xyz}$ 與 $\overline{zyx}$ 之和可被 c 整除，求此整數的最大可能值 d。
Let $d = \overline{xyz}$ be a three-digit integer that is **not** divisible by 10 .
If the sum of integers $\overline{xyz}$ and $\overline{zyx}$ is divisible by c ,
determine the greatest possible value of such an integer d.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.		
			+	Bonus score			Time		
			Total score					Min.	Sec.

Hong Kong Mathematics Olympiad (2015 – 2016)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 一個等邊三角形及一個正六邊形的周長比率為 1 : 1。
若三角形與六邊形的面積比率為 2 : *a*，求 *a* 的值。
Let the ratio of perimeter of an equilateral triangle to the perimeter of a regular hexagon be 1 : 1. If the ratio of the area of the triangle to the area of the hexagon is 2 : *a*, determine the value of *a* .

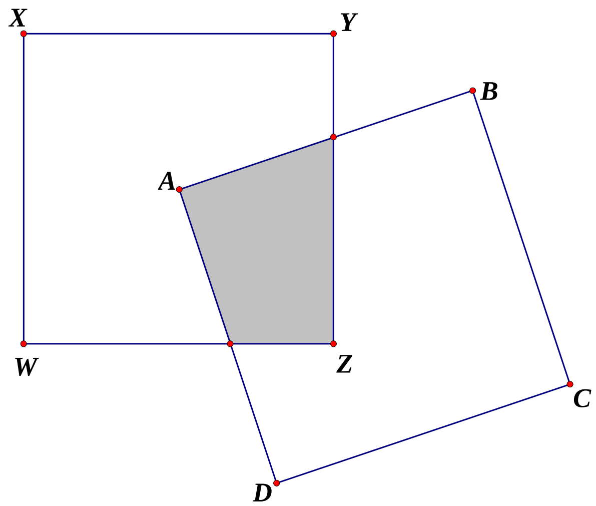
$a =$

2. 求 $b=\left[\log_2\left(a^2\right)+\log_4\left(\frac{1}{a^2}\right)\right]\times\left[\log_a 2+\log_{a^2}\left(\frac{1}{2}\right)\right]$ 的值。
Determine the value of $b=\left[\log_2\left(a^2\right)+\log_4\left(\frac{1}{a^2}\right)\right]\times\left[\log_a 2+\log_{a^2}\left(\frac{1}{2}\right)\right]$.

$b =$

3. 在下圖中，正方形 *ABCD* 及 *XYZW* 相等而且互相交疊使得頂點 *A* 位在 *XYZW* 的中心及綫段 *AB* 將綫段 *YZ* 邊分為 1 : 2。若 *XYZW* 的面積與交疊部分的面積比率為 *c* : 1，求 *c* 的值。
In the figure below, identical squares *ABCD* and *XYZW* overlap each other in such a way that the vertex is at the centre of *XYZW* and the line segment *AB* cuts line segment *YZ* into 1 : 2. If the ratio of the area of *XYZW* to the overlapped region is *c* : 1, determine the value of *c* .

$c =$



4. 若 76 與 *d* 的最小公倍數(L.C.M.)為 456 及 76 與 *d* 的最大公因數(H.C.F.)為 *c*，求正整數 *d* 的值。
If the least common multiples (L.C.M.) of 76 and *d* is 456 and the highest common factor (H.C.F.) of 76 and *d* is *c*, determine the value of the positive integer *d*.

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time		
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2015 – 2016)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 $f(x) = x^4 + x^3 + x^2 + x + 1$，求 $f(x^5)$ 除以 $f(x)$ 的餘值 a。
If $f(x) = x^4 + x^3 + x^2 + x + 1$, determine the remainder a of $f(x^5)$ divided by $f(x)$.

a =

2. 設 n 為整數。求 $n^a - n$ 除以 30 的餘值 b。
Let n be an integer. Determine the remainder b of $n^a - n$ divided by 30 .

b =

3. 若 $0 < x < 1$，求

$$c=\left(\frac{\sqrt{1+x}}{\sqrt{1+x}-\sqrt{1-x}}+\frac{1-x}{\sqrt{1-x^2}+x-1}\right)\times\left(\sqrt{\frac{1}{x^2-b^2}-1}-\frac{1}{x-b}\right)$$

的值。
If $0 < x < 1$, determine the value of

$$c=\left(\frac{\sqrt{1+x}}{\sqrt{1+x}-\sqrt{1-x}}+\frac{1-x}{\sqrt{1-x^2}+x-1}\right)\times\left(\sqrt{\frac{1}{x^2-b^2}-1}-\frac{1}{x-b}\right).$$

c =

4. 若實數 x 及 y 滿足方程 $2\log_{10}(x + 2cy) = \log_{10} x + \log_{10} y$，求 $d = \frac{x}{y}$ 的值。
If real numbers x and y satisfy the equation $2\log_{10}(x + 2cy) = \log_{10} x + \log_{10} y$, determine the value of $d = \frac{x}{y}$.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
		+	Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2015 – 2016)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 m 和 n 為正整數及 $a=\log_2\left[\left(\frac{m^4n^{-4}}{m^{-1}n}\right)^{-3}\div\left(\frac{m^{-2}n^2}{mn^{-1}}\right)^5\right]$，求 a 的值。

   If m and n are positive integers and $a=\log_2\left[\left(\frac{m^4n^{-4}}{m^{-1}n}\right)^{-3}\div\left(\frac{m^{-2}n^2}{mn^{-1}}\right)^5\right]$,

   determine the value of a.

   $a =$

2. 當整數 $1108+a$、1453、$1844+2a$ 及 2281 除以正整數 n (> 1)都得相同餘數 b，求 b 的值。
   When the integers $1108 + a$, 1453, $1844 + 2a$ and 2281 divided by some positive integer n (> 1), they all get the same remainder b. Determine the value of b.

   $b =$

3. 若 $\frac{6}{b}<x<\frac{10}{b}$，求 $c=\sqrt{x^2-2x+1}+\sqrt{x^2-6x+9}$ 的值。
   If $\frac{6}{b}<x<\frac{10}{b}$, determine the value of $c=\sqrt{x^2-2x+1}+\sqrt{x^2-6x+9}$.

   $c =$

4. 求 $1+3^c+(3^c)^2+(3^c)^3+(3^c)^4$ 除以 $1+3+3^2+3^3+3^4$ 的餘值 d。
   Determine the remainder d when $1+3^c+(3^c)^2+(3^c)^3+(3^c)^4$ is divided by $1+3+3^2+3^3+3^4$.

   $d =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]　　Team No. [ ]

+ Bonus score [ ]　　Time [ ] [ ]

Total score [ ]　　Min.　Sec.

Hong Kong Mathematics Olympiad (2015– 2016)
Final Event 1 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 一項工程包括三個項目：A、B 和 C。若項目 A 開始三天後，項目 B 才可開始進行。項目 C 亦必須在項目 B 開始四天後才可開始進行。若完成項目 A、B 和 C 分別需要四天、六天和五天，求最少天數 (P) 完成全項工程。
A project comprises of three tasks, A, B and C. Suppose task B must begin 3 days later than task A begins, and task C must begin 4 days later than task B begins. If the numbers of days to complete tasks A, B and C are 4, 6 and 5, respectively, determine the least number of days (P) to complete the project.

P =

2. 指示牌牌上掛有紅、黃、綠閃燈。紅、黃、綠閃燈分別每隔 3 秒、4 秒、8 秒閃爍一次。當 0 秒時，紅、黃、綠閃燈同時閃爍。若當 Q 秒時，第三次出現只有紅及黃閃燈同時閃爍，求 Q 的值。
There are 3 blinking lights, red, yellow and green, on a panel. Red, yellow and green lights blink at every 3, 4 and 8 seconds, respectively. Suppose each light blinks at the time $t = 0$. At time Q (in seconds), there is the third time at which only red and yellow lights blink, determine the value of Q .

Q =

3. 設 $f_{n+1}=\begin{cases} f_n+3 & \text{若 } n \text{ 是雙數} \\ f_n-2 & \text{若 } n \text{ 是單數} \end{cases}$。

若 $f_1 = 60$，求 n 的最少可能值，令當 $m \geq n$ 時，滿足 $f_m \geq 63$。

Let $f_{n+1}=\begin{cases} f_n+3 & \text{if } n \text{ is even} \\ f_n-2 & \text{if } n \text{ is odd} \end{cases}$.

If $f_1 = 60$, determine the smallest possible value of n satisfying $f_m \geq 63$ for all $m \geq n$.

n =

4. 求 $T=\left(3^{2^0}+1\right)\times\left(3^{2^1}+1\right)\times\left(3^{2^2}+1\right)\times\cdots\times\left(3^{2^{10}}+1\right)$ 的值。(答案以指數表示。)
Determine the value of $T=\left(3^{2^0}+1\right)\times\left(3^{2^1}+1\right)\times\left(3^{2^2}+1\right)\times\cdots\times\left(3^{2^{10}}+1\right)$.
(Leave your answer in index form.)

T =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.		
			+ Bonus score				Time		
			Total score					Min.	Sec.

Hong Kong Mathematics Olympiad (2015 – 2016)
Final Event 2 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 一個盒子有五個球，球面上分別印上號碼 3、4、6、9 或 10。由盒中同時隨機取出 2 個球，並得出其號碼的總和。若 A 為不同總和的數量，求 A 的值。
A box contains five distinctly marked balls with number markings being 3, 4, 6, 9 or 10. Two balls are randomly drawn without replacement from the box.
If A is the number of possible distinct sums of the selected numbers, determine the value of A.

A =

2. 設 $f_1 = 9$ 及 $f_n = \begin{cases} f_{n-1} + 3 & \text{若 } n \text{ 是3的倍數} \\ f_{n-1} - 1 & \text{若 } n \text{ 不是3的倍數} \end{cases}$。
若 B 為 k 的值的可能數量，使得 $f_k < 11$，求 B 的值。

Let $f_1 = 9$ and $f_n = \begin{cases} f_{n-1} + 3 & \text{if } n \text{ is a multiple of } 3 \\ f_{n-1} - 1 & \text{if } n \text{ is not a multiple of } 3 \end{cases}$.
If B is the number of possible values of k such that $f_k < 11$, determine the value of B.

B =

3. 設 a_1、a_2、a_3、a_4、a_5、a_6 為非負整數，並滿足
$$\begin{cases} a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 + 6a_6 = 26 \\ a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 5 \end{cases}。$$
若 c 為方程系統的解的數量，求 c 的值。

Let a_1, a_2, a_3, a_4, a_5, a_6 be non-negative integers and satisfy
$$\begin{cases} a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 + 6a_6 = 26 \\ a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 5 \end{cases}.$$
If c is the number of solutions to the system of equations, determine the value of c.

c =

4. 設 d 及 f 為正整數及 $a_1 = 0.9$。若 $a_{i+1} = a_i^2$ 及 $\prod_{i=1}^{4} a_i = \frac{3^d}{f}$，求 d 的最小可能值。
Let d and f be positive integers and $a_1 = 0.9$. If $a_{i+1} = a_i^2$ and $\prod_{i=1}^{4} a_i = \frac{3^d}{f}$, determine the smallest possible value of d.

d =

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

\+ Bonus score [ ]

Total score [ ]

Team No. [ ]

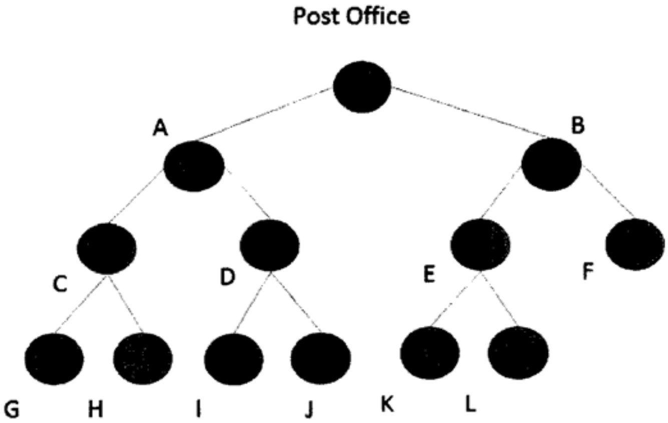
Time [ ] Min. [ ] Sec.

Hong Kong Mathematics Olympiad (2015 – 2016)
Final Event 3 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 下圖是郵差的送信路線圖：從郵局開始，到達十二個地點送信，最後返回郵局。若郵差從一地點步行到另一地點需要十分鐘及 K 為郵差需要的時數來完成整天路線，求 K 的最小可能值。
The figure below represents routes of a postman. Starting at the post office, the postman walks through all the 12 points and finally returns to the post office. If he takes 10 minutes from a point to another adjacent point by walk and K is the number of hours required for the postman to finish the routes, find the smallest possible value of K.

$K =$



2. 若 n 為正整數，$a_1 = 0.8$ 及 $a_{n+1} = a_n^2$，求 L 的最小值，滿足 $a_1 \times a_2 \times \cdots \times a_L < 0.3$。
If $a_1 = 0.8$ and $a_{n+1} = a_n^2$ for positive integers n, determine the least value of L satisfying $a_1 \times a_2 \times \cdots \times a_L < 0.3$.

$L =$

3. 若方程 $\sqrt[3]{5+\sqrt{x}}+\sqrt[3]{5-\sqrt{x}}=1$，求實數根 x。
Solve $\sqrt[3]{5+\sqrt{x}}+\sqrt[3]{5-\sqrt{x}}=1$ for real number x .

$x =$

4. 若 a、b 及 y 為實數，並滿足 $\begin{cases} a+b+y=5 \\ ab+by+ay=3 \end{cases}$，求 y 的最大值。
If a, b and y are real numbers and satisfy $\begin{cases} a+b+y=5 \\ ab+by+ay=3 \end{cases}$, determine the greatest possible value of y .

$y =$

FOR OFFICIAL USE

Score for accuracy [ ] × Mult. factor for speed [ ] = [ ]

\+ Bonus score [ ]

Total score [ ]

Team No. [ ]

Time [ ] Min. [ ] Sec.

Hong Kong Mathematics Olympiad (2015 – 2016)
Final Event 4 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 a 及 b 為整數，且 a^2 與 b^2 相差 144，求 $d = a + b$ 的最大值。
Let a and b are two integers and the difference between a^2 and b^2 is 144, determine the largest possible value of $d = a + b$.

d =

2. 若 n 為整數，n^2 的個位及 10 位分別為 u 及 7，求 u 的值。
If n is an integer, and the units and tens digits of n^2 are u and 7, respectively, determine the value of u.

u =

3. 求實數 $c=\dfrac{\left(4+\sqrt{15}\right)^{\frac{3}{2}}+\left(4-\sqrt{15}\right)^{\frac{3}{2}}}{\left(6+\sqrt{35}\right)^{\frac{3}{2}}-\left(6-\sqrt{35}\right)^{\frac{3}{2}}}$ 的值。

Determine the value of real number $c=\dfrac{\left(4+\sqrt{15}\right)^{\frac{3}{2}}+\left(4-\sqrt{15}\right)^{\frac{3}{2}}}{\left(6+\sqrt{35}\right)^{\frac{3}{2}}-\left(6-\sqrt{35}\right)^{\frac{3}{2}}}$.

c =

4. 求下列方程 $x=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}}}$ 的正實數解。

Determine the positive real root of the following equation: $x=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}}}$.

x =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2016 – 2017)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 a 為 $\frac{1}{(x+2)(x+3)}=\frac{1}{(x+1)(x+4)}$ 的實數解的數量，求 a 的值。

If a is the number of real roots of $\frac{1}{(x+2)(x+3)}=\frac{1}{(x+1)(x+4)}$, determine the value of a.

$a =$

2. 若 x 為實數及 b 為 $-|x-a-9|-|10-x|$ 的最大值，求 b 的值。
If x is a real number and b is the maximum value of $-|x-a-9|-|10-x|$, determine the value of b.

$b =$

3. 若實數 x 及 y 滿足 $4x^2+4y^2+9xy=-119b$，求 xy 的最大值 c。
If real numbers x and y satisfy $4x^2+4y^2+9xy=-119b$, determine c, the maximum value of xy.

$c =$

4. 若正實數 x 滿足方程 $x^2+\frac{1}{x^2}=c$，求 $d=x^3+\frac{1}{x^3}$。

If a positive real number x satisfies $x^2+\frac{1}{x^2}=c$, determine the value of $d=x^3+\frac{1}{x^3}$.

$d =$

FOR OFFICIAL USE

Score for accuracy $\square$ × Mult. factor for speed $\square$ = $\square$ Team No. $\square$

\+ Bonus score $\square$ Time $\square$ Min. $\square$ Sec.

Total score $\square$

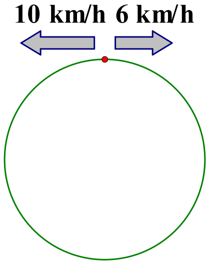
Hong Kong Mathematics Olympiad (2016 – 2017)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 兩個學生於長 1-km 的圓形跑道的起點開始分別以 10 km/h 及 6 km/h 的速率跑沿相反方向跑步。當他們於起點再相遇時便停止跑步。
若 a 為他們開始及停止前相互經過的次數，求 a 的值。

a =

Two students run in opposite directions from a starting point of a 1-km circular track at speeds of 10 km/h and 6 km/h, respectively. They stop running when they meet each other at the starting point again. If a is number of times they cross each other after they start and before they stop, determine the value of a .



2. 袋中有若干粒紅色及藍色的彈珠，紅色彈珠與藍色彈珠的比例為 3∶1。
若加入 a 粒藍色彈珠，紅色彈珠與藍色彈珠的比例則為 2∶1。求彈珠的總數 b。

b =

There is a set of red marbles and blue marbles. When a red marbles are added to the set, the ratio of red marbles to the blue marbles is 3 : 1. When a blue marbles are added, the ratio of red marbles to blue marbles becomes 2 : 1. Determine the total number of marbles, b .

3. 若 c 為 1 000 000 與一個平方數之最小的相差，其中此平方數為 b 的倍數，求 c 的值。
If c is the smallest difference between 1 000 000 and a square, where the square is a multiple of b, determine the value of c .

c =

4. 於一個月的時間完成建築一個水庫需要 d 個技工或 y 個勞工，當中 $d + y = c$。
若挑選 d 個勞工去建築一個同樣的水庫，所需要的時間是挑選 y 個技工的 4 倍，求 d 的值。

d =

The building of a reservoir takes d technicians, or alternatively y labours to complete in a month, where $d + y = c$. If d labours are employed to build the same reservoir, the time taken is 4 times as much as the time taken when y technicians are employed. Determine the value of d .

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2016 – 2017)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 $\{x_0, y_0, z_0\}$ 為以下方程組的解，求 $a = x_0 + y_0 + z_0$ 的值。

$$\begin{cases} 2x-2y+z=-15 \\ x+2y+2z=18 \\ 2x-y+2z=-5 \end{cases}$$

If $\{x_0, y_0, z_0\}$ is a solution to the set of simultaneous equations below, determine the value of $a = x_0 + y_0 + z_0$.

$$\begin{cases} 2x-2y+z=-15 \\ x+2y+2z=18 \\ 2x-y+2z=-5 \end{cases}$$

$a =$

2. 求 $b=\dfrac{\sqrt{6+2\sqrt{a}}+\sqrt{6-2\sqrt{a}}}{2}$ 的值。

Determine the value of $b=\dfrac{\sqrt{6+2\sqrt{a}}+\sqrt{6-2\sqrt{a}}}{2}$.

$b =$

3. 若 x 是正整數且 $\log_{10} b^x > 3$，求 x 的最小值 c。

If x is a positive integer and $\log_{10} b^x > 3$, determine c, the minimum value of x .

$c =$

4. 若 $f(x) = 2^0 + 2^1 + 2^2 + \cdots + 2^{x-2} + 2^{x-1}$，求 $d = f(c)$ 的值。

If $f(x) = 2^0 + 2^1 + 2^2 + \cdots + 2^{x-2} + 2^{x-1}$, determine the value of $d = f(c)$.

$d =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2016 – 2017)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 a 為正整數，求 a 的最大值使得 $ax^2-(a-3)x+(a-2)=0$ 有實根。
If a is a positive integer, determine the greatest value of a such that $ax^2-(a-3)x+(a-2)=0$ has real root(s).

a =

2. 若 x 及 y 為實數且 $1<y<x$ 及 $\log_x y+3\log_y x=\frac{13}{a}$，求 $b=\frac{x+y^4}{x^2+y^2}$ 的值。
If x and y are real numbers with $1<y<x$ and $\log_x y+3\log_y x=\frac{13}{a}$,
determine the value of $b=\frac{x+y^4}{x^2+y^2}$.

b =

3. 一個袋中有紅球 $b+2$ 個，白球 $b+3$ 個及藍球 $b+4$ 個，從袋中隨機抽出 3 個並不重新放進袋中。求三個抽出的球都是相同顏色的概率 c 的值。
A bag contains $b+2$ red balls, $b+3$ white balls and $b+4$ blue balls. Three balls are randomly drawn from the bag without replacement.
Determine the value of the probability, c, that the 3 balls are of the same colours.

c =

4. 若 $\cos 2\theta=c$，求 $d=\sin^4\theta+\cos^4\theta$ 的值。
If $\cos 2\theta=c$, determine the value of $d=\sin^4\theta+\cos^4\theta$.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time	Min.	Sec.
			Total score						

Hong Kong Mathematics Olympiad (2016– 2017)
Final Event 1 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若實數 x、y 及 z 滿足 $x+\frac{1}{y}=-1$，$y+\frac{1}{z}=-2$ 及 $z+\frac{1}{x}=-5$。求 $a=\frac{1}{xyz}$ 的值。

   If real numbers x, y and z satisfy $x+\frac{1}{y}=-1$, $y+\frac{1}{z}=-2$ and $z+\frac{1}{x}=-5$.

   Determine the value of $a=\frac{1}{xyz}$.

   a =

2. 若 $\left|x-\left|2x-1\right|\right|=\frac{1}{2}$ 為實數方程，求實根數量 b 的值。

   If $\left|x-\left|2x-1\right|\right|=\frac{1}{2}$ is a real equation,
   determine the value of b, the number of real solutions of the equation.

   b =

3. 若實數 x 及 y 滿足 $xy>0$ 及 $x+y=3$，求 $\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)$ 的最大值 c。

   If real numbers x and y satisfy $xy > 0$ and $x + y = 3$,

   find c, the maximum value of $\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)$.

   c =

4. 若實數 x 滿足 $x-\frac{1}{x}=3$，求 $d=x^5-\frac{1}{x^5}$ 的值。

   If a real number x satisfies $x-\frac{1}{x}=3$, determine the value of $d=x^5-\frac{1}{x^5}$.

   d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2016 – 2017)

Final Event 2 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 在六進制中，若 A 為 $12345_6 \div 13_6$ 的餘數，求 A 的值。
In base-6 system, if $12345_6 \div 13_6$ has remainder A, determine the value of A .

A =

2. 立方體的任意兩個頂點可相連成一線段。若 B 為最多所能夠相連成的直線的數量，求 B 的值。
Any two vertices in a cube can form a line segment. If B is the greatest number of line segments thus formed, determine the value of B .

B =

3. 若實數 x、y 及 z 滿足 $(x + y + z) = 30$ 及 $C = x^2 + y^2 + z^2$，求 C 的最小值。
If real numbers x, y and z satisfy $(x + y + z) = 30$ and $C = x^2 + y^2 + z^2$,
determine the least value of C .

C =

4. 已知 $D = (x - 1)^3 + 3$。當 $-3 \le x \le 3$，求 D 的最大值。
Given that $D = (x - 1)^3 + 3$. Determine the greatest value of D for $-3 \le x \le 3$.

D =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time	
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2016 – 2017)
Final Event 3 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設 a、b 及 c 為整數且 $1 < a < b < c$。若 $(ab - 1)(bc - 1)(ac - 1)$可被 abc 整除，求 $ab + bc + ac - 1$ 除以 abc 所得之餘數 R 的值。
Let a, b and c be integers with $1 < a < b < c$. If $(ab - 1)(bc - 1)(ac - 1)$ is divisible by abc, determine the value of the remainder R when $ab + bc + ac - 1$ is divided by abc.

$R =$

2. 若 $0 < x < 1$，求 $S=\left(\frac{\sqrt{1+x}}{\sqrt{1+x}-\sqrt{1-x}}+\frac{1-x}{\sqrt{1-x^2}+x-1}\right)\cdot\left(\sqrt{\frac{1}{x^2}-1}-\frac{1}{x}\right)$ 的值。

If $0 < x < 1$, determine the value of $S=\left(\frac{\sqrt{1+x}}{\sqrt{1+x}-\sqrt{1-x}}+\frac{1-x}{\sqrt{1-x^2}+x-1}\right)\cdot\left(\sqrt{\frac{1}{x^2}-1}-\frac{1}{x}\right)$.

$S =$

3. 求方程 $x^4 + (x - 4)^4 = 544$ 的實根之和 T 的值。
Determine the value of T, the sum of real roots of $x^4 + (x - 4)^4 = 544$.

$T =$

4. 在三角形 ABC 中，$BC = a$，$\angle ABC = \frac{\pi}{3}$ 及面積為 $\sqrt{3}a^2$ 。求 $U = \tan(\angle ACB)$的值。

In triangle ABC, $BC = a$, $\angle ABC = \frac{\pi}{3}$ and its area is $\sqrt{3}a^2$.

Determine the value of $U = \tan(\angle ACB)$.

$y =$

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2016 – 2017)
Final Event 4 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 製作某玩具，需要先倒模，後上色。甲先生每日可以為 3 件玩具倒模，或為 15 件玩具上色；乙先生每日則可以為 5 件玩具倒模，或為 15 件玩具上色。各人每日只能倒模或上色，而不能同做兩事。
若甲先生和乙先生合作，求最小多少日 P 才可以製作 120 件玩具。
To make a specific toy, it must be first moulded and then painted. Mr. A can mould 3 pieces of toys or paint 15 pieces of toys in one day, whereas Mr. B can mould 5 pieces or paint 15 pieces of toys in one day. Each of them can either mould or paint toys in one day, but not both. If Mr. A and Mr. B work together, determine the least number of days P to make 120 toys.

d =

2. 在一個射鴨子遊戲中一男孩射了 10 發子彈，該男孩每發子彈射中鴨子的概率為 0.5。求他於最後一發子彈射中第六隻鴨子的概率 Q。
In a duck shooting game, a boy fires 10 shots. The probability of him shooting down a duck with a shot is 0.5.
Determine the probability Q of him shooting down the 6th duck at the last shot.

Q =

3. 如圖 1，求按箭咀方向由 A 往 B 的路線總數 R。
As in Figure 1 below, determine the number of ways R getting from point A to B with the direction indicated by the arrows.

R =

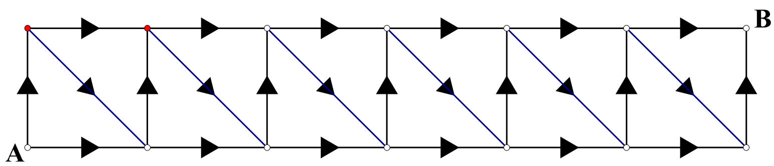
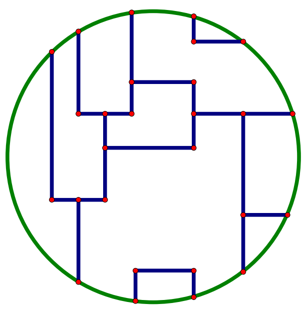


Figure 1/圖 1

4. 如果用 3 款顏料替下圖中所有區域著色，並且相鄰的區域不可用相同顏料。
求同一款顏料最多可用作上色的區域數目 S。
To shade all the regions inside the following circular map using 3 colours, for which adjacent regions must not be in the same colour.
Determine the maximum number S of regions being shaded by the same colour.

S =



FOR OFFICIAL USE

Score for accuracy × Mult. factor for speed =

\+ Bonus score

Total score

Team No.

Time

Min. Sec.

Hong Kong Mathematics Olympiad (2017 – 2018)
Final Event 1 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知 $x^2 = y^2 - 4y$，其中 x 及 y 為整數。求 $A = x + y$ 的最大值。
Given that $x^2 = y^2 - 4y$, where x and y are integers.
Determine the largest value of $A = x + y$.

A =

2. 已知 $y=\sqrt{9A^2-12A+4}\pm\sqrt{A^2-4A+4}\pm\sqrt{A^2+6A+9}$，且 B 是 y 的最小值，求 B 的值。
Given that $y=\sqrt{9A^2-12A+4}\pm\sqrt{A^2-4A+4}\pm\sqrt{A^2+6A+9}$,
and B is the least value of y, determine the value of B.

B =

3. 設 C 為正整數。已知 $144 + (B + 1)^C$ 為平方數，求 C 的值。
Let C be a positive integer. Given that $144 + (B + 1)^C$ is a perfect square, determine the value of C.

C =

4. 已知 $x+\frac{1}{x}=C$，求 $D = x^3+\frac{1}{x^3}$ 的值。
Given that $x+\frac{1}{x}=C$, determine the value of $D = x^3+\frac{1}{x^3}$.

D =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.		
			+	Bonus score			Time		
			Total score					Min.	Sec.

Hong Kong Mathematics Olympiad (2017 – 2018)
Final Event 2 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. $7778^2 - 2223^2$ 之值的所有數字之和是 $\boldsymbol{a}$，求 $\boldsymbol{a}$ 的值。
Determine the value of ***a***, where ***a*** is the sum of all digits of $7778^2 - 2223^2$.

a =

2. 若 b 是乘積 $\boldsymbol{a}\times(\boldsymbol{a}-1)\times(\boldsymbol{a}-2)\times\cdots\times2\times1$ 的尾隨零的數量。求 ***b*** 的值。
$\boldsymbol{a}\times(\boldsymbol{a}-1)\times(\boldsymbol{a}-2)\times\cdots\times2\times1 = \overline{\cdots*\underbrace{00\cdots0}_{\text{"0" 的數量是}\boldsymbol{b}}}$，* 代表非零數字。

If the number of trailing zeros of the product $\boldsymbol{a}\times(\boldsymbol{a}-1)\times(\boldsymbol{a}-2)\times\cdots\times2\times1$ is ***b***, determine the value of ***b*** .
$\boldsymbol{a}\times(\boldsymbol{a}-1)\times(\boldsymbol{a}-2)\times\cdots\times2\times1 = \overline{\cdots*\underbrace{00\cdots0}_{\text{The number of "0" is }\boldsymbol{b}}}$, * represents a non-zero digit.

b =

3. 若 ***c*** 是 $2^{10} - 2^8 + 2^6 - 2^4 + 2^2$ 除以 ***b*** 的餘數，求 ***c*** 的值。
If ***c*** is the remainder when $2^{10} - 2^8 + 2^6 - 2^4 + 2^2$ is divided by ***b***, determine the value of ***c*** .

c =

4. 求整數 ***d***，使得對於任何實數 $\boldsymbol{x}$， $\boldsymbol{x}^{13} + \boldsymbol{cx} + 90$ 可被 $\boldsymbol{x}^2 - \boldsymbol{x} + \boldsymbol{d}$ 整除。
Determine the integral value of ***d***, so that $\boldsymbol{x}^{13} + \boldsymbol{cx} + 90$ is divisible by $\boldsymbol{x}^2 - \boldsymbol{x} + \boldsymbol{d}$ for any real number $\boldsymbol{x}$.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+ Bonus score				Time		
			Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2017 – 2018)
Final Event 3 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知 a、b、c 為實數，且 $A = (3a - X)^2 + (3b - X)^2 + (3c - X)^2 + 6$.
若 $X = a + b + c$ 及 $X^2 = a^2 + b^2 + c^2$，求 A 的最小值。
Given that a, b, c are real numbers and $A = (3a - X)^2 + (3b - X)^2 + (3c - X)^2 + 6$.
If $X = a + b + c$ 及 $X^2 = a^2 + b^2 + c^2$, determine the least value of A .

A =

2. 假設,班中有 A 名男同學及 $30 - A$ 名女同學。若男同學的平均體重為 60 kg，女同學的平均體重為 45 kg 及全班同學的平均體重為 B kg，求 B 的值。
Suppose that there are A boys and $30 - A$ girls in a class. If the average weight of the boys is 60 kg, the average weight of the girls is 45 kg, and the average weight of the students in the class is B kg, determine the value of B .

B =

3. 若 n 是正整數、$a_1 = B$ 及 $a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{若 } a_n \text{ 是偶數；} \\ 3a_n + 1 & \text{若 } a_n \text{ 是奇數。} \end{cases}$ 求 $C = a_{2018}$ 的最值。

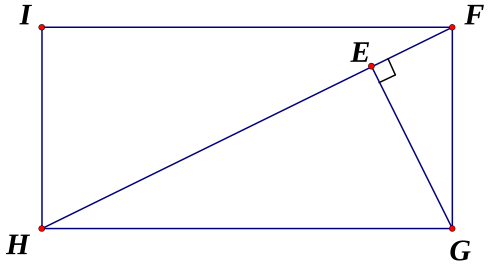
If n is a positive integer $a_1 = B$ and $a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{if } a_n \text{ is even;} \\ 3a_n + 1 & \text{if } a_n \text{ is odd.} \end{cases}$
determine the value of $C = a_{2018}$.

C =

4. 長方形 $FGHI$ 被直綫 FH 分為兩個直角三角形。三角形 ΔFGH 被直綫 EG 分為另外兩個直角三角形。若 $FH : FG = C : 1$ 及三角形 ΔEGH 與三角形 ΔFEG 的面積比為 $D : 1$，求 D 的值。

D =

Suppose that a rectangle $FGHI$ is divided into two right-angled triangles by line FH. The triangle ΔFGH is then divided into two right-angled triangles by line EG. If the ratio of lengths $FH : FG$ is $C : 1$ and the ratio of the areas of ΔEGH to ΔFEG is $D : 1$, determine the value of D.



FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.		
			+	Bonus score			Time		
				Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2017 – 2018)
Final Event 4 (Individual)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 若 a 為 $(1^{2018}+2^{2018}+3^{2018}+4^{2018})\div 5$ 的餘數，求 a 的值。
If a is the remainder of $(1^{2018}+2^{2018}+3^{2018}+4^{2018})\div 5$, determine the value of a.

a =

2. 若 x、y 為正整數及 b 為 x、y 組合的數量使得它們的乘積 $x\times y=\overline{1aa}$，求 b 的值。
If x, y are positive integers numbers and b is the number of groups of x, y such that the product $x\times y=\overline{1aa}$, determine the value of b.

b =

3. 若對於正整數 $x>y>z$， $xyz+xy+xz+yz+x+y+z+1=30b+87$。求 $c=x+y+z$ 的值。
If $xyz+xy+xz+yz+x+y+z+1=30b+87$ for positive integers $x>y>z$, determine the value of $c=x+y+z$.

c =

4. 若某長方形的面積為 d cm^2，它能被邊長為 $\frac{c}{3}$ cm 的正方形階磚密鋪，若該長方形亦能被闊度為 $\frac{c}{2}$ cm、長度為 7 cm 的長方形階磚密鋪，求 d 的最小值。
Let d cm^2 be the area of a rectangle that can be tessellated by square tiles with sides length of $\frac{c}{3}$ cm. If the rectangle can also be tessellated by rectangular tiles with width of $\frac{c}{2}$ cm and length of 7 cm, determine the least value of d.

d =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time		
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2017– 2018)
Final Event 1 (Group)

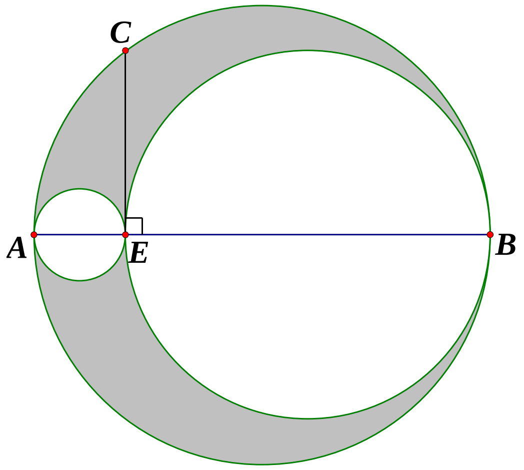
Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 瑪莉和小明在中文科、英文科及數學科獲得的分數為 s 或 t，及 $s > t > 0$。若瑪莉於中文科的分數比小明的高以及小明於英文的分數比瑪莉的高，而瑪莉和小明的總分分別為 12 分和 9 分。求 s 的值。
Suppose that Mary and Ming obtained a score of either s or t in each of the subjects: Chinese, English and Mathematics, where $s > t > 0$. It is known that Mary did better in Chinese but Ming did better in English. Mary's and Ming's total scores are 12 and 9 respectively. Determine the value of s.

s =

2. 已知兩圓的直徑為 AE 及 BE，內接於直徑為 AB 的圓中。若 $CE \perp AB$，$AB = 10$，$CE = 4$ 及陰影部份總面積為 $w\pi$，求 w 的值。
Given that the two circles, one with diameter AE and the other with diameter BE, are inscribed by a larger circle with diameter AB. If $CE \perp AB$ with $AB = 10$ and $CE = 4$, and the total area of the shaded regions is $w\pi$, determine the value of w.

w =



3. 設 m 及 r 為非負整數。若 $f(7m + r) = r$，求 $q = f(2^{2018})$ 的值。
Let m and r be non-negative integers.
If $f(7m + r) = r$, determine the value of $q = f(2^{2018})$.

q =

4. 在五進制中，若 v 為 $234234_5 \div 234_5$ 的餘數，求 v 的值。
In base 5 system, if v is the remainder of $234234_5 \div 234_5$, determine the value of v.

v =

FOR OFFICIAL USE

Score for accuracy × Mult. factor for speed =

Team No.

\+ Bonus score

Time

Total score

Min. Sec.

Hong Kong Mathematics Olympiad (2017 – 2018)
Final Event 2 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 已知 $\dfrac{1-2^{-1/u}}{2^{-1/u}-2^{-2/u}}=4$，求 $\boldsymbol{u}$ 的值。

Given that $\dfrac{1-2^{-1/u}}{2^{-1/u}-2^{-2/u}}=4$, determine the value of $\boldsymbol{u}$.

$\boldsymbol{u}$ =

2. 已知 $\boldsymbol{b}\ge 1$、$\boldsymbol{a}-12\boldsymbol{b}=15$ 及 $\boldsymbol{x}$ 是實數，求 $\boldsymbol{v}=\dfrac{(\boldsymbol{x}-\boldsymbol{a})^2}{2\boldsymbol{b}}+5\boldsymbol{x}$ 的最小值。

Given that $\boldsymbol{b}\ge 1$, $\boldsymbol{a}-12\boldsymbol{b}=15$ and $\boldsymbol{x}$ is a real number, determine the least value of $\boldsymbol{v}=\dfrac{(\boldsymbol{x}-\boldsymbol{a})^2}{2\boldsymbol{b}}+5\boldsymbol{x}$.

$\boldsymbol{v}$ =

3. 若班中有 20 位男同學及 15 位女同學參加兩次考試。已知 8 位同學在第一次考試中不合格，12 位同學在第二次考試中不合格，及 6 位同學於兩次考試均不合格。若 5 位男同學在第一次考試中不合格，7 位男同學在第二次考試中不合格，4 位男同學兩次考試均不合格及 $\boldsymbol{n}$ 位女同學兩次考試均合格，求 $\boldsymbol{n}$ 的值。

Suppose that there were 20 boys and 15 girls in a class taking two examinations. Given that 8 students failed in the first examinations, 12 students failed in the second examinations, and 6 students failed in both examinations. If 5 boys failed in the first examinations, 7 boys failed in the second examinations, 4 boys failed in both examinations, and $\boldsymbol{n}$ girls passed in both examinations, determine the value of $\boldsymbol{n}$.

$\boldsymbol{n}$ =

4. 求最小正整數 $\boldsymbol{m}$，使得 $\boldsymbol{m}^{200}>6^{300}$。

Determine the least positive integer $\boldsymbol{m}$ such that $\boldsymbol{m}^{200}>6^{300}$.

$\boldsymbol{m}$ =

FOR OFFICIAL USE

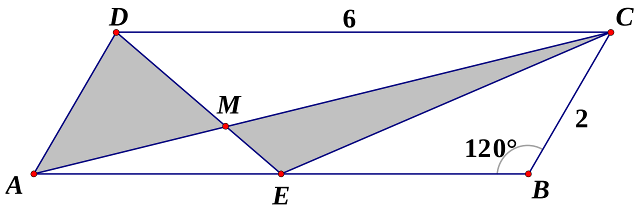
Score for accuracy		×	Mult. factor for speed		=		Team No.		
			+	Bonus score			Time		
				Total score				Min.	Sec.

Hong Kong Mathematics Olympiad (2017 – 2018)
Final Event 3 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. ***AC*** 是平行四邊形 ***ABCD*** 的對角綫，***CD*** = 6，***BC*** = 2 及 ∠***ABC*** = 120°。若 ***E*** 是 ***AB*** 的中點，***AC*** 與 ***DE*** 相交於 ***M*** 及陰影部分的總面積是α，求 α 的值。
ABCD is a parallelogram with diagonal ***AC***, ***CD*** = 6, ***BC*** = 2, and ∠***ABC*** = 120°. If ***E*** is the midpoint of ***AB***, ***AC*** and ***DE*** intersect at ***M***, and the total area of the shaded regions in α, determine the value of α.

α =



2. 設 β 為三位正整數且能被 11 整除，且其商相等於其值的各數字之和的三倍，求 β 的值。
If β is a 3-digit positive integer that is divisible by 11 and whose quotient when divided by 11 is 3 times the sum of its digits, determine the value of β.

β =

3. 求 φ 的最大實數值，使不等式 $\sqrt{1-\varphi}-\sqrt{1+\varphi}\ge 1$ 成立。
Determine the largest real value of φ such that the inequality $\sqrt{1-\varphi}-\sqrt{1+\varphi}\ge 1$ holds.

φ =

4. 設 θ 及 γ 為正整數，當中 θ < γ。若 $\frac{\theta+\gamma}{2}:\sqrt{\theta\gamma}=13:12$，求 γ 的最小值。
Suppose that θ and γ are positive integers, where θ < γ.
If $\frac{\theta+\gamma}{2}:\sqrt{\theta\gamma}=13:12$, determine the least value of γ.

γ =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time		
				Total score			Min.	Sec.

Hong Kong Mathematics Olympiad (2017 – 2018)
Final Event 4 (Group)

Unless otherwise stated, all answers should be expressed in numerals in their simplest form.
除非特別聲明，答案須用數字表達，並化至最簡。

1. 設 $X=\sqrt{2018-\sqrt{A}}$ 是正整數，求 A 的最大值。
Let $X=\sqrt{2018-\sqrt{A}}$ be a positive integer. Determine the largest value of A .

A =

2. 求方程 $(12x-1)(6x-1)(4x-1)(3x-1)=5$ 的所有實根之乘積 B 的值。
Determine the value of B, the product of all real roots of
$(12x-1)(6x-1)(4x-1)(3x-1)=5$.

B =

3. 求 $C=\cos\frac{\pi}{15}\times\cos\frac{2\pi}{15}\times\cos\frac{3\pi}{15}\times\cos\frac{4\pi}{15}\times\cos\frac{5\pi}{15}\times\cos\frac{6\pi}{15}\times\cos\frac{7\pi}{15}$ 的值。
Determine the value of

$$C=\cos\frac{\pi}{15}\times\cos\frac{2\pi}{15}\times\cos\frac{3\pi}{15}\times\cos\frac{4\pi}{15}\times\cos\frac{5\pi}{15}\times\cos\frac{6\pi}{15}\times\cos\frac{7\pi}{15}.$$

C =

4. 設 r、s 及 t 是正實數，且 $r^2+s^2+t^2=rs+st+rt$。若 $r=1$，求 $D=s+t$ 的值。
Let r, s and t be positive real numbers with $r^2+s^2+t^2=rs+st+rt$.
If $r=1$, determine the value of $D=s+t$.

D =

FOR OFFICIAL USE

Score for accuracy		×	Mult. factor for speed		=		Team No.	
			+	Bonus score			Time		
			Total score				Min.	Sec.

Part 2: Answers

Individual Events

I1	***a***	258	**I2**	***a***	17	**I3**	***a***	9	**I4**	***x***	$\frac{3}{2}$	**I5**	***a***	360
	b	15		***b***	136		***b***	15		***y***	1		***b***	36
	c	225		***c***	15		***c***	11		***z***	8		***c***	54
	d	75		**probability**	$\frac{1}{5}$		***d***	73.5		**$\log_z y$**	0		***d***	36

Group Events

G6	**l og 6**	$a + b$	**G7**	**surface area**	320π cm^2	**G8**	⊔ ∟ ⊐ ⊓ □	23485	**G9**	***A***	15	**G10**	**No. of digits**	10
	3.5*a*+3.5*c*	3.5		**volume**	$\frac{2000\pi}{3}$ cm^3		**ans**	┐ □ ┌		***B***	56		**smaller no.**	63
	$\frac{\log 30}{\log 15}$	$\frac{a+b+c}{b+c}$ or $\frac{b+1}{b+1-a}$		**volume**	$\frac{2}{3}\pi r^2 h$		**word**	CHRISTMAS		***C***	8		**larger no.**	65
	$(\log 15)^2 - \log 15$	$(b+c)(b+c-1)$ OR $(b-a+1)(b-a)$		**ratio**	3 : 1		**message**	JOIN US		***X***	0		**greatest no.**	$3^{\frac{1}{3}} \times 8^{\frac{1}{8}}$

Individual Event 1

I1.1 Find a if $a = 5 + 8 + 11 + \dots + 38$.
This is an arithmetic series with first term = 5, common difference = 3
Let n be the number of terms. $38 = 5 + (n - 1)(3) \Rightarrow n = 12$
$a = \frac{1}{2}(5 + 38)\cdot 12 = 258$

I1.2 Let b = the sum of the digits of the number a. Find b.
$b = 2 + 5 + 8 = 15$

I1.3 If $c = b^2$, find c.
$c = 15^2 = 225$

I1.4 Given that $3d = c$, find d.
$3d = 225 \Rightarrow d = 75$
Put $x = -4$ into the polynomial: $2(-4)^2 + 3(-4) + 4d = 0$
$d = -5$

Individual Event 2

I2.1 Two cards are drawn at random from a pack and not replaced. If the probability that both cards are hearts is $\frac{1}{a}$, find a.
P(both hearts) $= \frac{1}{a} = \frac{13}{52} \times \frac{12}{51} = \frac{1}{17}$
$a = 17$

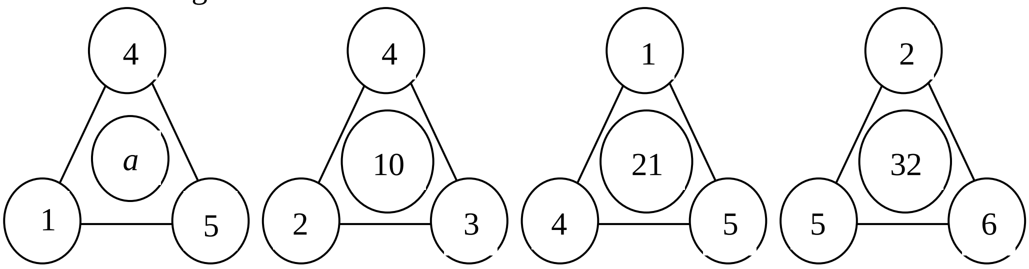
I2.2 If there are b ways of choosing 15 people from 'a' people, find b.
$b = C_{15}^{17} = \frac{17 \times 16}{2} = 136$

I2.3 If c signals can be made with $\frac{b}{2a}$ flags of different colours by raising at least one of the flags, without considering the arrangement of colours, find c.
$\frac{b}{2a} = \frac{136}{2 \cdot 17} = 4$
The following are different patterns:
0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111,
where '0' in the i$^{\text{th}}$ position represents the i$^{\text{th}}$ colour flag is put down and '1' represents the i$^{\text{th}}$ colour flag is raised.
$c = 15$

I2.4 There are c balls in a bag, of which 3 are red. What is the probability of drawing a red ball?
P(red ball) $= \frac{3}{15} = \frac{1}{5}$

Individual Event 3

I3.1 Find a in the figure.



$(1 \times 5) + 4 = 9$ $(2 \times 3) + 4 = 10$ $(4 \times 5) + 1 = 21$ $(5 \times 6) + 2 = 32$.

$a = 9$

I3.2 Find b if $\frac{\sin(4b)^\circ}{\cos(4b)^\circ} = \sqrt{\sqrt{a}}$ $(0 < 4b < 90)$

$\tan (4b)^\circ = \sqrt{3}$

$4b = 60 \Rightarrow b = 15$

I3.3 Find c from the sequence: $\frac{3}{12}, \frac{7}{34}, \frac{c}{56}, \frac{b}{78}$.

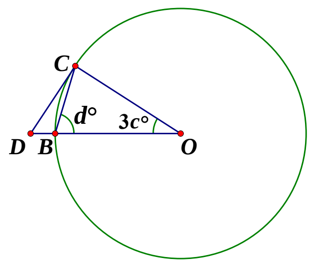
$\frac{3}{12}, \frac{7}{34}, \frac{c}{56}, \frac{15}{78}$

$12 + 22 = 34, 34 + 22 = 56, 56 + 22 = 78$

$3 + 4 = 7, 7 + 4 = 11, 11 + 4 = 15$

$c = 11$

I3.4 In the figure, O is the centre, B and C are points on the circumference. $\angle BOC = 3c^\circ$, $\angle OBC = d^\circ$. Find d.



$\angle BCO = d^\circ$ (base $\angle$s isos. Δ)

$2d + 33 = 180$ ($\angle$s sum of Δ)

$d = 73.5$

Individual Event 4

I4.1 Find x if $x = \frac{\log a^3}{\log a^2}$ where $a > 0$ and $a \neq 1$.

$x = \frac{\log a^3}{\log a^2} = \frac{3\log a}{2\log a} = \frac{3}{2}$

I4.2 If $y - 1 = \log x + \log 2 - \log 3$, find y.

$y - 1 = \log \frac{3}{2} + \log 2 - \log 3$

$y = \log\left(\frac{3}{2} \times \frac{2}{3}\right) + 1 = \log 1 + 1 = 1$

I4.3 What is Z if $\log_2 Z^y = 3$?

$\log_2 Z = 3 \Rightarrow Z = 2^3 = 8$

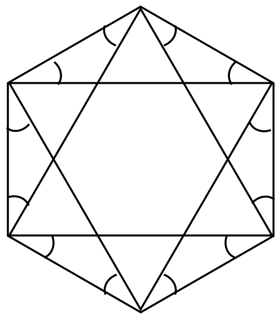
I4.4 Find $\log_z y$.

$\log_8 1 = 0$

Individual Event 5

I5.1 Let the sum of the marked angles be $a°$. Find a.

The figure shows two equilateral triangles inscribed in a regular hexagon.



Each interior angle of the hexagon = 120°

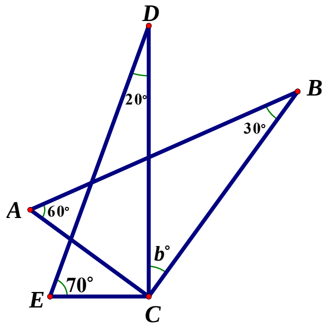
Each angle of an equilateral triangle = 60°

Each marked angle = (120° – 60°)÷ 2 = 30°

There are 12 marked angles.

$a = 12\times 30 = 360$

I5.2 $\angle ACE = \left(\frac{a}{10}\right)^\circ$. Find b.



$\angle DCE = 180° – 20° – 70° = 90°$ ($\angle$s sum of Δ)

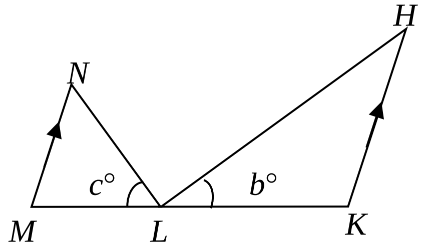
$\angle ACE = 36°$

$\angle ACD = 90° – 36°= 54°$

$\angle ACB = 180° – 30° – 60° = 90°$ ($\angle$s sum of Δ)

$b = 90 – 54 = 36$

I5.3 If $HK = KL$, $LM = MN$, $HK // MN$, find c.



$\angle KHL = b° = 36°$ (base $\angle$s isos. Δ)

$\angle LKH = 180° – 36° – 36° = 108°$ ($\angle$s sum of Δ)

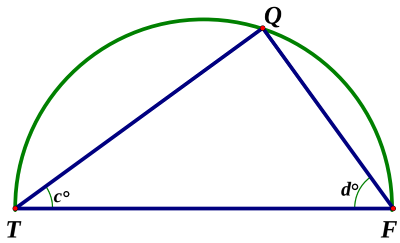
$\angle LMN = 180° – 108° = 72°$ (int. $\angle$s, $NM // HK$)

$\angle MNL = c°$ (base $\angle$s isos. Δ)

$c + c + 72 = 180$ ($\angle$s sum of Δ)

$c = 54$

I5.4 TQF is a semi-circle. Find d.



$\angle TQF = 90°$ ($\angle$s in semi-circle)

$c + d = 90$ ($\angle$s sum of Δ)

$54 + d = 90$

$d = 36$

Group Event 6

Let log 2 = a, log 3 = b, log 5 = c.

G6.1 Express log 6 in terms of a, b and c.

$\log 6 = \log 2 + \log 3 = a + b$

G6.2 Evaluate 3.5 a + 3.5 c.

$3.5\,a + 3.5\,c = 3.5 \log 2 + 3.5 \log 5$

$= 3.5 \log(2\times 5) = 3.5$

G6.3 Express $\frac{\log 30}{\log 15}$ in terms of a, b and c.

$$\frac{\log 30}{\log 15}=\frac{\log 3+\log 10}{\log 3+\log 10-\log 2}=\frac{b+1}{b+1-a} \text{ or } \frac{\log 30}{\log 15}=\frac{\log 2+\log 3+\log 5}{\log 3+\log 5}=\frac{a+b+c}{b+c}$$

G6.4 Express $(\log 15)^2 – \log 15$ in terms of a, b and c.

$(\log 15)^2 – \log 15 = \log 15(\log 15 – 1) = (\log 3 + \log 10 – \log 2) (\log 3 – \log 2)$

$= (b – a + 1)(b – a)$

OR $(\log 15)^2 – \log 15 = \log 15(\log 15 – 1) = (\log 3 + \log 5) (\log 3 + \log 5 – 1) = (b+c)(b+c–1)$

Group Event 7

G7.1 Figure 1 shows a cone and a hemisphere.
OB = 12 cm, r = 10 cm. Express the surface area of the solid in terms of π.
The surface area = $2\pi r^2 + \pi rL = 320\pi$ cm^2

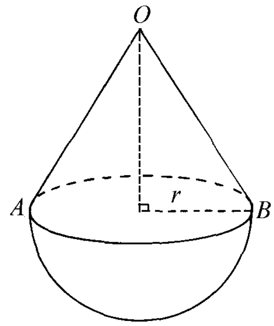


Figure 1

G7.2 What is the volume of the hemisphere shown in figure 1? Give your answer in terms of π.

Volume $= \frac{2}{3}\pi r^3 = \frac{2000\pi}{3}$ cm^3

G7.3 In figure 2, a right circular cone stands inside a right cylinder of same base radius r and height h. Express the volume of the space between them in terms of r and h.

Volume of space $= \pi r^2 h - \frac{1}{3}\pi r^2 h$

$= \frac{2}{3}\pi r^2 h$

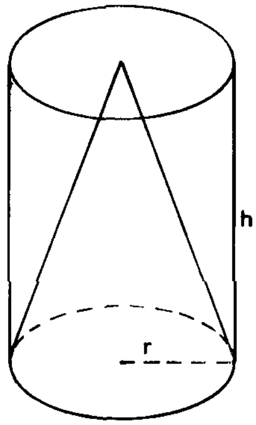


Figure 2

G7.4 Find the ratio of the volume of the cylinder to that of the cone.

Ratio $= \pi r^2 h : \frac{1}{3}\pi r^2 h = 3 : 1$

Group Event 8

Given that:

1	2	3
4	5	6
7	8	9

1 stands for A
2 stands for B
………………
25 stands for Y
26 stands for Z

G8.1 What number does the code ⊔ └ ⊐ ⊓ □ stand for?
⊔ └ ⊐ ⊓ □ stands for 23485

G8.2 Put Δ stands for zero. Calculate the following and give the answer in code.
(⊔Δ)(⊐Δ) + ┘┌ – └Δ
= 20×40 + 19 – 30 = 789
= ┐ □ ┌

G8.3 "3 8 18 9 19 20 13 1 19" stands for a word. What is it?
3 = C, 8 = H, 18 = R, 9 = I, 19 = S, 20 = T, 13 = M, 1 = A, 19 = S
The number stands for "CHRISTMAS"

G8.4 Decode the following message: (┘Δ ┘□ ┌ ┘⊐) (⊔┘ ┘┌)
There are two words in the message.
(┘Δ ┘□ ┌ ┘⊐) (⊔┘ ┘┌) = (10 15 9 14) (21 19) = JOIN US

Group Event 9

G9.1 Find A from the sequence: 0, 3, 8, A, 24, 35, …

$1^2 - 1, 2^2 - 1, 3^2 - 1, 4^2 - 1, 5^2 - 1, 6^2 - 1, \dots A = 4^2 - 1 = 15$

G9.2 The roots of the equation $x^2 - Ax + B = 0$ are 7 and C. Find B and C.

$x^2 - 15x + B = 0$

$7 + C = 15 \Rightarrow C = 8$

$B = 7C = 56$

G9.3 If $\log_7 B = \log_7 C + 7^X$; find X.

$\log_7 56 = \log_7 8 + 7^X$

$7^X = \log_7 (56/8) = \log_7 7 = 1$

$X = 0$

Group Event 10

G10.1 How many digits are there in the number N if $N = 2^{12} \times 5^8$?

Reference: 1992HI17, 2012 HI4

$N = 2^{12} \times 5^8 = 2^4 \times 10^8 = 16 \times 10^8$

There are 10 digits.

G10.2 If $(2^{48} - 1)$ is divisible by two whole numbers between 60 and 70, find them.

$2^{48} - 1 = (2^{24} + 1)(2^{24} - 1) = (2^{24} + 1)(2^{12} + 1)(2^{12} - 1) = (2^{24} + 1)(2^{12} + 1)(2^6 + 1)(2^6 - 1)$

Smaller number = $2^6 - 1 = 63$, larger number = $2^6 + 1 = 65$.

G10.3 Given $2^{\frac{1}{2}} \times 9^{\frac{1}{9}}$, $3^{\frac{1}{3}} \times 8^{\frac{1}{8}}$. What is the greatest number?

$2^{\frac{1}{2}} \times 9^{\frac{1}{9}} = 2^{\frac{1}{2}} \times 3^{\frac{2}{9}}$; $3^{\frac{1}{3}} \times 8^{\frac{1}{8}} = 3^{\frac{1}{3}} \times 2^{\frac{3}{8}}$

$$\frac{2^{\frac{1}{2}} \times 3^{\frac{2}{9}}}{3^{\frac{1}{3}} \times 2^{\frac{3}{8}}} = \frac{2^{\frac{1}{8}}}{3^{\frac{1}{9}}} = \frac{\left(2^9\right)^{\frac{1}{72}}}{\left(3^8\right)^{\frac{1}{72}}} = \left(\frac{512}{6561}\right)^{\frac{1}{72}} < 1$$

$\therefore$ $3^{\frac{1}{3}} \times 8^{\frac{1}{8}}$ is the greatest.

Individual Events

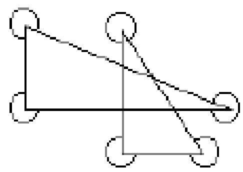
I1	***a***	1800	**I2**	missing	**I3**	***a***	10	**I4**	**no. of routes**	6	**I5**	***a***	$x^2 + 2x + 1$
	b	12				***b***	10		***b***	–2		***b***	–2
	c	*8 See the remark				***c***	30		***c***	3		***c***	2
	d	$\frac{1600}{3}$				***d***	90		**angle**	57°		***d***	1000

Group Events

G6	***a***	*R*	**G7**	**sum**	360	**G8**	***AC***	15 m	**G9**	***a***	$\frac{5}{4}$	**G10**	***A***	3578
	b	80		$S_{\Delta ABC}$	5 cm^2		***x***	60		**step**	2		***N***	10
	c	$\frac{1}{2}$		$a^3 + \frac{1}{a^3}$	18			$2x - 1$		***c***	–6		***∠OAB***	56°
	d	6			$\frac{8}{9}$		***d***	220		**Probability**	$\frac{144}{343}$		***X***	46

Individual Event 1

I.1.1 In the following figure, the sum of the marked angles is $a°$, find a.



Angle sum of a triangle = 180°, angles sum of 2 triangles = 360°

Angle at a point = 360°, angles sum at 6 vertices = 6×360° = 2160°

$\therefore a = 2160 - 360 = 1800$

I.1.2 The sum of the interior angles of a regular b-sided polygon is $a°$. Find b.

$180\times(b - 2) = 1800$

$b = 12$

I1.3 Find c, if $2^b = c^4$ and $c > 0$

$2^{12} = (2^3)^4 = 8^4$

$c = 8$

Remark Original question: Find c, if $2^b = c^4$.

$c = \pm 8$

I1.4 Find d, if $\frac{b}{c} = k$ and $c : d = k : 100$.

$k = \frac{12}{8} = \frac{3}{2}$

$8 : d = \frac{3}{2} : 100$

$\Rightarrow 8 : d = 3: 200$

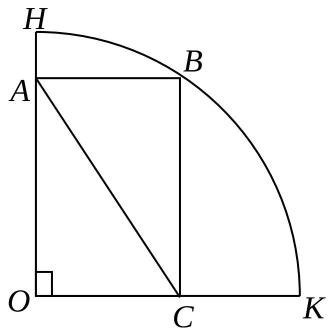
$d = \frac{200}{3}\times 8 = \frac{1600}{3}$

Individual Event 3

I3.1 If $a = 1.8 \times 5.0865 + 1 - 0.0865 \times 1.8$, find a.

$a = 1.8 \times (5 + 0.0865) + 1 - 0.0865 \times 1.8 = 9 + 1 = 10$

I3.2 In the diagram shown, $OH = OK = a$ units and $OABC$ is a rectangle. $AC = b$ units. What is b?



$b = OB = OH = a = 10$

I3.3 In the expression shown, what is c when it is expanded to the term with $x^{(b-2)}$ as the numerator?

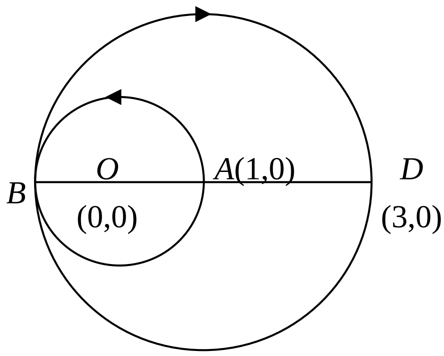
$$\cfrac{x^0}{2+\cfrac{x^1}{6+\cfrac{x^2}{10+\cfrac{x^3}{14+\cfrac{\cdots\cdots}{\cdots\cfrac{}{c+\cfrac{x^{(b-2)}}{\ddots}}}}}}}$$

$b - 2 = 10 - 2 = 8$

T(1) = 2, T(2) = 6, T(3) = 10, this is an arithmetic sequence with first term = 2, common difference = 4.

T(8) = 2 + (8 – 1)×4 = 30

I3.4 As shown a rabbit spends c minutes in travelling from A to B along half circle. With the same speed, it spends d minutes in travelling from $A \to B \to D$ along half circles. What is d?



Radius of the smaller circle = 1

Radius of the larger circle = 2

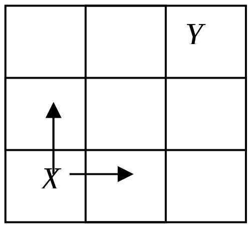
Circumference of the smaller semi-circle $A \to B = \pi$

Circumference of the larger semi-circle $B \to D = 2\pi$

Speed $= \frac{\pi}{c} = \frac{\pi + 2\pi}{d} \Rightarrow d = 3c = 90$

Individual Event 4

I4.1 The figure shows a board consisting of nine squares. A counter originally on square X can be moved either upwards or to the right one square at a time. By how many different routes may the counter be moved from X to Y?



Reference: 1998 HG6, 2000 HI4, 2007 HG5

By adding numbers on the right as shown (Pascal triangle), the number of different routes = 6

1	3	6
1	2	3
1	1	1

I4.2 Given $\sqrt{2a} = -b\tan\frac{\pi}{3}$. Find b.

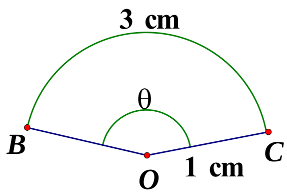
$\sqrt{12} = -b\cdot\sqrt{3}$

$b = -2$

I4.3 Given that $p * q = \frac{p-q}{p}$, find c if $c = (a + b) * (b - a)$.

$c = (6 - 2)*(-2 - 6) = 4*(-8) = \frac{4+8}{4} = 3$

I4.4 A wire of c cm is bent to form a sector of radius 1 cm. What is the angle of the sector in degrees (correct to the nearest degree)?



Let the angle at centre be θ radians.

$2 + 1\times\theta = 3$

$\theta = 1 \text{ radian} = \frac{180^\circ}{\pi} \approx 57.3^\circ = 57^\circ$ (correct to the nearest degree)

Individual Event 5

I5.1 If $a(x + 1) = x^3 + 3x^2 + 3x + 1$, find a in terms of x.

$a(x + 1) = (x + 1)^3$

$a = (x + 1)^2 = x^2 + 2x + 1$

I5.2 If $a - 1 = 0$, then the value of x is 0 or b, what is b?

$a = 1 \Rightarrow 1 = (x + 1)^2$

$x^2 + 2x = 0 \Rightarrow x = 0$ or $-2 \Rightarrow b = -2$

I5.3 If $pc^4 = 32$, $pc = b^2$ and c is positive, what is the value of c?

$pc^4 = 32$ …… (1)

$pc = (-2)^2 = 4$ …… (2)

(1)÷(2): $c^3 = 8$

$c = 2$

I5.4 P is an operation such that $P(A\cdot B) = P(A) + P(B)$.

$P(A) = y$ means $A = 10^y$. If $d = A\cdot B$, $P(A) = 1$ and $P(B) = c$, find d.

$P(A) = 1 \Rightarrow A = 10^1 = 10$

$P(B) = c \Rightarrow B = 10^2 = 100$

$d = A\cdot B = 10\cdot 100 = 1000$

Group Event 6

G6.1 The table shows the results of the operation * on P, Q, R, S taken two at a time.

*	P	Q	R	S
P	Q	R	S	P
Q	R	S	P	Q
R	S	P	Q	R
S	P	Q	R	S

Let a be the inverse of P. Find a.

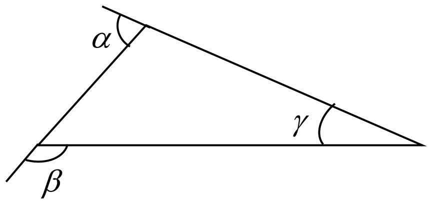
$P*S = P = S*P$, $Q*S = Q = S*Q$, $R*S = R = S*R$,

$S*S = S$

The identity element is S.

$P*R = S = R*P$, the inverse of P is R.

G6.2 The average of α and β is 105°, the average of α, β and γ is b°. Find b.



Reference: 1991 FG6.3

$(\alpha + \beta)\div 2 = 105° \Rightarrow \alpha + \beta = 210°$ …… (1)

$180° - \beta + \gamma = \alpha$ (adj. ∠s on st. line, ext. ∠ of Δ)

$\gamma = \alpha + \beta - 180°$ …… (2)

Sub. (1) into (2): $\gamma = 210° - 180° = 30°$

$b = (210 + 30)\div 3 = 80$

G6.3 The sum of two numbers is 10, their product is 20. The sum of their reciprocal is c. What is c?

Reference 1984 FSG.1, 1985 FSGI.1, 1986 FSG.1

Let the two numbers be x, y.

$x + y = 10$ …… (1)

$x\,y = 20$ …… (2)

$$c = \frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy} = \frac{10}{20} = \frac{1}{2}$$

G6.4 It is given that $\sqrt{90} = 9.49$, to 2 decimal places.

If $d < 7\sqrt{0.9} < d + 1$, where d is an integer, what is d?

$7\sqrt{0.9} = 0.7\sqrt{90} = 0.7\times 9.49$ (correct to 2 decimal places)

$= 6.643$

$d = 6$

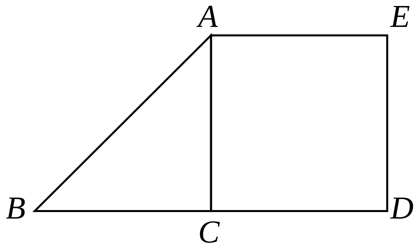
Group Event 7

G7.1 Find 3 + 6 + 9 + … + 45.

The above is an arithmetic series with first term = 3, common difference =3, no. of terms =15.

$S_{15}=\frac{15}{2}\cdot(3+45)=360$

G7.2 In the figure shown, *ACDE* is a square and *AC* = *BC*, $\angle ACB = 90°$. Find the area of ΔABC if the area of *ACDE* is 10 cm^2.



$\Delta ABC \cong \Delta CED \cong \Delta ECA$ (S.A.S.)

The area of $\Delta ABC = \frac{1}{2}\times$area of *ACDE* = 5 cm^2

G7.3 Given that $a+\frac{1}{a}=3$. Evaluate $a^3+\frac{1}{a^3}$.

Reference: 1996 FI1.2, 1998 FG5.2, 2010 FI3.2

$\left(a+\frac{1}{a}\right)^2=9$

$\Rightarrow a^2+\frac{1}{a^2}=7$

$a^3+\frac{1}{a^3}=\left(a+\frac{1}{a}\right)\left(a^2-1+\frac{1}{a^2}\right)$

$= 3\times(7-1) = 18$

G7.4 Given that $\sum_{y=1}^{n}\frac{1}{y}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$.

Find $\sum_{y=3}^{10}\frac{1}{y-2}-\sum_{y=3}^{10}\frac{1}{y-1}$. (Express your answer in fraction.)

Reference: 1991 FSG.1

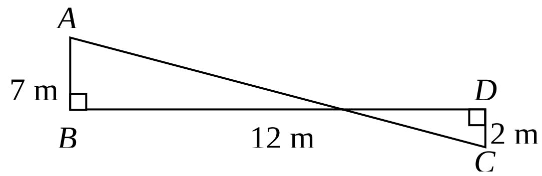
$\sum_{y=3}^{10}\frac{1}{y-2}-\sum_{y=3}^{10}\frac{1}{y-1}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{8}-\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{9}\right)$

$=1-\frac{1}{9}=\frac{8}{9}$

Group Event 8

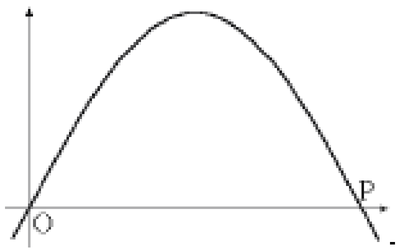
G8.1 Peter is standing at *A* and John is at *C*. The distance between *B* and *D* is 12 m. What is the shortest distance between John and Peter?

Reference: 1991 HG9, 1993 HI1, 1996 HG9



$AC = \sqrt{(7+2)^2 + 12^2}$ m = 15 m

G8.2 The following figure shows a part of the graph $y = \sin 3x°$. What is the x-coordinate of P?



$\sin 3x° = 0$

$3x° = 180°$

$x = 60$

G8.3 If $f(x) = x^2$, then express $f(x) - f(x - 1)$ in terms of x.

$f(x) - f(x - 1) = x^2 - (x - 1)^2 = 2x - 1$

G8.4 If *mnp*, *nmp*, *mmp* and *nnp* are numbers in base 10 composed of the digits *m*, *n* and *p*, such that: *mnp* – *nmp* = 180 and *mmp* – *nnp* = *d*. Find *d*.

$100m + 10n + p - (100n + 10m + p) = 180$

$100(m - n) - 10(m - n) = 180$

$m - n = 2$

$d = mmp - nnp$

$= 100m + 10m + p - (100n + 10n + p)$

$= 110(m - n)$

$= 220$

Group Event 9

G9.1 If $\sin\theta = \frac{3}{5}$, $a = \sqrt{\tan^2\theta + 1}$, find a.

$\cos^2\theta = 1 - \sin^2\theta = 1 - \frac{9}{25} = \frac{16}{25}$

$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} = \frac{\frac{9}{25}}{\frac{16}{25}} = \frac{9}{16}$

$a = \sqrt{\tan^2\theta + 1} = \sqrt{\frac{9}{16} + 1} = \frac{5}{4}$

G9.2 Examine the following proof carefully: To prove $\frac{1}{8} > \frac{1}{4}$.

<u>Steps</u>

1	$3 > 2$
2	Multiply both sides by $\log\left(\frac{1}{2}\right)$, then $3\log\left(\frac{1}{2}\right) > 2\log\left(\frac{1}{2}\right)$
3	$\log\left(\frac{1}{2}\right)^3 > \log\left(\frac{1}{2}\right)^2$
4	$\left(\frac{1}{2}\right)^3 > \left(\frac{1}{2}\right)^2$

$\therefore \frac{1}{8} > \frac{1}{4}$

Which step is incorrect?

Step 2 is incorrect because $\log\left(\frac{1}{2}\right) < 0$.

Multiply both sides by $\log\left(\frac{1}{2}\right)$, then $3\log\left(\frac{1}{2}\right) < 2\log\left(\frac{1}{2}\right)$.

G9.3 If the lines $2y + x + 3 = 0$ and $3y + cx + 2 = 0$ are perpendicular, find the value of c.

Reference: 1984 FSG.3, 1985 FI4.1, 1986 FSG.2, 1987 FG10.2, 1988 FG8.2

Product of slopes = –1

$-\frac{1}{2} \times \left(-\frac{c}{3}\right) = -1$

$c = -6$

G9.4 There are 4 red balls and 3 black balls in a box. If 3 balls are chosen one by one with replacement, what is the probability of choosing 2 red balls and 1 black ball?

$\text{P(2 red, 1 black)} = 3 \times \left(\frac{4}{7}\right)^2 \times \frac{3}{7} = \frac{144}{343}$

Group Event 10

G10.1 $1^2 - 1 = 0 \times 2$

$2^2 - 1 = 1 \times 3$

$3^2 - 1 = 2 \times 4$

$4^2 - 1 = 3 \times 5$

....

$A^2 - 1 = 3577 \times 3579$

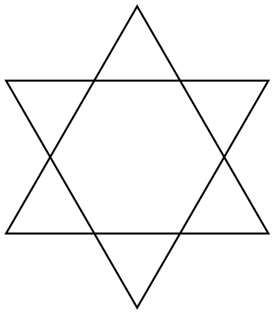
If $A > 0$, find A.

Reference: 1984 FSG..2, 1991 FI2.1

$A^2 - 1 = (3578 - 1) \times (3578 + 1)$

$A = 3578$

G10.2 The sides of an N-sided regular polygon are produced to form a "star". If the angle at each point of that "star" is 108°, find N. (For example, the "star" of a six-sided polygon is given as shown in the diagram.)



6-sided regular polygon.

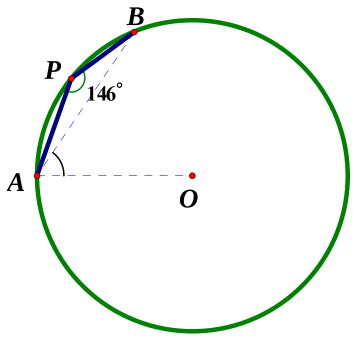
Consider an isosceles triangle formed by each point. The vertical angle is 108°.

Each of the base angle $= \frac{180^\circ - 108^\circ}{2} = 36^\circ$

$36N = 360$ (sum of ext. $\angle$s of polygon) $\Rightarrow N = 10$

G10.3 A, P, B are three points on a circle with centre O.

If $\angle APB = 146^\circ$, find $\angle OAB$.



Add a point Q as shown in the diagram.

$\angle AQB = 180^\circ - 146^\circ = 34^\circ$ (opp. $\angle$s cyclic quad.)

$\angle AOB = 2\times 34^\circ = 68^\circ$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

$OA = OB$ = radii

$\angle OAB = \angle OBA$ (base $\angle$s isos. Δ)

$= \frac{180^\circ - 68^\circ}{2} = 56^\circ$ ($\angle$s sum of Δ)

G10.4 A number X consists of 2 digits whose product is 24. By reversing the digits, the new number formed is 18 greater than the original one. What is X? (**Reference: 1991 FG6.1-2)**

Let the tens digit of X be a and the units digit be b.

$X = 10a + b$, reversed number $= 10b + a$

$ab = 24 \Rightarrow b = \frac{24}{a}$ ...... (1)

$10b + a - (10a + b) = 18 \Rightarrow b - a = 2$ ...... (2)

Sub. (1) into (2): $\frac{24}{a} - a = 2$

$24 - a^2 = 2a$

$a^2 + 2a - 24 = 0$

$(a - 4)(a + 6) = 0$

$a = 4$ or -6 (rejected)

$b = 6$

$X = 46$

Individual Events

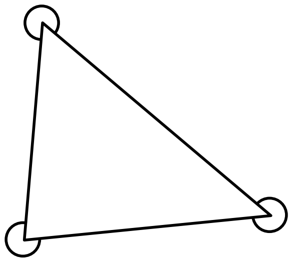
SI	***a***	900	**I1**	***a***	10	**I2**	***a***	$\frac{1}{2}$ (=0.5)	**I3**	***a***	–7	**I4**	***a***	15	**I5**	***a***	80
	b	7		***b***	1		***b***	5		***b***	6		***b***	8		***b***	4
	c	2		***c***	4		***c***	10		***x***	$\frac{1}{2}$ (=0.5)		***c***	4		***N***	10
	d	9		***d***	–5		***d***	15		***y***	–1		***d***	12		***x***	144

Group Events

SG	***a***	2	**G6**	***p***	10	**G7**	***p***	75	**G8**	***M***	1	**G9**	***x***	$\frac{1}{100}$	**G10**	***A***	50
	b	*136 see the remark		***q***	15		***q***	$\frac{1}{2}$ (=0.5)		***N***	6		***A***	52		***S***	2
	c	–6		***r***	24		***a***	2		***R***	8		***m***	501		***n***	7
	d	7		***s***	27		***m***	14		***Y***	2		***P***	36		***d***	5

Sample Individual Event (1988 Sample Individual Event)

SI.1 In the given diagram, the sum of the three marked angles is $a°$. Find a.



Reference: 1987 FSG.3, 1989 FSI.1

Sum of interior angles of a triangle = 180°

angle sum of three vertices = 3×360° = 1080°

$a = 1080 – 180 = 900$

SI.2 The sum of the interior angles of a regular b-sided polygon is $a°$. Find b.

Reference 1989 FSI.2

$a = 900 = 180×(b – 2)$

$b = 7$

SI.3 If $8^b = c^{21}$, find c.

$8^7 = c^{21}$

$2^{21} = c^{21}$

$c = 2$

SI.4 If $c = \log_d 81$, find d.

$2 = c = \log_d 81$ and $d > 0$

$d^2 = 81$

$d = 9$

Individual Event 1

I1.1 If $100a = 35^2 – 15^2$, find a.

Reference: 1987 FSG.1, 1988 FI2.2

$100a = (35 + 15)(35 – 15) = 50×20 = 1000$

$a = 10$

I1.2 If $(a – 1)^2 = 3^{4b}$, find b.

$9^2 = 3^{4b}$

$4b = 4$

$\Rightarrow b = 1$

I1.3 If b is a root of $x^2 + cx – 5 = 0$, find c.

Put $x = 1$ into the equation: $1 + c – 5 = 0$

$c = 4$

I1.4 If $x + c$ is a factor of $2x^2 + 3x + 4d$, find d.

$x + 4$ is a factor

Put $x = –4$ into the polynomial: $2(–4)^2 + 3(–4) + 4d = 0$

$d = –5$

Individual Event 2

I2.1 If α, β are roots of $x^2 - 10x + 20 = 0$, find a, where $a = \frac{1}{\alpha} + \frac{1}{\beta}$.

$\alpha + \beta = 10$, $\alpha\beta = 20$

$a = \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{10}{20} = \frac{1}{2}$

I2.2 If $\sin \theta = a$ ($0° < \theta < 90°$), and $10 \cos 2\theta = b$, find b.

$\sin \theta = \frac{1}{2}$

$\Rightarrow \theta = 30°$

$b = 10 \cos 60° = 5$

I2.3 The point $A(b, c)$ lies on the line $2y = x + 15$. Find c.

Reference: 1984 FI2.3

Put $x = b = 5$, $y = c$ into the line: $2c = 5 + 15$

$c = 10$

I2.4 If $x^2 - cx + 40 \equiv (x + k)^2 + d$, find d.

Reference: 1985 FG10.2, 1986 FG7.3, 1987 FSI.1, 1988 FG9.3

$x^2 - 10x + 40 \equiv (x - 5)^2 + 15$

$k = -5$, $d = 15$

Individual Event 3

I3.1 If a is the remainder when $2x^3 - 3x^2 + x - 1$ is divided by $x + 1$, find a.

$a = 2(-1)^3 - 3(-1)^2 - 1 - 1 = -7$

I3.2 If b cm^2 is the total surface area of a cube of side $(8 + a)$ cm, find b.

Similar Questions: 1984 FG9.2, 1985 FSI.2

$8 + a = 1$

$b = 6$

I3.3 One ball is taken at random from a bag containing $b + 4$ red balls and $2b - 2$ white balls. If x is the probability that the ball is white, find x.

There are $b + 4 = 10$ red balls and $2b - 2 = 10$ white balls

$x = \frac{1}{2}$

I3.4 If $\sin \theta = x$ ($90° < \theta < 180°$) and $\tan(\theta - 15°) = y$, find y.

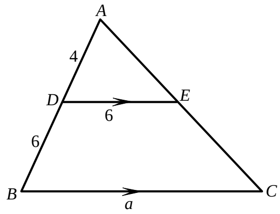
$\sin \theta = \frac{1}{2}$

$\Rightarrow \theta = 150°$

$y = \tan(\theta - 15°) = \tan 135° = -1$

Individual Event 4

I4.1 In figure 1, DE // BC. If $AD = 4$, $DB = 6$, $DE = 6$ and $BC = a$, find a.

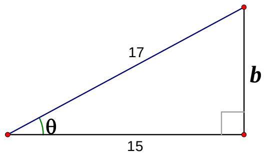


$\Delta ADE \sim \Delta ABC$ (equiangular)

$\frac{a}{6}=\frac{10}{4}$ (ratio of sides, ~Δ's)

$a = 15$

I4.2 θ is an acute angle such that $\cos\theta=\frac{a}{17}$. If $\tan\theta=\frac{b}{15}$, find b.



$b^2 = 17^2 - 15^2$

$b = 8$

I4.3 If $c^3 = b^2$, find c.

$c^3 = 8^2 = 64 = 4^3$

$\Rightarrow c = 4$

I4.4 The area of an equilateral triangle is $c\sqrt{3}$ cm^2. If its perimeter is d cm, find d.

Reference: 1985 FSI.4, 1986 FSG.3, 1987 FG6.2, 1988 FG9.1

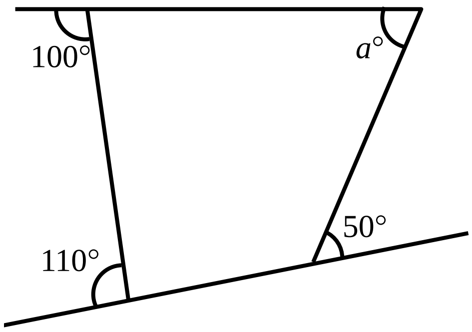
Each side $=\frac{d}{3}$ cm

$\frac{1}{2}\cdot\left(\frac{d}{3}\right)^2\sin 60^\circ = c\sqrt{3} = 4\sqrt{3}$

$d = 12$

Individual Event 5

I5.1 In Figure 2, find a.



$100 + (180 - a) + 50 + 110 = 360$ (sum of ext. ∠ of Δ)

$a = 80$

I5.2 If $b=\log_2\left(\frac{a}{5}\right)$, find b.

$2^b = 16$

$b = 4$

I5.3 A piece of string, 20 m long, is divided into 3 parts in the ratio of $b - 2 : b : b + 2$. If N m is the length of the longest portion, find N.

$b - 2 : b : b + 2 = 2 : 4 : 6 = 1 : 2 : 3$

$N = 20\times\frac{3}{1+2+3} = 10$

I5.4 Each interior angle of an N-sided regular polygon is $x°$. Find x.

$x=\frac{180\times(10-2)}{10} = 144$

Sample Group Event

SG.1 The sum of 2 numbers is 20, their product is 10. If the sum of their reciprocals is *a*, find *a*.

Reference: 1983 FG6.3, 1985 FSI.1, 1986 FSG.1

Let the 2 numbers be *x* and *y*.

$x + y = 20$ and $xy = 10$

$a = \frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy} = 2$

SG.2 $1^2 - 1 = 0\times2$, $2^2 - 1 = 1\times3$, $3^2 - 1 = 2\times4$, … , $b^2 - 1 = 135\times137$. If $b > 0$, find *b*.

Reference: 1983 FI10.1, 1991 FI2.1

$135\times137 = (136 - 1)\times(136 + 1) = 136^2 - 1$

$b = 136$

Remark The original question is:

$1^2 - 1 = 0\times2$, $2^2 - 1 = 1\times3$, $3^2 - 1 = 2\times4$, … , $b^2 - 1 = 135\times137$, find *b*.

$b = 136$ or -136, there are 2 different answers!

SG.3 If the lines $x + 2y + 1 = 0$ and $cx + 3y + 1 = 0$ are perpendicular, find *c*.

Reference: 1983 FG9.3, 1985 FI4.1, 1986 FSG.2, 1987 FG10.2, 1988 FG8.2

$-\frac{1}{2}\times\left(-\frac{c}{3}\right) = -1 \Rightarrow c = -6$

SG.4 The points $(2, -1)$, $(0, 1)$, (c, d) are collinear. Find *d*.

Reference: 1984 FG7.3, 1986FG6.2, 1987 FG7.4, 1989 HG8

$\frac{d-1}{-6} = \frac{1-(-1)}{0-2}$

$d = 7$

Group Event 6

G6.1 If $p = \frac{21^3 - 11^3}{21^2 + 21\times11 + 11^2}$, find *p*. **(Similar questions: 1985 FG7.1)**

$p = \frac{21^3 - 11^3}{21^2 + 21\times11 + 11^2} = \frac{(21-11)(21^2 + 21\times11 + 11^2)}{21^2 + 21\times11 + 11^2} = 10$

G6.2 If *p* men can do a job in 6 days and 4 men can do the same job in *q* days, find *q*.

10 men can do a job in 6 days.

1 man can do a job in 60 days

4 men can do a job in 15 days $\Rightarrow q = 15$

G6.3 If the q^{th} day of March in a year is Wednesday and the r^{th} day of March in the same year is Friday, where $18 < r < 26$, find *r*. **(Reference: 1985 FG9.3, 1987 FG6.4, 1988 FG10.2)**

15^{th} March is Wednesday

17^{th} March is Friday

24^{th} March is Friday $\Rightarrow r = 24$

G6.4 If $a*b = ab + 1$, and $s = (3*4)*2$, find *s*. **(Reference: 1985 FSG.1)**

$3*4 = 3\times4 + 1 = 13$

$s = (3*4)*2 = 13*2 = 13\times2 + 1 = 27$

Group Event 7 (1988 Sample Group Event)

G7.1 The acute angle between the 2 hands of a clock at 3:30 a.m. is $p°$. Find p.

Reference: 1985 FI3.1 1987 FG7.1, 1989 FI1.1, 1990 FG6.3, 2007 HI1

At 3:00 a.m., the angle between the arms of the clock = 90°

From 3:00 a.m. to 3:30 a.m., the hour-hand had moved $360°\times\frac{1}{12}\times\frac{1}{2}=15°$.

The minute hand had moved 180°.

$p = 180 - 90 - 15 = 75$

G7.2 In ΔABC, $\angle B = \angle C = p°$. If $q = \sin A$, find q.

$\angle B = \angle C = 75°$, $\angle A = 180° - 75° - 75° = 30°$

$q = \sin 30° = \frac{1}{2}$

G7.3 The 3 points (1, 3), (a, 5), (4, 9) are collinear. Find a.

Reference: 1984 FSG.4, 1986FG6.2, 1987 FG7.4, 1989 HG8

$\frac{9-5}{4-a}=\frac{9-3}{4-1}=2$

$\Rightarrow a = 2$

G7.4 The average of 7, 9, x, y, 17 is 10. If m is the average of $x + 3$, $x + 5$, $y + 2$, 8, y +18, find m.

$\frac{7+9+x+y+17}{5}=10$

$\Rightarrow x + y = 17$

$m=\frac{x+3+x+5+y+2+8+y+18}{5}$

$=\frac{2(x+y)+36}{5}=14$

Group Event 8

In the addition shown, each letter represents a different digit ranging from zero to nine. It is already known that

$S = 9$, O = zero, $E = 5$.

		S	E	N	D
+		M	O	R	E
	M	O	N	E	Y

Find the numbers represented by

(i) M, (ii) N, (iii) R, (iv) Y

Consider the thousands digit and the ten thousands digits.

$0 \le S, M \le 9$, $9 + M = 10M + 0$ or $9 + M + 1 = 10M + 0$

$\Rightarrow M = 1$ and there is no carry digit.

Consider the hundreds digit. $5 + 0 + 1 = N$

$\Rightarrow N = 6$ and there is a carry digit.

For the tens digit. $6 + R = 10 + 5$

$\Rightarrow R = 9$ (same as S, rejected) or $6 + R + 1 = 10 + 5$

$\Rightarrow R = 8$

There is a carry digit in the unit digit

$D + 5 = 10 + Y$, $(D, Y) = (7, 2) \Rightarrow Y = 2$

$\therefore$ $M = 1$, $N = 6$, $R = 8$, $Y = 2$

Group Event 9

G9.1 If $x=\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{100}\right)$, find x in the simplest fractional form.

Reference: 1985 FSG.3, 1986 FG10.4

$x=\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times\cdots\times\frac{99}{100}=\frac{1}{100}$

G9.2 The length, width and height of a rectangular block are 2, 3 and 4 respectively. Its total surface area is A, find A.

Similar Questions: 1984 FI3.2, 1985 FSI.2

$A = 2\times(2\times3 + 3\times4 + 2\times4) = 52$

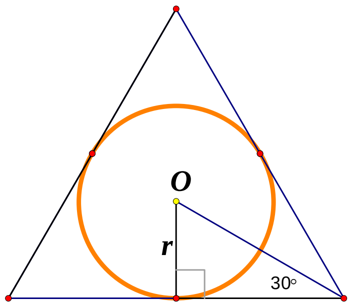
G9.3 The average of the integers 1, 2, 3, ... , 1001 is m. Find m.

$m=\frac{1}{1001}(1+2+3+\cdots+1001)$

$=\frac{1}{1001}\cdot\frac{(1+1001)\cdot1001}{2}=501$

G9.4 The area of a circle inscribed in an equilateral triangle is 12π. If P is the perimeter of this triangle, find P.

Reference: 1990 FI2.3



Let the radius be r and the centre be O.

$\pi r^2 = 12\pi$

$\Rightarrow r=2\sqrt{3}$

The length of one side of the equilateral triangle is $\frac{P}{3}$.

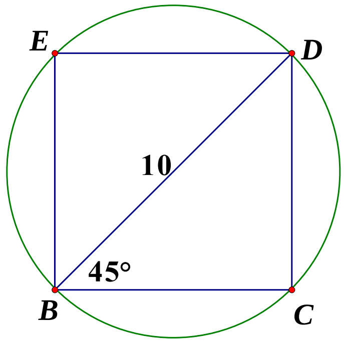
$\frac{P}{3} = 2r\cot 30°$

$=2\sqrt{3}r = 12$

$P = 36$

Group Event 10

G10.1 If A is the area of a square inscribed in a circle of diameter 10, find A.



Reference: 1985 FSG.4, 1989 FI3.3

Let the square be $BCDE$.

$BC = 10\cos 45° = 5\sqrt{2}$

$A = \left(5\sqrt{2}\right)^2 = 50$

G10.2 If $a + \frac{1}{a} = 2$, and $S = a^3 + \frac{1}{a^3}$, find S.

Reference: 1998 HG1

$a^2 + \frac{1}{a^2} = \left(a + \frac{1}{a}\right)^2 - 2 = 4 - 2 = 2$

$S = a^3 + \frac{1}{a^3}$

$= \left(a + \frac{1}{a}\right)\left(a^2 - 1 + \frac{1}{a^2}\right)$

$= 2(2 - 1) = 2$

G10.3 An n-sided convex polygon has 14 diagonals. Find n.

Reference: 1985 FG8.3, 1988 FG6.2, 1989 FG6.1, 1991 FI2.3, 2001 FI4.2, 2005 FI1.4

Number of diagonals $= C_2^n - n = \frac{n(n-1)}{2} - n = 14$

$n^2 - 3n - 28 = 0$

$(n - 7)(n + 4) = 0$

$\Rightarrow n = 7$

G10.4 If d is the distance between the 2 points (2, 3) and (–1, 7), find d.

Reference: 1986 FG9.4

$d = \sqrt{[2-(-1)]^2 + (3-7)^2} = 5$

Individual Events

SI	*a*	2	I1	*a*	5	I2	*a*	125	I3	*a*	4	I4	*a*	2	I5	*t*	8
	b	54		*b*	0		*b*	15		*b*	16		*b*	10		*u*	135
	c	2		*c*	–9		*c*	3		*c*	199		*c*	96		*v*	45
	d	1		*d*	2		*d*	16		*d*	4		*d*	95		*w*	70

Group Events

SG	*s*	19	G6	*x*	8	G7	*M*	100	G8	*M*	5	G9	*A*	60	G10	*k*	15
	n	8		*y*	25		*N*	59		*N*	2		*r*	3		*C*	6
	K	$\frac{1}{50}$		*d*	4		*x*	$\frac{24}{5}$		*x*	170		*n*	20		*R*	8
	A	200		*h*	$\frac{12}{5}$		*S*	1		*y*	5000		*x*	3240		*A*	243

Sample Individual Event (1994 Final Sample Individual Event)

SI.1 The sum of two numbers is 40, and their product is 20.

If the sum of their reciprocals is a, find a.

Reference: 1983 FG6.3, 1984 FSG.1, 1986 FSG.1

Let the two numbers be x and y.

$x + y = 40$ and $xy = 20$

$$a = \frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy} = 2$$

SI.2 If b cm^2 is the total surface area of a cube of side $(a + 1)$ cm, find b.

Similar Questions: 1984 FI3.2, 1984 FG9.2

$a + 1 = 3$

$b = 6\times 3^2 = 54$

SI.3 One ball is taken at random from a bag containing $b - 4$ white balls and $b + 46$ red balls.

If $\frac{c}{6}$ is the probability that the ball is white, find c.

There are $b - 4 = 50$ white balls and $b + 46 = 100$ red balls

$$\text{P(white ball)} = \frac{50}{50+100} = \frac{2}{6} \Rightarrow c = 2$$

SI.4 The length of a side of an equilateral triangle is c cm. If its area is $d\sqrt{3}$ cm^2, find d.

Reference: 1984FI4.4, 1986 FSG.3, 1987 FG6.2, 1988 FG9.1

$$d\sqrt{3} = \frac{1}{2}\cdot c^2 \sin 60^\circ = \sqrt{3}$$

$d = 1$

Individual Event 1

I1.1 Find a if $a=\log_5\frac{(125)(625)}{25}$.

$a=\log_5\frac{5^3\cdot 5^4}{5^2}=\log_5 5^5$

$a=5$

I1.2 If $\left(r+\frac{1}{r}\right)^2=a-2$ and $r^3+\frac{1}{r^3}=b$, find b.

Reference: 1990 HI12, 2017 FI1.4

$\left(r+\frac{1}{r}\right)^2=r^2+2+\frac{1}{r^2}=3\Rightarrow r^2+\frac{1}{r^2}=1$

$b=r^3+\frac{1}{r^3}=\left(r+\frac{1}{r}\right)\left(r^2-1+\frac{1}{r^2}\right)=\left(r+\frac{1}{r}\right)(1-1)=0$

I1.3 If one root of the equation $x^3+cx+10=b$ is 2, find c.

Put $x=2$ into the equation: $8+2c+10=0$

$c=-9$

I1.4 Find d if $9^{d+2}=(6489+c)+9^d$. **(Reference: 1986 FG7.4)**

$81\times 9^d=6480+9^d$

$80\times 9^d=6480\Rightarrow 9^d=81$

$d=2$

Individual Event 2

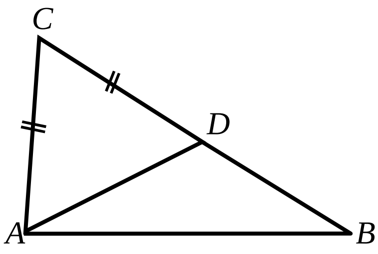
I2.1 Find a in the following sequence: 1, 8, 27, 64, a, 216, ……

$1^3, 2^3, 3^3, 4^3, a, 6^3$, ……

$a=5^3=125$

I2.2 In Figure 1, $AC=CD$ and $\angle CAB-\angle ABC=(a-95)°$.

If $\angle BAD=b°$, find b. **(Reference: 2010 HG3)**



Let $\angle CAD=\theta=\angle CDA$ (base $\angle$s isosceles Δ)

$\angle CAB=b°+\theta$

$\angle CAB-\angle ABC=30°\Rightarrow\angle ABC=b°+\theta-30°$

$\angle BAD+\angle ABC=\angle CDA$ (ext. $\angle$ of Δ)

$b°+b°+\theta-30°=\theta\Rightarrow b=15$

I2.3 A line passes through the points $(-1, 1)$ and $(3, b-6)$. If the y-intercept of the line is c, find c.

Similar question: 1986 FI1.4

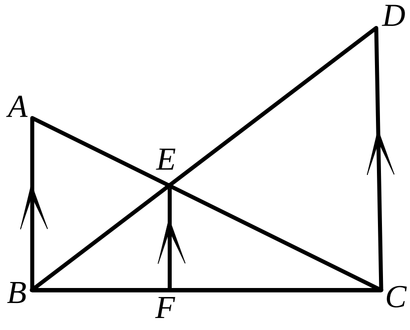
$b-6=9$

$\frac{c-9}{0-3}=\frac{9-1}{3-(-1)}$

$c=3$

I2.4 In Figure 2, $AB=c+17$, $BC=100$, $CD=80$.

If $EF=d$, find d. **(Reference: 1989 HG8, 1990 FG6.4)**



Let $BF=x$, then $FC=100-x$.

$\Delta BEF\sim\Delta BDC$ (equiangular)

$\Delta CEF\sim\Delta CAB$ (equiangular)

$\frac{x}{d}=\frac{100}{80}$ ……(1), $\frac{100-x}{d}=\frac{100}{3+17}$ ……(2)

(1) + (2): $\frac{100}{d}=100\cdot\left(\frac{1}{80}+\frac{1}{20}\right)\Rightarrow d=16$

Individual Event 3

I3.1 The acute angle formed by the hands of a clock at 2:15 is $\left(18\frac{1}{2}+a\right)^{\circ}$. Find a.

Reference: 1984 FG7.1, 1987 FG7.1, 1989 FI1.1, 1990 FG6.3, 2007 HI1

At 2:00, the angle between the arms of the clock = 60°

From 2:00 to 2:15, the hour-hand had moved $360°\times\frac{1}{12}\times\frac{1}{4}=7.5°$

The minute hand had moved 90°

$18.5 + a = 90 - (60 + 7.5) = 22.5$

$a = 4$

I3.2 If the sum of the coefficients in the expansion of $(x + y)^a$ is b, find b.

Put $x = 1$ and $y = 1$, then $b = (1 + 1)^4 = 16$

I3.3 If $f(x) = x - 2$, $F(x, y) = y^2 + x$ and $c = F(3, f(b))$, find c.

Reference: 1990 HI3, 2013 FI3.2, 2015 FI4.3

$f(b) = 16 - 2$

$= 14$

$c = F(3, 14)$

$= 14^2 + 3$

$= 199$

I3.4 x, y are real numbers. If $x + y = c - 195$ and d is the maximum value of xy, find d.

Reference: 1988 FI4.3

$x + y = 4$

$\Rightarrow y = 4 - x$

$xy = x(4 - x) = -(x - 2)^2 + 4 \le d$

$\Rightarrow d = 4$

Individual Event 4

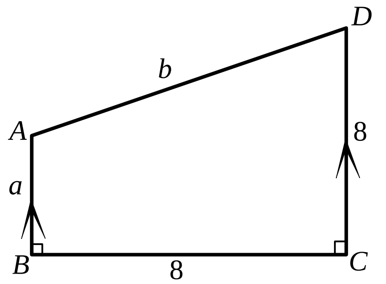
I4.1 If the lines $x + 2y + 3 = 0$ and $4x - ay + 5 = 0$ are perpendicular to each other, find a.

Reference: 1983 FG9.3, 1984 FSG.3, 1986 FSG.2, 1987 FG10.2, 1988 FG8.2

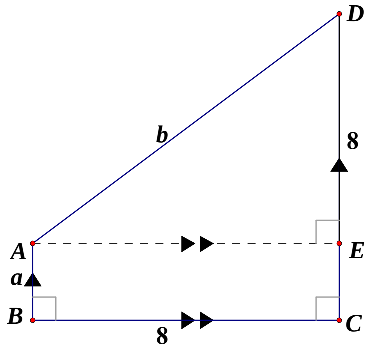
$-\frac{1}{2}\times\frac{4}{a}=-1$

$\Rightarrow a = 2$

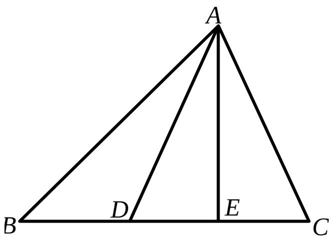
I4.2 In Figure 1, $ABCD$ is a trapezium with AB parallel to DC and $\angle ABC = \angle DCB = 90°$. If $AB = a$, $BC = CD = 8$ and $AD = b$, find b.



Draw a line segment AE // BC, cutting DC at E.
$\angle BAE = 90° = \angle AEC$ (int. $\angle$s, AE // BC)
$ABCE$ is a rectangle
$AE = 8$, $CE = a = 2$ (opp. sides, //-gram)
$DE = 8 - a = 6$
$b^2 = 8^2 + 6^2 = 100$ (Pythagoras' theorem on ΔADE)
$b = 10$



I4.3 In Figure 2, $BD = \frac{b}{2}$, $DE = 4$, $EC = 3$. If the area of ΔAEC is 24 and the area of ΔABC is c, find c.



ΔABD, ΔADE and ΔACE have the same height.

The area of $\Delta ABC = c = 24\times\frac{5+4+3}{3} = 96$

I4.4 If $3x^3 - 2x^2 + dx - c$ is divisible by $x - 1$, find d.

$3 - 2 + d - 96 = 0$

$d = 95$

Individual Event 5

I5.1 If $1 + 2 + 3 + 4 + \ldots + t = 36$, find t.

$\frac{1}{2}\cdot t(t+1) = 36$

$t = 8$ or -9 (rejected)

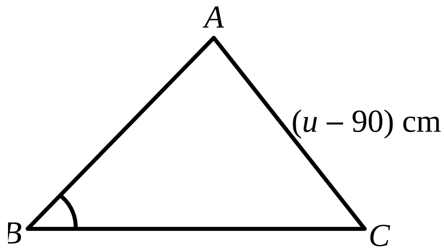
I5.2 If $\sin u° = \frac{2}{\sqrt{t}}$ and $90 < u < 180$, find u.

$\sin u° = \frac{1}{\sqrt{2}}$

$\Rightarrow u = 135$

I5.3 In Figure 1, $\angle ABC = 30°$ and $AC = (u - 90)$ cm.

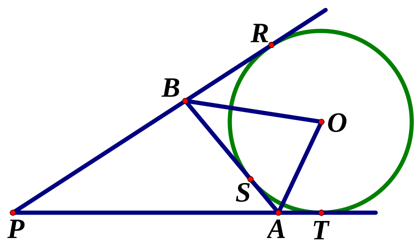
If the radius of the circumcircle of ΔABC is v cm, find v.



$\frac{135-90}{\sin 30^\circ} = 2v$ (Sine formula)

$v = 45$

I5.4 In Figure 2, ΔPAB is formed by the 3 tangents of the circle with centre O. If $\angle APB = (v - 5)°$ and $\angle AOB = w°$, find w.



$\angle APB = 40°$

$OT \perp PA$, $OS \perp AB$, $OR \perp PB$ (tangent ⊥ radius)

$\angle ROT = 360° - 40° - 90° - 90° = 140°$ (∠s sum of polygon)

$\angle ROB = \angle SOB$, $\angle TOA = \angle SOA$ (tangent from ext. pt.)

$\angle AOB = 140° \div 2 = 70°$

$\Rightarrow w = 70$

Sample Group Event (1994 Sample Group Event)

SG.1 If $a*b = ab + 1$, and $s = (2*4)*2$, find s.

Reference: 1984 FG6.4

$2*4 = 2\times 4 + 1 = 9$

$s = (2*4)*2 = 9*2$

$= 9\times 2 + 1 = 19$

SG.2 If the n^{th} prime number is s, find n.

Reference: 1989 FSG.3, 1990 FI5.4

2, 3, 5, 7, 11, 13, 17, 19

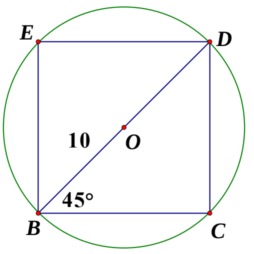
$n = 8$

SG.3 If $K = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{50}\right)$, find K in the simplest fractional form.

Reference: 1984 FG9.1, 1986 FG10.4

$K = \frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times\cdots\times\frac{49}{50} = \frac{1}{50}$

SG.4 If A is the area of a square inscribed in a circle of radius 10, find A.



Reference: 1984 FG10.1, 1989 FI3.3

Let the square be $BCDE$.

$BC = 20\cos 45° = 10\sqrt{2}$

$A = \left(10\sqrt{2}\right)^2 = 200$

Group Event 6

G6.1 The average of p, q, r is 4. The average of p, q, r, x is 5. Find x.

Reference: 1986 FG6.4, 1987 FG10.1, 1988 FG9.2

$p + q + r = 12$

$p + q + r + x = 20$

$x = 8$

G6.2 A wheel of a truck travelling at 60 km/h makes 4 revolutions per second.

If its diameter is $\frac{y}{6\pi}$ m, find y.

60 km/h $= \frac{60000}{3600}$ m/s $= \frac{50}{3}$ m/s

$\frac{y}{6\pi} \times \pi \times 4 = \frac{50}{3}$

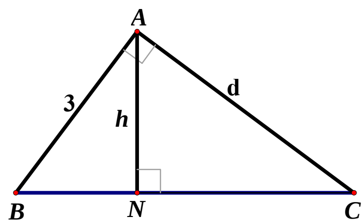
$\Rightarrow y = 25$

G6.3 If $\sin(55 - y)° = \frac{d}{x}$, find d.

$\sin 30° = \frac{d}{8} = \frac{1}{2}$

$\Rightarrow d = 4$

G6.4 In the figure, $BA \perp AC$ and $AN \perp BC$. If $AB = 3$, $AC = d$, $AN = h$, find h.



Reference: 1992 FI5.3

$BC^2 = 3^2 + 4^2$ (Pythagoras' theorem)

$\Rightarrow BC = 5$

Area of $\Delta ABC = \frac{1}{2} \cdot 5 \times h = \frac{1}{2} \cdot 3 \times 4$

$\Rightarrow h = \frac{12}{5}$

Group Event 7

G7.1 Let $M=\frac{78^3+22^3}{78^2-78\times 22+22^2}$. Find M.

Similar questions: 1984 FG6.1

$$M=\frac{78^3+22^3}{78^2-78\times 22+22^2}$$
$$=\frac{(78+22)(78^2-78\times 22+22^2)}{78^2-78\times 22+22^2}$$
$$= 100$$

G7.2 When the positive integer N is divided by 6, 5, 4, 3 and 2, the remainders are 5, 4, 3, 2 and 1 respectively. Find the least value of N.

Reference: 1990 HI13, 2013 FG4.3

$N + 1$ is divisible by 6, 5, 4, 3 and 2.

The L.C.M. of 6, 5, 4, 3 and 2 is 60.

$\therefore$ The least value of N is 59.

G7.3 A man travels 10 km at a speed of 4 km/h and another 10 km at a speed of 6 km/h. If the average speed of the whole journey is x km/h, find x.

$$x=\frac{20}{\frac{10}{4}+\frac{10}{6}}=\frac{24}{5}$$

G7.4 If $S = 1 + 2 - 3 - 4 + 5 + 6 - 7 - 8 + \dots + 1985$, find S.

Reference: 1988 FG6.4, 1990 FG10.1, 1991 FSI.1, 1992 FI1.4

$S = 1 + (2 - 3 - 4 + 5) + (6 - 7 - 8 + 9) + \dots + (1982 - 1983 - 1984 + 1985) = 1$

Group Event 8

Similar Questions 1988 FG7.1-2, 1990 FG7.3-4

M, *N* are positive integers less than 10 and 258024M8 × 9 = 2111110N × 11.

G8.1 Find *M*.

11 and 9 are relatively prime ⇒ 258024M8 is divisible by 11

$\Rightarrow 2 + 8 + 2 + M - (5 + 0 + 4 + 8)$ is divisible by 11

$\Rightarrow M - 5 = 11k$

$\Rightarrow M = 5$

G8.2 Find *N*.

2111110N is divisible by 9

$\Rightarrow 2 + 1 + 1 + 1 + 1 + 1 + N = 9t$

$\Rightarrow N = 2$

G8.3 A convex 20-sided polygon has *x* diagonals. Find *x*.
Reference: 1984 FG10.3, 1988 FG6.2, 1989 FG6.1, 1991 FI2.3, 2001 FI4.2, 2005 FI1.4

$x = C_2^{20} - 20$

$= \frac{20 \times 19}{2} - 20$

$= 170$

G8.4 If $y = ab + a + b + 1$ and $a = 99$, $b = 49$, find *y*.

Reference: 1986 FG9.3, 1988 FG6.3, 1990 FG9.2

$y = (a + 1)(b + 1)$

$= (99 + 1)(49 + 1)$

$= 5000$

Group Event 9

G9.1 The lengths of the 3 sides of ΔLMN are 8, 15 and 17 respectively.

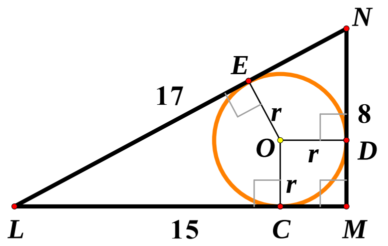
If the area of ΔLMN is A, find A.

$8^2 + 15^2 = 64 + 225 = 289 = 17^2$

$\therefore$ ΔLMN is a right-angled triangle

$A = \dfrac{8\times 15}{2} = 60$

G9.2 If r is the length of the radius of the circle inscribed in ΔLMN, find r.



Reference: 1989 HG9

Let O be the centre and the radius of the circle be r, which touches the triangle at C, D and E.

$OC \perp LM$, $OD \perp MN$, $OE \perp LN$ (tangent $\perp$ radius)

$ODMC$ is a rectangle (which consists of 3 right angles)

$OC = r = OD$ (radii)

$\Rightarrow$ $OCMD$ is a square.

$CM = MD = r$ (opp. sides, rectangle)

$LC = 15 - r$, $ND = 8 - r$

$LE = LC = 15 - r$, $NE = ND = 8 - r$ (tangent from ext. pt.)

$LE + NE = LN$

$\Rightarrow 15 - r + 8 - r = 17$

$\Rightarrow r = 3$

G9.3 If the r^{th} day of May in a year is Friday and the n^{th} day of May in the same year is Monday, where $15 < n < 25$, find n.

Reference: 1984 FG6.3, 1987 FG8.4, 1988 FG10.2

3^{rd} May is Friday

17^{th} May is Friday

$\Rightarrow 20^{\text{th}}$ May is Monday

$\Rightarrow n = 20$

G9.4 If the sum of the interior angles of an n-sided convex polygon is $x°$, find x.

$x = 180\times(20 - 2) = 3240$ ($\angle$s sum of polygon)

Group Event 10

G10.1 The sum of 3 consecutive odd integers (the smallest being k) is 51. Find k.

$k + k + 2 + k + 4 = 51$

$\Rightarrow k = 15$

G10.2 If $x^2 + 6x + k \equiv (x + a)^2 + C$, where a, C are constants, find C.

Reference: 1984 FI2.4, 1986 FG7.3, 1987 FSI.1, 1988 FG9.3

$x^2 + 6x + 15 \equiv (x + 3)^2 + 6$

$C = 6$

G10.3 If $\frac{p}{q} = \frac{q}{r} = \frac{r}{s} = 2$ and $R = \frac{p}{s}$, find R.

$R = \frac{p}{s}$

$= \frac{p}{q} \times \frac{q}{r} \times \frac{r}{s}$

$= 2^3 = 8$

G10.4 If $A = \frac{3^n \cdot 9^{n+1}}{27^{n-1}}$, find A.

$A = \frac{3^n \cdot 9^{n+1}}{27^{n-1}}$

$= \frac{3^n \cdot 3^{2n+2}}{3^{3n-3}}$

$= 3^6 = 243$

Individual Events

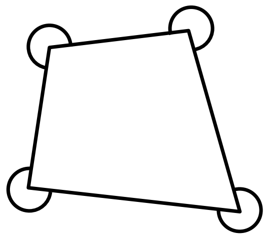
SI	***a***	1080	**I1**	***a***	6	**I2**	***h***	4	**I3**	***m***	2900	**I4**	***n***	39	**I5**	***a***	36
	b	8		***b***	2		***k***	32		***x***	8		***m***	78		***b***	48
	c	3		***c***	7		***p***	3		***y***	12		***p***	4		***p***	4
	d	64		***d***	–16		****q***	16 see the remark		***n***	33		***q***	7		***q***	6

Group Events

SG	***a***	2	**G6**	***p***	7	**G7**	***r***	2	**G8**	***A***	4	**G9**	***C***	93	**G10**	***P***	10
	b	–3		***q***	5		***s***	7		***B***	2		***n***	6		***x***	9
	p	60		***r***	–96		***a***	5		***C***	8		***S***	5000		***k***	2
	q	136		***t***	18		***p***	$\frac{1}{2}=0.5$		***D***	5		***d***	17		***S***	$\frac{11}{20}$

Sample Individual Event

SI.1 In the given figure, the sum of the four marked angles is $a°$. Find a.



Sum of interior angles of a quadrilateral = 360°

angle sum of four vertices = 4×360° = 1440°

$a = 1440 - 360 = 1080$

SI.2 The sum of the interior angles of a regular b-sided polygon is $a°$. Find b.

$180(b - 2) = 1080 = 180\times6$

$\Rightarrow b = 8$

SI.3 If $b^5 = 32^c$, find c.

$8^5 = 32^c \Rightarrow 2^{15} = 2^{5c}$

$\Rightarrow c = 3$

SI.4 If $c = \log_4 d$, find d.

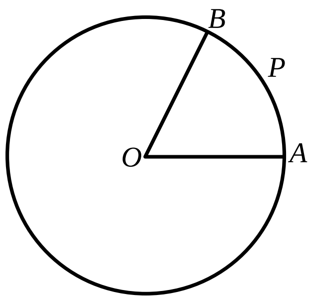
$3 = \log_4 d$

$\Rightarrow d = 4^3 = 64$

Individual Event 1

I1.1 The given figure shows a circle of radius 18 cm, centre *O*.

If $\angle AOB = \frac{\pi}{3}$ and the length of arc *APB* is $a\pi$ cm, find *a*.



$a\pi = 18\times\frac{\pi}{3}$

$a = 6$

I1.2 If the solution of the inequality $2x^2 - ax + 4 < 0$ is $1 < x < b$, find *b*.

$2x^2 - 6x + 4 < 0$

$\Rightarrow 2(x^2 - 3x + 2) < 0$

$2(x - 1)(x - 2) < 0$

$1 < x < 2$

$\Rightarrow b = 2$

I1.3 If $b(2x - 5) + x + 3 \equiv 5x - c$, find *c*.

$2(2x - 5) + x + 3 \equiv 5x - c$

$\Rightarrow c = 7$

I1.4 The line through (2, 6) and (5, *c*) cuts the *x*-axis at (*d*, 0). Find *d*.

Similar question: 1985 FI2.3

They have the same slope

$\Rightarrow \frac{0-6}{d-2} = \frac{7-6}{5-2}$

$d - 2 = -18$

$d = -16$

Individual Event 2

I2.1 If the equation $3x^2 - 4x + \frac{h}{3} = 0$ has equal roots, find h.

$\Delta = (-4)^2 - 4(3)\cdot\frac{h}{3} = 0$

$h = 4$

I2.2 If the height of a cylinder is doubled and the new radius is h times the original, then the new volume is k times the original. Find k.

Let the original radius be r, the original height be p. Then the new radius is $4r$, the new height is $2p$.

$\pi(4r)^2(2p) = k\pi r^2\cdot p$

$k = 32$

I2.3 If $\log_{10}210 + \log_{10}k - \log_{10}56 + \log_{10}40 - \log_{10}120 + \log_{10}25 = p$, find p.

$p = \log_{10}\frac{210\times 32\times 40\times 25}{56\times 120}$

$= \log_{10}(10\times 4\times 25)$

$= 3$

I2.4 If $\sin A = \frac{p}{5}$ and $\frac{\cos A}{\tan A} = \frac{q}{15}$, find q.

$\sin A = \frac{3}{5}$

$\frac{\cos A}{\tan A} = \frac{q}{15}$

$\frac{\cos^2 A}{\sin A} = \frac{q}{15}$

$\frac{1-\sin^2 A}{\sin A} = \frac{1-\left(\frac{3}{5}\right)^2}{\frac{3}{5}} = \frac{16}{15} = \frac{q}{15}$

$q = 16$

Remark: A type-writing mistake is found in the original version:

If $\sin A = \frac{p}{3}$ and $\frac{\cos A}{\tan A} = \frac{q}{15}$, find q.

Using the given condition: $\sin A = \frac{p}{3} = 1$

We arrives at the conclusion that $A = 180°n + 90°$, where n is an integer.

So that $\frac{\cos A}{\tan A}$ will be undefined because the denominator becomes undefined.

As HKMO 1990 Final Sample Individual Event is the same as HKMO 1986 Final Individual Event 2, the error is found and corrected.

Individual Event 3

I3.1 The monthly salaries of 100 employees in a company are as shown:

Sa1aries ($)	6000	4000	2500
No. of employees	5	15	80

If the mean salary is $*m*, find *m*.

$m = \dfrac{6000\times5+4000\times15+2500\times80}{5+15+80}$

$= \dfrac{290000}{100} = 2900$

I3.2 If $8\sin^2(m+10)° + 12\cos^2(m+25)° = x$, find x.

$x = 8\sin^2 2910° + 12\cos^2 2925°$

$= 8\sin^2(360\times8+30)° + 12\cos^2(360\times8+45)°$

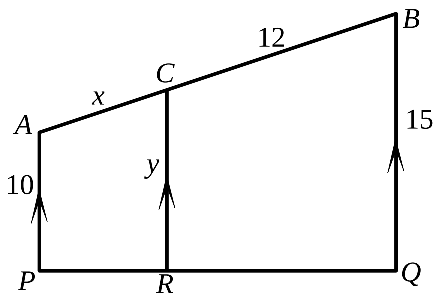
$= 8\sin^2 30° + 12\cos^2 45°$

$= 8\cdot\left(\dfrac{1}{2}\right)^2 + 12\cdot\left(\dfrac{1}{\sqrt{2}}\right)^2$

$= 8$

Remark: A mistake was found in the solution by a student, Nicholas Ng (吳庭俊). The mistake was corrected. Thanks for Mr. Ng pointing out the mistake.

I3.3 In the figure, AP // CR // BQ, $AC = x$, $CB = 12$, $AP = 10$, $BQ = 15$ and $CR = y$. Find y.



Reference: 1991 FI5.3

Draw ADE // PRQ, cutting CR at D, BQ at E respectively.

$APRD$, $APQE$ are //-grams

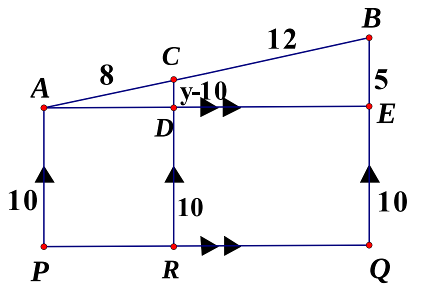
$DR = 10 = EQ$ (opp. sides, //-gram)

$CD = y - 10$, $BE = 15 - 10 = 5$

$\Delta ACD \sim \Delta ABE$ (equiangular)

$(y - 10) : 8 = 5 : 20$ (corr. of sides, ~Δ's)

$y = 12$



I3.4 Define $(a, b, c)\cdot(p, q, r) = ap + bq + cr$, where a, b, c, p, q, r are real numbers.

If $(3, 4, 5)\cdot(y, -2, 1) = n$, find n.

Reference: 1989 HI13

$n = 3y - 8 + 5$

$= 3\times12 - 3$

$= 33$

Individual Event 4

I4.1 It is known that $\begin{cases}1=1^2\\1+3=2^2\\1+3+5=3^2\\1+3+5+7=4^2\end{cases}$. If 1 + 3 + 5 + … + n = 20^2, find n.

1 + 3 + 5 + … + (2m – 1) = m^2 = 20^2

m = 20

n = 2(20) – 1 = 39

I4.2 If the lines $x + 2y = 3$ and $nx + my = 4$ are parallel, find m.

Reference: 1987 FSG.4, 1989 FSG.2

$-\frac{1}{2}=-\frac{n}{m}$

$\Rightarrow \frac{1}{2}=\frac{39}{m}$

m = 78

I4.3 If a number is selected from the whole numbers 1 to m, and if each number has an equal chance of being selected, the probability that the number is a factor of m is $\frac{p}{39}$, find p.

Let S be the sample space, $n(S)$ = 78

Favourable outcomes = {1, 2, 3, 6, 13, 26, 39, 78}

$\frac{p}{39}=\frac{8}{78}$

$\Rightarrow p = 4$

I4.4 A boy walks from home to school at a speed of p km/h and returns home along the same route at a speed of 3 km/h. If the average speed for the double journey is $\frac{24}{q}$ km/h, find q.

Let the distance of the single journey be x km.

$\frac{24}{q}=\frac{2x}{\frac{x}{4}+\frac{x}{3}}$

$\frac{24}{q}=\frac{24}{7}$

$\Rightarrow q = 7$

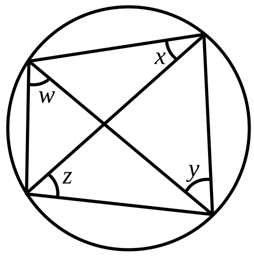
Individual Event 5

I5.1 A die is rolled. If the probability of getting a prime number is $\frac{a}{72}$, find a.

Favourable outcomes = {2, 3, 5}

$\frac{a}{72}=\frac{3}{6}\Rightarrow a = 36$

I5.2 In the figure, $x = a°$, $y = 44°$, $z = 52°$ and $w = b°$. Find b.



$x + w + y + z = 180$ (∠s in the same segment, ∠s sum of Δ)

$36 + b + 44 + 52 = 180$

$b = 48$

I5.3 A, B are two towns b km apart. Peter cycles at a speed of 7 km/h from A to B and at the same time John cycles from B to A at a speed of 5 km/h. If they meet after p hours, find p.

$p = 48÷(5 + 7) = 4$

I5.4 The base of a pyramid is a triangle with sides 3 cm, p cm and 5 cm. If the height and volume of the pyramid are q cm and 12 cm^3 respectively, find q.

$3^2 + 4^2 = 5^2 \Rightarrow$ the triangle is a right-angled triangle (Converse, Pythagoras' theorem)

Base area $=\frac{1}{2}\cdot 3\cdot 4$ cm^2 = 6 cm^2

Volume $=\frac{1}{3}\cdot 6\cdot q$ cm^3 = 12 cm^3 $\Rightarrow q = 6$

Sample Group Event

SG.1 The sum of two numbers is 50, and their product is 25.
If the sum of their reciprocals is a, find a.

Reference: 1983 FG6.3, 1984 FSG.1, 1985 FSI.1

Let the two numbers be x and y.

$x + y = 50$ and $xy = 25$

$a=\frac{1}{x}+\frac{1}{y}=\frac{x+y}{xy}=2$

SG.2 If the lines $ax + 2y + 1 = 0$ and $3x + by + 5 = 0$ are perpendicular, find b.

Reference: 1983 FG9.3, 1984 FSG.3, 1985 FI4.1, 1987 FG10.2, 1988 FG8.2

$-\frac{a}{2}\times\left(-\frac{b}{3}\right)=-1\Rightarrow b=-3$

SG.3 The area of an equilateral triangle is $100\sqrt{3}$ cm^2. If its perimeter is p cm, find p.

Reference: 1984FI4.4, 1985 FSI.4, 1987 FG6.2, 1988 FG9.1

Each side $=\frac{p}{3}$ cm

$\frac{1}{2}\cdot\left(\frac{p}{3}\right)^2\sin 60^\circ=100\sqrt{3}$

$p = 60$

SG.4 If $x^3 - 2x^2 + px + q$ is divisible by $x + 2$, find q.

$(-2)^3 - 2(-2)^2 + 60(-2) + q = 0$

$q = 136$

Group Event 6

G6.1 If $12345 \times 6789 = a \times 10^p$ where p is a positive integer and $1 \le a < 10$, find p.

$12345 \times 6789 = 1.2345 \times 10^4 \times 6.789 \times 10^3 = a \times 10^7$,

where $a = 1.2345 \times 6.789$

$\approx 1.2 \times 6.8 = 8.16$

$1 \le a < 10$

$p = 7$

G6.2 If (p, q), $(5, 3)$ and $(1, -1)$ are collinear, find q.

Reference: 1984 FSG.4, 1984 FG7.3, 1987 FG7.4, 1989 HI8

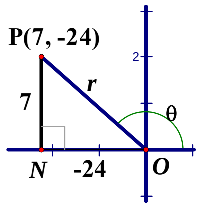
$$\frac{q-3}{7-5} = \frac{3-(-1)}{5-1}$$

$q - 3 = 2$

$\Rightarrow q = 5$

G6.3 If $\tan \theta = \frac{-7}{24}$, $90° < \theta < 180°$ and $100 \cos \theta = r$, find r.

In the figure, $r^2 = 7^2 + (-24)^2$ (Pythagoras' theorem)



$r = 25$

$r = 100 \cos \theta$

$= 100 \times \frac{-24}{25}$

$= -96$

G6.4 The average of x, y, z is 10. The average of x, y, z, t is 12. Find t.

Reference: 1985 FG6.1, 1987 FG10.1, 1988 FG9.2

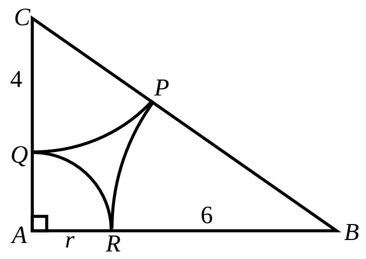
$x + y + z = 30$

$x + y + z + t = 48$

$t = 18$

Group Event 7

G7.1 In the figure, QR, RP, PQ are 3 arcs, centres at A, B, C respectively, touching one another at R, P, Q. If $AR = r$, $RB = 6$, $QC = 4$, $\angle A = 90°$, find r.



Reference: 1990 FG9.4

$AQ = r$, $CP = 4$, $BP = 6$

$AB = r + 6$, $AC = r + 4$, $BC = 4 + 6 = 10$

$AB^2 + AC^2 = BC^2$

$\Rightarrow (r + 6)^2 + (r + 4)^2 = 10^2$

$2r^2 + 20r - 48 = 0$

$r^2 + 10r - 24 = 0$

$(r - 2)(r + 12) = 0$

$r = 2$

G7.2 M, N are the points (3, 2) and (9, 5) respectively.

If $P(s, t)$ is a point on MN such that $MP : PN = 4 : r$, find s.

$MP : PN = 4 : 2 = 2 : 1$

$s = \dfrac{3\times1+9\times2}{2+1} = 7$

G7.3 $x^2 + 10x + t \equiv (x + a)^2 + k$, where t, a, k are constants. Find a.

Reference: 1984 FI2.4, 1985 FG10.2, 1987 FSI.1, 1988 FG9.3

$x^2 + 10x + t \equiv (x + 5)^2 + t - 25$

$a = 5$

G7.4 If $9^{p+2} = 240 + 9^p$, find p.

Reference: 1985 FI1.4

$81\times9^p = 240 + 9^p$

$80\times9^p = 240$

$9^p = 3$

$p = \dfrac{1}{2}$

Group Event 8

In the given multiplication, different letters represent different integers whose possible values are 2, 4, 5, 6, 7, 8, 9. (**Reference: 2000 HI8**)

G8.1 Find *A*.
G8.2 Find *B*.
G8.3 Find *C*.
G8.4 Find *D*.

$$\begin{array}{rcccccc} & 1 & A & B & C & D & E \\ \times & & & & & & 3 \\ \hline & A & B & C & D & E & 1 \end{array}$$

E = 7 and the carry digit is 20.

$$\begin{array}{rcccccc} & 1 & A & B & C & D & 7 \\ \times & & & & & & 3 \\ \hline & A & B & C & D & 7 & 1 \end{array}$$

3D + 2 ≡ 7 (mod 10)
D = 5 and the carry digit is 10

$$\begin{array}{rcccccc} & 1 & A & B & C & 5 & 7 \\ \times & & & & & & 3 \\ \hline & A & B & C & 5 & 7 & 1 \end{array}$$

3C + 1 ≡ 5 (mod 10)
C = 8 and the carry digit is 20.

$$\begin{array}{rcccccc} & 1 & A & B & 8 & 5 & 7 \\ \times & & & & & & 3 \\ \hline & A & B & 8 & 5 & 7 & 1 \end{array}$$

3B + 2 ≡ 8 (mod 10)
B = 2 and there is no carry digit.

$$\begin{array}{rcccccc} & 1 & A & 2 & 8 & 5 & 7 \\ \times & & & & & & 3 \\ \hline & A & 2 & 8 & 5 & 7 & 1 \end{array}$$

3A ≡ 2 (mod 10)
A = 4 and the carry digit is 10.
3 + 1 = 4
∴ A = 4, B = 2, C = 8, D = 5

$$\begin{array}{rcccccc} & 1 & 4 & 2 & 8 & 5 & 7 \\ \times & & & & & & 3 \\ \hline & 4 & 2 & 8 & 5 & 7 & 1 \end{array}$$

Group Event 9

G9.1 7 oranges and 5 apples cost \$13. 3 oranges and 4 apples cost \$8. 37 oranges and 45 apples cost \$*C*. Find *C*.

Let the cost of an orange be \$$x$ and the cost of an apple be \$$y$.

$7x + 5y = 13 \cdots\cdots (1)$

$3x + 4y = 8 \cdots\cdots (2)$

4(1) – 5(2): $13x = 12$

$\Rightarrow x = \dfrac{12}{13}$

7(2) – 3(1): $13y = 17$

$\Rightarrow y = \dfrac{17}{13}$

$C = 37x + 45y = \dfrac{12\times 37 + 17\times 45}{13} = 93$

G9.2 There are exactly n values of θ satisfying the equation $(\sin^2\theta - 1)(2\sin^2\theta - 1) = 0$, where $0° \le \theta \le 360°$. Find n.

$\sin\theta = \pm 1, \pm\dfrac{1}{\sqrt{2}}$

$\theta = 90°, 270°, 45°, 135°, 225°, 315°$

$n = 6$

G9.3 If $S = ab + a - b - 1$ and $a = 101$, $b = 49$, find S.

Reference: 1985 FG8.4, 1988 FG6.3, 1990 FG9.1

$S = (a - 1)(b + 1) = 100\times 50 = 5000$

G9.4 If d is the distance between the points (13, 5) and (5, –10), find d.

Reference: 1984 FG10.4

$d = \sqrt{(13-5)^2 + [5-(-10)]^2} = 17$

Group Event 10

G10.1 If $b + c = 3$ …… (1), $c + a = 6$ …… (2), $a + b = 7$ …… (3) and $P = abc$, find P.

Reference: 1989 HI15, 1990 HI7

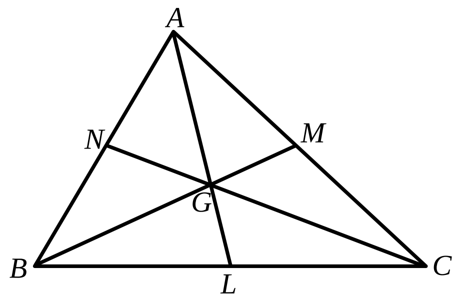
(1) + (2) – (3): $2c = 2 \Rightarrow c = 1$

(1) + (3) – (2): $2b = 4 \Rightarrow b = 2$

(2) + (3) – (1): $2a = 10 \Rightarrow a = 5$

$P = 1\times2\times5 = 10$

G10.2 The medians AL, BM, CN of ΔABC meet at G. If the area of ΔABC is 54 cm^2 and the area of ΔANG is x cm^2. Find x.



$\because BL : LC = 1 : 1$

$\therefore$ area of $\Delta ABL = \frac{1}{2}\cdot 54$ cm^2 = 27 cm^2

$\because AG : GL = 2 : 1$

$\therefore$ area of $\Delta ABG = \frac{2}{3}\cdot 27$cm^2 = 18 cm^2

$\because AN : NB = 1 : 1$

$\therefore$ area of $\Delta ANG = \frac{1}{2}\cdot 18$ cm^2 = 9 cm^2

$x = 9$

G10.3 If $k = \frac{3\sin\theta + 5\cos\theta}{2\sin\theta + \cos\theta}$ and $\tan\theta = 3$, find k.

Reference: 1987 FG8.1, 1989 FSG.4, 1989 FG10.3, 1990 FG7.2

$$k = \frac{(3\sin\theta + 5\cos\theta)\div\cos\theta}{(2\sin\theta + \cos\theta)\div\cos\theta}$$

$$= \frac{3\tan\theta + 5}{2\tan\theta + 1}$$

$$= \frac{3\times3+5}{2\times3+1} = 2$$

G10.4 If $S = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{10^2}\right)$, find S.

Reference: 1999 FIS.4, 2014 FG3.1

$$S = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{10}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{1}{10}\right)$$

$$= \left(\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times\cdots\times\frac{9}{10}\right)\times\left(\frac{3}{2}\times\frac{4}{3}\times\frac{5}{4}\times\cdots\times\frac{11}{10}\right)$$

$$= \frac{1}{10}\times\frac{11}{2}$$

$$= \frac{11}{20}$$

Individual Events

SI	a	10	I1	A	380	I2	r	3	I3	x	30	I4	a	8	I5	a	1968
	b	280		B	70		x	1		a	6		b	17		b	2
	c	400		n	60		y	15		c	8		d	287		c	25
	d	120		m	5		p	40		$f(4)$	63		K	280		d	95

Group Events

SG	A	10	G6	p	60	G7	A	75	G8	A	2	G9	A	1	G10	A	20
	B	4		k	100		B	3		B	52		B	0		B	30
	C	900		N	9		C	2		N	5		C	8		C	2
	D	18		M	999999		D	5		K	16		D	9		D	18

Sample Individual Event

SI.1 If $x^2 - 8x + 26 \equiv (x + k)^2 + a$, find a.

Reference: 1984 FI2.4, 1985 FG10.2, 1986 FG7.3, 1988 FG9.3

$x^2 - 8x + 26 \equiv (x - 4)^2 + 26 - 16$

$a = 10$

SI.2 If $\sin a° = \cos b°$, where $270 < b < 360$, find b.

$\sin 10° = \cos b°$

$\cos b° = \cos 80°$

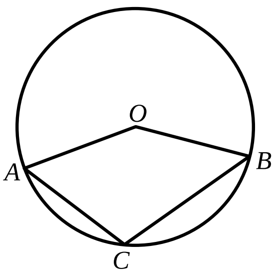
$b = 360 - 80 = 280$

SI.3 X sold an article to Y for \$$b$ at a loss of 30%. If the cost price of the article for X is \$$c$, find c.

$c\cdot(1 - 30\%) = 280$

$c = 400$

SI.4 In the figure, O is the centre of the circle. If $\angle ACB = \frac{3c°}{10}$ and $\angle AOB = d°$, find d.



$\angle ACB = 120°$

reflex $\angle AOB = 240°$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

$\angle AOB = 120°$ ($\angle$s at a point)

$d = 120$

Individual Event 1

I1.1 If $A = 11 + 12 + 13 + \dots + 29$, find A.

$$A = \frac{1}{2}(11 + 29)\cdot 19 = 380$$

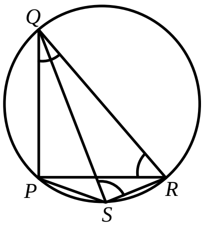
I1.2 If $\sin A° = \cos B°$, where $0 < B < 90$, find B.

$\sin 380° = \cos B°$

$\sin 20° = \cos B°$

$B = 70$

I1.3 In the given figure, $\angle PQR = B°$, $\angle PRQ = 50°$. If $\angle QSR = n°$, find n.



$\angle PQR = 70°$

$\angle QPR = 60°$ ($\angle$s sum of Δ)

$n = 60$ ($\angle$s in the same segment)

I1.4 n cards are marked from 1 to n and one is drawn at random. If the chance of it being a multiple of 5 is $\frac{1}{m}$, find m.

Favourable outcome = {5, 10, … , 60}

$$\frac{1}{m} = \frac{12}{60} = \frac{1}{5}$$

$\Rightarrow m = 5$

Individual Event 2

I2.1 The volume of a sphere with radius r is 36π, find r.

$\frac{4\pi}{3}r^3 = 36\pi$

$r = 3$

I2.2 If $r^x + r^{1-x} = 4$ and $x > 0$, find x.

$3^x + \frac{3}{3^x} = 4$ (It is straight forward by guessing $x = 1$)

$(3^x)^2 - 4\cdot 3^x + 3 = 0$

$(3^x - 1)(3^x - 3) = 0$

$3^x = 1$ or 3

$x = 0$ (rejected, as $x > 0$) or 1

I2.3 In $a : b = 5 : 4$, $b : c = 3 : x$ and $a : c = y : 4$, find y.

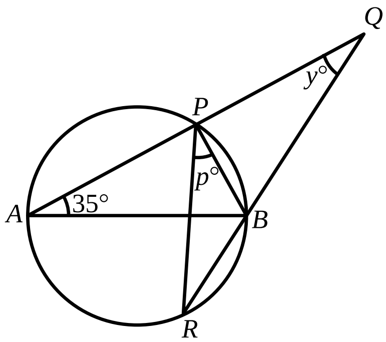
$a : b : c = 15 : 12 : 4$

$a : c = 15 : 4$

$\Rightarrow y = 15$

I2.4 In the figure, AB is a diameter of the circle. APQ and RBQ are straight lines.

If $\angle PAB = 35°$, $\angle PQB = y°$ and $\angle RPB = p°$, find p.



$\angle ABR = 35° + y° = 50°$ (ext. $\angle$ of Δ)

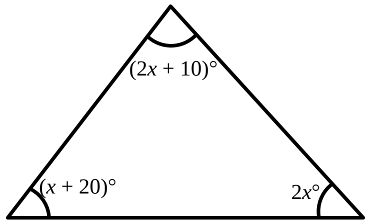
$\angle APR = \angle ABR = 50°$ ($\angle$s in the same segment)

$p + 50 = 90$ ($\angle$ in semi-circle)

$p = 40$

Individual Event 3

I3.1 In the figure, find x.



$x + 20 + 2x + 10 + 2x = 180$ (∠s sum of Δ)

$x = 30$

I3.2 The coordinates of the points P and Q are $(a, 2)$ and $(x, -6)$ respectively.
If the coordinates of the mid-point of PQ is $(18, b)$, find a.

$\frac{1}{2}(a + 30) = 18$

$a = 6$

I3.3 A man travels from X to Y at a uniform speed of a km/h and returns at a uniform speed of $2a$ km/h. If his average speed is c km/h, find c.

Let the distance between X and Y be s km.

$c = \frac{2s}{\frac{s}{a} + \frac{s}{2a}} = \frac{2}{\frac{1}{6} + \frac{1}{12}} = 8$

I3.4 If $f(y) = 2y^2 + cy - 1$, find $f(4)$.

$f(4) = 2(4)^2 + 8(4) - 1 = 63$

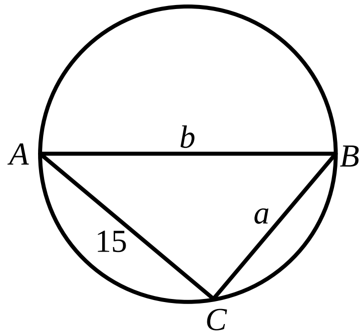
Individual Event 4

I4.1 If the curve $y = 2x^2 - 8x + a$ touches the x-axis, find a.

$\Delta = (-8)^2 - 4(2)a = 0$

$a = 8$

I4.2 In the figure, AB is a diameter of the circle.
If $AC = 15$, $BC = a$ and $AB = b$, find b.



$\angle ACB = 90°$ (∠ in semi-circle)

$b^2 = 15^2 + 8^2$ (Pythagoras' theorem)

$b = 17$

I4.3 The line $5x + by + 2 = d$ passes through $(40, 5)$. Find d.

Reference: 1984 FI2.3

$d = 5(40) + 17(5) + 2 = 287$

I4.4 X sold an article to Y for $\$d$ at a profit of 2.5%.
If the cost price of the article for X is $\$K$, find K.

$K = 287 \div (1 + 2.5\%) = 280$

Individual Event 5

I5.1 Let $x = 19.\dot{8}\dot{7}$. If $19.\dot{8}\dot{7} = \frac{a}{99}$, find a. (Hint: $99x = 100x - x$)

$99x = 100x - x = 1987 + 0.\dot{8}\dot{7} - 19.\dot{8}\dot{7} = 1968$

$x = \frac{1968}{99}$

$\Rightarrow a = 1968$

I5.2 If $f(y) = 4 \sin y°$ and $f(a - 18) = b$, find b.

$b = f(a - 18) = f(1950)$

$= 4 \sin 1950°$

$= 4 \sin(360°\times 5 + 150°)$

$= 4 \sin 150° = 2$

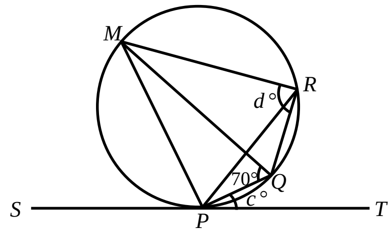
I5.3 If $\frac{\sqrt{3}}{b\sqrt{7} - \sqrt{3}} = \frac{2\sqrt{21} + 3}{c}$, find c.

$\frac{\sqrt{3}}{2\sqrt{7} - \sqrt{3}} \cdot \frac{2\sqrt{7} + \sqrt{3}}{2\sqrt{7} + \sqrt{3}} = \frac{2\sqrt{21} + 3}{c}$

$c = 4(7) - 3 = 25$

I5.4 In the figure, ST is a tangent to the circle at P.

If $\angle MQP = 70°$, $\angle QPT = c°$ and $\angle MRQ = d°$, find d.



$\angle MRP = 70°$ ($\angle$s in the same segment)

$\angle PRQ = c° = 25°$ ($\angle$ in alt. segment)

$d = 70 + 25 = 95$

Sample Group Event

SG.1 If $100A = 35^2 - 15^2$, find A.

Reference: 1984 FI1.1

$100A = (35 - 15)(35 + 15) = 1000$

$A = 10$

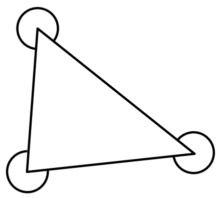
SG.2 If $(A - 1)^6 = 27^B$, find B.

$(10 - 1)^6 = 27^B$

$3^{12} = 3^{3B}$

$\Rightarrow B = 4$

SG.3 In the given diagram, the sum of the three marked angles is $C°$. Find C.



Reference: 1984 FSI.1, 1989 FSI.1

Sum of interior angles of a triangle = 180°

angle sum of three vertices = 3×360° = 1080°

$C = 1080 - 180 = 900$

SG.4 If the lines $x + 2y + 1 = 0$ and $9x + Dy + 1 = 0$ are parallel, find D.

Reference: 1986 FI4.2, 1989 FSG.2

$-\frac{1}{2} = -\frac{9}{D}$

$\Rightarrow D = 18$

Group Event 6

G6.1 If α, β are the roots of $x^2 - 10x + 20 = 0$, and $p = \alpha^2 + \beta^2$, find p.

$\alpha + \beta = 10$, $\alpha\beta = 20$

$p = (\alpha + \beta)^2 - 2\alpha\beta$

$= 10^2 - 2(20) = 60$

G6.2 The perimeter of an equilateral triangle is p. If its area is $k\sqrt{3}$, find k.

Reference: 1984FI4.4, 1985 FSI.4, 1986 FSG.3, 1988 FG9.1

Length of one side = 20

$\frac{1}{2}\cdot 20^2 \sin 60^\circ = k\sqrt{3}$

$k = 100$

G6.3 Each interior angle of an N-sided regular polygon is 140°. Find N.

Reference: 1997 FI4.1

Each exterior angle = 40° (adj. ∠s on st. line)

$\frac{360^\circ}{N} = 40^\circ$ (sum of ext. ∠s of polygon)

$\Rightarrow N = 9$

G6.4 If $M = (10^2 + 10\times1 + 1^2)(10^2 - 1^2)(10^2 - 10\times1 + 1^2)$, find M.

$M = (10^2 + 10\times1 + 1^2)(10 - 1)(10 + 1)(10^2 - 10\times1 + 1^2)$

$= (10^3 - 1)(10^3 + 1)$

$= 10^6 - 1 = 999999$

Group Event 7

G7.1 The acute angle formed by the hands of a clock at 3:30 p.m. is $A°$. Find A.

Reference 1984 FG7.1, 1985 FI3.1, 1989 FI1.1, 1990 FG6.3, 2007 HI1

At 3:00 p.m., the angle between the arms of the clock = 90°

From 3:00 p.m. to 3:30 p.m., the hour-hand had moved $360°\times\frac{1}{12}\times\frac{1}{2}=15°$.

The minute hand had moved 180°.

$A = 180 - 90 - 15 = 75$

G7.2 If $\tan(3A + 15)° = \sqrt{B}$, find B.

$\tan(225 + 15)° = \sqrt{B}$

$\Rightarrow B = 3$

G7.3 If $\log_{10}AB = C \log_{10}15$, find C.

$\log_{10} (75\times3) = C \log_{10} 15$

$\log_{10} 225 = C \log_{10} 15$

$\Rightarrow C = 2$

G7.4 The points (1, 3), (4, 9) and (2, D) are collinear. Find D.

Reference: 1984 FSG.4, 1984 FG7.3, 1986 FG6.2, 1989 HI8

$$\frac{D-9}{2-4}=\frac{9-3}{4-1}$$

$D - 9 = -4$

$\Rightarrow D = 5$

Group Event 8

G8.1 If $A=\frac{5\sin\theta+4\cos\theta}{3\sin\theta+\cos\theta}$ and $\tan\theta = 2$, find A.

Reference: 1986 FG10.3, 1989 FSG.4, 1989 FG10.3, 1990 FG7.2

$A=\frac{(5\sin\theta+4\cos\theta)\div\cos\theta}{(3\sin\theta+\cos\theta)\div\cos\theta}$

$=\frac{5\tan\theta+4}{3\tan\theta+1}$

$=\frac{5(2)+4}{3(2)+1}=2$

G8.2 If $x+\frac{1}{x}=2A$, and $x^3+\frac{1}{x^3}=B$, find B.

Reference: 1983 FG7.3, 1984 FG10.2, 1985 FI1.2, 1989 HI1, 1990 HI12, 2002 FG2.2

$x+\frac{1}{x}=4$

$\Rightarrow x^2+\frac{1}{x^2}=4^2-2=14$

$B=x^3+\frac{1}{x^3}$

$=\left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right)$

$=4(14-1)=52$

G8.3 There are exactly N values of α satisfying the equation $\cos^3\alpha - \cos\alpha = 0$, where $0°\le\alpha\le 360°$. Find N.

$\cos\alpha(\cos\alpha+1)(\cos\alpha-1)=0$

$\cos\alpha = 0, -1$ or 1

$\alpha = 90, 270, 180, 0, 360$

$\Rightarrow N = 5$

G8.4 If the N^{th} day of May in a year is Thursday and the K^{th} day of May in the same year is Monday, where $10 < K < 20$, find K.

Reference: 1984 FG6.3, 1985 FG9.3, 1988 FG10.2

5^{th} May is Thursday

9^{th} May is Monday

16^{th} May is Monday

$\Rightarrow K = 16$

Group Event 9

In the given multiplication, different letters represent different integers ranging from 0 to 9.

	A	B	C	D
×				9
	D	C	B	A

G9.1 Find *A*.
G9.2 Find *B*.
G9.3 Find *C*.
G9.4 Find *D*.

Reference: 1994 HI6

	1	B	C	9
×				9
	9	C	B	1

As there is no carry digit in the thousands digit multiplication, A = 1, D = 9

Consider the tens digit: 9C + 8 ≡ B (mod 10) …… (1)

As there is no carry digit in the thousands digit, let the carry digit in the hundreds digit be x.

9B + x = C and B, C are distinct integers different from 1 and 9

⇒ B = 0, C = x

Sub. B = 0 into (1): 9C + 8 ≡ 0 (mod 10)

⇒ 9C ≡ 2 (mod 10)

⇒ C = 8

∴ A = 1, B = 0, C = 8, D = 9

Group Event 10

G10.1 The average of p, q, r and s is 5. The average of p, q, r, s and A is 8. Find A.

Reference: 1985 FG6.1, 1986 FG6.4, 1988 FG9.2

$p + q + r + s = 20$

$p + q + r + s + A = 40$

$A = 20$

G10.2 If the lines $3x - 2y + 1 = 0$ and $Ax + By + 1 = 0$ are perpendicular, find B.

Reference: 1983 FG9.3, 1984 FSG.3, 1985 FI4.1, 1986 FSG.2, 1988 FG8.2

$\frac{3}{2}\times\left(-\frac{20}{B}\right) = -1 \Rightarrow B = 30$

G10.3 When $Cx^3 - 3x^2 + x - 1$ is divided by $x + 1$, the remainder is –7. Find C.

$C(-1) - 3 - 1 - 1 = -7$

$C = 2$

G10.4 If P, Q are positive integers such that $P + Q + PQ = 90$ and $D = P + Q$, find D.

(Hint: Factorise $1 + P + Q + PQ$)

Reference: 2002 HG9, 2012 FI4.2

WLOG assume $P \le Q$, $1 + P + Q + PQ = 91$

$(1 + P)(1 + Q) = 1\times91 = 7\times13$

$1 + P = 1 \Rightarrow P = 0$ (rejected)

or $1 + P = 7 \Rightarrow P = 6$

$1 + Q = 13 \Rightarrow Q = 12$

$D = 6 + 12 = 18$

Individual Events

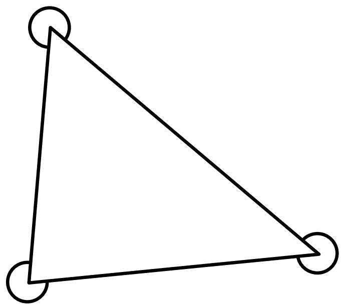
SI	***a***	900	**I1**	***P***	100	**I2**	***k***	4	**I3**	***h***	3	**I4**	***a***	18	**I5**	***a***	495
	b	7		***Q***	8		***m***	58		***k***	6		***r***	3		***b***	2
	p	2		***R***	50		***a***	2		***m***	4		***M***	$\frac{9}{4}$		***x***	99
	q	9		***S***	3		***b***	3		***p***	$\frac{15}{16}$		***w***	1		***Y***	109

Group Events

SG	***p***	75	**G6**	***x***	125	**G7**	***M***	5	**G8**	***S***	27	**G9**	***p***	60	**G10**	***n***	18
	q	0.5		***n***	10		***N***	6		***T***	135		***t***	10		***k***	22
	a	9		***y***	1000		***a***	8		***A***	9		***K***	43		***t***	96
	m	14		***K***	1003		***k***	4		***B***	0		***C***	9		***h***	$\frac{168}{25}$

Sample Individual Event (1984 Sample Individual Event)

SI.1 In the given diagram, the sum of the three marked angles is $a°$. Find a.



Sum of interior angles of a triangle = 180°

angle sum of three vertices = 3×360° = 1080°

$a = 1080 - 180 = 900$

SI.2 The sum of the interior angles of a regular b-sided polygon is $a°$. Find b.

$a = 900 = 180\times(b - 2)$

$b = 7$

SI.3 If $8^b = p^{21}$, find p.

$8^7 = p^{21}$

$2^{21} = p^{21}$

$\Rightarrow p = 2$

SI.4 If $p = \log_q 81$, find q.

$2 = p = \log_q 81$ and $q > 0$

$q^2 = 81$

$\Rightarrow q = 9$

Individual Event 1

I1.1 If $N(t) = 100\times18^t$ and $P = N(0)$, find P.

$P = 100\times18^0 = 100$

I1.2 A fox ate P grapes in 5 days, each day eating 6 more than on the previous day.

If he ate Q grapes on the first day, find Q.

$Q + (Q + 6) + (Q + 12) + (Q + 18) + (Q + 24) = P = 100$

$5Q + 60 = 100$

$\Rightarrow Q = 8$

I1.3 If Q% of $\frac{25}{32}$ is $\frac{1}{Q}$% of R, find R.

$\frac{25}{32}\times\frac{8}{100} = R\times\frac{1}{100\times8}$

$\Rightarrow R = 50$

I1.4 If one root of the equation $3x^2 - ax + R = 0$ is $\frac{50}{9}$ and the other root is S, find S.

$\frac{50}{9}\times S = \text{product of roots} = \frac{R}{3} = \frac{50}{3}$

$\Rightarrow S = 3$

Individual Event 2

I2.1 If $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ and $\begin{vmatrix} 3 & 4 \\ 2 & k \end{vmatrix} = k$, find k.

$3k - 8 = k$

$\Rightarrow k = 4$

I2.2 If $50m = 54^2 - k^2$, find m.

Reference: 1984 FI1.1, 1987 FSG.1

$50m = 54^2 - 4^2 = (54 + 4)(54 - 4) = 58\times50$

$\Rightarrow m = 58$

I2.3 If $(m + 6)^a = 2^{12}$, find a.

$(58 + 6)^a = 2^{12}$

$\Rightarrow 64^a = 2^{12}$

$\Rightarrow 2^{6a} = 2^{12}$

$\Rightarrow a = 2$

I2.4 A, B and C are the points $(a, 5)$, $(2, 3)$ and $(4, b)$ respectively. If $AB \perp BC$, find b.

$A(2, 5)$, $B(2, 3)$, $C(4, b)$.

AB is parallel to y-axis

$\Rightarrow BC$ is parallel to x-axis

$\Rightarrow b = 3$

Individual Event 3

I3.1 If $\frac{\sqrt{3}}{2\sqrt{7}-\sqrt{3}}=\frac{2\sqrt{21}+h}{25}$, find h.

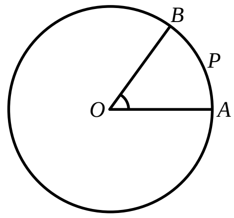
$\frac{\sqrt{3}}{2\sqrt{7}-\sqrt{3}}\cdot\frac{2\sqrt{7}+\sqrt{3}}{2\sqrt{7}+\sqrt{3}}=\frac{2\sqrt{21}+h}{25}$

$2\sqrt{21}+3=2\sqrt{21}+h$

$\Rightarrow h = 3$

I3.2 The given figure shows a circle of radius $2h$ cm, centre O.

If $\angle AOB=\frac{\pi}{3}$, and the area of sector $AOBP$ is $k\pi$ cm^2, find k.



$\frac{1}{2}\cdot(2\cdot3)^2\cdot\frac{\pi}{3}=k\pi$

$k = 6$

I3.3 A can do a job in k days, B can do the same job in $(k + 6)$ days.
If they work together, they can finish the job in m days. Find m.

$\frac{1}{m}=\frac{1}{k}+\frac{1}{k+6}$

$\Rightarrow\frac{1}{m}=\frac{1}{6}+\frac{1}{12}$

$\Rightarrow m = 4$

I3.4 m coins are tossed. If the probability of obtaining at least one head is p, find p.

P(at least one head) = 1 – P(all tail)

$=1-\left(\frac{1}{2}\right)^4=\frac{15}{16}$

Individual Event 4

I4.1 If $f(t) = 2-\frac{t}{3}$, and $f(a) = -4$, find a.

$f(a) = 2-\frac{a}{3}=-4$

$\Rightarrow a = 18$

I4.2 If $a + 9 = 12Q + r$, where Q, r are integers and $0 < r < 12$, find r.

$18 + 9 = 27 = 12\times2 + 3 = 12Q + r$

$r = 3$

I4.3 x, y are real numbers. If $x + y = r$ and M is the maximum value of xy, find M.

Reference: 1985 FI3.4

$x + y = 3$

$\Rightarrow y = 3 - x$

$xy = x(3 - x) = 3x - x^2 = -(x - 1.5)^2 + 2.25$

$M = 2.25 = \frac{9}{4}$

I4.4 If w is a real number and $2^{2w}-2^w-\frac{8}{9}M=0$, find w.

$2^{2w}-2^w-\frac{8}{9}\cdot\frac{9}{4}=0$

$\Rightarrow (2^w)^2-2^w-2=0$

$(2^w+1)(2^w-2)=0$

$\Rightarrow w = 1$

Individual Event 5

I5.1 If $0.3\dot{5}\dot{7}=\frac{177}{a}$, find a.

$$0.3\dot{5}\dot{7}=\frac{3}{10}+0.0\dot{5}\dot{7}$$
$$=\frac{3}{10}+\frac{57}{990}$$
$$=\frac{297+57}{990}$$
$$=\frac{354}{990}=\frac{59}{165}=\frac{177}{495}$$
$$=\frac{177}{a}$$

$a = 495$

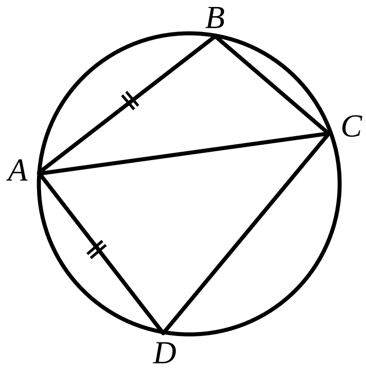
I5.2 If $\tan^2 a° + 1 = b$, find b.

$b = \tan^2 495° + 1$

$= \tan^2(180°\times3 - 45°) + 1$

$= 1 + 1 = 2$

I5.3 In the figure, $AB = AD$, $\angle BAC = 26° + b°$, $\angle BCD = 106°$.
If $\angle ABC = x°$, find x.



$\angle BCA = \angle DCA = \frac{1}{2}\angle BCD = 53°$ (eq. chords eq. ∠s)

$\angle BAC = 28°$

$x° = \angle ABC = 180° - 28° - 53° = 99°$ (∠s sum of Δ)

$x = 99$

I5.4 If $\begin{pmatrix} h & k \end{pmatrix}\begin{pmatrix} m & p \\ n & q \end{pmatrix}=\begin{pmatrix} hm+kn & hp+kq \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & x \\ 4 & 5 \end{pmatrix}=\begin{pmatrix} 11 & Y \end{pmatrix}$, find Y.

$\begin{pmatrix} 1\times3+2\times4 & x+2\times5 \end{pmatrix}=\begin{pmatrix} 11 & Y \end{pmatrix}$

$\Rightarrow Y = 109$

Sample Group Event (1984 Group Event 7)

SG.1 The acute angle between the 2 hands of a clock at 3:30 p.m. is $p°$. Find p.

At 3:00 p.m., the angle between the arms of the clock = 90°

From 3:00 p.m. to 3:30 p.m., the hour-hand had moved $360° \times \frac{1}{12} \times \frac{1}{2} = 15°$.

The minute hand had moved 180°.

$p = 180 - 90 - 15 = 75$

SG.2 In ΔABC, $\angle B = \angle C = p°$. If $q = \sin A$, find q.

$\angle B = \angle C = 75°$, $\angle A = 180° - 75° - 75° = 30°$

$q = \sin 30° = \frac{1}{2}$

SG.3 The 3 points (1, 3), (2, 5), (4, a) are collinear. Find a.

$\frac{9-5}{4-2} = \frac{a-3}{4-1} = 2$

$\Rightarrow a = 9$

SG.4 The average of 7, 9, x, y,17 is 10. If the average of x +3, x +5, y + 2, 8 and y + 18 is m, find m.

$\frac{7+9+x+y+17}{5} = 10$

$\Rightarrow x + y = 17$

$m = \frac{x+3+x+5+y+2+8+y+18}{5}$

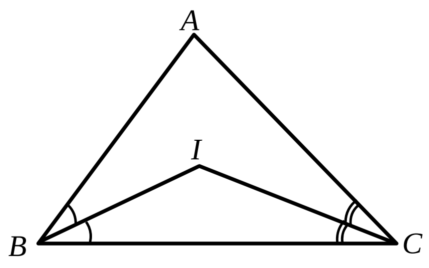
$= \frac{2(x+y)+36}{5}$

$= 14$

Group Event 6

G6.1 In the figure, the bisectors of $\angle B$ and $\angle C$ meet at I.

If $\angle A = 70°$ and $\angle BIC = x°$, find x.



Let $\angle ABI = b = \angle CBI$, $\angle ACI = c = \angle BCI$

$2b + 2c + 70 = 180$ ($\angle$s sum of Δ)

$b + c = 55$

In ΔBCI, $b + c + x = 180$ ($\angle$s sum of Δ)

$x = 180 - 55 = 125$

G6.2 A convex n-sided polygon has 35 diagonals. Find n.

Reference: 1984 FG10.3, 1985 FG8.3, 1989 FG6.1, 1991 FI2.3, 2001 FI4.2, 2005 FI1.4

$C_2^n - n = 35$

$\Rightarrow \frac{n(n-3)}{2} = 35$

$n^2 - 3n - 70 = 0$

$\Rightarrow (n - 10)(n + 7) = 0$

$n = 10$

G6.3 If $y = ab - a + b - 1$ and $a = 49$, $b = 21$, find y.

Reference: 1985 FG8.4, 1986 FG9.3, 1990 FG9.1

$y = (a + 1)(b - 1) = (49 + 1)(21 - 1) = 50\times 20 = 1000$

G6.4 If $K = 1 + 2 - 3 - 4 + 5 + 6 - 7 - 8 + \dots + 1001 + 1002$, find K.

Reference: 1985 FG7.4, 1990 FG10.1, 1991 FSI.1, 1992 FI1.4

$K = 1 + (2 - 3 - 4 + 5) + (6 - 7 - 8 + 9) + \dots + (998 - 999 - 1000 + 1001) + 1002 = 1003$

Group Event 7 (**Similar Questions 1985 FG8.1-2, 1990 FG7.3-4**)

M, N are positive integers less than 10 and $8M420852 \times 9 = N9889788 \times 11$.

G7.1 Find M.

11 and 9 are relatively prime

$\Rightarrow 8M420852$ is divisible by 11

$\Rightarrow 8 + 4 + 0 + 5 - (M + 2 + 8 + 2)$ is divisible by 11

$\Rightarrow 5 - M = 11k$

$\Rightarrow M = 5$

G7.2 Find N.

$N9889788$ is divisible by 9

$\Rightarrow N + 9 + 8 + 8 + 9 + 7 + 8 + 8 = 9t$

$\Rightarrow N = 6$

G7.3 The equation of the line through (4, 3) and (12, –3) is $\frac{x}{a}+\frac{y}{b}=1$. Find a.

$$\frac{y-3}{x-4}=\frac{3-(-3)}{4-12}$$

$3x - 12 + 4y - 12 = 0$

$\Rightarrow 3x + 4y = 24$

$$\frac{x}{8}+\frac{y}{6}=1$$

$\Rightarrow a = 8$

G7.4 If $x + k$ is a factor of $3x^2 + 14x + a$, find k. (k is an integer.)

$3(-k)^2 + 14(-k) + 8 = 0$

$\Rightarrow 3k^2 - 14k + 8 = 0$

$(3k - 2)(k - 4) = 0$

$\Rightarrow k = 4$ (reject $\frac{2}{3}$)

Group Event 8

G8.1 If $\log_9 S = \frac{3}{2}$, find S.

$S = 9^{3/2} = 27$

G8.2 If the lines $x + 5y = 0$ and $Tx - Sy = 0$ are perpendicular to each other, find T.

Reference: 1983 FG9.3, 1984 FSG.3, 1985 FI4.1, 1986 FSG.2, 1987 FG10.2

$-\frac{1}{5} \times \frac{T}{27} = -1$

$T = 135$

The 3-digit number AAA, where $A \neq 0$, and the 6-digit number $AAABBB$ satisfy the following equality: $AAA \times AAA + AAA = AAABBB$.

G8.3 Find A.

$A(111) \times A(111) + A(111) = A(111000) + B(111)$

$111A^2 + A = 1000A + B$

Consider the thousands digit: $9 < A^2 \leq 81$

$\Rightarrow A = 4, 5, 6, 7, 8, 9$

When $A = 4$: $111\times16 + 4 = 4000 + B$ (rejected)

When $A = 5$: $111\times25 + 5 = 5000 + B$ (rejected)

When $A = 6$: $111\times36 + 6 = 6000 + B$ (rejected)

When $A = 7$: $111\times49 + 7 = 7000 + B$ (rejected)

When $A = 8$: $111\times64 + 8 = 8000 + B$ (rejected)

When $A = 9$: $111\times81 + 9 = 9000 + B$

$\therefore A = 9$

G8.4 Find B.

$B = 0$

Group Event 9

G9.1 The area of an equilateral triangle is $50\sqrt{12}$. If its perimeter is *p*, find *p*.

Reference: 1984FI4.4, 1985 FSI.4, 1986 FSG.3, 1987 FG6.2

Each side $= \frac{p}{3}$

$$\frac{1}{2}\cdot\left(\frac{p}{3}\right)^2 \sin 60^\circ = 50\sqrt{12} = 100\sqrt{3}$$

$p = 60$

G9.2 The average of *q*, *y*, *z* is 14. The average of *q*, *y*, *z*, *t* is 13. Find *t*.

Reference: 1985 FG6.1, 1986 FG6.4, 1987 FG10.1

$$\frac{q+y+z}{3} = 14$$

$\Rightarrow q + y + z = 42$

$$\frac{q+y+z+t}{4} = 13$$

$$\Rightarrow \frac{42+t}{4} = 13$$

$t = 10$

G9.3 If $7 - 24x - 4x^2 \equiv K + A(x + B)^2$, where *K*, *A*, *B* are constants, find *K*.

Reference: 1984 FI2.4, 1985 FG10.2, 1986 FG7.3, 1987 FSI.1

$7 - 24x - 4x^2 \equiv -4(x^2 + 6x) + 7 \equiv -4(x + 3)^2 + 43$

$K = 43$

G9.4 If $C = \frac{3^{4n}\cdot 9^{n+4}}{27^{2n+2}}$, find *C*.

$$C = \frac{3^{4n}\cdot 3^{2n+8}}{3^{6n+6}} = 9$$

Group Event 10

G10.1 Each interior angle of an n-sided regular polygon is 160°. Find n.

Each exterior angle = 20° (adj. ∠s on st. line)

$\frac{360^\circ}{n} = 20^\circ$

$\Rightarrow n = 18$

G10.2 The n^{th} day of May in a year is Friday. The k^{th} day of May in the same year is Tuesday, where $20 < k < 26$. Find k.

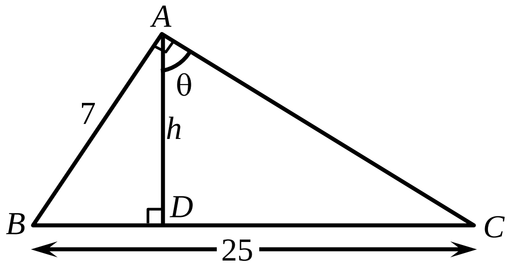
Reference: 1984 FG6.3, 1985 FG9.3, 1987 FG8.4

18^{th} May is Friday

22^{nd} May is Tuesday

$\Rightarrow k = 22$

In the figure, $AD \perp BC$, $BA \perp CA$, $AB = 7$, $BC = 25$, $AD = h$ and $\angle CAD = \theta$.



G10.3 If $100 \sin \theta = t$, find t.

$AC^2 + 7^2 = 25^2$ (Pythagoras' theorem)

$AC = 24$

$\angle ACD = 90° - \theta$ (∠s sum of Δ)

$\angle ABC = \theta$ (∠s sum of Δ)

$t = 100 \sin \theta = 100 \times \frac{24}{25} = 96$

G10.4 Find h.

Area of $\Delta ABC = \frac{1}{2} \cdot 7 \times 24 = \frac{1}{2} \cdot 25h$

$h = \frac{168}{25}$

Method 2

In ΔABD,

$h = AB \sin \theta$

$= 7 \times \frac{24}{25} = \frac{168}{25}$

Individual Events

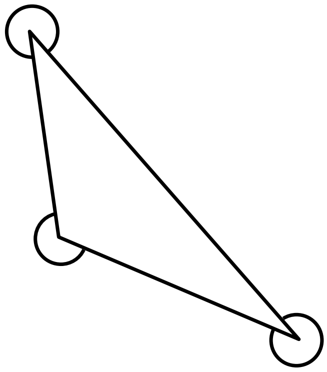
SI	***a***	900	**I1**	***a***	35	**I2**	***a***	7	**I3**	α	8	**I4**	***t***	13	**I5**	***a***	30
	b	7		***b***	7		***b***	3		***b***	16		***s***	4		***b***	150
	c	3		***c***	10		***c***	9		***A***	128		***a***	3		***n***	12
	d	5		***d***	2		***d***	5		***d***	7		***c***	12		***k***	24

Group Events

SG	***a***	2	**G6**	***n***	8	**G7**	***G***	1	**G8**	***y***	7	**G9**	***x***	40	**G10**	***a***	6
	b	9		***k***	5		***D***	8		***k***	–96		***y***	3		***x***	3
	p	23		***u***	35		***L***	2		***a***	1		***k***	8		***k***	2
	k	3		***a***	1		***E***	5		***m***	2		***r***	5		***y***	4

Sample Individual Event

SI.1 In the given diagram, the sum of the three marked angles is $a°$. Find the value of a.



Reference: 1984 FSI.1 1987 FSG.3

Sum of interior angles of a triangle = 180°

angle sum of three vertices = 3×360° = 1080°

$a = 1080 - 180 = 900$

SI.2 The sum of the interior angles of a convex b-sided polygon is $a°$. Find the value of b.

Reference 1984 FSI.2

$a = 900 = 180\times(b - 2)$

$b = 7$

SI.3 If $27^{b-1} = c^{18}$, find the value of c.

$3^{3(7-1)} = c^{18}$

$c = 3$

SI.4 If $c = \log_d 125$, find the value of d.

$3 = c = \log_d 125$

$d^3 = 125$

$d = 5$

Individual Event 1

I1.1 The obtuse angle formed by the hands of a clock at 10:30 is $(100 + a)°$. Find the value of a.

Reference 1984 FG7.1, 1985 FI3.1, 1987 FG7.1, 1990 FG6.3, 2007 HI1

At 10:00, the angle between the arms of the clock = 60°

From 10:00 to 10:30, the hour-hand had moved $360°\times\frac{1}{12}\times\frac{1}{2} = 15°$.

The minute hand had moved 180°.

$100 + a = 180 - 60 + 15 = 135 \Rightarrow a = 35$

I1.2 The lines $ax + by = 0$ and $x - 5y + 1 = 0$ are perpendicular to each other. Find the value of b.

$-\frac{35}{b}\times\frac{1}{5}=-1$

$\Rightarrow b = 7$

I1.3 If $(b + 1)^4 = 2^{c+2}$, find the value of c.

$8^4 = 2^{c+2}$

$2^{3(4)} = 2^{c+2}$

$\Rightarrow c = 10$

I1.4 If $c - 9 = \log_c (6d - 2)$, find the value of d.

$10 - 9 = 1 = \log_{10} (6d - 2)$

$\Rightarrow 6d - 2 = 10$

$\Rightarrow d = 2$

Individual Event 2

I2.1 If $1000a = 85^2 - 15^2$, find the value of a.

$1000a = (85 + 15)(85 - 15) = 100\times70$

$\Rightarrow a = 7$

I2.2 The point (a, b) lies on the line $5x + 2y = 41$. Find the value of b.

$5(7) + 2b = 41$

$\Rightarrow b = 3$

I2.3 $x + b$ is a factor of $x^2 + 6x + c$. Find the value of c.

Put $x = -3$ into $x^2 + 6x + c = 0$

$(-3)^2 + 6(-3) + c = 0$

$\Rightarrow c = 9$

I2.4 If d is the distance between the points $(c, 1)$ and $(5, 4)$, find the value of d.

$d^2 = (9 - 5)^2 + (1 - 4)^2 = 25$

$\Rightarrow d = 5$

Individual Event 3

I3.1 If $\alpha + \beta = 11$, $\alpha\beta = 24$ and $\alpha > \beta$, find the value of α.

α and β are the roots of the equation $x^2 - 11x + 24 = 0$

$(x - 3)(x - 8) = 0$

$\because \alpha > \beta$

$\therefore \alpha = 8$

I3.2 If $\tan\theta = \frac{-\alpha}{15}$, $90° < \theta < 180°$ and $\sin\theta = \frac{b}{34}$, find the value of b.

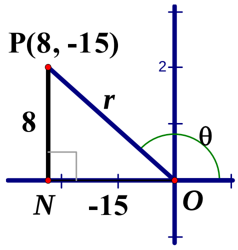
In the figure, $P = (8, -15)$

$r^2 = 8^2 + (-15)^2$ (Pythagoras' theorem)

$r = 17$

$\sin\theta = \frac{8}{17} = \frac{16}{34}$

$b = 16$



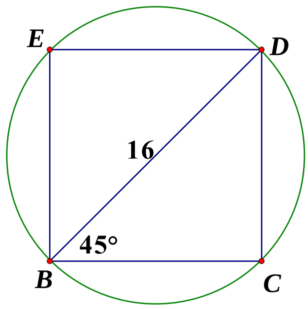
I3.3 If A is the area of a square inscribed in a circle of diameter b, find the value of A.

Reference: 1984 FG10.1, 1985 FSG.4

Let the square be *BCDE*.

$BC = 16\cos 45° = 8\sqrt{2}$

$A = \left(8\sqrt{2}\right)^2 = 128$



I3.4 If $x^2 + 22x + A \equiv (x + k)^2 + d$, where k, d are constants, find the value of d.

$x^2 + 22x + 128 \equiv (x + 11)^2 + 7$

$d = 7$

Individual Event 4

I4.1 The average of p, q, r is 12. The average of $p, q, r, t, 2t$ is 15. Find the value of t.

$p + q + r = 36$

$p + q + r + t + 2t = 75$

$3t = 75 - 36 = 39$

$t = 13$

I4.2 k is a real number such that $k^4 + \frac{1}{k^4} = t + 1$, and $s = k^2 + \frac{1}{k^2}$. Find the value of s.

$k^4 + \frac{1}{k^4} = 14$

$k^4 + 2 + \frac{1}{k^4} = 16$

$(k^2 + \frac{1}{k^2})^2 = 16$

$\Rightarrow s = k^2 + \frac{1}{k^2} = 4$

I4.3 M and N are the points (1, 2) and (11, 7) respectively. $P(a, b)$ is a point on MN such that $MP : PN = 1 : s$. Find the value of a.

$MP : PN = 1 : 4$

$a = \frac{4+11}{1+4} = 3$

I4.4 If the curve $y = ax^2 + 12x + c$ touches the x-axis, find the value of c.

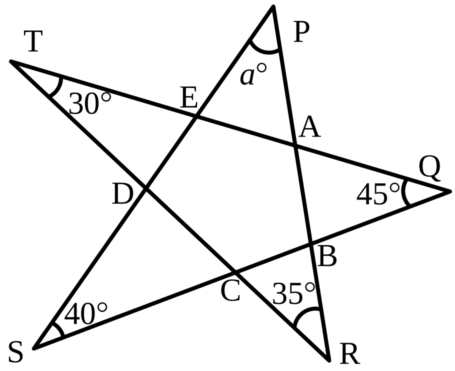
$y = 3x^2 + 12x + c$

$\Delta = 12^2 - 4(3)c = 0$

$\Rightarrow c = 12$

Individual Event 5

I5.1 In the figure, find the value of a.



Reference: 1997 FG1.1, 2005 FI2.3

Label the vertices *A*, *B*, *C*, *D*, *E*, *P*, *Q*, *R*, *S*, *T* as shown.

$\angle AEP = 40° + 45° = 85°$ (ext. ∠ of ΔSQE)

$\angle EAP = 30° + 35° = 65°$ (ext. ∠ of ΔTRA)

In ΔAEP, $85° + 65° + a° = 180°$ (∠s sum of Δ)

$a = 30$

I5.2 If $\sin(a° + 210°) = \cos b°$, and $90° < b < 180°$, find the value of b.

$\sin 240° = -\frac{\sqrt{3}}{2} = \cos b°$

$b = 150$

I5.3 Each interior angle of an n-sided regular polygon is $b°$. Find the value of n.

Each exterior angle = 30° (adj. ∠s on st. line)

$\frac{360}{n} = 30$ (sum of exterior angles of polygon)

$\Rightarrow n = 12$

I5.4 If the n^{th} day of March in a year is Friday. The k^{th} day of March in the same year is Wednesday, where $20 < k < 25$, find the value of k.

12^{th} March is Friday

17^{th} March is Wednesday

24^{th} March is Wednesday

$\Rightarrow k = 24$

Sample Group Event

SG.1 If $2at^2 + 12t + 9 = 0$ has equal roots, find the value of a.

$(12)^2 - 4(2a)(9) = 0$

$\Rightarrow a = 2$

SG.2 If $ax + by = 1$ and $4x + 18y = 3$ are parallel, find the value of b.

Reference: 1986 FI4.2, 1987 FSG.4

$$-\frac{2}{b} = -\frac{4}{18}$$

$\Rightarrow b = 9$

SG.3 The b^{th} prime number is p. Find the value of p.

Reference: 1985 FSG.2, 1990 FI5.4

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, $\cdots$

$p = 23$

SG.4 If $k = \frac{4\sin\theta + 3\cos\theta}{2\sin\theta - \cos\theta}$ and $\tan\theta = 3$, find the value of k.

Reference: 1986 FG10.3, 1987 FG8.1, 1989 FG10.3, 1990 FG7.2

$$k = \frac{(4\sin\theta + 3\cos\theta) \div \cos\theta}{(2\sin\theta - \cos\theta) \div \cos\theta}$$

$$= \frac{4\tan\theta + 3}{2\tan\theta - 1}$$

$$= \frac{4(3) + 3}{2(3) - 1}$$

$= 3$

Group Event 6

G6.1 An n-sided convex polygon has 20 diagonals. Find the value of n.

Reference: 1984 FG10.3, 1985 FG8.3, 1988 FG6.2, 1991 FI2.3, 2001 FI4.2, 2005 FI1.4

Number of diagonals $= C_2^n - n = \frac{n(n-1)}{2} - n = 20$

$n^2 - 3n - 40 = 0$

$(n - 8)(n + 5) = 0$

$\Rightarrow n = 8$

G6.2 Two dice are thrown. The probability of getting a total of n is $\frac{k}{36}$. Find the value of k.

Total = 8

Favourable outcomes = {(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)}

P(total = 8) $= \frac{5}{36}$

$k = 5$

G6.3 A man drives at 25 km/h for 3 hours and then at 50 km/h for 2 hours.
His average speed for the whole journey is u km/h. Find the value of u.

$u = \frac{25\times 3 + 50\times 2}{3+2} = 35$

G6.4 If $a\Delta b = ab + 1$ and $(2\Delta a)\Delta 3 = 10$, find the value of a.

$2\Delta a = 2a + 1$

$(2\Delta a)\Delta 3 = (2a + 1)\Delta 3 = 3(2a + 1) + 1 = 10$

$6a + 4 = 10$

$a = 1$

Group Event 7

In the attached calculation, different letters represent different integers ranging from 1 to 9.

If the letters O and J represent 4 and 6 respectively, find the values of

G7.1 G.

G7.2 D.

G7.3 L.

G7.4 E.

	G	O	L	D	E	N
$\times$						J
	D	E	N	G	O	L

	1	4	L	8	E	N
$\times$						6
	8	E	N	1	4	L

Carry digit in the 100000 digit is 2

$G = 1, D = 8$

Carry digit in the hundreds digit is 3

$E = 5$

Carry digit in the tens digit is 4

$N = 7, L = 2$

$\therefore G = 1, D = 8, L = 2, E = 5$

	1	4	2	8	5	7
$\times$						6
	8	5	7	1	4	2

Group Event 8

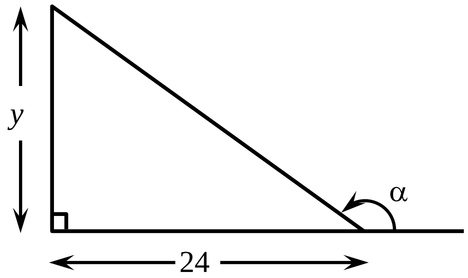
G8.1 If y is the greatest value of $\dfrac{14}{5+3\sin\theta}$, find the value of y.

$2 \le 5 + 3\sin\theta \le 8$

$\dfrac{14}{8} \le \dfrac{14}{5+3\sin\theta} \le \dfrac{14}{2}$

$\Rightarrow y = 7$

G8.2 In the figure, $100\cos\alpha = k$. Find the value of k.



Hypotenuse = 25

$k = -100\cos(\alpha - 180°)$

$= -100 \cdot \dfrac{24}{25} = -96$

G8.3 When $3x^2 + 4x + a$ is divided by $x + 2$, the remainder is 5. Find the value of a.

$3(-2)^2 + 4(-2) + a = 5$

$a = 1$

G8.4 The solution for $3t^2 - 5t - 2 < 0$ is $-\dfrac{1}{3} < t < m$. Find the value of m.

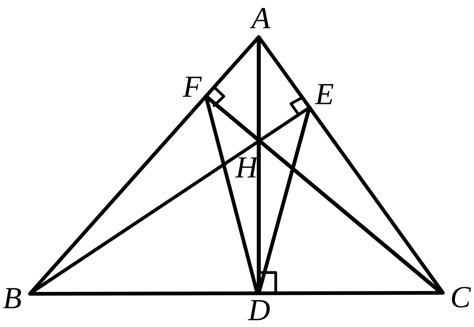
$(3t + 1)(t - 2) < 0$

$\Rightarrow -\dfrac{1}{3} < t < 2$

$\Rightarrow m = 2$

Group Event 9

G9.1 In the figure, $\angle BAC = 70°$ and $\angle FDE = x°$. Find the value of x.



$\angle AFC = 90° = \angle ADC$ (given)

$ACDF$ is a cyclic quad (converse, $\angle$s in the same seg.)

$\angle BDF = \angle BAC = 70°$ (ext. $\angle$, cyclic quad.)

$\angle AEB = 90° = \angle ADB$ (given)

$ABDE$ is a cyclic quad (converse, $\angle$s in the same seg.)

$\angle CDE = \angle BAC = 70°$ (ext. $\angle$, cyclic quad.)

$\angle FDE = 180° - \angle BDF - \angle CDE$ (adj. $\angle$s on st. line)

$= 180° - 70° - 70° = 40°$

$\Rightarrow x = 40$

G9.2 A cuboid is y cm wide, 6 cm long and 5 cm high. Its surface area is 126 cm^2. Find the value of y.

$2(5y + 6y + 5\times6) = 126$

$11y = 33$

$y = 3$

G9.3 If $\log_9(\log_2 k) = \frac{1}{2}$, find the value of k.

$\log_2 k = \sqrt{9} = 3$

$k = 2^3 = 8$

G9.4 If $a : b = 3 : 8$, $b : c = 5 : 6$ and $a : c = r : 16$, find the value of r.

$a : b : c = 15 : 40 : 48$

$a : c = 15 : 48 = 5 : 16$

$\Rightarrow r = 5$

Group Event 10

G10.1 If $\frac{6\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}=3\sqrt{a}+6$, find the value of a.

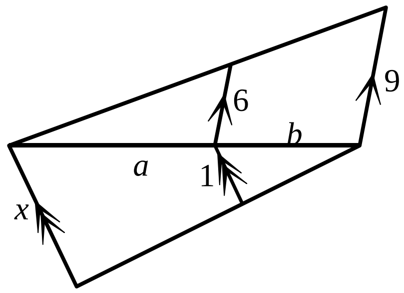
Reference: 2014 FI4.1

$\frac{6\sqrt{3}\left(3\sqrt{2}+2\sqrt{3}\right)}{18-12}=3\sqrt{a}+6$

$3\sqrt{6}+6=3\sqrt{a}+6$

$a = 6$

G10.2 In the figure, find the value of x.



Reference: 1994 FI4.3

By similar triangles

$6 : 9 = a : (a + b)$

$a = 2k, b = k$

$x : 1 = (a + b) : b = 3 : 1$

$x = 3$

G10.3 If $k=\frac{6\cos^2\theta+2\sin\theta\cos\theta+\sin^2\theta}{\cos^2\theta+\sin\theta\cos\theta+\sin^2\theta}$ and $\tan\theta = 2$, find the value of k.

Reference: 1986 FG10.3, 1987 FG8.1, 1989 FSG.4, 1990 FG7.2

$k=\frac{\left(6\cos^2\theta+2\sin\theta\cos\theta+\sin^2\theta\right)\div\cos^2\theta}{\left(\cos^2\theta+\sin\theta\cos\theta+\sin^2\theta\right)\div\cos^2\theta}$

$=\frac{6+2\tan\theta+\tan^2\theta}{1+\tan\theta+\tan^2\theta}$

$=\frac{6+2(2)+2^2}{1+2+2^2}$

$= 2$

G10.4 If $y=\frac{3\left(2^k\right)-4\left(2^{k-2}\right)}{2^k-2^{k-1}}$, find the value of y.

$y=\frac{3\left(2^k\right)-4\left(2^{k-2}\right)}{2^k-2^{k-1}}$

$=\frac{3-1}{1-\frac{1}{2}}$

$= 4$

Individual Events

SI	***h***	4	**I1**	***a***	5	**I2**	***p***	3	**I3**	***a***	1000	**I4**	***a***	5	**I5**	***a***	17
	k	32		***b***	4		***q***	36		***b***	8		***b***	12		***b***	5
	p	3		***c***	10		***k***	12		***c***	16		***c***	4		***c***	23
	q	16		***d***	34		***m***	150		***d***	1		***d***	12		***d***	9

Group Events

SG	***a***	2	**G6**	***a***	150	**G7**	***C***	47	**G8**	***A***	2	**G9**	***S***	1000	**G10**	***A***	1584
	b	–3		***b***	10		***K***	2		***B***	3		***K***	98		***k***	14
	p	60		***k***	37.5		***A***	1		***C***	7		***t***	20		***x***	160
	q	136		***d***	6		***B***	5		***k***	9		***d***	5		***n***	15

Sample Individual Event (1986 Final Individual Event 2)

SI.1 Given that $3x^2 - 4x + \frac{h}{3} = 0$ has equal roots, find h.

$\Delta = (-4)^2 - 4(3)\cdot\frac{h}{3} = 0$

$h = 4$

SI.2 If the height of a cylinder is doubled and the new radius is h times the original, then the new volume is k times the original. Find k.

Let the old height be x, old radius be r, then the old volume is $\pi r^2 x$.

The new height is $2x$, the new radius is $4r$,

then the new volume is $\pi(4r)^2(2x) = 32\pi r^2 x$

$k = 32$

SI.3 If $\log_{10} 210 + \log_{10}k - \log_{10}56 + \log_{10}40 - \log_{10}120 + \log_{10}25 = p$, find p.

$p = \log_{10}\left(\frac{210\times 32\times 40\times 25}{56\times 120}\right)$

$= \log_{10} 1000 = 3$

SI.4 If $\sin A = \frac{p}{5}$ and $\frac{\cos A}{\tan A} = \frac{q}{15}$, find q.

$\sin A = \frac{3}{5}$

$\frac{\cos A}{\tan A} = \frac{q}{15}$

$\frac{\cos^2 A}{\sin A} = \frac{q}{15}$

$\frac{1-\sin^2 A}{\sin A} = \frac{1-\left(\frac{3}{5}\right)^2}{\frac{3}{5}} = \frac{16}{15} = \frac{q}{15}$

$q = 16$

Individual Event 1

I1.1 Find a if $2t + 1$ is a factor of $4t^2 + 12t + a$.

Let $f(t) = 4t^2 + 12t + a$

$$f\left(-\frac{1}{2}\right) = 4\left(-\frac{1}{2}\right)^2 + 12\left(-\frac{1}{2}\right) + a = 0$$

$a = 5$

I1.2 $\sqrt{K}$ denotes the nonnegative square root of K, where $K \ge 0$. If b is the root of the equation $\sqrt{a-x} = x - 3$, find b.

$\left(\sqrt{5-x}\right)^2 = (x-3)^2$

$\Rightarrow 5 - x = x^2 - 6x + 9$

$\Rightarrow x^2 - 5x + 4 = 0$

$\Rightarrow x = 1$ or 4

When $x = 1$, LHS $= 2 \ne -1 =$ RHS

When $x = 4$, LHS $= 1 =$ RHS.

$\therefore\ x = b = 4$

I1.3 If c is the greatest value of $\dfrac{20}{b + 2\cos\theta}$, find c.

$$\frac{20}{b+2\cos\theta} = \frac{20}{4+2\cos\theta} = \frac{10}{2+\cos\theta}$$

c = the greatest value $= \dfrac{10}{2-1} = 10$

I1.4 A man drives a car at $3c$ km/h for 3 hours and then $4c$ km/h for 2 hours. If his average speed for the whole journey is d km/h, find d.

Total distance travelled = $(30\times3 + 40\times2)$ km = 170 km

$d = \dfrac{170}{3+2} = 34$

Individual Event 2

I2.1 If $0° \le \theta < 360°$, the equation in θ: $3\cos\theta + \dfrac{1}{\cos\theta} = 4$ has p roots. Find p.

$3\cos^2\theta + 1 = 4\cos\theta$

$\Rightarrow 3\cos^2\theta - 4\cos\theta + 1 = 0$

$\Rightarrow \cos\theta = \dfrac{1}{3}$ or 1

$p = 3$

I2.2 If $x - \dfrac{1}{x} = p$ and $x^3 - \dfrac{1}{x^3} = q$, find q.

Reference: 2009 FI2.3

$x - \dfrac{1}{x} = 3$; $\left(x - \dfrac{1}{x}\right)^2 = 9$

$\Rightarrow x^2 + \dfrac{1}{x^2} = 11$

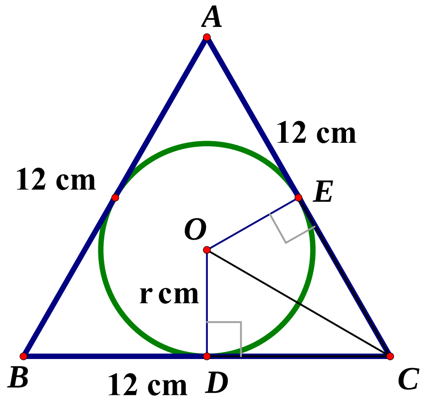
$q = x^3 - \dfrac{1}{x^3}$

$= \left(x - \dfrac{1}{x}\right)\left(x^2 + 1 + \dfrac{1}{x^2}\right)$

$= 3(11 + 1) = 36$

I2.3 A circle is inscribed in an equilateral triangle of perimeter q cm. If the area of the circle is $k\pi$ cm^2, find k.

Reference: 1984 FG9.4



Let the equilateral triangle be ABC, the centre of the inscribed circle is O, which touches the triangle at D and E, with radius r cm

Perimeter = 36 cm

⇒ Each side = 12 cm

$\angle ACB = 60°$ (∠s of an equilateral Δ)

$\angle ODC = 90°$ (tangent ⊥ radius)

$\angle OCD = 30°$ (tangent from ext. pt.)

$CD = 6$ cm (tangent from ext. pt.)

$r = 6\tan 30° = 2\sqrt{3}$

Area of circle $= \pi\left(2\sqrt{3}\right)^2$ cm^2 = 12π cm^2

$k = 12$

I2.4 Each interior angle of a regular polygon of k sides is $m°$. Find m.

Angle sum of 12-sides polygon = 180°(12 – 2) = 1800°

Each interior angle = $m°$ = 1800° ÷ 12 = 150°

$m = 150$

Individual Event 3

I3.1 If $998a + 1 = 999^2$, find a.

$998a = 999^2 - 1$

$= (999 - 1)(999 + 1)$

$= 998\times 1000$

$a = 1000$

I3.2 If $\log_{10}a = \log_2 b$, find b.

$\log_{10} 1000 = \log_2 b$

$\log_2 b = 3$

$\Rightarrow b = 2^3 = 8$

I3.3 The area of the triangle formed by the x-axis, the y-axis and the line $2x + y = b$ is c sq. units. Find c.

Reference: 1994 FI5.3

$2x + y = 8$; x-intercept = 4, y-intercept = 8

$c = \text{area} = \frac{1}{2}\cdot 4\times 8 = 16$

I3.4 If $64t^2 + ct + d$ is a perfect square, find d.

$64t^2 + 16t + d$ has a double root

$\Delta = 16^2 - 4\times 64d = 0$

$d = 1$

Individual Event 4

I4.1 Solve for a in the equation $2^{a+1} + 2^a + 2^{a-1} = 112$.

$2^a \cdot (2 + 1 + \frac{1}{2}) = 112$

$2^a = 32$

$a = 5$

Method 2

$112 = 64 + 32 + 16 = 2^6 + 2^5 + 2^4$

$a = 5$

I4.2 If a is one root of the equation $x^2 - bx + 35 = 0$, find b.

One root of $x^2 - bx + 35 = 0$ is 5

$\Rightarrow 5^2 - 5b + 35 = 0$

$\Rightarrow b = 12$

I4.3 If $\sin \theta = \frac{-b}{15}$, where $180° < \theta < 270°$, and $\tan \theta = \frac{c}{3}$, find c.

$\sin \theta = -\frac{12}{15} = -\frac{4}{5}$

$\Rightarrow \tan \theta = \frac{4}{3}$

$\Rightarrow c = 4$

I4.4 The probability of getting a sum of c in throwing two dice is $\frac{1}{d}$. Find d.

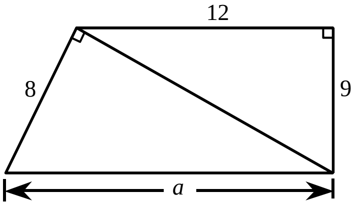
P(sum = 4) = P((1,3), (2, 2), (3, 1))

$= \frac{3}{36} = \frac{1}{12} = \frac{1}{d}$

$\Rightarrow d = 12$

Individual Event 5

I5.1 In the figure, find *a*.



$a^2 - 8^2 = 12^2 + 9^2$ (Pythagoras' Theorem)

$a = 17$

I5.2 If the lines $ax + by = 1$ and $10x - 34y = 3$ are perpendicular to each other, find *b*.

$17x + by = 1$ is perpendicular to $10x - 34y = 3$

$\Rightarrow$ product of slopes = –1

$$-\frac{17}{b}\times\frac{10}{34}=-1$$

$\Rightarrow b = 5$

I5.3 If the b^{th} day of May in a year is Friday and the c^{th} day of May in the same year is Tuesday, where $16 < c < 24$, find *c*.

5^{th} May is a Friday

$\Rightarrow 9^{th}$ May is Tuesday

$\Rightarrow 16^{th}$ May is Tuesday

$\Rightarrow 23^{rd}$ May is Tuesday

$c = 23$

I5.4 *c* is the d^{th} prime number. Find *d*.

Reference: 1985 FSG.2, 1989 FSG.3

The first few prime numbers are: 2, 3, 5, 7, 11, 13, 17, 19, 23

23 is the 9^{th} prime number

$d = 9$

Sample Group Event (1986 Sample Group Event)

SG.1 The sum of two numbers is 50, and their product is 25.

If the sum of their reciprocals is a, find a.

Let the 2 numbers be x, y.

$x + y = 50$, $xy = 25$

$\Rightarrow a = \frac{1}{x} + \frac{1}{y}$

$= \frac{x+y}{xy}$

$= \frac{50}{25} = 2$

SG.2 If the lines $ax + 2y + 1 = 0$ and $3x + by + 5 = 0$ are perpendicular, find b.

$2x + 2y + 1 = 0$ is $\perp$ to $3x + by + 5 = 0$

$\Rightarrow$ product of slopes = –1

$-\frac{2}{2} \times \frac{-3}{b} = -1$

$\Rightarrow b = -3$

SG.3 The area of an equilateral triangle is $100\sqrt{3}$ cm^2. If its perimeter is p cm, find p.

Let the length of one side be x cm.

$\frac{1}{2}x^2 \sin 60^\circ = 100\sqrt{3}$

$\Rightarrow x = 20$

$\Rightarrow p = 60$

SG.4 If $x^3 - 2x^2 + px + q$ is divisible by $x + 2$, find q.

Let $f(x) = x^3 - 2x^2 + 60x + q$

$f(-2) = -8 - 8 - 120 + q = 0$

$q = 136$

Group Event 6

G6.1 If $a = \dfrac{(68^3-65^3)\cdot(32^3+18^3)}{(32^2-32\times 18+18^2)\cdot(68^2+68\times 65+65^2)}$, find a.

$a = \dfrac{(32+18)(32^2-32\times 18+18^2)\cdot(68-65)(68^2+68\times 65+65^2)}{(32^2-32\times 18+18^2)\cdot(68^2+68\times 65+65^2)}$

$= 50\times 3 = 150$

G6.2 If the 3 points (a, b), (10, –4) and (20, –3) are collinear, find b.

The slopes are equal: $\dfrac{b+4}{150-10} = \dfrac{-3+4}{20-10}$

$\Rightarrow b = 10$

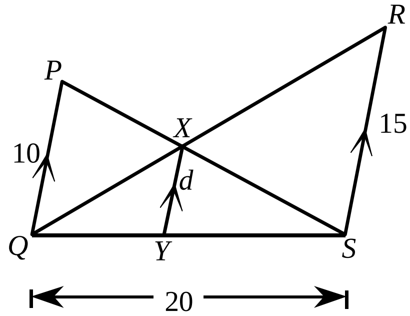
G6.3 If the acute angle formed by the hands of a clock at 4:15 is $k°$, find k.

Reference 1984 FG7.1, 1985 FI3.1, 1987 FG7.1, 1989 FI1.1, 2007 HI1

$k = 30 + 30\times\dfrac{1}{4} = 37.5$

G6.4 In the figure, $PQ = 10$, $RS = 15$, $QS = 20$. If $XY = d$, find d.

Reference: 1985 FI2.4, 1989 HG8



$\dfrac{1}{d} = \dfrac{1}{10} + \dfrac{1}{15} = \dfrac{25}{150} = \dfrac{1}{6}$

$d = 6$

Group Event 7

G7.1 2 apples and 3 oranges cost 6 dollars.

4 apples and 7 oranges cost 13 dollars.

16 apples and 23 oranges cost C dollars. Find C.

Let the cost of one apple be \$$x$ and one orange be \$$y$.

$2x + 3y = 6$ …… (1)

$4x + 7y = 13$…… (2)

(2) – 2(1): $y = 1$, $x = 1.5$

$C = 16x + 23y = 24 + 23 = 47$

G7.2 If $K = \dfrac{6\cos\theta + 5\sin\theta}{2\cos\theta + 3\sin\theta}$ and tan θ = 2, find K.

Reference: 1986 FG10.3, 1987 FG8.1, 1989 FSG.4, 1989 FG10.3

$$K = \frac{6\frac{\cos\theta}{\cos\theta} + 5\frac{\sin\theta}{\cos\theta}}{2\frac{\cos\theta}{\cos\theta} + 3\frac{\sin\theta}{\cos\theta}}$$

$$= \frac{6 + 5\tan\theta}{2 + 3\tan\theta}$$

$$= \frac{6 + 5\times 2}{2 + 3\times 2} = 2$$

G7.3 and G7.4 A, B are positive integers less than 10 such that $21A104 \times 11 = 2B8016 \times 9$.

Similar Questions 1985 FG8.1-2, 1988 FG8.3-4

G7.3 Find A.

11 and 9 are relatively prime, $21A104$ is divisible by 9.

$2 + 1 + A + 1 + 0 + 4 = 9m$

$\Rightarrow 8 + A = 9m$

$\Rightarrow A = 1$

G7.4 Find B.

$2B8016$ is divisible by 11.

$2 + 8 + 1 – (B + 0 + 6) = 11n$

$\Rightarrow 11 – (B + 6) = 11n$

$\Rightarrow B = 5$

Group Event 8

In the multiplication shown, the letters A, B, C and K ($A < B$) represent different integers from 1 to 9.

$$\begin{array}{rcc} & A & C \\ \times) & B & C \\ \hline K & K & K \end{array}$$

(Hint: $KKK = K\times 111$.)

G8.1 Find A.

$1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$, $6^2 = 36$, $7^2 = 49$, $8^2 = 64$, $9^2 = 81$

Possible $K = 1, 4, 5, 6, 9$

$100K + 10K + K = 111K = 3\times 37K$, 37 is a prime number

Either $10A + C$ or $10B + C$ is divisible by 37

$10B + C = 37$ or 74

When $B = 3$, $C = 7$, $K = 9$

$999 \div 37 = 27$

$\therefore A = 2$

G8.2 Find B.

$B = 3$

G8.3 Find C.

$C = 7$

G8.4 Find K.

$K = 9$

Group Event 9

G9.1 If $S = ab - 1 + a - b$ and $a = 101$, $b = 9$, find S.

Reference: 1985 FG8.4, 1986 FG9.3, 1988 FG6.3

$S = (a - 1)(b + 1) = 100\times10 = 1000$

G9.2 If $x = 1.9\dot{8}\dot{9}$ and $x - 1 = \frac{K}{99}$, find K.

$x = 1.9 + \frac{89}{990}$

$x - 1 = \frac{K}{99} = \frac{9}{10} + \frac{89}{990}$

$= \frac{9\times99+89}{990} = \frac{980}{990} = \frac{98}{99}$

$K = 98$

G9.3 The average of p, q and r is 18. The average of $p + 1$, $q - 2$, $r + 3$ and t is 19. Find t.

$\frac{p+q+r}{3} = 18$

$\Rightarrow p + q + r = 54$

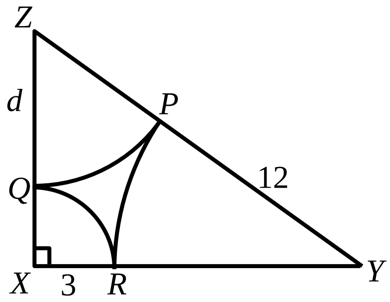
$\frac{p+1+q-2+r+3+t}{4} = 19$

$\Rightarrow p + q + r + 2 + t = 76$

$\Rightarrow 54 + 2 + t = 76$

$t = 20$

G9.4 In the figure, $\overset{\frown}{QR}$, $\overset{\frown}{RP}$, $\overset{\frown}{PQ}$ are 3 arcs, centres at X, Y and Z respectively, touching one another at P, Q and R. If $ZQ = d$, $XR = 3$, $YP = 12$, $\angle X = 90°$, find d.



Reference: 1986 FG7.1

$XZ = 3 + d$, $XY = 3 + 12 = 15$, $YZ = 12 + d$

$XZ^2 + XY^2 = YZ^2$ (Pythagoras' theorem)

$(3 + d)^2 + 15^2 = (12 + d)^2$

$9 + 6d + d^2 + 225 = 144 + 24d + d^2$

$18d = 90$

$\Rightarrow d = 5$

Group Event 10

G10.1 If $A = 1 + 2 - 3 + 4 + 5 - 6 + 7 + 8 - 9 + \ldots + 97 + 98 - 99$, find A.

Reference: 1985 FG7.4, 1988 FG6.4, 1991 FSI.1, 1992 FI1.4

$A = (1 + 2 - 3) + (4 + 5 - 6) + (7 + 8 - 9) + \ldots + (97 + 98 - 99)$

$A = 0 + 3 + 6 + \ldots + 96 = \frac{3+96}{2}\times 32 = 99\times 16 = 1584$

G10.2 If $\log_{10}(k - 1) - \log_{10}(k^2 - 5k + 4) + 1 = 0$, find k.

$10(k - 1) = k^2 - 5k + 4$

$k^2 - 15k + 14 = 0$

$k = 1$ or 14

When $k = 1$, LHS is undefined $\therefore$ rejected

When $k = 14$, LHS $= \log_{10} 13 - \log_{10}(14 - 1)(14 - 4) + 1 =$ RHS

$\therefore k = 14$

G10.3 and **G10.4** One interior angle of a convex n-sided polygon is $x°$. The sum of the remaining interior angles is $2180°$.

Reference: 1989 HG2, 1992 HG3, 2002 FI3.4, 2013 HI6

G10.3 Find x.

$2180 + x = 180(n - 2)$ ($\angle$s sum of polygon)

$2160 + 20 + x = 180\times 12 + 20 + x = 180(n - 2)$

$\because x < 180$

$\therefore 20 + x = 180$

$x = 160$

G10.4 Find n.

$n - 2 = 12 + 1$

$n = 15$

Individual Events

SI	***a***	50	**I1**	***a***	15	**I2**	***a***	124	**I3**	***a***	7	**I4**	***a***	11	**I5**	***a***	1080
	b	10		***b***	3		***b***	50		***b***	125		***b***	5		***n***	21
	c	5		***c***	121		***n***	12		***c***	40		****c***	9 see the remark		***x***	25
	d	2		***d***	123		***d***	–10		***d***	50		***d***	5		***K***	6

Group Events

SG	***a***	64	**G6**	***M***	4	**G7**	***n***	5	**G8**	H_5	61	**G9**	**Area of Δ*BDF***	30	**G10**	***A***	3
	b	7		***N***	5		***c***	2		***a***	3		**Area of Δ*FDE***	75		***B***	1
	h	30		***z***	4		***x***	60		***t***	12		**Area of Δ*ABC***	28		***C***	5
	k	150		***r***	70		***y***	20		***m***	7		***x***	44		***D***	7

Sample Individual Event

SI.1 If $a = -1 + 2 - 3 + 4 - 5 + 6 - \dots + 100$, find a.

Reference: 1998 FI2.4

$a = (-1 + 2 - 3 + 4) + (-5 + 6 - 7 + 8) + \dots + (-97 + 98 - 99 + 100)$

$= 2 + 2 + \dots + 2$ (25 terms) $= 50$

SI.2 The sum of the first b positive odd numbers is $2a$. Find b.

$1 + 3 + \dots + (2b - 1) = 2a = 100$

$$\frac{b}{2}\left[2+2(b-1)\right]=100$$

$b^2 = 100$

$b = 10$

SI.3 A bag contains b white balls and 3 black balls. Two balls are drawn from the bag at random.

If the probability of getting 2 balls of different colours is $\frac{c}{13}$, find c.

The bag contains 10 white balls and 3 black balls.

$$\text{P(2 different colours)} = 2\times\frac{10}{13}\times\frac{3}{12}=\frac{5}{13}=\frac{c}{13}$$

$c = 5$

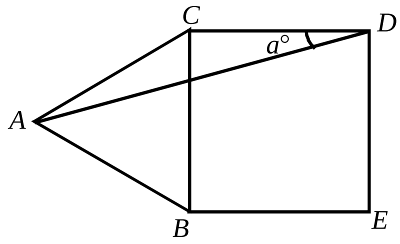
SI.4 If the lines $cx + 10y = 4$ and $dx - y = 5$ are perpendicular to each other, find d.

$$-\frac{5}{10}\times\frac{d}{1}=-1$$

$\Rightarrow d = 2$

Individual Event 1

I1.1 In the figure, ABC is an equilateral triangle and $BCDE$ is a square. If $\angle ADC = a°$, find a. (**Reference 2014 FG3.3)**



$\angle ACD = (60 + 90)° = 150°$

$AC = CD$

$\angle CAD = a°$ (base, $\angle$s isos. Δ)

$a + a + 150 = 180$ ($\angle$s sum of Δ)

$a = 15$

I1.2 If $rb = 15$ and $br^4 = 125a$, where r is an integer, find b.

$br \cdot r^3 = 15r^3 = 125 \times 15$

$\Rightarrow r^3 = 125$

$\Rightarrow r = 5$

$rb = 15$

$\Rightarrow b = 3$

I1.3 If the positive root of the equation $bx^2 - 252x - 13431 = 0$ is c, find c.

$3x^2 - 252x - 13431 = 0$

$\Rightarrow x^2 - 84x - 4477 = 0$, $4477 = 11\times11\times37$ and $-84 = -121 + 37$

$\Rightarrow (x - 121)(x + 37) = 0$

$\Rightarrow x = c = 121$

I1.4 Given $x \# y = \frac{y-1}{x} - x + y$. If $d = 10 \# c$, find d.

$d = 10 \# c$

$= \frac{121-1}{10} - 10 + 121$

$= 12 + 111 = 123$

Individual Event 2

I2.1 If $a^2 - 1 = 123\times125$ and $a > 0$, find a.

Reference: 1983 FI10.1, 1984 FSG.2

$a^2 - 1 = (124 - 1)\times(124 + 1)$

$= 124^2 - 1$

$a = 124$

I2.2 If the remainder of $x^3 - 16x^2 - 9x + a$ when divided by $x - 2$ is b, find b.

$b = 2^3 - 16(2)^2 - 9(2) + 124 = 50$

I2.3 If an n-sided polygon has $(b + 4)$ diagonals, find n.

Reference: 1984 FG10.3, 1985 FG8.3, 1988 FG6.2, 1989 FG6.1, 2001 FI4.2, 2005 FI1.4

$C_2^n - n = 50 + 4$

$n(n - 3) = 108$

$n^2 - 3n - 108 = 0$

$(n - 12)(n + 9) = 0$

$\Rightarrow n = 12$

I2.4 If the points $(3, n)$, $(5, 1)$ and $(7, d)$ are collinear, find d.

$\frac{12-1}{3-5} = \frac{d-1}{7-5}$

$d - 1 = -11$

$\Rightarrow d = -10$

Individual Event 3

I3.1 If the 6-digit number $168a26$ is divisible by 3, find the greatest possible value of a.

$1 + 6 + 8 + a + 2 + 6 = 3k$, where k is an integer.

The greatest possible value of $a = 7$

I3.2 A cube with edge a cm long is painted red on all faces. It is then cut into cubes with edge 1 cm long. If the number of cubes with all the faces not painted is b, find b.

Reference: 1994 HG2

The number of cubes with all the faces not painted is $b = (7 - 1 - 1)^3 = 125$

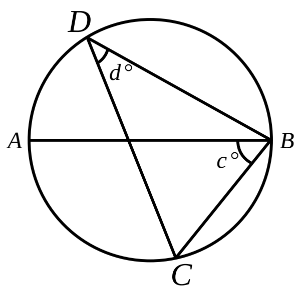
I3.3 If $(x - 85)(x - c) \equiv x^2 - bx + 85c$, find c.

$(x - 85)(x - c) \equiv x^2 - (85 + c)x + 85c$

$85 + c = b = 125$

$\Rightarrow c = 40$

I3.4 In the figure, AB is a diameter of the circle. Find d.



Label the vertices as shown.

$\angle CAB = d°$ ($\angle$ in the same segment)

$c + d = 90$ ($\angle$ in semi-circle)

$d = 50$

Individual Event 4

I4.1 Given $x - \frac{1}{x} = 3$. If $a = x^2 + \frac{1}{x^2}$, find a.

$a = (x - \frac{1}{x})^2 + 2$

$= 9 + 2 = 11$

I4.2 If $f(x) = \log_2 x$ and $f(a + 21) = b$, find b.

$b = f(11 + 21) = f(32)$

$= \log_2 32 = \log_2 2^5 = 5$

I4.3 If $\cos\theta = \frac{8b}{41}$, where θ is an acute angle, and $c = \frac{1}{\sin\theta} + \frac{1}{\tan\theta}$, find c.

$\cos\theta = \frac{40}{41}$

$\Rightarrow \sin\theta = \frac{9}{41}$, $\tan\theta = \frac{9}{40}$

$\Rightarrow c = \frac{41}{9} + \frac{40}{9} = 9$

Remark: Original question was .... where θ is **a positive** acute angle ....
Acute angle must be positive, the words "a positive" is replaced by "an".

I4.4 Two dice are tossed. If the probability of getting a sum of 7 or c is $\frac{d}{18}$, find d.

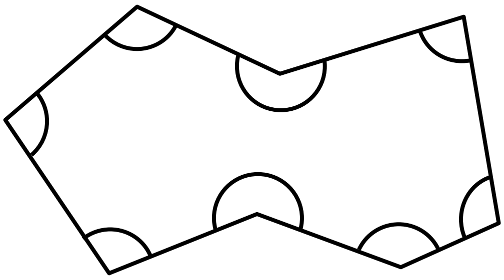
P(sum = 7 or 9) = P(7) + P(9)

$= \frac{6}{36} + \frac{4}{36} = \frac{5}{18}$

$\Rightarrow d = 5$

Individual Event 5

I5.1 In Figure 1, if the sum of the interior angles is $a°$, find a.



$a = 180×(8 – 2)$ (∠s sum of polygon)

$a = 1080$

I5.2 If the n^{th} term of the arithmetic progression 80, 130, 180, 230, 280, … is a, find n.

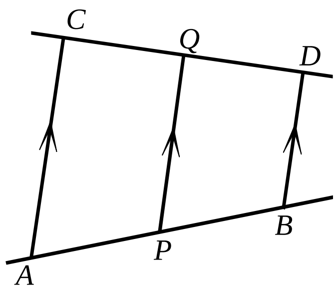
First term = 80, common difference = 50

$80 + (n – 1)·50 = 1080$

$\Rightarrow n = 21$

I5.3 In Figure 2, $AP : PB = 2 : 1$.

If $AC = 33$ cm, $BD = n$ cm, $PQ = x$ cm, find x.



Reference: 1986 FI3.3

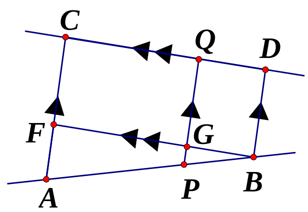
From B, draw a line segment FGB // CQD, cutting AC, PQ at F and G respectively.

$CDBF$, $BDQG$ are parallelograms (2 pairs of // lines)

$CF = QG = DB = 21$ cm (opp. sides //-gram)

$AF = (33 – 21)$cm = 12 cm

$\Delta BPG \sim \Delta BAF$ (equiangular)



$\frac{PG}{AF} = \frac{PB}{AP + PB}$ (ratio of sides, ~Δs)

$\frac{PG}{12\text{ cm}} = \frac{1}{3}$

$\Rightarrow PG = 4$ cm

$PQ = PG + GQ = (4 + 21)$ cm = 25 cm

$x = 25$

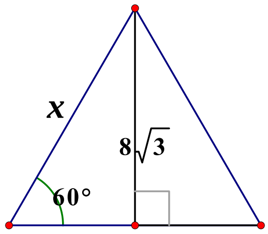
I5.4 If $K = \frac{\sin 65^\circ \tan^2 60^\circ}{\tan 30^\circ \cos 30^\circ \cos x^\circ}$, find K.

$$K = \frac{\sin 65^\circ \tan^2 60^\circ}{\tan 30^\circ \cos 30^\circ \cos 25^\circ}$$

$$= \frac{\sin 65^\circ \cdot \left(\sqrt{3}\right)^2}{\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} \cdot \sin 65^\circ} = 6$$

Sample Group Event

SG.1 The height of an equilateral triangle is $8\sqrt{3}$ cm and the area of the triangle is $a\sqrt{3}$ cm^2. Find a.



Let the length of a side be x cm.

In the figure, $x \sin 60° = 8\sqrt{3}$

$\Rightarrow x = 16$

$$\text{Area} = \frac{1}{2}\cdot x^2 \sin 60^\circ$$

$$= \frac{1}{2}\cdot 16^2 \cdot \frac{\sqrt{3}}{2} = a\sqrt{3}$$

$\Rightarrow a = 64$

SG.2 Given that $\sum_{x=1}^{n}\frac{1}{x} = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$, and $\sum_{x=4}^{10}\frac{1}{x-2} - \sum_{x=4}^{10}\frac{1}{x-1} = \frac{b}{18}$. Find b.

Reference: 1983 FG7.4

$$\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)-\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}\right)$$

$$= \frac{1}{2}-\frac{1}{9} = \frac{b}{18}$$

$\Rightarrow b = 7$

SG.3-SG.4 A boy tries to find the area of a parallelogram by multiplying together the lengths of two adjacent sides. His answer is double the correct answer. If the acute angle and the obtuse angle of the figure are $h°$ and $k°$ respectively,

Reference: 1989 HI7

SG.3 find h.

Let the two adjacent sides be x and y.

$xy = 2\cdot xy \sin h°$

$\Rightarrow \sin h° = \frac{1}{2}$

$\Rightarrow h = 30$

SG.4 find k.

$k = 180 - 30 = 150$ (int. ∠s, // lines)

Group Event 6

G6.1-6.2 A 2-digit number x has M as the units digit and N as the tens digit. Another 2-digit number y has N as the units digit and M as the tens digit. If $x > y$ and their sum is equal to eleven times their differences,

Reference: 1983 FG10.4

G6.1 find M. **G6.2** find N.

$x = 10N + M, y = 10M + N$

$x > y \Rightarrow N > M > 0$

$x + y = 11(x - y)$

$10N + M + 10M + N = 11(10N + M - 10M - N)$

$M + N = 9N - 9M$

$10M = 8N$

$5M = 4N$

M is a multiple of 4 and N is a multiple of 5.

$N = 5, M = 4$

G6.3 The sum of two numbers is 20 and their product is 5.

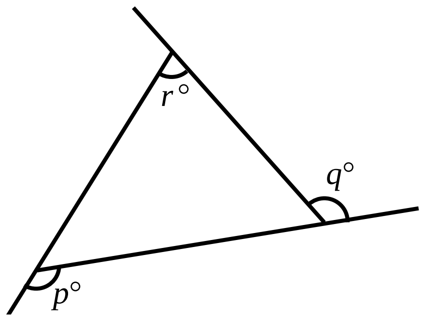
If the sum of their reciprocals is z, find z.

Let the 2 numbers be x and y.

$x + y = 20$ and $xy = 5$

$z = \frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy} = 4$

G6.4 In the figure, the average of p and q is $121 + z$. Find r.



Reference: 1983 FG6.2

The exterior angle of $r°$ is $180° - r°$ (adj. ∠s on st. line)

$p + q + (180 - r) = 360$ (sum of ext. ∠s of polygon)

$p + q - r = 180$ …… (1)

$\frac{p+q}{2} = 121 + z = 125$

$\Rightarrow p + q = 250$ …… (2)

Sub. (2) into (1): $250 - r = 180$

$\Rightarrow r = 70$

Group Event 7

G7.1 5 printing machines can print 5 books in 5 days.

If n printing machines are required in order to have 100 books printed in 100 days, find n.

100 printing machines can print 100 books in 5 days.

5 printing machines can print 100 books in 100 days

$\Rightarrow n = 5$

G7.2 If the equation $x^2 + 2x + c = 0$ has no real root and c is an integer less than 3, find c.

$\Delta = 2^2 - 4c < 0$

$\Rightarrow c > 1$ and c is an integer less than 3

$\Rightarrow c = 2$

G7.3-G7.4 Chicken eggs cost \$0.50 each, duck eggs cost \$0.60 each and goose eggs cost \$0.90 each. A man sold x chicken eggs, y duck eggs, z goose eggs and received \$60. If x, y, z are all positive numbers with $x + y + z = 100$ and two of the values x, y, z are equal,

G7.3 find x. **G7.4** find y.

$0.5x + 0.6y + 0.9z = 60$

$\Rightarrow 5x + 6y + 9z = 600$ …… (1)

$x + y + z = 100$ …… (2)

If $x = z$, then $14x + 6y = 600$

$\Rightarrow 7x + 3y = 300$ …… (3) and $2x + y = 100$ …… (4)

(3) – 3(4): $x = 0$ (rejected)

If $x = y$, then $11x + 9z = 600$ …… (5) and $2x + z = 100$ …… (6)

9(6) – (5): $7x = 300$, x is not an integer, rejected.

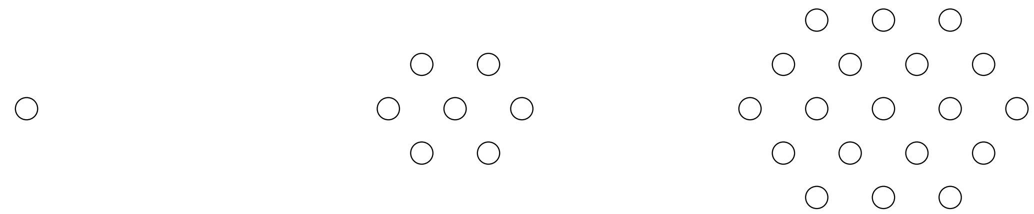
(1) – 5(2): $y + 4z = 100$ …… (7)

If $y = z$, then $y = z = 20$, $x = 60$

Group Event 8

Reference: 1992 FG9.3-4

G8.1-G8.2 Consider the following hexagonal numbers :



$H_1 = 1$ $H_2 = 7$ $H_3 = 19$

G8.1 Find H_5.

$H_2 - H_1 = 6\times1$, $H_3 - H_2 = 12 = 6\times2$

$H_4 - H_3 = 18 = 6\times3$

$\Rightarrow H_4 = 19 + 18 = 37$

$H_5 - H_4 = 6\times4 = 24$

$\Rightarrow H_5 = 24 + 37 = 61$

G8.2 If $H_n = an^2 + bn + c$, where n is any positive integer, find a.

$H_1 = a + b + c = 1 \ldots\ldots (1)$

$H_2 = 4a + 2b + c = 7 \ldots\ldots (2)$

$H_3 = 9a + 3b + c = 19 \ldots\ldots (3)$

(2) – (1): $3a + b = 6 \ldots\ldots (4)$

(3) – (2): $5a + b = 12 \ldots\ldots (5)$

(5) – (4): $2a = 6$

$\Rightarrow a = 3$

G8.3 If $p : q = 2 : 3$, $q : r = 4 : 5$ and $p : q : r = 8 : t : 15$, find t.

$p : q : r = 8 : 12 : 15$

$\Rightarrow t = 12$

G8.4 If $\frac{1}{x} : \frac{1}{y} = 4 : 3$ and $\frac{1}{x+y} : \frac{1}{x} = 3 : m$, find m.

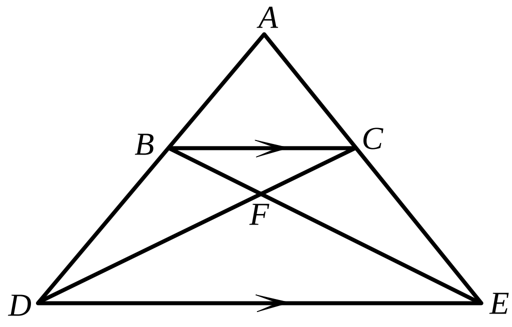
$$x : y = \frac{1}{4} : \frac{1}{3} = 3 : 4$$

$$\frac{1}{x+y} : \frac{1}{x} = \frac{1}{3+4} : \frac{1}{3} = 3 : 7$$

$m = 7$

Group Event 9

G9.1-G9.3



In the figure, BC is parallel to DE.

If $AB : BC : BF : CF : FE = 5 : 4 : 2 : 3 : 5$

and the area of ΔBCF is 12, find

G9.1 the area of ΔBDF,

G9.2 the area of ΔFDE,

G9.3 the area of ΔABC.

G9.1 $\Delta BCF \sim \Delta EDF$ (equiangular)

$DF : EF : DE = CE : FB : BC$ (ratio of sides, ~Δs)

$DF = 3\times\frac{5}{2} = 7.5$, $DE = 4\times\frac{5}{2} = 10$

The area of $\Delta BDF = 12\times\frac{7.5}{3} = 30$

G9.2 The area of $\Delta FDE = 30\times\frac{5}{2} = 75$

G9.3 The area of $\Delta CEF = 12\times\frac{5}{2} = 30$

The area of $BCED = 12 + 30 + 30 + 75 = 147$

$\Delta ABC \sim \Delta ADE$ (equiangular)

Area of ΔABC : area of $\Delta ADE = BC^2 : DE^2 = 4^2 : 10^2 = 4 : 25$

Let the area of ΔABC be y

$y : (y + 147) = 4 : 25$

$4y + 588 = 25y$

$21y = 588$

$y =$ Area of $\Delta ABC = 28$

G9.4 If the volume of a sphere is increased by 72.8%, then the surface area of the sphere is increased by x%. Find x.

Let the original radius of the sphere be r and the new radius be R

$$\frac{4}{3}\pi R^3 = \frac{4}{3}\pi r^3 \cdot (1 + 72.8\%)$$

$$\left(\frac{R}{r}\right)^3 = 1.728 = 1.2^3$$

$\Rightarrow R = 1.2r$

$\Rightarrow x = 20$

Group Event 10

In the attached division

G10.1 find A,

G10.2 find B,

G10.3 find C,

G10.4 find D.

					1	D	E
2	1	5	A	7	B	9	C
			F	G	H		
			J	5	K	9	
			L	5	M	5	
					N	4	P
					Q	R	S

$\overline{FGH} = 215$

$D \geq 5$

$D = 5, 7, 9$

$215\times 5 = 1055 \neq \overline{L5M5}$ (rejected)

$215\times 7 = 1505 = \overline{L5M5}$ (accepted)

$215\times 9 = 1935 \neq \overline{L5M5}$ (rejected)

$\therefore D = 7, L = 1, M = 0$

					1	D	E
2	1	5	A	7	B	9	C
			2	1	5		
			J	5	K	9	
			L	5	M	5	
					N	4	P
					Q	R	S

$J = 1, A = 3$

$E = 2, 3$ or 4

$\overline{N4P} = \overline{QRS}$

$215\times 2 = 430 \neq \overline{N4P}$ (rejected)

$215\times 3 = 645 = \overline{N4P}$ (accepted)

$215\times 4 = 860 \neq \overline{N4P}$ (rejected)

$\therefore E = 3$

					1	7	E
2	1	5	A	7	B	9	C
			2	1	5		
			J	5	K	9	
			1	5	0	5	
					N	4	P
					Q	R	S

$215\times 173 = 37195$

$\therefore A = 3, B = 1, C = 5, D = 7$

					1	7	3
2	1	5	3	7	1	9	5
			2	1	5		
			1	5	6	9	
			1	5	0	5	
					6	4	5
					6	4	5

Individual Events

SI			I1			I2			I3			I4			I5		
	A	20		*n*	10		*a*	48		*a*	2		*A*	40		*a*	45
	B	4		*a*	25		*b*	144		*b*	–3		*B*	6		*b*	15
	C	5		*z*	205		*c*	4		*c*	12		*C*	198		*c*	12
	D	$\frac{5}{2}$		*S*	1		*d*	572		*d*	140		*D*	7		*d*	2

Group Events

SG			G6			G7			G8			G9			G10		
		2550		*a*	1		*a*	–8		*A*	2		*x*	6		*c*	3
		2452		*b*	52		*b*	10		*b*	171		*y*	6		*a*	–2
	P	2501		*c*	13		area	116		*c*	3		T_{10}	200		*b*	5
	Q	10001		*d*	3		tan θ	2		*d*	27		*n*	19		*d*	5

Sample Individual Event

SI.1 Given $A = (b^m)^n + b^{m+n}$. Find the value of A when $b = 4$, $m = n = 1$.

$A = (4^1)^1 + 4^{1+1} = 4 + 16 = 20$

SI.2 If $2^A = B^{10}$ and $B > 0$, find the value of B.

$2^{20} = 4^{10}$

$\Rightarrow B = 4$

SI.3 Solve for C in the following equation: $\sqrt{\frac{20B+45}{C}} = C$.

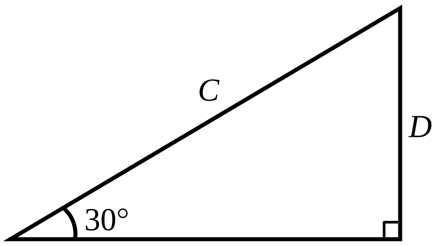
$\sqrt{\frac{20\times 4+45}{C}} = C$

$125 = C^3$

$\Rightarrow C = 5$

SI.4 Find the value of D in the figure.

$D = C \sin 30° = \frac{5}{2}$



Individual Event 1

I1.1 If the sum of the interior angles of an n-sided polygon is 1440°, find the value of n.

$180°\times(n – 2) = 1440°$

$\Rightarrow n = 10$

I1.2 If $x^2 – nx + a = 0$ has 2 equal roots, find the value of a.

$(–10)^2 – 4a = 0$

$\Rightarrow a = 25$

I1.3 In the figure, if $z = p + q$, find the value of z.

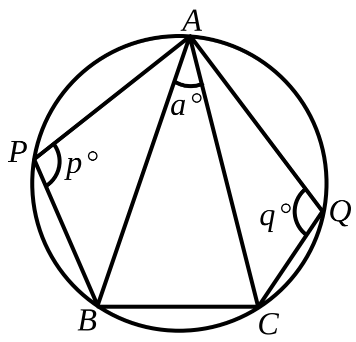
Reference: 1989 HI19

$\angle ACB = 180° – p°$ (opp. ∠s cyclic quad.)

$\angle ABC = 180° – q°$ (opp. ∠s cyclic quad.)

$180 – p + 180 – q + a = 180$ (∠s sum of Δ)

$z = p + q = 180 + a = 205$



I1.4 If $S = 1 + 2 – 3 – 4 + 5 + 6 – 7 – 8 + \ldots + z$, find the value of S.

Reference: 1985 FG7.4, 1988 FG6.4, 1990 FG10.1, 1991 FSI.1

$S = 1 + (2 – 3 – 4 + 5) + (6 – 7 – 8 + 9) + \ldots + (202 – 203 – 204 + 205) = 1$

Individual Event 2

I2.1 If $ar = 24$ and $ar^4 = 3$, find the value of a.

$r^3 = \frac{ar^4}{ar} = \frac{3}{24} = \frac{1}{8}$

$\Rightarrow r = \frac{1}{2}$

$ar = 24$

$\Rightarrow \frac{1}{2}a = 24$

$\Rightarrow a = 48$

I2.2 If $\left(x + \frac{a}{4}\right)^2 = x^2 + \frac{a}{2}\cdot x + b$, find the value of b.

$(x + 12)^2 = x^2 + 24x + 144$

$\Rightarrow b = 144$

I2.3 If $c = \log_2 \frac{b}{9}$, find the value of c.

$c = \log_2 \frac{144}{9}$

$= \log_2 16$

$= 4$

I2.4 If $d = 12^c - 142^2$, find the value of d.

$d = 12^4 - 142^2$

$= 144^2 - 142^2$

$= (144 + 142)(144 - 142)$

$= 2(286) = 572$

Individual Event 3

I3.1 If $a=\frac{\sin 15^\circ}{\cos 75^\circ}+\frac{1}{\sin^2 75^\circ}-\tan^2 15^\circ$, find the value of a.

$a=\frac{\sin 15^\circ}{\sin 15^\circ}+\sec^2 15^\circ-\tan^2 15^\circ$

$= 1 + 1 = 2$

I3.2 If the lines $ax + 2y + 1 = 0$ and $3x + by + 5 = 0$ are perpendicular to each other, find the value of b.

$-\frac{a}{2}\times\left(-\frac{3}{b}\right)=-1$

$\Rightarrow b = -3$

I3.3 The three points $(2, b)$, $(4, -b)$ and $(5, \frac{c}{2})$ are collinear. Find the value of c.

The three points are $(2, -3)$, $(4, 3)$ and $(5, \frac{c}{2})$, so their slopes are equal.

$\frac{3-(-3)}{4-2}=\frac{\frac{c}{2}-3}{5-4}$

$\Rightarrow \frac{c}{2}-3=3$

$\Rightarrow c = 12$

I3.4 If $\frac{1}{x}:\frac{1}{y}:\frac{1}{z}=3:4:5$ and $\frac{1}{x+y}:\frac{1}{y+z}=9c:d$, find the value of d.

$x:y:z=\frac{1}{3}:\frac{1}{4}:\frac{1}{5}$

$=\frac{20}{60}:\frac{15}{60}:\frac{12}{60}$

$= 20 : 15 : 12$

$x = 20k$, $y = 15k$, $z = 12k$

$\frac{1}{x+y}:\frac{1}{y+z}=\frac{1}{20k+15k}:\frac{1}{15k+12k}$

$= 27 : 35$

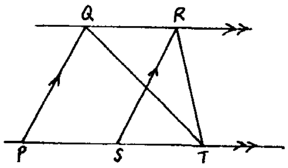
$= 108 : 140 = 9c : d$

$\Rightarrow d = 140$

Individual Event 4

I4.1 In the figure, the area of *PQRS* is 80 cm^2.

If the area of ΔQRT is A cm^2, find the value of A .



ΔQRT has the same base and same height as the parallelogram *PQRS*.

$A = \frac{1}{2} \cdot 80 = 40$

I4.2 If $B = \log_2\left(\frac{8A}{5}\right)$, find the value of B .

$B = \log_2\left(\frac{8 \cdot 40}{5}\right)$

$= \log_2 64$

$= \log_2 2^6$

$= 6$

I4.3 Given $x + \frac{1}{x} = B$. If $C = x^3 + \frac{1}{x^3}$, find the value of C .

$x + \frac{1}{x} = 6$

$x^2 + \frac{1}{x^2} = (x + \frac{1}{x})^2 - 2$

$= 6^2 - 2 = 34$

$C = x^3 + \frac{1}{x^3}$

$= \left(x + \frac{1}{x}\right)\left(x^2 + \frac{1}{x^2} - 1\right)$

$= 6(34 - 1) = 198$

I4.4 Let $(p, q) = qD + p$. If $(C, 2) = 212$, find the value of D .

$2D + C = 212$

$\Rightarrow 2D = 212 - 198 = 14$

$\Rightarrow D = 7$

Individual Event 5

I5.1 Let p, q be the roots of the quadratic equation $x^2 - 3x - 2 = 0$ and $a = p^3 + q^3$.
Find the value of a.

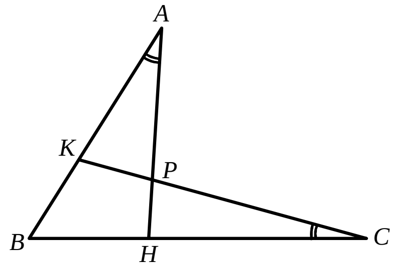
$p + q = 3$, $pq = -2$

$a = (p + q)(p^2 - pq + q^2)$

$= 3[(p + q)^2 - 3pq]$

$= 3[3^2 - 3(-2)] = 45$

I5.2 If $AH = a$, $CK = 36$, $BK = 12$ and $BH = b$, find the value of b.

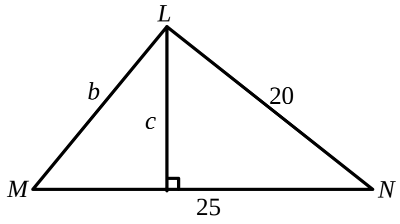


$\Delta ABH \sim \Delta CBK$ (equiangular)

$\frac{b}{12} = \frac{45}{36}$ (ratio of sides, ~Δs)

$b = 15$

I5.3 Find the value of c.



Reference: 1985 FG6.4

$15^2 + 20^2 = 25^2$

$\Rightarrow ML \perp LN$ (converse, Pythagoras' theorem)

Area of $\Delta MNL = \frac{1}{2}\cdot 15\cdot 20 = \frac{1}{2}\cdot 25c$

$c = 12$

I5.4 Let $\sqrt{2x+23}+\sqrt{2x-1}=c$ and $d=\sqrt{2x+23}-\sqrt{2x-1}$. Find the value of d.

Reference: 2014 HG1

$cd = \left(\sqrt{2x+23}+\sqrt{2x-1}\right)\left(\sqrt{2x+23}-\sqrt{2x-1}\right)$

$12d = (2x + 23) - (2x - 1) = 24$

$\Rightarrow d = 2$

Sample Group Event Reference HKCEE Mathematics 1990 Paper 1 Q14

Consider the following groups of numbers:

(2)
(4, 6)
(8, 10, 12)
(14, 16, 18, 20)
(22, 24, 26, 28, 30)
……………………………

SG.1 Find the last number of the 50th group.

$2 = 2\times1$

$6 = 2(1 + 2)$

$12 = 2(1 + 2 + 3)$

$20 = 2(1 + 2 + 3 + 4)$

$30 = 2(1 + 2 + 3 + 4 + 5)$

The last number of the 50th group

$= 2(1 + 2 + \ldots + 50)$

$= 2\cdot\frac{1}{2}\cdot 50\cdot(1+50) = 2550$

SG.2 Find the first number of the 50th group.

There are 50 numbers in the 50th group.

The first number of the 50th group = $2550 - 2(50 - 1) = 2452$

SG.3 Find the value of P if the sum of the numbers in the 50th group is $50P$.

$2452 + 2454 + \ldots + 2550 = 50P$

$\frac{1}{2}\cdot 50\cdot(2452+2550) = 50P$

$P = 2501$

SG.4 Find the value of Q if the sum of the numbers in the 100th group is $100Q$.

The last number in the 100th group = $2(1 + 2 + \ldots + 100) = 2\cdot\frac{1}{2}\cdot 100\cdot(1+100) = 10100$

The first number of the 100th group = $10100 - 2(100 - 1) = 9902$

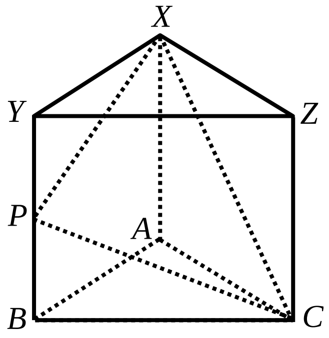
$9902 + 9904 + \ldots + 10100 = 100P$

$\frac{1}{2}\cdot 100\cdot(9902+10100) = 100P$

$P = 10001$

Group Event 6

As shown in the figure, ΔABC and ΔXYZ are equilateral triangles and are ends of a right prism. P is the mid-point of BY and $BP = 3$ cm, $XY = 4$ cm.



G6.1 If $a = \dfrac{CP}{PX}$, find the value of a.

$CP = \sqrt{3^2 + 4^2}$ cm = 5 cm = PX (Pythagoras' theorem)

$a = 1$

G6.2 If $CX = \sqrt{b}$ cm, find the value of b.

$CX = \sqrt{6^2 + 4^2}$ cm = $\sqrt{52}$ cm (Pythagoras' theorem)

$b = 52$

G6.3 If $\cos\angle PCX = \dfrac{\sqrt{c}}{5}$, find the value of c.

$\cos\angle PCX = \dfrac{\sqrt{52} \div 2}{5} = \dfrac{\sqrt{13}}{5}$

$\Rightarrow c = 13$

G6.4 If $\sin\angle PCX = \dfrac{2\sqrt{d}}{5}$, find the value of d.

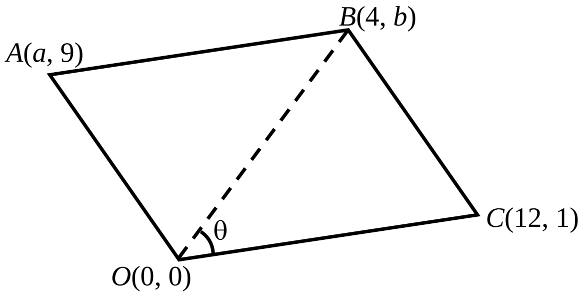
$\sin^2\angle PCX = 1 - \cos^2\angle PCX = \dfrac{12}{25}$

$\sin\angle PCX = \dfrac{2\sqrt{3}}{5}$

$\Rightarrow d = 3$

Group Event 7

In the figure, *OABC* is a parallelogram.



G7.1 Find the value of a .

$a - 0 = 4 - 12$

$\Rightarrow a = -8$

G7.2 Find the value of b .

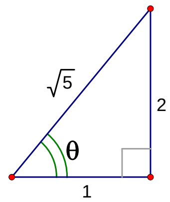
$b - 1 = 9 - 0$

$\Rightarrow b = 10$

G7.3 Find the area of *OABC*.

$$\text{Area} = 2\cdot\frac{1}{2}\begin{vmatrix} 0 & 0 \\ 12 & 1 \\ 4 & 10 \\ 0 & 0 \end{vmatrix} = 116$$

G7.4 Find the value of $\tan\theta$.



$OC = \sqrt{145}$

$OB = \sqrt{116}$

$BC = \sqrt{(12-4)^2 + (1-10)^2} = \sqrt{145}$

$$\cos\theta = \frac{\sqrt{145}^2 + \sqrt{116}^2 - \sqrt{145}^2}{2\left(\sqrt{145}\right)\left(\sqrt{116}\right)} = \frac{1}{\sqrt{5}}$$

$\tan\theta = 2$

Method 2

$$m_{OC} = \frac{1-0}{12-0} = \frac{1}{12}$$

$$m_{OB} = \frac{10-0}{4-0} = \frac{5}{2}$$

$$\tan\theta = \frac{\frac{5}{2} - \frac{1}{12}}{1 + \frac{5}{2}\cdot\frac{1}{12}} = 2$$

Group Event 8

G8.1 The area of an equilateral triangle of side A cm is $\sqrt{3}$ cm^2. Find the value of A.

$$\frac{1}{2}\cdot A^2\sin 60^\circ=\sqrt{3}$$

$\Rightarrow A = 2$

G8.2 If $19\times 243^{\frac{A}{5}}=b$, find the value of b.

$$b=19\times\left(3^5\right)^{\frac{2}{5}}=171$$

G8.3 The roots of the equation $x^3 - 173x^2 + 339x + 513 = 0$ are -1, b and c. Find the value of c.

$-1 + 171 + c$ = sum of roots = 173

$\Rightarrow c = 3$

G8.4 The base of a triangular pyramid is an equilateral triangle of side $2c$ cm.

If the height of the pyramid is $\sqrt{27}$ cm, and its volume is d cm^3, find the value of d.

$$d=\frac{1}{3}\cdot\frac{1}{2}\cdot\left(6^2\cdot\sin 60^\circ\right)\cdot\sqrt{27}=27$$

Group Event 9

If the area of a regular hexagon $ABCDEF$ is $54\sqrt{3}$ cm^2 and $AB = x$ cm, $AC = y\sqrt{3}$ cm,

G9.1 find the value of x .

The hexagon can be cut into 6 identical equilateral triangles

$$6\cdot\frac{1}{2}\cdot\left(x^2\cdot\sin 60^\circ\right)=54\sqrt{3}$$

$\Rightarrow x = 6$

G9.2 find the value of y .

$\angle ABC = 120°$

$AC^2 = (x^2 + x^2 - 2x^2 \cos 120°)$ cm^2

$= [6^2 + 6^2 - 2(6)^2\cdot\left(-\frac{1}{2}\right)]$ cm^2

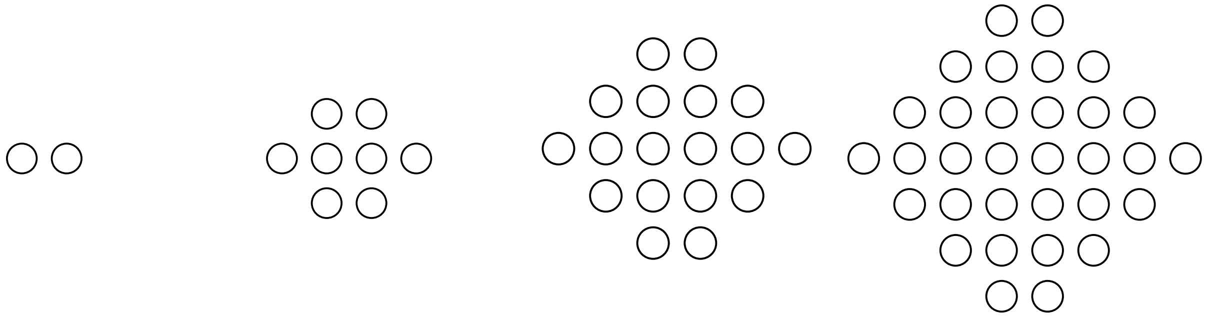
$= 3\times 6^2$ cm^2

$y\sqrt{3}=6\sqrt{3}$

$\Rightarrow y = 6$

G9.3 - G9.4 (Reference: 1991 FG8.1-2)

Consider the following number pattern:



$T_1 = 2$ $T_2 = 8$ $T_3 = 18$ $T_4 = 32$

G9.3 Find the value of T_{10} .

8 – 2 = 6, 18 – 8 = 10, 32 – 18 = 14

$\Rightarrow T_1 = 2$, $T_2 = 2 + 6$, $T_3 = 2 + 6 + 10$, $T_4 = 2 + 6 + 10 + 14$

$$T_{10}=\frac{10}{2}\cdot[2(2)+(10-1)\cdot 4]=200$$

G9.4 If $T_n = 722$, find the value of n .

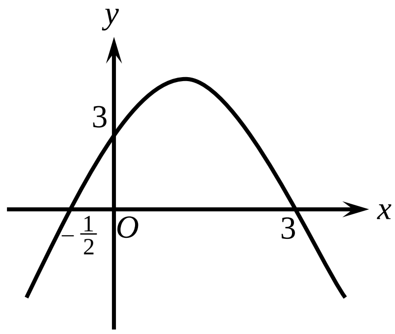
$$\frac{n}{2}\cdot[2(2)+(n-1)\cdot 4]=722$$

$n^2 = 361$

$n = 19$

Group Event 10

The following shows the graph of $y = ax^2 + bx + c$.



G10.1 Find the value of c .

$x = 0$, $y = c = 3$

G10.2 Find the value of a .

$y = a(x + \frac{1}{2})(x - 3)$

Sub. $x = 0$, $y = 3$

$\Rightarrow -\frac{3}{2}a = 3$

$a = -2$

G10.3 Find the value of b .

$3 - \frac{1}{2} = \text{sum of roots} = -\frac{b}{(-2)}$

$b = 5$

G10.4 If $y = x + d$ is tangent to $y = ax^2 + bx + c$, find the value of d .

Sub. $y = x + d$ into $y = ax^2 + bx + c$

$-2x^2 + 5x + 3 = x + d$

$2x^2 - 4x + d - 3 = 0$

$\Delta = (-4)^2 - 4(2)(d - 3) = 0$

$4 - 2d + 6 = 0$

$d = 5$

Individual Events

I1	***a***	2	**I2**	***a***	136	**I3**	***a***	4	**I4**	***a***	8	**I5**	***a***	20
	b	2		***b***	–2620		***b***	24		***b***	9		***b***	2
	c	2		***c***	100		***c***	50		***c***	4		***c***	257
	d	1		***d***	50		***d***	500		***d***	54		***d***	7

Group Events

G6	***p***	–2	**G7**	***a***	36	**G8**	***m***	–2	**G9**	***a***	9	**G10**	***a***	50
	m	8		***b***	18		***d***	3		***b***	3		***b***	10
	r	1		***c***	2		***n***	96		***x***	11		***c***	15
	s	–2		***d***	6		***s***	95856		***y***	10		***d***	60

Individual Event 1

I1.1 Given that $7^{2x} = 36$ and $7^{-x} = (6)^{-\frac{a}{2}}$, find the value of a.

$7^x = 6$

$\Rightarrow 7^{-x} = (6)^{-\frac{a}{2}} = 6^{-1}$

$\Rightarrow a = 2$

I1.2 Find the value of b if $\log_2\{\log_2[\log_2(2b) + a] + a\} = a$.

$\log_2\{\log_2[\log_2(2b) + 2] + 2\} = 2$

$\log_2[\log_2(2b) + 2] + 2 = 2^2 = 4$

$\log_2[\log_2(2b) + 2] = 2$

$\log_2(2b) + 2 = 2^2 = 4$

$\Rightarrow \log_2(2b) = 2$

$2b = 2^2 = 4$

$\Rightarrow b = 2$

I1.3 If c is the total number of positive roots of the equation $(x - b)(x - 2)(x + 1) = 3(x - b)(x + 1)$, find the value of c.

$(x - 2)(x - 2)(x + 1) - 3(x - 2)(x + 1) = 0$

$(x - 2)(x + 1)[(x - 2) - 3] = 0$

$(x - 2)(x + 1)(x - 5) = 0$

$x = 2, -1$ or 5

$\Rightarrow$ Number of positive roots $= c = 2$

I1.4 If $\sqrt{3-2\sqrt{2}} = \sqrt{c} - \sqrt{d}$, find the value of d.

Reference: 1999 HG3, 2001 FG2.1, 2011 HI7, 2015 FI4.2, 2015 FG3.1

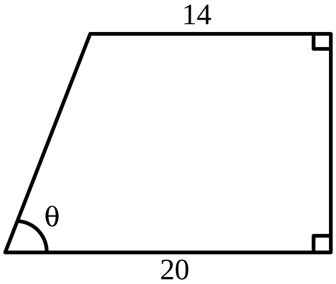
$\sqrt{3-2\sqrt{2}} = \sqrt{1-2\sqrt{2}+2}$

$= \sqrt{\left(\sqrt{1}\right)^2 - 2\sqrt{2} + \left(\sqrt{2}\right)^2}$

$= \sqrt{\left(\sqrt{2}-1\right)^2} = \sqrt{2} - 1$

$\Rightarrow d = 1$

Individual Event 2



I2.1 If $\sin\theta = \frac{4}{5}$, find $\boldsymbol{a}$, the area of the quadrilateral.

Let the height be $\boldsymbol{h}$.

$\tan\theta = \frac{4}{3} = \frac{\boldsymbol{h}}{6}$

$\Rightarrow \boldsymbol{h} = 8$

Area $= \frac{1}{2}(14+20)\cdot 8 = 136$

I2.2 If $\boldsymbol{b} = 126^2 - \boldsymbol{a}^2$, find $\boldsymbol{b}$.

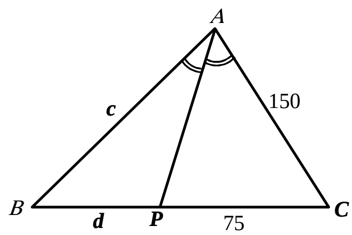
$\boldsymbol{b} = 126^2 - \boldsymbol{a}^2$

$= (126 - 136)(126 + 136) = -2620$

I2.3 Dividing \$(3000 + $\boldsymbol{b}$) in a ratio 5 : 6 : 8, the smallest part is \$$\boldsymbol{c}$. Find $\boldsymbol{c}$.

Sum of money = \$(3000 – 2620) = \$380

$\boldsymbol{c} = \frac{5}{5+6+8}\cdot 380 = \frac{5}{19}\cdot 380 = 100$



I2.4 In the figure, $\boldsymbol{AP}$ bisects $\angle \boldsymbol{BAC}$. Given that $\boldsymbol{AB} = \boldsymbol{c}$, $\boldsymbol{BP} = \boldsymbol{d}$, $\boldsymbol{PC} = 75$ and $\boldsymbol{AC} = 150$, find $\boldsymbol{d}$.

Let $\angle \boldsymbol{BAP} = \theta = \angle \boldsymbol{CAP}$, $\angle \boldsymbol{APC} = \alpha$, $\angle \boldsymbol{BPC} = 180° - \alpha$

$\frac{\boldsymbol{d}}{\sin\theta} = \frac{100}{\sin(180^\circ - \alpha)}$ … (1) and $\frac{75}{\sin\theta} = \frac{150}{\sin\alpha}$ … (2)

(1) ÷ (2) $\Rightarrow \boldsymbol{d} = 50$

Individual Event 3

I3.1 If a is the remainder when 2614303940317 is divided by 13, find a.

$2614303939000 = 13\times21100303000$

$2614303940317 = 13\times21100303000 + 1317 = 13\times21100303000 + 1313 + 4$

$a = 4$

I3.2 Let $P(x, b)$ be a point on the straight line $x + y = 30$ such that slope of $OP = a$ (O is the origin). Determine b. **(Reference: 1994 FI1.4)**

$x + b = 30$

$\Rightarrow x = 30 - b$

$m_{OP} = \frac{b}{30-b} = 4$

$\Rightarrow b = 120 - 4b$

$\Rightarrow b = 24$

I3.3 Two cyclists, initially (b + 26) km apart travelling towards each other with speeds 40 km/h and 60 km/h respectively. A fly flies back and forth between their noses at 100 km/h. If the fly flied c km before crushed between the cyclists, find c.

The velocity of one cyclist relative to the other cyclist is (40 + 60) km/h = 100 km/h.

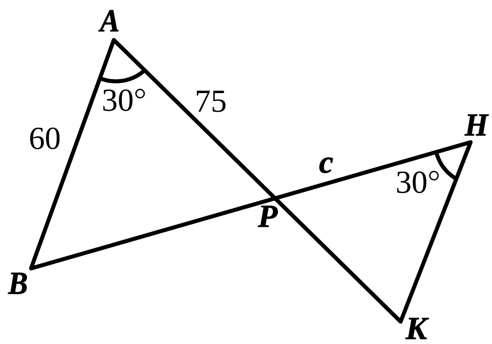
Distance between the two cyclists = (24 + 26) km = 50 km

Time for the two cyclists meet $= \frac{50}{100}\text{ h} = \frac{1}{2}\text{ h}$

The distance the fly flied $= \frac{1}{2}\times100\text{ km} = 50\text{ km}$

$\Rightarrow c = 50$

I3.4 In the figure, APK and BPH are straight lines. If d = area of triangle HPK, find d.



$\angle BAP = \angle KHP = 30°$ (given)

$\angle APB = \angle KPH$ (vert. opp. $\angle$s)

$\Delta ABP \sim \Delta HKP$ (equiangular)

$\frac{HK}{60} = \frac{50}{75}$

$\Rightarrow HK = 40$

$d = \frac{1}{2}\times50\times40\cdot\sin30^\circ = 500$

Individual Event 4

I4.1 Given that the means of x and y, y and z, z and x are respectively 5, 9, 10. If a is the mean of x, y, z, find the value of a.

$\frac{x+y}{2} = 5 \ldots (1)$; $\frac{y+z}{2} = 9 \ldots (2)$; $\frac{z+x}{2} = 10 \ldots (3)$

(1) + (2) + (3): $x + y + z = 24$

$\Rightarrow a = 8$

I4.2 The ratio of two numbers is 5 : a. If 12 is added to each of them, the ratio becomes 3 : 4. If b is the difference of the original numbers and $b > 0$, find the value of b.

Let the two numbers be $5k$, $8k$.

$\frac{5k+12}{8k+12} = \frac{3}{4}$

$\Rightarrow 20k + 48 = 24k + 36$

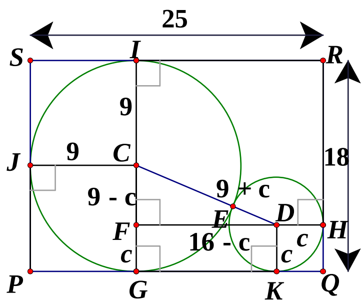
$\Rightarrow 4k = 12$

$\Rightarrow k = 3$

$5k = 15$, $8k = 24$

$b = 24 - 15 = 9$

I4.3 ***PQRS*** is a rectangle. If ***c*** is the radius of the smaller circle, find the value of ***c***.



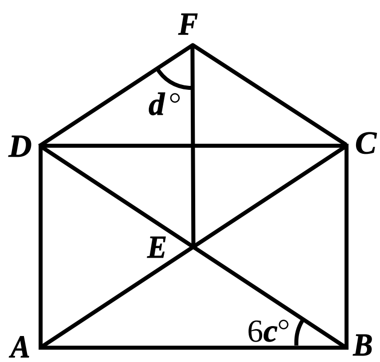
Let the centres of the two circles be ***C*** and ***D***, with radius 9 and ***c*** respectively.
Suppose the circles touch each other at ***E***.
Further, assume that the circle with centre at ***C*** touches ***SR***, ***PS***, ***PQ*** at ***I***, ***J*** and ***G*** respectively. Let the circle with centre at ***D*** touches ***PQ***, ***QR*** at ***K*** and ***H*** respectively.
Join ***CI***, ***CJ***, ***CG***, ***CED***, ***DF***, ***DK***, ***DH***.

CI ⊥ ***SR***, ***CJ*** ⊥ ***PS***, ***CG*** ⊥ ***PQ***, ***DK*** ⊥ ***PQ***, ***DH*** ⊥ ***PR*** (tangent ⊥ radius)
DK // ***HQ*** (corr. ∠s eq.)
∠***FDK*** = 90° (corr. ∠s, ***DK*** // ***HQ***)
DFGK is a rectangle (3 angles = 90°)
∴ ∠***DFG*** = 90° (∠s sum of polygon)
∠***DFC*** = 90° (adj. ∠s on st. line)

C, ***E***, ***D*** are collinear (∵ the two circles touch each other at ***E***)

CI = ***CJ*** = ***CG*** = ***CE*** = 9 (radii of the circle with centre at ***C***)
DH = ***DK*** = ***DE*** = ***c*** (radii of the circle with centre at ***D***)
CD = ***c*** + 9
FG = ***DK*** = ***c*** (opp. sides of rectangle ***DFGK***)
CF = 9 – ***c***
FD = ***GK*** (opp. sides of rectangle ***DFGK***)
  = ***PD*** – ***PG*** – ***KQ***
  = 25 – 9 – ***c*** (opp. sides of rectangle)
  = 16 – ***c***
$CF^2 + DF^2 = CD^2$ (Pythagoras' theorem)
$(9 - c)^2 + (16 - c)^2 = (9 + c)^2$
$81 - 18c + c^2 + 256 - 32c + c^2 = 81 + 18c + c^2$
$c^2 - 68c + 256 = 0$
$(c - 4)(c - 64) = 0$
$c = 4$ or 64 (> 18, rejected)

I4.4 ***ABCD*** is a rectangle and ***CEF*** is an equilateral triangle, ∠***ABD*** = 6***c***°, find the value of ***d***.



Reference: HKCEE MC 1982 Q51
∠***ABD*** = 24° (given)
∠***CAB*** = 24° (diagonals of rectangle)
∠***AEB*** = 132° (∠s sum of Δ)
∠***CED*** = 132° (vert. opp. ∠s)
∠***CEF*** = 60° (∠ of an equilateral triangle)
∠***DEF*** = 132° – 60° = 72°
ED = ***EC*** = ***EF*** (diagonals of rectangle, sides of equilateral Δ)
∴ Δ***DEF*** is isosceles (2 sides equal)
∠***EFD*** = ∠***EDF*** (base ∠s isos. Δ)
d = (180 – 72)÷ 2 = 54 (∠s sum of isos. Δ)

Individual Event 5

I5.1 Two opposite sides of a rectangle are increased by 50% while the other two are decreased by 20%. If the area of the rectangle is increased by a%, find a.

Let the length and width be x and y respectively.

$1.5x \times 0.8y = 1.2xy$

$\Rightarrow a = 20$

I5.2 Let $f(x) = x^3 - 20x^2 + x - a$ and $g(x) = x^4 + 3x^2 + 2$. If $h(x)$ is the highest common factor of $f(x)$ and $g(x)$, find $b = h(1)$.

Reference: 1992 HI5

$f(x) = x^3 - 20x^2 + x - 20 = (x^2 + 1)(x - 20)$

$g(x) = x^4 + 3x^2 + 2 = (x^2 + 1)(x^2 + 2)$

$h(x) = \text{H.C.F.} = x^2 + 1$

$b = h(1) = 2$

I5.3 It is known that $b^{16} - 1$ has four distinct prime factors, determine the largest one, denoted by c

$2^{16} - 1 = (2 - 1)(2 + 1)(2^2 + 1)(2^4 + 1)(2^8 + 1) = 3\times5\times17\times257$

$c = 257$

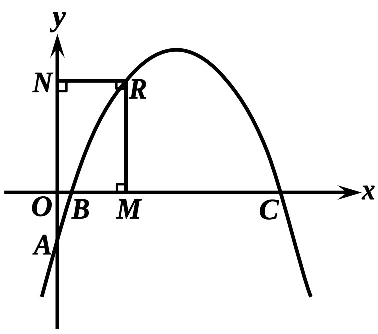
I5.4 When c is represented in binary scale, there are d '0's. Find d.

$257_{(x)} = 100000001_{(ii)}$

$\Rightarrow d = 7$

Group Event 6

The following shows the graph of $y = px^2 + 5x + p$. $A = (0, -2)$, $B = \left(\frac{1}{2}, 0\right)$, $C = (2, 0)$, $O = (0, 0)$.



G6.1 Find the value of p.

$y = p\left(x - \frac{1}{2}\right)(x - 2)$

It passes through $A(0, -2)$: $-2 = p\left(-\frac{1}{2}\right)(-2)$.

$p = -2$

G6.2 If $\frac{9}{m}$ is the maximum value of y, find the value of m.

$y = -2x^2 + 5x - 2$

$\frac{9}{m} = \frac{4(-2)(-2) - 5^2}{4(-2)}$

$\Rightarrow m = 8$

G6.3 Let R be a point on the curve such that $OMRN$ is a square. If r is the x-coordinate of R, find the value of r.

$R(r, r)$ lies on $y = -2x^2 + 5x - 2$

$r = -2r^2 + 5r - 2$

$2r^2 - 4r + 2 = 0$

$\Rightarrow r = 1$

G6.4 A straight line with slope $= -2$ passes through the origin cutting the curve at two points E and F. If $\frac{7}{s}$ is the y-coordinate of the midpoint of EF, find the value of s.

Sub. $y = -2x$ into $y = -2x^2 + 5x - 2$

$-2x = -2x^2 + 5x - 2$

$2x^2 - 7x + 2 = 0$

Let $E = (x_1, y_1)$, $F = (x_2, y_2)$.

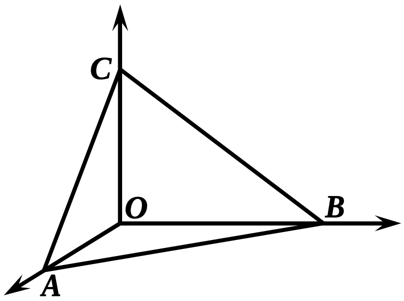
$x_1 + x_2 = \frac{7}{2}$

$\frac{7}{s} = \frac{y_1 + y_2}{2} = \frac{-2x_1 - 2x_2}{2} = -(x_1 + x_2) = \frac{7}{-2}$

$s = -2$

Group Event 7

OABC is a tetrahedron with ***OA***, ***OB*** and ***OC*** being mutually perpendicular. Given that ***OA*** = ***OB*** = ***OC*** = 6***x***.



G7.1 If the volume of ***OABC*** is $\boldsymbol{a}\boldsymbol{x}^3$, find ***a***.

$$\boldsymbol{a}\boldsymbol{x}^3=\frac{1}{3}\cdot\frac{1}{2}(6\boldsymbol{x})^2\cdot(6\boldsymbol{x})=36\boldsymbol{x}^3$$

$\Rightarrow \boldsymbol{a} = 36$

G7.2 If the area of $\Delta\boldsymbol{ABC}$ is $\boldsymbol{b}\sqrt{3}\boldsymbol{x}^2$, find ***b***.

$$\boldsymbol{AB}=\boldsymbol{BC}=\boldsymbol{AC}=\sqrt{(6\boldsymbol{x})^2+(6\boldsymbol{x})^2}=6\boldsymbol{x}\sqrt{2}$$

$\Delta\boldsymbol{ABC}$ is equilateral

$\angle\boldsymbol{BAC} = 60°$

$$\text{Area of }\Delta\boldsymbol{ABC}=\boldsymbol{b}\sqrt{3}\boldsymbol{x}^2=\frac{1}{2}\left(6\boldsymbol{x}\sqrt{2}\right)^2\sin 60^\circ=18\sqrt{3}\boldsymbol{x}^2$$

$\boldsymbol{b} = 18$

G7.3 If the distance from ***O*** to $\Delta\boldsymbol{ABC}$ is $\boldsymbol{c}\sqrt{3}\boldsymbol{x}$, find ***c***.

By finding the volume of ***OABC*** in two different ways.

$$\frac{1}{3}\cdot 18\sqrt{3}\boldsymbol{x}^2\times\left(\boldsymbol{c}\sqrt{3}\boldsymbol{x}\right)=36\boldsymbol{x}^3$$

$\boldsymbol{c} = 2$

G7.4 If θ is the angle of depression from ***C*** to the midpoint of ***AB*** and $\sin\theta=\frac{\sqrt{\boldsymbol{d}}}{3}$, find ***d***.

$$\frac{1}{3}\cdot 18\sqrt{3}\boldsymbol{x}^2\times\left(\boldsymbol{c}\sqrt{3}\boldsymbol{x}\right)=36\boldsymbol{x}^3$$

Let the midpoint of ***AB*** be ***M***.

$$\boldsymbol{OC}=6\boldsymbol{x},\quad \frac{\boldsymbol{OM}\times\boldsymbol{AB}}{2}=\frac{\boldsymbol{OA}\times\boldsymbol{OB}}{2}$$

$$\Rightarrow 6\boldsymbol{x}\sqrt{2}\cdot\boldsymbol{OM}=(6\boldsymbol{x})^2$$

$$\Rightarrow \boldsymbol{OM}=3\sqrt{2}\boldsymbol{x}$$

$$\boldsymbol{CM}=\sqrt{\boldsymbol{OM}^2+\boldsymbol{OC}^2}$$

$$=\sqrt{\left(3\sqrt{2}\boldsymbol{x}\right)^2+(6\boldsymbol{x})^2}$$

$$=3\sqrt{6}\boldsymbol{x}$$

$$\sin\theta=\frac{\sqrt{\boldsymbol{d}}}{3}=\frac{\boldsymbol{OC}}{\boldsymbol{CM}}$$

$$=\frac{6\boldsymbol{x}}{3\sqrt{6}\boldsymbol{x}}=\frac{\sqrt{6}}{3}$$

$\boldsymbol{d} = 6$

Group Event 8

Given that the equation $\boldsymbol{x}^2 + (\boldsymbol{m} + 1)\boldsymbol{x} - 2 = 0$ has 2 integral roots $(\alpha + 1)$ and $(\beta + 1)$ with $\alpha < \beta$ and $\boldsymbol{m} \neq 0$. Let $\boldsymbol{d} = \beta - \alpha$.

G8.1 Find the value of $\boldsymbol{m}$.

$(\alpha + 1)(\beta + 1) = -2$

$\Rightarrow \alpha + 1 = -1, \beta + 1 = 2$ or $\alpha + 1 = -2, \beta + 1 = 1$

$\Rightarrow (\alpha, \beta) = (-2, 1), (-3, 0)$

When $(\alpha, \beta) = (-3, 0)$, sum of roots $= (\alpha + 1) + (\beta + 1) = -(\boldsymbol{m} + 1) \Rightarrow \boldsymbol{m} = 0$ (rejected)

When $(\alpha, \beta) = (-2, 1)$, sum of roots $= (\alpha + 1) + (\beta + 1) = -(\boldsymbol{m} + 1) \Rightarrow \boldsymbol{m} = -2$

G8.2 Find the value of $\boldsymbol{d}$.

$\boldsymbol{d} = \beta - \alpha = 1 - (-2) = 3$

Let $\boldsymbol{n}$ be the total number of integers between 1 and 2000 such that each of them gives a remainder of 1 when it is divided by 3 or 7.

Reference: 1994 FG8.1-2, 1998 HI6, 2015 FI3.1

G8.3 Find the value of $\boldsymbol{n}$.

These numbers give a remainder of 1 when it is divided by 21.

They are $1, 21 + 1, 21\times2 + 1, \dots , 21\times95 + 1$ (= 1996)

$\boldsymbol{n} = 96$

G8.4 If $\boldsymbol{s}$ is the sum of all these $\boldsymbol{n}$ integers, find the value of $\boldsymbol{s}$.

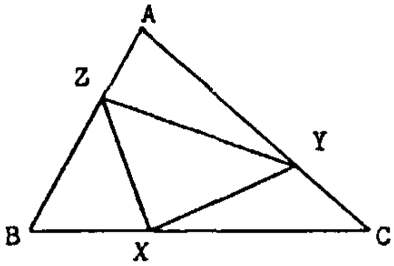
$\boldsymbol{s} = 1 + 22 + 43 + \dots + 1996 = \frac{1}{2}(1+1996)\cdot 96 = 95856$

Group Event 9

BC, ***CA***, ***AB*** are divided respectively by the points ***X***, ***Y***, ***Z*** in the ratio 1 : 2. Let

area of ΔAZY : area of $\Delta ABC = 2 : a$ and

area of ΔAZY : area of $\Delta XYZ = 2 : b$.



G9.1 Find the value of ***a***.

area of $\Delta AZY = \frac{2}{3}$ area of ΔACZ (same height)

$= \frac{2}{3} \times \frac{1}{3}$ area of ΔABC (same height)

$\Rightarrow a = 9$

G9.2 Find the value of ***b***.

Reference: 2000 FI5.3

Similarly, area of $\Delta BZX = \frac{2}{9}$ area of ΔABC; area of $\Delta CXY = \frac{2}{9}$ area of ΔABC

area of ΔXYZ = area of ΔABC – area of ΔAZY – area of ΔBZX – area of ΔCXY

$= \frac{1}{3}$ area of ΔABC

$2 : b$ = area of ΔAZY : area of $\Delta XYZ = \frac{2}{9} : \frac{1}{3}$

$\Rightarrow b = 3$

A die is thrown 2 times. Let $\frac{x}{36}$ be the probability that the sum of numbers obtained is 7 or 8 and $\frac{y}{36}$ be the probability that the difference of numbers obtained is 1.

G9.3 Find the value of ***x***.

Favourable outcomes are (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6,1), (2,6), (3,5), (4,4), (5,3), (6,2)

P(7 or 8) $= \frac{x}{36}$

$\Rightarrow x = 11$

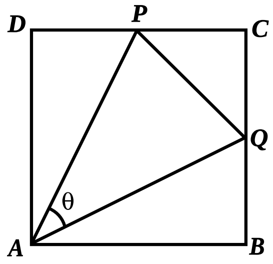
G9.4 Find the value of ***y***.

Favourable outcomes are (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6,5), (5, 4), (4, 3), (3, 2), (2, 1).

P(difference is 1) $= \frac{y}{36}$

$\Rightarrow y = 10$

Group Event 10



ABCD is a square of side length $20\sqrt{5}\boldsymbol{x}$. ***P***, ***Q*** are midpoints of ***DC*** and ***BC*** respectively.

G10.1 If ***AP*** = ***ax***, find ***a***.

$$AP = \sqrt{AD^2 + DP^2}$$
$$= \sqrt{\left(20\sqrt{5}x\right)^2 + \left(10\sqrt{5}x\right)^2} = 50x$$
$$\Rightarrow a = 50$$

G10.2 If $PQ = b\sqrt{10}x$, find ***b***.

$$PQ = \sqrt{CP^2 + CQ^2} = 10\sqrt{10}x$$
$$\Rightarrow b = 10$$

G10.3 If the distance from ***A*** to ***PQ*** is $c\sqrt{10}x$, find ***c***.

$c\sqrt{10}x$ = ***AC*** – distance from ***C*** to ***PQ***

$$= 20\sqrt{5}x \cdot \sqrt{2} - 10\sqrt{5}x \cdot \left(\frac{1}{\sqrt{2}}\right)$$
$$= 15\sqrt{10}x$$
$$\Rightarrow c = 15$$

G10.4 If $\sin\theta = \frac{d}{100}$, find ***d***.

$$\text{Area of } \Delta APQ = \frac{1}{2} \cdot AP \cdot AQ \sin\theta = \frac{1}{2} \cdot PQ \cdot \left(c\sqrt{10}x\right)$$
$$\Leftrightarrow \frac{1}{2} \cdot (50x)^2 \sin\theta = \frac{1}{2} \cdot 10\sqrt{10}x \cdot 15\sqrt{10}x$$
$$\sin\theta = \frac{d}{100} = \frac{3}{5}$$
$$\Rightarrow d = 60$$

Individual Events

SI	***a***	2	**I1**	***a***	6	**I2**	***A***	$\frac{5}{16}$	**I3**	***a***	6	**I4**	***a***	8	**I5**	***A***	1
	b	54		***b***	12		***B***	60		***b***	*9 see the remarks		***b***	10		***B***	36
	c	2		***c***	10		***C***	15		***c***	16		***c***	3		***C***	$\frac{1}{2}$
	d	1		***d***	20		***D***	68		***d***	48		***d***	203		***D***	50

Group Events

SG	***a***	19	**G6**	***a***	$4, \frac{1}{2}$	**G7**	***a***	24	**G8**	***a***	56	**G9**	***A***	2	**G10**	***a***	10
	b	8		***b***	16		***b***	1024		***b***	83		***B***	1		***b***	$\sqrt{37}$
	c	$\frac{1}{50}$		***c***	$\frac{3}{7}$		***c***	2		***c***	256		***C***	4		***c***	$\frac{1}{16}$
	d	200		***d***	186		***d***	–1		***d***	711040		***D***	9		***d***	4

Sample Individual Event (1985 Final Sample Individual Event)

SI.1 The sum of two numbers is 40, their product is 20 .

If the sum of their reciprocals is a, find the value of a .

Let the two numbers be x, y .

$x + y = 40$; $xy = 20$

$$a = \frac{1}{x} + \frac{1}{y}$$

$$= \frac{x+y}{xy} = \frac{40}{20}$$

$= 2$

SI.2 If b cm^2 is the total surface area of a cube of side $(a+1)$ cm, find the value of b .

$b = 6(2 + 1)^2 = 54$

SI.3 One ball is taken at random from a bag containing $(b – 4)$ white balls and $(b + 46)$ red balls.

If $\frac{c}{6}$ is the probability that the ball is white, find the value of c .

There are 50 white balls and 100 red balls.

$$\text{P(white ball)} = \frac{50}{150} = \frac{1}{3} = \frac{2}{6} = \frac{c}{6}$$

$\Rightarrow c = 2$

SI.4 The length of a side of an equilateral triangle is c cm.

If its area is $d\sqrt{3}$ cm^2, find the value of d .

$$\frac{1}{2}(2)^2 \sin 60^\circ = d\sqrt{3}$$

$$\sqrt{3} = d\sqrt{3}$$

$\Rightarrow d = 1$

Individual Event 1

I1.1 The equation $x^2 - ax + (a + 3) = 0$ has equal roots. Find a, if a is a positive integer.

$\Delta = (-a)^2 - 4(a + 3) = 0$

$a^2 - 4a - 12 = 0$

$(a - 6)(a + 2) = 0$

$a = 6$ or $a = -2$ (rejected)

I1.2 In a test, there are 20 questions. a marks will be given to a correct answer and 3 marks will be deducted for each wrong answer. A student has done all the 20 questions and scored 48 marks. Find b, the number of questions that he has answered correctly.

Reference: 1998 HG10

$6b - 3(20 - b) = 48$

$9b = 108$

$\Rightarrow b = 12$

I1.3 If $x : y = 2 : 3$, $x : z = 4 : 5$, $y : z = b : c$, find the value of c .

$x : y : z = 4 : 6 : 5$

$y : z = 6 : 5 = 12 : 10$

$\Rightarrow c = 10$

I1.4 Let $P(x, d)$ be a point on the straight line $x + y = 22$ such that the slope of OP equals to c (O is the origin). Determine the value of d .

Reference: 1993 FI3.2

$x + d = 22$

$\Rightarrow x = 22 - d$

$m_{OP} = \frac{d}{x} = c$

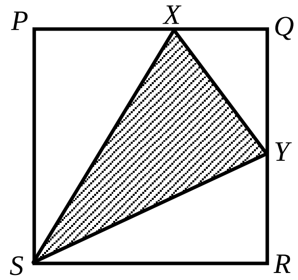
$\Rightarrow \frac{d}{22-d} = 10$

$d = 220 - 10d$

$\Rightarrow d = 20$

Individual Event 2

I2.1 In square *PQRS*, *Y* is the mid-point of the side *QR* and $PX = \frac{3}{4}PQ$.



If *A* is the ratio of the area of the shaded triangle to the area of the square, find *A*.

Let $PQ = 4x$, $PX = 3x$, $QX = x$, $QY = YR = 2x$

$$A = \frac{(4x)^2 - \frac{1}{2}\cdot 4x(3x) - \frac{1}{2}\cdot x(2x) - \frac{1}{2}\cdot 4x(2x)}{(4x)^2}$$

$$= \frac{5}{16}$$

I2.2 A man bought a number of ping-pong balls where a 16*A*% sales tax is added. If he did not have to pay tax he could have bought 3 more balls for the same amount of money. If *B* is the total number of balls that he bought, find *B*.

Let the price of 1 ping-pong ball be *x*. Sales tax = 5%

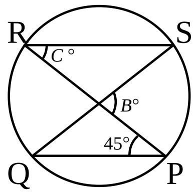
$Bx(1 + 5\%) = (B + 3)x$

$\frac{21}{20}B = B + 3$

$\Rightarrow 21B = 20B + 60$

$\Rightarrow B = 60$

I2.3 Refer to the diagram, find *C*.



$\angle PQS = C°$ ($\angle$s in the same segment)

$C + 45 = B$ (ext. $\angle$ of Δ)

$C = 60 - 45 = 15$

I2.4 The sum of 2*C* consecutive even numbers is 1170. If *D* is the largest of them, find *D*.

$$\frac{30}{2}[2D + (30-1)\cdot(-2)] = 1170$$

$\Rightarrow D = 68$

Individual Event 3

I3.1 If $183a8$ is a mu1tiple of 287, find the value of a .

$287 = 7\times41$ and $2614\times7 = 18298$

$183a8 - 18298 = 10(a + 1)$, a multiple of 7

$a = 6$

Method 2 The quotient $\dfrac{183a8}{287}$ should be a two digit number.

The first digit of $\dfrac{183}{3}$ (approximate value) is 6. The last digit must be 4 ($\because$ $7\times4 = 28$)

$\therefore$ $287\times64 = 18368$

$\Rightarrow a = 6$

I3.2 The number of positive factors of a^2 is b, find the value of b .

Reference 1993 HI8, 1997 HI3, 1998 HI10, 1998 FI1.4, 2002 FG4.1, 2005 FI4.4

Remark: The original question is: The number of factors of a^2 ⋯⋯, which may include negative factors.

$6^2 = 2^2\times3^2$

Factors of 36 are in the form $2^x\times3^y$, where $0 \le x \le 2$, $0 \le y \le 2$.

The number of factors = $(1 + 2)(1 + 2) = 9$

I3.3 In an urn, there are c balls, b of them are either black or red, $(b + 2)$ of them are either red or white and 12 of them are either black or white. Find the value of c .

Suppose there are x black balls, y red balls, z white balls.

$x + y = 9$ …… (1)

$y + z = 11$ …… (2)

$z + x = 12$ …… (3)

(1) + (2) + (3): $2(x + y + z) = 32$

$c = x + y + z = 16$

I3.4 Given $f(3 + x) = f(3 - x)$ for all values of x, and the equation $f(x) = 0$ has exactly c distinct roots. Find d, the sum of these roots.

Reference: 2010 FG3.4

Let one root be $3 + \alpha$.

$f(3 + \alpha) = 0 = f(3 - \alpha)$

$\Rightarrow 3 - \alpha$ is also a root.

$3 + \alpha + 3 - \alpha = 6$

$\therefore$ Sum of a pair of roots = 6

There are 16 roots, i.e. 8 pairs of roots

Sum of all roots = $8\times6 = 48$

Individual Event 4

I4.1 The remainder when $x^6 - 8x^3 + 6$ is divided by $(x - 1)(x - 2)$ is $7x - a$, find a.

Let $f(x) = x^6 - 8x^3 + 6$

$f(1) = 1 - 8 + 6 = 7 - a$

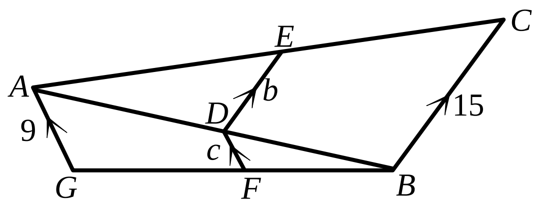
$a = 8$

I4.2 If $x^2 - x + 1 = 0$ and $b = x^3 - 3x^2 + 3x + a$, find b.

$b = x(x^2 - x + 1) - 2(x^2 - x + 1) + 10$

$= 10$

I4.3 Refer to the diagram, find c.



Reference: 1989 FG10.2

$\Delta ADE \sim \Delta ABC$, $AD : AB = 10 : 15 = 2 : 3$

$AD : DB = 2 : 1$

$BD : AB = 1 : 3$

$\Delta BDF \sim \Delta BAG$, $c : 9 = 1 : 3$

$c = 3$

I4.4 If c boys were all born in June 1990 and the probability that their birthdays are all different is $\frac{d}{225}$, find the value of d.

P(3 boys were born in different days) $= 1\times\frac{29}{30}\times\frac{28}{30} = \frac{d}{225}$

$d = 203$

Individual Event 5

I5.1 Given $1-\frac{4}{x}+\frac{4}{x^2}=0$. If $A=\frac{2}{x}$, find the value of A.

Reference: 1999 FI5.2

$$\left(1-\frac{2}{x}\right)^2=0$$

$$\Rightarrow A=\frac{2}{x}=1$$

I5.2 If B circular pipes each with an internal diameter of A cm carry the same amount of water as a pipe with an internal diameter 6 cm, find the value of B.

$\pi(1)^2 \cdot B = \pi(6)^2$

$\Rightarrow B = 36$

I5.3 If C is the area of the triangle formed by x-axis, y-axis and the line $Bx + 9y = 18$, find the value of C.

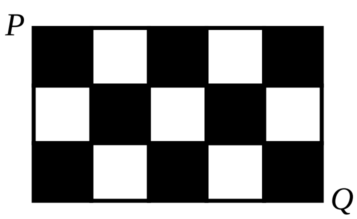
Reference: 1990 FI3.3

$36x + 9y = 18$

x-intercept $=\frac{1}{2}$, y-intercept $= 2$

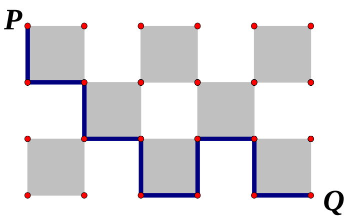
$C=\frac{1}{2}\cdot\frac{1}{2}\cdot 2=\frac{1}{2}$

I5.4 Fifteen square tiles with side $10C$ units long are arranged as shown. An ant walks along the edges of the tiles, always keeping a black tile on its left. Find the shortest distance D that the ant would walk in going from P to Q.



Length of a square = $10C$ = 5

As shown in the figure, $D = 10(10C) = 50$



Sample Group Event (1985 Sample Group Event)

SG.1 If $x*y = xy + 1$ and $a = (2*4)*2$, find the value of a.

$2*4 = 2(4) + 1 = 9$

$(2 * 4)*2 = 9*2 = 9(2) + 1 = 19$

SG.2 If the b^{th} prime number is a, find the value of b.

List the prime number in ascending order: 2, 3, 5, 7, 11, 13, 17, 19 .

$b = 8$

SG.3 If $c = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{50}\right)$, find the value of c in the simplest fractional form.

$c = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\cdots\cdot\frac{49}{50} = \frac{1}{50}$

SG.4 If d is the area of a square inscribed in a circle of radius 10 , find the value of d.

Diameter = 20 = diagonal of the square

Let the side of the square be x.

By Pythagoras' Theorem, $2x^2 = 20^2 = 400$

$d = x^2 = 200$

Group Event 6

G6.1 If $\log_2 a - 2\log_a 2 = 1$, find the value of a.

$$\frac{\log a}{\log 2} - \frac{2\log 2}{\log a} = 1$$

$(\log a)^2 - 2(\log 2)^2 = \log 2 \log a$

$(\log a)^2 - \log 2 \log a - 2(\log 2)^2 = 0$

$(\log a - 2\log 2)(\log a + \log 2) = 0$

$\log a = 2\log 2$ or $-\log 2$

$a = 4$ or $\frac{1}{2}$

G6.2 If $b = \log_3[2(3+1)(3^2+1)(3^4+1)(3^8+1)+1]$, find the value of b.

Reference: 2016 FG1.4

$b = \log_3[(3-1)(3+1)(3^2+1)(3^4+1)(3^8+1)+1]$

$= \log_3[(3^2-1)(3^2+1)(3^4+1)(3^8+1)+1]$

$= \log_3[(3^4-1)(3^4+1)(3^8+1)+1]$

$= \log_3[(3^8-1)(3^8+1)+1]$

$= \log_3(3^{16}-1+1) = 16$

G6.3 If a 31-day month is taken at random, find c, the probability that there are 5 Sundays in the month.

1^{st} day = Sunday $\Rightarrow 29^{th}$ day = 5^{th} Sunday

1^{st} day = Saturday $\Rightarrow 30^{th}$ day = 5^{th} Sunday

1^{st} day = Friday $\Rightarrow 31^{st}$ day = 5^{th} Sunday

Probability $= \frac{3}{7}$

G6.4 A group of 5 people is to be selected from 6 men and 4 women. Find d, the number of ways that there are always more men than women.

3 men and 2 women, number of combinations $= C_3^6 \cdot C_2^4 = 120$

4 men and 1 woman, number of combinations $= C_4^6 \cdot C_1^4 = 60$

5 men, number of combinations $= C_5^6 = 6$

Total number of ways = 120 + 60 + 6 = 186

Group Event 7

G7.1 There are a zeros at the end of the product $1\times2\times3\times\ldots\times100$. Find the value of a.

Reference: 1990 HG6, 1996 HI3, 2004 FG1.1, 2011 HG7, 2012 FI1.4, 2012 FG1.3

When each factor of 5 is multiplied by 2, a trailing zero will appear in the product.
The number of factors of 2 is clearly more than the number of factors of 5 in 100!
It is sufficient to find the number of factors of 5.
5, 10, 15, … , 100; altogether 20 numbers, have at least one factor of 5.
25, 50, 75, 100; altogether 4 numbers, have two factors of 5.
$\therefore$ Total number of factors of 5 is 20 + 4 = 24
There are 24 trailing zeros of 100!
$\Rightarrow a = 24$

G7.2 Find b, if b is the remainder when 1998^{10} is divided by 10^4.

$1998^{10} = (2000 - 2)^{10}$

$= \sum_{k=0}^{10} C_k^{10} \cdot 2000^{10-k} \cdot 2^k$

$= \sum_{k=0}^{9} C_k^{10} \cdot 2000^{10-k} \cdot 2^k + 2^{10}$

$\equiv 2^{10} \bmod 10^4$ ($\because C_9^{10} \cdot 2000^{10-9} \cdot 2^1 = 10000m$, where m is an integer)

$\equiv 1024 \bmod 10^4$

$\Rightarrow b = 1024$

G7.3 Find the largest value of c, if $c = 2 - x + 2\sqrt{x-1}$ and $x > 1$.

$(c + x - 2)^2 = 4(x - 1)$
$c^2 + x^2 + 4 + 2cx - 4x - 4c = 4x - 4$
$x^2 + 2(c - 4)x + (c^2 - 4c + 8) = 0$
For real values of x, $\Delta \geq 0$
$4(c - 4)^2 - 4(c^2 - 4c + 8) \geq 0$
$c^2 - 8c + 16 - c^2 + 4c - 8 \geq 0$
$8 \geq 4c$
$\Rightarrow c \leq 2$
$\Rightarrow$ The largest value $c = 2$

Method 2

Let $y = \sqrt{x-1}$, then $y^2 = x - 1$
$\Rightarrow x = y^2 + 1$
$c = 2 - (y^2 + 1) + 2y = 2 - (1 - y)^2 \leq 2$
The largest value of $c = 2$.

G7.4 Find the least value of d, if $\left|\frac{3-2d}{5}+2\right| \leq 3$.

$-3 \leq \frac{3-2d}{5} + 2 \leq 3$

$\Leftrightarrow -5 \leq \frac{3-2d}{5} \leq 1$

$\Leftrightarrow -25 \leq 3 - 2d \leq 5$
$\Leftrightarrow -28 \leq -2d \leq 2$
$\Leftrightarrow 14 \geq d \geq -1$
The least value of $d = -1$

Group Event 8

G8.1 From 1 to 121, there are a numbers which are multiplies of 3 or 5. Find the value of a .

Reference: 1993 FG8.3-4, 1998 HI6, 2015 FI3.1

Number of multiples of 3 = 40 (120 = 3×40)

Number of multiples of 5 = 24 (120 = 5×24)

Number of multiples of 15 = 8 (120 = 15×8)

Number of multiples of 3 or 5 = a = 40 + 24 – 8 = 56

G8.2 From 1 to 121, there are b numbers which are not divisible by 5 nor 7. Find the value of b .

Number of multiples of 5 = 24 (120 = 5×24)

Number of multiples of 7 = 17 (119 = 7×17)

Number of multiples of 35 = 3 (105 = 35×3)

Number of multiples of 5 or 7 = 24 + 17 – 3 = 38

Number which are not divisible by 5 nor 7 = 121 – 38 = 83

From the digits 1, 2, 3, 4, when each digit can be used repeatedly, 4-digit numbers are formed. Find

G8.3 c, the number of 4-digit numbers that can be formed.

$c = 4^4 = 256$

G8.4 d, the sum of all these 4-digit numbers.

Reference: 2002 HI4

∵ There are 256 different numbers

∴ 1, 2, 3, 4 each appears 64 times in the thousands, hundreds, tens and units digit.

$d = [1000(1 + 2 + 3 + 4) + 100(1 + 2 + 3 + 4) + 10(1 + 2 + 3 + 4) + 1 + 2 + 3 + 4]\times 64$

$= 1111(10)\times 64 = 711040$

Group Event 9

A, B, C, D are different integers ranging from 0 to 9 and
$(ABA)^2 = (CCDCC) < 100000$.
Find the values of A, B, C and D .

$$\begin{array}{r} A\ B\ A \\ \times\quad A\ B\ A \\ \hline C\ C\ D\ C\ C \end{array}$$

$(ABA) < \sqrt{100000} < 316$

$A = 1, 2$ or 3

When $A = 1$, then $A^2 = 1 = C$ contradict that A and C must be different $\therefore$ rejected

When $A = 2$, $C = 4$

$(202 + 10B)^2 = 44044 + 100D$

$40804 + 4040B + 100B^2 = 44044 + 100D$

$4040B + 100B^2 = 3240 + 100D$

$404B + 10B^2 = 324 + 10D$

$\therefore B = 1, 414 = 324 + 10D$

$\Rightarrow D = 9$

When $A = 3$, $C = 9$

$(303 + 10B)^2 = 99099 + 100D$

$91809 + 6060B + 100B^2 = 99099 + 100D$

$606B + 10B^2 = 729 + 10D$

$\therefore B = 1, 616 = 729 + 10D$

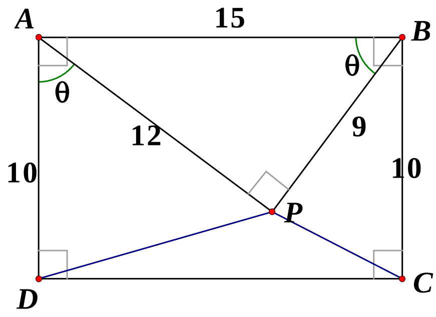
$\Rightarrow$ no solution for D

$\therefore A = 2, B = 1, C = 4, D = 9$

Group Event 10

In rectangle *ABCD*, *AD* = 10, *CD* = 15, *P* is a point inside the rectangle such that *PB* = 9, *PA* = 12. Find **(Reference: 2001 FG2.2, 2003 FI3.4, 2018 HI7)**

G10.1 *a*, the length of *PD* and



$AP^2 + BP^2 = 12^2 + 9^2 = 144 + 81 = 225 = 15^2 = AB^2$

∴ $\angle APB = 90°$ (Converse, Pythagoras' theorem)

Let $\angle ABP = \theta$, then $\cos\theta = \frac{9}{15} = \frac{3}{5}$, $\sin\theta = \frac{4}{5}$

$\angle BAP = 90° - \theta$ (∠s sum of Δ)

$\angle DAP = \theta$

$\angle PBC = 90° - \theta$

$a = PD = \sqrt{10^2 + 12^2 - 2\cdot 10\cdot 12\cdot \frac{3}{5}}$ (Cosine rule on Δ*ADP*)

$a = 10$

G10.2 *b*, the length of *PC*.

$b = CP = \sqrt{10^2 + 9^2 - 2\cdot 10\cdot 9\cdot \frac{4}{5}} = \sqrt{37}$

G10.3 It is given that sin 2θ = 2 sin θ cos θ. Find *c*, if $c = \frac{\sin 20^\circ \cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ}{\sin 160^\circ}$.

$$C = \frac{2\sin 20^\circ \cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ}{2\sin 160^\circ} = \frac{2\sin 40^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ}{4\sin 160^\circ}$$

$$= \frac{2\sin 80^\circ \cos 60^\circ \cos 80^\circ}{8\sin 160^\circ} = \frac{\sin 160^\circ \times \frac{1}{2}}{8\sin 160^\circ} = \frac{1}{16}$$

G10.4 It is given that $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$. Find *d*, if

$d = (1 + \tan 21°)(1 + \tan 22°)(1 + \tan 23°)(1 + \tan 24°)$.

If $A + B = 45°$, $1 = \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

$1 - \tan A \tan B = \tan A + \tan B$

$2 = 1 + \tan A + \tan B + \tan A \tan B$

$(1 + \tan A)(1 + \tan B) = 2$

$d = (1 + \tan 21°)(1 + \tan 24°)(1 + \tan 22°)(1 + \tan 23°)$

$= 2\times 2 = 4$

Individual Events

I1	***a***	$\frac{1}{2}$	**I2**	***x***	0	**I3**	***a***	$\frac{1}{2}$	**I4**	***r***	3	**I5**	***a***	2
	b	$3\sqrt{2}$		***y***	3		***b***	8		***s***	4		***b***	2
	c	3		***z***	2		***c***	2		***t***	5		***c***	12
	d	1		***w***	$\frac{1}{6}$		***d***	120		***u***	41		***d***	$16\sqrt{3}$

Group Events

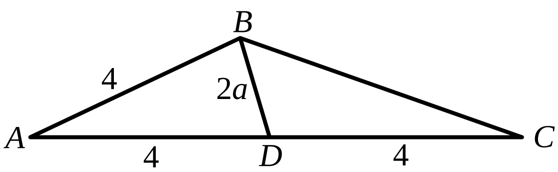
G6	***a***	5	**G7**	***a***	$\frac{1}{2}$	**G8**	***V***	1	**G9**	***A***	9	**G10**	***a***	4
	b	2		***b***	$2\sqrt{2}$		***V***	0		***B***	6		***b***	13
	c	$\frac{1}{4}$		***c***	700		***r***	3		***C***	8		***c***	16
	d	$\frac{1}{1995}$		***d***	333		***V***	35		***D***	2		***d***	$\frac{1}{10}$

Individual Event 1

I1.1 Find a, if $a = \log_{\frac{1}{4}} \frac{1}{2}$.

$$a = \log_{\frac{1}{4}} \frac{1}{2} = \log_{\frac{1}{4}} \left(\frac{1}{4}\right)^{\frac{1}{2}} = \frac{1}{2}$$

I1.2 In the figure, $AB = AD = DC = 4$, $BD = 2a$. Find b, the length of BC.



Let $\angle ADB = \theta$, $\angle CDB = 180° - \theta$ (adj. ∠s on st. line)

In ΔABD, $\cos \theta = \frac{a}{4} = \frac{1}{8}$

Apply cosine formula on ΔBCD.

$b^2 = (2a)^2 + 4^2 - 2(2a)\cdot 4\cdot\cos(180° - \theta)$

$b^2 = 1 + 16 - 2\cdot 4\cdot(-\cos\theta) = 17 + 8\times\frac{1}{8} = 18$

$b = 3\sqrt{2}$

I1.3 It is given that $f(x) = px^3 + qx + 5$ and $f(-7) = \sqrt{2}\,b + 1$. Find c, if $c = f(7)$.
Reference: 2006 FG2.2

$p(-7)^3 + q(-7) + 5 = \sqrt{2}\cdot 3\sqrt{2} + 1 = 7$

$-[p(7)^3 + q(7)] = 2$

$c = f(7)$

$\quad = p(7)^3 + q(7) + 5$

$\quad = -2 + 5 = 3$

I1.4 Find the least positive integer d, such that $d^c + 1000$ is divisible by $10 + c$.

$d^3 + 1000$ is divisible by 13

$13\times 77 = 1001 = 1000 + 1^3$

$d = 1$

Individual Event 2

I2.1 If $\frac{x}{(x-1)(x-4)}=\frac{x}{(x-2)(x-3)}$, find x.

Reference: 1998 HI3

$x = 0$ or $(x - 1)(x - 4) = (x - 2)(x - 3)$

$x = 0$ or $x^2 - 5x + 4 = x^2 - 5x + 6$

$x = 0$ or $4 = 6$

$x = 0$

I2.2 If $f(t) = 3\times52^t$ and $y = f(x)$, find y.

$y = f(0) = 3\times52^0 = 3$

I2.3 A can finish a job in y days, B can finish a job in $(y + 3)$ days. If they worked together, they can finish the job in z days, find z.

$\frac{1}{z}=\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$

$z = 2$

I2.4 The probability of throwing z dice to score 7 is w, find w.

P(sum of 2 dice = 7) = P((1,6), (2,5), (3,4), (4,3), (5,2), (6, 1)) $=\frac{6}{36}=\frac{1}{6}$

$w=\frac{1}{6}$

Individual Event 3

I3.1 If a = sin 30° + sin 300° + sin 3000°, find a.

$$a=\frac{1}{2}-\frac{\sqrt{3}}{2}+\sin(360°\times 8+120°)=\frac{1}{2}-\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}=\frac{1}{2}$$

I3.2 It is given that $\frac{x+y}{2}=\frac{z+x}{3}=\frac{y+z}{4}$ and $x + y + z = 36a$. Find the value of b, if $b=x+y$.

$x + y = 2k$ …… (1)

$z + x = 3k$ …… (2)

$y + z = 4k$ …… (3)

(1) + (2) + (3): $2(x + y + z) = 9k$

$2(36)(\frac{1}{2}) = 9k$

$k = 4$

$b = x + y$

$= 2k$

$= 2(4) = 8$

I3.3 It is given that the equation $x + 6 + 8k = k(x + b)$ has positive integral solution.
Find c, the least value of k.

$x + 6 + 8k = k(x + 8)$

$(k – 1)x = 6$

If $k = 1$, the equation has no solution

If $k \neq 1$, $x=\frac{6}{k-1}$

The positive integral solution, 6 must be divisible by $k – 1$.

The least positive factor of 6 is 1, $c = 2$

I3.4 A car has already travelled 40% of its journey at an average speed of $40c$ km/h. In order to make the average speed of the whole journey become 100 km/h, the speed of the car is adjusted to d km/h to complete the rest of the journey. Find d.

Let the total distance be s.

$$\frac{s}{\frac{0.4s}{40(2)}+\frac{0.6s}{d}}=100$$

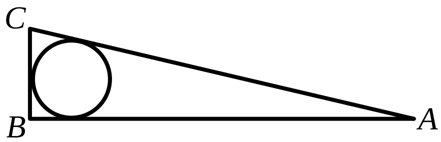
$$\Rightarrow \frac{1}{200}+\frac{3}{5d}=\frac{1}{100}$$

$$\Rightarrow \frac{120}{200d}=\frac{1}{200}$$

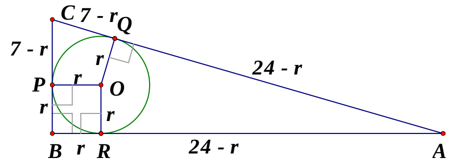
$$\Rightarrow d = 120$$

Individual Event 4

I4.1 In triangle ABC, $\angle B = 90°$, $BC = 7$ and $AB = 24$. If r is the radius of the inscribed circle, find r.



Let O be the centre of the inscribed circle, which touches BC, CA, AB at P, Q, R respectively.



$OP \perp BC$, $OQ \perp AC$, $OR \perp AB$ (tangent $\perp$ radius)

$ORBP$ is a rectangle (it has 3 right angles)

$BR = r$, $BP = r$ (opp. sides of rectangle)

$CP = 7 - r$, $AR = 24 - r$

$AC^2 = AB^2 + BC^2$ (Pythagoras' Theorem)

$= 24^2 + 7^2 = 625$

$AC = 25$

$CQ = 7 - r$, $AQ = 24 - r$ (tangent from ext. point)

$CQ + AQ = AC$

$7 - r + 24 - r = 25$

$r = 3$

I4.2 If $x^2 + x - 1 = 0$ and $s = x^3 + 2x^2 + r$, find s.

By division, $s = x^3 + 2x^2 + 3 = (x + 1)(x^2 + x - 1) + 4 = 4$

I4.3 It is given that $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$, where $n \geq 3$. If $F_t = s + 1$, find t.

$F_t = 4 + 1 = 5$

$F_3 = 1 + 1 = 2$, $F_4 = 2 + 1 = 3$, $F_5 = 3 + 2 = 5$

$t = 5$

I4.4 If $u = \sqrt{t(t+1)(t+2)(t+3)+1}$, find u.

Reference: 1993 HG6, 1996 FG10.1, 2000 FG3.1, 2004 FG3.1, 2012 FI2.3

$u = \sqrt{5\times6\times7\times8+1} = \sqrt{40\times42+1}$

$= \sqrt{(41-1)\times(41+1)+1} = \sqrt{41^2-1+1}$

$u = 41$

Individual Event 5

I5.1 It is given that $\log_7(\log_3(\log_2 x)) = 0$. Find a, if $a = x^{\frac{1}{3}}$.

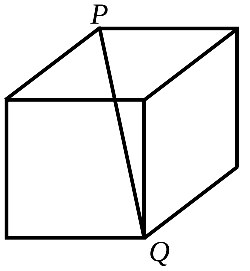
$\log_3(\log_2 x) = 1$

$\log_2 x = 3$

$x = 2^3 = 8$

$a = x^{\frac{1}{3}} = 2$

I5.2 In the figure, PQ is a diagonal of the cube and $PQ = \frac{a}{2}$.



Find b, if b is the total surface area of the cube.

Reference: 1992 HI14, 2003 HI7

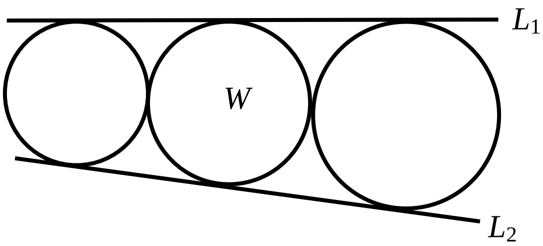
Let the length of the cube be x. $PQ = 1$

$x^2 + x^2 + x^2 = 1$ (Pythagoras' Theorem)

$3x^2 = 1$

The total surface area $= b = 6x^2 = 2$

I5.3 In the figure, L_1 and L_2 are tangents to the three circles. If the radius of the largest circle is 18 and the radius of the smallest circle is $4b$, find c, where c is the radius of the circle W.



Let the centres of the 3 circles be A, B, C as shown in the figure.

L_1 touches the circles at D, E, F as shown.

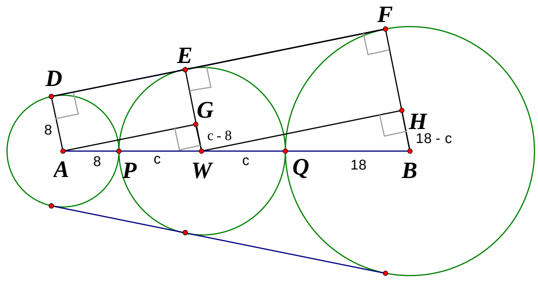
$AD \perp L_1$, $WE \perp L_1$, $BF \perp L_1$ (tangent $\perp$ radius)

Let AB intersects the circle W at P and Q.

$AD = AP = 4b = 8$, $EW = WQ = PW = c$

$QB = BF = 18$ (radii of the circle)

Draw $AG \parallel DE$, $WH \parallel EF$ as shown



$EW \parallel FB$ (int. $\angle$ supp.)

$\angle AWG = \angle WBH$ (corr. $\angle$s $EW \parallel FB$)

$AG \perp GW$, $WH \perp HB$ (by construction)

$\Delta AGW \sim \Delta WHB$ (equiangular)

$GW = c - 8$, $BH = c + 18$ (opp. sides of rectangle)

$\frac{c-8}{c+8} = \frac{18-c}{c+18}$ (ratio of sides, ~ Δ)

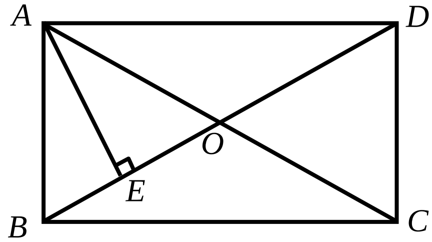
$(c - 8)(c + 18) = (c + 8)(18 - c)$

$c^2 + 10c - 144 = -c^2 + 10c + 144$

$2c^2 = 2(144)$

$c = 12$

I5.4 Refer to the figure, $ABCD$ is a rectangle. $AE \perp BD$ and $BE = EO = \frac{c}{6}$. Find d, the area of the rectangle $ABCD$.



$BO = 4 = OD = AO = OC$ (diagonal of rectangle)

$AE^2 = OA^2 - OE^2 = 4^2 - 2^2 = 12$ (Pythagoras' Theorem)

$AE = 2\sqrt{3}$

$\Delta ABD \cong \Delta CDB$ (R.H.S.)

$d = 2\times$area of ΔABD

$= \frac{2\times(4+4)\cdot 2\sqrt{3}}{2}$

$= 16\sqrt{3}$

Group Event 6

G6.1 $2^a \cdot 9^b$ is a four digit number and its thousands digit is 2, its hundreds digit is a, its tens digit is 9 and its units digit is b, find a, b.

$2^a \cdot 9^b = 2000 + 100a + 90 + b$

If $a = 0$, $9^b = 2090 + b$

$9^3 = 729$, $9^4 = 6561$

$\Rightarrow$ No solution for a

$\therefore$ $a > 0$ and $0 \le b \le 3$, $2000 + 100a + 90 + b$ is divisible by 2

$b = 0$ or 2

If $b = 0$, $2^a = 2090 + 100a$

$2^{10} = 1024$, $2^{11} = 2048$, $2^{12} = 4096$ and $0 \le a \le 9$

$\Rightarrow$ No solution for a

$\therefore$ $b = 2$, $2000 + 100a + 92$ is divisible by 9

$2 + a + 9 + 2 = 9m$, where m is a positive integer

$a = 5$, $b = 2$

Check: $2^5 \cdot 9^2 = 32\times81 = 2592 = 2000 + 100(5) + 90 + 2$

G6.2 Find c, if $c = \left(1+\frac{1}{2}+\frac{1}{3}\right)\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\left(\frac{1}{2}+\frac{1}{3}\right)$.

Reference: 2006 FI4.1

Let $x = 1+\frac{1}{2}+\frac{1}{3}$, $y = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$, then $c = x(y-1) - y(x-1) = -x + y = \frac{1}{4}$

G6.3 Find d, if

$$d = \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1994}\right)\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1995}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1995}\right)\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1994}\right)$$

$x = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1994}$, $y = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1995}$

$\Rightarrow d = x(y-1) - y(x-1)$

$= -x + y = \frac{1}{1995}$

Group Event 7

G7.1 Let p, q, r be the three sides of triangle PQR. If $p^4 + q^4 + r^4 = 2r^2(p^2 + q^2)$, find a, where $a = \cos^2 R$ and R denotes the angle opposite r.

$\cos R = \frac{p^2+q^2-r^2}{2pq}$

$a = \cos^2 R$

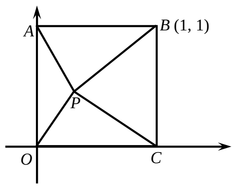
$= \frac{(p^2+q^2-r^2)^2}{4p^2q^2}$

$= \frac{p^4+q^4+r^4+2p^2q^2-2p^2r^2-2q^2r^2}{4p^2q^2}$

$= \frac{2r^2(p^2+q^2)+2p^2q^2-2p^2r^2-2q^2r^2}{4p^2q^2}$

$= \frac{2p^2q^2}{4p^2q^2} = \frac{1}{2}$

G7.2 Refer to the diagram, P is any point inside the square $OABC$ and b is the minimum value of $PO + PA + PB + PC$, find b.



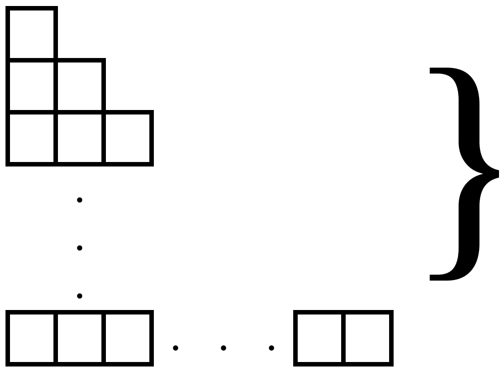
$PO + PA + PB + PC \ge OB + AC$ (triangle inequality)

$= 2\ OB$

$= 2\sqrt{1^2+1^2}$

$\Rightarrow b = 2\sqrt{2}$

G7.3 Identical matches of length 1 are used to arrange the following pattern, if c denotes the total length of matches used, find c.



1^{st} row = 4

1^{st} row + 2^{nd} row = 4 + 6 = 10

$1^{st} + 2^{nd} + 3^{rd} = 4 + 6 + 8 = 18$

....

$c = 1^{st} + \ldots + 25^{th}$ row

$= 4 + 6 + 8 + \ldots + [4 + (25 - 1)\cdot 2]$

$= \frac{n[2a+(n-1)d]}{2}$

$= \frac{25[2(4)+(24)(2)]}{2}$

$= 700$

G7.4 Find d, where $d = \sqrt{111111-222}$.

Reference: 2000 FI2.4

$111111 - 222 = 111(1001 - 2)$

$= 111\times 999$

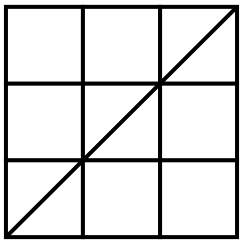
$= 3^2 \times 111^2$

$= 333^2$

$\Rightarrow d = 333$

Group Event 8

Rectangles of length ℓ and breadth b where ℓ, b are positive integers, are drawn on square grid paper. For each of these rectangles, a diagonal is drawn and the number of vertices V intersected (excluding the two end points) is counted (see figure).



$\ell = b = 3$

$V = 2$

G8.1 Find V, when $\ell = 6$, $b = 4$.

Intersection point = (3, 2)
$V = 1$

G.8.2 Find V, when $\ell = 5$, $b = 3$

As 3 and 5 are relatively prime, there is no intersection $\Rightarrow V = 0$

G8.3 When $\ell = 12$ and $1 < b < 12$, find r, the number of different values of b that makes $V = 0$?

$b = 5, 7, 11$ are relatively prime to 12.
The number of different values of $b = 3$

G8.4 Find V, when $\ell = 108$, $b = 72$.

H.C.F. (108, 72) = 36, 108 = 36×3, 72 = 36×2

Intersection points = (3, 2), (6, 4), (9, 6), $\cdots$, (105, 70)

$\Rightarrow V = 35$

Group Event 9

A, B, C, D are different integers ranging from 0 to 9 and

Find A, B, C and D.

$$\begin{array}{rcccc} & A & A & B & C \\ - & B & A & C & B \\ \hline & D & A & C & D \end{array}$$

If $A = 0$, then $B \geq 1$, $(AABC) - (BACB) < 0$ rejected

$\therefore A > 0$, consider the hundreds digit:

If there is no borrow digit in the hundreds digit, then $A - A = A$

$\Rightarrow A = 0$ rejected

$\therefore$ There is a borrow digit in the hundreds digit. Also, there is a borrow digit in the thousands digit

$10 + A - 1 - A = A$

$\Rightarrow A = 9$

Consider the thousands digit: $A - 1 - B = D$

$\Rightarrow B + D = 8$ …… (1)

Consider the units digit:

If $C < B$, then $10 + C - B = D$

$\Rightarrow 10 + C = B + D$

$\Rightarrow 10 + C = 8$ by (1)

$\Rightarrow C = -2$ (rejected)

$\therefore C > B$ and there is no borrow digit in the tens digit

Consider the tens digit: $10 + B - C = C$

$10 + B = 2C$ …… (2)

Consider the units digit, $\because C > B \therefore C - B = D$

$C = B + D$

$\Rightarrow C = 8$ by (1)

Sub. $C = 8$ into (2)

$10 + B = 16$

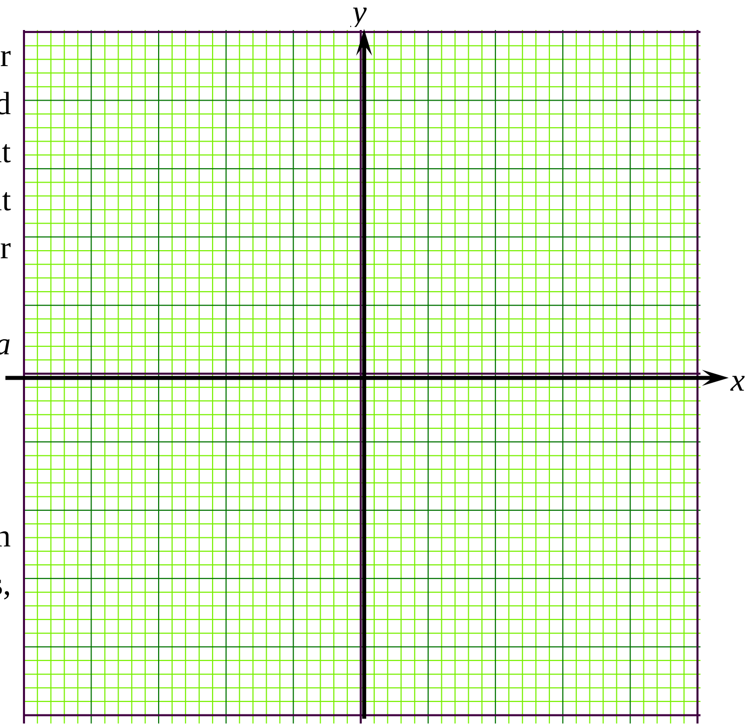
$\Rightarrow B = 6$

Sub. $B = 6$ into (1), $6 + D = 8$

$\Rightarrow D = 2$

$\therefore A = 9, B = 6, C = 8, D = 2$

Group Event 10



Lattice points are points on a rectangular coordinate plane having both *x*- and *y*-coordinates being integers. A moving point *P* is initially located at (0, 0). It moves 1 unit along the coordinate lines (in either directions) in a single step.

G10.1 If *P* moves 1 step then *P* can reach *a* different lattice points, find *a*.

(1, 0), (–1, 0), (0, 1), (0, –1)

$a = 4$

G10.2 If *P* moves not more than 2 steps then *P* can reach *b* different lattice points, find *b*.

(1, 0), (–1, 0), (0, 1), (0, –1),

(1, 1), (1, (1, –1), (–1, 1), (–1, –1)

(2, 0), (–2, 0), (0, 2), (0, –2), (0, 0)

$b = 13$

G10.3 If *P* moves 3 steps then *P* can reach *c* different lattice points, find *c*.

(1, 0), (–1, 0), (0, 1), (0, –1), (3, 0), (2, 1), (1, 2), (0, 3), (–1, 2), (–2, 1), (–3, 0), (–2, –1), (–1, –2), (0, –3), (1, –2), (2, –1); $c = 4 + 12 = 16$

G10.4 If *d* is the probability that *P* lies on the straight line $x + y = 9$ when *P* advances 9 steps, find *d*.

Total number of outcomes = 4 + 12 + 20 + 28 + 36 = 100

Favourable outcomes = {(0,9), (1,8), (2,7), … , (9,0)}, number = 10

Probability $= \frac{1}{10}$

Individual Events

I1	***a***	3	**I2**	***a***	9	**I3**	***a***	5	**I4**	***a***	$\frac{7}{2}$	**I5**	***a***	$\frac{11}{16}$	**ISpare**	***a***	17
	b	18		***b***	3		***b***	3		***b***	2		***b***	2		***b***	9
	c	36		***c***	4		***c***	2		***c***	10		***c***	$\frac{10}{21}$		***c***	1
	d	$\frac{5}{36}$		***d***	15		***d***	16		***d***	$\frac{9}{10}$		***d***	$\frac{23}{28}$		***d***	258

Group Events

G6	***a***	7	**G7**	***a***	2 *see the remark	**G8**	***a***	8	**G9**	***a***	2	**G10**	***a***	1003001
	b	7		***b***	333		***b***	4		***b***	2		***b***	10
	c	12		***c***	1		***c***	7		***c***	39923992		***c***	35
	d	757		***d***	9		***d***	$-\frac{3}{2}$		***d***	885		***d***	92

Individual Event 1

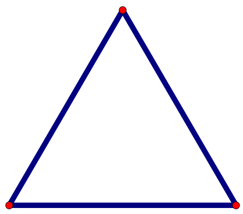
I1.1 The perimeter of an equilateral triangle is exactly the same in length as the perimeter of a regular hexagon. The ratio of the areas of the triangle and the hexagon is 2 : a, find the value of a. **Reference: 2014 FI4.3, 2016 FI2.1**

Let the length of the equilateral triangle be x, and that of the regular hexagon be y.

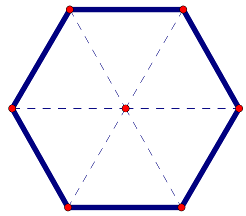
Since they have equal perimeter, $3x = 6y$

$\therefore x = 2y$

The hexagon can be divided into 6 identical equilateral triangles.



x



y

Ratio of areas $=\frac{1}{2}x^2\sin 60^\circ : 6\times\frac{1}{2}y^2\sin 60^\circ = 2 : a$

$x^2 : 6y^2 = 2 : a$

$(2y)^2 : 6y^2 = 2 : a$

$\Rightarrow a = 3$

I1.2 If $5^x + 5^{-x} = a$ and $5^{3x} + 5^{-3x} = b$, find the value of b.

Reference: 1983 FG7.3, 1998 FG5.2, 2010 FI3.2

$(5^x + 5^{-x})^2 = 3^2$

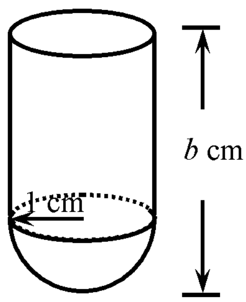
$5^{2x} + 2 + 5^{-2x} = 9$

$5^{2x} + 5^{-2x} = 7$

$b = (5^x + 5^{-x})(5^{2x} - 1 + 5^{-2x})$

$= 3(7 - 1) = 18$

I1.3 The figure shows an open cylindrical tube (radius = 1 cm) with a hemispherical bottom of radius 1 cm. The height of the tube is b cm and the external surface area of the tube is $c\pi$ cm^2. Find the value of c.



$c\pi = 2\pi r\ell + 2\pi r^2 = 2\pi(1)(17) + 2\pi(1^2) = 36\pi$

$c = 36$

I1.4 Two fair dice are thrown. Let d be the probability of getting the sum of scores to be $\frac{c}{6}$. Find the value of d.

Sum = 6, d = P(6) = P((1,5), (2, 4), (3, 3), (4, 2), (5, 1)) $=\frac{5}{36}$.

Individual Event 2

I2.1 It is given that m, $n > 0$ and $m + n = 1$. If the minimum value of $\left(1+\frac{1}{m}\right)\left(1+\frac{1}{n}\right)$ is a, find the value of a.

$(m + n)^2 = 1$
$\Rightarrow m^2 + n^2 + 2mn = 1$
$\Rightarrow m^2 + n^2 = 1 - 2mn$
$(m - n)^2 \geq 0$
$\Rightarrow m^2 - 2mn + n^2 \geq 0$
$\Rightarrow 1 - 2mn - 2mn \geq 0$
$\Rightarrow mn \leq \frac{1}{4} \Rightarrow \frac{1}{mn} \geq 4$
$\left(1+\frac{1}{m}\right)\left(1+\frac{1}{n}\right) = \frac{(1+m)(1+n)}{mn} = \frac{1+m+n+mn}{mn} = \frac{1+1+mn}{mn} = \frac{2}{mn} + 1 \geq 2\times4 + 1 = 9$

I2.2 If the roots of the equation $x^2 - (10 + a)x + 25 = 0$ are the square of the roots of the equation $x^2 + bx = 5$, find the positive value of b. **(Reference: 2001 FI3.4)**

$x^2 - 19x + 25 = 0$, roots $= \alpha, \beta$; $\alpha + \beta = 19$, $\alpha\beta = 25$
$x^2 + bx = 5$, roots r, s; $r + s = -b$, $rs = -5$
Now $r^2 = \alpha$, $s^2 = \beta$
$19 = \alpha + \beta = r^2 + s^2 = (r + s)^2 - 2rs = b^2 - 2(-5)$
$b^2 = 9$
$\Rightarrow b = 3$ (positive root)

Method 2 Replace x by $\sqrt{x}$ in $x^2 + bx = 5$
$x + b\sqrt{x} = 5$
$b\sqrt{x} = 5 - x$
$b^2x = 25 - 10x + x^2$
$x^2 - (10 + b^2)x + 25 = 0$, which is identical to $x^2 - 19x + 25 = 0$
$\Rightarrow b = 3$ (positive root)

I2.3 If $(xy - 2)^{b-1} + (x - 2y)^{b-1} = 0$ and $c = x^2 + y^2 - 1$, find the value of c.
Reference: 2005FI4.1, 2006 FI4.2, 2009 FG1.4, 2013 FI1.4, 2015 HG4, 2015 FI1.1

$(xy - 2)^2 + (x - 2y)^2 = 0$
$\Rightarrow xy = 2$ and $x = 2y$
$\Rightarrow 2y^2 = 2$
$\Rightarrow y = \pm1, x = \pm2$
$c = x^2 + y^2 - 1 = 4 + 1 - 1 = 4$

I2.4 If $f(x)$ is a polynomial of degree two, $f(f(x)) = x^4 - 2x^2$ and $d = f(c)$, find the value of d.

Let $f(x) = px^2 + qx + r$
$f(f(x)) = p(px^2 + qx + r)^2 + q(px^2 + qx + r) + r \equiv x^4 - 2x^2$
Compare coefficient of x^4: $p = 1$
Compare coefficient of x^3: $2q = 0 \Rightarrow q = 0$
Compare the constant: $r^2 + qr + r = 0 \Rightarrow r = 0$ or $r + q + 1 = 0$ …… (1)
Compare coefficient of x: $2qr + q^2 = 0 \Rightarrow q = 0$ or $2r + q = 0$ …… (2)
Sub. $q = 0$ into (1): $r = 0$ or -1
But $q = 0$, $r = -1$ does not satisfy $2r + q = 0$ in (2)
$\therefore (p, q, r) = (1, 0, 0)$ or $(1, 0, -1)$
Sub. $(p, q, r) = (1, 0, 0)$ into $f(f(x)) = x^4 - 2x^2 \equiv x^4$, which is a contradiction $\therefore$ rejected
Sub. $(p, q, r) = (1, 0, -1)$ into $f(f(x)) = x^4 - 2x^2 \equiv (x^2 - 1)^2 - 1$
RHS $= x^4 - 2x^2 + 1 - 1 =$ LHS
$\therefore f(x) = x^2 - 1$; $d = f(4) = 4^2 - 1 = 15$

Individual Event 3

I3.1 If a is a real number and $2a^3 + a^2 - 275 = 0$, find the value of a.

Let $f(a) = 2a^3 + a^2 - 275$; $275 = 5\times5\times11$

$f(5) = 2\times125 + 25 - 275 = 0$

$f(a) = (a - 5)(2a^2 + 11a + 55)$

Δ of $2a^2 + 11a + 55$ is $11^2 - 4(2)(55) < 0$

$\therefore a = 5$

$$\begin{array}{r|l} & 2a^2 + 11a + 55 \\ \hline a-5 & 2a^3 + a^2 \qquad -275 \\ & \underline{2a^3 - 10a^2} \\ & 11a^2 \\ & \underline{11a^2 - 55a} \\ & 55a - 275 \\ & \underline{55a - 275} \end{array}$$

I3.2 Find the value of b if $3^2\cdot3^5\cdot3^8\ldots3^{3b-1} = 27^a$.

$3^{2+5+8+\ldots(3b-1)} = 3^{3\times5}$

$\because 2 + 5 + 8 = 15$

$\therefore 3b - 1 = 8$

$b = 3$

I3.3 Find the value of c if $\log_b(b^c - 8) = 2 - c$.

$\log_3(3^c - 8) = 2 - c$

$\Rightarrow 3^c - 8 = 3^{2-c}$

Let $y = 3^c$; then $3^{2-c} = 3^2\cdot3^{-c} = \frac{9}{y}$

$y - 8 = \frac{9}{y}$

$\Rightarrow y^2 - 8y - 9 = 0$

$\Rightarrow (y - 9)(y + 1) = 0$

$\Rightarrow y = 9$ or -1

$\Rightarrow 3^c = 9$ or -1 (rejected)

$c = 2$

I3.4 If $[(4^c)^c]^c = 2^d$, find the value of d.

$[(4^2)^2]^2 = 2^d$

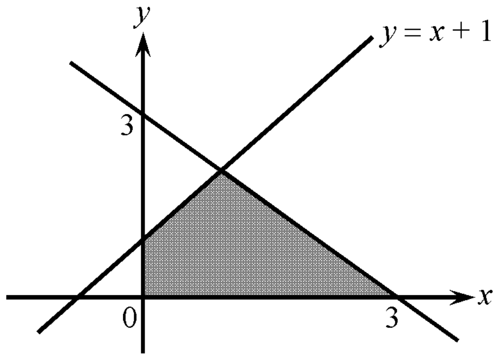
$\Rightarrow 4^8 = 2^d$

$\Rightarrow 2^{16} = 2^d$

$d = 16$

Individual Event 4

I4.1 In the figure, the area of the shaded region is a. Find the value of a.



Equation of the other straight line is $x + y = 3$

The intersection point is (1, 2)

y-intercept of $y = x + 1$ is 1

Shaded area = area of big triangle – area of small Δ

$$= \frac{1}{2}\cdot 3\times 3 - \frac{1}{2}(3-1)\cdot 1 = \frac{7}{2}$$

I4.2 If $8^b = 4^a - 4^3$, find the value of b.

$8^b = 4^{3.5} - 4^3 = 4^3\cdot(2 - 1) = 64$; $b = 2$

I4.3 Given that c is the positive root of the equation $x^2 - 100b + \frac{10000}{x^2} = 0$, find the value of c.

$x^4 - 200x^2 + 10000 = 0$

$\Rightarrow (x^2 - 100)^2 = 0$

$\Rightarrow x = 10$

I4.4 If $d = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{(c-1)\times c}$, find the value of d.

$$d = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{9\times 10}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{9} - \frac{1}{10}\right)$$

$$= 1 - \frac{1}{10} = \frac{9}{10}$$

Individual Event 5

I5.1 Four fair dice are thrown. Let a be the probability of getting at least half of the outcome of the dice to be even. Find the value of a.

P(at least half of the outcome of the dice to be even)

= P(even, even, odd, odd) + P(even, even, even, odd) + P(even, even, even, even)

$= 6\times\frac{1}{16}+4\times\frac{1}{16}+\frac{1}{16}=\frac{11}{16}$

I5.2 It is given that $f(x)=\frac{3}{8}x^2(81)^{-\frac{1}{x}}$ and $g(x) = 4\log_{10}(14x) - 2\log_{10} 49$.

Find the value of $b = f\{g[16(1-a)]\}$.

$g[16(1-a)] = g[16(1-\frac{11}{16})] = g(5)$

$= 4\log 70 - 2\log 49$

$= 4\log 70 - 4\log 7 = 4\log 10 = 4$

$b = f(g(5)) = f(4) = \frac{3}{8}(4)^2(81)^{-\frac{1}{4}} = 6\times 9^{-\frac{1}{2}} = \frac{6}{3} = 2$

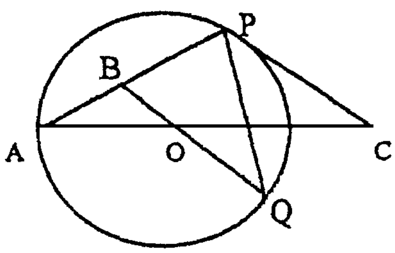
I5.3 Let $c=\frac{1}{b^2-1}+\frac{1}{(2b)^2-1}+\frac{1}{(3b)^2-1}+\cdots+\frac{1}{(10b)^2-1}$, find the value of c.

Hint: $\frac{1}{x^2-1}=\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)$

$c=\frac{1}{2^2-1}+\frac{1}{4^2-1}+\frac{1}{6^2-1}+\cdots+\frac{1}{20^2-1}=\frac{1}{2}\left(1-\frac{1}{3}\right)+\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right)+\frac{1}{2}\left(\frac{1}{5}-\frac{1}{7}\right)+\cdots+\frac{1}{2}\left(\frac{1}{19}-\frac{1}{21}\right)$

$=\frac{1}{2}\left(1-\frac{1}{21}\right)=\frac{10}{21}$

I5.4 In the following diagram, PC is a tangent to the circle (centre O) at the point P, and ΔABO is an isosceles triangle, $AB = OB$, $\angle PCO = c$ ($=\frac{10}{21}$) and $d = \angle QPC$, where c, d are radian measures. Find the value of d. (Take $\pi=\frac{22}{7}$)



$\because AB = OB$, let $\angle BAO = \angle AOB = \theta$ (base $\angle$s, isos. Δ)

Join OP. $\angle OPC = \frac{\pi}{2}$ (tangent $\perp$ radius)

$\because OA = OP$ (radii), $\angle OPA = \theta$ (base $\angle$s, isos. Δ)

In ΔAPC, $\theta + \theta + \frac{\pi}{2} + c = \pi$ ($\angle$s sum of Δ)

$2\theta = \frac{\pi}{2}-\frac{10}{21}=\frac{11}{7}-\frac{10}{21}=\frac{23}{21} \Rightarrow \theta=\frac{23}{42}$

$\angle COQ = \theta = \frac{23}{42}$ (vert. opp. $\angle$s)

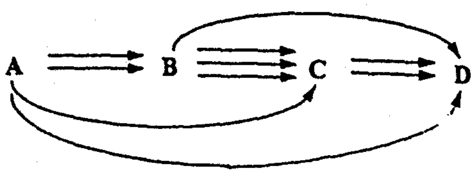
Join AQ. $\angle QAO = \frac{\theta}{2}=\frac{23}{84}$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

$d = \angle BAO + \angle QAO$ ($\angle$ in alt. segment)

$= \theta + \frac{\theta}{2} = \frac{23}{42}+\frac{23}{84}=\frac{23}{28}$

Spare Event (Individual)

IS.1 From the following figure, determine the number of routes a from A to D.



$a = 2\times3\times2 + 1\times2 + 1 + 2\times1 = 17$

IS.2 If $\sin(2b° + 2a°) = \cos(6b° – 16°)$, where $0 < b < 90$, find the value of b.

$\cos(90° – 2b° – 34°) = \cos(6b° – 16°)$

$56 – 2b = 6b – 16$

$72 = 8b$

$b = 9$

IS.3 The lines $(bx – 6y + 3) + k(x – y + 1) = 0$, where k is any real constant, pass through a fixed point $P(c, m)$, find the value of c.

The fixed point is the intersection of $9x – 6y + 3 = 0$ … (1) and $x – y + 1 = 0$ … (2)

(1)÷3 – 2(2): $x – 1 = 0$

$x = 1 \Rightarrow c = 1$

IS.4 It is known that $d^2 – c = 257\times259$. Find the positive value of d.

$d^2 – 1 = 257\times259$

$= (258 – 1)(258 + 1)$

$= 258^2 – 1$

$d = 258$

Group Event 6

G6.1 The number of eggs in a basket was a. Eggs were given out in three rounds. In the first round half of egg plus half an egg were given out. In the second round, half of the remaining eggs plus half an egg were given out. In the third round, again, half of the remaining eggs plus half an egg were given out. The basket then became empty. Find a.

$$\frac{a}{2}+\frac{1}{2}+\frac{1}{2}\left(\frac{a}{2}-\frac{1}{2}\right)+\frac{1}{2}+\frac{1}{2}\left\{\frac{a}{2}-\frac{1}{2}-\left[\frac{1}{2}\left(\frac{a}{2}-\frac{1}{2}\right)+\frac{1}{2}\right]\right\}+\frac{1}{2}=a$$

$$\frac{3}{2}+\frac{1}{2}\left(\frac{a}{2}-\frac{1}{2}\right)+\frac{1}{2}\left\{\frac{a}{2}-\frac{1}{2}-\left[\frac{1}{2}\left(\frac{a}{2}-\frac{1}{2}\right)+\frac{1}{2}\right]\right\}=\frac{a}{2}$$

$$3+\left(\frac{a}{2}-\frac{1}{2}\right)+\left\{\frac{a}{2}-\frac{1}{2}-\left[\frac{1}{2}\left(\frac{a}{2}-\frac{1}{2}\right)+\frac{1}{2}\right]\right\}=a$$

$$6+a-1+a-1-\left[\left(\frac{a}{2}-\frac{1}{2}\right)+1\right]=2a$$

$$3-\left(\frac{a}{2}-\frac{1}{2}\right)=0$$

$a = 7$

G6.2 If $p – q = 2$; $p – r = 1$ and $b = (r – q)[(p – q)^2 + (p – q)(p – r) + (p – r)^2]$. Find the value of b.

$b = [p – q – (p – r)][(p – q)^2 + (p – q)(p – r) + (p – r)^2]$

$= (2 – 1)[2^2 + 2\cdot 1 + 1^2]$

$= 2^3 – 1^3 = 7$

G6.3 If n is a positive integer, $m^{2n} = 2$ and $c = 2m^{6n} – 4$, find the value of c.

$c = 2m^{6n} – 4$

$= 2(m^{2n})^3 – 4$

$= 2\times 2^3 – 4 = 12$

G6.4 If r, s, t, u are positive integers and $r^5 = s^4$, $t^3 = u^2$, $t – r = 19$ and $d = u – s$, find the value of d.

Reference: 1998 HG4

Let $w=u^{\frac{1}{15}}$, $v=s^{\frac{1}{15}}$, then $t=u^{\frac{2}{3}}=u^{\frac{10}{15}}=w^{10}$, $r=s^{\frac{4}{5}}=s^{\frac{12}{15}}=v^{12}$

$t – r = 19 \Rightarrow w^{10} – v^{12} = 19$

$\Rightarrow (w^5 + v^6)(w^5 – v^6) = 19\times 1$

$\because$ 19 is a prime number, $w^5 + v^6 = 19$, $w^5 – v^6 = 1$

Solving these equations give $w^5 = 10$, $v^6 = 9 \Rightarrow w^5 = 10$, $v^3 = 3$

$u = w^{15} = 1000$, $s = v^{15} = 3^5 = 729$

$d = u – s = 1000 – 243 = 757$

Group Event 7

G7.1 If the two distinct roots of the equation $ax^2 - mx + 1996 = 0$ are primes, find the value of a.

Reference: 1996 HG8, 2001 FG4.4, 2005 FG1.2, 2012 HI6

Let the roots be α, β. $\alpha + \beta = \frac{m}{a}$, $\alpha\beta = \frac{1996}{a}$

1996 = 4×499 and 499 is a prime

$\therefore$ a = 1, 2, 4, 499, 998 or 1996

When $a = 1$, $\alpha\beta = 1996$, which cannot be expressed as a product of two primes $\therefore$ rejected

When $a = 2$, $\alpha\beta = 998$; $\alpha = 2$, $\beta = 499$ (accepted)

When $a = 4$, $\alpha\beta = 499$, which cannot be expressed as a product of two primes $\therefore$ rejected

When $a = 499$, $\alpha\beta = 4$, $\alpha = 2$, $\beta = 2$ (not distinct roots, rejected)

When $a = 998$, $\alpha\beta = 2$, which cannot be expressed as a product of two primes $\therefore$ rejected

When $a = 1996$, $\alpha\beta = 1$, which cannot be expressed as a product of two primes $\therefore$ rejected

Remark: the original question is:

If the two roots of the equation $ax^2 - mx + 1996 = 0$ are primes, find the value of a.

a = 2 or 499 (Not unique solution)

G7.2 A six-digit figure 111*aaa* is the product of two consecutive positive integers b and $b + 1$, find the value of b.

Reference: 2001 FG2.3 Given that $111111222222 = c\times(c + 1)$

111222 = 111000 + 222 = 111×1000 + 2×111 = 111×1002 = 111×3×334 = 333×334; $b = 333$

G7.3 If p, q, r are non-zero real numbers;

$p^2 + q^2 + r^2 = 1$, $p\left(\frac{1}{q}+\frac{1}{r}\right)+q\left(\frac{1}{r}+\frac{1}{p}\right)+r\left(\frac{1}{p}+\frac{1}{q}\right)+3=0$ and $c = p + q + r$, find the largest value of c.

The second equation becomes: $\frac{p^2(r+q)+q^2(p+r)+r^2(q+p)+3pqr}{pqr} = 0$

$p^2(c - p) + q^2(c - q) + r^2(c - r) + 3pqr = 0$

$c(p^2 + q^2 + r^2) - (p^3 + q^3 + r^3 - 3pqr) = 0$

$c - (p + q + r)(p^2 + q^2 + r^2 - pq - qr - pr) = 0$

$c - c[1 - (pq + qr + pr)] = 0$

$c(pq + qr + pr) = 0$

$\frac{c}{2}[(p + q + r)^2 - (p^2 + q^2 + r^2)] = 0$

$c(c^2 - 1) = 0$

c = 0, 1 or –1

Maximum $c = 1$

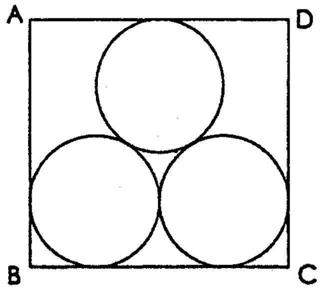
G7.4 If the units digit of 7^{14} is d, find the value of d.

$7^1 = 7$, $7^2 = 49$, $7^3 = 343$, $7^4 = 2401$; the units digit repeat in the pattern 7, 9, 3, 1, …

$7^{14} = (7^4)^3 \times 7^2$ $\therefore$ $d = 9$

Group Event 8 In this question, all unnamed circles are unit circles.

G8.1 If the area of the rectangle *ABCD* is $a + 4\sqrt{3}$, find the value of a.



The lines joining the centres form an equilateral triangle, side = 2.

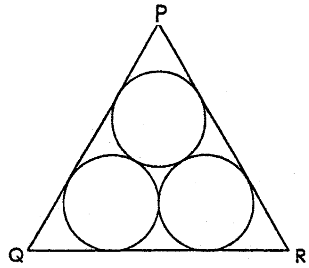
$AB = 2 + 2\sin 60° = 2 + \sqrt{3}$

$BC = 4$

Area of $ABCD = (2 + \sqrt{3})\times 4 = 8 + 4\sqrt{3}$

$a = 8$

G8.2 If the area of the equilateral triangle *PQR* is $6 + b\sqrt{3}$, find the value of b.



Reference: 1997 HG9

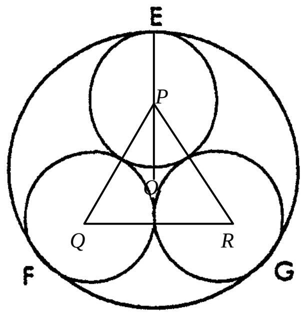
$PQ = 2 + 2\tan 60° = 2 + 2\sqrt{3}$

Area of $PQR = \frac{1}{2}\left(2+2\sqrt{3}\right)^2 \sin 60^\circ = 2\left(1+2\sqrt{3}+3\right)\cdot\frac{\sqrt{3}}{2}$

$6 + b\sqrt{3} = 6 + 4\sqrt{3}$

$b = 4$

G8.3 If the area of the circle *EFG* is $\frac{\left(c+4\sqrt{3}\right)\pi}{3}$, find the value of c.



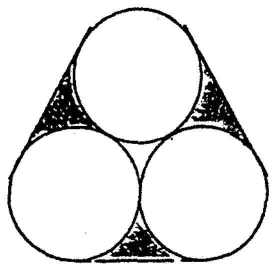
Let the centre be O, the equilateral triangle formed by the lines joining the centres be PQR, the radius be r.

$r = OE = OP + PE = 1\sec 30° + 1 = \frac{2}{\sqrt{3}} + 1 = \frac{2+\sqrt{3}}{\sqrt{3}}$

Area of circle $= \pi\cdot\frac{\left(2+\sqrt{3}\right)^2}{3} = \frac{\left(7+4\sqrt{3}\right)\pi}{3}$

$c = 7$

G8.4 If all the straight lines in the diagram below are common tangents to the two circles, and the area of the shaded part is $6 + d\pi$, find the value of d.



There are three identical shaded regions.

One shaded part = area of rectangle – area of semi-circle

$$= 2\cdot 1 - \frac{1}{2}\pi(1)^2 = 2 - \frac{\pi}{2}$$

Total shaded area $= 3\times\left(2-\frac{\pi}{2}\right) = 6 - \frac{3}{2}\pi$

$d = -\frac{3}{2}$

Group Event 9

G9.1 If $(1995)^a + (1996)^a + (1997)^a$ is divisible by 10, find the least possible integral value of a.

The unit digit of $(1995)^a + (1996)^a + (1997)^a$ is 0.

For any positive integral value of a, the units digit of 1995^a is 5, the units digit of 1996^a is 6.

The units digit of 1997^a repeats in the pattern 7, 9, 3, 1, …

$5 + 6 + 9 = 20$

So the least possible integral value of a is 2.

G9.2 If the expression $(x^2 + y^2)^2 \le b(x^4 + y^4)$ holds for all real values of x and y, find the least possible integral value of b.

$b(x^4 + y^4) - (x^2 + y^2)^2 = b(x^4 + y^4) - (x^4 + y^4 + 2x^2y^2) = (b - 1)x^4 - 2x^2y^2 + (b - 1)y^4$

If $b = 1$, the expression $= -2x^2y^2$ which cannot be positive for all values of x and y.

If $b \ne 1$, discriminant $= (-2)^2 - 4(b - 1)^2 = 4(1 - b^2 + 2b - 1) = -4b(b - 2)$

In order that the expression is always non-negative, $(b - 1) > 0$ and discriminant ≤ 0

$b > 1$ and $-4b(b - 2) \le 0$

$b > 1$ and $(b \le 0$ or $b \ge 2)$

$\therefore\ b \ge 2$, the least possible integral value of b is 2.

G9.3 If $c = 1996\times19971997 - 1995\times19961996$, find the value of c.

Reference: 1998 FG2.2

$c = 1996\times1997\times1001 - 1995\times1996\times1001 = 1001\times1996\times(1997 - 1995)$

$= 3992\times1001 = 39923992$

G9.4 Find the sum d where

$$d=\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{60}\right)+\left(\frac{2}{3}+\frac{2}{4}+\frac{2}{5}+\cdots+\frac{2}{60}\right)+\left(\frac{3}{4}+\frac{3}{5}+\cdots+\frac{3}{60}\right)+\cdots+\left(\frac{58}{59}+\frac{58}{60}\right)+\frac{59}{60}$$

Reference: 1995 HG3, 2004 HG1, 2018 HG9

$$d=\frac{1}{2}+\left(\frac{1}{3}+\frac{2}{3}\right)+\left(\frac{1}{4}+\frac{2}{4}+\frac{3}{4}\right)+\left(\frac{1}{5}+\frac{2}{5}+\frac{3}{5}+\frac{4}{5}\right)+\cdots+\left(\frac{1}{60}+\frac{2}{60}+\cdots+\frac{59}{60}\right)$$

$$=\frac{1}{2}+\frac{\frac{3\times2}{2}}{3}+\frac{\frac{4\times3}{2}}{4}+\frac{\frac{5\times4}{2}}{5}+\cdots+\frac{\frac{60\times59}{2}}{60}$$

$$=\frac{1}{2}+\frac{2}{2}+\frac{3}{2}+\frac{4}{2}+\cdots+\frac{59}{2}$$

$$=\frac{1}{2}(1+2+3+4+\cdots+59)$$

$$=\frac{1}{2}\times\frac{1}{2}\cdot60\cdot59=885$$

Group Event 10

G10.1 It is given that $3\times4\times5\times6 = 19^2 - 1$

$4\times5\times6\times7 = 29^2 - 1$

$5\times6\times7\times8 = 41^2 - 1$

$6\times7\times8\times9 = 55^2 - 1$

If $a^2 = 1000\times1001\times1002\times1003 + 1$, find the value of a.

Reference: 1993 HG6, 1995 FI4.4, 2000 FG3.1, 2004 FG3.1, 2012 FI2.3

$19 = 4\times5 - 1$; $29 = 5\times6 - 1$; $41 = 6\times7 - 1$; $55 = 5\times6 - 1$

$a^2 = (1001\times1002 - 1)^2 - 1 + 1 = (1001\times1002 - 1)^2$

$a = 1003002 - 1 = 1003001$

G10.2 Let $f(x) = x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$. When $f(x^{10})$ is divided by $f(x)$, the remainder is b. Find the value of b.

Reference: 2016 FI3.1

Consider the roots of $f(x) = x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$

$f(x)$ can be rewritten as $\frac{x^{10}-1}{x-1} = 0$, where $x \neq 1$

There are 9 roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9$: where $\alpha_i^{10} = 1$and $\alpha_i \neq 1$ for $1 \leq i \leq 9$

Let $f(x^{10}) = f(x)\,Q(x) + a_8x^8 + a_7x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.

$f(\alpha_i^{10}) = f(\alpha_i)\,Q(\alpha_i) + a_8\alpha_i^8 + a_7\alpha_i^7 + a_6\alpha_i^6 + a_5\alpha_i^5 + a_4\alpha_i^4 + a_3\alpha_i^3 + a_2\alpha_i^2 + a_1\alpha_i + a_0$, $1 \leq i \leq 9$

$f(1) = 0\cdot Q(\alpha_i) + a_8\alpha_i^8 + a_7\alpha_i^7 + a_6\alpha_i^6 + a_5\alpha_i^5 + a_4\alpha_i^4 + a_3\alpha_i^3 + a_2\alpha_i^2 + a_1\alpha_i + a_0$ for $1 \leq i \leq 9$

$a_8\alpha_i^8 + a_7\alpha_i^7 + a_6\alpha_i^6 + a_5\alpha_i^5 + a_4\alpha_i^4 + a_3\alpha_i^3 + a_2\alpha_i^2 + a_1\alpha_i + a_0 = 10$ for $1 \leq i \leq 9$

$\therefore \alpha_i$ $(1\leq i\leq 9)$ are the roots of $a_8x^8 + a_7x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 - 10 = 0$

Since a polynomial of degree 8 has at most 8 roots and it is satisfied by α_i for $1 \leq i \leq 9$.

$\therefore$ $a_8x^8 + a_7x^7 + a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 - 10$ must be a zero polynomial.

$a_8 = 0$, $a_7 = 0$, $a_6 = 0$, $a_5 = 0$, $a_4 = 0$, $a_3 = 0$, $a_2 = 0$, $a_1 = 0$, $a_0 = 10$

The remainder when $f(x^{10})$ is divided by $f(x)$ is $a_0 = 10$.

Method 2 (Provided by Pui Ching Middle School 李國柱老師)

$f(x^{10}) = x^{90} + x^{80} + x^{70} + x^{60} + x^{50} + x^{40} + x^{30} + x^{20} + x^{10} + 1$

$= x^{90} - 1 + x^{80} - 1 + x^{70} - 1 + x^{60} - 1 + x^{50} - 1 + x^{40} - 1 + x^{30} - 1 + x^{20} - 1 + x^{10} - 1 + 10$

$= (x^{10} - 1)g_1(x) + (x^{10} - 1)g_2(x) + (x^{10} - 1)g_3(x) + \dots + (x^{10} - 1)g_9(x) + 10$

where $g_1(x) = x^{80} + x^{70} + \dots + x^{10} + 1$, $g_2(x) = x^{70} + x^{60} + \dots + x^{10} + 1$, … , $g_9(x) = 1$

$f(x^{10}) = (x^{10} - 1)[g_1(x) + g_2(x) + \dots + g_9(x)] + 10$

$= f(x)(x - 1)[g_1(x) + g_2(x) + \dots + g_9(x)] + 10$

The remainder is 10.

Method 3 (Provided by Pui Ching Middle School 李國柱老師)

Clearly $f(1) = 10$.

By division algorithm, $f(x) = (x - 1)Q(x) + 10$, where $Q(x)$ is a polynomial

$f(x^{10}) = (x^{10} - 1)Q(x^{10}) + 10$

$= (x - 1)(x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)Q(x^{10}) + 10$

$= f(x)(x - 1)Q(x^{10}) + 10$

The remainder is 10.

G10.3 The fraction $\frac{p}{q}$ is in its simplest form. If $\frac{7}{10}<\frac{p}{q}<\frac{11}{15}$ where q is the smallest possible positive integer and $c = pq$. Find the value of c.

Reference 2005 HI1, 2010 HG7

$\frac{7}{10}<\frac{p}{q}<\frac{11}{15} \Rightarrow 1-\frac{7}{10}>1-\frac{p}{q}>1-\frac{11}{15}$

$\frac{3}{10}>\frac{q-p}{q}>\frac{4}{15} \Rightarrow \frac{10}{3}<\frac{q}{q-p}<\frac{15}{4}$

$\frac{1}{3}<\frac{q}{q-p}-3<\frac{3}{4} \Rightarrow \frac{1}{3}<\frac{3p-2q}{q-p}<\frac{3}{4}$

$\frac{3p-2q}{q-p}=\frac{1}{2} \Rightarrow 3p-2q=1,\ q-p=2$

Solving the equations gives $p = 5$, $q = 7$, $\frac{7}{10}<\frac{5}{7}<\frac{11}{15}$, $c = 35$

G10.4 A positive integer d when divided by 7 will have 1 as its remainder; when divided by 5 will have 2 as its remainder and when divided by 3 will have 2 as its remainder. Find the least possible value of d.

$d = 7m + 1 = 5n + 2 = 3r + 2$

$7m - 5n = 1$ …… (1)

$5n = 3r$ …… (2)

From (2), $n = 3k$, $r = 5k$

Sub. $n = 3k$ into (1), $7m - 5(3k) = 1$

$\Rightarrow 7m - 15k = 1$

$-14 + 15 = 1 \Rightarrow$ A possible solution is $m = -2$, $k = -1$

$m = -2 + 15t$, $k = -1 + 7t$

When $t = 1$, $m = 13$, $k = 6$, $n = 18$, $r = 30$.

The least possible value of $d = 3(30) + 2 = 92$

Individual Events

I1	***a***	$\frac{2}{3}$	**I2**	***a***	12	**I3**	***P***	8	**I4**	***n***	9	**I5**	***a***	6
	b	0		***b***	36		***Q***	12		***b***	3		***b***	30
	c	3		***c***	12		***R***	4		***c***	8		***c***	4
	d	–6		***d***	5		***S***	70		***d***	62		***d***	4

Group Events

G1	***a***	180	**G2**	***a***	1	**G3**	***m***	–3	**G4**	***a***	99999919	**G5**	***a***	10	**Group Spare**	***a***	4
	b	7		***b***	2		***b***	1		***b***	1		***b***	9		***k***	2
	c	9		***c***	1		***c***	1.6		***c***	2		***c***	55		***d***	8.944
	d	4		***d***	120		***d***	2		***d***	1891		***d***	16		***r***	$\frac{25}{24}$

Individual Event 1 (1998 Sample Individual Event)

I1.1 Given that $\frac{3}{a}+\frac{1}{u}=\frac{7}{2}$ and $\frac{2}{a}-\frac{3}{u}=6$ are simultaneous equations in *a* and *u*. Solve for *a*.

3(1) + (2): $\frac{11}{a}=\frac{33}{2}$

$a=\frac{2}{3}$

I1.2 Three solutions of the equation $px + qy + bz = 1$ are (0, 3*a*, 1), (9*a*, –1, 2) and (0, 3*a*, 0). Find the value of the coefficient *b* .

$$\begin{cases} 3aq+b=1\cdots\cdots(1) \\ 9ap-q+2b=1\cdots\cdots(2) \\ 3aq=1\cdots\cdots(3) \end{cases}$$

Sub. (3) into (1): $1 + b = 1$

$\Rightarrow b = 0$

I1.3 Find the value of *c* so that the graph of $y = mx + c$ passes through the two points (*b* + 4, 5) and (–2, 2) .

The 2 points are: (4, 5) and (–2, 2). The slope is $\frac{5-2}{4-(-2)}=\frac{1}{2}$.

The line $y=\frac{1}{2}x+c$ passes through (–2, 2): $2 = -1 + c$

$\Rightarrow c = 3$

I1.4 The solution of the inequality $x^2 + 5x - 2c \le 0$ is $d \le x \le 1$. Find the value of *d* .

$x^2 + 5x - 6 \le 0$

$\Rightarrow (x + 6)(x - 1) \le 0$

$-6 \le x \le 1$

$d = -6$

Individual Event 2

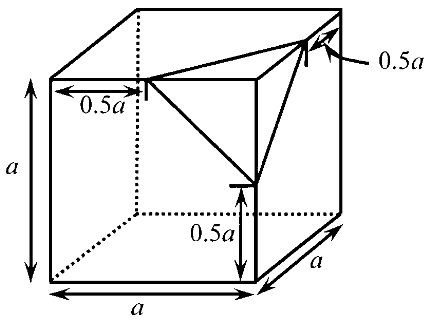
I2.1 By considering: $\frac{1^2}{1}=1$, $\frac{1^2+2^2}{1+2}=\frac{5}{3}$, $\frac{1^2+2^2+3^2}{1+2+3}=\frac{7}{3}$, $\frac{1^2+2^2+3^2+4^2}{1+2+3+4}=3$, find the value of a such that $\frac{1^2+2^2+\cdots+a^2}{1+2+\cdots+a}=\frac{25}{3}$.

The given is equivalent to: $\frac{1^2}{1}=\frac{3}{3}$, $\frac{1^2+2^2}{1+2}=\frac{5}{3}$, $\frac{1^2+2^2+3^2}{1+2+3}=\frac{7}{3}$, $\frac{1^2+2^2+3^2+4^2}{1+2+3+4}=\frac{9}{3}$

and 2×1 + 1 = 3, 2×2 + 1 = 5, 2×3 + 1 = 7, 2×4 + 1 = 9; so $2a + 1 = 25$

$\Rightarrow a = 12$

I2.2 A triangular pyramid is cut from a corner of a cube with side length a cm as the figure shown. If the volume of the pyramid is b cm^3, find the value of b.



$b=\frac{1}{3}\text{base area}\times\text{height}=\frac{1}{3}\left(\frac{\frac{1}{2}a\times\frac{1}{2}a}{2}\right)\times\frac{1}{2}a$

$=\frac{1}{48}a^3=\frac{1}{48}\cdot 12^3=36$

I2.3 If the value of $x^2 + cx + b$ is not less than 0 for all real number x, find the maximum value of c

$x^2 + cx + 36 \geq 0$

$\Delta = c^2 - 4(36) \leq 0$

$\Rightarrow c \leq 12$

The maximum value of $c = 12$.

I2.4 If the units digit of 1997^{1997} is $c - d$, find the value of d.

$1997^{1997} \equiv 7^{1997} \equiv 7^{4(499)+1} \equiv 7 \pmod{10}$

The units digit of 1997^{1997} is 7

$12 - d = 7$

$d = 5$

Individual Event 3

I3.1 The average of a, b c and d is 8. If the average of a, b, c, d and P is P, find the value of P.

$$\frac{a+b+c+d}{4}=8$$

$\Rightarrow a + b + c + d = 32$

$$\frac{a+b+c+d+P}{5}=P$$

$\Rightarrow 32 + P = 5P$

$P = 8$

I3.2 If the lines $2x + 3y + 2 = 0$ and $Px + Qy + 3 = 0$ are parallel, find the value of Q.

Their slopes are equal: $-\frac{2}{3}=-\frac{8}{Q}$

$Q = 12$

I3.3 The perimeter and the area of an equilateral triangle are Q cm and $\sqrt{3}R\,\text{cm}^2$ respectively. Find the value of R.

Perimeter = 12 cm, side = 4 cm

Area $=\frac{1}{2}\cdot 4^2\sin 60^\circ=4\sqrt{3}$

$R = 4$

I3.4 If $(1 + 2 + \ldots + R)^2 = 1^2 + 2^2 + \ldots + R^2 + S$, find the value of S.

$(1 + 2 + 3 + 4)^2 = 1^2 + 2^2 + 3^2 + 4^2 + S$

$100 = 30 + S$

$S = 70$

Individual Event 4

I4.1 If each interior angle of a n-sided regular polygon is 140°, find the value of n .

Reference: 1987 FG6.3

Each exterior angle is 40° (adj. ∠s on st. line)

$$\frac{360^\circ}{n} = 40^\circ$$

$n = 9$

I4.2 If the solution of the inequality $2x^2 - nx + 9 < 0$ is $k < x < b$, find the value of b .

$2x^2 - 9x + 9 < 0$

$(2x - 3)(x - 3) < 0$

$$\frac{3}{2} < x < 3$$

$\Rightarrow b = 3$

I4.3 If $cx^3 - bx + x - 1$ is divided by $x + 1$, the remainder is -7, find the value of c .

$f(x) = cx^3 - 3x + x - 1$

$f(-1) = -c + 3 - 1 - 1 = -7$

$c = 8$

I4.4 If $x + \frac{1}{x} = c$ and $x^2 + \frac{1}{x^2} = d$, find d .

$$x + \frac{1}{x} = 8$$

$$\Rightarrow \left(x + \frac{1}{x}\right)^2 = 64$$

$$x^2 + \frac{1}{x^2} + 2 = 64$$

$d = 62$

Individual Event 5

I5.1 The volume of a hemisphere with diameter a cm is 18π cm^3, find the value of a .

$$\frac{1}{2}\cdot 4\pi\left(\frac{a}{2}\right)^2 = 18\pi$$

$a = 6$

I5.2 If $\sin 10a° = \cos(360° - b°)$ and $0 < b < 90$, find the value of b .

$\sin 60° = \cos(360° - b°)$

$360° - b° = 330°$

$b = 30$

I5.3 The triangle is formed by the x-axis and y-axis and the line $bx + 2by = 120$.

If the bounded area of the triangle is c, find the value of c .

$30x + 60y = 120$

$\Rightarrow x + 2y = 4$

x-intercept = 4, y-intercept = 2

$c = \frac{1}{2}\cdot 4\cdot 2 = 4$

I5.4 If the difference of the two roots of the equation $x^2 - (c + 2)x + (c + 1) = 0$ is d , find the value of d .

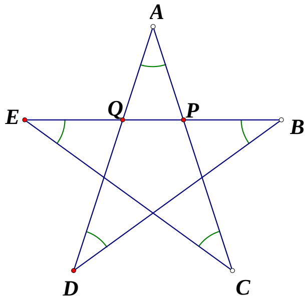
$x^2 - 6x + 5 = 0$

$\Rightarrow (x - 1)(x - 5) = 0$

$x = 1$ or 5

$\Rightarrow d = 5 - 1 = 4$

Group Event 1



G1.1 In the given diagram, $\angle A + \angle B + \angle C + \angle D + \angle E = a°$, find the value of a .

Reference: 1989 FI5.1, 2005 FI2.3

In ΔAPQ, $\angle B + \angle D = \angle AQP$ ……(1) (ext. ∠ of Δ)

$\angle C + \angle E = \angle APQ$ ……(2) (ext. ∠ of Δ)

$\angle A+\angle B + \angle C + \angle D +\angle E = \angle A + \angle AQP +\angle APQ$ (by (1) and (2))

$= 180°$ (∠s sum of Δ)

$\therefore a = 180$

G1.2 There are x terms in the algebraic expression $x^6 + x^6 + x^6 + \dots + x^6$ and its sum is x^b. Find the value of b .

$x \cdot x^6 = x^b$

$x^7 = x^b$

$b = 7$

G1.3 If $1 + 3 + 3^2 + 3^3 + \dots +3^8 = \frac{3^c - 1}{2}$, find the value of c .

$$\frac{3^9 - 1}{2} = \frac{3^c - 1}{2}$$

$c = 9$

G1.4 16 cards are marked from 1 to 16 and one is drawn at random.

If the chance of it being a perfect square number is $\frac{1}{d}$, find the value of d .

Reference: 1995 HI4

Perfect square numbers are 1, 4, 9, 16.

Probability $= \frac{4}{16} = \frac{1}{d}$

$d = 4$.

Group Event 2

G2.1 If the sequence 1, 6 + 2a, 10 + 5a, … forms an A.P., find the value of a .

$$6 + 2a = \frac{1+10+5a}{2}$$

$12 + 4a = 11 + 5a$

$\Rightarrow a = 1$

G2.2 If $(0.0025\times 40)^b = \frac{1}{100}$, find the value of b .

$$\left(\frac{1}{400}\times 40\right)^b = \frac{1}{100}$$

$$\Rightarrow \frac{1}{10^b} = \frac{1}{10^2}$$

$b = 2$

G2.3 If c is an integer and $c^3 + 3c + \frac{3}{c} + \frac{1}{c^3} = 8$, find the value of c .

$$\left(c + \frac{1}{c}\right)^3 = 2^3$$

$$\Rightarrow \left(c + \frac{1}{c} - 2\right)\left[\left(c + \frac{1}{c}\right)^2 + 2\left(c + \frac{1}{c}\right) + 4\right] = 0$$

$c^2 - 2c + 1 = 0$ or $\left(c + \frac{1}{c}\right)^2 + 2\left(c + \frac{1}{c}\right) + 4 = 0$

$\Rightarrow c = 1$ or no real solution ($\because\ \Delta = 2^2 - 4(2)(4) < 0$)

$\therefore\ c = 1$

G2.4 There are d different ways for arranging 5 girls in a row. Find the value of d .

First position has 5 choices; 2nd position has 4 choices, …, the last position has 1 choice.

$d = 5\times4\times3\times2\times1 = 120$

Group Event 3

G3.1 Let m be an integer satisfying the inequality $14x - 7(3x - 8) < 4(25 + x)$.
Find the least value of m.
$14x - 21x + 56 < 100 + 4x$
$-44 < 11x$
$\Rightarrow -4 < x$
$m = -3$

G3.2 It is given that $f(x) = \frac{1}{3}x^3 - 2x^2 + \frac{2}{3}x^3 + 3x^2 + 5x + 7 - 4x$. If $f(-2) = b$, find the value of b.
$f(x) = x^3 + x^2 + x + 7$
$b = f(-2) = -8 + 4 - 2 + 7 = 1$

G3.3 It is given that $\log\frac{x}{2} = 0.5$ and $\log\frac{y}{5} = 0.1$. If $\log xy = c$, find the value of c.
$\log\frac{x}{2} + \log\frac{y}{5} = 0.5 + 0.1$
$\log xy - 1 = 0.6$
$\Rightarrow c = \log xy = 1.6$

G3.4 Three prime numbers d, e and f which are all less than 10, satisfy the two conditions $d + e = f$ and $d < e$. Find the value of d.
Possible prime numbers are 2, 3, 5, 7.
$2 + 3 = 5$ or $2 + 5 = 7$
$\therefore d = 2$

Group Event 4

G4.1 It is given that $a = 103\times97\times10009$, find the value of a .

$a = (100 + 3)(100 - 3)\times10009$

$= (10000 - 9)\times(10000 + 9)$

$= 100000000 - 81$

$a = 99999919$

G4.2 It is given that $1 + x + x^2 + x^3 + x^4 = 0$. If $b = 2 + x + x^2 + x^3 + x^4 + \cdots + x^{1989}$, find the value of b.

Reference: 2014 HI7

$b = 1 + (1 + x + x^2 + x^3 + x^4) + x^5(1 + x + x^2 + x^3 + x^4) + \cdots + x^{1985}(1 + x + x^2 + x^3 + x^4) = 1$

G4.3 It is given that m and n are two natural numbers and both are not greater than 10.

If c is the number of pairs of m and n satisfying the equation $mx = n$, where $\frac{1}{4} < x < \frac{1}{3}$,

find the value of c .

$\frac{1}{4} < \frac{m}{n} < \frac{1}{3} \Rightarrow \frac{n}{4} < m < \frac{n}{3}$

$\begin{cases} 4m - n > 0 \\ 3m - n < 0 \end{cases}$

$3m < n < 4m$

$1 \le m \Rightarrow 3 \le 3m < n < 4m \le 4\times10 = 40$

Possible $n = 4, 5, 6, \dots , 10$

when $n = 4$, $\frac{4}{4} < m < \frac{4}{3}$ no solution

when $n = 5$, $\frac{5}{4} < m < \frac{5}{3}$ no solution

when $n = 6$, $\frac{6}{4} < m < \frac{6}{3}$ no solution

when $n = 7$, $\frac{7}{4} < m < \frac{7}{3} \Rightarrow m = 2, x = \frac{2}{7}$

when $n = 8$, $\frac{8}{4} < m < \frac{8}{3}$ no solution

when $n = 9$, $\frac{9}{4} < m < \frac{9}{3}$ no solution

when $n = 10$, $\frac{10}{4} < m < \frac{10}{3} \Rightarrow m = 3, x = \frac{3}{10}$

$c = 2$ (There are 2 solutions.)

G4.4 Let x and y be real numbers and define the operation * as $x*y = px^y + q + 1$.

It is given that $1*2 = 869$ and $2*3 = 883$. If $2*9 = d$, find the value of d .

$\begin{cases} p + q + 1 = 869 \\ 8p + q + 1 = 883 \end{cases}$

(2) – (1): $7p = 14$

$p = 2$, $q = 866$

$\Rightarrow d = 2\times2^9 + 866 + 1 = 1891$

Group Event 5

G5.1 If a is a positive multiple of 5, which gives remainder 1 when divided by 3, find the smallest possible value of a . **(Reference: 1998 FSG.1)**

$a = 5k = 3m + 1$

$5\times2 = 3\times3 + 1$

The smallest possible $a = 10$.

G5.2 If $x^3 + 6x^2 + 12x + 17 = (x + 2)^3 + b$, find the value of b .

Reference: 1998 FG1.4

$(x + 2)^3 + b = x^3 + 6x^2 + 12x + 8 + b$

$b = 9$

G5.3 If c is a 2 digit positive integer such that sum of its digits is 10 and product of its digit is 25, find the value of c . **(Reference: 1998 FSG.3)**

$c = 10x + y$, where $0 < x < 10$, $0 \le y < 10$.

$x + y = 10$

$xy = 25$

Solving these two equations gives $x = y = 5$; $c = 55$

G5.4 Let $S_1, S_2,\dots, S_{10}$ be the first ten terms of an A.P., which consists of positive integers.

If $S_1 + S_2 +\dots+ S_{10} = 55$ and $(S_{10} - S_8) + (S_9 - S_7) +\cdots\cdots+ (S_3 - S_1) = d$, find the value of d .

Reference: 1998 FSG.4

Let the general term be $S_n = a + (n - 1)t$

$$\frac{10}{2}[2a+(10-1)t]=55$$

$\Rightarrow 2a + 9t = 11$

$\because$ a, t are positive integers, $a = 1$, $t = 1$

$d = (S_{10} - S_8) + (S_9 - S_7) + \cdots\cdots + (S_3 - S_1)$

$= [a + 9t - (a + 7t)] + [a + 8t - (a + 6t)] + \cdots\cdots + (a + 2t - a)$

$d = 2t + 2t + 2t + 2t + 2t + 2t + 2t + 2t = 16t = 16$

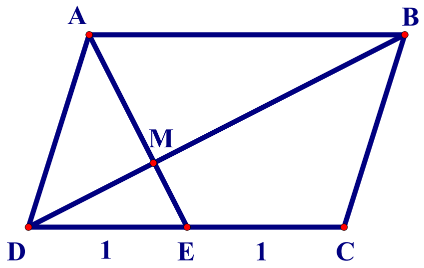
Group Spare

GS.1 *ABCD* is a parallelogram and *E* is the midpoint of *CD*. If the ratio of the area of the triangle *ADE* to the area of the parallelogram *ABCD* is $1 : a$, find the value of a .

$1 : a = 1 : 4$

$a = 4$

GS.2 *ABCD* is a parallelogram and *E* is the midpoint of *CD*. *AE* and *BD* meet at *M*. If $DM : MB = 1 : k$, find the value of k .



It is easy to show that $\Delta ABM \sim \Delta EDM$ (equiangular)

$DM : MB = DE : AB = 1 : 2$

$k = 2$

GS.3 If the square root of 5 is approximately 2.236, the square root of 80 with the same precision is d. Find the value of d .

$\sqrt{80} = \sqrt{16\times 5} = 4\sqrt{5} = 4\times 2.236 = 8.944$

GS.4 A square is changed into a rectangle by increasing its length by 20% and decreasing its width by 20% .

If the ratio of the area of the rectangle to the area of the square is $1 : r$, find the value of r .

Let the side of the square be x .

Ratio of areas $= 1.2x\cdot 0.8x : x^2$

$= 0.96 : 1 = 1 : \frac{25}{24}$

$r = \frac{25}{24}$

Individual Events

SI			I1			I2			I3			I4			I5		
	a	$\frac{2}{3}$		***a***	*1 see the remark		***a***	38		***a***	10		***p***	15		***a***	4
	b	0		***b***	2		***b***	104		***b***	27		***q***	4		***b***	5
	c	3		***c***	4		***c***	100		***c***	*23 see the remark		***r***	57		***c***	24
	d	–6		***d***	24		***d***	–50		***d***	26		***s***	3		***d***	57

Group Events

SG			G1			G2			G3			G4			G5			GS		
	a	10		***p***	4		***a***	1110		***a***	90		***a***	0.13717421		***a***	4290		***s***	6
	b	73		***q***	3		***b***	1		***b***	1		***b***	90		***b***	18		***b***	10
	c	55		***r***	2		***c***	0		***c***	0		***c***	$\frac{665}{729}$		***c***	67		***c***	81
	d	16		***a***	9		***d***	6		***d***	1		***d***	50		***d***	30		***d***	50

Sample Individual Event (1997 Final Individual Event 1)

SI.1 Given that $\frac{3}{a}+\frac{1}{u}=\frac{7}{2}$ and $\frac{2}{a}-\frac{3}{u}=6$ are simultaneous equations in ***a*** and ***u***. Solve for ***a***.

3(1) + (2): $\frac{11}{a}=\frac{33}{2}$

$a=\frac{2}{3}$

SI.2 Three solutions of the equation ***px*** + ***qy*** + ***bz*** = 1 are (0, 3***a***, 1), (9***a***, –1, 2) and (0, 3***a***, 0). Find the value of the coefficient ***b***.

$$\begin{cases} 3aq+b=1 \\ 9ap-q+2b=1 \\ 3aq=1 \end{cases}$$

Sub. (3) into (1): 1 + ***b*** = 1

⇒ ***b*** = 0

SI.3 Find ***c*** so that the graph of ***y = mx + c*** passes through the two points (***b*** + 4, 5) and (–2, 2).

The 2 points are: (4, 5) and (–2, 2). The slope is $\frac{5-2}{4-(-2)}=\frac{1}{2}$.

The line $y=\frac{1}{2}x+c$ passes through (–2, 2): 2 = –1 + ***c***

⇒ ***c*** = 3

SI.4 The solution of the inequality $x^2+5x-2c\le 0$ is $d\le x\le 1$. Find ***d***.

$x^2+5x-6\le 0$

⇒ $(x+6)(x-1)\le 0$

$-6\le x\le 1$

d = –6

Individual Event 1

I1.1 If $\boldsymbol{a}$ is the maximum value of $\frac{1}{2}\sin^2 3\theta - \frac{1}{2}\cos 2\theta$, find the value of $\boldsymbol{a}$.

$-1 \le \sin 3\theta \le 1$ and $-1 \le \cos 2\theta \le 1$

$\frac{1}{2}\sin^2 3\theta \le \frac{1}{2}$ and $-\frac{1}{2}\cos 2\theta \le \frac{1}{2}$

$\frac{1}{2}\sin^2 3\theta - \frac{1}{2}\cos 2\theta \le \frac{1}{2} + \frac{1}{2} = 1 = \boldsymbol{a}$,

Maximum occur when $\sin^2 3\theta = 1$ and $-\cos 2\theta = 1$

i.e. $3\theta = 90° + 180°\boldsymbol{n}$ and $2\theta = 360°\boldsymbol{m} + 180°$, where $\boldsymbol{m}$, $\boldsymbol{n}$ are integers.

$\theta = 30° + 60°\boldsymbol{n} = 180°\boldsymbol{m} + 90° \Rightarrow 60°\boldsymbol{n} = 180°\boldsymbol{m} + 60° \Rightarrow \boldsymbol{n} = 3\boldsymbol{m} + 1$; let $\boldsymbol{m} = 1$, $\boldsymbol{n} = 4, \theta = 270°$

Remark: the original question is

If $\boldsymbol{a}$ is the maximum value of $\frac{1}{2}\sin^2 \theta + \frac{1}{2}\cos 3\theta$, find the value of $\boldsymbol{a}$.

Maximum occur when $\sin^2 \theta = 1$ and $\cos 3\theta = 1$

i.e. $\theta = 90° + 180°\boldsymbol{n}$ and $3\theta = 360°\boldsymbol{m}$, where $\boldsymbol{m}$, $\boldsymbol{n}$ are integers.

$\theta = 90° + 180°\boldsymbol{n} = 120°\boldsymbol{m} \Rightarrow 3 + 6\boldsymbol{n} = 4\boldsymbol{m}$, LHS is odd and RHS is even, contradiction.

The question was wrong because we cannot find any θ to make the expression a maximum.

I1.2 If $\begin{cases} \boldsymbol{x} + \boldsymbol{y} = 2 \\ \boldsymbol{xy} - \boldsymbol{z}^2 = \boldsymbol{a} \\ \boldsymbol{b} = \boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z} \end{cases}$, find the value of $\boldsymbol{b}$.

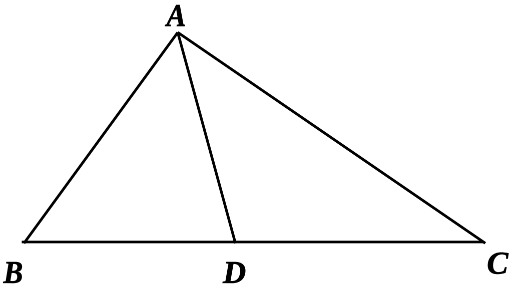
(2), $\boldsymbol{xy} = 1 + \boldsymbol{z}^2 > 0$; together with (1) we have $\boldsymbol{x} > 0$ and $\boldsymbol{y} > 0$

by A.M. ≥ G.M. in (1) $\boldsymbol{x} + \boldsymbol{y} \ge 2\sqrt{\boldsymbol{xy}} \Rightarrow 2 \ge 2\sqrt{1 + \boldsymbol{z}^2}$

After simplification, $0 \ge \boldsymbol{z}^2 \Rightarrow \boldsymbol{z} = 0$

(3): $\boldsymbol{b} = \boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z} = 2 + 0 = 2$

I1.3 In the figure, $\boldsymbol{BD} = \boldsymbol{b}$ cm, $\boldsymbol{DC} = \boldsymbol{c}$ cm and area of $\Delta\boldsymbol{ABD} = \frac{1}{3} \times$area of $\Delta\boldsymbol{ABC}$, find the value of $\boldsymbol{c}$.



Let the common height be $\boldsymbol{h}$ cm

$\frac{1}{2}\boldsymbol{BD} \times \boldsymbol{h}\text{cm} = \frac{1}{3}\cdot\frac{1}{2}\boldsymbol{BC} \times \boldsymbol{h}\text{cm}$

$2 = \frac{1}{3}(2 + \boldsymbol{c}) \Rightarrow \boldsymbol{c} = 4$

I1.4 Suppose $\boldsymbol{d}$ is the number of positive factors of $500 + \boldsymbol{c}$, find the value of $\boldsymbol{d}$.

Reference 1993 HI8, 1994 FI3.2, 1997 HI3, 1998 HI10, 2002 FG4.1, 2005 FI4.4

$500 + \boldsymbol{c} = 504 = 2^3 \times 3^2 \times 7$

A positive factor is in the form $2^{\boldsymbol{i}} \times 3^{\boldsymbol{j}} \times 7^{\boldsymbol{k}}$, where $0 \le \boldsymbol{i} \le 3$, $0 \le \boldsymbol{j} \le 2$, $0 \le \boldsymbol{k} \le 1$

The total number of positive factors are $(1 + 3)(1 + 2)(1 + 1) = 24$

Individual Event 2

I2.1 If $\boldsymbol{A}(1, 3)$, $\boldsymbol{B}(5, 8)$ and $\boldsymbol{C}(29, \boldsymbol{a})$ are collinear, find the value of $\boldsymbol{a}$.

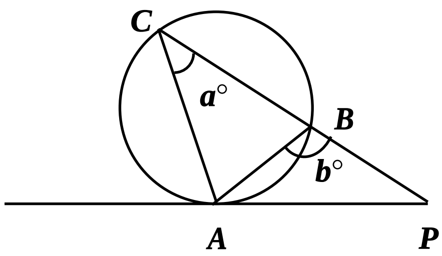
The slopes are equal: $\frac{8-3}{5-1}=\frac{\boldsymbol{a}-8}{29-5}$

$\frac{\boldsymbol{a}-8}{24}=\frac{5}{4}$

$\Rightarrow \boldsymbol{a} - 8 = 30$

$\boldsymbol{a} = 38$

I2.2 In the figure, $\boldsymbol{PA}$ touches the circle $\boldsymbol{ABC}$ at $\boldsymbol{A}$, $\boldsymbol{PBC}$ is a straight line, $\boldsymbol{AB} = \boldsymbol{PB}$, $\angle \boldsymbol{ACB} = \boldsymbol{a}°$. If $\angle \boldsymbol{ABP} = \boldsymbol{b}°$, find the value of $\boldsymbol{b}$.



$\angle \boldsymbol{BAP} = \boldsymbol{a}° = 38°$ ($\angle$ in alt. seg.)

$\angle \boldsymbol{BPA} = 38°$ (base $\angle$s isos. Δ)

$38 + 38 + \boldsymbol{b} = 180$ ($\angle$ sum of Δ)

$\boldsymbol{b} = 104$

I2.3 If $\boldsymbol{c}$ is the minimum value of the quadratic function $\boldsymbol{y} = \boldsymbol{x}^2 + 4\boldsymbol{x} + \boldsymbol{b}$, find the value of $\boldsymbol{c}$.

$\boldsymbol{y} = \boldsymbol{x}^2 + 4\boldsymbol{x} + 104 = (\boldsymbol{x} + 2)^2 + 100 \geq 100 = \boldsymbol{c}$

I2.4 If $\boldsymbol{d} = 1 - 2 + 3 - 4 + \ldots - \boldsymbol{c}$, find the value of $\boldsymbol{d}$.

Reference: 1991 FSI.1

$\boldsymbol{d} = (1 - 2) + (3 - 4) + \ldots + (99 - 100)$

$= -1 - 1 - \ldots - 1$ (50 times)

$= -50$

Individual Event 3

I3.1 If $\{p, q\} = q \times a + p$ and $\{2, 5\} = 52$, find the value of a.

$\{2, 5\} = 5 \times a + 2 = 52$

$a = 10$

I3.2 If a, $\frac{37}{2}$, b is an arithmetic progression, find the value of b.

$\frac{a+b}{2} = \frac{37}{2}$

$b = 27$

I3.3 If $b^2 - c^2 = 200$ and $c > 0$, find the value of c.

$27^2 - c^2 = 200$

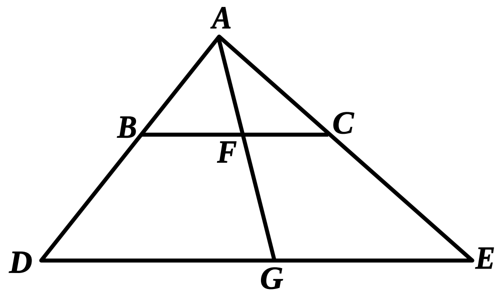
$c^2 = 729 - 200 = 529$

$c = 23$

Remark: Original question is: If $b^2 - c^2 = 200$, find the value of c.

$c = \pm 23$, c is not unique.

I3.4 Given that in the figure, $BC // DE$, $BC : DE = 10 : c$ and $AF : FG = 20 : d$, find the value of d.



By similar triangles, $AF : AG = AC : AE = BC : DE$

$20 : (20 + d) = 10 : 23$

$d = 26$

Individual Event 4

I4.1 Given that $\frac{10\boldsymbol{x}-3\boldsymbol{y}}{\boldsymbol{x}+2\boldsymbol{y}}=2$ and $\boldsymbol{p}=\frac{\boldsymbol{y}+\boldsymbol{x}}{\boldsymbol{y}-\boldsymbol{x}}$, find the value of $\boldsymbol{p}$.

$10\boldsymbol{x} - 3\boldsymbol{y} = 2(\boldsymbol{x} + 2\boldsymbol{y})$

$8\boldsymbol{x} = 7\boldsymbol{y}$

$\boldsymbol{p}=\frac{\boldsymbol{y}+\boldsymbol{x}}{\boldsymbol{y}-\boldsymbol{x}}$

$=\frac{8\boldsymbol{y}+8\boldsymbol{x}}{8\boldsymbol{y}-8\boldsymbol{x}}$

$=\frac{8\boldsymbol{y}+7\boldsymbol{y}}{8\boldsymbol{y}-7\boldsymbol{y}}=15$

I4.2 Given that $\boldsymbol{a} \neq \boldsymbol{b}$ and $\boldsymbol{ax} = \boldsymbol{bx}$. If $\boldsymbol{p} + \boldsymbol{q} = 19(\boldsymbol{a} - \boldsymbol{b})^{\boldsymbol{x}}$, find the value of $\boldsymbol{q}$.

$\boldsymbol{a} \neq \boldsymbol{b}$ and $\boldsymbol{ax} = \boldsymbol{bx} \Rightarrow \boldsymbol{x} = 0$

$\boldsymbol{p} + \boldsymbol{q} = 19(\boldsymbol{a} - \boldsymbol{b})^{\boldsymbol{x}}$

$\Rightarrow 15 + \boldsymbol{q} = 19$

$\boldsymbol{q} = 4$

I4.3 Given that the sum of $\boldsymbol{q}$ consecutive numbers is 222, and the largest of these consecutive numbers is $\boldsymbol{r}$, find the value of $\boldsymbol{r}$.

The smallest integer is $\boldsymbol{r} - \boldsymbol{q} + 1$

$\frac{\boldsymbol{q}}{2}(\boldsymbol{r}-\boldsymbol{q}+1+\boldsymbol{r})=222$

$\Rightarrow 2(2\boldsymbol{r} - 3) = 222$

$\boldsymbol{r} = 57$

I4.4 If $\tan^2(\boldsymbol{r} + \boldsymbol{s})^\circ = 3$ and $0 \leq \boldsymbol{r} + \boldsymbol{s} \leq 90$, find the value of $\boldsymbol{s}$.

$\tan^2(57 + \boldsymbol{s})^\circ = 3$

$57 + \boldsymbol{s} = 60$

$\boldsymbol{s} = 3$

Individual Event 5

I5.1 If the sum of roots of $5x^2 + ax - 2 = 0$ is twice the product of roots, find the value of a.

$\alpha + \beta = 2\alpha\beta$

$-\frac{a}{5} = 2\left(-\frac{2}{5}\right)$

$a = 4$

I5.2 Given that $y = ax^2 - bx - 13$ passes through (3, 8), find the value of b.

$8 = 4(3)^2 - b(3) - 13$

$b = 5$

I5.3 If there are c ways of arranging b girls in a circle, find the value of c.

Reference: 2000 FG4.4, 2011 FI1.4

First arrange the 5 girls in a line, the number of ways = $5\times4\times3\times2\times1 = 120$

Next, join the first girl and the last girl to form a circle. There are 5 repetitions.

The number of ways = $c = 120 \div 5 = 24$

I5.4 If $\frac{c}{4}$ straight lines and 3 circles are drawn on a paper, and d is the largest numbers of points of intersection, find the value of d.

For the 3 circles, there are 6 intersections.

If each straight line is drawn not passing through these intersections, it intersects the 3 circles at 6 other points. The 6 straight lines intersect each other at 1 + 2 + 3 + 4 + 5 points.

$\therefore$ d = the largest numbers of points of intersection = $6 + 6\times6 + 15 = 57$

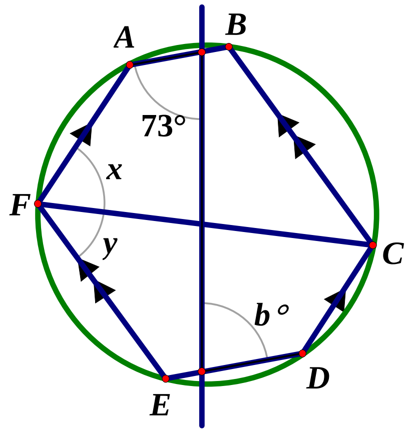
Sample Group Event

SG.1 If $\boldsymbol{a}$ is the smallest positive integer which gives remainder 1 when divided by 3 and is a multiple of 5, find the value of $\boldsymbol{a}$. **(Reference: 1997 FG5.1)**

$a = 5k = 3m + 1$

The smallest possible $a = 10$.

SG.2 In the following diagram, ***FA***//***DC*** and ***FE***//***BC***. Find the value of $\boldsymbol{b}$.



Join ***AD*** and ***CF***.

Let $\angle CFE = x$, $\angle AFC = y$

$\angle BCF = x$ (alt. ∠s, ***FE*** // ***BC***)

$\angle DCF = y$ (alt. ∠s, ***FA*** // ***DC***)

$\angle BCD = x + y$

$\angle BAD = 180° - x - y = \angle ADE$ (opp. ∠ cyclic quad.)

∴ ***AB*** // ***ED*** (alt. ∠s eq.)

$b = 73$ (alt. ∠s ***AB*** // ***ED***)

SG.3 If $\boldsymbol{c}$ is a 2 digit positive integer such that sum of its digits is 10 and product of its digit is 25, find the value of $\boldsymbol{c}$. **(Reference: 1997 FG5.3)**

$c = 10x + y$, where $0 < x < 10$, $0 \le y < 10$.

$x + y = 10$

$xy = 25$

Solving these two equations gives $x = y = 5$; $c = 55$

SG.4 Let $S_1, S_2, \ldots, S_{10}$ be the first ten terms of an A.P., which consists of positive integers.
If $S_1 + S_2 + \ldots + S_{10} = 55$ and $(S_{10} - S_8) + (S_9 - S_7) + \ldots + (S_3 - S_1) = d$, find $\boldsymbol{d}$.
Reference: 1997 FG5.4

Let the general term be $S_n = a + (n - 1)t$

$$\frac{10}{2}[2a + (10-1)t] = 55$$

$\Rightarrow 2a + 9t = 11$

∵ a, t are positive integers, $a = 1$, $t = 1$

$d = (S_{10} - S_8) + (S_9 - S_7) + \ldots + (S_3 - S_1)$

$= [a + 9t - (a + 7t)] + [a + 8t - (a + 6t)] + \ldots + (a + 2t - a)$

$d = 2t + 2t + 2t + 2t + 2t + 2t + 2t + 2t$

$= 16t = 16$

Group Event 1

G1.1 If the area of a given sector $s = 4$ cm^2, the radius of this sector $r = 2$ cm and the arc length of this sector $A = p$ cm, find the value of p.

By the formula $A = \frac{1}{2}rs$, where A is the sector area, r is the radius and s is the arc length

$4 = \frac{1}{2}(2)p$

$p = 4$

G1.2 Given that $\frac{a}{2b+c} = \frac{b}{2c+a} = \frac{c}{2a+b}$ and $a + b + c \neq 0$. If $q = \frac{2b+c}{a}$, find the value of q.

Reference 2010 FG1.2

Let $\frac{a}{2b+c} = \frac{b}{2c+a} = \frac{c}{2a+b} = k$

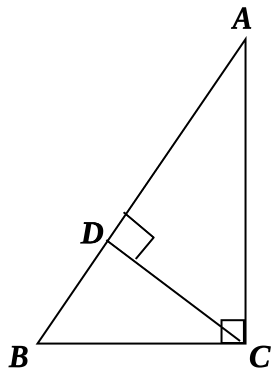
$a = (2b + c)k$; $b = (2c + a)k$; $c = (2a + b)k$

$a + b + c = (2b + c + 2c + a + 2a + b)k$

$a + b + c = (3a + 3b + 3c)k \Rightarrow k = \frac{1}{3}$

$q = \frac{2b+c}{a} = \frac{1}{k} = 3$

G1.3 Let ABC be a right-angled triangle, CD is the altitude on AB, $AC = 3$, $DB = \frac{5}{2}$, $AD = r$, find the value of r.



Reference: 1999 FG5.4

$AD = AC \cos A = \frac{3AC}{AB} = \frac{9}{\frac{5}{2} + AD}$

$\frac{5}{2}AD + AD^2 = 9$

$2AD^2 + 5AD - 18 = 0$

$(2AD + 9)(AD - 2) = 0$

$AD = r = 2$

G1.4 If $x^3 + px^2 + qx + 17 \equiv (x + 2)^3 + a$, find the value of a.

Reference: 1997 FG5.2

Compare the constant term: $17 = 8 + a$

$a = 9$

Group Event 2

G2.1 If $\frac{137}{a} = 0.1\dot{2}3\dot{4}$, find the value of a.

$\frac{137}{a} = 0.1\dot{2}3\dot{4} = 0.1 + \frac{234}{9990} = \frac{999+234}{9990} = \frac{1233}{9990} = \frac{137}{1110}$

$a = 1110$

G2.2 If $b = 1999\times19981998 - 1998\times19991999 + 1$, find the value of b.

Reference: 1996 FG9.3

$b = 1999\times1998\times1001 - 1998\times1999\times1001 + 1 = 1$

G2.3 If the parametric equation $\begin{cases} x = \sqrt{3-t^2} \\ y = t - 3 \end{cases}$ can be transformed into $x^2 + y^2 + cx + dy + 6 = 0$, find the values of c and d.

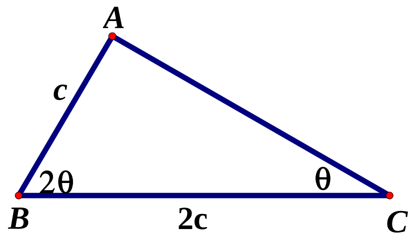
$(1)^2 + (2)^2 : x^2 + y^2 = -6t + 12 = -6(y + 3) + 12$

$c = 0$, $d = 6$

Group Event 3

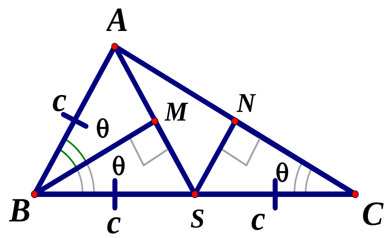
G3.1 In ΔABC, $\angle ABC = 2\angle ACB$, $BC = 2AB$.
If $\angle BAC = a°$, find the value of a.

Reference: 2001 HG8



Let $\angle ACB = \theta$, $\angle ABC = 2\theta$ (given)
$AB = c$, $BC = 2c$
$\angle BAC = 180° - \theta - 2\theta$ ($\angle$s sum of Δ)
By sine formula, $\dfrac{c}{\sin\theta} = \dfrac{2c}{\sin(180° - 3\theta)}$
$\sin 3\theta = 2\sin\theta$
$3\sin\theta - 4\sin^3\theta = 2\sin\theta$
$4\sin^2\theta - 1 = 0$
$\sin\theta = \dfrac{1}{2}$; $\theta = 30°$, $\angle BAC = 180° - 3\theta = 90°$; $a = 90$

Method 2 Let $\angle ACB = \theta$, $\angle ABC = 2\theta$ (given)
Let S be the mid-point of BC.
Let N and M be the feet of perpendiculars drawn from S on AC and B from AS respectively.



$\Delta BSM \cong \Delta BAM$ (RHS)
$\angle RQN = \theta = \angle SQN$ (corr. $\angle$s, $\cong \Delta$'s)
$\Delta CSN \cong \Delta BSM \cong \Delta BAM$ (AAS)
$NS = MS = AM$ (corr. sides $\cong \Delta$'s)
$\sin\angle NAS = \dfrac{NS}{AS} = \dfrac{1}{2}$; $\angle NAS = 30°$;
$\angle ASN = 60°$ ($\angle$s sum of ΔASN)
$90° - \theta + 60° + 90° - \theta = 180°$ (adj. $\angle$s on st. line BSC)
$\theta = 30°$
$\angle BAC = 180° - 3\theta = 90°$ ($\angle$s sum of ΔABC)
$a = 90$

G3.2 Given that $x + \dfrac{1}{x} = \sqrt{2}$, $\dfrac{x^2}{x^4 + x^2 + 1} = b$, find the value of b.

$\left(x + \dfrac{1}{x}\right)^2 = 2 \Rightarrow x^2 + 2 + \dfrac{1}{x^2} = 2$

$\Rightarrow x^2 + \dfrac{1}{x^2} = 0$ (remark: x is a complex number)

$b = \dfrac{x^2}{x^4 + x^2 + 1} = \dfrac{1}{x^2 + 1 + \frac{1}{x^2}} = 1$

G3.3 If the number of positive integral root(s) of the equation $x + y + 2xy = 141$ is c, find the value of c.
$2x + 2y + 4xy = 282 \Rightarrow 2x + 2y + 4xy + 1 = 283$, which is a prime number
$(2x + 1)(2y + 1) = 1\times283$
$2x + 1 = 1$, $2y + 1 = 283$ (or $2x + 1 = 283$, $2y + 1 = 1$)
Solving the above equations, there is no positive integral roots.
$c = 0$

G3.4 Given that $x + y + z = 0$, $x^2 + y^2 + z^2 = 1$ and $d = 2(x^4 + y^4 + z^4)$, find the value of d.

Let $x + y + z = 0$ …… (1), $x^2 + y^2 + z^2 = 1$ …… (2)

From (1), $(x + y + z)^2 = 0$

$\Rightarrow x^2 + y^2 + z^2 + 2(xy + yz + zx) = 0$

Sub. (2) into the above equation, $xy + yz + zx = -\frac{1}{2}$ …… (3)

From (3), $(xy + yz + zx)^2 = \frac{1}{4}$

$\Rightarrow x^2y^2 + y^2z^2 + z^2x^2 + 2xyz(x + y + z) = \frac{1}{4}$

Sub. (1) into the above equation, $x^2y^2 + y^2z^2 + z^2x^2 = \frac{1}{4}$ …… (4)

From (2), $(x^2 + y^2 + z^2)^2 = 1$

$\Rightarrow x^4 + y^4 + z^4 + 2(x^2y^2 + y^2z^2 + z^2x^2) = 1$

Sub. (4) into the above equation, $x^4 + y^4 + z^4 = \frac{1}{2}$ …… (5)

Sub. (5) into $d \Rightarrow d = 2(x^4 + y^4 + z^4) = 2\times\frac{1}{2} = 1$

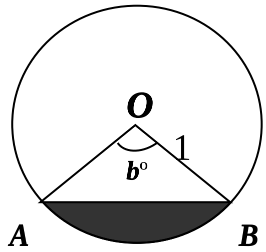
Group Event 4

G4.1 If $0.\dot{1}+0.0\dot{2}+0.00\dot{3}+\ldots+0.00000000\dot{9}=\boldsymbol{a}$, find the value of $\boldsymbol{a}$ (Give your answer in decimal)

$$\boldsymbol{a}=\frac{1}{9}+\frac{2}{90}+\frac{3}{900}+\cdots+\frac{9}{900000000}=\frac{100000000+20000000+3000000+\cdots+9}{900000000}$$

$$\boldsymbol{a}=\frac{123456789}{900000000}=\frac{13717421}{100000000}=0.13717421$$

G4.2 The circle in the figure has centre ***O*** and radius 1, ***A*** and ***B*** are points on the circle. Given that $\dfrac{\text{Area of shaded part}}{\text{Area of unshaded part}}=\dfrac{\pi-2}{3\pi+2}$ and $\angle \boldsymbol{AOB}=\boldsymbol{b}°$, find the value of ***b***.



$$\frac{\text{Area of shaded part}}{\text{Area of the circle}}=\frac{\pi-2}{\pi-2+3\pi+2}=\frac{\pi-2}{4\pi}$$

$$\frac{\pi(1)^2\cdot\frac{\boldsymbol{b}}{360}-\frac{1}{2}(1)^2\sin\boldsymbol{b}°}{\pi(1)^2}=\frac{\pi-2}{4\pi}\Rightarrow\frac{\pi\boldsymbol{b}}{90}-2\sin\boldsymbol{b}°=\pi-2;\ \boldsymbol{b}=90$$

G4.3 A sequence of figures S_0, S_1, S_2, … are constructed as follows. S_0 is obtained by removing the middle third of [0,1] interval; S_1 by removing the middle third of each of the two intervals in S_0; S_2 by removing the middle third of each of the four intervals in S_1; S_3, S_4, … are obtained similarly. Find the total length ***c*** of the intervals removed in the construction of S_5 (Give your answer in fraction).

|——————( )——————| S_0

$0 \quad \frac{1}{3} \quad \frac{2}{3} \quad 1$

|——( )——( )——( )——| S_1

$0 \quad \frac{1}{9} \quad \frac{2}{9} \quad \frac{1}{3} \quad \frac{2}{3} \quad \frac{7}{9} \quad \frac{8}{9} \quad 1$

|–( )–( )–( )–( )–( )–( )–( )–| S_2

The total length in $\boldsymbol{S}_0=\frac{2}{3}$

The total length in $\boldsymbol{S}_1=4\times\frac{1}{9}=\frac{4}{9}$

The total length in $\boldsymbol{S}_2=8\times\frac{1}{27}=\frac{8}{27}$

Deductively, the total length in $\boldsymbol{S}_5=2^6\times\frac{1}{3^6}=\frac{64}{729}$

The total length removed in $\boldsymbol{S}_5=1-\frac{64}{729}=\frac{665}{729}$

G4.4 All integers are coded as shown in the following table. If the sum of all integers coded from 101 to 200 is ***d***, find the value of ***d***.

Integer	…	…	–3	–2	–1	0	1	2	3	…	…
Code	…	…	7	5	3	1	2	4	6	…	…

Sum of integers code as 102, 104, … , 200 is 51 + 52 + … + 100
Sum of integers code as 101, 103, … , 199 is –50 – 51 – … – 99
Sum of all integers = 1 + 1 + … + 1 (50 times) = 50

Group Event 5

G5.1 If 1×2×3 + 2×3×4 + 3×4×5 +…+ 10×11×12 = $\boldsymbol{a}$, find the value of $\boldsymbol{a}$.

$$a=\frac{1}{4}n(n+1)(n+2)(n+3)$$

$$=\frac{1}{4}10(11)(12)(13)=4290$$

G5.2 Given that $5^x + 5^{-x} = 3$. If $5^{3x} + 5^{-3x} = \boldsymbol{b}$, find the value of $\boldsymbol{b}$.

Reference: 1983 FG7.3, 1996FI1.2, 2010 FI3.2

$(5^x + 5^{-x})^2 = 9$

$\Rightarrow 5^{2x} + 2 + 5^{-2x} = 9$

$\Rightarrow 5^{2x} + 5^{-2x} = 7$

$b = 5^{3x} + 5^{-3x}$

$= (5^x + 5^{-x})(5^{2x} - 1 + 5^{-2x})$

$= 3(7 - 1) = 18$

G5.3 Given that the roots of equation $x^2 + mx + n = 0$ are 98 and 99 and $y = x^2 + mx + n$. If x takes on the values of 0, 1, 2, … , 100, then there are c values of y that can be divisible by 6. Find the value of c.

$m = -98 - 99 = -197$; $n = 98\times99 = 49\times33\times6$, which is divisible by 6

$y = x^2 - 197x + 98\times99$

$= x^2 + x - 198x + 49\times33\times6$

$= x(x + 1) - 6(33x + 49\times33)$

If y is divisible by 6, then $x(x + 1)$ is divisible by 6

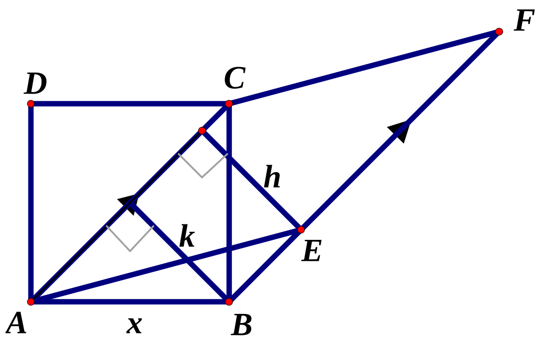
One of x, $x + 1$ must be even. If it is divisible by 6, then one of x, $x + 1$ must be divisible by 3.

We count the number of possible x for which y cannot be divisible by 6

These x may be 1, 4, 7, 10, … , 97, 100; totally 34 possible x.

$c = 101 - 34 = 67$

G5.4 In the figure, ***ABCD*** is a square, ***BF*** // ***AC***, and ***AEFC*** is a rhombus. If $\angle EAC = d°$, find the value of d.



Reference HKCEE Mathematics 1992 P2 Q54

From B and E draw 2 lines $h, k \perp AC$

$h = k$ (∵ BF // AC)

Let $AB = x$, $\angle CAB = 45°$

$$k = x\sin 45° = \frac{x}{\sqrt{2}} = h$$

$AC = x \div \cos 45°$

$$=\sqrt{2}x = AE \text{ (∵ } AEFC \text{ is a rhombus)}$$

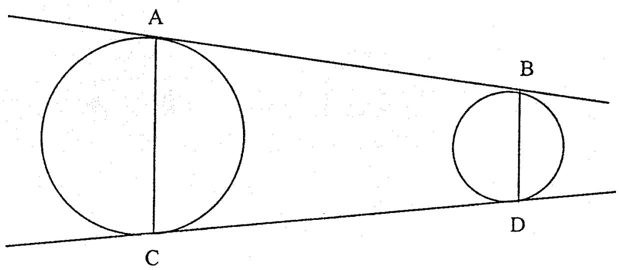
$$\sin\angle EAC = \frac{h}{AE}$$

$$=\frac{\frac{x}{\sqrt{2}}}{\sqrt{2}x}=\frac{1}{2}$$

$d = 30$

Group Spare Event

GS.1 In the figure, there are two common tangents. These common tangents meet the circles at points A, B, C and D. If AC = 9 cm, BD = 3 cm, $\angle BAC = 60°$ and $AB = s$ cm, find the value of s.



Produce AB and CD to meet at E.

$AE = CE$, $BE = DE$ (tangent from ext. pt.)

ΔEAC and ΔEBD are isosceles triangles

$\angle ECA = \angle BAC = 60°$ (base $\angle$s isos. Δ)

$\angle AEC = 60°$ ($\angle$ sum of Δ)

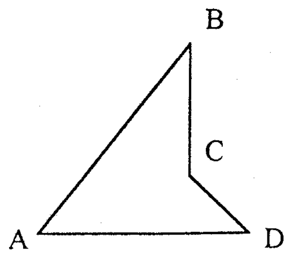
$\angle EBD = \angle EDB = 60°$ ($\angle$ sum of Δ, base $\angle$s isos. Δ)

$\therefore$ ΔEAC and ΔEBD are equilateral triangles

$EB = BD = 3$ cm, $EA = AC = 9$ cm (sides of equilateral triangles)

$s = 9 - 3 = 6$

GS.2 In the figure, $ABCD$ is a quadrilateral, where the interior angles $\angle A$, $\angle B$ and $\angle D$ are all equal to 45°. When produced, BC is perpendicular to AD. If AC = 10 and $BD = b$, find the value of b.



reflex $\angle BCD = 360° - 45° - 45° - 45° = 225°$ ($\angle$s sum of polygon)

$\angle BCD = 360° - 225° = 135°$ ($\angle$s at a point)

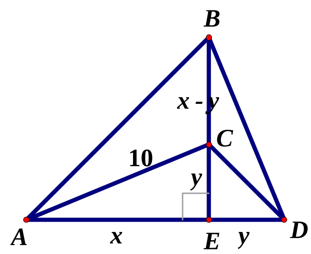
Produce BC to meet AD at E, $\angle AEB = 90°$ (given)

$\angle BAE = 45° = \angle ABE$ (given)

ΔABE and ΔCDE are right angled isosceles triangles

Let $AE = x$, $DE = y$, then $BE = x$, $CE = y$, $BC = x - y$

In ΔACE, $x^2 + y^2 = 10^2$ … (1) (Pythagoras' theorem)



$CD = \sqrt{y^2 + y^2} = \sqrt{2}y$ (Pythagoras' theorem)

Apply cosine rule on ΔBCD

$BD^2 = (x - y)^2 + 2y^2 - 2(x - y)\sqrt{2}y \cos 135°$

$BD^2 = x^2 - 2xy + y^2 + 2y^2 + 2(x - y)y = x^2 + y^2 = 10^2$

$\Rightarrow BD = b = 10$

GS.3 If $\log_c 27 = 0.75$, find the value of c.

$c^{0.75} = 27$

$\Rightarrow c = \left(3^3\right)^{\frac{4}{3}} = 81$

GS.4 If the mean, mode and median of the data 30, 80, 50, 40, d are all equal, find the value of d.

Mean $= \frac{30+80+50+40+d}{5} = 40 + \frac{d}{5}$ = mode

By trial and error, $d = 50$

Individual Events

I1	P	4	I2	a	8	I3	a	6	I4	a	23	I5	a	2	IS	a	2
	Q	8		b	10		b	7		b	2		b	1	spare	b	770
	R	11		c	1		c	2		c	2		c	0		c	57
	S	10		d	2000		d	9902		d	8		d	6		d	58

Group Events

G1	a	1	G2	a	–1	G3	a	2	G4	a	4	G5	P	35	GS	P	4
	b	15		b	0		b	7		b	0		Q	6	spare	Q	6
	c	80		c	13		c	0		c	3		R	11		R	35
	d	1		d	5		d	*6 see the remark		d	3		S	150		S	8

Individual Event 1

I1.1 If the interior angles of a $\boldsymbol{P}$-sided polygon form an Arithmetic Progression and the smallest and the largest angles are 20° and 160° respectively. Find the value of $\boldsymbol{P}$.

Sum of all interior angles $= \frac{\boldsymbol{P}}{2}(20° + 160°) = 180°(\boldsymbol{P} - 2)$

$90\boldsymbol{P} = 180\boldsymbol{P} - 360$

$\Rightarrow \boldsymbol{P} = 4$

I1.2 In $\Delta \boldsymbol{ABC}$, $\boldsymbol{AB} = 5$, $\boldsymbol{AC} = 6$ and $\boldsymbol{BC} = \boldsymbol{P}$. If $\frac{1}{\boldsymbol{Q}} = \cos 2\boldsymbol{A}$, find the value of $\boldsymbol{Q}$.

(<u>Hint</u>: $\cos 2\boldsymbol{A} = 2\cos^2 \boldsymbol{A} - 1$)

$\cos \boldsymbol{A} = \frac{6^2 + 5^2 - 4^2}{2\times 6\times 5} = \frac{3}{4}$

$\cos 2A = 2\cos^2 \boldsymbol{A} - 1$

$= 2\times\left(\frac{3}{4}\right)^2 - 1 = \frac{1}{8}$

$\boldsymbol{Q} = 8$

I1.3 If $\log_2 \boldsymbol{Q} + \log_4 \boldsymbol{Q} + \log_8 \boldsymbol{Q} = \frac{\boldsymbol{R}}{2}$, find the value of $\boldsymbol{R}$.

$\frac{\boldsymbol{R}}{2} = \log_2 8 + \log_4 8 + \log_8 8$

$= 3 + \frac{3}{2} + 1 = \frac{11}{2}$

$\boldsymbol{R} = 11$

I1.4 If the product of the numbers $\boldsymbol{R}$ and $\frac{11}{\boldsymbol{S}}$ is the same as their sum, find the value of $\boldsymbol{S}$.

$11\times\frac{11}{\boldsymbol{S}} = 11 + \frac{11}{\boldsymbol{S}}$

$\Rightarrow \frac{110}{\boldsymbol{S}} = 11$

$\boldsymbol{S} = 10$

Individual Event 2

I2.1 If x, y and z are positive real numbers such that $\frac{x+y-z}{z}=\frac{x-y+z}{y}=\frac{-x+y+z}{x}$ and $a=\frac{(x+y)\cdot(y+z)\cdot(z+x)}{xyz}$, find the value of a.

Reference: 1992 HG2

Let $\frac{x+y-z}{z}=k$, $\frac{x-y+z}{y}=k$, $\frac{-x+y+z}{x}=k$.

$$\begin{cases} x+y-z=kz\cdots\cdots(1)\\ x-y+z=ky\cdots\cdots(2)\\ -x+y+z=kx\cdots\cdots(3)\end{cases}$$

(1) + (2) + (3): $x+y+z=k(x+y+z)$

$\Rightarrow k=1$

From (1), $x+y=2z$, (2): $x+z=2y$, (3): $y+z=2x$

$\therefore a=\frac{(x+y)\cdot(y+z)\cdot(z+x)}{xyz}=\frac{8xyz}{xyz}=8$

I2.2 Let u and t be positive integers such that $u+t+ut=4a+2$. If $b=u+t$, find the value of b.

$u+t+ut=34\Rightarrow 1+u+t+ut=35$

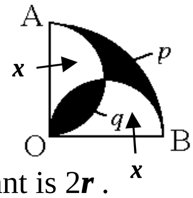
$\Rightarrow (1+u)(1+t)=35$

$\Rightarrow 1+u=5$, $1+t=7$

$u=4$, $t=6$

$\Rightarrow b=4+6=10$

I2.3 In Figure 1, OAB is a quadrant of a circle and semi-circles are drawn on OA and OB. If p, q denotes the areas of the shaded regions, where $p=(b-9)$ cm^2 and $q=c$ cm^2, find the value of c.



$p=1$, let the area of each of two unshaded regions be x cm^2

Let the radius of each of the smaller semicircles be r. The radius of the quadrant is $2r$.

$x+q$ = area of one semi-circle $=\frac{\pi r^2}{2}$; $2x+p+q$ = area of the quadrant $=\frac{1}{4}\pi(2r)^2=\pi r^2$

$2\times(1)=(2)$, $2x+2q=2x+p+q\Rightarrow q=p$; $c=1$

I2.4 Let $f_0(x)=\frac{1}{c-x}$ and $f_n(x)=f_0(f_{n-1}(x))$, $n=1, 2, 3, \ldots$. If $f_{2000}(2000)=d$, find the value of d.

Reference: 2009 HI6

$f_0(x)=\frac{1}{1-x}$, $f_1(x)=f_0(\frac{1}{1-x})=\frac{1}{1-\frac{1}{1-x}}=\frac{1-x}{-x}=\frac{x-1}{x}=1-\frac{1}{x}$

$f_2(x)=f_0(1-\frac{1}{x})=\frac{1}{1-\left(1-\frac{1}{x}\right)}=x$, which is an identity function.

So $f_5(x)=f_2(x)=x$, $\cdots$, $f_{2000}(x)=x$;

$f_{2000}(2000)=2000=d$

Individual Event 3 (2000 Sample Individual Event)

I3.1 For all integers $\boldsymbol{m}$ and $\boldsymbol{n}$, $\boldsymbol{m} \otimes \boldsymbol{n}$ is defined as: $\boldsymbol{m} \otimes \boldsymbol{n} = \boldsymbol{m}^{\boldsymbol{n}} + \boldsymbol{n}^{\boldsymbol{m}}$.

If $2 \otimes \boldsymbol{a} = 100$, find the value of $\boldsymbol{a}$.

Reference: 1990 HI4

$2^{\boldsymbol{a}} + \boldsymbol{a}^2 = 100$

$64 + 36 = 2^6 + 6^2 = 100$

$\boldsymbol{a} = 6$

I3.2 If $\sqrt[3]{13\boldsymbol{b}+6\boldsymbol{a}+1} - \sqrt[3]{13\boldsymbol{b}-6\boldsymbol{a}-1} = \sqrt[3]{2}$, where $\boldsymbol{b} > 0$, find the value of $\boldsymbol{b}$.

Reference: 2005 FI2.2, 2016 FG3.3, 2019 HI10

$\left(\sqrt[3]{13\boldsymbol{b}+37} - \sqrt[3]{13\boldsymbol{b}-37}\right)^3 = 2$

$13\boldsymbol{b} + 37 - 3\sqrt[3]{(13\boldsymbol{b}+37)^2}\sqrt[3]{13\boldsymbol{b}-37} + 3\sqrt[3]{(13\boldsymbol{b}-37)^2}\sqrt[3]{13\boldsymbol{b}+37} - (13\boldsymbol{b} - 37) = 2$

$24 = \sqrt[3]{(13\boldsymbol{b})^2 - 37^2}\sqrt[3]{13\boldsymbol{b}+37} \quad - \sqrt[3]{(13\boldsymbol{b})^2 - 37^2}\sqrt[3]{13\boldsymbol{b}-37}$

$24 = \sqrt[3]{(13\boldsymbol{b})^2 - 37^2}\ \sqrt[3]{2}$; $\qquad (\because \sqrt[3]{13\boldsymbol{b}+37} - \sqrt[3]{13\boldsymbol{b}-37} = \sqrt[3]{2})$

$13824 = [(13\boldsymbol{b})^2 - 1369] \times 2$

$6912 + 1369 = 169\ \boldsymbol{b}^2$

$\boldsymbol{b}^2 = 49$

$\Rightarrow \boldsymbol{b} = 7$

Method 2 $\sqrt[3]{13\boldsymbol{b}+37} - \sqrt[3]{13\boldsymbol{b}-37} = \sqrt[3]{2}$

We look for the difference of multiples of $\sqrt[3]{2}$

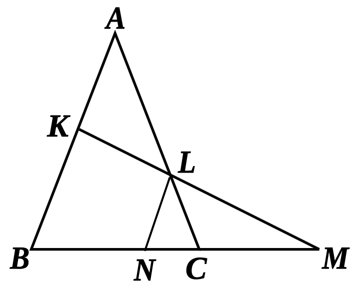
$\sqrt[3]{8\times 2} - \sqrt[3]{2} = \sqrt[3]{2} \Rightarrow 13\boldsymbol{b} + 37 = 16,\ 13\boldsymbol{b} - 37 = 2$, no solution

$\sqrt[3]{27\times 2} - \sqrt[3]{8\times 2} = \sqrt[3]{2} \Rightarrow 13\boldsymbol{b} + 37 = 54,\ 13\boldsymbol{b} - 37 = 16$, no solution

$\sqrt[3]{64\times 2} - \sqrt[3]{27\times 2} = \sqrt[3]{2} \Rightarrow 13\boldsymbol{b} + 37 = 128,\ 13\boldsymbol{b} - 37 = 54$

$\Rightarrow \boldsymbol{b} = 7$

I3.3 In figure 2, $\boldsymbol{AB} = \boldsymbol{AC}$ and $\boldsymbol{KL} = \boldsymbol{LM}$. If $\boldsymbol{LC} = \boldsymbol{b} - 6$ cm and $\boldsymbol{KB} = \boldsymbol{c}$ cm, find the value of $\boldsymbol{c}$.



Reference: 1992 HG6

Draw $\boldsymbol{LN}$ // $\boldsymbol{AB}$ on $\boldsymbol{BM}$.

$\boldsymbol{BN} = \boldsymbol{NM}$ intercept theorem

$\angle \boldsymbol{LNC} = \angle \boldsymbol{ABC} = \angle \boldsymbol{LCN}$ (corr. ∠s, $\boldsymbol{AB}$ // $\boldsymbol{LN}$, base ∠s, isos. Δ)

$\boldsymbol{LN} = \boldsymbol{LC} = \boldsymbol{b} - 6$ cm = 1 cm (sides opp. eq. ∠s)

$\boldsymbol{c}$ cm = $\boldsymbol{KB} = 2\ \boldsymbol{LN}$ = 2 cm (mid point theorem)

I3.4 The sequence $\{\boldsymbol{a}_{\boldsymbol{n}}\}$ is defined as $\boldsymbol{a}_1 = \boldsymbol{c}$, $\boldsymbol{a}_{\boldsymbol{n}+1} = \boldsymbol{a}_{\boldsymbol{n}} + 2\boldsymbol{n}$ ($\boldsymbol{n} \geq 1$). If $\boldsymbol{a}_{100} = \boldsymbol{d}$, find the value of $\boldsymbol{d}$.

$\boldsymbol{a}_1 = 2$, $\boldsymbol{a}_2 = 2 + 2$, $\boldsymbol{a}_3 = 2 + 2 + 4$, $\cdots$,

$\boldsymbol{a}_{100} = 2 + 2 + 4 + \cdots + 198$

$\quad = 2 + \frac{1}{2}(2 + 198)\cdot 99 = 9902 = \boldsymbol{d}$

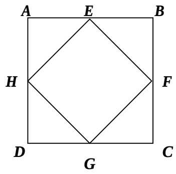
Individual Event 4

I4.1 Mr. Lee is a years old, $a < 100$.
If the product of a and his month of birth is 253, find the value of a .
$253 = 11\times 23$
11 = his month of birth and $a = 23$

I4.2 Mr. Lee has $a + b$ sweets. If he divides them equally among 10 persons, 5 sweets will be remained. If he divides them equally among 7 persons, 3 more sweets are needed. Find the minimum value of b .
$10m + 5 = 7n - 3 = 23 + b$
$7n - 10m = 8$
By trial and error $n = 4$, $m = 2$
$23 + b = 7\times 4 - 3 = 25$
$b = 2$

I4.3 Let c be a positive real number. If $x^2 + 2\sqrt{c}\,x + b = 0$ has one real root only, find the value of c .
$x^2 + 2\sqrt{c}\,x + 2 = 0$
$\Delta = 4(c - 2) = 0$
$\Rightarrow c = 2$

I4.4 In figure 3, the area of the square $ABCD$ is equal to d. If E, F, G, H are the mid-points of AB, BC, CD and DA respectively and $EF = c$, find the value of d.



Area of $EFGH = c^2 = 2^2 = 4$
Area of $ABCD = 2 \times$ area of $EFGH = 8$
$\Rightarrow d = 8$

Individual Event 5

I5.1 If $144^p = 10$, $1728^q = 5$ and $a = 12^{2p-3q}$, find the value of a .
$a = 12^{2p-3q} = 144^p \div 1728^q = 10 \div 5 = 2$

I5.2 If $1 - \frac{4}{x} + \frac{4}{x^2} = 0$, $b = \frac{a}{x}$, find b .
Reference: 1994 FI5.1
$\left(1 - \frac{2}{x}\right)^2 = 0$; $x = 2$, $b = \frac{2}{2} = 1$

I5.3 If the number of real roots of the equation $x^2 - bx + 1 = 0$ is c, find the value of c .
$x^2 - x + 1 = 0$
$\Delta = 1^2 - 4 < 0$
c = number of real roots = 0

I5.4 Let $f(1) = c + 1$ and $f(n) = (n - 1)\, f(n - 1)$, where $n > 1$. If $d = f(4)$, find the value of d .
Reference: 2009 FI1.4
$f(1) = 1$
$f(2) = f(1) = 1$
$f(3) = 2f(2) = 2$
$f(4) = 3f(3) = 3\times 2 = 6$

Individual Event (Spare)

IS.1 If $\boldsymbol{a}$ is the smallest prime number which can divide the sum $3^{11} + 5^{13}$, find the value of $\boldsymbol{a}$.

Reference: 2010 FG3.1

3^{11} is an odd number

5^{13} is also an odd number

So $3^{11} + 5^{13}$ is an even number, which is divisible by 2 .

IS.2 For all real number $\boldsymbol{x}$ and $\boldsymbol{y}$, $\boldsymbol{x} \oplus \boldsymbol{y}$ is defined as: $\boldsymbol{x} \oplus \boldsymbol{y} = \frac{1}{\boldsymbol{xy}}$.

If $\boldsymbol{b} = 4 \oplus (\boldsymbol{a} \oplus 1540)$, find the value of $\boldsymbol{b}$.

$$\boldsymbol{a} \oplus 1540 = \frac{1}{2\times 1540} = \frac{1}{3080}$$

$$\boldsymbol{b} = 4 \oplus (\boldsymbol{a} \oplus 1540) = \frac{3080}{4} = 770$$

IS.3 $\boldsymbol{W}$ and $\boldsymbol{F}$ are two integers which are greater than 20. If the product of $\boldsymbol{W}$ and $\boldsymbol{F}$ is $\boldsymbol{b}$ and the sum of $\boldsymbol{W}$ and $\boldsymbol{F}$ is $\boldsymbol{c}$, find the value of $\boldsymbol{c}$.

$$\begin{cases} \boldsymbol{WF} = 770 \cdots\cdots(1) \\ \boldsymbol{W} + \boldsymbol{F} = \boldsymbol{c} \quad \cdots\cdots(2) \end{cases}$$

$770 = 22\times 35$

$\boldsymbol{W} = 22$, $\boldsymbol{F} = 35$

$\boldsymbol{c} = 22 + 35 = 57$

IS.4 If $\frac{\boldsymbol{d}}{114} = \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{\boldsymbol{c}^2}\right)$, find the value of $\boldsymbol{d}$.

Reference: 1986 FG10.4, 2014 FG3.1

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{57^2}\right) = \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{57}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{57}\right)$$

$$= \frac{1}{2}\cdot\frac{2}{3}\cdots\cdots\frac{56}{57}\times\frac{3}{2}\cdot\frac{4}{3}\cdots\cdots\frac{58}{57} = \frac{1}{57}\times\frac{58}{2} = \frac{58}{114}$$

$\boldsymbol{d} = 58$

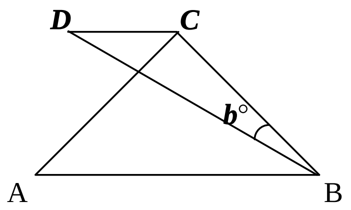
Group Event 1 (2000 Final Sample Group Event)

G1.1 Let $x * y = x + y - xy$, where x, y are real numbers. If $a = 1 * (0 * 1)$, find the value of a.

$0 * 1 = 0 + 1 - 0 = 1$

$a = 1 * (0 * 1)$
$= 1 * 1$
$= 1 + 1 - 1 = 1$

G1.2 In figure 1, AB is parallel to DC, $\angle ACB$ is a right angle, $AC = CB$ and $AB = BD$. If $\angle CBD = b°$, find the value of b.



ΔABC is a right angled isosceles triangle.

$\angle BAC = 45°$ ($\angle$s sum of Δ, base $\angle$s isos. Δ)

$\angle ACD = 45°$ (alt. $\angle$s, AB // DC)

$\angle BCD = 135°$

Apply sine law on ΔBCD,

$$\frac{BD}{\sin 135°} = \frac{BC}{\sin D}$$

$$AB\sqrt{2} = \frac{AB \sin 45°}{\sin D}$$, given that $AB = BD$

$\sin D = \frac{1}{2}$; $D = 30°$

$\angle CBD = 180° - 135° - 30° = 15°$ ($\angle$s sum of ΔBCD)

$b = 15$

G1.3 Let x, y be non-zero real numbers. If x is 250% of y and $2y$ is c% of x, find the value of c.

$x = 2.5y$ …… (1)

$2y = \frac{c}{100} \cdot x$ ……(2)

Sub. (1) into (2): $2y = \frac{c}{100} \cdot 2.5y$

$c = 80$

G1.4 If $\log_p x = 2$, $\log_q x = 3$, $\log_r x = 6$ and $\log_{pqr} x = d$, find the value of d.

Reference: 2001 FG1.4, 2015 HI7

$$\frac{\log x}{\log p} = 2;\ \frac{\log x}{\log q} = 3;\ \frac{\log x}{\log r} = 6$$

$$\frac{\log p}{\log x} = \frac{1}{2};\ \frac{\log q}{\log x} = \frac{1}{3};\ \frac{\log r}{\log x} = \frac{1}{6}$$

$$\frac{\log p}{\log x} + \frac{\log q}{\log x} + \frac{\log r}{\log x} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$$

$$\frac{\log pqr}{\log x} = 1$$

$$\Rightarrow \frac{\log x}{\log pqr} = 1$$

$d = \log_{pqr} x = 1$

Group Event 2

G2.1 If $\boldsymbol{a} = \boldsymbol{x}^4 + \boldsymbol{x}^{-4}$ and $\boldsymbol{x}^2 + \boldsymbol{x} + 1 = 0$, find the value of $\boldsymbol{a}$.

$$\frac{x^2+x+1}{x}=0$$

$$\Rightarrow x+\frac{1}{x}=-1$$

$$\Rightarrow \left(x+\frac{1}{x}\right)^2=1$$

$$\Rightarrow x^2+\frac{1}{x^2}=-1$$

$$\Rightarrow \left(x^2+\frac{1}{x^2}\right)^2=1$$

$$a=x^4+\frac{1}{x^4}=-1$$

G2.2 If $6^{\boldsymbol{b}} + 6^{\boldsymbol{b}+1} = 2^{\boldsymbol{b}} + 2^{\boldsymbol{b}+1} + 2^{\boldsymbol{b}+2}$, find the value of $\boldsymbol{b}$.

$6^{\boldsymbol{b}}\cdot(1 + 6) = 2^{\boldsymbol{b}}\cdot(1 + 2 + 4)$

$\Rightarrow \boldsymbol{b} = 0$

G2.3 Let $\boldsymbol{c}$ be a prime number. If $11\boldsymbol{c} + 1$ is the square of a positive integer, find the value of $\boldsymbol{c}$.

$11\boldsymbol{c} + 1 = \boldsymbol{m}^2$

$\Rightarrow \boldsymbol{m}^2 - 1 = 11\boldsymbol{c}$

$\Rightarrow (\boldsymbol{m} + 1)(\boldsymbol{m} - 1) = 11\boldsymbol{c}$

$\Rightarrow \boldsymbol{m} - 1 = 11$ and $\boldsymbol{m} + 1 = \boldsymbol{c}$

$\boldsymbol{m} = 13$

G2.4 Let $\boldsymbol{d}$ be an odd prime number. If $89 - (\boldsymbol{d} + 3)^2$ is the square of an integer, find the value of $\boldsymbol{d}$.

$\because$ $\boldsymbol{d}$ is odd, $\boldsymbol{d} + 3$ must be even, $89 - (\boldsymbol{d} + 3)^2$ must be odd.

$89 = (\boldsymbol{d} + 3)^2 + \boldsymbol{m}^2$

By trial and error, $\boldsymbol{m} = 5$, $89 = 8^2 + 5^2$

$\Rightarrow \boldsymbol{d} + 3 = 8$

$\Rightarrow \boldsymbol{d} = 5$

Group Event 3

G3.1 Let $\boldsymbol{a}$ be the number of positive integers less than 100 such that they are both square and cubic numbers, find the value of $\boldsymbol{a}$.

1998 HG4

The positive integers less than 100 such that they are both square and cubic numbers are:

1 and $2^6 = 64$ only, so there are only 2 numbers satisfying the condition.

G3.2 The sequence $\{a_k\}$ is defined as: $a_1 = 1$, $a_2 = 1$ and $a_k = a_{k-1} + a_{k-2}$ ($k > 2$) .

If $a_1 + a_2 + \dots + a_{10} = 11\,a_b$, find the value of $\boldsymbol{b}$.

$a_1 = 1$, $a_2 = 1$, $a_3 = 2$, $a_4 = 3$, $a_5 = 5$, $a_6 = 8$, $a_7 = 13$, $a_8 = 21$, $a_9 = 34$, $a_{10} = 55$

$a_1 + a_2 + \dots + a_{10} = 1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 + 34 + 55 = 143 = 11\times 13 = 11a_7$

$b = 7$

G3.3 If $\boldsymbol{c}$ is the maximum value of log(sin x), where $0 < x < \pi$, find the value of $\boldsymbol{c}$.

$0 < \sin x \le 1$

$\log(\sin x) \le \log 1 = 0$

$\Rightarrow c = 0$

G3.4 Let $x \ge 0$ and $y \ge 0$. Given that $x + y = 18$. If the maximum value of $\sqrt{x} + \sqrt{y}$ is $\boldsymbol{d}$,

find the value of $\boldsymbol{d}$. **(Reference: 1999 FGS.2)**

$$x + y = \left(\sqrt{x} + \sqrt{y}\right)^2 - 2\sqrt{xy}$$

$$\Rightarrow \left(\sqrt{x} + \sqrt{y}\right)^2 = 18 + 2\sqrt{xy} \le 18 + 2\left(\frac{x+y}{2}\right) = 36 \quad \text{(GM} \le \text{AM)}$$

$\sqrt{x} + \sqrt{y} \le 6 = d$ (It is easy to get the answer by letting $x = y$ in $x + y = 18$)

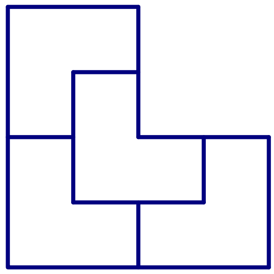
Remark The original question is Given that $x + y = 18$. If the maximum value of $\sqrt{x} + \sqrt{y}$ $\cdots$

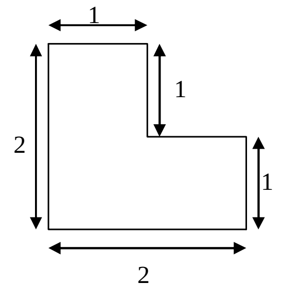
$\sqrt{x} + \sqrt{y}$ is undefined for $x < 0$ or $y < 0$.

Group Event 4

G4.1 If $\boldsymbol{a}$ tiles of L-shape are used to form a larger similar figure (figure 2) without overlapping, find the least possible value of $\boldsymbol{a}$.

From the figure, $\boldsymbol{a} = 4$.





G4.2 Let α, β be the roots of $x^2 + \boldsymbol{b}x - 2 = 0$.

If $\alpha > 1$ and $\beta < -1$, and $\boldsymbol{b}$ is an integer, find the value of $\boldsymbol{b}$.

$\alpha - 1 > 0$ and $\beta + 1 < 0$

$\Rightarrow (\alpha - 1)(\beta + 1) < 0$

$\Rightarrow \alpha\beta + \alpha - \beta - 1 < 0$

$\Rightarrow \alpha - \beta < 3$

$\Rightarrow (\alpha - \beta)^2 < 9$

$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta < 9$

$\Rightarrow \boldsymbol{b}^2 + 8 < 9$

$\Rightarrow -1 < \boldsymbol{b} < 1$

$\because$ $\boldsymbol{b}$ is an integer

$\therefore$ $\boldsymbol{b} = 0$

G4.3 Given that $\boldsymbol{m}$, $\boldsymbol{c}$ are positive integers less than 10 .

If $\boldsymbol{m} = 2\boldsymbol{c}$ and $0.\dot{\boldsymbol{m}}\dot{\boldsymbol{c}} = \frac{\boldsymbol{c}+4}{\boldsymbol{m}+5}$, find the value of $\boldsymbol{c}$.

$0.\dot{\boldsymbol{m}}\dot{\boldsymbol{c}} = \frac{10\boldsymbol{m}+\boldsymbol{c}}{99} = \frac{\boldsymbol{c}+4}{\boldsymbol{m}+5}$

$\Rightarrow \frac{20\boldsymbol{c}+\boldsymbol{c}}{99} = \frac{\boldsymbol{c}+4}{2\boldsymbol{c}+5}$

$\Rightarrow \frac{7\boldsymbol{c}}{33} = \frac{\boldsymbol{c}+4}{2\boldsymbol{c}+5}$

$\Rightarrow 14\boldsymbol{c}^2 + 35\boldsymbol{c} = 33\boldsymbol{c} + 132$

$14\boldsymbol{c}^2 + 2\boldsymbol{c} - 132 = 0$

$\Rightarrow 7\boldsymbol{c}^2 + \boldsymbol{c} - 66 = 0$

$\Rightarrow (7\boldsymbol{c} + 22)(\boldsymbol{c} - 3) = 0$

$\Rightarrow \boldsymbol{c} = 3$

G4.4 A bag contains $\boldsymbol{d}$ balls of which $\boldsymbol{x}$ are black, $\boldsymbol{x} + 1$ are red and $\boldsymbol{x} + 2$ are white. If the probability of drawing a black ball randomly from the bag is less than $\frac{1}{6}$, find the value of $\boldsymbol{d}$.

$\frac{\boldsymbol{x}}{3\boldsymbol{x}+3} < \frac{1}{6}$

$\Rightarrow \frac{\boldsymbol{x}}{\boldsymbol{x}+1} < \frac{1}{2}$

$\Rightarrow 2\boldsymbol{x} < \boldsymbol{x} + 1$

$\Rightarrow \boldsymbol{x} < 1$

$\Rightarrow \boldsymbol{x} = 0$

$\Rightarrow \boldsymbol{d} = 3\boldsymbol{x} + 3 = 3$

Group Event 5

G5.1 If the roots of $x^2 - 2x - P = 0$ differ by 12, find the value of P.

Reference: 1999 FGS.3

$\alpha + \beta = 2$, $\alpha\beta = -P$

$\alpha - \beta = 12$

$\Rightarrow (\alpha - \beta)^2 = 144$

$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = 144$

$\Rightarrow 4 + 4P = 144$

$\Rightarrow P = 35$

G5.2 Given that the roots of $x^2 + ax + 2b = 0$ and $x^2 + 2bx + a = 0$ are both real and $a, b > 0$. If the minimum value of $a + b$ is Q, find the value of Q.

$a^2 - 8b \ge 0$ and $4b^2 - 4a \ge 0$

$a^2 \ge 8b$ and $b^2 \ge a$

$\Rightarrow a^4 \ge 64b^2 \ge 64a$

$\Rightarrow a^4 - 64a \ge 0$

$\Rightarrow a(a^3 - 64) \ge 0$

$\Rightarrow a^3 \ge 64$

$\Rightarrow a \ge 4$

Minimum $a = 4$, $b^2 \ge a$

$\Rightarrow b^2 \ge 4 \Rightarrow$ minimum $b = 2$

Q = minimum value of $a + b = 4 + 2 = 6$

G5.3 If $R^{2000} < 5^{3000}$, where R is a positive integer, find the largest value of R.

Reference: 1996 HI4, 2008 FI4.3, 2018 FG2.4

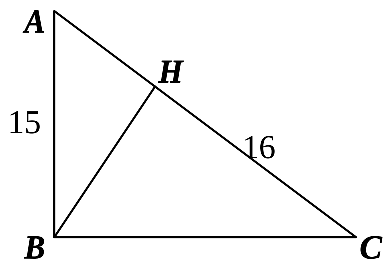
$(R^2)^{1000} < (5^3)^{1000}$

$\Rightarrow R^2 < 5^3 = 125$

$\Rightarrow R < \sqrt{125} < 12$

The largest integral value of $R = 11$

G5.4 In figure 3, ΔABC is a right-angled triangle and $BH \perp AC$. If $AB = 15$, $HC = 16$ and the area of ΔABC is S, find the value of S.



Reference: 1998 FG1.3

It is easy to show that $\Delta ABH \sim \Delta BCH \sim \Delta ACB$.

Let $\angle ABH = \theta = \angle BCH$

In ΔABH, $BH = 15\cos\theta$

In ΔBCH, $CH = BH \div \tan\theta \Rightarrow 16\tan\theta = 15\cos\theta$

$16\sin\theta = 15\cos^2\theta \Rightarrow 16\sin\theta = 15 - 15\sin^2\theta$

$15\sin^2\theta + 16\sin\theta - 15 = 0$

$(3\sin\theta + 5)(5\sin\theta - 3) = 0$

$\sin\theta = \frac{3}{5}$; $\tan\theta = \frac{3}{4}$

$BC = AB \div \tan\theta = 15 \times \frac{4}{3} = 20$

Area of $\Delta ABC = \frac{1}{2} \cdot 15 \times 20 = 150 = S$

Group Event (Spare)

GS.1 If a number N is chosen randomly from the set of positive integers, the probability of the unit digit of N^4 being unity is $\frac{P}{10}$, find the value of P.

If the unit digit of N^4 is 1, then the unit digit of N may be 1, 3, 7, 9. So the probability $=\frac{4}{10}$

$P = 4$

GS.2 Let $x \ge 0$ and $y \ge 0$. Given that $x + y = 18$.

If the maximum value of $\sqrt{x}+\sqrt{y}$ is d, find the value of d.

Reference: 1999 FG3.4

$x+y=\left(\sqrt{x}+\sqrt{y}\right)^2-2\sqrt{xy}$

$\Rightarrow \left(\sqrt{x}+\sqrt{y}\right)^2=18+2\sqrt{xy}\le 18+2\left(\frac{x+y}{2}\right)=36$ (G.M. ≤ A.M.)

$\sqrt{x}+\sqrt{y} \le 6 = d$

GS.3 If the roots of $x^2 - 2x - R = 0$ differs by 12, find the value of R.

Reference: 1999 FG5.1

$\alpha + \beta = 2$, $\alpha\beta = -R$

$\alpha - \beta = 12$

$\Rightarrow (\alpha - \beta)^2 = 144$

$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = 144$

$\Rightarrow 4 + 4R = 144$

$\Rightarrow R = 35$

GS.4 If the product of a 4-digit number ab$S$d and 9 is equal to another 4-digit number dSba, find the value of S.

Reference: 1987 FG9, 1994HI6

$a = 1$, $d = 9$, Let the carry digit in the hundred digit be x. Then $9S + 8 = 10x + b$ ....(1)

$9b + x = S$ ....(2); $x = S - 9b$ ....(3)

Sub. (3) into (1): $9s + 8 = 10(S - 9b) + b \Rightarrow 8 = S - 89b$

$\Rightarrow S = 8$, $b = 0$

Individual Events

SI	***P***	6	**I1**	***P***	25	**I2**	***P***	16	**I3**	***P***	1	**I4**	***P***	2	**I5**	***P***	2
	Q	7		***Q***	8		***Q***	81		***Q***	2		***Q***	12		***Q***	1
	R	2		***R***	72		***R***	1		***R***	3996		***R***	12		***R***	1
	S	9902		***S***	6		***S***	333332		***S***	666		***S***	2		***S***	0

Group Events

SG	***a***	1	**G1**	***a***	243	**G2**	***a***	9025	**G3**	***a***	3994001	**G4**	***a***	504	**G5**	***a***	729000
	b	15		***b***	25		***b***	9		***b***	5		***b***	3		***b***	12
	c	80		***c***	4		***c***	6		***c***	3		***c***	60		***c***	26
	d	1		***d***	3		***d***	–40		***d***	38		***d***	48		***d***	3

Sample Individual Event (1999 Individual Event 3)

SI.1 For all integers m and n, $m \otimes n$ is defined as $m \otimes n = m^n + n^m$. If $2 \otimes P = 100$, find the value of P.

$2^P + P^2 = 100$

$64 + 36 = 2^6 + 6^2 = 100$, $P = 6$

SI.2 If $\sqrt[3]{13Q+6P+1} - \sqrt[3]{13Q-6P-1} = \sqrt[3]{2}$, where $Q > 0$, find the value of Q.

$\left(\sqrt[3]{13Q+37} - \sqrt[3]{13Q-37}\right)^3 = 2$

$13Q + 37 - 3\sqrt[3]{(13Q+37)^2}\sqrt[3]{13Q-37} + 3\sqrt[3]{(13Q-37)^2}\sqrt[3]{13Q+37} - (13Q-37) = 2$

$24 = \sqrt[3]{(13Q)^2 - 37^2}\sqrt[3]{13Q+37} - \sqrt[3]{(13Q)^2 - 37^2}\sqrt[3]{13Q-37}$

$24 = \sqrt[3]{(13Q)^2 - 37^2}\ \sqrt[3]{2}$; $\quad (\because \sqrt[3]{13Q+37} - \sqrt[3]{13Q-37} = \sqrt[3]{2})$

$13824 = [(13Q)^2 - 1369] \times 2$

$6912 + 1369 = 169\,Q^2$

$Q^2 = 49 \Rightarrow Q = 7$

Method 2 $\sqrt[3]{13b+37} - \sqrt[3]{13b-37} = \sqrt[3]{2}$,

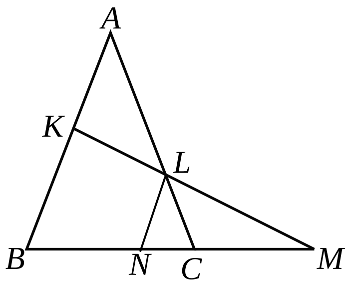
We look for the difference of multiples of $\sqrt[3]{2}$

$\sqrt[3]{8\times2} - \sqrt[3]{2} = \sqrt[3]{2} \Rightarrow 13b + 37 = 16$, $13b - 37 = 2$, no solution

$\sqrt[3]{27\times2} - \sqrt[3]{8\times2} = \sqrt[3]{2} \Rightarrow 13b + 37 = 54$, $13b - 37 = 16$, no solution

$\sqrt[3]{64\times2} - \sqrt[3]{27\times2} = \sqrt[3]{2} \Rightarrow 13b + 37 = 128$, $13b - 37 = 54 \Rightarrow b = 7$

SI.3 In figure 1, $AB = AC$ and $KL = LM$. If $LC = Q - 6$ cm and $KB = R$ cm, find the value of R.



Draw LN // AB on BM.

$BN = NM$ intercept theorem

$\angle LNC = \angle ABC = \angle LCN$ (corr. $\angle$s, AB // LN, base $\angle$s, isos. Δ)

$LN = LC = Q - 6$ cm $= 1$ cm (sides opp. eq. $\angle$s)

R cm $= KB = 2\,LN = 2$ cm (mid point theorem)

SI.4 The sequence $\{a_n\}$ is defined as $a_1 = R$, $a_{n+1} = a_n + 2n$ ($n \geq 1$). If $a_{100} = S$, find the value of S.

$a_1 = 2$, $a_2 = 2 + 2$, $a_3 = 2 + 2 + 4$, …

$a_{100} = 2 + 2 + 4 + \ldots + 198$

$= 2 + \frac{1}{2}(2+198)\cdot 99 = 9902 = S$

Individual Event 1

I1.1 Let [x] represents the integral part of the decimal number x.

Given that $[3.126] + [3.126+\frac{1}{8}]+[3.126+\frac{2}{8}]+\ldots+[3.126+\frac{7}{8}] = P$, find the value of P.

$P = [3.126] + [3.126+\frac{1}{8}]+[3.126+\frac{2}{8}]+\ldots+[3.126+\frac{7}{8}]$

$= 3 + 3 + 3 + 3 + 3 + 3 + 3 + 4 = 25$

I1.2 Let $a + b + c = 0$. Given that $\frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ac}+\frac{c^2}{2c^2+ab} = P - 3Q$, find the value of Q.

$a = -b - c$ $\quad \frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ac}+\frac{c^2}{2c^2+ab}$

$=\frac{(b+c)^2}{2b^2+5bc+2c^2}+\frac{b^2}{2b^2-bc-c^2}+\frac{c^2}{2c^2-bc-b^2}$

$=\frac{a^2}{(2b+c)(b+2c)}+\frac{b^2}{(2b+c)(b-c)}+\frac{c^2}{(b+2c)(c-b)}$

$=\frac{(b+c)^2(b-c)+b^2(b+2c)-c^2(2b+c)}{(2b+c)(b+2c)(b-c)}$

$=\frac{(b+c)^2(b-c)+b^3-c^3+2bc(b-c)}{(2b+c)(b+2c)(b-c)}$

$=\frac{(b-c)(b^2+2bc+c^2+b^2+bc+c^2+2bc)}{(2b+c)(b+2c)(b-c)}$

$=\frac{(2b^2+5bc+2c^2)}{(2b+c)(b+2c)} = 1 = 25 - 3Q \Rightarrow Q = 8$

Method 2

$\because \frac{a^2}{2a^2+bc}+\frac{b^2}{2b^2+ac}+\frac{c^2}{2c^2+ab}$ =25–3Q

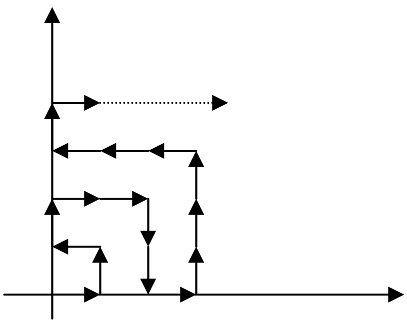
$\therefore$ The above is an identity which holds for all values of a, b and c, provided that $a+b+c = 0$

Let $a = 0$, $b = 1$, $c = -1$, then

$0+\frac{1}{2}+\frac{1}{2} = 25 - 3Q$.

$Q = 8$

I1.3 In the first quadrant of the rectangular co-ordinate plane, all integral points are numbered as follows,



point (0, 0) is numbered as 1,
point (1, 0) is numbered as 2,
point (1, 1) is numbered as 3,
point (0, 1) is numbered as 4,
point (0, 2) is numbered as 5,
point (1, 2) is numbered as 6,
point (2, 2) is numbered as 7,
point (2, 1) is numbered as 8,
……………………………

Given that point (Q–1,Q) is numbered as R, find the value of R.

point (0, 1) is numbered as $4 = 2^2$
point (2, 0) is numbered as $9 = 3^2$
point (0, 3) is numbered as $16 = 4^2$
point (4, 0) is numbered as $25 = 5^2$
……………………………
point (0, 7) is numbered as $64 = 8^2$
point (0, 8) is numbered as 65, point (1, 8) is numbered as 66, point (2, 8) is numbered as 67
……………………………
(Q–1, Q) = (7, 8) is numbered as 72

I1.4 When $x + y = 4$, the minimum value of $3x^2 + y^2$ is $\frac{R}{S}$, find the value of S.

$3x^2 + y^2 = 3x^2 + (4 - x)^2 = 4x^2 - 8x + 16 = 4(x - 1)^2 + 12$, min $= 12 = \frac{72}{S}$; $S = 6$

Individual Event 2

I2.1 If $\log_2(\log_4 P) = \log_4(\log_2 P)$ and $P \neq 1$, find the value of P.

$$\frac{\log(\log_4 P)}{\log 2} = \frac{\log(\log_2 P)}{\log 4}$$

$$\frac{\log(\log_4 P)}{\log 2} = \frac{\log(\log_2 P)}{2\log 2}$$

$2\log(\log_4 P) = \log(\log_2 P)$

$\Rightarrow \log(\log_4 P)^2 = \log(\log_2 P)$

$(\log_4 P)^2 = \log_2 P$

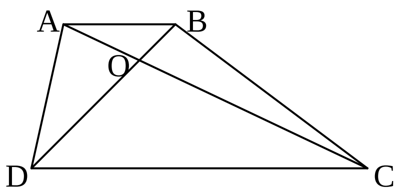
$$\left(\frac{\log P}{\log 4}\right)^2 = \frac{\log P}{\log 2}$$

$P \neq 1$, $\log P \neq 0 \Rightarrow \dfrac{\log P}{(2\log 2)^2} = \dfrac{1}{\log 2}$

$\log P = 4\log 2 = \log 16$

$P = 16$

I2.2 In the trapezium $ABCD$, $AB // DC$. AC and BD intersect at O. The areas of triangles AOB and COD are P and 25 respectively. Given that the area of the trapezium is Q, find the value of Q.



Reference 1993 HI2, 1997 HG3, 2002 FI1.3, 2004 HG7, 2010HG4, 2013 HG2

$\Delta AOB \sim \Delta COD$ (equiangular)

$$\frac{\text{area of }\Delta\text{AOB}}{\text{area of }\Delta\text{COD}} = \left(\frac{OA}{OC}\right)^2;\ \frac{16}{25} = \left(\frac{OA}{OC}\right)^2$$

$OA : OC = 4 : 5$

$\dfrac{\text{area of }\Delta\text{AOB}}{\text{area of }\Delta\text{BOC}} = \dfrac{4}{5}$ (the two triangles have the same height, but different bases.)

Area of $\Delta BOC = 16\times\dfrac{5}{4} = 20$

Similarly, area of $\Delta AOD = 20$

Q = the area of the trapezium = 16 + 25 + 20 + 20 = 81

I2.3 When 1999^Q is divided by 7, the remainder is R. Find the value of R.

$1999^{81} = (7\times285+4)^{81}$

$= 7m + 4^{81}$

$= 7m + (4^3)^{27}$

$= 7m + (7\times9+1)^{27}$

$= 7m + 7n + 1$, where m and n are integers

$R = 1$

I2.4 If $111111111111 - 222222 = (R + S)^2$ and $S > 0$, find the value of S.

Reference: 1995 FG7.4

$111111111111 - 222222 = (1 + S)^2$

$111111(1000001 - 2) = (1 + S)^2$

$111111\times999999 = (1 + S)^2$

$3^2 \times 111111^2 = (1 + S)^2$

$1 + S = 333333$

$S = 333332$

Individual Event 3

I3.1 Given that the units digit of 1+2+3+⋯+1997+1998+1999+1998+1997+⋯+3+2+1 is P, find the value of P.

1+2+3+⋯+1997+1998+1999+1998+1997+⋯+3+2+1

$= 2(1 + 2 + \cdots + 1998) + 1999$

$= (1 + 1998)\times 1998 + 1999$

P = units digit = 1

I3.2 Given that $x+\frac{1}{x}=P$. If $x^6+\frac{1}{x^6}=Q$, find the value of Q.

$x+\frac{1}{x}=1$

$\left(x+\frac{1}{x}\right)^2=1$

$\Rightarrow x^2+\frac{1}{x^2}=-1$

$\left(x^2+\frac{1}{x^2}\right)^3=-1$

$\Rightarrow x^6+\frac{1}{x^6}+3\left(x^2+\frac{1}{x^2}\right)=-1$

$\Rightarrow x^6+\frac{1}{x^6}=2$

$\therefore Q = 2$

I3.3 Given that $\frac{\mathrm{Q}}{\sqrt{\mathrm{Q}}+\sqrt{2\mathrm{Q}}}+\frac{\mathrm{Q}}{\sqrt{2\mathrm{Q}}+\sqrt{3\mathrm{Q}}}+\cdots+\frac{\mathrm{Q}}{\sqrt{1998\mathrm{Q}}+\sqrt{1999\mathrm{Q}}}=\frac{\mathrm{R}}{\sqrt{\mathrm{Q}}+\sqrt{1999\mathrm{Q}}}$, find the value of R.

$\frac{2}{\sqrt{2}+\sqrt{4}}+\frac{2}{\sqrt{4}+\sqrt{6}}+\cdots+\frac{2}{\sqrt{3996}+\sqrt{3998}}=\frac{\mathrm{R}}{\sqrt{2}+\sqrt{3998}}$

$2\left(\frac{\sqrt{4}-\sqrt{2}}{4-2}+\frac{\sqrt{6}-\sqrt{4}}{6-4}+\cdots+\frac{\sqrt{3998}-\sqrt{3996}}{3998-3996}\right)=\frac{\mathrm{R}}{\sqrt{2}+\sqrt{3998}}$

$\sqrt{3998}-\sqrt{2}=\frac{R}{\sqrt{3998}+\sqrt{2}}$

$R=\left(\sqrt{3998}-\sqrt{2}\right)\left(\sqrt{3998}+\sqrt{2}\right)=3996$

I3.4 Let $f(0) = 0$; $f(n) = f(n-1) + 3$ when $n = 1, 2, 3, 4, \ldots$. If $2f(S) = R$, find the value of S.

$f(1) = 0 + 3 = 3$, $f(2) = 3 + 3 = 3\times 2$, $f(3) = 3\times 3$, … , $f(n) = 3n$

$R = 3996 = 2f(S) = 2\times 3S$

$S = 666$

Individual Event 4

I4.1 Suppose $a+\frac{1}{a+1}=b+\frac{1}{b-1}-2$, where $a \neq -1$, $b \neq 1$, and $a - b + 2 \neq 0$.

Given that $ab - a + b = P$, find the value of P.

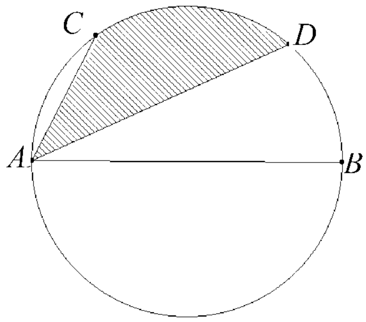
$a-b+2+\frac{1}{a+1}-\frac{1}{b-1}=0$

$(a-b+2)[1-\frac{1}{(a+1)(b-1)}]=0$

$\Rightarrow ab + b - a - 2 = 0$

$P = 2$

I4.2 In the following figure, AB is a diameter of the circle. C and D divide the arc AB into three equal parts. The shaded area is P.



If the area of the circle is Q, find the value of Q.

Reference: 2004 HI9, 2005 HG7, 2018 HI12

Let O be the centre.

Area of ΔACD = area of ΔOCD

(same base, same height) and $\angle COD = 60°$

Shaded area = area of sector COD = 2

$\therefore$ area of the circle = $6 \times 2 = 12$

I4.3 Given that there are R odd numbers in the digits of the product of the two Q-digit numbers $1111\cdots11$ and $9999\cdots99$, find the value of R.

Reference: 2015 FI1.2

Note that $99 \times 11 = 1089$; $999\times111 = 110889$.

Deductively, $999999999999 \times 111111111111 = 111111111110888888888889$

$R = 12$ odd numbers in the digits.

I4.4 Let $a_1, a_2, \cdots, a_R$ be positive integers such that $a_1 < a_2 < a_3 < \cdots < a_{R-1} < a_R$. Given that the sum of these R integers is 90 and the maximum value of a_1 is S, find the value of S.

$a_1 + a_2 + \cdots + a_{12} = 90$

$a_1 + (a_1 + 1) + (a_1 + 2) + \cdots + (a_1 + 11) \leq 90$

$12a_1 + 55 \leq 90$

$a_1 \leq 2.9167$

S = maximum value of $a_1 = 2$

Individual Event 5

I5.1 If $\left(\frac{1\times2\times4+2\times4\times8+3\times6\times12+\cdots+1999\times3998\times7996}{1^3+2^3+3^3+\cdots+1999^3}\right)^{\frac{1}{3}}$ = P, find the value of P.

Reference: 2015 FG1.1

$$P=\left(\frac{1\times2\times4+2\times4\times8+3\times6\times12+\cdots+1999\times3998\times7996}{1^3+2^3+3^3+\cdots+1999^3}\right)^{\frac{1}{3}}$$

$$=\left[\frac{1\times2\times4\left(1^3+2^3+3^3+\cdots+1999^3\right)}{1^3+2^3+3^3+\cdots+1999^3}\right]^{\frac{1}{3}}$$

$$=8^{\frac{1}{3}}=2$$

I5.2 If $(x-P)(x-2Q)-1=0$ has two integral roots, find the value of Q.

Reference: 2001 FI2.1, 2010 FI2.2, 2011 FI3.1, 2013 HG1

$(x-2)(x-2Q)-1=0$

$x^2-2(1+Q)x+4Q-1=0$

Two integral roots $\Rightarrow \Delta$ is perfect square

$\Delta=4[(1+Q)^2-(4Q-1)]$

$=4(Q^2-2Q+2)$

$=4(Q-1)^2+4$

It is a perfect square $\Rightarrow Q-1=0$, $Q=1$

Method 2

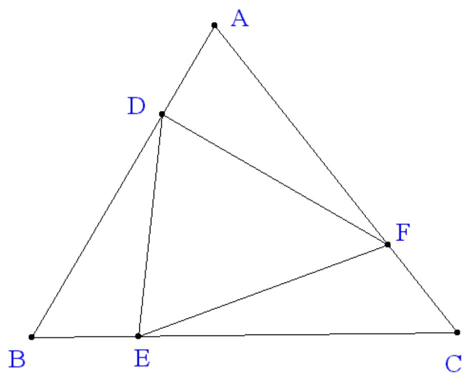
$(x-2)(x-2Q)=1$

$(x-2=1$ and $x-2Q=1)$ or $(x-2=-1$ and $x-2Q=-1)$

$(x=3$ and $Q=1)$ or $(x=1$ and $Q=1)$

$\therefore Q=1$

I5.3 Given that the area of the ΔABC is $3Q$; D, E and F are the points on AB, BC and CA respectively such that $AD=\frac{1}{3}AB$, $BE=\frac{1}{3}BC$, $CF=\frac{1}{3}CA$. If the area of ΔDEF is R, find the value of R. (**Reference: 1993 FG9.2)**



$R=3-$ area $\Delta ADF-$ area $\Delta BDE-$ area ΔCEF

$=3-\left(\frac{1}{2}AD\cdot AF\sin A+\frac{1}{2}BE\cdot BD\sin B+\frac{1}{2}CE\cdot CF\sin C\right)$

$=3-\frac{1}{2}\left(\frac{c}{3}\cdot\frac{2b}{3}\sin A+\frac{2c}{3}\cdot\frac{a}{3}\sin B+\frac{2a}{3}\cdot\frac{b}{3}\sin C\right)$

$=3-\frac{2}{9}\left(\frac{1}{2}\cdot bc\sin A+\frac{1}{2}\cdot ac\sin B+\frac{1}{2}\cdot ab\sin C\right)$

$=3-\frac{2}{9}\left(3\times\text{area of }\Delta\text{ABC}\right)$

$=3-\frac{2}{9}\times9=1$

I5.4 Given that $(Rx^2-x+1)^{1999}\equiv a_0+a_1x+a_2x^2+\cdots+a_{3998}x^{3998}$.

If $S=a_0+a_1+a_2+\cdots+a_{3997}$, find the value of S.

$(x^2-x+1)^{1999}\equiv a_0+a_1x+a_2x^2+\cdots+a_{3998}x^{3998}$

Compare coefficients of x^{3998} on both sides, $a_{3998}=1$

Put $x=1$, $1^{1999}=a_0+a_1+a_2+\cdots+a_{3998}$

$S=a_0+a_1+a_2+\cdots+a_{3997}$

$=\left(a_0+a_1+a_2+\cdots+a_{3998}\right)-a_{3998}$

$=1-1=0$

Sample Group Event (1999 Final Group Event 1)

SG.1 Let $x * y = x + y - xy$, where x, y are real numbers. If $a = 1 * (0 * 1)$, find the value of a.

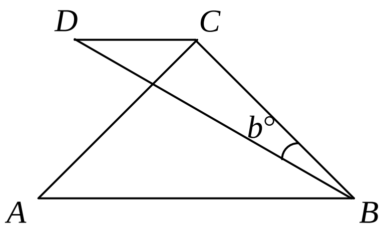
$0 * 1 = 0 + 1 - 0 = 1$

$a = 1 * (0 * 1)$

$= 1 * 1$

$= 1 + 1 - 1 = 1$

SG.2 In figure 1, AB is parallel to DC, $\angle ACB$ is a right angle, $AC = CB$ and $AB = BD$. If $\angle CBD = b°$, find the value of b.



ΔABC is a right angled isosceles triangle.

$\angle BAC = 45°$ ($\angle$s sum of Δ, base $\angle$s isos. Δ)

$\angle ACD = 45°$ (alt. $\angle$s, AB // DC)

$\angle BCD = 135°$

Apply sine law on ΔBCD,

$$\frac{BD}{\sin 135°} = \frac{BC}{\sin D}$$

$$AB\sqrt{2} = \frac{AB \sin 45°}{\sin D}$$, given that $AB = BD$

$\sin D = \frac{1}{2}$; $D = 30°$

$\angle CBD = 180° - 135° - 30° = 15°$ ($\angle$s sum of ΔBCD)

$b = 15$

SG.3 Let x, y be non-zero real numbers. If x is 250% of y and $2y$ is c% of x, find the value of c.

$x = 2.5y$ ....(1)

$2y = \frac{c}{100} \cdot x$ ....(2)

sub. (1) into (2): $2y = \frac{c}{100} \cdot 2.5y$

$c = 80$

SG.4 If $\log_p x = 2$, $\log_q x = 3$, $\log_r x = 6$ and $\log_{pqr} x = d$, find the value of d.

$$\frac{\log x}{\log p} = 2;\ \frac{\log x}{\log q} = 3;\ \frac{\log x}{\log r} = 6$$

$$\frac{\log p}{\log x} = \frac{1}{2};\ \frac{\log q}{\log x} = \frac{1}{3};\ \frac{\log r}{\log x} = \frac{1}{6}$$

$$\frac{\log p}{\log x} + \frac{\log q}{\log x} + \frac{\log r}{\log x} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$$

$$\frac{\log pqr}{\log x} = 1$$

$$\frac{\log x}{\log pqr} = 1$$

$d = \log_{pqr} x = 1$

Group Event 1

G1.1 Given that when 81849, 106392 and 124374 are divided by an integer n, the remainders are equal. If a is the maximum value of n, find a.

Reference: 2016 FI4.2

$81849 = pn + k$ …… (1)

$106392 = qn + k$ …… (2)

$124374 = rn + k$ …… (3)

(2) – (1): $24543 = (q - p)n$ …… (4)

(3) – (2): $17982 = (r - q)n$ …… (5)

(4): $243 \times 101 = (q - p)n$

(5): $243 \times 74 = (r - q)n$

a = maximum value of $n = 243$

G1.2 Let $x = \frac{1-\sqrt{3}}{1+\sqrt{3}}$ and $y = \frac{1+\sqrt{3}}{1-\sqrt{3}}$. If $b = 2x^2 - 3xy + 2y^2$, find the value of b.

$b = 2x^2 - 3xy + 2y^2 = 2x^2 - 4xy + 2y^2 + xy = 2(x - y)^2 + xy$

$$= 2\left(\frac{1-\sqrt{3}}{1+\sqrt{3}} - \frac{1+\sqrt{3}}{1-\sqrt{3}}\right)^2 + \frac{1-\sqrt{3}}{1+\sqrt{3}} \cdot \frac{1+\sqrt{3}}{1-\sqrt{3}}$$

$$= 2\left[\frac{\left(1-\sqrt{3}\right)^2 - \left(1+\sqrt{3}\right)^2}{1-3}\right]^2 + 1$$

$$= 2\left(\frac{-4\sqrt{3}}{-2}\right)^2 + 1 = 25$$

G1.3 Given that c is a positive number. If there is only one straight line which passes through point A(1, c) and meets the curve C: $x^2 + y^2 - 2x - 2y - 7 = 0$ at only one point, find the value of c.

The curve is a circle.

There is only one straight line which passes through point A and meets the curve at only one point $\Rightarrow$ the straight line is a tangent and the point $A(1, c)$ lies on the circle.

(otherwise two tangents can be drawn if A lies outside the circle)

Put $x = 1$, $y = c$ into the circle.

$1 + c^2 - 2 - 2c - 7 = 0$

$c^2 - 2c - 8 = 0$

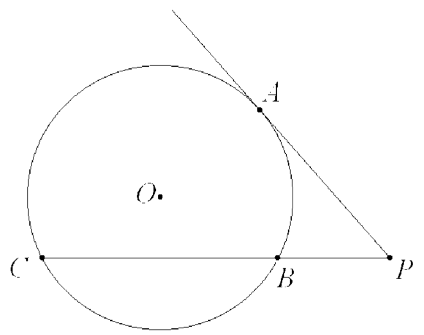
$(c - 4)(c + 2) = 0$

$c = 4$ or $c = -2$ (rejected)

G1.4 In Figure 1, PA touches the circle with centre O at A.

If $PA = 6$, $BC = 9$, $PB = d$, find the value of d.

It is easy to show that $\Delta PAB \sim \Delta PCA$



$\frac{PA}{PB} = \frac{PC}{PA}$ (ratio of sides, ~Δ's)

$\frac{6}{d} = \frac{9+d}{6}$

$36 = 9d + d^2$

$d^2 + 9d - 36 = 0$

$(d - 3)(d + 12) = 0$

$d = 3$ or -12 (rejected)

Group Event 2

G2.1 If 191 is the difference of two consecutive perfect squares, find the value of the smallest square number, a.

Let $a = t^2$, the larger perfect square is $(t+1)^2$

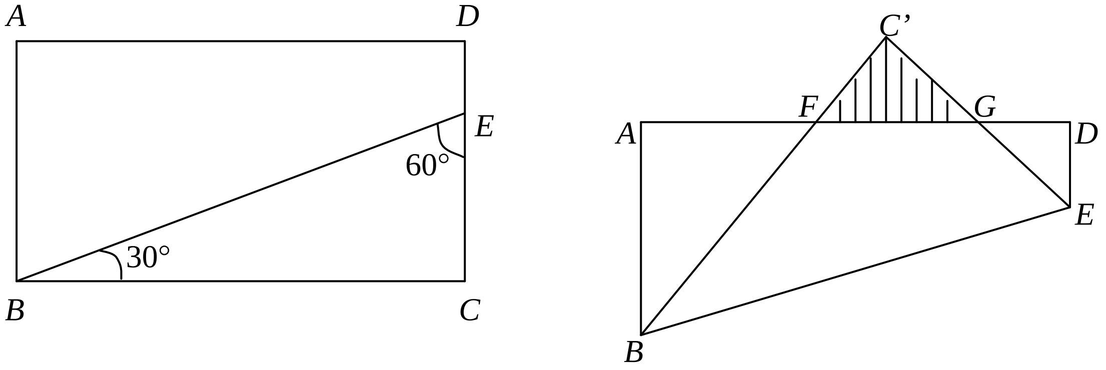
$(t + 1)^2 - t^2 = 191$

$2t + 1 = 191$

$t = 95$

$a = 95^2 = 9025$

G2.2 In Figure 2(a), $ABCD$ is a rectangle. DE:EC = 1: 5, and $DE = 12^{\frac{1}{4}}$. ΔBCE is folded along the side BE. If b is the area of the shaded part as shown in Figure 2(b), find the value of b.



Let $DE = t$, then $CE = 5t$. Suppose BC' intersects AD at F, $C'E$ intersects AD at G.

$BC = BC' = AD = 5t \tan 60° = 5\sqrt{3}\,t$

$\angle C'ED = 60°$, $\angle ABC' = 30°$, $\angle C'FG = 60°$, $\angle C'GF = 30°$

$AF = 6t \tan 30° = 2\sqrt{3}\,t$, $DG = t \tan 60° = \sqrt{3}\,t$

$FG = 5\sqrt{3}\,t - 2\sqrt{3}\,t - \sqrt{3}\,t = 2\sqrt{3}\,t$

$C'F = 2\sqrt{3}\,t \cos 60° = \sqrt{3}\,t$, $C'G = 2\sqrt{3}\,t \cos 30° = 3t$

Area of $\Delta C'FG = \frac{1}{2}\sqrt{3}t \times 3t = \frac{3\sqrt{3}}{2}t^2 = \frac{3\sqrt{3}}{2}\sqrt{12} = 9$

G2.3 Let the curve $y = x^2 - 7x + 12$ intersect the x-axis at points A and B, and intersect the y-axis at C. If c is the area of ΔABC, find the value of c.

$x^2 - 7x + 12 = (x - 3)(x - 4)$

The x-intercepts of 3, 4.

Let $x = 0$, $y = 12$

$c = \frac{1}{2}(4-3)\cdot 12 = 6$ sq. units

G2.4 Let $f(x) = 41x^2 - 4x + 4$ and $g(x) = -2x^2 + x$. If d is the smallest value of k such that $f(x) + kg(x) = 0$ has a single root, find d.

$41x^2 - 4x + 4 + k(-2x^2 + x) = 0$

$(41 - 2k)x^2 + (k - 4)x + 4 = 0$

It has a single root $\Rightarrow \Delta = 0$ or $41 - 2k = 0$

$(k - 4)^2 - 4(41 - 2k)(4) = 0$ or $k = \frac{41}{2}$

$k^2 - 8 + 16 - 16\times 41 + 32k = 0$ or $k = \frac{41}{2}$

$k^2 + 24k - 640 = 0$ or $k = \frac{41}{2}$

$k = 16$ or -40 or $\frac{41}{2}$, d = the smallest value of $k = -40$

Group Event 3

G3.1 Let $a=\sqrt{1997\times1998\times1999\times2000+1}$, find the value of a.

Reference: 1993 HG6, 1995 FI4.4, 1996 FG10.1, 2004 FG3.1, 2012 FI2.3

Let t = 1998.5, then 1997 = t – 1.5, 1998 = t – 0.5, 1999 = t + 0.5, 2000 = t + 1.5

$$\sqrt{1997\times1998\times1999\times2000+1}=\sqrt{(t-1.5)\times(t-0.5)\times(t+0.5)\times(t+1.5)+1}$$

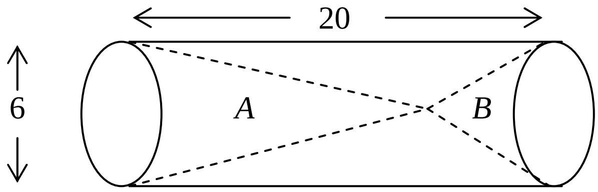
$$=\sqrt{(t^2-2.25)\times(t^2-0.25)+1}=\sqrt{\left(t^2-\frac{9}{4}\right)\times\left(t^2-\frac{1}{4}\right)+1}$$

$$=\sqrt{t^4-\frac{10}{4}t^2+\frac{25}{16}}=\sqrt{\left(t^2-\frac{5}{4}\right)^2}=t^2-1.25$$

$= 1998.5^2 - 1.25 = (2000 - 1.5)^2 - 1.25$

$= 4000000 - 6000 + 2.25 - 1.25 = 3994001$

G3.2 In Figure 3, A and B are two cones inside a cylindrical tube with length of 20 and diameter of 6. If the volumes of A and B are in the ratio 3:1 and b is the height of the cone B, find the value of b.



$$\frac{1}{3}\pi\cdot3^2(20-b):\frac{1}{3}\pi\cdot3^2b=3:1$$

$20 - b = 3b$

$b = 5$

G3.3 If c is the largest slope of the tangents from the point $A\left(\frac{\sqrt{10}}{2},\frac{\sqrt{10}}{2}\right)$ to the circle C: $x^2+y^2=1$, find the value of c.

Let the equation of tangent be $y-\frac{\sqrt{10}}{2}=c\left(x-\frac{\sqrt{10}}{2}\right)$

$$cx-y+\frac{\sqrt{10}}{2}(1-c)=0$$

Distance form centre (0, 0) to the straight line = radius

$$\left|\frac{0-0+\frac{\sqrt{10}}{2}(1-c)}{\sqrt{c^2+(-1)^2}}\right|=1$$

$$\frac{5}{2}(1-c)^2=c^2+1$$

$5 - 10c + 5c^2 = 2c^2 + 2$

$3c^2 - 10c + 3 = 0$

$(3c - 1)(c - 3) = 0$

$c=\frac{1}{3}$ or 3. The largest slope = 3.

G3.4 P is a point located at the origin of the coordinate plane. When a dice is thrown and the number n shown is even, P moves to the right by n. If n is odd, P moves upward by n. Find the value of d, the total number of tossing sequences for P to move to the point (4, 4).

Possible combinations of the die:

2,2,1,1,1,1. There are ${}_6C_2$ permutations, i.e. 15.

4,1,1,1,1. There are ${}_5C_1$ permutations, i.e. 5.

2,2,1,3. There are ${}_4C_2\times 2$ permutations, i.e. 12.

4,1,3. There are 3! permutations, i.e. 6.

Total number of possible ways = 15 + 5 + 12 + 6 = 38.

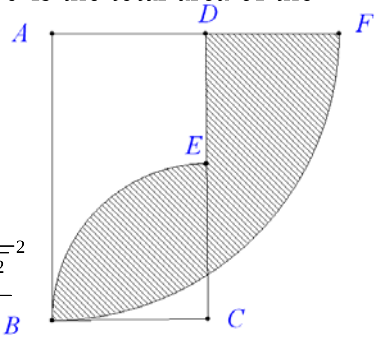
Group Event 4

G4.1 Let a be a 3-digit number. If the 6-digit number formed by putting a at the end of the number 504 is divisible by 7, 9, and 11, find the value of a.

Reference: 2010 HG1

Note that 504 is divisible by 7 and 9. We look for a 3-digit number which is a multiple of 63 and that 504000 + a is divisible by 11. 504504 satisfied the condition.

G4.2 In Figure 4, $ABCD$ is a rectangle with $AB=\sqrt{\dfrac{8+\sqrt{64-\pi^2}}{\pi}}$ and $BC=\sqrt{\dfrac{8-\sqrt{64-\pi^2}}{\pi}}$. BE and BF are the arcs of circles with centres at C and A respectively. If b is the total area of the shaded parts, find the value of b.



$AB = AF$, $BC = CE$

Shaded area = sector ABF – rectangle $ABCD$ + sector BCE

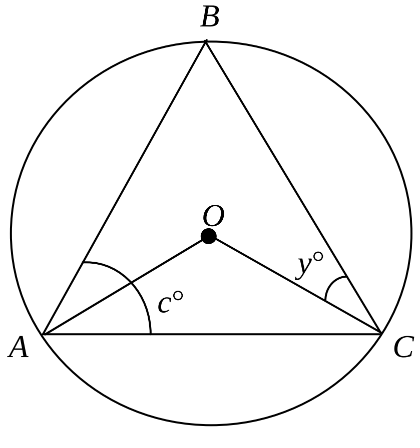
$$=\frac{\pi}{4}AB^2-AB\cdot BC+\frac{\pi}{4}BC^2$$

$$=\frac{\pi}{4}\sqrt{\frac{8+\sqrt{64-\pi^2}}{\pi}}^2-\sqrt{\frac{8+\sqrt{64-\pi^2}}{\pi}}\cdot\sqrt{\frac{8-\sqrt{64-\pi^2}}{\pi}}+\frac{\pi}{4}\sqrt{\frac{8-\sqrt{64-\pi^2}}{\pi}}^2$$

$$=\frac{\pi}{4}\left(\frac{8+\sqrt{64-\pi^2}}{\pi}+\frac{8-\sqrt{64-\pi^2}}{\pi}\right)-\sqrt{\frac{64-\left(64-\pi^2\right)}{\pi^2}}$$

$$=\frac{\pi}{4}\left(\frac{16}{\pi}\right)-\sqrt{\frac{\pi^2}{\pi^2}}=4-1=3=b$$

G4.3 In Figure 5, O is the centre of the circle and $c° = 2y°$. Find the value of c.



$\angle BOC = 2c°$ ($\angle$ at centre twice $\angle$ at $\odot^{ce}$)

$y + y + 2c = 180$ ($\angle$s sum of ΔOBC)

$2y + 2c = 180$

$c + 2c = 180$

$c = 60$

G4.4 A, B, C, D, E, F, G are seven people sitting around a circular table. If d is the total number of ways that B and G must sit next to C, find the value of d.

Reference: 1998 FI5.3, 2011 FI1.4

If B, C, G are neighbours, we can consider these persons bound together as one person. So, there are 5 persons sitting around a round table. The number of ways should be 5!. Since it is a round table, every seat can be counted as the first one. That is, $ABCDE$ is the same as $BCDEA$, $CDEAB$, $DEABC$, $EABCD$. Therefore every 5 arrangements are the same. The number of arrangement should be 5! ÷ 5 = 4! = 24. But B and G can exchange their seats.

$\therefore$ Total number of arrangements = 24 × 2 = 48.

Group Event 5

G5.1 If a is the smallest cubic number divisible by 810, find the value of a.

Reference: 2002 HI2

$810 = 2\times 3^4\times 5$

$a = 2^3\times 3^6\times 5^3 = 729000$

G5.2 Let b be the maximum of the function $y = |x^2 - 4| - 6x$ (where $-2 \le x \le 5$), find the value of b.

When $-2 \le x \le 2$, $y = 4 - x^2 - 6x = -(x + 3)^2 + 13$

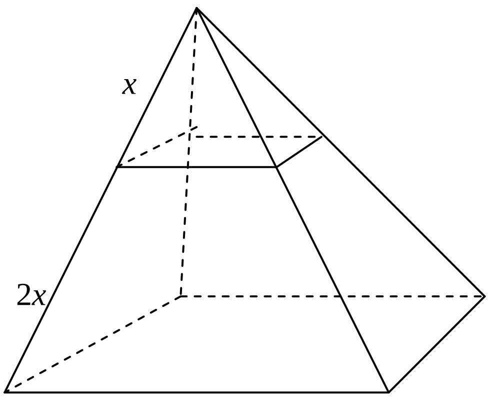
Maximum value occurs at $x = -2$, $y = -(-2 + 3)^2 + 13 = 12$

When $2 \le x \le 5$, $y = x^2 - 4 - 6x = (x - 3)^2 - 13$

Maximum value occurs at $x = 5$, $y = -9$

Combing the two cases, $b = 12$

G5.3 In Figure 6, a square-based pyramid is cut into two shapes by a cut running parallel to the base and made $\frac{2}{3}$ of the way up. Let $1 : c$ be the ratio of the volume of the small pyramid to that of the truncated base, find the value of c.



Reference: 2001 HG5

The two pyramids are similar.

$$\frac{\text{volume of the small pyramid}}{\text{volume of the big pyramid}} = \left(\frac{x}{3x}\right)^3 = \frac{1}{27}$$

$c = 27 - 1 = 26$

G5.4 If $\cos^6\theta + \sin^6\theta = 0.4$ and $d = 2 + 5\cos^2\theta\sin^2\theta$, find the value of d.

$(\cos^2\theta + \sin^2\theta)(\cos^4\theta - \sin^2\theta\cos^2\theta + \cos^4\theta) = 0.4$

$\cos^4\theta + 2\sin^2\theta\cos^2\theta + \cos^4\theta - 3\sin^2\theta\cos^2\theta = 0.4$

$(\cos^2\theta + \sin^2\theta)^2 - 3\sin^2\theta\cos^2\theta = 0.4$

$1 - 0.4 = 3\sin^2\theta\cos^2\theta$

$\sin^2\theta\cos^2\theta = 0.2$

$d = 2 + 5\cos^2\theta\sin^2\theta = 2 + 5\times 0.2 = 3$

Individual Events

I1	***P***	1	**I2**	***P***	12	**I3**	***P***	4	**I4**	***P***	35
	Q	4		***Q***	14		***Q***	7		***Q***	10
	R	2		***R***	1		***R***	14		***R***	10
	S	32		***S***	2		***S***	34		***S***	222

Group Events

G1	***a***	4	**G2**	***a***	2	**G3**	***a***	3	**G4**	***a***	3840
	b	1001		***b***	3		***b***	20		***b***	1
	c	8		***c***	333333		***c***	14		***c***	3
	d	3		***d***	46		***d***	15		***d***	1853

Individual Event 1

I1.1 a, b and c are the lengths of the opposite sides $\angle A$, $\angle B$ and $\angle C$ of the ΔABC respectively.

If $\angle C = 60°$ and $\frac{a}{b+c}+\frac{b}{a+c} = P$, find the value of P.

$c^2 = a^2 + b^2 - 2ab\cos 60° = a^2 + b^2 - ab \Rightarrow a^2 + b^2 = c^2 + ab$

$P = \frac{a}{b+c}+\frac{b}{a+c}=\frac{a(a+c)+b(b+c)}{(b+c)(a+c)}$

$P = \frac{a^2+ac+b^2+bc}{ab+ac+bc+c^2}=\frac{ab+ac+bc+c^2}{ab+ac+bc+c^2} = 1$

I1.2 Given that $f(x) = x^2 + ax + b$ is the common factor of $x^3 + 4x^2 + 5x + 6$ and $2x^3 + 7x^2 + 9x + 10$

If $f(P) = Q$, find the value of Q.

Let $g(x) = x^3 + 4x^2 + 5x + 6$; $h(x) = 2x^3 + 7x^2 + 9x + 10$

$g(-3) = -27 + 36 - 15 + 6 = 0$, $(x + 3)$ is a factor of $g(x)$; by division, $g(x) = (x + 3)(x^2 + x + 2)$

$h(-2.5)= -31.25+43.75-22.5+10 = 0$, $(2x+5)$ is a factor of $h(x)$; by division, $h(x) = (2x+5)(x^2+x+2)$

$f(x)$ = common factor = $(x^2 + x + 2)$

$Q = f(P) = f(1) = 1 + 1 + 2 = 4$

I1.3 Given that $\frac{1}{a}+\frac{1}{b}=\frac{Q}{a+b}$ and $\frac{a}{b}+\frac{b}{a} = R$, find the value of R.

$\frac{1}{a}+\frac{1}{b}=\frac{4}{a+b}$

$\Rightarrow (a + b)^2 = 4ab$

$\Rightarrow a^2 + 2ab + b^2 = 4ab$

$\Rightarrow a^2 - 2ab + b^2 = 0$

$\Rightarrow (a - b)^2 = 0$

$a = b$

$\Rightarrow R = \frac{a}{b}+\frac{b}{a} = 2$

I1.4 Given that $\begin{cases} a+b=R \\ a^2+b^2=12 \end{cases}$ and $a^3 + b^3 = S$, find the value of S.

$\begin{cases} a+b=2 \cdots\cdots(1) \\ a^2+b^2=12\cdots\cdots(2) \end{cases}$

$(1)^2 - (2)$: $2ab = -8$

$\Rightarrow \begin{cases} ab=-4 \\ a+b=2 \end{cases}$

$S = a^3 + b^3 = (a + b)(a^2 - ab + b^2) = 2(12 + 4) = 32$

Individual Event 2

I2.1 Suppose P is an integer and $5 < P < 20$. If the roots of the equation $x^2 - 2(2P - 3)x + 4P^2 - 14P + 8 = 0$ are integers, find the value of P.

Reference: 2000 FI5.2, 2010 FI2.2, 2011 FI3.1, 2013 HG1

$\Delta = 4(2P - 3)^2 - 4(4P^2 - 14P + 8) = m^2$

$\left(\frac{m}{2}\right)^2 = 4P^2 - 12P + 9 - 4P^2 + 14P - 8 = 2P + 1$

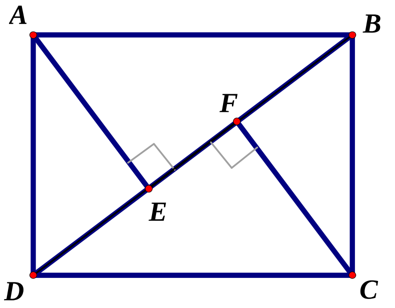
$\because 5 < P < 20 \therefore 11 < 2P + 1 < 41$

The only odd square lying in this interval is 25

$\Rightarrow 2P + 1 = 25 = 5^2$

$\therefore P = 12$

I2.2 $ABCD$ is a rectangle. $AB = 3P + 4$, $AD = 2P + 6$. AE and CF are perpendiculars to the diagonal BD. If $EF = Q$, find the value of Q.



$AB = 40$, $AD = 30$, $BD = 50$, let $\angle ADB = \theta$, $\cos\theta = \frac{3}{5}$

$DE = AD\cos\theta = 30\times\frac{3}{5} = 18 = BF$

$EF = 50 - 18 - 18 = 14$

I2.3 There are less than $4Q$ students in a class. In a mathematics test, $\frac{1}{3}$ of the students got grade A, $\frac{1}{7}$ of the students got grade B, half of the students got grade C, and the rest failed. Given that R students failed in the mathematics test, find the value of R.

$4Q = 56$, let the number of students be x, then x is divisible by 2, 3 and 7.

i.e. x is divisible by 42, as $x < 56$, so $x = 42$

R = number of students failed in mathematics $= 42\times\left(1-\frac{1}{3}-\frac{1}{7}-\frac{1}{2}\right) = 1$; $R = 1$

I2.4 $[a]$ represents the largest integer not greater than a. For example, $\left[2\frac{1}{3}\right] = 2$.

Given that the sum of the roots of the equation $[3x + R] = 2x + \frac{3}{2}$ is S, find the value of S.

Reference: 1994 HG9

$[3x + 1] = 2x + \frac{3}{2} \Rightarrow 3x + 1 = 2x + \frac{3}{2} + a$, where $0 \le a < 1$

$a = x - \frac{1}{2} \Rightarrow 0 \le x - \frac{1}{2} < 1 \Rightarrow 2.5 \le 2x + \frac{3}{2} < 4.5$

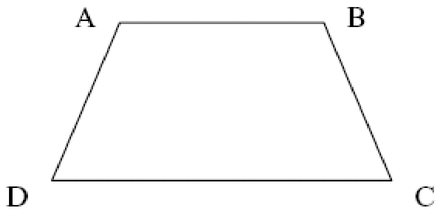
$\because 2x + \frac{3}{2}$ is an integer $\therefore 2x + \frac{3}{2} = 4$ or 3

$x = 0.75$ or 1.25

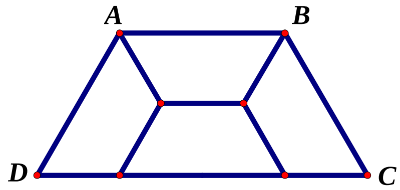
$S = 0.75 + 1.25 = 2$

Individual Event 3

I3.1 *ABCD* is a trapezium such that $\angle ADC = \angle BCD = 60°$ and $AB = BC = AD = \frac{1}{2}CD$. If this trapezium is divided into P equal portions ($P > 1$) and each portion is similar to trapezium *ABCD* itself, find the minimum value of P.



From the graph, $P = 4$



I3.2 The sum of tens and units digits of $(P + 1)^{2001}$ is Q. Find the value of Q.

$5^{2001} = 100a + 25$, where a is a positive integer.

$Q = 2 + 5 = 7$.

I3.3 If $\sin 30° + \sin^2 30° + \dots + \sin^Q 30° = 1 - \cos^R 45°$, find the value of R.

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^7}=1-\frac{1}{\sqrt{2}^R}$$

$$1-\frac{1}{2^7}=1-\frac{1}{2^{\frac{R}{2}}}$$

$R = 14$

I3.4 Let α and β be the roots of the equation $x^2 - 8x + (R + 1) = 0$. If $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$ are the roots of the equation $225x^2 - Sx + 1 = 0$, find the value of S.

Reference: 1996 FI2.2

$x^2 - 8x + 15 = 0$, $\alpha = 3$, $\beta = 5$

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{1}{9}+\frac{1}{25}=\frac{34}{225}=\frac{S}{225}$$

$S = 34$

Individual Event 4

I4.1 Let $a^{\frac{2}{3}}+b^{\frac{2}{3}}=17\frac{1}{2}$, $x=a+3a^{\frac{1}{3}}b^{\frac{2}{3}}$ and $y=b+3a^{\frac{2}{3}}b^{\frac{1}{3}}$. If $P=(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}$, find the value of P.

$P=\left(a+3a^{\frac{2}{3}}b^{\frac{1}{3}}+3a^{\frac{1}{3}}b^{\frac{2}{3}}+b\right)^{\frac{2}{3}}+\left(a-3a^{\frac{2}{3}}b^{\frac{1}{3}}+3a^{\frac{1}{3}}b^{\frac{2}{3}}-b\right)^{\frac{2}{3}}$

$=\left(a^{\frac{1}{3}}+b^{\frac{1}{3}}\right)^{3\times\frac{2}{3}}+\left(a^{\frac{1}{3}}-b^{\frac{1}{3}}\right)^{3\times\frac{2}{3}}$

$P=\left(a^{\frac{1}{3}}+b^{\frac{1}{3}}\right)^{2}+\left(a^{\frac{1}{3}}-b^{\frac{1}{3}}\right)^{2}$

$=a^{\frac{2}{3}}+2a^{\frac{1}{3}}b^{\frac{1}{3}}+a^{\frac{2}{3}}+a^{\frac{2}{3}}-2a^{\frac{1}{3}}b^{\frac{1}{3}}+a^{\frac{2}{3}}$

$=2\left(a^{\frac{2}{3}}+a^{\frac{2}{3}}\right)$

$= 2\times17.5 = 35$

I4.2 If a regular Q-sided polygon has P diagonals, find the value of Q.

Reference: 1984 FG10.3, 1985 FG8.3, 1988 FG6.2, 1989 FG6.1, 1991 FI2.3, 2005 FI1.4

The number of diagonals = $C_2^Q - Q = 35$

$\frac{Q(Q-1)}{2}-Q=35$

$Q^2 - 3Q - 70 = 0$

$Q = 10$

I4.3 Let $x=\sqrt{\frac{Q}{2}+\sqrt{\frac{Q}{2}}}$ and $y=\sqrt{\frac{Q}{2}-\sqrt{\frac{Q}{2}}}$. If $R=\frac{x^6+y^6}{40}$, find the value of R.

$$R=\frac{\left(x^2+y^2\right)\left(x^4+y^4-x^2y^2\right)}{40}$$

$$=\frac{\left(\frac{Q}{2}+\sqrt{\frac{Q}{2}}+\frac{Q}{2}-\sqrt{\frac{Q}{2}}\right)\left[\left(\frac{Q}{2}+\sqrt{\frac{Q}{2}}\right)^2+\left(\frac{Q}{2}-\sqrt{\frac{Q}{2}}\right)^2-\left(\frac{Q}{2}+\sqrt{\frac{Q}{2}}\right)\left(\frac{Q}{2}-\sqrt{\frac{Q}{2}}\right)\right]}{40}$$

$$=\frac{Q\left[2\left(\frac{Q}{2}\right)^2+2\left(\sqrt{\frac{Q}{2}}\right)^2-\left(\frac{Q}{2}\right)^2+\left(\sqrt{\frac{Q}{2}}\right)^2\right]}{40}$$

$$=\frac{10\left(5^2+3\times5\right)}{40}= 10$$

I4.4 $[a]$ represents the largest integer not greater than a. For example, $[2.5] = 2$.

If $S=\left[\frac{2001}{R}\right]+\left[\frac{2001}{R^2}\right]+\left[\frac{2001}{R^3}\right]+\cdots$, find the value of S.

$S=\left[\frac{2001}{10}\right]+\left[\frac{2001}{100}\right]+\left[\frac{2001}{1000}\right]+\cdots$

$= 200 + 20 + 2 + 0 + \cdots = 222$

Group Event 1

G1.1 Given that $(a + b + c)^2 = 3(a^2 + b^2 + c^2)$ and $a + b + c = 12$, find the value of a.

Sub. (2) into (1), $12^2 = 3(a^2 + b^2 + c^2)$

$\Rightarrow a^2 + b^2 + c^2 = 48$ ……(3)

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$\Rightarrow 12^2 = 48 + 2(ab + bc + ca)$

$\Rightarrow ab + bc + ca = 48$

$2[a^2 + b^2 + c^2 - (ab + bc + ca)] = (a - b)^2 + (b - c)^2 + (c - a)^2$

$2[48 - 48] = 0 = (a - b)^2 + (b - c)^2 + (c - a)^2$

$\Rightarrow a = b = c$

$a + b + c = 3a = 12$

$\Rightarrow a = 4$

G1.2 Given that $b\left[\frac{1}{1\times 3}+\frac{1}{3\times 5}+\cdots+\frac{1}{1999\times 2001}\right]=2\times\left[\frac{1^2}{1\times 3}+\frac{2^2}{3\times 5}+\cdots+\frac{1000^2}{1999\times 2001}\right]$, find the value of b.

Note that $\frac{1}{(2r-1)\times(2r+1)}=\frac{1}{2}\left(\frac{1}{2r-1}-\frac{1}{2r+1}\right)$ and $\frac{r^2}{(2r-1)\times(2r+1)}=\frac{1}{4}+\frac{1}{8}\left(\frac{1}{2r-1}-\frac{1}{2r+1}\right)$

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\cdots+\frac{1}{1999\times 2001}=\frac{1}{2}\left[1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\cdots+\frac{1}{1999}-\frac{1}{2001}\right]=\frac{1}{2}\left(1-\frac{1}{2001}\right)=\frac{1000}{2001}$$

$$\frac{1^2}{1\times 3}+\frac{2^2}{3\times 5}+\cdots+\frac{1000^2}{1999\times 2001}=\frac{1}{4}+\frac{1}{8}\left(1-\frac{1}{3}\right)+\frac{1}{4}+\frac{1}{8}\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\frac{1}{4}+\frac{1}{8}\left(\frac{1}{1999}-\frac{1}{2001}\right)\text{ (1000 terms)}$$

$$=\frac{1000}{4}+\frac{1}{8}\left(1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\cdots+\frac{1}{1999}-\frac{1}{2001}\right)=\frac{1000}{4}+\frac{1}{8}\left(1-\frac{1}{2001}\right)$$

$$=\frac{1000}{4}+\frac{1}{8}\cdot\frac{2000}{2001}=250+\frac{250}{2001}=250\left(1+\frac{1}{2001}\right)=\frac{250\cdot 2002}{2001}$$

The given equation becomes: $b\cdot\frac{1000}{2001}=2\cdot\frac{250\cdot 2002}{2001}\Rightarrow b=1001$

G1.3 A six-digit number $1234xy$ is divisible by both 8 and 9. Given that $x + y = c$, find the value of c.

Reference: 2003 FI4.1, 2017 HI1

The number formed by last 3 digits must be divisible by 8 and the sum of digits must be divisible by 9. i.e. $400 + 10x + y$ is divisible by 8 and $1 + 2 + 3 + 4 + x + y = 9m$

$10x + y = 8n$ ……(1); $x + y = 9m - 10$ ……(2)

(1) – (2): $9x = 8n - 9m + 9 + 1$

$\Rightarrow n = 1$ or 10

When $n = 1$, (1) has no solution; when $n = 10$, $x = 8$, $y = 0$; $c = x + y = 8$

G1.4 Suppose $\log_x t = 6$, $\log_y t = 10$ and $\log_z t = 15$. If $\log_{xyz} t = d$, find the value of d.

Reference: 1999 FG1.4, 2015 HI7

$$\frac{\log t}{\log x}=6,\quad \frac{\log t}{\log y}=10,\quad \frac{\log t}{\log z}=15$$

$$\Rightarrow \frac{\log x}{\log t}=\frac{1}{6},\quad \frac{\log y}{\log t}=\frac{1}{10},\quad \frac{\log z}{\log t}=\frac{1}{15}$$

$$\frac{\log x}{\log t}+\frac{\log y}{\log t}+\frac{\log z}{\log t}=\frac{1}{6}+\frac{1}{10}+\frac{1}{15}=\frac{10}{30}=\frac{1}{3}$$

$$\frac{\log x+\log y+\log z}{\log t}=\frac{1}{3}$$

$$\frac{\log xyz}{\log t}=\frac{1}{3}$$

$$d=\frac{\log t}{\log xyz}=3$$

Group Event 2

G2.1 Given that $x=\sqrt{7-4\sqrt{3}}$ and $\frac{x^2-4x+5}{x^2-4x+3}=a$, find the value of a.

Reference: 1993 FI1.4, 1999 HG3, 2011 HI7, 2015 FI4.2, 2015 FG3.1

Reference: 1993 HI9, 2000HG1, 2007 HG3, 2009HG2

$x=\sqrt{7-4\sqrt{3}}=\sqrt{4-2\sqrt{12}+3}$

$=\sqrt{\sqrt{4}^2-2\sqrt{4}\sqrt{3}+\sqrt{3}^2}$

$=\sqrt{\left(\sqrt{4}-\sqrt{3}\right)^2}=\sqrt{4}-\sqrt{3}=2-\sqrt{3}$

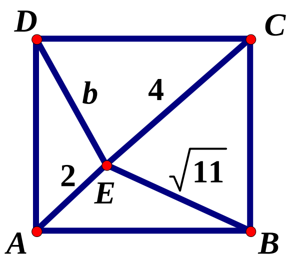
$\sqrt{3}=2-x$

$\Rightarrow 3=(2-x)^2$

$\Rightarrow x^2-4x+1=0$

$a=\frac{x^2-4x+5}{x^2-4x+3}=\frac{x^2-4x+1+4}{x^2-4x+1+2}=2$

G2.2 E is an interior point of the rectangle $ABCD$. Given that the lengths of EA, EB, EC and ED are 2, $\sqrt{11}$, 4 and b respectively, find the value of b.



Reference: 1994 FG10.1-2, 2003 FI3.4, 2018 HI7

Let P, Q R and S be the foot of perpendiculars drawn from E onto AB, BC, CD and DA respectively. $PE=p$, $QE=q$, $RE=r$, $SE=s$.

Using Pythagoras' Theorem, it can be proved that

$p^2+s^2=4$ ....(1)

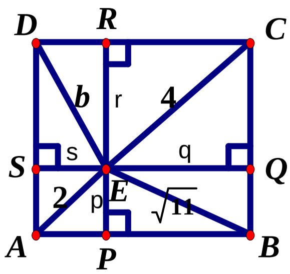
$p^2+q^2=11$ ....(2)

$q^2+r^2=16$ ....(3)

$r^2+s^2=b^2$ ....(4)

(1) + (3) – (2) – (4): $0=4+16-11-b^2$

$b=3$



G2.3 Given that 111111222222 = $c\times(c+1)$, find the value of c.

Reference 1996 FG7.2 111aaa = $b\times(b+1)$ ……

111111222222 = 111111000000 + 222222

= 111111×1000000 + 2×111111

= 111111×1000002

111111222222 = 111111×3×333334 = 333333 × 333334

c = 333333

G2.4 Given that cos 16° = sin 14° + sin d° and $0<d<90$, find the value of d.

sin d° = cos 16° – sin 14°

sin d° = sin 74° – sin 14°

sin d° = $2\cos\frac{74^\circ+14^\circ}{2}\sin\frac{74^\circ-14^\circ}{2}$

sin d° = cos 44° = sin 46°

d = 46

Group Event 3

G3.1 Given that the solution of the equation $\sqrt{3x+1}+\sqrt{3x+6}=\sqrt{4x-2}+\sqrt{4x+3}$ is a, find the value of a.

$\sqrt{3x+6}-\sqrt{4x-2}=\sqrt{4x+3}-\sqrt{3x+1}$

$\left(\sqrt{3x+6}-\sqrt{4x-2}\right)^2=\left(\sqrt{4x+3}-\sqrt{3x+1}\right)^2$

$3x+6+4x-2-2\sqrt{12x^2+18x-12}=4x+3+3x+1-2\sqrt{12x^2+13x+3}$

$\sqrt{12x^2+18x-12}=\sqrt{12x^2+13x+3}$

$12x^2 + 18x - 12 = 12x^2 + 13x + 3$

$x = 3$

G3.2 Suppose the equation $x^2y - x^2 - 3y - 14 = 0$ has only one positive integral solution (x_0, y_0). If $x_0 + y_0 = b$, find the value of b.

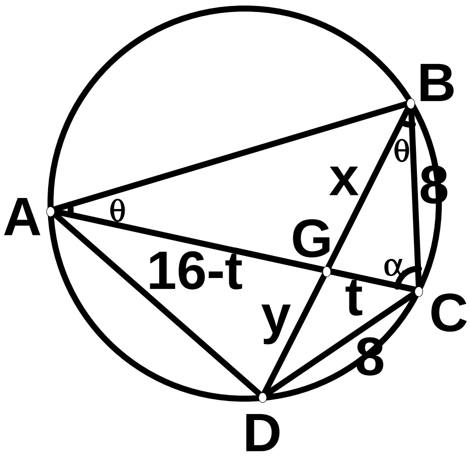
$(y - 1)x^2 = 3y + 14$

$x^2=\frac{3y+14}{y-1}=\frac{3y-3+17}{y-1}=3+\frac{17}{y-1}=3+1$

$y = 18$, $x = 2$

$b = 20$

G3.3 *ABCD* is a cyclic quadrilateral. *AC* and *BD* intersect at *G*. Suppose *AC* =16 cm, *BC* = *CD* = 8 cm, *BG* = *x* cm and *GD* = *y* cm. If *x* and *y* are integers and $x + y = c$, find the value of c.



As shown in the figure, let $CG = t$, $AG = 16 - t$.

Let $\angle CBG = \theta$, $\angle ACB = \alpha$.

Then $\angle CAB = \theta$ (eq. chords eq. ∠s)

Then $\Delta BCG \sim \Delta ACB$ (equiangular)

$t : 8 = 8 : 16$ (ratio of sides, ~Δs)

$t = 4$

It is easy to see that $\Delta ADG \sim \Delta BCG$ (equiangular)

$(16 - t) : y = x : t$ (ratio of sides, ~Δs)

$(16 - 4)\times 4 = xy$

$xy = 48$

Assume that x and y are integers, then possible pairs of (x, y) are (1,48), (2, 24), ..., (6, 8), ..., (48, 1).

Using triangle inequality $x + t > 8$ and $8 + t > x$ in ΔBCG, the only possible combinations are:

$(x, y) = (6, 8)$ or $(8, 6)$

$c = x + y = 14$

G3.4 Given that $5^{\log 30}\times\left(\frac{1}{3}\right)^{\log 0.5}=d$, find the value of d.

$\log 30 \log 5 + \log 0.5 \log\frac{1}{3} = \log d$

$\log (3\times 10) \log\frac{10}{2} + (-\log 2)(-\log 3) = \log d$

$(\log 3 + 1)(1 - \log 2) + \log 2 \log 3 = \log d$

$\log 3 + 1 - \log 3 \log 2 - \log 2 + \log 2 \log 3 = \log d$

$\log d = \log 3 + 1 - \log 2 = \log\frac{3\times 10}{2}$

$d = 15$

Group Event 4

G4.1 $x_1 = 2001$. When $n > 1$, $x_n = \dfrac{n}{x_{n-1}}$. Given that $x_1x_2x_3\ldots x_{10} = a$, find the value of a .

$x_2 = \dfrac{2}{x_1} \Rightarrow x_1x_2 = 2$

$x_4 = \dfrac{4}{x_3} \Rightarrow x_3x_4 = 4$

$x_6 = \dfrac{6}{x_5} \Rightarrow x_5x_6 = 6$

$x_8 = \dfrac{8}{x_7} \Rightarrow x_7x_8 = 8$

$x_{10} = \dfrac{10}{x_9} \Rightarrow x_9x_{10} = 10$

Multiply these equations gives $a = x_1x_2x_3\ldots x_{10} = 2\times4\times6\times8\times10 = 32\times120 = 3840$

G4.2 Given that the units digit of $1^3 + 2^3 + 3^3 + \ldots + 2001^3$ is b, find the value of b .
Arrange the numbers in groups of 10 in ascending order, the units digit of sum each group is the same (except the last number, 2001^3) .

$1^3 + 2^3 + \cdots + 10^3 \equiv \cancel{1} + \cancel{8} + \cancel{7} + \cancel{4} + 5 + \cancel{6} + \cancel{3} + \cancel{2} + \cancel{9} + 0 \pmod{10}$

$\equiv 5 \pmod{10}$

$1^3 + 2^3 + \cdots + 2000^3 + 2001^3 \equiv 200(5) + 1 \pmod{10}$

So $b = 1$

G4.3 A and B ran around a circular path with constant speeds. They started from the same place and at the same time in opposite directions. After their first meeting, B took 1 minute to go back to the starting place. If A and B need 6 minutes and c minutes respectively to complete one round of the path, find the value of c.

In one minute, A and B ran $\dfrac{1}{6} + \dfrac{1}{c} = \dfrac{c+6}{6c}$ of the total distance.

They will meet at the first time after $\dfrac{6c}{c+6}$ minutes.

After 1 more minute, (i.e. total time elapsed $= \dfrac{6c}{c+6} + 1$ minutes), B retuned to the starting point. So $\left(\dfrac{6c}{c+6} + 1\right) \times \dfrac{1}{c} = 1$

$6c + c + 6 = c^2 + 6c$

$c^2 - c - 6 = 0$

$(c - 3)(c + 2) = 0$

$c = 3$

G4.4 The roots of the equation $x^2 - 45x + m = 0$ are prime numbers. Given that the sum of the squares of the roots is d, find the value of d.

Reference: 1996 HG8, 1996FG7.1, 2005 FG1.2, 2012 HI6

Let the roots be α, β. $\alpha + \beta = 45$, $\alpha\beta = m$

The sum of two prime numbers $\alpha + \beta = 45$

$\alpha = 2$, $\beta = 43$ (2 is the only even prime number)

$d = \alpha^2 + \beta^2 = 4 + 43^2 = 1853$

Individual Events

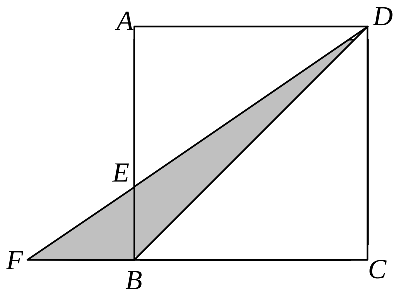
I1	***P***	40	**I2**	***P***	$\frac{99}{100}$	**I3**	***P***	12	**I4**	***P***	4
	Q	72		***Q***	1		***Q***	1		***Q***	8
	R	648		***R***	3		***R***	615		***R***	4
	S	40.5		***S***	$\frac{1}{12}$		***S***	60		***S***	10

Group Events

G1	***a***	21	**G2**	***a***	24	**G3**	***a***	2005	**G4**	***a***	4032
	b	2.5		***b***	52		***b***	2		***b***	2
	c	19		***c***	2005003		***c***	649		***c***	1
	d	300°		***d***	3		***d***	8		***d***	2

Individual Event 1

I1.1 In the following figure, *ABCD* is a square of length 10 cm. *AEB*, *FED* and *FBC* are straight lines. The area of ΔAED is larger than that of ΔFEB by 10 cm^2. If the area of ΔDFB is *P* cm^2, find the value of *P*.



Let the area of ΔBDE be *x*.

Then area of $\Delta AED + x$ – (area of $\Delta BEF + x$) = 10

area of ΔABD – area of ΔBDF = 10

$\frac{1}{2}\cdot 10\times 10$ – area of ΔBDF = 10

area of ΔBDF = 40

Method 2

Area of ΔADE – area of ΔBFE = 10 (given)

$\Rightarrow$ Area of ΔADE + area of ΔAEF – area of ΔBFE – area of ΔAEF = 10

$\Rightarrow$ Area of ΔADF – area of ΔAFB = 10

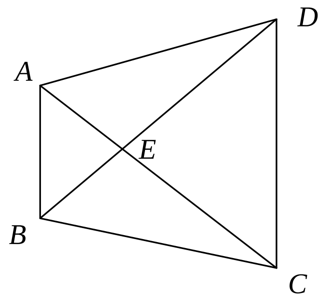
$\Rightarrow \frac{1}{2}\cdot 10\times 10$ – area of ΔDFB = 10 $\Rightarrow$ Area of ΔDFB = 50 – 10 = 40

I1.2 Workman A needs 90 days to finish a task independently while workman B needs *Q* days for the same task. If they only need *P* days to finish the task when working together, find the value of *Q*.

$$\frac{1}{90}+\frac{1}{Q}=\frac{1}{40}$$

$Q = 72$

I1.3 In the following figure, *AB*//*CD*, the area of trapezium *ABCD* is *R* cm^2. Given that the areas of ΔABE and ΔCDE are *Q* cm^2 and 4*Q* cm^2 respectively, find the value of *R*.



Reference: 1993 HI2, 1997 HG3, 2000 FI2.2, 2004 HG7, 2010HG4, 2013 HG2

It is easy to show that $\Delta ABE \sim \Delta CDE$ (equiangular)

$Q : 4Q = (AB)^2 : (CD)^2 \Rightarrow AB : CD = 1 : 2$

$AE : EC = AE : EC = BE : ED = 1 : 2$ (ratio of sides, ~ Δ's)

$S_{\Delta AEB} : S_{\Delta AED} = BE : ED = 1 : 2$ (the 2 Δs have the same heights)

$S_{\Delta AED} = 2Q$

$S_{\Delta AEB} : S_{\Delta BEC} = AE : EC = 1:2$ (the 2 Δs have the same heights)

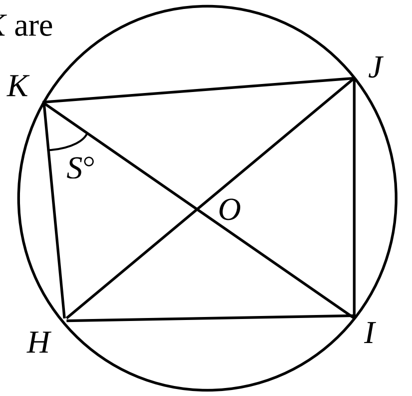
$S_{\Delta BEC} = 2Q$

$S_{ABCD} = Q + 4Q + 2Q + 2Q = 9Q = 648$

$R = 648$

I1.4 In the following figure, O is the centre of the circle, HJ and IK are diameters and $\angle HKI = S°$.

Given that $\angle HKI + \angle HOI + \angle HJI = \frac{1}{4}R°$, find the value of S.



$S° + 2S° + S° = \frac{1}{4} \times 648°$

$\Rightarrow 4S° = 162°$ ($\angle$ at centre = 2$\angle$ at $\odot^{ce}$)

$S = 40.5$

Individual Event 2

I2.1 Given that $P = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{99\times 100}$, find the value of P.

$P = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{99} - \frac{1}{100}$

$= 1 - \frac{1}{100} = \frac{99}{100}$

I2.2 Given that $99Q = P \times (1 + \frac{99}{100} + \frac{99^2}{100^2} + \frac{99^3}{100^3} + \ldots)$, find the value of Q.

$99Q = \frac{99}{100} \times (1 + \frac{99}{100} + \frac{99^2}{100^2} + \frac{99^3}{100^3} + \ldots)$

$= \frac{99}{100} \times \frac{1}{1 - \frac{99}{100}} = 99$

$Q = 1$

I2.3 Given that x and R are real numbers and $\frac{2x^2 + 2Rx + R}{4x^2 + 6x + 3} \le Q$ for all x, find the maximum value of R.

$4x^2 + 6x + 3 = (2x + 1.5)^2 + 0.75 > 0$

$\frac{2x^2 + 2Rx + R}{4x^2 + 6x + 3} \le 1$

$2x^2 + 2Rx + R \le 4x^2 + 6x + 3$

$2x^2 + 2(3 - R)x + 3 - R \ge 0$

$\Delta \le 0$

$(3 - R)^2 - 2(3 - R) \le 0$

$(3 - R)(1 - R) \le 0$

$1 \le R \le 3$

The maximum value of $R = 3$

I2.4 Given that $S = \log_{144} \sqrt[R]{2} + \log_{144} \sqrt[2R]{R}$, find the value of S.

$S = \frac{\frac{1}{3}\log 2}{\log 144} + \frac{\frac{1}{6}\log 3}{\log 144} = \frac{2\log 2 + \log 3}{6\log 144} = \frac{\log 12}{6\log 12^2} = \frac{\log 12}{12\log 12} = \frac{1}{12}$

Method 2

$S = \log_{144} \sqrt[R]{2} + \log_{144} \sqrt[2R]{R}$

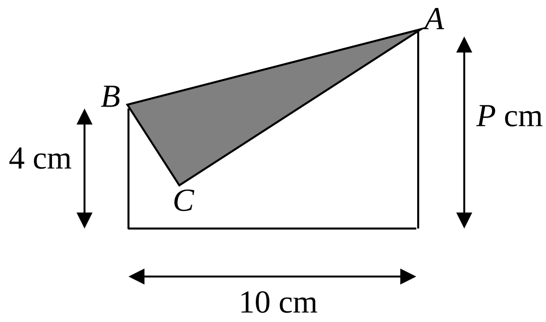
$= \log_{144}(\sqrt[3]{2} \cdot \sqrt[6]{3})$

$= \log_{144}(\sqrt[6]{12})$

$= \log_{144}(\sqrt[12]{144}) = \frac{1}{12}$

Individual Event 3

I3.1 A rectangular piece of paper is folded into the following figure. If the area of $\triangle ABC$ is $\frac{1}{3}$ of the area of the original rectangular piece of paper, find the value of P.



$BC = P - 4$, $AC = 10$, $\angle ACB = 90°$

$$\frac{(P-4)\cdot 10}{2} = \frac{1}{3}\times P\times 10$$

$\Rightarrow P = 12$

I3.2 If Q is the positive integral solution of the equation $\frac{P}{2}(4^x + 4^{-x}) - 35(2^x + 2^{-x}) + 62 = 0$, find the value of Q.

Let $t = 2^x + 2^{-x}$, then $t^2 = 4^x + 4^{-x} + 2$

$\Rightarrow 4^x + 4^{-x} = t^2 - 2$

The equation becomes $6(t^2 - 2) - 35t + 62 = 0$

$6t^2 - 35t + 50 = 0$

$(2t - 5)(3t - 10) = 0$

$t = \frac{5}{2}$ or $\frac{10}{3}$

$2^x + 2^{-x} = \frac{5}{2}$ or $2^x + 2^{-x} = \frac{10}{3}$

$2^x + \frac{1}{2^x} = \frac{5}{2}$ or $2^x + \frac{1}{2^x} = \frac{10}{3}$

$2(2^x)^2 + 2 = 5(2^x)$ or $3(2^x)^2 + 3 = 10(2^x)$

$2(2^x)^2 - 5(2^x) + 2 = 0$ or $3(2^x)^2 - 10(2^x) + 3 = 0$

$(2\cdot 2^x - 1)(2^x - 2) = 0$ or $(3\cdot 2^x - 1)(2^x - 3) = 0$

$2^x = \frac{1}{2}, 2, \frac{1}{3}$ or 3

For positive integral solution $x = 1$; $Q = 1$

I3.3 Let $[a]$ be the largest integer not greater than a. For example, $[2.5] = 2$.

If $R = [\sqrt{1}] + [\sqrt{2}] + \ldots + [\sqrt{99Q}]$, find the value of R.

$$R = [\sqrt{1}] + [\sqrt{2}] + \ldots + [\sqrt{99}] = 1+1+1+\underbrace{2+\cdots+2}_{5\text{ times}}+\underbrace{3+\cdots+3}_{7\text{ times}}+\cdots+\underbrace{9+\cdots+9}_{19\text{ times}}$$

$R = 3\times1 + 5\times2 + 7\times3 + \ldots + 19\times9$

$R = (2\times1+1)\times1 + (2\times2+1)\times2+(2\times3+1)\times3 + \ldots + (2\times9+1)\times9$

$R = 2\times1^2 + 1 + 2\times2^2 + 2 + 2\times3^2 + 3 + \ldots + 2\times9^2 + 9$

$R = 2\times(1^2 + 2^2 + 3^2 + \ldots + 9^2) + (1 + 2 + 3 + \ldots + 9)$

$$R = 2\times\frac{1}{6}\cdot 9(9+1)(2\times9+1)+\frac{(1+9)9}{2} = 3\times10\times19 + 45 = 570 + 45 = 615$$

I3.4 In a convex polygon, other than the interior angle A, the sum of all the remaining interior angles is equal to $4R°$. If $\angle A = S°$, find the value of S.

Reference: 1989 HG2, 1990 FG10.3-4, 1992 HG3, 2013 HI6

$4\times615 + S = 180\times(n - 2)$

$S = 180(n - 2) - 2460$

$\because$ The polygon is convex

$\therefore S < 180$. $S = 180(14) - 2460 = 60$

Individual Event 4

I4.1 Given that $f(x) = (x^2 + x - 2)^{2002} + 3$ and $f\left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right) = P$, find the value of P.

$x = \frac{\sqrt{5}}{2} - \frac{1}{2}$

$\Rightarrow 2x = \sqrt{5} - 1$

$\Rightarrow (2x + 1) = \sqrt{5}$

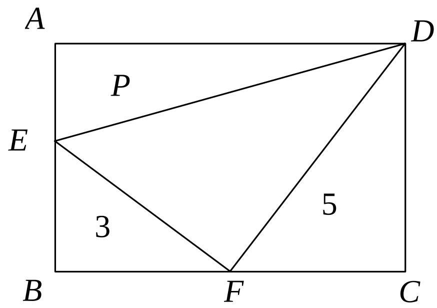
$\Rightarrow (2x + 1)^2 = 5$

$\Rightarrow 4x^2 + 4x - 4 = 0$

$\Rightarrow x^2 + x = 1$

$f(x) = (x^2 + x - 2)^{2002} + 3 = (1 - 2)^{2002} + 3 = 1 + 3 = 4$

I4.2 In the following figure, $ABCD$ is a rectangle. E and F are points on AB and BC respectively. The areas of triangles AED, EBF and FCD are P, 3 and 5 respectively. If the area of ΔEFD is Q, find the value of Q.



Let $AE = x$, $CF = y$, $AD = b$, $CD = a$.

Then $BE = a - x$, $BF = b - y$

Given the area of $\Delta ADE = 4 \Rightarrow bx = 8$ ………..(1)

the area of $\Delta CDF = 5 \Rightarrow ay = 10$……….(2)

The area of $\Delta BEF = 3 \Rightarrow (a - x)(b - y) = 6$

$\Rightarrow ab - bx - ay + xy = 6$ ……..(3)

Sub. (1), (2) into (3) $ab - 8 - 10 + xy = 6$

Sub. (1), (2) into the equation again: $ab - 18 + \frac{80}{ab} = 6$

Solving for ab, $ab = 20$ or 4 (rejected)

The area of $\Delta DEF = 20 - 3 - 4 - 5 = 8$

$Q = 8$

I4.3 It is given that x and y are positive integers. If the number of solutions (x, y) of the inequality $x^2 + y^2 \le Q$ is R, find the value of R. **(Reference: 2007 FG1.2)**

$x^2 + y^2 \le 8$

$\Rightarrow (x, y) = (1, 1), (1, 2), (2, 1), (2, 2)$

$R = 4$

I4.4 It is given that α and β are roots of the equation $x^2 - ax + a - R = 0$, where a is real. If the minimum value of $(\alpha + 1)^2 + (\beta + 1)^2$ is S, find the value of S.

$x^2 - ax + a - 4 = 0$; $\alpha + \beta = a$, $\alpha\beta = a - 4$

$(\alpha+1)^2 + (\beta+1)^2 = \alpha^2 + 2\alpha + 1 + \beta^2 + 2\beta + 1$

$= (\alpha + \beta)^2 - 2\alpha\beta + 2(\alpha + \beta) + 2$

$= a^2 - 2(a - 4) + 2a + 2 = a^2 + 10 \ge 10$

The minimum value is $S = 10$.

Group Event 1

G1.1 Assume that the curve $x^2 + 3y^2 = 12$ and the straight line $mx + y = 16$ intersect at only one point. If $a = m^2$, find the value of a.

Sub. $y = 16 - mx$ into $x^2 + 3y^2 = 12$

$\Rightarrow x^2 + 3(16 - mx)^2 = 12$

$x^2 + 3(256 - 32mx + m^2x^2) = 12$

$\Rightarrow (1 + 3m^2)x^2 - 96mx + 756 = 0$

The straight line is a tangent $\Rightarrow \Delta = (-96m)^2 - 4(1 + 3m^2)756 = 0$

$576m^2 - 189(1 + 3m^2) = 0$

$\Rightarrow 64m^2 - 21(1 + 3m^2) = 0$

$\Rightarrow a = m^2 = 21$

G1.2 It is given that $x + y = 1$ and $x^2 + y^2 = 2$. If $x^3 + y^3 = b$, find the value of b.

Reference: 2011 FI2.2

$(x + y)^2 = 1 \Rightarrow x^2 + y^2 + 2xy = 1$

$\Rightarrow 2 + 2xy = 1$

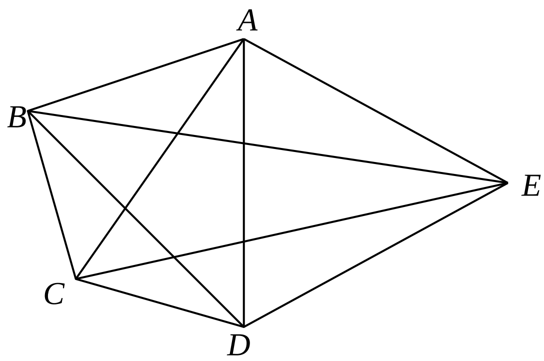
$\Rightarrow xy = -\frac{1}{2}$

$b = x^3 + y^3$

$= (x + y)(x^2 + y^2 - xy)$

$= 1(2 + \frac{1}{2}) = \frac{5}{2}$

G1.3 In the following figure, $AC = AD = AE = ED = DB$ and $\angle BEC = c°$. Given that $\angle BDC = 26°$ and $\angle ADB = 46°$, find the value of c.



ΔADE is an equilateral triangle.

$\angle DAE = \angle ADE = \angle AED = 60°$

$\because BD = DE$ and $\angle BDE = 46° + 60° = 106°$

$\therefore \angle BED = (180° - 106°) \div 2 = 37°$ ($\angle$s sum of Δ)

$\angle AEB = 60° - 37° = 23°$

$\angle ADC = 26° + 46° = 72°$

$\because AC = AD$ and $\angle ADC = 72° = \angle ACD$ (base $\angle$, isos. Δ)

$\therefore \angle CAD = 180° - 72° \times 2 = 36°$ ($\angle$s sum of Δ)

$\because AC = AE$ and $\angle CAE = 36° + 60° = 96°$

$\therefore \angle AEC = (180° - 96°) \div 2 = 42°$ ($\angle$s sum of Δ)

$\angle CED = 60° - 42° = 18°$

$\angle BCE = 60° - 18° - 23° = 19°$

$c = 19$

G1.4 It is given that $4\cos^4\theta + 5\sin^2\theta - 4 = 0$, where $0° < \theta < 360°$. If the maximum value of θ is d, find the value of d.

$4\cos^4\theta + 5\sin^2\theta - 4 = 0 \Rightarrow 4\cos^4\theta + 5(1 - \cos^2\theta) - 4 = 0 \Rightarrow 4\cos^4\theta - 5\cos^2\theta + 1 = 0$

$(4\cos^2\theta - 1)(\cos^2\theta - 1) = 0$

$\cos^2\theta = \frac{1}{4}$ or 1

$\Rightarrow \cos\theta = \frac{1}{2}, -\frac{1}{2}, 1$ or -1.

$\theta = 60°, 300°, 120°, 180°, 240°$.

The maximum value of $\theta = 300°$

$d = 300°$

Group Event 2

G2.1 It is given that the lengths of the sides of a triangle are 6, 8, and 10.
If the area of the triangle is a, find the value of a.

$6^2 + 8^2 = 36 + 64 = 100 = 10^2$

It is a right angled triangle.

The area of the triangle = $6 \times 8 \div 2 = 24$

$a = 24$

G2.2 Given that $f\left(x+\frac{1}{x}\right)=x^3+\frac{1}{x^3}$ and $f(4) = b$, find the value of b.

Reference: 1987 FG8.2, 2002 HI10

Let $y = x + \frac{1}{x}$

$y^2 - 2 = x^2 + \frac{1}{x^2}$

$x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right)$

$= y(y^2 - 3) = y^3 - 3y$

$f(y) = y^3 - 3y$

$b = f(4) = 4^3 - 3(4) = 52$

G2.3 Given that $2002^2 - 2001^2 + 2000^2 - 1999^2 + \ldots + 4^2 - 3^2 + 2^2 - 1^2 = c$, find the value of c.

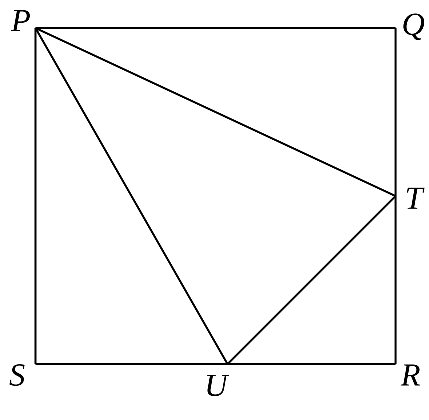
Reference: 1997 HI5, 2004 HI1, 2015 FI3.2, 2015 FG4.1

$c = (2002+ 2001)(2002 - 2001) + (2000 + 1999)(2000 - 1999) + \ldots + (4+3)(4-3) + (2+1)(2-1)$

$c = 4003 + 3999 + \ldots + 7 + 3$

$= \frac{4003+3}{2} \times 1001 = 2005003$

G2.4 $PQRS$ is a square, PTU is an isosceles triangle, and $\angle TPU = 30°$. Points T and U lie on QR and RS respectively. The area of ΔPTU is 1. If the area of $PQRS$ is d, find the value of d.



Let $PT = a = PU$

$\frac{1}{2}a^2 \sin 30^\circ = 1$

$\Rightarrow a = 2$

$\Delta PSU \cong \Delta PQT$ (RHS)

Let $PS = x = PQ$; $SU = y = QT$

$\angle SPU = \angle QPT = 30°$ (corr. $\angle$s $\cong \Delta$)

$x = PU \cos 30° = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$

d = area of $PQRS = \sqrt{3}^2 = 3$

Group Event 3

G3.1 If $\frac{2002^3 + 4\times 2002^2 + 6006}{2002^2 + 2002} = a$, find the value of a.

$a = \frac{2002(2002^2 + 4\times 2002 + 3)}{2002(2002+1)}$

$= \frac{(2002+1)(2002+3)}{2002+1}$

$= 2005$

G3.2 It is given that the real numbers x and y satisfy the relation $y = \frac{x}{2x-1}$.

If the minimum value of $\frac{1}{x^2} + \frac{1}{y^2}$ is b, find the value of b.

$\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{x^2} + \frac{(2x-1)^2}{x^2}$

$= \frac{4x^2 - 4x + 2}{x^2}$

Let $T = \frac{4x^2 - 4x + 2}{x^2}$

$Tx^2 = 4x^2 - 4x + 2$

$(T - 4)x^2 + 4x - 2 = 0$

$\Delta = 4^2 + 4\times 2(T - 4) \geq 0$

$2 + T - 4 \geq 0$

$\Rightarrow T \geq 2$

The minimum value is 2

$b = 2$

G3.3 Suppose two different numbers are chosen randomly from the 50 positive integers 1, 2, 3, … , 50, and the sum of these two numbers is not less than 50. If the number of ways of choosing these two numbers is c, find the value of c.

Reference: 2011 FG2.2

Possible combinations may be: (1, 49), (1, 50),
(2, 48), (2, 49), (2, 50),
(3, 47), (3, 48), (3, 49), (3, 50),
……………………………………………
(24, 26), (24, 27), … , (24, 50),
(25, 26), (25, 27), … , (25, 50),
(26, 27), … , (26, 50)
……………………………
(49, 50)

Total number of combinations = (2 + 3 + … + 25) + 25 + (24 + 23 + … + 1)
= (1 + 2 + … + 24)×2 + 24 + 25
= 25×24 + 49 = 649

G3.4 Given that $x - y = 1 + \sqrt{5}$, $y - z = 1 - \sqrt{5}$. If $x^2 + y^2 + z^2 - xy - yz - zx = d$, find the value of d.

$2d = (x - y)^2 + (y - z)^2 + (z - x)^2 = (1 + \sqrt{5})^2 + (1 - \sqrt{5})^2 + [(z - y) - (x - y)]^2$

$2d = 1 + 2\sqrt{5} + 5 + 1 - 2\sqrt{5} + 5 + [-1 + \sqrt{5} - (1 + \sqrt{5})]^2 = 12 + 4 = 16$

$d = 8$

Group Event 4

G4.1 If a is the sum of all the positive factors of 2002, find the value of a.

Reference 1993 HI8, 1994 FI3.2, 1997 HI3, 1998 HI10, 1998 FI1.4, 2005 FI4.4

$2002 = 2\times7\times11\times13$

The positive factors may be $2^a7^b11^c13^d$, where $0 \le a, b, c, d \le 1$ are integers.

The sum of all positive factors are $(1 + 2)(1 + 7)(1 + 11)(1 + 13) = 3\times8\times12\times14 = 4032 = a$

G4.2 It is given that $x > 0$, $y > 0$ and $\sqrt{x}(\sqrt{x}+\sqrt{y})=3\sqrt{y}(\sqrt{x}+5\sqrt{y})$. If $b=\frac{2x+\sqrt{xy}+3y}{x+\sqrt{xy}-y}$,

find the value of b.

$\sqrt{x}(\sqrt{x}+\sqrt{y})=3\sqrt{y}(\sqrt{x}+5\sqrt{y})$

$\Rightarrow x+\sqrt{xy} = 3\sqrt{xy} + 15y$

$\Rightarrow x - 2\sqrt{xy} - 15y = 0$

$\left(\sqrt{x}+3\sqrt{y}\right)\left(\sqrt{x}-5\sqrt{y}\right)=0$

$\Rightarrow \sqrt{x}=5\sqrt{y}$ and $x = 25y$

$$b=\frac{2x+\sqrt{xy}+3y}{x+\sqrt{xy}-y}=\frac{50y+\sqrt{25y^2}+3y}{25y+\sqrt{25y^2}-y}=\frac{58y}{29y}=2$$

$b = 2$

G4.3 Given that the equation $||x-2|-1| = c$ has only 3 integral solutions, find the value of c.

Reference: 2005 FG4.2, 2009 HG9, 2012 FG4.2

$|x-2| - 1 = \pm c$

$\Rightarrow |x-2| = 1\pm c$

In order that it has only 3 integral solutions

$c = 1$

G4.4 If d is the positive real root of the equation $\frac{1}{2}\left\{\frac{1}{2}\left[\frac{1}{2}\left(\frac{1}{2}x^2+2\right)+2\right]+2\right\}=2$,

find the value of d.

$\frac{1}{2}\left\{\frac{1}{2}\left[\frac{1}{2}\left(\frac{1}{2}x^2+2\right)+2\right]+2\right\}=2$

$\Rightarrow\left\{\frac{1}{2}\left[\frac{1}{2}\left(\frac{1}{2}x^2+2\right)+2\right]+2\right\}=4$

$\Rightarrow\frac{1}{2}\left[\frac{1}{2}\left(\frac{1}{2}x^2+2\right)+2\right]=2$

$\left[\frac{1}{2}\left(\frac{1}{2}x^2+2\right)+2\right]=4$

$\Rightarrow\frac{1}{2}\left(\frac{1}{2}x^2+2\right)=2$

$\Rightarrow\frac{1}{2}x^2+2=4$

$\Rightarrow\frac{1}{2}x^2=2$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm2$

d = the positive real root = 2

Individual Events

I1	P	5	I2	P	23	I3	P	4	I4	P	12
	Q	4		Q	4		Q	33		Q	$\frac{2}{3}$
	R	1		R	8		R	3		R	4
	S	62		S	8		S	$3\sqrt{2}$		S	144

Group Events

G1	a	29	G2	a	12	G3	a	334501	G4	α	$\frac{180}{7}$
	b	7		b	6		b	$\frac{1}{3}$		b	$\frac{1}{5}$
	c	100		c	16		c	$1+\sqrt{2}$		c	10
	d	206		d	$\frac{44}{125}$		d	3		d	$\frac{1+\sqrt{5}}{2}$

Individual Event 1

I1.1 Let P be the units digit of $3^{2003} \times 5^{2002} \times 7^{2001}$. Find the value of P.

$3^{2003} \times 7^{2001}$ is an odd number, and the units digit of 5^{2002} is 5; $P = 5$

I1.2 If the equation $(x^2 - x - 1)^{x+P-1} = 1$ has Q integral solutions, find the value of Q.

The equation is $(x^2 - x - 1)^{x+4} = 1$

Either $x^2 - x - 1 = 1$ ....(1) or $x + 4 = 0$ ....(2) or ($x^2 - x - 1 = -1$ and $x + 4$ is even) ......(3)

(1): $x = 2$ or -1; (2): $x = 4$; (3): $x = 0$ or 1 and x is even $\Rightarrow x = 0$ only

Conclusion: $x = -4, -1, 0, 2$

$Q = 4$

I1.3 Let x, y be real numbers and $xy = 1$.

If the minimum value of $\frac{1}{x^4}+\frac{1}{Qy^4}$ is R, find the value of R.

$$\frac{1}{x^4}+\frac{1}{Qy^4}=\frac{1}{x^4}+\frac{1}{4y^4}\ge 2\sqrt{\frac{1}{x^4}\cdot\frac{1}{4y^4}}=1=R \text{ (A.M.} \ge \text{G.M.)}$$

I1.4 Let $x_R, x_{R+1}, \ldots, x_K$ ($K > R$) be $K - R + 1$ distinct positive integers and $x_R + x_{R+1}+\ldots+ x_K = 2003$.

If S is the maximum possible value of K, find the value of S. **(Reference: 2004 HI4)**

$x_1 + x_2 + \ldots + x_K = 2003$

For maximum possible value of K, $x_1 = 1, x_2 = 2, \ldots , x_{K-1} = K - 1$

$1 + 2 + \ldots + K - 1 + x_K = 2003$

$\frac{(K-1)K}{2} + x_K = 2003$, $x_K \ge K$

$2003 \ge \frac{(K-1)K}{2} + K$

$4006 \ge K^2 + K$

$K^2 + K - 4006 \le 0$

$$\left(K-\frac{-1-\sqrt{1+4\times4006}}{2}\right)\left(K-\frac{-1+\sqrt{1+4\times4006}}{2}\right)\le 0$$

$$0\le K\le\frac{-1+\sqrt{1+4\times4006}}{2}$$

$$\frac{-1+\sqrt{1+4\times4006}}{2}\approx\frac{-1+\sqrt{4\times4006}}{2}=\sqrt{4006}-0.5\ge\sqrt{3969}-0.5=\sqrt{63^2}-0.5=62.5$$

Maximum possible $K = 62 = S$

$1 + 2 + \ldots + 62 = 1953 = 2003 - 50$; $1 + 2 + \ldots + 61 + 112 = 2003$

Individual Event 2

I2.1 If the 50^{th} power of a two-digit number $\boldsymbol{P}$ is a 69-digit number, find the value of $\boldsymbol{P}$.
(Given that log 2 = 0.3010, log 3 = 0.4771, log 11 = 1.0414.)

Reference: 1995 HG5 … 37^{100} … 157-digit number, 37^{15} … $\boldsymbol{n}$-digit ….

$P^{50} = y$, $10 < P \le 99$, $10^{68} \le y < 10^{69}$

$P = y^{\frac{1}{50}}$; $10^{68\div 50} < P < 10^{69\div 50}$

$1.34 < \log P < 1.38$

$\log 22 = \log 2 + \log 11 = 1.3424$; $\log 24 = 3\log 2 + \log 3 = 1.3801$

$\log 22 < \log P < \log 24$, $P = 23$

I2.2 The roots of the equation $x^2 + ax - P + 7 = 0$ are α and β, whereas the roots of the equation $x^2 + bx - r = 0$ are $-\alpha$ and $-\beta$. If the positive root of the equation $(x^2+ax-P+7) + (x^2+bx-r) = 0$ is $\boldsymbol{Q}$, find the value of $\boldsymbol{Q}$.

$\alpha + \beta = -a$, $\alpha\beta = -16$; $-\alpha - \beta = -b$, $(-\alpha)(-\beta) = -r$

$\therefore b = -a$, $r = 16$

$(x^2 + ax - P + 7) + (x^2 + bx - r) = 0$ is equivalent to $(x^2 + ax - 16) + (x^2 - ax - 16) = 0$

$2x^2 - 32 = 0$

$x = 4$ or -4

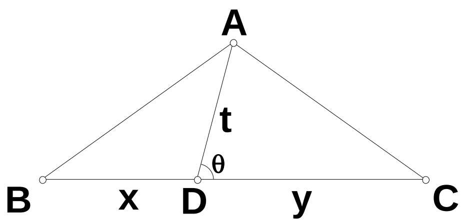
Q = positive root = 4

I2.3 Given that ΔABC is an isosceles triangle, $AB = AC = \sqrt{2}$, and $D_1, D_2, \dots, D_Q$ are Q points on BC. Let $m_i = AD_i^2 + BD_i \times D_iC$. If $m_1 + m_2 + m_3 + \dots + m_Q = R$, find the value of R.

Reference: 2010 HIS

As shown in the figure, $AB = AC = \sqrt{2}$

$BD = x$, $CD = y$, $AD = t$, $\angle ADC = \theta$



Apply cosine formula on ΔABD and ΔACD

$$\cos\theta = \frac{t^2 + y^2 - 2}{2ty}$$

$$\cos(180^\circ - \theta) = \frac{t^2 + x^2 - 2}{2tx}$$

since $\cos(180° - \theta) = -\cos\theta$

Add these equations and multiply by $2txy$:

$x(t^2 + y^2 - 2) + y(t^2 + x^2 - 2) = 0$

$(x + y)t^2 + (x + y)xy - 2(x + y) = 0$

$(x + y)(t^2 + xy - 2) = 0$

$t^2 + xy - 2 = 0$

$AD^2 + BD \cdot DC = 2$

$R = m_1 + m_2 + m_3 + m_4 = 2 + 2 + 2 + 2 = 8$

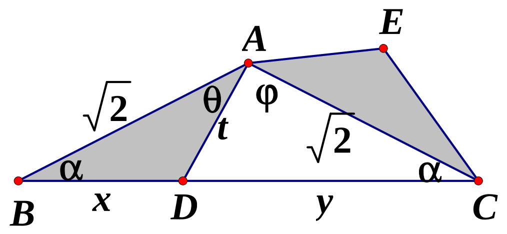
Method 2

Let $BD = x$, $CD = y$, $AD = t$, $\angle ABC = \alpha = \angle ACD$, $\angle BAD = \theta$, $\angle CAD = \varphi$.

Rotate AD anticlockwise about A to AE so that $\angle DAE = \angle BAC$.

$\angle CAE = \angle DAE - \varphi = \angle BAD = \theta$

By the property of rotation, $AE = AD = t$.



$\Delta CAE \cong \Delta BAD$	(S.A.S.)
$CE = BD = x$	(corr. sides, $\cong\Delta$s)
$\angle ACE = \angle ABD = \alpha$	(corr. $\angle$s, $\cong\Delta$s)
$\angle DAE + \angle DCE = \theta + \varphi + 2\alpha = 180°$	($\angle$s sum of Δ)

$\Rightarrow 2\alpha = 180° - (\theta + \varphi) \cdots\cdots\cdots (*)$

The area of $ADCE = S_{\Delta ADE} + S_{\Delta CDE}$ = the area of ΔABC

$$\frac{1}{2}t^2\sin(\varphi + \theta) + \frac{1}{2}xy\sin 2\alpha = \frac{1}{2}\sqrt{2}^2\sin(\varphi + \theta)$$

$t^2\sin(\theta + \varphi) + xy\sin[180° - (\theta + \varphi)] = 2\sin(\theta + \varphi)$ by (*)

$\because \sin[180° - (\theta + \varphi)] = \sin(\theta + \varphi)$ $\therefore t^2 + xy = 2$

$AD^2 + BD \cdot DC = 2$

$R = m_1 + m_2 + m_3 + m_4 = 2 + 2 + 2 + 2 = 8$

I2.4 There are 2003 bags arranged from left to right. It is given that the leftmost bag contains R balls, and every 7 consecutive bags contains 19 balls altogether. If the rightmost bag contains S balls, find the value of S.

The leftmost bag contains 8 balls.

Starting from left to right, the total number of balls from 2^{nd} bag to the 7^{th} bag is 11.

The number of balls in the 8^{th} bag is therefore 8.

Similarly, the number of balls in the 15^{th} bag, 22^{th} bag, 29^{th} bag, … are all 8.

$2003 = 7 \times 286 + 1$, the rightmost bag should have the same number of balls as the leftmost bag.

$S = 8$

Individual Event 3

I3.1 Given that $\begin{cases} wxyz = 4 \\ w - xyz = 3 \end{cases}$ and $w > 0$. If the solution of w is P, find the value of P.

From (2), $xyz = w - 3$.............(3), sub. into (1)
$w(w - 3) = 4$
$w^2 - 3w - 4 = 0$
$w = 4$ or $w = -1$ (rejected)
$P = 4$

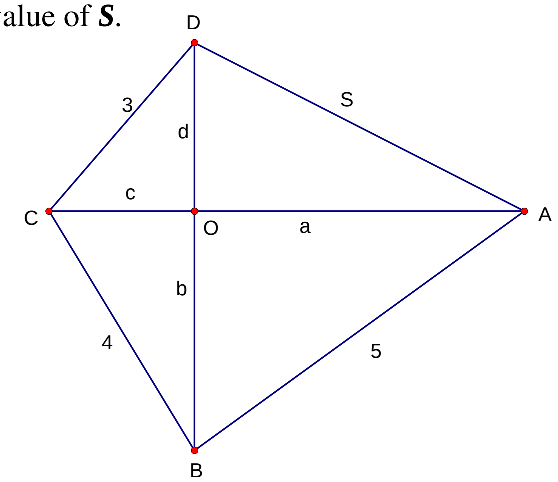
I3.2 Let $[y]$ represents the integral part of the decimal number y. For example, $[3.14] = 3$.
If $\left[\left(\sqrt{2}+1\right)^P\right] = Q$, find the value of Q. **(Reference: HKAL PM 1991 P1 Q11, 2005 HG5)**

Note that $0 < \sqrt{2} - 1 < 1$ and $0 < (\sqrt{2} - 1)^4 < 1$
$(\sqrt{2} + 1)^4 + (\sqrt{2} - 1)^4 = 2(\sqrt{2}^4 + 6\sqrt{2}^2 + 1) = 2(4 + 12 + 1) = 34$
$33 < (\sqrt{2} + 1)^4 < 34$
$Q = \left[\left(\sqrt{2}+1\right)^4\right] = 33$

I3.3 Given that $x_0y_0 \neq 0$ and $Qx_0^2 - 22\sqrt{3}\,x_0y_0 + 11y_0^2 = 0$. If $\dfrac{6x_0^2 + y_0^2}{6x_0^2 - y_0^2} = R$, find the value of R.

$33x_0^2 - 22\sqrt{3}\,x_0y_0 + 11y_0^2 = 0$
$3x_0^2 - 2\sqrt{3}\,x_0y_0 + y_0^2 = 0$
$(\sqrt{3}\,x_0 - y_0)^2 = 0$
$y_0 = \sqrt{3}\,x_0$
$R = \dfrac{6x_0^2 + y_0^2}{6x_0^2 - y_0^2} = \dfrac{6x_0^2 + 3x_0^2}{6x_0^2 - 3x_0^2} = 3$

I3.4 The diagonals AC and BD of a quadrilateral $ABCD$ are perpendicular to each other.
Given that $AB = 5$, $BC = 4$, $CD = R$. If $DA = S$, find the value of S.
Reference 1994 FG10.1-2, 2001 FG2.2, 2018HI7
Suppose AC and BD intersect at O.
Let $OA = a$, $OB = b$, $OC = c$, $OD = d$.
$a^2 + b^2 = 5^2$ ....(1)
$b^2 + c^2 = 4^2$...........(2)
$c^2 + d^2 = 3^2$...........(3)
$d^2 + a^2 = S^2$ ....(4)
(1) + (3) – (2): $S^2 = d^2 + a^2 = 5^2 + 3^2 - 4^2 = 18$
$S = 3\sqrt{2}$



Individual Event 4

I4.1 Suppose the 9-digit number $\overline{32x35717y}$ is a multiple of 72, and $P = xy$, find the value of P.

72 = 8×9, the number is divisible by 8 and 9. **(Reference: 2001 FG1.3, 2017 HI1)**

$\overline{17y}$ is divisible by 8, i.e. $y = 6$.

$3 + 2 + x + 3 + 5 + 7 + 1 + 7 + 6 = 9m$, where m is an integer.

$34 + x = 9m$, $x = 2$

$P = xy = 2\times6 = 12$

I4.2 Given that the lines $4x + y = \frac{P}{3}$, $mx + y = 0$ and $2x - 3my = 4$ cannot form a triangle. Suppose that $m > 0$ and Q is the minimum possible value of m, find Q.

Slope of $L_1 = -4$, slope of $L_2 = -m$, slope of $L_3 = \frac{2}{3m}$

If $L_1 // L_2$: $m = 4$; if $L_2 // L_3$: $m^2 = -\frac{2}{3}$ (no solution); if $L_1 // L_3$: $m = -\frac{1}{6}$ (rejected, $\because m > 0$)

If they are concurrent: $\begin{cases} 4x + y = 4 & \cdots\cdots(1) \\ mx + y = 0 & \cdots\cdots(2) \\ 2x - 3my = 4 & \cdots\cdots(3) \end{cases}$

Solve (1), (2) gives: $x = \frac{4}{4-m}; y = \frac{-4m}{4-m}$

Sub. into (3): $\frac{2\times4}{4-m} - \frac{3m(-4m)}{4-m} = 4$

$3m^2 + m - 2 = 0$

$(m + 1)(3m - 2) = 0$

$m = \frac{2}{3}$ (rejected -1, $\because m > 0$)

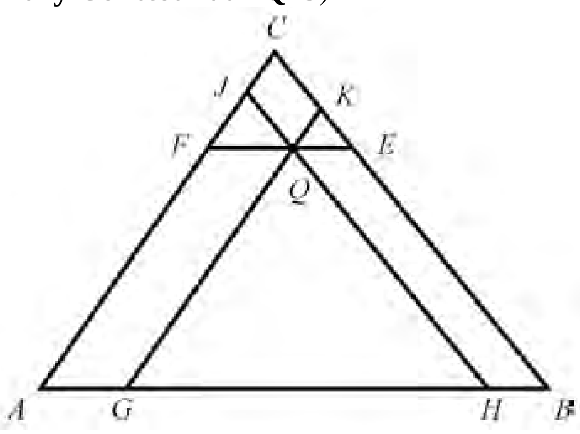
Minimum positive $m = \frac{2}{3}$

I4.3 Given that R, x, y, z are integers and $R > x > y > z$. If R, x, y, z satisfy the equation $2^R + 2^x + 2^y + 2^z = \frac{495Q}{16}$, find the value of R.

$2^R + 2^x + 2^y + 2^z = \frac{495\cdot\frac{2}{3}}{16} = \frac{165}{8} = 20 + \frac{5}{8} = 2^4 + 2^2 + \frac{1}{2} + \frac{1}{2^3}$

$R = 4$

I4.4 In Figure 1, Q is the interior point of ΔABC. Three straight lines passing through Q are parallel to the sides of the triangle such that $FE // AB$, $GK // AC$ and $HJ // BC$. Given that the areas of ΔKQE, ΔJFQ and ΔQGH are R, 9 and 49 respectively. If the area of ΔABC is S, find the value of S. (**Reference: IMO (HK) Preliminary Contest 2001 Q13**)



It is easy to show that all triangles are similar.

By the ratio of areas of similar triangles,

$S_{\Delta KQE} : S_{\Delta JFQ} : S_{\Delta QGH} = (QE)^2 : (FQ)^2 : (GH)^2$

$4 : 9 : 49 = (QE)^2 : (FQ)^2 : (GH)^2$

$QE : FQ : GH = 2 : 3 : 7$

Let $QE = 2t$, $FQ = 3t$, $GH = 7t$

$AFQG$ and $BEQH$ are parallelograms.

$AG = 3t$, $BH = 2t$ (opp. sides of //-gram)

$AB = 3t + 7t + 2t = 12t$

$S_{\Delta ABC} = 4\times\left(\frac{12}{2}\right)^2 = 144 = S$

Group Event 1

G1.1 Given that $\boldsymbol{n}$ and $\boldsymbol{k}$ are natural numbers and $1 < \boldsymbol{k} < \boldsymbol{n}$. If $\frac{(1+2+3+\cdots+\boldsymbol{n})-\boldsymbol{k}}{\boldsymbol{n}-1}=10$ and $\boldsymbol{n} + \boldsymbol{k} = \boldsymbol{a}$, find the value of $\boldsymbol{a}$.

$\frac{\boldsymbol{n}(\boldsymbol{n}+1)}{2}-\boldsymbol{k}=10\boldsymbol{n}-10 \Rightarrow \boldsymbol{n}^2 - 19\boldsymbol{n} + 2(10 - \boldsymbol{k}) = 0$

$\Delta = 281 + 8\boldsymbol{k}$, $\boldsymbol{n}$ is an integer $\Rightarrow \Delta$ is a perfect square.

$281 + 8\boldsymbol{k} = 289, 361, 441, \ldots \Rightarrow \boldsymbol{k} = 1, 10, 20, \ldots$ Given $1 < \boldsymbol{k} < \boldsymbol{n}$, $\therefore \boldsymbol{k} = 10, 20, \ldots$

when $\boldsymbol{k} = 10$, $\boldsymbol{n} = 19$; $\boldsymbol{a} = \boldsymbol{n} + \boldsymbol{k} = 29$; when $\boldsymbol{k} = 20$, $\boldsymbol{n} = 20$ rejected.

G1.2 Given that $(\boldsymbol{x} - 1)^2 + \boldsymbol{y}^2 = 4$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are real numbers. If the maximum value of $2\boldsymbol{x} + \boldsymbol{y}^2$ is $\boldsymbol{b}$, find the value of $\boldsymbol{b}$. **(Reference: 2009 HI5, 2011 HI2)**

$2\boldsymbol{x} + \boldsymbol{y}^2 = 2\boldsymbol{x} + 4 - (\boldsymbol{x} - 1)^2$

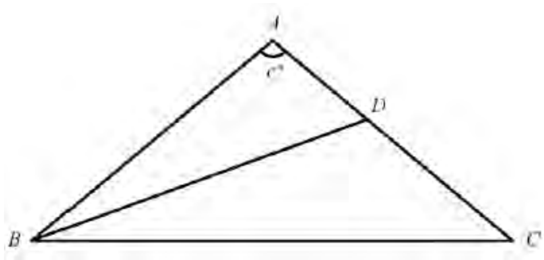
$= -\boldsymbol{x}^2 + 2\boldsymbol{x} - 1 + 2\boldsymbol{x} + 4$

$= -\boldsymbol{x}^2 + 4\boldsymbol{x} + 3$

$= -(\boldsymbol{x}^2 - 4\boldsymbol{x} + 4) + 7$

$= -(\boldsymbol{x} - 2)^2 + 7 \le 7 = \boldsymbol{b}$

G1.3 In Figure 1, $\Delta\boldsymbol{ABC}$ is an isosceles triangle and $\boldsymbol{AB} = \boldsymbol{AC}$. Suppose the angle bisector of $\angle\boldsymbol{B}$ meets $\boldsymbol{AC}$ at $\boldsymbol{D}$ and $\boldsymbol{BC} = \boldsymbol{BD} + \boldsymbol{AD}$. Let $\angle\boldsymbol{A} = \boldsymbol{c}°$, find the value of $\boldsymbol{c}$.

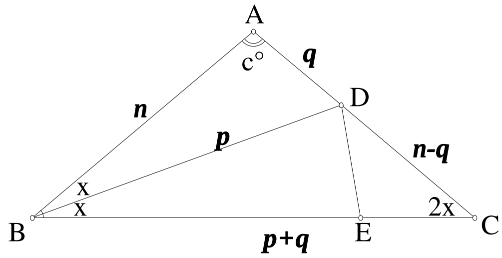


Let $\boldsymbol{AB} = \boldsymbol{n} = \boldsymbol{AC}$; $\boldsymbol{AD} = \boldsymbol{q}$, $\boldsymbol{BD} = \boldsymbol{p}$, $\boldsymbol{CD} = \boldsymbol{n} - \boldsymbol{q}$

$\angle\boldsymbol{ABD} = \boldsymbol{x} = \angle\boldsymbol{CBD}$; $\angle\boldsymbol{ACB} = 2\boldsymbol{x}$.

Let $\boldsymbol{E}$ be a point on $\boldsymbol{BC}$ such that $\boldsymbol{BE} = \boldsymbol{p}$, $\boldsymbol{EC} = \boldsymbol{q}$

Apply sine formula on $\Delta\boldsymbol{ABD}$ and $\Delta\boldsymbol{BCD}$.



$\frac{\boldsymbol{n}}{\sin\angle\boldsymbol{ADB}}=\frac{\boldsymbol{q}}{\sin\boldsymbol{x}};\frac{\boldsymbol{p}+\boldsymbol{q}}{\sin\angle\boldsymbol{BDC}}=\frac{\boldsymbol{n}-\boldsymbol{q}}{\sin\boldsymbol{x}}$

$\because \sin\angle\boldsymbol{ADB} = \sin\angle\boldsymbol{BDC}$

Dividing the above two equations

$\frac{\boldsymbol{n}}{\boldsymbol{p}+\boldsymbol{q}}=\frac{\boldsymbol{q}}{\boldsymbol{n}-\boldsymbol{q}}$

$\frac{\boldsymbol{AB}}{\boldsymbol{BC}}=\frac{\boldsymbol{EC}}{\boldsymbol{CD}}$ and $\angle\boldsymbol{ABC} = \angle\boldsymbol{ECD} = 2\boldsymbol{x}$

$\Delta\boldsymbol{ABC} \sim \Delta\boldsymbol{ECD}$ (2 sides proportional, included angle)

$\therefore \angle\boldsymbol{CDE} = 2\boldsymbol{x}$ (corr. $\angle$s, ~ Δ's)

$\angle\boldsymbol{BED} = 4\boldsymbol{x}$ (ext. $\angle$ of $\Delta\boldsymbol{CDE}$)

$\angle\boldsymbol{BDE} = 4\boldsymbol{x}$ ($\boldsymbol{BD} = \boldsymbol{BE} = \boldsymbol{p}$, base $\angle$s, isos. Δ)

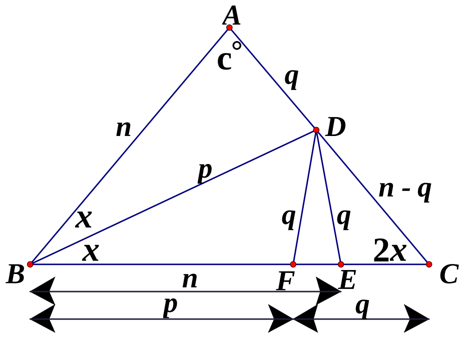
$\angle\boldsymbol{ADB} = 3\boldsymbol{x}$ (ext. $\angle$ of $\Delta\boldsymbol{BCD}$)

$2\boldsymbol{x} + 3\boldsymbol{x} + 4\boldsymbol{x} = 180°$ (adj. $\angle$s on st. line $\boldsymbol{ADC}$)

$\boldsymbol{x} = 20°$

$\boldsymbol{c}° = 180° - 4\boldsymbol{x} = 100°$ ($\angle$ sum of $\Delta\boldsymbol{ABC}$)

Method 2



Claim $c° > 90°$

Proof: Otherwise, either $c° < 90°$ or $c° = 90°$

If $c° < 90°$, then locate a point E on BC so that $BE = n$

$\Delta ABD \cong \Delta EBD$ (S.A.S.)

$DE = q$ (corr. sides $\cong \Delta$s)

$\angle DEB = c°$ (corr. $\angle$s $\cong \Delta$s)

Locate a point F on BE so that $DF = q$

ΔDEF is isosceles

$\angle DFE = c°$ (base $\angle$s isos. Δ) …… (1)

$\angle ABD = x = \angle CBD$, $\angle ACB = 2x$ …… (2)

Consider ΔABC and ΔFCD

$\angle BAC = c° = \angle CFD$ (by (1))

$\angle ABC = 2x = \angle FCD$ (by (2))

$\therefore \Delta ABC \sim \Delta FCD$ (equiangular)

$CF : FD = BA : AC$ (corr. sides, $\sim\Delta$s)

$CF : FD = 1 : 1$ ($\because \Delta ABC$ is isosceles)

$\therefore CF = FD = q$

$BF = BC - CF = (p + q) - q = p$

$\because BF = p = BD$

$\therefore \Delta BDF$ is isosceles

$\angle BFD = \angle BDF$ (base $\angle$s isos. Δ)

$= \dfrac{180° - x}{2}$ ($\angle$ sum of Δ)

$< 90°$

$180° = \angle BFD + \angle EFD < 90° + 90° = 180°$, which is a contradiction

If $c° = 90°$, we use the same working steps as above, with $E = F$.

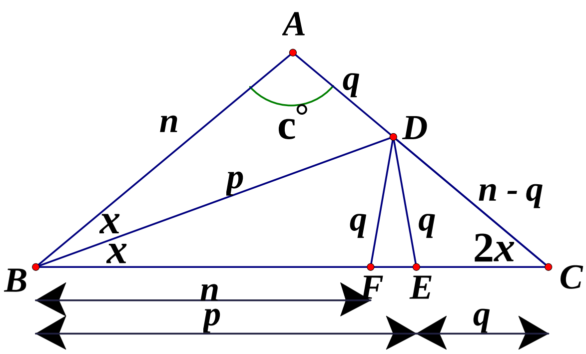
$\Delta ABC \sim \Delta FCD$ (equiangular)

$BE = n = BF = p$

$\therefore \Delta BDF$ is isosceles

$c° = 90° = \angle BFD = \dfrac{180° - x}{2} < 90°$, which is a contradiction

Conclusion: $c° > 90°$



Locate a point F on BC so that $BF = n$

$\Delta ABD \cong \Delta FBD$ (S.A.S.)

$DF = q$ (corr. sides $\cong \Delta$s)

$\angle DFB = c°$ (corr. $\angle$s $\cong \Delta$s)

$\angle DFC = 180° - c° < 90°$ (adj. $\angle$s on st. line)

Locate a point E on FC so that $DE = q$

ΔDEF is isosceles

$\angle DEF = 180° - c°$ (base $\angle$s isos. Δs)

$\angle DEC = c°$ (adj. $\angle$s on st. line) …… (3)

$\angle ABD = x = \angle CBD$; $\angle ACB = 2x$ …… (4)

Consider ΔABC and ΔECD

$\angle BAC = c° = \angle CED$ (by (3))

$\angle ABC = 2x = \angle ECD$ (by (4))

$\therefore \Delta ABC \sim \Delta ECD$ (equiangular)

$CE : ED = BA : AC$ (corr. sides, $\sim\Delta$s)

$CE : ED = 1 : 1$ ($\because \Delta ABC$ is isosceles)

$\therefore CE = ED = q$

$BE = BC - CE = (p + q) - q = p$

$\because BE = p = BD$

$\therefore \Delta BDE$ is isosceles

$\angle BED = \angle BDE = 180° - c°$ (base $\angle$s isos. Δ)

In ΔBDE, $x + 2(180° - c°) = 180°$ ($\angle$ sum of Δ)

$\Rightarrow x = 2c° - 180°$ …… (5)

In ΔABC, $c° + 4x = 180°$ ($\angle$ sum of Δ) …… (6)

Sub. (5) into (6), $c° + 4(2c° - 180°) = 180°$

$c = 100$

G1.4 Given that the sum of two prime numbers is 105. If the product of these prime numbers is d, find the value of d.

"2" is the only prime number which is an even integer.

The sum of two prime number is 105, which is odd

$\Rightarrow$ One prime is odd and the other prime is even

$\Rightarrow$ One prime is odd and the other prime is 2

$\Rightarrow$ One prime is 103 and the other prime is 2

$d = 2\times 103 = 206$

Group Event 2

G2.1 Given that the equation $ax(x+1)+bx(x+2)+c(x+1)(x+2)=0$ has roots 1 and 2. If $a+b+c=2$, find the value of a.

Expand and rearrange the terms in descending orders of x:

$(a+b+c)x^2+(a+2b+3c)x+2c=0$

$2x^2+(a+b+c+b+2c)x+2c=0$

$2x^2+(b+2c+2)x+2c=0$

It is identical to $2(x-1)(x-2)=0$

$\therefore b+2c+2=-6;\ 2c=4$

Solving these equations give $c=2$, $b=-12$, $a=12$

G2.2 Given that $48^x=2$ and $48^y=3$. If $8^{\frac{x+y}{1-x-y}}=b$, find the value of b.

Reference: 2001 HI1, 2004 FG4.3, 2005 HI9, 2006 FG4.3

Take logarithms on the two given equations: $x\log 48=\log 2$, $y\log 48=\log 3$

$$\therefore\ x=\frac{\log 2}{\log 48};\ y=\frac{\log 3}{\log 48}$$

$$\frac{x+y}{1-x-y}=\frac{\frac{\log 2}{\log 48}+\frac{\log 3}{\log 48}}{1-\frac{\log 2}{\log 48}-\frac{\log 3}{\log 48}}$$

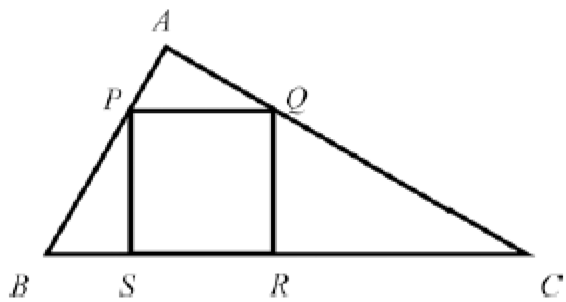
$$=\frac{\log 2+\log 3}{\log 48-\log 2-\log 3}$$

$$=\frac{\log 6}{\log 8}\Rightarrow b=8^{\frac{x+y}{1-x-y}}$$

$$\log b=\log\left(8^{\frac{x+y}{1-x-y}}\right)=\frac{x+y}{1-x-y}\log 8$$

$$=\frac{\log 6}{\log 8}\cdot\log 8=\log 6\Rightarrow b=6$$

G2.3 In Figure 1, the square $PQRS$ is inscribed in ΔABC. The areas of ΔAPQ, ΔPBS and ΔQRC are 4, 4 and 12 respectively. If the area of the square is c, find the value of c.



Let $BC=a$, $PS=x$, the altitude from A onto $BC=h$.

Area of $\Delta BPS=\frac{1}{2}x\cdot BS=4\Rightarrow BS=\frac{8}{x}$

Area of $\Delta CQR=\frac{1}{2}x\cdot CR=12\Rightarrow CR=\frac{24}{x}$

$BC=BS+SR+RC=\frac{8}{x}+x+\frac{24}{x}=x+\frac{32}{x}$ ....(1)

Area of $\Delta APQ=\frac{1}{2}x(h-x)=4\Rightarrow h=\frac{8}{x}+x$ ....(2)

Area of $\Delta ABC=\frac{1}{2}h\cdot BC=4+4+12+x^2=20+x^2$ ....(3)

Sub. (1) and (2) into (3): $\frac{1}{2}\left(\frac{8}{x}+x\right)\cdot\left(x+\frac{32}{x}\right)=20+x^2$

$(8+x^2)(x^2+32)=40x^2+2x^4$

$x^4+40x^2+256=40x^2+2x^4$

$x^4=256$

$c=$ area of the square $=x^2=16$

G2.4 In $\Delta \boldsymbol{ABC}$, $\cos \boldsymbol{A} = \frac{4}{5}$ and $\cos \boldsymbol{B} = \frac{7}{25}$. If $\cos \boldsymbol{C} = \boldsymbol{d}$, find the value of $\boldsymbol{d}$.

(Reference: 2012 FI3.2)

$\sin \boldsymbol{A} = \frac{3}{5}$, $\sin \boldsymbol{B} = \frac{24}{25}$

$\cos \boldsymbol{C} = \cos(180° - \boldsymbol{A} - \boldsymbol{B}) = -\cos(\boldsymbol{A} + \boldsymbol{B}) = -\cos \boldsymbol{A} \cos \boldsymbol{B} + \sin \boldsymbol{A} \sin \boldsymbol{B}$

$= -\frac{4}{5} \cdot \frac{7}{25} + \frac{3}{5} \cdot \frac{24}{25} = \frac{44}{125}$

Group Event 3

G3.1 Let f be a function such that $f(1) = 1$ and for any integers m and n, $f(m + n) = f(m) + f(n) + mn$. If $a = \frac{f(2003)}{6}$, find the value of a.

$f(n+1) = f(n) + n + 1 = f(n-1) + n + n + 1 = f(n-2) + n-1 + n + n+1 = \ldots = 1 + 2 + \ldots + n + n+1$

$\therefore f(n) = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$

$\frac{f(2003)}{6} = \frac{2004\times 2003}{12} = 334501$

G3.2 Suppose $x^{\frac{1}{2}} + x^{-\frac{1}{2}} = 3$, $b = \frac{x^{\frac{3}{2}} + x^{-\frac{3}{2}} - 3}{x^2 + x^{-2} - 2}$, find the value of b.

$\left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)^2 = 9 \Rightarrow x + x^{-1} = 7 \Rightarrow \left(x + x^{-1}\right)^2 = 49 \Rightarrow x^2 + x^{-2} = 47$

$\left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)\left(x + x^{-1}\right) = 3\times 7 \Rightarrow x^{\frac{3}{2}} + x^{-\frac{3}{2}} + x^{\frac{1}{2}} + x^{-\frac{1}{2}} = 21 \Rightarrow x^{\frac{3}{2}} + x^{-\frac{3}{2}} = 18$

$b = \frac{x^{\frac{3}{2}} + x^{-\frac{3}{2}} - 3}{x^2 + x^{-2} - 2} = \frac{18-3}{47-2} = \frac{1}{3}$

G3.3 Given that $f(n) = \sin\frac{n\pi}{4}$, where n is an integer. If $c = f(1) + f(2) + \ldots + f(2003)$, find the value of c.

$f(1) + f(2) + f(3) + f(4) + f(5) + f(6) + f(7) + f(8)$

$= \frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}} + 0 - \frac{1}{\sqrt{2}} - 1 - \frac{1}{\sqrt{2}} + 0 = 0$

and the function repeats for every multiples of 8.

$c = f(2001) + f(2002) + f(2003) = \frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}} = 1 + \sqrt{2}$

G3.4 Given that $f(x) = \begin{cases} -2x+1, \text{ when } x<1 \\ x^2 - 2x, \text{ when } x \ge 1 \end{cases}$. If d is the maximum integral solution of $f(x) = 3$, find the value of d.

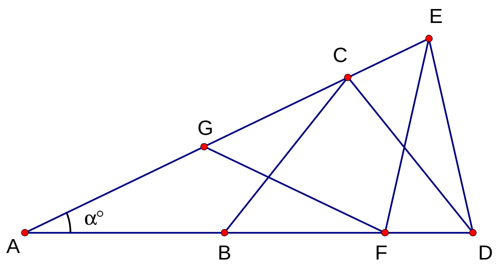
When $x \ge 1$, $f(x) = 3 \Rightarrow x^2 - 2x = 3 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow x = 3$ or -1 (rejected)

When $x < 1$, $-2x + 1 = 3 \Rightarrow 2x = -2 \Rightarrow x = -1$

$\therefore d = 3$

Group Event 4

G4.1 In Figure 1, AE and AD are two straight lines and $AB = BC = CD = DE = EF = FG = GA$.
If $\angle DAE = \alpha°$, find the value of α.



$\angle AFG = \alpha° = \angle ACB$ (base ∠s isos. Δ)
$\angle CBD = 2\alpha° = \angle FGE$ (ext. ∠ of Δ)
$\angle FEG = 2\alpha° = \angle BDC$ (base ∠s isos. Δ)
$\angle DFE = 3\alpha° = \angle DCE$ (ext. ∠ of Δ)
$\angle ADE = 3\alpha° = \angle AED$ (base ∠s isos. Δ)
$\alpha° + 3\alpha° + 3\alpha° = 180°$ (∠s sum of Δ)
$\alpha = \frac{180}{7}$

G4.2 Suppose $P(x) = a_0 + a_1x + a_2x^2 + \dots + a_8x^8$ is a polynomial of degree 8 with real coefficients $a_0, a_1, \dots, a_8$. If $P(k) = \frac{1}{k}$ when $k = 1, 2, \dots, 9$, and $b = P(10)$, find the value of b.

Reference: 2018 HG1

$P(k) = \frac{1}{k}$, for $k = 1, 2, \dots, 9$.
Let $F(x) = x\,P(x) - 1$, then $F(k) = k\,P(k) - 1 = 0$, for $k = 1, 2, \dots, 9$.
$F(x)$ is a polynomial of degree 9 and the roots are 1, 2, … , 9.
$F(x) = x\,P(x) - 1 = c(x-1)(x-2)\dots(x-9)$
$P(x) = \frac{c(x-1)(x-2)\cdots(x-9)+1}{x}$, which is a polynomial of degree 8.
Compare the constant term of $c(x-1)(x-2)\dots(x-9) + 1 = 0$:
$-c\cdot 9! + 1 = 0$
$c = \frac{1}{9!} \Rightarrow P(x) = \frac{(x-1)(x-2)\cdots(x-9)+9!}{9!x}$
$P(10) = \frac{9!+9!}{9!\cdot 10} = \frac{1}{5}$

G4.3 Given two positive integers x and y, $xy - (x + y) = \text{HCF}(x, y) + \text{LCM}(x, y)$, where HCF$(x, y)$ and LCM(x, y) are respectively the greatest common divisor and the least common multiple of x and y. If c is the maximum possible value of $x + y$, find c.
Without loss of generality assume $x \ge y$.
Let the H.C.F. of x and y be m and $x = ma$, $y = mb$ where the H.C.F. of a, b is 1.
L.C.M. of x and $y = mab$. $a \ge b$.
$xy - (x + y) = \text{HCF} + \text{LCM} \Rightarrow m^2ab - m(a + b) = m + mab$
$ab(m - 1) = a + b + 1$
$m - 1 = \frac{1}{a} + \frac{1}{b} + \frac{1}{ab}$
$1 \le m - 1 = \frac{1}{a} + \frac{1}{b} + \frac{1}{ab} \le 3$
$m = 2, 3$ or 4
when $m = 2$, $\frac{1}{a} + \frac{1}{b} + \frac{1}{ab} = 1 \Rightarrow a + b + 1 = ab \Rightarrow ab - a - b - 1 = 0$
$ab - a - b + 1 = 2$
$(a - 1)(b - 1) = 2$
$a = 3, b = 2, m = 2, x = 6, y = 4, c = x + y = 10$
When $m = 3$, $\frac{1}{a} + \frac{1}{b} + \frac{1}{ab} = 2 \Rightarrow a + b + 1 = 2ab \Rightarrow 2ab - a - b - 1 = 0$

$4ab - 2a - 2b + 1 = 3$

$(2a - 1)(2b - 1) = 3$

$a = 2, b = 1, m = 3, x = 6, y = 3, c = x + y = 9$

When $m = 4$, $\frac{1}{a}+\frac{1}{b}+\frac{1}{ab} = 3 \Rightarrow a + b + 1 = 3ab \Rightarrow 3ab - a - b - 1 = 0$

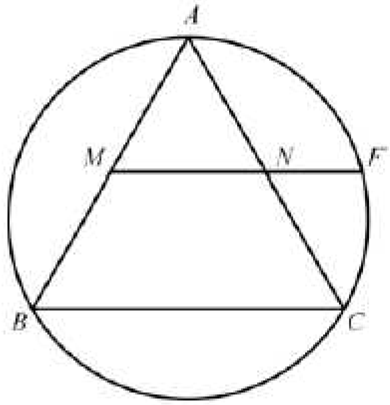
$9ab - 3a - 3b + 1 = 4$

$(3a - 1)(3b - 1) = 4$

$a = 1, b = 1, m = 4, x = 4, y = 4, c = x + y = 8$

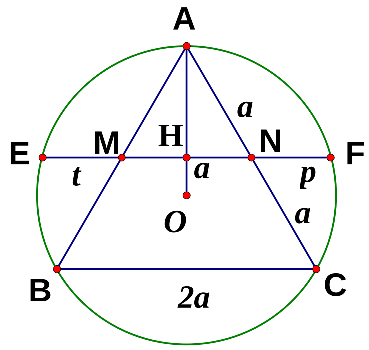
Maximum $c = 10$

G4.4 In Figure 2, ΔABC is an equilateral triangle, points M and N are the midpoints of sides AB and AC respectively, and F is the intersection of the line MN with the circle ABC.



If $d = \frac{MF}{MN}$, find the value of d.

Let O be the centre, join AO.
Suppose MN intersects AO at H.
Produce FNM to meet the circle at E.
Then it is easy to show that:
$MN \parallel BC$ (mid-point theorem)
$\Delta AMO \cong \Delta ANO$ (SSS)
$\Delta AMH \cong \Delta ANH$ (SAS)
$AO \perp MN$ and $MH = HN$ ($\cong \Delta$'s)
$EH = HF$ ($\perp$ from centre bisect chords)
Let $EM = t$, $MN = a$, $NF = p$.
$t = EH - MH = HF - HN = p$
By intersecting chords theorem,
$AN \times NC = FN \times NE$



$a^2 = p(p + a)$

$p^2 + ap - a^2 = 0$

$$\left(\frac{p}{a}\right)^2 + \frac{p}{a} - 1 = 0$$

$$\frac{p}{a} = \frac{-1+\sqrt{5}}{2} \text{ or } \frac{-1-\sqrt{5}}{2} \text{ (rejected)}$$

$$d = \frac{MF}{MN} = \frac{a+p}{a}$$

$$= 1 + \frac{-1+\sqrt{5}}{2}$$

$$= \frac{1+\sqrt{5}}{2}$$

Individual Events

I1	***a***	6	**I2**	***P***	2	**I3**	***a***	–2	**I4**	***a***	2	**IS**	***P***	84
	b	$2+4\sqrt{2}$		***Q***	6		***b***	9		***b***	11		***Q***	8
	c	7		***R***	56		***c***	$\frac{1}{24}$		***c***	462		***R***	$\frac{\sqrt{3}}{4}$
	d	11		***S***	2352		***d***	$-\frac{7}{18}$		***d***	*334 see the remark		***S***	$1+\sqrt{2}$

Group Events

G1	***a***	47	**G2**	***a***	*2 see the remark	**G3**	***a***	–10	**G4**	***P***	500	**GS**	***a***	16
	b	101		***b***	4.5		***b***	0		***Q***	15		***b***	1
	c	43		***c***	15		***c***	2005		***R***	$\frac{1}{2}$		***c***	12
	d	0		***d***	1		***d***	2005		***S***	1		***d***	9

Individual Event 1

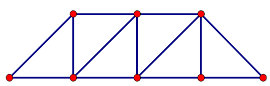
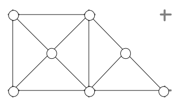
I1.1 Given that there are a positive integers less than 200 and each of them has exactly three positive factors, find the value of a.

If $x = rs$, where r and s are positive integers, then the positive factors of x may be 1, r, s and x.

In order to have exactly three positive factors, $r = s =$ a prime number.

Possible $x = 4, 9, 25, 49, 121, 169$. $a = 6$.

I1.2 If a copies of a right-angled isosceles triangle with hypotenuse $\sqrt{2}$ cm can be assembled to form a trapezium with perimeter equal to b cm, find the least possible value of b. (give the answer in surd form).


or


The perimeter $= 6 + 2\sqrt{2} \approx 8.8$ or $2 + 4\sqrt{2} \approx 7.7$

The least possible value of $b = 2 + 4\sqrt{2}$

I1.3 If $\sin(c^2 - 3c + 17)° = \frac{4}{b-2}$, where $0 < c^2 - 3c + 17 < 90$ and $c > 0$, find the value of c.

$\sin(c^2 - 3c + 17)° = \frac{4}{2+4\sqrt{2}-2} = \frac{1}{\sqrt{2}}$

$c^2 - 3c + 17 = 45$

$c^2 - 3c - 28 = 0$

$(c - 7)(c + 4) = 0$

$c = 7$ or -4 (rejected)

I1.4 Given that the difference between two 3-digit numbers $\overline{xyz}$ and $\overline{zyx}$ is $700 - c$, where $x > z$. If d is the greatest value of $x + z$, find the value of d.

$\overline{xyz} - \overline{zyx} = 700 - c$

$100x + 10y + z - (100z + 10y + x) = 700 - 7$

$99x - 99z = 693$

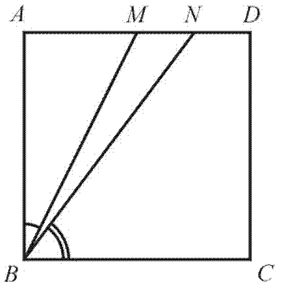
$x - z = 7$

Possible answers: $x = 8, z = 1$ or $x = 9, z = 2$

d is the greatest value of $x + z = 9 + 2 = 11$

Individual Event 2

I2.1 In Figure 1, *ABCD* is a square, *M* is the mid-point of *AD* and *N* is the mid-point of *MD*. If $\angle CBN : \angle MBA = P : 1$, find the value of *P*.



Let $\angle ABM = \theta$, $\angle CBM = P\theta$. Let $AB = 4$, $AM = 2$, $MN = 1 = ND$.

$$\tan\theta = \frac{1}{2}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2} = \frac{4}{3} = \tan P\theta,\ P = 2$$

I2.2 Given that *ABCD* is a rhombus on a Cartesian plane, and the co-ordinates of its vertices are *A*(0, 0), *B*(*P*, 1), *C*(*u*, *v*) and *D*(1, *P*) respectively. If $u + v = Q$, find the value of *Q*.

$\because$ *ABCD* is a rhombus, $\therefore$ It is also a parallelogram

By the property of parallelogram, the diagonals bisect each other

Mid point of *B*, *D* = mid point of *AC*

$$\left(\frac{1+2}{2},\frac{2+1}{2}\right) = \left(\frac{0+u}{2},\frac{0+v}{2}\right)$$

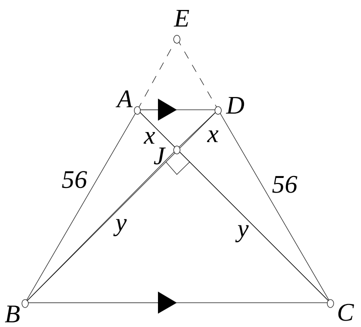
$u = 3$, $v = 3 \Rightarrow Q = u + v = 6$

I2.3 If $1 + (1 + 2) + (1 + 2 + 3) + \ldots + (1 + 2 + 3 + \ldots + Q) = R$, find the value of *R*.

$R = 1 + (1 + 2) + (1 + 2 + 3) + \ldots + (1 + 2 + 3 + \ldots + 6)$

$R = 1 + 3 + 6 + 10 + 15 + 21 = 56$

I2.4 In the figure, *EBC* is an equilateral triangle, and *A*, *D* lie on *EB* and *EC* respectively. Given that *AD*//*BC*, $AB = CD = R$ and $AC \perp BD$. If the area of the trapezium *ABCD* is *S*, find the value of *S*.



$\angle ABC = \angle BCD = 60°$, *AC* intersects *BD* at *J*, $AC \perp BD$.

$\Delta ACD \cong \Delta DBA$ (S.A.S.)

$AC = BD$ (corr. sides, $\cong$ Δ's)

$\angle ABD = \angle DCA$ (corr. $\angle$s, $\cong$ Δ's)

$\therefore \angle JBC = 60° - \angle ABD = \angle JCB$

Δ*JBC* is a right-angled isosceles triangle.

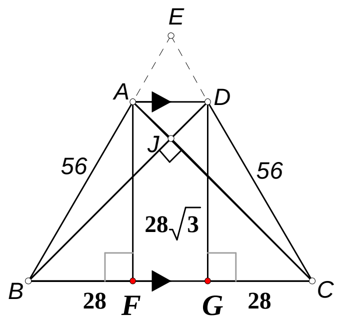
$\angle JBC = \angle JCB = 45°$

$BJ = CJ = y$, $AJ = AC - y = BD - y = DJ = x$

$\angle JAD = \angle JDA = 45°$, $\angle ADC = 120°$

Apply sine formula on Δ*ACD*, $\frac{56}{\sin 45^\circ} = \frac{x+y}{\sin 120^\circ}$

$x + y = 28\sqrt{6}$, area $= \frac{1}{2}(x+y)^2\sin 90^\circ = \frac{1}{2}(784)(6) = 2352$



Method 2 $\angle EBC = 60° = \angle ECB$

Draw $AF \perp BC$ and $DG \perp BC$, cutting *BC* at *F* and *G*.

$BF = CG = 56\cos 60° = 28$

$AF = DG = 56\sin 60° = 28\sqrt{3}$

AC intersects *BD* at *J*, $AC \perp BD$.

$\Delta ACD \cong \Delta DBA$ (S.A.S.)

$\angle ABD = \angle DCA$ (corr. $\angle$s, $\cong$ Δ's)

$\therefore \angle JBC = 60° - \angle ABD = \angle JCB$

Δ*JBC* is a right-angled isosceles triangle.

$\angle JBC = \angle JCB = 45°$

$CF = AF\cot 45° = 28\sqrt{3}$

$FG = CF - CG = 28\left(\sqrt{3}-1\right) = AD$

Area of the trapezium $ABCD = S = \frac{28\left(\sqrt{3}-1\right)\times 2 + 28\times 2}{2}\cdot\left(28\sqrt{3}\right) = 2352$

Individual Event 3

I3.1 Let $x\neq\pm1$ and $x\neq-3$.If a is the real root of the equation $\frac{1}{x-1}+\frac{1}{x+3}=\frac{2}{x^2-1}$,find the value of a

$$\frac{1}{x+3}=\frac{2}{x^2-1}-\frac{1}{x-1}\Rightarrow\frac{1}{x+3}=\frac{2-(x+1)}{(x-1)(x+1)}$$

$$\frac{1}{x+3}=\frac{1-x}{(x-1)(x+1)}\Rightarrow\frac{1}{x+3}=-\frac{1}{x+1}$$

$x+1=-x-3\Rightarrow x=-2=a$

I3.2 If $b>1$，$f(b)=\frac{-a}{\log_2 b}$ and $g(b)=1+\frac{1}{\log_3 b}$. If b satisfies the equation $|f(b)-g(b)|+f(b)+g(b)=3$, find the value of b.

Similar question: 2007 HI9

$$\left|\frac{2\log 2}{\log b}-1-\frac{\log 3}{\log b}\right|+\frac{2\log 2}{\log b}+1+\frac{\log 3}{\log b}=3\Rightarrow\left|\frac{\log \frac{4}{3b}}{\log b}\right|+\frac{\log 12}{\log b}=2\Rightarrow\left|\log\frac{4}{3b}\right|=\log\frac{b^2}{12}$$

$$\log\frac{4}{3b}=\pm\log\frac{b^2}{12}$$

$$\log\frac{4}{3b}=\log\frac{b^2}{12}\quad\text{or}\quad\log\frac{4}{3b}=\log\frac{12}{b^2}$$

$b^3=16$ or $b=9$

When $b^3=16$, $(b^3)^2=256<1728=12^3\Rightarrow b^2<12\Rightarrow\left|\log\frac{4}{3b}\right|=\log\frac{b^2}{12}<0$ rejected.

When $b=9$, $\left|\log\frac{4}{3b}\right|=\log\frac{b^2}{12}=\log\frac{81}{12}>0$ accepted.

Method 2

Define the maximum function of x, y as: $\text{Max}(x,y)=\frac{x+y+|x-y|}{2}$

Similarly, the minimum function of x, y is: $\text{Min}(x,y)=\frac{x+y-|x-y|}{2}$

$$f(b)=\frac{-a}{\log_2 b}=\frac{2\log 2}{\log b}=\frac{\log 4}{\log b},\ g(b)=1+\frac{1}{\log_3 b}=\frac{\log b}{\log b}+\frac{\log 3}{\log b}=\frac{\log 3b}{\log b}$$

The given equation is equivalent to 2 Max($f(b)$, $g(b)$) = 3

If $f(b)>g(b)$, i.e. $\frac{\log 4}{\log b}>\frac{\log 3b}{\log b}\Rightarrow b<\frac{4}{3}$, then the equation is $2f(b)=3$

$$\frac{2\log 4}{\log b}=3\Rightarrow\log 16=\log b^3\Rightarrow b^3=16<\left(\frac{4}{3}\right)^3\Rightarrow 1<\frac{4}{27}$$

$\Rightarrow 27<4$, which is a contradiction; $\therefore$ rejected

If $f(b)\leq g(b)$, the equation is equivalent to $2\,g(b)=3$

i.e. $\frac{2\log 3b}{\log b}=3\Rightarrow\log 9b^2=\log b^3\Rightarrow 9b^2=b^3\Rightarrow b=9$

I3.3 Given that x_0 satisfies the equation $x^2-5x+(b-8)=0$. If $c=\frac{x_0^2}{x_0^4+x_0^2+1}$, find the value of c.

$x^2-5x+1=0$, $x^2+1=5x$, $x^4+2x^2+1=25x^2$, $x^4+x^2+1=24x^2$

$$c=\frac{x_0^2}{x_0^4+x_0^2+1}=\frac{x_0^2}{24x_0^2}=\frac{1}{24}$$

I3.4 If –2 and 216c are the roots of the equation $px^2 + dx = 1$, find the value of d.

–2 and 9 are roots of $px^2 + dx - 1 = 0$

Product of roots $= -\dfrac{1}{p} = -2\times 9 \Rightarrow p = \dfrac{1}{18}$

Sum of roots $= -\dfrac{d}{p} = -2+9 \Rightarrow -18d = 7 \Rightarrow d = -\dfrac{7}{18}$

Individual Event 4

I4.1 Let a be a real number.

If a satisfies the equation $\log_2(4^x + 4) = x + \log_2(2^{x+1} - 3)$, find the value of a.

$\log_2(4^x + 4) = \log_2 2^x + \log_2(2^{x+1} - 3)$

$4^x + 4 = 2^x\cdot(2^{x+1} - 3)$

$(2^x)^2 + 4 = 2\cdot(2^x)^2 - 3\cdot 2^x$

$0 = (2^x)^2 - 3\cdot 2^x - 4$

$(2^x - 4)(2^x + 1) = 0$

$2^x = 4$, $x = 2 = a$

I4.2 Given that n is a natural number.

If $b = n^3 - 4an^2 - 12n + 144$ is a prime number, find the value of b. **(Reference: 2011 FI3.3)**

Let $f(n) = n^3 - 8n^2 - 12n + 144$

$f(6) = 6^3 - 8\cdot 6^2 - 12\cdot 6 + 144 = 216 - 288 - 72 + 144 = 0$

$\therefore f(6)$ is a factor

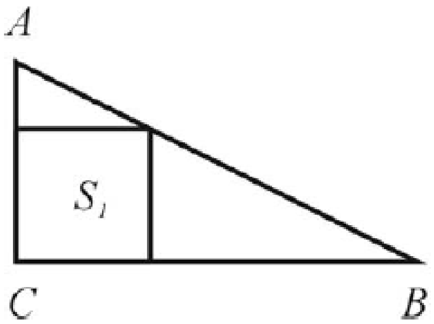
By division, $f(n) = (n - 6)(n - 6)(n + 4)$

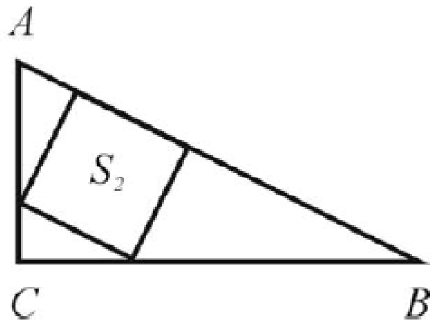
$b = n^3 - 8n^2 - 12n + 144$, it is a prime $\Rightarrow n - 6 = 1$, $n = 7$, $b = 11$

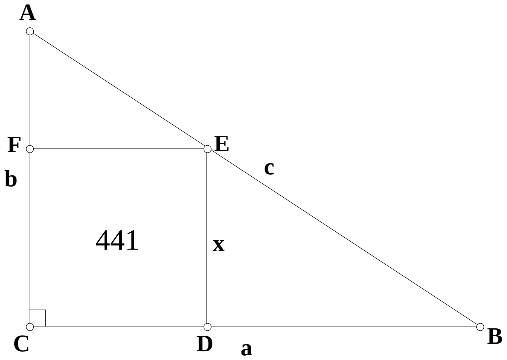
I4.3 In Figure 1, S_1 and S_2 are two different inscribed squares of the right-angled triangle ABC.

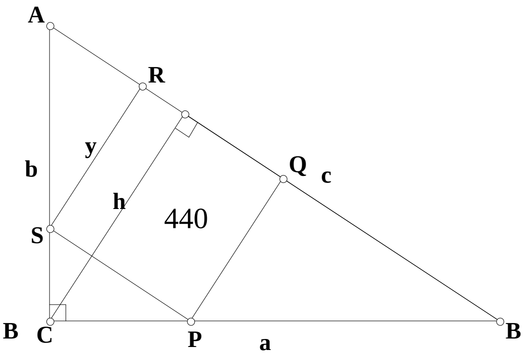
If the area of S_1 is $40b + 1$, the area of S_2 is $40b$ and $AC + CB = c$, find the value of c.

Reference: American Invitation Mathematics Examination 1987 Q15









Add the label D, E, F, P, Q, R, S as shown. $CDEF$, $PQRS$ are squares.

Let $DE = x$, $SR = y$, then $x = \sqrt{441} = 21$, $y = \sqrt{440}$. Let $BC = a$, $AC = b$, $AB = c = \sqrt{a^2+b^2}$.

Let the height of the triangle drawn from C onto AB be h, then $ab = ch = 2$ area of Δ .....(*)

$\Delta AFE \sim \Delta ACB$: $\frac{b-x}{b} = \frac{x}{a} \Rightarrow x = \frac{ab}{a+b} = 21$ ....(1)

$\Delta CSP \sim \Delta CAB$: $\frac{\text{height of } \Delta\text{CSP from C}}{SP} = \frac{h}{c} \Rightarrow \frac{h-y}{y} = \frac{h}{c} \Rightarrow y = \frac{ch}{c+h} = \sqrt{440}$

By (*), $\frac{ab}{c+\frac{ab}{c}} = \sqrt{440} \Rightarrow \sqrt{440} = \frac{ab\sqrt{a^2+b^2}}{a^2+ab+b^2}$ ......(2)

From (1) $ab = 21(a + b)$ .... (3), sub. (3) into (2):

$$\sqrt{440} = \frac{21(a+b)\sqrt{(a+b)^2-2ab}}{(a+b)^2-ab} = \frac{21(a+b)\sqrt{(a+b)^2-42(a+b)}}{(a+b)^2-21(a+b)} = \frac{21\sqrt{(a+b)^2-42(a+b)}}{(a+b)-21}$$

Cross multiplying and squaring both sides:

$440[(a + b)^2 – 42(a + b) + 441] = 441[(a + b)^2 – 42(a + b)]$

$(a + b)^2 – 42(a + b) – 440\times441 = 0$

$(a + b – 462)(a + b + 420) = 0$

$AC + CB = a + b = 462$

I4.4 Given that $241c + 214 = d^2$, find the positive value of d.

$d^2 = 241\times462 + 214 = 111556$

$d = \sqrt{111556}$

Reference: 昌爸工作坊圖解直式開平方

Divide 111556 into 3 groups of numbers 11, 15, 56.

Find the maximum integer p such that $p^2 \le 11 \Rightarrow p = 3$

$11 – p^2 = 2$

$3 + 3 = 6$

Find the maximum integer q such that $(60 + q)q \le 215$

$\Rightarrow q = 3$

$215 – 63\times3 = 26$

$60 + q + q = 66$

Find the maximum integer r such that $(660 + r)r \le 2656$

$\Rightarrow r = 4$

$d = 334$

Method 2: $d^2 = 241\times462 + 214 = 111556$

$300^2 = 90000 < 111556 < 160000 = 400^2 \Rightarrow 300 < d < 400$

$330^2 = 108900 < 111556 < 115600 = 340^2 \Rightarrow 330 < d < 340$

The unit digit of d^2 is 6 $\Rightarrow$ the unit digit of d is 4 or 6

$335^2 = 112225 \Rightarrow d = 334$

or 111556 is not divisible by 3, but 336 is divisible by 3

$\therefore d = 334$

Method 3

Observe the number patterns:

$34^2 = 1156$

$334^2 = 111556$

$3334^2 = 11115556$

....

$\therefore d = 334$

Also, $33^2 = 1089$

$333^2 = 110889$

$3333^2 = 11108889$

....

$3\times4 = 12$

$33\times34 = 1122$

$333\times334 = 111222$

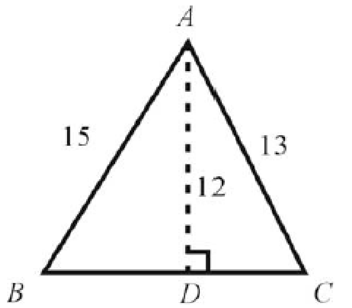
$3333\times3334 = 11112222$

....

Remark: Original question: ⋯⋯, find the value of d. $\Rightarrow d = \pm334$

Individual Event (Spare)

IS.1 In figure 1, ΔABC is an acute triangle, $AB = 15$, $AC = 13$, and its altitude $AD = 12$. If the area of the ΔABC is P, find the value of P.



$BD = \sqrt{15^2 - 12^2} = 9$

$CD = \sqrt{13^2 - 12^2} = 5$

$P = \text{area of } \Delta = \frac{1}{2}(9+5)\times 12 = 84$

IS.2 Given that x and y are positive integers. If $x^4 + y^4$ is divided by $x + y$, the quotient is $P + 13$ and the remainder is Q, find the value of Q.

$P + 13 = 84 + 13 = 97$

$x^4 + y^4 = 97(x + y) + Q$, $0 \le Q < x + y$

Without loss of generality, assume $x \ge y$,

$x^4 < x^4 + y^4 = 97(x + y) + Q < 98(x + y) \le 98(2x) = 196x$

$x^3 < 196$

$x \le 5$

On the other hand, $x^4 + y^4 = 97(x + y) + Q = x^3(x + y) - y(x^3 - y^3)$

$\Rightarrow (x^3 - 97)(x + y) = y(x^3 - y^3) + Q$

$\text{RHS} \ge 0 \Rightarrow \text{LHS} \ge 0 \Rightarrow x^3 \ge 97$

$x \ge 5$

$\therefore x = 5$

$5^4 + y^4 = 625 + y^4$ is divided by $5 + y$, the quotient is 97 and the remainder is Q, $1 \le y \le 5$

$625 + y^4 = 97(5 + y) + Q$, $0 \le Q < 5 + y \le 10$

$y^4 + 140 = 97y + Q$

$97y \le y^4 + 140 \le 97y + 9$

$131 \le 97y - y^4 \le 140$

By putting $y = 1, 2, 3, 4$ and 5 into the above inequalities, only $y = 4$ satisfies it.

$5^4 + 4^4 = 881 = (5 + 4)\times 97 + 8$, $Q = 8$

IS.3 Given that the perimeter of an equilateral triangle equals to that of a circle with radius $\frac{12}{Q}$ cm. If the area of the triangle is $R\pi^2$ cm^2, find the value of R.

Radius of circle $= \frac{12}{8}$ cm $= 1.5$ cm $\Rightarrow$ circumference $= 2\times\pi\times 1.5$ cm $= 3\pi$ cm

Side of the equilateral triangle $= \pi$ cm

Area of the triangle $= \frac{1}{2}\pi^2 \sin 60^\circ \text{ cm}^2 = \frac{\sqrt{3}}{4}\pi^2 \text{ cm}^2 \Rightarrow R = \frac{\sqrt{3}}{4}$

IS.4 Let $W = \frac{\sqrt{3}}{2R}$, $S = W + \cfrac{1}{W + \cfrac{1}{W + \cfrac{1}{W + \cdots}}}$, find the value of S.

$W = 2$, $S = 2 + \frac{1}{S} \Rightarrow S^2 - 2S - 1 = 0$, $S = \frac{1 \pm \sqrt{2}}{2}$, $S > 0$, $\therefore S = 1 + \sqrt{2}$ only

Group Event 1

G1.1 Given that a is an integer. If 50! is divisible by 2^a, find the largest possible value of a.

2, 4, 6, 8, … , 50 are divisible by 2, there are 25 even integers.

Reference: 1990 HG6, 1994 FG7.1, 1996 HI3, 2004 FG1.1, 2011 HG7, 2012 FI1.4, 2012 FG1.3

4, 8, … , 48 are divisible by 4, there are 12 multiples of 4.

8, …, 48 are divisible by 8, there are 6 multiples of 8.

16, …, 48 are divisible by 16, there are 3 multiples of 16.

32 is the only multiple of 32.

$a = 25 + 12 + 6 + 3 + 1 = 47$

G1.2 Let $[x]$ be the largest integer not greater than x. For example, $[2.5] = 2$.

If $b = \left[100\times\frac{11\times77+12\times78+13\times79+14\times80}{11\times76+12\times77+13\times78+14\times79}\right]$, find the value of b.

$$100\times\frac{11\times77+12\times78+13\times79+14\times80}{11\times76+12\times77+13\times78+14\times79}$$

$$=100\times\frac{11\times76+12\times77+13\times78+14\times79+11+12+13+14}{11\times76+12\times77+13\times78+14\times79}$$

$$=100\times\left(1+\frac{11+12+13+14}{11\times76+12\times77+13\times78+14\times79}\right)$$

$$= 100 + \frac{11\times100+12\times100+13\times100+14\times100}{11\times76+12\times77+13\times78+14\times79}$$

$11\times76+12\times77+13\times78+14\times79 < 11\times100+12\times100+13\times100+14\times100 < 2(11\times76+12\times77+13\times78+14\times79)$

$$1<\frac{11\times77+12\times78+13\times79+14\times80}{11\times76+12\times77+13\times78+14\times79}<2$$

$$101<100\times\frac{11\times77+12\times78+13\times79+14\times80}{11\times76+12\times77+13\times78+14\times79}<102,\ b = 101$$

G1.3 If there are c multiples of 7 between 200 and 500, find the value of c.

$\frac{200}{7} = 28.6$, the least multiple of 7 is $7\times29 = 203$

$\frac{500}{7} = 71.4$, the greatest multiple of 7 is $7\times71 = 497$

$497 = a + (c - 1)d = 203 + (n - 1)\cdot7$; $c = 43$

G1.4 Given that $0 \le x_0 \le \frac{\pi}{2}$ and x_0 satisfies the equation $\sqrt{\sin x+1}-\sqrt{1-\sin x}=\sin\frac{x}{2}$. If $d = \tan x_0$, find the value of d.

$$\left(\sqrt{\sin x+1}-\sqrt{1-\sin x}\right)^2=\left(\sin\frac{x}{2}\right)^2$$

$$1 + \sin x + 1 - \sin x - 2\sqrt{1-\sin^2 x}=\frac{1-\cos x}{2}$$

$2(2 - 2 \cos x) = 1 - \cos x$

$\cos x = 1$

$x_0 = 0$

$d = \tan x_0 = 0$

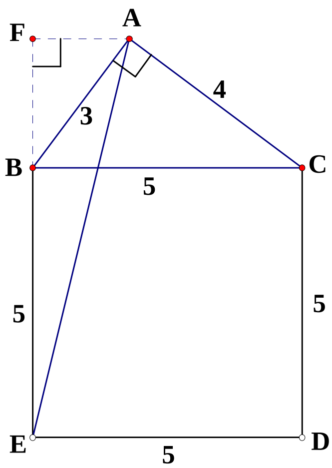
Group Event 2

G2.1 If the tens digit of 5^{5^5} is a, find the value of a.

$5^{5^5} = \cdots\cdots 125$, the tens digit = a = 2

Remark Original question: If the tenth-place digit $\cdots\cdots$, this is the position of the first digit to the right of the decimal point.

G2.2 In Figure 1, ΔABC is a right-angled triangle, $AB = 3$ cm, $AC = 4$ cm and $BC = 5$ cm. If $BCDE$ is a square and the area of ΔABE is b cm^2, find the value of b.



$\cos B = \frac{AB}{BC} = \frac{3}{5}$

Height of ΔABE from A

$= AF = AB \sin(90° - B) = 3 \times \frac{3}{5} = \frac{9}{5}$

$b = \frac{1}{2} \cdot AF \times BE$

$= \frac{1}{2} \cdot 5 \times \frac{9}{5} = \frac{9}{2}$

G2.3 Given that there are c prime numbers less than 100 such that their unit digits are not square numbers, find the values of c.

The prime are: {2, 3, 5, 7, 13, 17, 23, 37, 43, 47, 53, 67, 73, 83, 97}

$c = 15$

G2.4 If the lines $y = x + d$ and $x = -y + d$ intersect at the point $(d - 1, d)$, find the value of d.

$x = -(x + d) + d$

$x = 0 = d - 1$

$d = 1$

Group Event 3

G3.1 If a is the smallest real root of the equation $\sqrt{x(x+1)(x+2)(x+3)+1}=71$, find the value of a.

Reference: 1993 HG6, 1995 FI4.4, 1996 FG10.1, 2000 FG3.1, 2012 FI2.3

Let $t = x + 1.5$, then the equation becomes $\sqrt{(t-1.5)(t-0.5)(t+0.5)(t+1.5)+1}=71$

$$\sqrt{\left(t^2-\frac{9}{4}\right)\left(t^2-\frac{1}{4}\right)+1}=71$$

$$\sqrt{t^4-\frac{5}{2}t^2+\frac{9}{16}+1}=71$$

$$\Rightarrow\sqrt{t^4-\frac{5}{2}t^2+\frac{25}{16}}=71$$

$$\Rightarrow\sqrt{\left(t^2-\frac{5}{4}\right)^2}=71$$

$$t^2-\frac{5}{4}=71\Rightarrow t^2=\frac{289}{4}\Rightarrow t=\frac{17}{2}\quad\text{or}\quad t=-\frac{17}{2}$$

$x = t - 1.5 = \pm\frac{17}{2}-\frac{3}{2} = 7$ or -10

a = the smallest root = -10

G3.2 Given that p and q are prime numbers satisfying the equation $18p + 30q = 186$.

If $\log_8\frac{p}{3q+1} = b \geq 0$, find the value of b.

$18p + 30q = 186 \Rightarrow 3p + 5q = 31$

Note that the number "2" is the only prime number which is even.

$3p + 5q = 31$ = odd number $\Rightarrow$ either $p = 2$ or $q = 2$

If $p = 2$, then $q = 5$; $b=\log_8\frac{p}{3q+1}=\log_8\frac{2}{3\times5+1}=\log_8\frac{1}{16}<0$ (rejected)

If $q = 2$, then $p = 7$; $b=\log_8\frac{p}{3q+1}=\log_8\frac{7}{3\times2+1}=0$

G3.3 Given that for any real numbers x, y and z, $\oplus$ is an operation satisfying

(i) $x \oplus 0 = 1$, and

(ii) $(x \oplus y) \oplus z = (z \oplus xy) + z$.

If $1\oplus2004 = c$, find the value of c.

$c = 1 \oplus 2004$

$= (1\oplus0) \oplus 2004$

$= (2004\oplus0) + 2004$

$= 1 + 2004$

$= 2005$

G3.4 Given that $f(x) = (x^4 + 2x^3 + 4x - 5)^{2004} + 2004$. If $f\left(\sqrt{3}-1\right) = d$, find the value of d.

$x=\sqrt{3}-1$

$(x + 1)^2 = \left(\sqrt{3}\right)^2$

$x^2 + 2x - 2 = 0$

By division, $x^4 + 2x^3 + 4x - 5 = (x^2 + 2x - 2)(x^2 + 2) - 1 = -1$

$d = f\left(\sqrt{3}-1\right)$

$= f(x)$

$= (x^4 + 2x^3 + 4x - 5)^{2004} + 2004$

$= (-1)^{2004} + 2004 = 2005$

Group Event 4

G4.1 If $f(x)=\frac{4^x}{4^x+2}$ and $P = f\left(\frac{1}{1001}\right)+f\left(\frac{2}{1001}\right)+\cdots+f\left(\frac{1000}{1001}\right)$, find the value of P.

Reference: 2011 HG5, 2012 FI2.2

$$f(x)+f(1-x)=\frac{4^x}{4^x+2}+\frac{4^{1-x}}{4^{1-x}+2}=\frac{4+4+2\cdot 4^x+2\cdot 4^{1-x}}{4+4+2\cdot 4^x+2\cdot 4^{1-x}}=1$$

$$P = f\left(\frac{1}{1001}\right)+f\left(\frac{2}{1001}\right)+\cdots+f\left(\frac{1000}{1001}\right)$$

$$= f\left(\frac{1}{1001}\right)+f\left(\frac{1000}{1001}\right)+f\left(\frac{2}{1001}\right)+f\left(\frac{999}{1001}\right)+\cdots+f\left(\frac{500}{1001}\right)+f\left(\frac{501}{1001}\right)=500$$

G4.2 Let $f(x) = |x - a| + |x - 15| + |x - a - 15|$, where $a \le x \le 15$ and $0 < a < 15$.
If Q is the smallest value of $f(x)$, find the value of Q.

Reference: 1994 HG1, 2001 HG9, 2008 HI8, 2008 FI1.3, 2010 HG6, 2011 FGS.1, 2012 Final G2.3

$f(x) = x - a + 15 - x + 15 - x + a = 30 - x \ge 30 - 15 = 15 = Q$

G4.3 If $2^m = 3^n = 36$ and $R=\frac{1}{m}+\frac{1}{n}$, find the value of R.

Reference: 2001 HI1, 2003 FG2.2, 2004 FG4.3, 2005 HI9, 2006 FG4.3

$\log 2^m = \log 3^n = \log 36$

$m \log 2 = n \log 3 = \log 36$

$m=\frac{\log 36}{\log 2}$; $n=\frac{\log 36}{\log 3}$

$$\frac{1}{m}+\frac{1}{n}=\frac{\log 2}{\log 36}+\frac{\log 3}{\log 36}=\frac{\log 6}{\log 36}=\frac{\log 6}{2\log 6}=\frac{1}{2}$$

G4.4 Let $[x]$ be the largest integer not greater than x, for example, $[2.5] = 2$.

If $a=1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{2004^2}$ and $S = [a]$, find the value of a.

$$1<a=1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{2004^2}<1+\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{2003\times 2004}$$

$$=1+1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\cdots+\frac{1}{2003}-\frac{1}{2004}$$

$$=2-\frac{1}{2004}<2$$

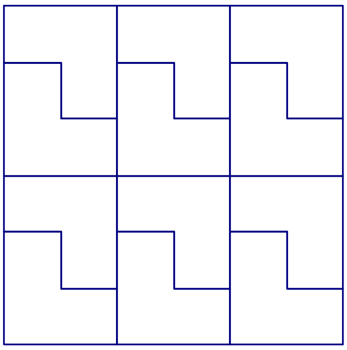
$S = [a] = 1$

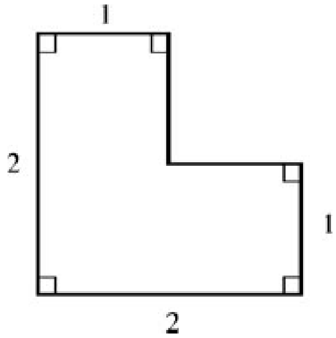
Group Event (Spare)

GS.1 For all integers n, F_n is defined by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$ and $F_1 = 1$.
If $a = F_{-5} + F_{-4} + \dots + F_4 + F_5$, find the value of a.
$F_2 = 0 + 1 = 1$, $F_3 = 1 + 1 = 2$, $F_4 = 1 + 2 = 3$, $F_5 = 2 + 3 = 5$
$F_{-1} + 0 = 1 \Rightarrow F_{-1} = 1$, $F_{-2} + 1 = 0 \Rightarrow F_{-2} = -1$, $F_{-3} + (-1) = 1 \Rightarrow F_{-3} = 2$,
$F_{-4} + 2 = -1 \Rightarrow F_{-4} = -3$, $F_{-5} + (-3) = 2 \Rightarrow F_{-5} = 5$
$a = F_{-5} + F_{-4} + \dots + F_4 + F_5 = 5 - 3 + 2 - 1 + 1 + 0 + 1 + 1 + 2 + 3 + 5 = 16$

GS.2 Given that x_0 satisfies the equation $x^2 + x + 2 = 0$. If $b = x_0^4 + 2x_0^3 + 3x_0^2 + 2x_0 + 1$, find the value of b.
By division, $b = x_0^4 + 2x_0^3 + 3x_0^2 + 2x_0 + 1$
$= (x_0^2 + x_0 + 2)(x_0^2 + x_0) + 1 = 1$

GS.3 Figure 1 shows a tile. If C is the minimum number of tiles required to tile a square, find the value of C.





$C = 12$

GS.4 If the line $5x + 2y - 100 = 0$ has d points whose x and y coordinates are both positive integers, find the value of d.
$(x, y) = (18, 5), (16, 10), (14, 15), (12, 20), (10, 25), (8, 30), (6, 35), (4, 40), (2, 45)$
$d = 9$

Individual Events (Remark: The Individual Events are interchanged with Group Events)

I1	***a***	40	**I2**	***a***	7	**I3**	***a***	1	**I4**	***a***	1
	b	70		***b***	5		****b*** see the remark	0.0625 or $\frac{1}{16}$		***b***	9
	C	4		***C***	35		***c***	0.5 or $\frac{1}{2}$		***c***	20
	d	*20 see the remark		***d***	6		***d***	6		***d***	6

Group Events (Remark: The Group Events are interchanged with Individual Events)

G1	***a***	9	**G2**	***a***	15	**G3**	***A***	$\frac{\sqrt{7}}{4}$	**G4**	***a***	3
	b	125		***b***	999985		***b***	$\frac{\sqrt{3}}{3}$ or $\frac{1}{\sqrt{3}}$		***b***	2
	c	5		***c***	4		***c***	7		***c***	6
	d	$2\frac{1}{2}$ or 2.5		***d***	$\frac{509}{256}$		***d***	$\frac{100}{11}$		***d***	171

Individual Event 1

I1.1 There are a camels in a zoo. The number of one-hump camels exceeds that of two-hump camels by 10. If there have 55 humps altogether, find the value of a.

Suppose there are x one-hump camels, y two-hump camels.

$x - y = 10$ .... (1)

$x + 2y = 55$ .... (2)

(2) – (1) $3y = 45 \Rightarrow y = 15$

sub. $y = 15$ into (1): $x - 15 = 10 \Rightarrow x = 25$

$a = x + y = 25 + 15 = 40$

I1.2 If LCM(a, b) = 280 and HCF(a, b) = 10, find the value of b.

Reference: 2016 FI2.4

HCF $\times$ LCM = ab

$2800 = 40b$

$b = 70$

I1.3 Let C be a positive integer less than $\sqrt{b}$. If b is divided by C, the remainder is 2; when divided by $C + 2$, the remainder is C, find the value of C.

$C < \sqrt{70} \Rightarrow C \le 8$ ....(1)

$70 = mC + 2$ ....(2)

$70 = n(C + 2) + C$ ....(3)

From (2), $mC = 68$ $\because 2 < C \le 8$, $\therefore C = 4$ ($C \ne 1, 2$, otherwise remainder > divisor !!!)

I1.4 A regular 2C-sided polygon has d diagonals, find the value of d.

Reference: 1984 FG10.3, 1985 FG8.3, 1988 FG6.2, 1989 FG6.1, 1991 FI2.3, 2001 FI4.2

The number of diagonals of a convex n-sided polygon is $\frac{n(n-3)}{2}$.

$d = \frac{8\times 5}{2} = 20$

Remark: The following note was put at the end of the original question:

(註：對角線是連接兩個不在同一邊上的頂點的直線。)

(NB: a diagonal is a straight line joining two vertices not on the same side.)

The note is very confusing. As the definition of diagonal is well known, there is no need to add this note.

Individual Event 2

I2.1 Mr. Chan has 8 sons and $\boldsymbol{a}$ daughters. Each of his sons has 8 sons and $\boldsymbol{a}$ daughters. Each of his daughters has $\boldsymbol{a}$ sons and 8 daughters. It is known that the number of his grand sons is one more than the number of his grand daughters and $\boldsymbol{a}$ is a prime number, find the value of $\boldsymbol{a}$.

Grandsons = $8\times 8 + \boldsymbol{a}\times\boldsymbol{a} = \boldsymbol{a}^2 + 64$

Grand daughters = $8\times\boldsymbol{a} + \boldsymbol{a}\times 8 = 16\boldsymbol{a}$

$\boldsymbol{a}^2 + 64 = 16\boldsymbol{a} + 1$

$\boldsymbol{a}^2 - 16\boldsymbol{a} + 63 = 0$

$(\boldsymbol{a} - 7)(\boldsymbol{a} - 9) = 0$

$\boldsymbol{a} = 7$ or $\boldsymbol{a} = 9$

$\boldsymbol{a}$ is a prime number, $\boldsymbol{a} = 7$

I2.2 Let $\frac{\boldsymbol{a}}{7} = \sqrt[3]{2+\sqrt{\boldsymbol{b}}} + \sqrt[3]{2-\sqrt{\boldsymbol{b}}}$. Find the value of $\boldsymbol{b}$.

Reference: 1999 FI3.2, 2016 FG3.3, 2019 HI10

$1 = \sqrt[3]{2+\sqrt{\boldsymbol{b}}} + \sqrt[3]{2-\sqrt{\boldsymbol{b}}}$

$1 = \left(\sqrt[3]{2+\sqrt{\boldsymbol{b}}} + \sqrt[3]{2-\sqrt{\boldsymbol{b}}}\right)^3$

$1 = 2+\sqrt{\boldsymbol{b}} + 3\left(2+\sqrt{\boldsymbol{b}}\right)^{2/3}\left(2-\sqrt{\boldsymbol{b}}\right)^{1/3} + 3\left(2+\sqrt{\boldsymbol{b}}\right)^{1/3}\left(2-\sqrt{\boldsymbol{b}}\right)^{2/3} + 2-\sqrt{\boldsymbol{b}}$

$1 = 4 + 3\,(4-\boldsymbol{b})^{1/3}\left(2+\sqrt{\boldsymbol{b}}\right)^{1/3} + 3(4-\boldsymbol{b})^{1/3}\left(2-\sqrt{\boldsymbol{b}}\right)^{1/3}$

$0 = 3 + 3\,(4-\boldsymbol{b})^{1/3}\left[\left(2+\sqrt{\boldsymbol{b}}\right)^{1/3} + \left(2-\sqrt{\boldsymbol{b}}\right)^{1/3}\right]$

$0 = 1 + (4-\boldsymbol{b})^{1/3}$

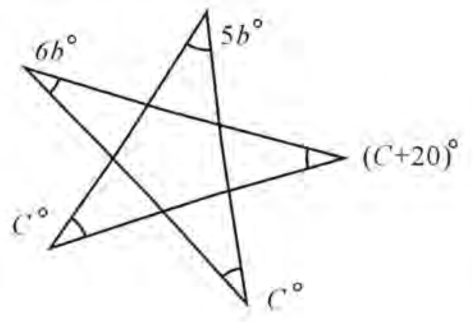
$(4-\boldsymbol{b})^{1/3} = -1$

$4 - \boldsymbol{b} = -1$

$\boldsymbol{b} = 5$

I2.3 In Figure 1, find the value of $\boldsymbol{C}$.

Reference: 1989 FI5.1, 1997 FG1.1



$6\boldsymbol{b}° + 5\boldsymbol{b}° + \boldsymbol{C}° + \boldsymbol{C}° + (\boldsymbol{C} + 20)° = 180°$

$11\times 5 + 3\boldsymbol{C} + 20 = 180$

$\boldsymbol{C} = 35$

I2.4 Given that $\boldsymbol{P}_1, \boldsymbol{P}_2, \ldots, \boldsymbol{P}_{\boldsymbol{d}}$ are $\boldsymbol{d}$ consecutive prime numbers.

If $\boldsymbol{P}_1 + \boldsymbol{P}_2 + \ldots + \boldsymbol{P}_{\boldsymbol{d}-2} = \boldsymbol{P}_{\boldsymbol{d}-1} + \boldsymbol{P}_{\boldsymbol{d}} = \boldsymbol{C} + 1$, find the value of $\boldsymbol{d}$.

By trial and error $5 + 7 + 11 + 13 = 17 + 19 = 36$, $\boldsymbol{d} = 6$.

Individual Event 3

I3.1 Given that $\boldsymbol{a}$ is a positive real root of the equation $2^{x+1}=8^{\frac{1}{x}-\frac{1}{3}}$. Find the value of $\boldsymbol{a}$.

$2^{x+1}=8^{\frac{1}{x}-\frac{1}{3}} \Rightarrow 2^{x+1}=2^{\frac{3}{x}-1} \Rightarrow x+1=\frac{3}{x}-1$

$x^2+2x-3=0$

$(x+3)(x-1)=0$

$x=-3$ or 1, $\boldsymbol{a}=1$ is a real positive root.

I3.2 The largest area of the rectangle with perimeter $\boldsymbol{a}$ meter is $\boldsymbol{b}$ square meter, find the value of $\boldsymbol{b}$.

The perimeter = 1 m.

Let the length of the rectangle be x m, the width is $\frac{1}{2}(1-2x)$m.

Its area is $\frac{1}{2}(1-2x)\cdot x\text{ m}^2=-\left(x^2-\frac{1}{2}x\right)\text{m}^2=\left[-\left(x-\frac{1}{4}\right)^2+\frac{1}{16}\right]\text{m}^2$.

$\boldsymbol{b}=\frac{1}{16}=0.0625$

Remark: The original version is: The area of the largest rectangle …

It is ambiguous to define the largest rectangle. It should be changed to "The largest area of the rectangle ……"

I3.3 If $\boldsymbol{c}=(1234^3-1232\times(1234^2+2472))\times\boldsymbol{b}$, find the value of $\boldsymbol{c}$.

$\boldsymbol{c}=(1234^3-1232\times(1234^2+2472))\times\frac{1}{16}$, let $x=1234$

$=\frac{1}{16}\{x^3-(x-2)\times[x^2+2(x+2)]\}=\frac{1}{16}\{x^3-(x-2)\times(x^2+2x+4)\}$

$=\frac{1}{16}\{x^3-(x^3-8)\}=\frac{8}{16}=\frac{1}{2}$

I3.4 If $\frac{1}{(c+1)(c+2)}+\frac{1}{(c+2)(c+3)}+\cdots+\frac{1}{(c+d)(c+d+1)}=\frac{8}{15}$, find the value of $\boldsymbol{d}$.

$\frac{1}{(c+1)(c+2)}+\frac{1}{(c+2)(c+3)}+\cdots+\frac{1}{(c+d)(c+d+1)}=\frac{8}{15}$

$\left(\frac{1}{c+1}-\frac{1}{c+2}\right)+\left(\frac{1}{c+2}-\frac{1}{c+3}\right)+\cdots+\left(\frac{1}{c+d}-\frac{1}{c+d+1}\right)=\frac{8}{15}$

$\frac{1}{c+1}-\frac{1}{c+d+1}=\frac{8}{15}\Rightarrow \frac{1}{\frac{1}{2}+1}-\frac{1}{\frac{1}{2}+d+1}=\frac{8}{15}$

$\frac{2}{3}-\frac{8}{15}=\frac{2}{3+2d}$

$\frac{2}{15}=\frac{2}{3+2d}$

$3+2d=15$

$d=6$

Individual Event 4

I4.1 If $A^2 + B^2 + C^2 = AB + BC + CA = 3$ and $a = A^2$, find the value of a.

Reference: 2018 FG4.4

$2[A^2 + B^2 + C^2 - (AB + BC + CA)] = 6 - 6 = 0$

$A^2 - 2AB + B^2 + B^2 - 2BC + C^2 + C^2 - 2AC + A^2 = 0$

$(A - B)^2 + (B - C)^2 + (C - A)^2 = 0$ (sum of three non-negative numbers = 0)

$A - B = B - C = C - A = 0$

$A = B = C = 1$

$a = A^2 = B^2 = C^2 = 1$

I4.2 Given that n and b are integers satisfying the equation $29n + 42b = a$. If $5 < b < 10$, find the value b.

$42 = 29 + 13 \Rightarrow 13 = 42 - 29$ ....(1)

$29 = 13\times 2 + 3 \Rightarrow 3 = 29 - 13\times 2$ ....(2)

$13 = 3\times 4 + 1 \Rightarrow 1 = 13 - 3\times 4$ ....(3)

Sub. (1) into (2): $3 = 29 - (42 - 29)\times 2 = 29\times 3 - 42\times 2$ ......(4)

Sub. (1), (4) into (3) $1 = 42 - 29 - (29\times 3 - 42\times 2)\times 4$

$1 = 29\times(-13) + 42\times 9$

$\therefore n = -13$, $b = 9$

Method 2

$b = 6, 7, 8, 9$.

By trial and error,

when $b = 9$,

$29n + 42\times 9 = 1$

$n = -13$

$\therefore$ $b = 9$ satisfies the equation.

I4.3 If $\dfrac{\sqrt{3}-\sqrt{5}+\sqrt{7}}{\sqrt{3}+\sqrt{5}+\sqrt{7}} = \dfrac{c\sqrt{21}-18\sqrt{15}-2\sqrt{35}+b}{59}$, find the value of c.

$$\frac{\sqrt{3}-\sqrt{5}+\sqrt{7}}{\sqrt{3}+\sqrt{5}+\sqrt{7}} = \frac{c\sqrt{21}-18\sqrt{15}-2\sqrt{35}+9}{59}$$

$$\frac{\sqrt{3}+\sqrt{7}-\sqrt{5}}{\sqrt{3}+\sqrt{7}+\sqrt{5}}\cdot\frac{\sqrt{3}+\sqrt{7}-\sqrt{5}}{\sqrt{3}+\sqrt{7}-\sqrt{5}} = \frac{c\sqrt{21}-18\sqrt{15}-2\sqrt{35}+9}{59}$$

$$\frac{\left(\sqrt{3}+\sqrt{7}\right)^2-2\sqrt{5}\left(\sqrt{3}+\sqrt{7}\right)+5}{\left(\sqrt{3}+\sqrt{7}\right)^2-5} = \frac{c\sqrt{21}-18\sqrt{15}-2\sqrt{35}+9}{59}$$

$$\frac{3+7+2\sqrt{21}-2\sqrt{15}-2\sqrt{35}+5}{3+7+2\sqrt{21}-5} = \frac{c\sqrt{21}-18\sqrt{15}-2\sqrt{35}+9}{59}$$

$$\frac{15+2\sqrt{21}-2\sqrt{15}-2\sqrt{35}}{5+2\sqrt{21}}\cdot\frac{2\sqrt{21}-5}{2\sqrt{21}-5} = \frac{c\sqrt{21}-18\sqrt{15}-2\sqrt{35}+9}{59}$$

$$\frac{30\sqrt{21}+84-12\sqrt{35}-28\sqrt{15}-75-10\sqrt{21}+10\sqrt{15}+10\sqrt{35}}{4\times 21-25} = \frac{c\sqrt{21}-18\sqrt{15}-2\sqrt{35}+9}{59}$$

$$\frac{20\sqrt{21}-18\sqrt{15}-2\sqrt{35}+9}{59} = \frac{c\sqrt{21}-18\sqrt{15}-2\sqrt{35}+9}{59}$$

$c = 20$

Method 2 Cross multiplying $59\left(\sqrt{3}-\sqrt{5}+\sqrt{7}\right) = \left(\sqrt{3}+\sqrt{5}+\sqrt{7}\right)\cdot\left(c\sqrt{21}-18\sqrt{15}-2\sqrt{35}+9\right)$

Compare coefficient of $\sqrt{3}$: $9 - 90 + 7c = 59 \Rightarrow c = 20$

I4.4 If c has d positive factors, find the value of d.

Reference 1993 HI8, 1994 FI3.2, 1997 HI3, 1998 HI10, 1998 FI1.4, 2002 FG4.1

The positive factors of 20 are 1, 2, 4, 5, 10 and 20.

$d = 6$

Group Event 1

G1.1 Suppose there are $\boldsymbol{a}$ numbers between 1 and 200 that can be divisible by 3 and 7, find the value of $\boldsymbol{a}$.

The number which can be divisible be 3 and 7 are multiples of 21. 200÷21 = 9.5, $\boldsymbol{a}$ = 9

G1.2 Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be prime numbers that are the two distinct roots of the equation $\boldsymbol{x}^2 - 13\boldsymbol{x} + \boldsymbol{R} = 0$, where $\boldsymbol{R}$ is a real number. If $\boldsymbol{b} = \boldsymbol{p}^2 + \boldsymbol{q}^2$, find the value of $\boldsymbol{b}$.

Reference: 1996 HG8, 1996FG7.1, 2001 FG4.4, 2012 HI6

$\boldsymbol{x}^2 - 13\boldsymbol{x} + \boldsymbol{R} = 0$, roots $\boldsymbol{p}$ and $\boldsymbol{q}$ are prime numbers. $\boldsymbol{p} + \boldsymbol{q} = 13$, $\boldsymbol{pq} = \boldsymbol{R}$

The sum of two prime numbers is 13, so one is odd and the other is even, $\boldsymbol{p} = 2$, $\boldsymbol{q} = 11$

$\boldsymbol{b} = \boldsymbol{p}^2 + \boldsymbol{q}^2 = 2^2 + 11^2 = 125$

G1.3 Given that $\tan\alpha = -\frac{1}{2}$. If $\boldsymbol{c} = \frac{2\cos\alpha - \sin\alpha}{\sin\alpha + \cos\alpha}$, find the value of $\boldsymbol{c}$.

$\tan\alpha = -\frac{1}{2}$. $\boldsymbol{c} = \frac{2\cos\alpha - \sin\alpha}{\sin\alpha + \cos\alpha} = \frac{2 - \tan\alpha}{\tan\alpha + 1} = \frac{2 + \frac{1}{2}}{-\frac{1}{2} + 1} = 5$

G1.4 Let $\boldsymbol{r}$ and $\boldsymbol{s}$ be the two distinct real roots of the equation $2\left(\boldsymbol{x}^2 + \frac{1}{\boldsymbol{x}^2}\right) - 3\left(\boldsymbol{x} + \frac{1}{\boldsymbol{x}}\right) = 1$. If $\boldsymbol{d} = \boldsymbol{r} + \boldsymbol{s}$, find the value of $\boldsymbol{d}$.

$2\left(\boldsymbol{x}^2 + \frac{1}{\boldsymbol{x}^2}\right) - 3\left(\boldsymbol{x} + \frac{1}{\boldsymbol{x}}\right) = 1$, real roots $\boldsymbol{r}$, $\boldsymbol{s}$. Let $\boldsymbol{t} = \boldsymbol{x} + \frac{1}{\boldsymbol{x}}$, then $\boldsymbol{x}^2 + \frac{1}{\boldsymbol{x}^2} = \boldsymbol{t}^2 - 2$.

$2(\boldsymbol{t}^2 - 2) - 3\boldsymbol{t} = 1$

$2\boldsymbol{t}^2 - 3\boldsymbol{t} - 5 = 0$

$(2\boldsymbol{t} - 5)(\boldsymbol{t} + 1) = 0$

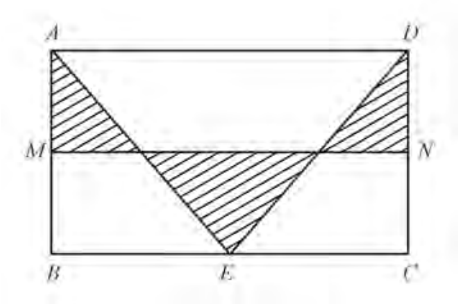
$\boldsymbol{t} = \frac{5}{2}$ or -1

$\boldsymbol{x} + \frac{1}{\boldsymbol{x}} = \frac{5}{2}$ or $\boldsymbol{x} + \frac{1}{\boldsymbol{x}} = -1$

$\boldsymbol{x} = 2$ or $\frac{1}{2}$ $\Rightarrow \boldsymbol{r} = 2$, $\boldsymbol{s} = \frac{1}{2} \Rightarrow \boldsymbol{d} = \boldsymbol{r} + \boldsymbol{s} = \frac{5}{2}$

Group Event 2

G2.1 In Figure 1, ***ABCD*** is a rectangle, ***AB*** = 6 cm and ***BC*** = 10 cm. ***M*** and ***N*** are the midpoints of ***AB*** and ***DC*** respectively. If the area of the shaded region is ***a*** cm^2, find the value of ***a***.



$\boldsymbol{a} = \frac{1}{4}$ area of rectangle $= \frac{1}{4} \times 6 \times 10 = 15$

G2.2 Let $\boldsymbol{b} = 89 + 899 + 8999 + 89999 + 899999$, find the value of ***b***.

$\boldsymbol{b} = 89 + 899 + 8999 + 89999 + 899999$

$= (90 - 1)+(900 - 1)+ (9000 - 1) + (90000 - 1) + (900000 - 1)$

$= 999990 - 5 = 999985$

G2.3 Given that $2\boldsymbol{x} + 5\boldsymbol{y} = 3$. If $\boldsymbol{c} = \sqrt{4^{x+\frac{1}{2}} \times 32^{y}}$, find the value of ***c***.

$2\boldsymbol{x} + 5\boldsymbol{y} = 3$. $\boldsymbol{c} = \sqrt{4^{x+\frac{1}{2}} \times 32^{y}} = \sqrt{2^{2x+1} \times 2^{5y}} = \sqrt{2^{2x+5y+1}} = \sqrt{2^{3+1}} = 4$

G2.4 Let $\boldsymbol{d} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \ldots + \frac{10}{2^{10}}$, find the value of ***d***.

Reference: 2005 HI7, 2007 FG2.1

$\boldsymbol{d} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \ldots + \frac{10}{2^{10}}$, $2\boldsymbol{d} = 1 + 1 + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \ldots + \frac{10}{2^{9}}$

$$2\boldsymbol{d} - \boldsymbol{d} = 1 + 1 + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \ldots + \frac{10}{2^{9}} - \left(\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \ldots + \frac{10}{2^{10}}\right)$$

$$= 1 + (1 - \frac{1}{2}) + (\frac{3}{4} - \frac{2}{4}) + (\frac{4}{8} - \frac{3}{8}) + (\frac{5}{16} - \frac{4}{16}) + \ldots\ldots.. + (\frac{10}{2^9} - \frac{9}{2^9}) - \frac{10}{2^{10}}$$

$$\boldsymbol{d} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\ldots\ldots + \frac{1}{2^9} - \frac{10}{1024} = \frac{1 - \frac{1}{2^{10}}}{1 - \frac{1}{2}} - \frac{5}{512}$$

$$= \frac{1023}{512} - \frac{5}{512} = \frac{1018}{512} = \frac{509}{256}$$

Group Event 3

G3.1 Let 0° < α < 45°. If $\sin\alpha\cos\alpha = \frac{3\sqrt{7}}{16}$ and ***A*** = sin α, find the value of ***A***.

Method 1 $2\sin\alpha\cos\alpha = \frac{3\sqrt{7}}{8}$

$\sin 2\alpha = \frac{3\sqrt{7}}{8}$

$\cos 2\alpha = \sqrt{1-\sin^2 2\alpha} = \sqrt{1-\left(\frac{3\sqrt{7}}{8}\right)^2}$

$= \frac{1}{8}\sqrt{64-63} = \frac{1}{8}$

$1 - 2\sin^2\alpha = \frac{1}{8} \Rightarrow \sin\alpha = \frac{\sqrt{7}}{4}$

Method 2 $2\sin\alpha\cos\alpha = \frac{3\sqrt{7}}{8}$,

$\sin 2\alpha = \frac{3\sqrt{7}}{8} \Rightarrow \tan 2\alpha = 3\sqrt{7}$

$t = \tan\alpha$, $\tan 2\alpha = \frac{2t}{1-t^2} = 3\sqrt{7}$

$2t = 3\sqrt{7} - 3\sqrt{7}\,t^2$

$3\sqrt{7}\,t^2 + 2t - 3\sqrt{7} = 0$

$(3t - \sqrt{7})(\sqrt{7}\,t + 3) = 0$

$t = \frac{\sqrt{7}}{3}$ or $-\frac{3}{\sqrt{7}}$ (rejected)

$\tan\alpha = \frac{\sqrt{7}}{3}$

$A = \sin\alpha = \frac{\sqrt{7}}{4}$

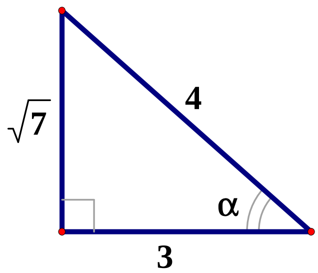
Method 3

$\sin\alpha\cos\alpha = \frac{3\sqrt{7}}{16} = \frac{3}{4}\times\frac{\sqrt{7}}{4}$

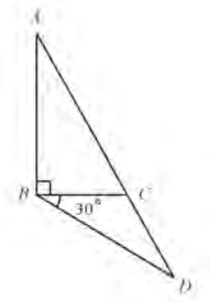
$\sin\alpha = \frac{3}{4}$, $\cos\alpha = \frac{\sqrt{7}}{4}$

or $\sin\alpha = \frac{\sqrt{7}}{4}$, $\cos\alpha = \frac{3}{4}$

∵ 0° < α < 45°, ∴ sin α < cos α, $\sin\alpha = \frac{\sqrt{7}}{4}$



G3.2 In figure 1, ***C*** lies on ***AD***, ***AB*** = ***BD*** = 1 cm, ∠***ABC*** = 90° and ∠***CBD*** = 30°.
If ***CD*** = ***b*** cm, find the value of ***b***.



AB = ***BD*** = 1 cm, Δ***ABD*** is isosceles.

∠***BAD*** = ∠***BDA*** = (180° – 90° – 30°) ÷ 2 = 30° (∠s sum of isosceles Δ)

Δ***BCD*** is also isosceles.

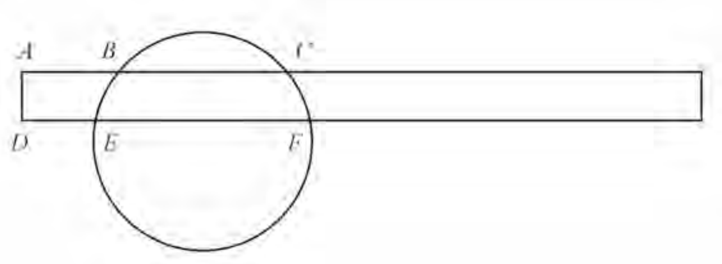
CD = ***b*** cm = ***BC***

= ***AB*** tan ∠***BAD***

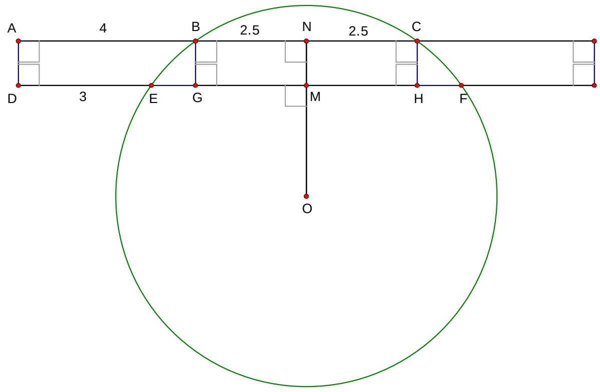
= 1 tan 30° cm

$= \frac{1}{\sqrt{3}}$ cm

G3.3 In Figure 2, a rectangle intersects a circle at points ***B***, ***C***, ***E*** and ***F***. Given that ***AB*** = 4 cm, ***BC*** = 5 cm and ***DE*** = 3 cm. If ***EF*** = ***c*** cm, find the value of ***c***.



Draw ***BG*** ⊥ ***DF***, ***CH*** ⊥ ***DF***
DG = ***AB*** = 4 cm, ***GH*** = ***BC*** = 5 cm
EG = ***DG*** – ***DE*** = 4 cm – 3 cm = 1cm
Let ***O*** be the centre.
Let ***M*** be the foot of perpendicular of ***O*** on ***EF*** and produce ***OM*** to ***N*** on ***BC***.
ON ⊥ ***BC*** (corr. ∠s ***AC*** // ***DF***)
BN = ***NC*** = 2.5 cm (⊥ from centre bisect chord)
MF = ***EM*** (⊥ from centre bisect chords)
= ***EG*** + ***GM*** = 1 cm + ***BN***
= 1 cm + 2.5cm = 3.5 cm
EF = 2***EM*** = 7 cm



G3.4 Let ***x*** and ***y*** be two positive numbers that are inversely proportional to each other. If ***x*** is increased by 10 %, ***y*** will be decreased by ***d*** %, find the value of ***d***.

$xy = k$, $x_1 = 1.1x$

$x_1y_1 = xy$

$\Rightarrow 1.1xy_1 = xy$

$y_1 = \frac{10y}{11}$

Percentage decrease $= \frac{y - \frac{10y}{11}}{y} \times 100\%$

$= \frac{100}{11}\%$

$d = \frac{100}{11}$

Group Event 4

G4.1 If $\boldsymbol{a} = \log_{\frac{1}{2}} 0.125$, find the value of $\boldsymbol{a}$.

$$\boldsymbol{a} = \log_{\frac{1}{2}} 0.125 = \frac{\log 0.125}{\log \frac{1}{2}} = \frac{\log \frac{1}{8}}{\log \frac{1}{2}} = \frac{\log 2^{-3}}{\log 2^{-1}} = \frac{-3\log 2}{-\log 2} = 3$$

G4.2 Suppose there are $\boldsymbol{b}$ distinct solutions of the equation $\left| x - |2x+1| \right| = 3$, find the value of $\boldsymbol{b}$.

Reference: 2002 FG4.3, 2009 HG9, 2012 FG4.2

$\left| x - |2x+1| \right| = 3$

$x - |2x + 1| = 3$ or $x - |2x + 1| = -3$

$x - 3 = |2x + 1|$ or $x + 3 = |2x + 1|$

$x - 3 = 2x + 1$ or $3 - x = 2x + 1$ or $x + 3 = 2x + 1$ or $2x + 1 = -x - 3$

$x = -4$ or $\frac{2}{3}$ or -2 or $-\frac{4}{3}$

Check: when $x = -4$ or $\frac{2}{3}$, $x - 3 = |2x + 1| \ge 0$, no solution

When $x = -2$ or $-\frac{4}{3}$, $x + 3 = |2x + 1| \ge 0$, accepted

There are 2 real solutions.

G4.3 If $\boldsymbol{c} = 2\sqrt{3} \times \sqrt[3]{1.5} \times \sqrt[6]{12}$, find the value of $\boldsymbol{c}$.

$$\boldsymbol{c} = 2\sqrt{3} \times \sqrt[3]{1.5} \times \sqrt[6]{12} = 2\times 3^{\frac{1}{2}} \times \left(\frac{3}{2}\right)^{\frac{1}{3}} \times \left(2^2\times 3\right)^{\frac{1}{6}}$$

$$= 2^{1-\frac{1}{3}+\frac{2}{6}} \times 3^{\frac{1}{2}+\frac{1}{3}+\frac{1}{6}} = 2\times 3 = 6$$

G4.4 Given that $f_1 = 0$, $f_2 = 1$, and for any positive integer $n \ge 3$, $f_n = f_{n-1} + 2f_{n-2}$. If $d = f_{10}$, find the value of $\boldsymbol{d}$.

The characteristic equation: $x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x + 1)(x - 2) = 0 \Rightarrow x = -1$ or 2

$f_n = A(-1)^n + B\times 2^n$, $n = 1, 2, 3, \cdots\cdots\cdots$

$f_1 = -A + 2B = 0$ ……… (1)

$f_2 = A + 4B = 1$ ………(2)

(1) + (2) $6B = 1$, $B = \frac{1}{6}$

Sub. into (1): $-A + \frac{1}{3} = 0$, $A = \frac{1}{3}$

$f_n = \frac{1}{3}(-1)^n + \frac{1}{6}\times 2^n$, $d = f_{10} = \frac{1}{3} + \frac{1}{6}\times 1024 = \frac{513}{3} = 171$

Method 2: $f_1 = 0$, $f_2 = 1$

$f_3 = f_2 + 2f_1 = 1 + 0 = 1$; $f_4 = f_3 + 2f_2 = 1 + 2 = 3$

$f_5 = f_4 + 2f_3 = 3 + 2\times1 = 5$; $f_6 = f_5 + 2f_4 = 5 + 2\times3 = 11$

$f_7 = f_6 + 2f_5 = 11 + 2\times5 = 21$; $f_8 = f_7 + 2f_6 = 21 + 2\times11 = 43$

$f_9 = f_8 + 2f_7 = 43 + 2\times21 = 85$; $f_{10} = f_9 + 2f_8 = 85 + 2\times43 = 171$

$d = 171$

Individual Events

I1	***a***	1	**I2**	***R***	$*\frac{9}{25}$ see the remark	**I3**	***S***	10	**I4**	***k***	1
	b	$\frac{1}{2}$		***S***	1		***R***	30		***r***	2
	C	10		***T***	1		***T***	6		***s***	$\frac{1}{\sqrt{2}}(=\frac{\sqrt{2}}{2})$
	D	–2		***W***	$\sqrt{5}$		***P***	$\sqrt{7}+2$		***w***	9

Group Events

G1	***k***	1	**G2**	***w***	45	**G3**	***r***	2006	**G4**	***R***	$12\sqrt{3}$
	B	$\frac{16}{15}$		***z***	–13		***x***	$\frac{7}{4}$ (=1.75)		***S***	*8 see the remark
	C	$\frac{1}{4}$		***s***	$\frac{1}{4}$		***z***	30		***T***	$\frac{1}{2}$
	a	1		***t***	14		***R***	$\frac{15}{4}$ (= 3.75)		***W***	2013021

Individual Event 1

I1.1 If a is a real number satisfying $\log_2(x + 3) - \log_2(x + 1) = 1$, find the value of a.

$$\log_2\frac{x+3}{x+1} = \log_2 2$$

$x + 3 = 2x + 2$

$x = 1 \Rightarrow a = 1$

I1.2 In Figure 1, O is the centre of the circle with radius 1 cm. If the length of the arc AB is equal to a cm and the area of the shaded sector OAB is equal to b cm^2, find the value of b. (Take $\pi = 3$)

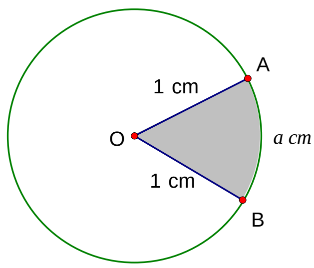


Figure 1

$$b = \frac{1}{2}rs = \frac{1}{2}1\cdot 1 = \frac{1}{2}$$

I1.3 An interior angle of a regular C-sided polygon is $288b°$, find the value of C.

Each interior angle = $288b° = 144°$

Each exterior angle = $36° = \frac{360°}{C}$

$C = 10$

I1.4 Given that C is a root of the equation $kx^2 + 2x + 5 = 0$, where k is a constant.

If D is another root, find the value of D.

$100k + 20 + 5 = 0 \Rightarrow k = -\frac{1}{4}$

$C + D$ = sum of roots $= -\frac{2}{k}$

$10 + D = 8 \Rightarrow D = -2$

Individual Event 2

I2.1 Given that $a : b : c = 6 : 3 : 1$. If $R = \frac{3b^2}{2a^2 + bc}$, find the value of R.

Let $a = 6k$, $b = 3k$, $c = k$, then $R = \frac{3(3k)^2}{2(6k)^2 + (3k)k} = \frac{9}{25}$

Remark: original question is Given that a, b and c are three numbers not equal to 0 and $a : b : c = 6 : 3 : 1$ ……, the condition $a : b : c = 6 : 3 : 1$ already implies that a, b and c are not zero.

I2.2 Given that $\frac{|k + R|}{|R|} = 0$. If $S = \frac{|k + 2R|}{|2k + R|}$, find the value of S.

$k + \frac{9}{25} = 0 \Rightarrow k = -\frac{9}{25}$

$S = \frac{|k + 2R|}{|2k + R|} = \frac{\left|-\frac{9}{25} + \frac{18}{25}\right|}{\left|-\frac{18}{25} + \frac{9}{25}\right|} = 1$

I2.3 Given that $T = \sin 50°\times(S + \sqrt{3}\times\tan 10°)$, find the value of T.

$T = \sin 50°\times(1 + \sqrt{3}\cdot\frac{\sin 10^\circ}{\cos 10^\circ})$

$= \frac{\sin 50^\circ}{\cos 10^\circ}\cdot\left(\cos 10^\circ + \sqrt{3}\sin 10^\circ\right)$

$= \frac{2\sin 50^\circ}{\cos 10^\circ}\left(\frac{1}{2}\cos 10^\circ + \frac{\sqrt{3}}{2}\sin 10^\circ\right)$

$= \frac{2\sin 50^\circ}{\cos 10^\circ}\left(\cos 60^\circ\cdot\cos 10^\circ + \sin 60^\circ\cdot\sin 10^\circ\right)$

$= \frac{2\sin 50^\circ\cos 50^\circ}{\cos 10^\circ} = \frac{\sin 100^\circ}{\cos 10^\circ} = 1$

I2.4 Given that x_0 and y_0 are real numbers satisfying the system of equations $\begin{cases} y = \frac{T}{x} \\ y = |x| + T \end{cases}$.

If $W = x_0 + y_0$, find the value of W.

$\begin{cases} y = \frac{1}{x} \\ y = |x| + 1 \end{cases}$

$\frac{1}{x} = |x| + 1$ ....(*)

$\frac{1}{x} = x + 1$ or $\frac{1}{x} = -x + 1$

$1 = x^2 + x$ or $1 = -x^2 + x$

$x^2 + x - 1 = 0$ or $x^2 - x + 1 = 0$

$x = \frac{-1\pm\sqrt{5}}{2}$ or no solution

Check: sub. $x = \frac{-1-\sqrt{5}}{2}$ into (*)

LHS $= \frac{2}{-1-\sqrt{5}} < 0$, RHS > 0 (rejected)

When $x = \frac{-1+\sqrt{5}}{2}$; LHS $= \frac{2}{-1+\sqrt{5}} = \frac{\sqrt{5}+1}{2}$; RHS $= \frac{-1+\sqrt{5}}{2} + 1 = \frac{1+\sqrt{5}}{2}$ (accepted)

$y = \frac{1+\sqrt{5}}{2} \Rightarrow W = x_0 + y_0 = \frac{-1+\sqrt{5}}{2} + \frac{1+\sqrt{5}}{2} = \sqrt{5}$

Individual Event 3

I3.1 Given that $\frac{2x-3}{x^2-x}=\frac{A}{x-1}+\frac{B}{x}$, where A and B are constants. If $S=A^2+B^2$, find the value of S

$\frac{2x-3}{x^2-x}=\frac{Ax+B(x-1)}{(x-1)x}$

$A+B=2$, $-B=-3$

$A=-1$, $B=3$

$S=(-1)^2+3^2=10$

I3.2 In Figure 1, $ABCD$ is an inscribed rectangle, $AB=(S-2)$ cm and $AD=(S-4)$ cm. If the circumference of the circle is R cm, find the value of R. (Take $\pi=3$)

$AB=8$ cm, $CD=6$ cm

$AC=10$ cm (Pythagoras' theorem)

$R=10\pi=30$

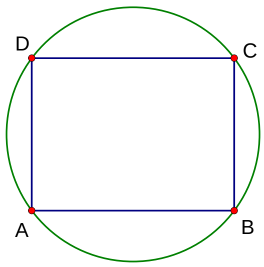


Figure 1

I3.3 Given that x and y are integers satisfying the equation $\frac{R}{2}xy=21x+20y-13$.

If $T=xy$, find the value of T.

$15xy=21x+20y-13$

$(3x-4)(5y-7)=15$

$\begin{cases}3x-4=1\\5y-7=15\end{cases}$ or $\begin{cases}3x-4=3\\5y-7=5\end{cases}$ or $\begin{cases}3x-4=5\\5y-7=3\end{cases}$ or $\begin{cases}3x-4=15\\5y-7=1\end{cases}$ or $\begin{cases}3x-4=-1\\5y-7=-15\end{cases}$ or

$\begin{cases}3x-4=-3\\5y-7=-5\end{cases}$ or $\begin{cases}3x-4=-5\\5y-7=-3\end{cases}$ or $\begin{cases}3x-4=-15\\5y-7=-1\end{cases}$

For integral solution: $3x-4=5$, $5y-7=3$

$x=3$, $y=2\Rightarrow T=3\times2=6$

I3.4 Let a be the positive root of the equation $x^2-2x-T=0$.

If $P=3+\cfrac{T}{2+\cfrac{T}{2+\cfrac{T}{2+\cfrac{T}{a}}}}$, find the value of P.

$x^2-2x-6=0$

$a=1+\sqrt{7}$

$a^2-2a-6=0\Rightarrow a^2=2a+6\Rightarrow a=2+\frac{6}{a}$

$2+\frac{T}{a}=2+\frac{6}{a}=a\Rightarrow 2+\cfrac{T}{2+\cfrac{T}{a}}=2+\frac{6}{a}=a\Rightarrow 2+\cfrac{T}{2+\cfrac{T}{2+\cfrac{T}{a}}}=2+\frac{6}{a}=a$

$P=3+\frac{6}{a}=1+2+\frac{6}{a}=1+a=1+1+\sqrt{7}=2+\sqrt{7}$

Individual Event 4

I4.1 Let $\frac{k}{4}=\left(1+\frac{1}{2}+\frac{1}{3}\right)\times\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\times\left(\frac{1}{2}+\frac{1}{3}\right)$, find the value of k.

Reference: 1995 FG6.2

Let $x=1+\frac{1}{2}+\frac{1}{3}$, $y=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$, then $\frac{k}{4}=x(y-1)-y(x-1)=-x+y=\frac{1}{4}\Rightarrow k=1$

I4.2 Let x and y be real numbers satisfying the equation $y^2+4y+4+\sqrt{x+y+k}=0$.

If $r=|xy|$, find the value of r.

Reference: 2005 FI4.1, 2009 FG1.4, 2011 FI4.3, 2013 FI1.4, 2015 HG4, 2015 FI1.1

The equation is equivalent to $(y+2)^2+\sqrt{x+y+k}=0$

which is a sum of two non-negative numbers.

$\Rightarrow y+2=0$ and $x+y+1=0$

$y=-2$ and $x-2+1=0$, $x=1$

$r=|-2\times 1|=2$

I4.3 In Figure 1, there are eight positive numbers in series. Starting from the 3rd number, each number is the product of the previous two numbers. Given that the 5th number is $\frac{1}{r}$ and the 8th number is $\frac{1}{r^4}$. If the first number is s, find the value of s.

s				$\frac{1}{r}$			$\frac{1}{r^4}$

Let the 2nd number be a, then other numbers are as follows:

s	a	sa	sa^2	$s^2a^3=\frac{1}{r}$	s^3a^5	s^5a^8	$s^8a^{13}=\frac{1}{r^4}$

$s^2a^3=\frac{1}{r}$ .... (1)

$s^8a^{13}=\frac{1}{r^4}$ .... (2)

(2)÷(1)^4: $a=1$

Sub. $a=1$ into (1): $s^2=\frac{1}{r}=\frac{1}{2}$; $s>0\Rightarrow s=\frac{1}{\sqrt{2}}$

I4.4 Let $[x]$ be the largest integer not greater than x. For example, $[2.5]=2$.

Let $w=1+[10\times s^2]+[10\times s^4]+[10\times s^6]+\ldots+[10\times s^{2n}]+\ldots$, find the value of w.

$w=1+[10\times\frac{1}{2}]+[10\times\frac{1}{4}]+[10\times\frac{1}{8}]+\ldots+[10\times\frac{1}{2^n}]+\ldots$

$=1+5+2+1+0+0+\ldots=9$

Group Event 1

G1.1 Given that k is a real number. If $x^2 + 2kx - 3k^2$ can be divisible by $x - 1$, find the greatest value of k.

By factor theorem, $1^2 + 2k - 3k^2 = 0$

$3k^2 - 2k - 1 = 0$

$(3k + 1)(k - 1) = 0$

$k = -\frac{1}{3}$ or 1

Greatest value of $k = 1$

G1.2 Given that $x = x_0$ and $y = y_0$ satisfy the system of equations $\begin{cases}\frac{x}{3}+\frac{y}{5}=1\\ \frac{x}{5}+\frac{y}{3}=1\end{cases}$. If $B = \frac{1}{x_0}+\frac{1}{y_0}$, find the value of B.

$\frac{x}{3}+\frac{y}{5}=\frac{x}{5}+\frac{y}{3}=1 \Rightarrow \frac{x}{3}-\frac{x}{5}=\frac{y}{3}-\frac{y}{5} \Rightarrow x = y$

Sub. $x = y$ into the first equation: $\frac{x}{3}+\frac{x}{5}=1 \Rightarrow x = y = \frac{15}{8} \Rightarrow B = \frac{16}{15}$

G1.3 Given that $x = 2 + \sqrt{3}$ is a root of the equation $x^2 - (\tan\alpha + \cot\alpha)x + 1 = 0$.

If $C = \sin\alpha \times \cos\alpha$, find the value of C.

Let the other root be y, $(2 + \sqrt{3})y$ = product of roots $= 1 \Rightarrow y = 2 - \sqrt{3}$

$\tan\alpha + \cot\alpha$ = sum of roots $= 2 + \sqrt{3} + 2 - \sqrt{3} = 4$

$\frac{\sin\alpha}{\cos\alpha}+\frac{\cos\alpha}{\sin\alpha}=4 \Rightarrow \frac{1}{\sin\alpha\cos\alpha}=4 \Rightarrow C = \sin\alpha \times \cos\alpha = \frac{1}{4}$

G1.4 Let a be an integer. If the inequality $|x + 1| < a - 1.5$ has no integral solution, find the greatest value of a.

$\because |x+ 1| \ge 0$, In order that the equation has no integral solution, it is sufficient that $a - 1.5 < 0$

$a < 1.5$

Greatest integral value of $a = 1$

Group Event 2

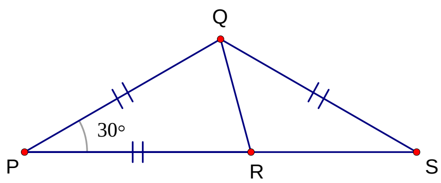
G2.1 In Figure 1, PRS is a straight line, $PQ = PR = QS$ and $\angle QPR = 30°$. If $\angle RQS = w°$, find the value of w.

$\angle QPR = \angle QSP = 30°$ (base $\angle$s isos. Δ)

$\angle PQS = 120°$ ($\angle$s sum of Δ)

$\angle PQR = \angle PRQ = (180° - 30°) \div 2 = 75°$ ($\angle$s sum of isos. Δ)

$\angle RQS = 120° - 75° = 45° \Rightarrow w = 45$



圖一
Figure 1

G2.2 Let $f(x) = px^7 + qx^3 + rx - 5$, where p, q and r are real numbers.

If $f(-6) = 3$ and $z = f(6)$, find the value of z . **(Reference: 1995 FI1.3)**

$f(-6) = 3 \Rightarrow -p\times 6^7 - q\times 6^3 - 6r - 5 = 3$

$f(6) = p\times 6^7 + q\times 6^3 + 6r - 5 = -(-p\times 6^7 - q\times 6^3 - 6r - 5) - 10 = -3 - 10 = -13$

G2.3 If $n \neq 0$ and $s = \left(\frac{20}{2^{2n+4}+2^{2n+2}}\right)^{\frac{1}{n}}$, find the value of s.

$s = \left(\frac{20}{16\cdot 2^{2n}+4\cdot 2^{2n}}\right)^{\frac{1}{n}} = \left(\frac{20}{20\cdot 2^{2n}}\right)^{\frac{1}{n}} = \frac{1}{4}$

G2.4 Given that x and y are positive integers and $x + y + xy = 54$. If $t = x + y$, find the value of t.

$1 + x + y + xy = 55$

$(1 + x)(1 + y) = 55$

$1 + x = 5$, $1 + y = 11$ or $1 + x = 11$, $1 + y = 5$

$x = 4$, $y = 10$ or $x = 10$, $y = 4$

$t = 14$

Group Event 3

G3.1 Given that $r = 2006 \times \frac{\sqrt{8} - \sqrt{2}}{\sqrt{2}}$, find the value of r.

It is easy to show that $r = 2006$.

G3.2 Given that $6^{x+y} = 36$ and $6^{x+5y} = 216$, find the value of x.

$x + y = 2$ .... (1)

$x + 5y = 3$ .... (2)

5(1) – (2): $4x = 7 \Rightarrow x = \frac{7}{4}$

G3.3 Given that $\tan x + \tan y + 1 = \cot x + \cot y = 6$. If $z = \tan(x + y)$, find the value of z.

$\tan x + \tan y + 1 = \frac{\tan y + \tan x}{\tan x \tan y} = 6$

$\tan x + \tan y = 5$; $\tan x \tan y = \frac{5}{6}$

$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{5}{1 - \frac{5}{6}} = 30$

G3.4 In Figure 1, $ABCD$ is a rectangle, F is the midpoint of CD and $BE : EC = 1 : 3$. If the area of the rectangle $ABCD$ is 12 cm^2 and the area of $BEFD$ is R cm^2, find the value of R.

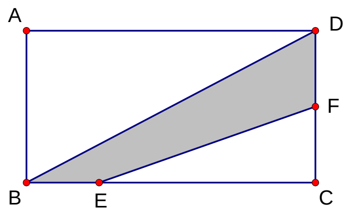


Figure 1

Area of ΔBCD = 6 cm^2

Area of $\Delta CEF = \frac{3}{4} \cdot \frac{1}{2} \cdot 6\text{ cm}^2 = \frac{9}{4}\text{ cm}^2$

Area of $BEFD = (6 - \frac{9}{4})\text{ cm}^2 = \frac{15}{4}\text{ cm}^2$

Group Event 4

G4.1 In Figure 1, *ABCD* is a parallelogram, $BE \perp CD$, $BF \perp AD$, $CE = 2$ cm, $DF = 1$ cm and $\angle EBF = 60°$.
If the area of the parallelogram *ABCD* is R cm^2, find the value of R.

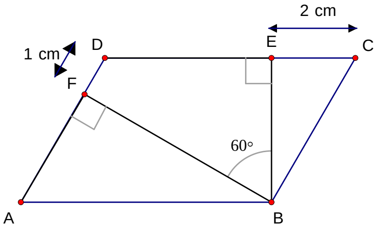


Figure 1

$\angle EDF = 360° - 90° - 90° - 60° = 120°$ ($\angle$s sum of polygon)
$\angle BAD = \angle BCD = 180° - 120° = 60°$ (int. $\angle$s //-lines)

$BC = \frac{2}{\cos 60^\circ}$ cm = 4 cm = AD

$BE = 2\tan 60° = 2\sqrt{3}$ cm
$AF = (4 - 1)$ cm = 3 cm

$AB = \frac{3}{\cos 60^\circ}$ cm = 6 cm

Area of $ABCD = AB \times BE = 6\times 2\sqrt{3}$ cm^2 = $12\sqrt{3}$ cm^2

G4.2 Given that a and b are positive numbers and $a + b = 2$. If $S = \left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2$, find the minimum value S. (**Reference: HKAL Pure Mathematics 1964 Paper 1 Q5 (b)**)

$\left(\sqrt{a} - \sqrt{b}\right)^2 \ge 0 \Rightarrow a + b - 2\sqrt{ab} \ge 0 \Rightarrow 1 \ge \sqrt{ab} \Rightarrow 1 \ge ab$ $\cdots\cdots$ (1)

$a^2 + b^2 = (a + b)^2 - 2ab = 4 - 2ab \ge 4 - 2 = 2$ $\cdots\ \cdots$ (2)

$\frac{1}{ab} \ge 1 \Rightarrow \frac{1}{a^2b^2} \ge 1$ $\cdots\cdots$ (3)

$$S = \left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 = a^2 + b^2 + \frac{1}{a^2} + \frac{1}{b^2} + 4$$

$$= a^2 + b^2 + \frac{a^2 + b^2}{a^2b^2} + 4$$

$$= \left(a^2 + b^2\right)\left(1 + \frac{1}{a^2b^2}\right) + 4$$

$\ge 2\times(1 + 1) + 4 = 8$ (by (2) and (3))

Remark: original question $\cdots$ a and b are positive real numbers $\cdots$

Positive numbers must be real, there is no need to emphasise the word 'real'.

G4.3 Let $2^x = 7^y = 196$. If $T = \frac{1}{x} + \frac{1}{y}$, find the value of T.

Reference: 2001 HI1, 2003 FG2.2, 2004 FG4.3, 2005 HI9

$x \log 2 = y \log 7 = \log 196$

$x = \frac{\log 196}{\log 2}$, $y = \frac{\log 196}{\log 7}$

$$T = \frac{1}{x} + \frac{1}{y} = \frac{\log 2 + \log 7}{\log 196} = \frac{\log 14}{\log 14^2} = \frac{1}{2}$$

Method 2 (provided by Denny)

$2 = 196^{\frac{1}{x}}$, $7 = 196^{\frac{1}{y}}$

$2\times 7 = 14 = \sqrt{196} = 196^{\frac{1}{x}} \times 196^{\frac{1}{y}} = 196^{\frac{1}{x}+\frac{1}{y}}$

$\frac{1}{x} + \frac{1}{y} = \frac{1}{2}$

G4.4 If $W = 2006^2 - 2005^2 + 2004^2 - 2003^2 + \ldots + 4^2 - 3^2 + 2^2 - 1^2$, find the value of W.

$W = (2006 + 2005)(2006–2005) + (2004 + 2003)(2004–2003) + \ldots + (4+3)(4–3) + (2+1)(2–1)$
$= 2006 + 2005 + 2004 + \ldots + 4 + 3 + 2 + 1$
$= \frac{2006}{2}(2006 + 1) = 1003\times 2007 = 2013021$

Individual Events

I1			I2			I3			I4		
	a	2		*a*	16		*a*	–1		*A*	16
	b	1		*b*	160		*b*	17		*b*	$-\frac{1}{2}$
	c	–6		*c*	3		*c*	8		*c*	$\frac{3}{2}$
	d	$\frac{50}{11}$		*d*	$\frac{8}{27}$		*d*	18		*d*	6

Group Events

G1			G2			G3			G4		
	W	$\frac{1+\sqrt{5}}{2}$		*R*	18434		*b*	40		*x*	137
	T	29		*x*	6		*t*	$\frac{12}{5}(=2.4)$		*R*	$\frac{1}{2}$
	S	106		*y*	12100		*x*	$10\sqrt{3}$		*z*	77
	k	4		*Q*	9		*S*	25		*r*	6

Individual Event 1

I1.1 Let *a* be a real number and $\sqrt{a}=\sqrt{7+\sqrt{13}}-\sqrt{7-\sqrt{13}}$. Find the value of *a*.

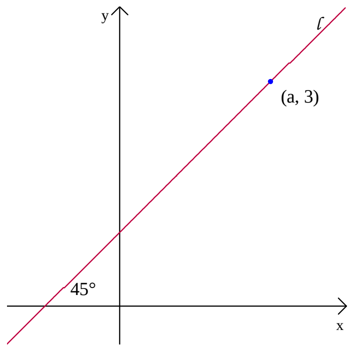
Reference: 2016 FI1.2

$$\left(\sqrt{a}\right)^2=\left(\sqrt{7+\sqrt{13}}-\sqrt{7-\sqrt{13}}\right)^2$$

$$a=7+\sqrt{13}-2\sqrt{7^2-\sqrt{13}^2}+7-\sqrt{13}$$

$$=14-2\sqrt{36}=2$$

I1.2 In Figure 1, the straight line ℓ passes though the point (*a*, 3), and makes an angle 45° with the *x*-axis. If the equation of ℓ is $x + my + n = 0$ and $b = |1 + m + n|$, find the value of *b*.



$$\ell:\ \frac{y-3}{x-2}=\tan 45^\circ$$

$y - 3 = x - 2$

$x - y + 1 = 0$, $m = -1$, $n = 1$

$b = |1 - 1 + 1| = 1$

I1.3 If $x - b$ is a factor of $x^3 - 6x^2 + 11x + c$, find the value of *c*.

$f(x) = x^3 - 6x^2 + 11x + c$

$f(1) = 1 - 6 + 11 + c = 0$

$c = -6$

I1.4 If $\cos x + \sin x = -\frac{c}{5}$ and $d = \tan x + \cot x$, find the value of *d*.

Reference: 1992 HI20, 1993 HG10, 1995 HI5, 2007 HI7, 2014 HG3

$$\cos x+\sin x=\frac{6}{5}$$

$$(\cos x+\sin x)^2=\frac{36}{25}$$

$$1+2\sin x\cos x=\frac{36}{25}$$

$$2\sin x\cos x=\frac{11}{25}\Rightarrow \sin x\cos x=\frac{11}{50}$$

$$d=\tan x+\cot x=\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}=\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}=\frac{1}{\sin x\cos x}=\frac{50}{11}$$

Individual Event 2

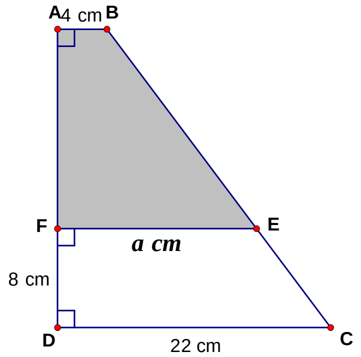
I2.1 Let $n = 1 + 3 + 5 + ... + 31$ and $m = 2 + 4 + 6 ... + 32$. If $a = m - n$, find the value of a.

$a = 2 + 4 + 6 \dots + 32 - (1 + 3 + 5 + \dots + 31)$

$= (2 - 1) + (4 - 3) + \dots + (32 - 31)$

$= 1 + 1 + \dots + 1 = 16$

I2.2 If Figure 1, $ABCD$ is a trapezium, $AB = 4$ cm, $EF = a$ cm, $CD = 22$ cm and $FD = 8$ cm, if the area of $ABEF$ is b cm^2, find the value of b.



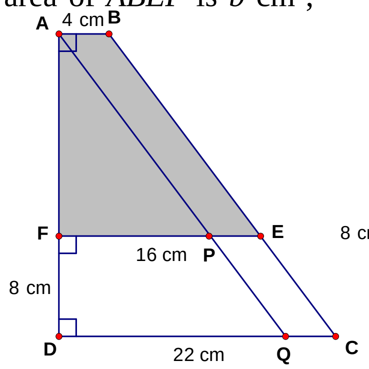
From A, draw a line APQ parallel to BC, cutting FE at P and DC at Q.

$FP = 12$ cm, $DQ = 18$ cm

$\Delta AFP \sim \Delta ADQ$

Let $AF = x$ cm $\Rightarrow \dfrac{x}{12} = \dfrac{x+8}{18}$, $x = 16$

$b = \dfrac{(4+16)16}{2} = 160$

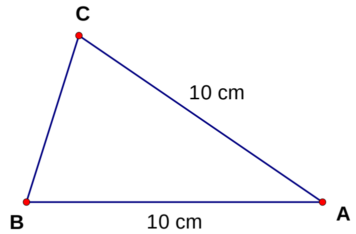


I2.3 In Figure 2, ΔABC is a triangle, $AB = AC = 10$ cm and $\angle ABC = b° - 100°$. If ΔABC has c axis of symmetry, find the value of c.

$\angle ABC = 160° - 100° = 60° = \angle ACB = \angle BAC$

ΔABC is an equilateral triangle.

It has 3 axis of symmetry.



I2.4 Let d be the least real root of the $cx^{\frac{2}{3}} - 8x^{\frac{1}{3}} + 4 = 0$, find the value of d.

$3x^{\frac{2}{3}} - 8x^{\frac{1}{3}} + 4 = 0 \Rightarrow \left(3x^{\frac{1}{3}} - 2\right)\left(x^{\frac{1}{3}} - 2\right) = 0$

$x^{\frac{1}{3}} = \dfrac{2}{3}$ or 2

$x = \dfrac{8}{27}$ or 8, the least real root is $\dfrac{8}{27}$.

Individual Event 3

I3.1 Suppose that $a = \cos^4\theta - \sin^4\theta - 2\cos^2\theta$, find the value of a.

$a = (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta) - 2\cos^2\theta$

$= \cos^2\theta - \sin^2\theta - 2\cos^2\theta = -(\sin^2\theta + \cos^2\theta) = -1$

I3.2 If $x^y = 3$ and $b = x^{3y} + 10a$, find the value of b.

$b = (x^y)^3 - 10 = 3^3 - 10 = 27 - 10 = 17$

I3.3 If there is (are) c positive integer(s) n such that $\dfrac{n+b}{n-7}$ is also a positive integer, find the value of c.

$\dfrac{n+17}{n-7} = 1 + \dfrac{24}{n-7}$

$n - 7 = 1, 2, 3, 4, 6, 8, 12, 24$

$c = 8$

I3.4 Suppose that $d = \log_4 2 + \log_4 4 + \log_4 8 + \cdots + \log_4 2^c$, find the value of d.

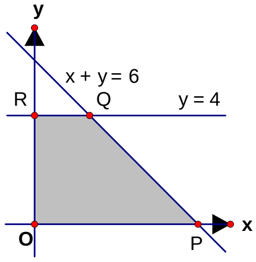
$d = \log_4 2 + \log_4 4 + \log_4 8 + \cdots + \log_4 2^8$

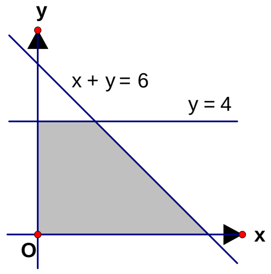
$= \log_4(2\times4\times8\times\cdots\times2^8) = \log_4(2^{1+2+3+\dots+8})$

$= \log_4(2^{36}) = \dfrac{\log 2^{36}}{\log 4} = \dfrac{36\log 2}{2\log 2} = 18$

Individual Event 4

I4.1 In Figure 1, let the area of the closed region bounded by the straight line $x + y = 6$ and $y = 4$, $x = 0$ and $y = 0$ be A square units, find the value of A.





As shown in the figure, the intersection points

$P(6, 0)$, $Q(2, 4)$, $R(0, 6)$

$OP = 6$, $OR = 4$, $QR = 2$

Area $= A = \frac{1}{2}(6+2)\cdot 4 = 16$

I4.2 Let $[x]$ be the largest integer not greater than x. For example, $[2.5] = 2$.

If b satisfies the system of equations $\begin{cases} Ax^2 - 4 = 0 \\ 3 + 2(x + [x]) = 0 \end{cases}$, find the value of b.

$\begin{cases} 16x^2 - 4 = 0 \\ 3 + 2(x + [x]) = 0 \end{cases}$ from the first equation $x = \frac{1}{2}$ or $-\frac{1}{2}$.

Substitute $x = \frac{1}{2}$ into the second equation: LHS $= 3 + 2(\frac{1}{2} + 0) = 4 \neq$ RHS

Substitute $x = -\frac{1}{2}$ into the second equation: LHS $= 3 + 2(-\frac{1}{2} - 1) = 0 =$ RHS

$\therefore b = -\frac{1}{2}$

I4.3 Let c be the constant term in the expansion of $(2x + \frac{b}{\sqrt{x}})^3$. Find the value of c.

$(2x + \frac{b}{\sqrt{x}})^3 = 8x^3 + 12bx\sqrt{x} + 6b^2 + \frac{b^3}{x\sqrt{x}}$

c = the constant term

$= 6b^2$

$= 6(-\frac{1}{2})^2$

$= \frac{3}{2}$

I4.4 If the number of integral solutions of the inequality $\left|\frac{x}{2} - \sqrt{2}\right| < c$ is d, find the value of d.

$\left|\frac{x}{2} - \sqrt{2}\right| < \frac{3}{2}$

$-\frac{3}{2} < \frac{x}{2} - \sqrt{2} < \frac{3}{2}$

$2\sqrt{2} - 3 < x < 2\sqrt{2} + 3$

$2(1.4) - 3 < x < 2(1.4) + 3$

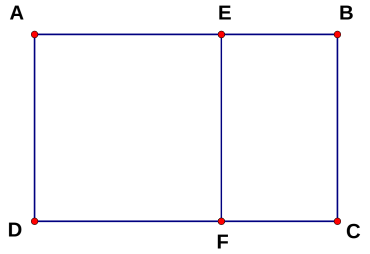
$-0.2 < x < 5.8$

$x = 0, 1, 2, 3, 4, 5$

$d = 6$

Group Event 1

G1.1 In Figure 1, *AEFD* is a unit square. The ratio of the length of the rectangle *ABCD* to its width is equal to the ratio of the length of the rectangle *BCFE* to its width. If the length of *AB* is *W* units, find the value of *W*.

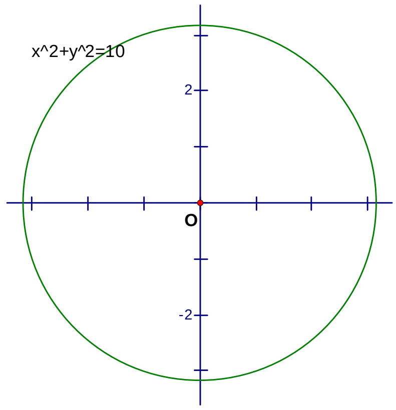


$$\frac{W}{1}=\frac{1}{W-1}$$

$$W^2 - W - 1 = 0 \Rightarrow W=\frac{1+\sqrt{5}}{2}$$

G1.2 On the coordinate plane, there are T points (x, y), where x, y are integers, satisfying $x^2 + y^2 < 10$, find the value of T. **(Reference: 2002 FI4.3)**

T = number of integral points inside the circle $x^2 + y^2 = 10$.



We first count the number of integral points in the first quadrant:

$x = 1$; $y = 1, 2$

$x = 2$; $y = 1, 2$

Next, the number of integral points on the x-axis and y-axis

$= 3 + 3 + 3 + 3 + 1 = 13$

$T = 4\times4 + 3 + 3 + 3 + 3 + 1$

$= 29$

G1.3 Let P and $P + 2$ be both prime numbers satisfying $P(P + 2) \le 2007$.

If S represents the sum of such possible values of P, find the value of S.

$P^2 + 2P - 2007 \le 0$

$(P + 1)^2 - 2008 \le 0$

$(P + 1 + \sqrt{2008})(P + 1 - \sqrt{2008}) \le 0$

$(P + 1 + 2\sqrt{502})(P + 1 - 2\sqrt{502}) \le 0$

$-1 - 2\sqrt{502} \le P \le -1 + 2\sqrt{502}$

P is a prime $\Rightarrow 0 < P \le -1 + 2\sqrt{502}$

$22 = \sqrt{484} < \sqrt{502} < \sqrt{529} = 23$

$43 < -1 + 2\sqrt{502} < 45$

$\therefore (P, P + 2) = (3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43)$

$S = 3 + 5 + 11 + 17 + 29 + 41 = 106$

G1.4 It is known that $\log_{10}(2007^{2006} \times 2006^{2007}) = a\times10^k$, where $1 \le a < 10$ and k is an integer. Find the value of k.

$a\times10^k = 2006 \log 2007 + 2007 \log 2006 = 2006\times(\log 2007 + \log 2006) + \log 2006$

$2006\times(\log 2006 + \log 2006) + \log 2006 < a\times10^k < 2006\times(\log 2007 + \log 2007) + \log 2007$

$4013 \log 2006 < a\times10^k < 4013 \log 2007$

$4013 \log(2.006\times10^3) < a\times10^k < 4013 \log(2.007\times10^3)$

$4013 (\log 2.006 + 3) < a\times10^k < 4013(\log 2.007 + 3)$

$4013 \log 2 + 4013\times3 < a\times10^k < 4013 \log 3 + 3$

$1.32429\times10^4 = 4013\times0.3 + 4013\times3 < a\times10^k < 4013\times0.5 + 4013\times3 = 1.40455\times10^4$

$k = 4$

Group Event 2

G2.1 If $R = 1\times2 + 2\times2^2 + 3\times2^3 + \ldots + 10\times2^{10}$, find the value of R.

Reference: 2005 HI7, 2005 FG2.4

$2R = \quad 1\times2^2 + 2\times2^3 + \ldots + 9\times2^{10} + 10\times2^{11}$

$R - 2R = 2 + 2^2 \quad + 2^3 + \ldots\ldots + 2^{10} \quad - 10\times2^{11}$

$-R = \dfrac{a(R^n - 1)}{R-1} - 10\times2^{11} = \dfrac{2(2^{10}-1)}{2-1} - 10\times2048$

$R = 20480 - 2(1023) = 18434$

G2.2 If integer x satisfies $x \ge 3+\sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3}}}}}$, find the minimum value of x.

Let $y = 3+\sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3+\cdots}}}}$ (to infinity)

$(y-3)^2 = 3+\sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3+\cdots}}}} = y$

$y^2 - 7y + 9 = 0$

$y = \dfrac{7+\sqrt{13}}{2}$ or $\dfrac{7-\sqrt{13}}{2}$

Clearly $y > 3$ and $\dfrac{7-\sqrt{13}}{2} < 3$

$\therefore y = \dfrac{7+\sqrt{13}}{2}$ only

$5 = \dfrac{7+\sqrt{9}}{2} < \dfrac{7+\sqrt{13}}{2} < \dfrac{7+\sqrt{16}}{2} = 5.5$

$3+\sqrt{3+\sqrt{3}} > 3+\sqrt{3+1.7} > 3+\sqrt{4.41} = 3 + 2.1 = 5.1$

$5.1 < 3+\sqrt{3+\sqrt{3}} < 3+\sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3}}}}} < 3+\sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3+\cdots}}}} < 5.5$

$x = 6$

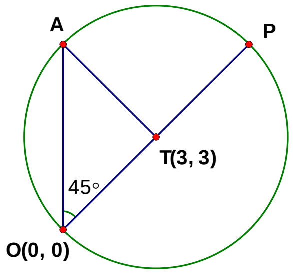
G2.3 Let $y = \dfrac{146410000 - 12100}{12099}$, find the value of y.

$y = \dfrac{12100^2 - 12100}{12100-1}$

$= \dfrac{12100(12100-1)}{12100-1}$

$= 12100$

G2.4 On the coordinate plane, a circle with centre $T(3, 3)$ passes through the origin $O(0, 0)$. If A is a point on the circle such that $\angle AOT = 45°$ and the area of ΔAOT is Q square units, find the value of Q.



$OT = \sqrt{3^2 + 3^2} = 3\sqrt{2}$

$OT = AT =$ radii

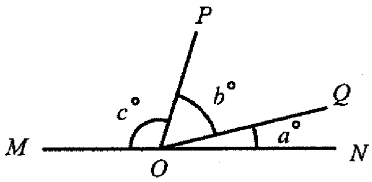
$\angle OAT = 45°$ (side opp. eq. $\angle$s)

$\angle ATO = 90°$ ($\angle$s sum of Δ)

$Q = \dfrac{1}{2} OT\cdot AT = \dfrac{1}{2}\cdot\left(3\sqrt{2}\right)^2 = 9$

Group Event 3

G3.1 In figure 1, MN is a straight line, $\angle QON = a°$, $\angle POQ = b°$ and $\angle POM = c°$. If $b : a = 2 : 1$ and $c : b = 3 : 1$, find the value of b.



$b = 2a$, $c = 3b = 6a$

$a + b + c = 180$ (adj. ∠s on st. line)

$a + 2a + 6a = 180 \Rightarrow a = 20$

$b = 2a = 40$

G3.2 It is known that $\sqrt{\frac{50+120+130}{2}\times(150-50)\times(150-120)\times(150-130)} = \frac{50\times130\times k}{2}$.

If $t = \frac{k}{\sqrt{1-k^2}}$, find the value of t.

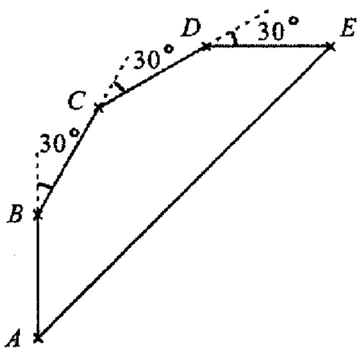
The question is equivalent to: given a triangle with sides 50, 120, 130, find its area.

$\cos C = \frac{50^2 + 130^2 - 120^2}{2\cdot 50\cdot 130} = \frac{5}{13}$

Using the formula $\frac{1}{2}ab\sin C = \frac{50\times130\times k}{2}$, $k = \sin C = \sqrt{1-\cos^2 C} = \frac{12}{13}$

$t = \frac{k}{\sqrt{1-k^2}} = \frac{\sin C}{\cos C} = \tan C = \frac{12}{5}$

G3.3 In Figure 2, an ant runs ahead straightly for 5 sec 15° cm from point A to point B. It then turns 30° to the right and run 5 sec 15° cm to point C. Again it repeatedly turns 30° to the right and run 5 sec 15° cm twice to reach the points D and E respectively. If the distance of AE is x cm, find the value of x.

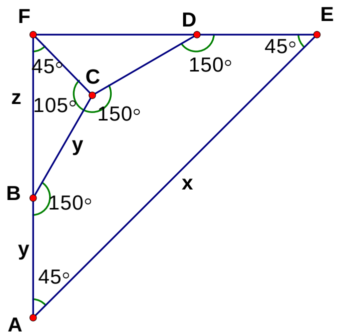


By symmetry $\angle BAE = \angle DEA = [180°\times(5-2) - 150°\times3] \div 2$

$= 45°$ (∠ sum of polygon)

Produce AB and ED to intersect at F.

$\angle AFE = 180° - 45° - 45° = 90°$ (∠s sum of Δ)



By symmetry, $\angle BFC = \angle DFC = 45°$

$\angle BCF = \angle DCF = (360° - 150°) \div 2 = 105°$ (∠s at a pt.)

Let $AB = y = 5$ sec 15° cm $= CD = DE$, let $z = BF$.

Apply Sine rule on ΔABC, $\frac{z}{\sin 105°} = \frac{y}{\sin 45°}$

$z = \sqrt{2}\sin 105°\, y$

In ΔAEF, $x = (y + z)\sec 45° = \sqrt{2}\,(y + \sqrt{2}\sin 105°\, y)$

$= y\sqrt{2}\left(1+\sqrt{2}\sin 105°\right)$

$= 5\sec 15°\cdot 2\left(\frac{1}{\sqrt{2}} + \sin 105°\right) = 10\sec 15°\,(\sin 105° + \sin 45°)$

$= 10\sec 15°(2\sin 75°\cos 30°) = 20\sec 15°\cdot\cos 15°\ \frac{\sqrt{3}}{2} = 10\sqrt{3}$

Method 2

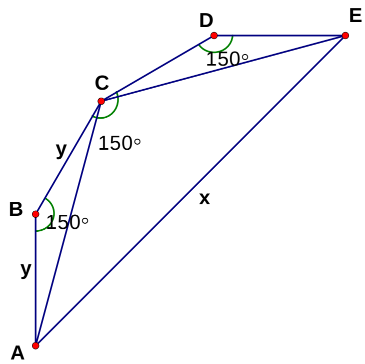
Join AC, CE. With similar working steps, $\angle BAE = \angle DEA = 45°$

$\angle BAC = \angle BCA = 15° = \angle DCE = \angle DEC$ (∠s sum of isos. Δ)

$\angle CAE = 45° - 15° = 30° = \angle CEA$

$AC = CE = 2y\cos 15° = 2\times5\sec 15° \times \cos 15° = 10$

$x = 2\,AC\cos 30° = 20\times\frac{\sqrt{3}}{2} = 10\sqrt{3}$



G3.4 There are 4 problems in a mathematics competition. The scores of each problem are allocated in the following ways: 2 marks will be given for a correct answer, 1 mark will be deducted from a wrong answer and 0 mark will be given for a blank answer. To ensure that 3 candidates will have the same scores, there should be at least S candidates in the competition. Find the value of S.

We shall tabulate different cases:

case no.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	marks for each question
correct	4	3	3	2	2	2	1	1	1	1	0	0	0	0	0	2
blank	0	1	0	2	0	1	3	2	1	1	4	3	2	1	0	0
wrong	0	0	1	0	2	1	0	1	2	3	0	1	2	3	4	–1
Total	8	6	5	4	2	3	2	1	0	–1	0	–1	–2	–3	–4	

The possible total marks for one candidate to answer 4 questions are:

8, 6, 5, 4, 3, 2, 1, 0, –1, –2, –3, –4; altogether 12 possible combinations.

To ensure **3** candidates will have the same scores, we consider the worst scenario:

Given that there are 24 candidates. 2 candidates score 8 marks, 2 candidates score 6 marks, ..., 2 candidates score –4 marks, then the 25th candidate will score the same as the other two candidates.

Group Event 4

G4.1 Let x be the number of candies satisfies the inequalities $120 \le x \le 150$. 2 candies will be remained if they are divided into groups of 5 candies each; 5 candies will be remained if they are divided into groups of 6 candies each. Find the value of x.

$x = 5m + 2 = 6n + 5$, where m and n are integers.

$5m - 6n = 3$

$5\times3 - 6\times2 = 15 - 12 = 3$

$\therefore m = 3, n = 2$ is a pair of solution

The general solution is $m = 3 + 6t$, $n = 2 + 5t$, where t is any integer.

$x = 5m + 2 = 5(3 + 6t) + 2 = 30t + 17$

$120 \le x \le 150 \Rightarrow 120 \le 30t + 17 \le 150$

$103 \le 30t \le 133$

$3.43 < t < 4.43 \Rightarrow t = 4$

$x = 30\times4 + 17 = 137$

G4.2 On the coordinate plane, the points $A(3, 7)$ and $B(8, 14)$ are reflected about the line $y = kx + c$, where k and c are constants, their images are $C(5, 5)$ and $D(12,10)$ respectively. If $R = \frac{k}{c}$, find the value of R.

By the property of reflection, the line $y = kx + c$ is the perpendicular bisector of A, C and B, D.

That is to say, the mid points of A, C and B, D lies on the line $y = kx + c$

M = mid point of A, $C = (4, 6)$, N = mid point of B, $D = (10, 12)$

By two points form, $\frac{y-6}{x-4} = \frac{12-6}{10-4}$

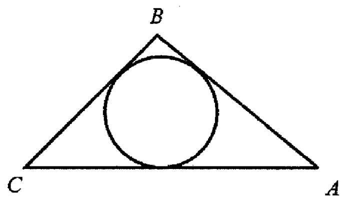
$y = x + 2 \Rightarrow k = 1, c = 2, R = \frac{1}{2}$

G4.3 Given that $z = \sqrt[3]{456533}$ is an integer, find the value of z.

$70 = \sqrt[3]{343000} < \sqrt[3]{456533} < \sqrt[3]{512000} = 80$

By considering the cube of the unit digit, the only possible answer for z is 77.

G4.4 In Figure 1, ΔABC is an isosceles triangle, $AB = BC = 20$ cm and $\tan \angle BAC = \frac{4}{3}$. If the length of radius of the inscribed circle of ΔABC is r cm, find the value of r. **Reference: 2013 HG8**



$\angle BAC = \angle BCA$; $\sin \angle BAC = \frac{4}{5}$, $\cos \angle BAC = \frac{3}{5}$.

$AC = 2\times20 \cos \angle BAC = 40\times\frac{3}{5} = 24$, the height of ΔABC from $B = 20 \sin \angle BAC = 16$

Area of $\Delta ABC = \frac{1}{2}\cdot 24\cdot 16 = 192 = \frac{r}{2}(20 + 20 + 24)$

$r = 6$

Individual Events

SI	k	2	I1	A	15	I2	P	3	I3	A	2	I4	P	4	IS	a	2
	d	1		B	3		Q	4		B	3		Q	300		b	2
	a	–6		C	4		R	10		C	45		R	2		c	–3
	t	$\frac{50}{11}$		D	8		S	112.5		D	7		S	$2\sqrt{3}$		d	35

Group Events

SG	W	$\frac{1+\sqrt{5}}{2}$	G1	m	8	G2	z	540	G3	k	$\sqrt{33}$	G4	m	13	GS	value	–1
	T	29		h	$\sqrt{13}$		R	6		v	6		n	6		$x^4+\frac{1}{x^4}$	4036079
	S	106		$x+y+z$	11		k	5		value	106		$abc+def$	72		$\cot\alpha$	$\frac{99}{20}$
	k	4		Number	72		xyz	1		r	27405		$p+q$	2		value	$\frac{6023}{6022}$

Sample Individual Event (2007 Final Individual Event 1)

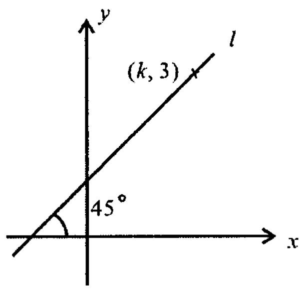
SI.1 Let $\sqrt{k}=\sqrt{7+\sqrt{13}}-\sqrt{7-\sqrt{13}}$, find the value of k.

$$\sqrt{k}^2=\left(\sqrt{7+\sqrt{13}}-\sqrt{7-\sqrt{13}}\right)^2$$

$$k=7+\sqrt{13}-2\sqrt{7^2-\sqrt{13}^2}+7-\sqrt{13}$$

$$=14-2\sqrt{36}=2$$

SI.2 In Figure 1, the straight line ℓ passes though the point (k, 3) and makes an angle 45° with the x-axis. If the equation of ℓ is $x + by + c = 0$ and $d = |1 + b + c|$, find the value of d.



$$\ell: \frac{y-3}{x-2}=\tan 45^\circ$$

$y - 3 = x - 2$

$x - y + 1 = 0$, $b = -1$, $c = 1$

$d = |1 - 1 + 1| = 1$

SI.3 If $x - d$ is a factor of $x^3 - 6x^2 + 11x + a$, find the value of a.

$f(x) = x^3 - 6x^2 + 11x + a$

$f(1) = 1 - 6 + 11 + a = 0$

$a = -6$

SI.4 If $\cos x + \sin x = -\frac{a}{5}$ and $t = \tan x + \cot x$, find the value of t.

$$\cos x+\sin x=\frac{6}{5}$$

$$(\cos x+\sin x)^2=\frac{36}{25}$$

$$1+2\sin x\cos x=\frac{36}{25}$$

$$2\sin x\cos x=\frac{11}{25}\Rightarrow \sin x\cos x=\frac{11}{50}$$

$$d=\tan x+\cot x$$

$$=\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}$$

$$=\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}$$

$$\frac{1}{\sin x\cos x}=\frac{50}{11}$$

Individual Event 1

I1.1 Let A = 15×tan 44°×tan45°×tan46°, find the value of A.
Similar question: 2012 FG2.1

$$A = 15\times\tan 44^\circ\times 1\times\frac{1}{\tan 44^\circ} = 15$$

I1.2 Let n be a positive integer and $\overbrace{20082008\cdots2008}^{n\ 2008's}15$ is divisible by A.

If the least possible value of n is B, find the value of B.

Reference: 2010 HG2

The given number is divisible by 15. Therefore it is divisible by 3 and 5.

The last 2 digits of the given number is 15, which is divisible by 15.

The necessary condition is: $\overbrace{20082008\cdots2008}^{n\ 2008's}$ must be divisible by 3.

2 + 0 + 0 + 8 = 10 which is not divisible by 3.

The least possible n is 3: 2+0+0+8+2+0+0+8+2+0+0+8 = 30 which is divisible by 3.

I1.3 Given that there are C integers that satisfy the equation $|x - 2| + |x + 1| = B$, find the value of C
Reference: 1994 HG1, 2001 HG9, 2004 FG4.2, 2008 HI8, 2008 FI1.3, 2010 HG6, 2012 FG2.3
$|x - 2| + |x + 1| = 3$
If $x < -1$, $2 - x - x - 1 = 3$
$\Rightarrow x = -1$ (rejected)
If $-1 \le x \le 2$, $2 - x + x + 1 = 3$
$\Rightarrow 3 = 3$, always true $\therefore$
$-1 \le x \le 2$
If $2 < x$, $x - 2 + x + 1 = 3$
$\Rightarrow x = 2$ (reject)
$\therefore -1 \le x \le 2$ only
$\because$ x is an integer
$\therefore x = -1, 0, 1, 2$; $C = 4$

I1.4 In the coordinate plane, the distance from the point $(-C, 0)$ to the straight line $y = x$ is $\sqrt{D}$, find the value of D.

The distance from P(x_0, y_0) to the straight line $Ax + By + C = 0$ is $\left|\frac{Ax_0 + By_0 + C}{\sqrt{A^2 + B^2}}\right|$.

The distance from $(-4, 0)$ to $x - y = 0$ is

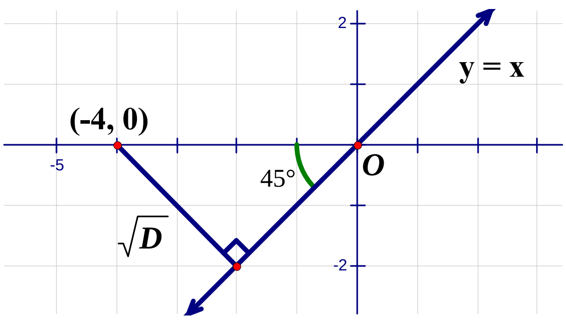
$$\sqrt{D} = \left|\frac{-4-0+0}{\sqrt{1^2+(-1)^2}}\right| = \frac{4}{\sqrt{2}} = 2\sqrt{2} = \sqrt{8}$$

$D = 8$

Method 2

$\sqrt{D} = 4\sin 45^\circ = \frac{4}{\sqrt{2}}$

$D = \frac{16}{2} = 8$



Individual Event 2

I2.1 Given that $P=\left[\sqrt[3]{6}\times\left(\sqrt[3]{\frac{1}{162}}\right)\right]^{-1}$, find the value of P.

$P=\left[\sqrt[3]{\frac{6}{162}}\right]^{-1}$

$=\sqrt[3]{\frac{162}{6}}$

$=\sqrt[3]{27}=3$

I2.2 Let a, b and c be real numbers with ratios $b:(a+c)=1:2$ and $a:(b+c)=1:P$.

If $Q=\frac{a+b+c}{a}$, find the value of Q.

$2b=a+c$ …… (1), $3a=b+c$ …… (2)

(1) – (2): $2b-3a=a-b\Rightarrow 3b=4a\Rightarrow a:b=3:4$

Let $a=3k$, $b=4k$, sub. into (1): $2(4k)=3k+c\Rightarrow c=5k$

$Q=\frac{a+b+c}{a}=\frac{3k+4k+5k}{3k}=4$

I2.3 Let $R=\left(\sqrt{\sqrt{3}+\sqrt{2}}\right)^{Q}+\left(\sqrt{\sqrt{3}-\sqrt{2}}\right)^{Q}$. Find the value of R.

$R=\left(\sqrt{\sqrt{3}+\sqrt{2}}\right)^{4}+\left(\sqrt{\sqrt{3}-\sqrt{2}}\right)^{4}$

$=\left(\sqrt{3}+\sqrt{2}\right)^{2}+\left(\sqrt{3}-\sqrt{2}\right)^{2}$

$=3+2\sqrt{6}+2+3-2\sqrt{6}+2=10$

I2.4 Let $S=(x-R)^2+(x+5)^2$, where x is a real number. Find the minimum value of S.

$S=(x-10)^2+(x+5)^2$

$=x^2-20x+100+x^2+10x+25$

$=2x^2-10x+125$

$=2(x^2-5x)+125$

$=2(x-2.5)^2+125-2\times2.5^2$

$=2(x-2.5)^2+112.5\geq 112.5$

The minimum value of S is 112.5.

Method 2

$S=(10-x)^2+(x+5)^2$

Let $a=10-x$, $b=x+5$

$a+b=15$, which is a constant

$\therefore a^2+b^2$ reaches its minimum when $a=b=7.5$

$\therefore$ Minimum $S=7.5^2+7.5^2$

$=112.5$

Individual Event 3

I3.1 Given that $\frac{1-\sqrt{3}}{2}$ satisfies the equation $x^2 + px + q = 0$, where p and q are rational numbers. If $A = |p| + 2|q|$, find the value of A.

For an equation with rational coefficients, conjugate roots occur in pairs.

That is, the other root is $\frac{1+\sqrt{3}}{2}$.

$\frac{1-\sqrt{3}}{2}+\frac{1+\sqrt{3}}{2}=-p$

$\Rightarrow p = -1$

$\frac{1-\sqrt{3}}{2}\times\frac{1+\sqrt{3}}{2}=q$

$\Rightarrow q=-\frac{1}{2}$

$A = |-1| + 2\left|-\frac{1}{2}\right| = 2$

I3.2 Two bags U_1 and U_2 contain identical red and white balls. U_1 contains A red balls and 2 white balls. U_2 contains 2 red balls and B white balls. Take two balls out of each bag. If the probability of all four balls are red is $\frac{1}{60}$, find the value of B.

U_1 contains 2 red and 2 white, total 4 balls. U_2 contains 2 red and B white, total $2 + B$ balls.

P(all 4 are red) $=\frac{2}{4}\times\frac{1}{3}\times\frac{2}{2+B}\times\frac{1}{1+B}=\frac{1}{60}$

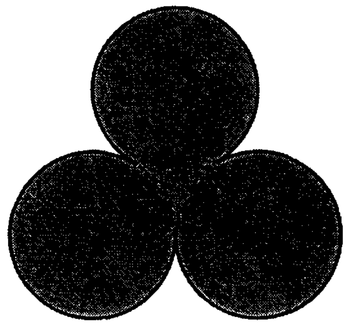
$20 = (2 + B)(1 + B)$

$B^2 + 3B - 18 = 0$

$(B - 3)(B + 6) = 0$

$B = 3$

I3.3 Figure 1 is formed by three identical circles touching one another, the radius of each circle is B cm. If the perimeter of the shaded region is C cm, find the value of C. (Take $\pi = 3$)

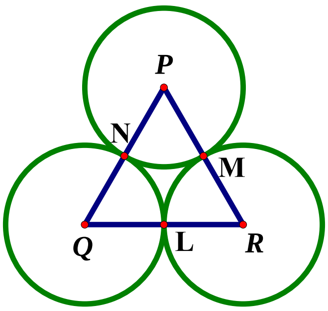


Let the centres of the circles be P, Q and R respectively.

Then $PQ = 2B = 6 = QR = PR$

ΔPQR is an equilateral Δ.

$\angle P = \angle Q = \angle R = 60°$



Perimeter $= 3\times 2\pi\times 3 - 3\times \overset{\frown}{MN}$

$= 18\pi - 3\times 2\pi\times 3\times\frac{60}{360}$

$= 15\pi = 45$

I3.4 Let D be the integer closest to $\sqrt{C}$, find the value of D.

$6=\sqrt{36}<\sqrt{45}<\sqrt{49}=7$

$6 < D < 7$

$6.5^2 = 42.25 < 45$

$6.5 < D < 7$

$D = 7$

Method 2

$6=\sqrt{36}<\sqrt{45}<\sqrt{49}=7 \Rightarrow 6 < D < 7$

$\because |45 - 36| = 9, |45 - 49| = 4$

$\therefore$ 45 is closer to 49

$\therefore \sqrt{45}$ is closer to 7

$D = 7$

Individual Event 4

I4.1 Given that x and y are real numbers such that $|x| + x + y = 10$ and $|y| + x - y = 10$. If $P = x + y$, find the value of P.

If $x \ge 0, y \ge 0$; $\begin{cases} 2x + y = 10 \\ x = 10 \end{cases}$, solving give $x = 10$, $y = -10$ (reject)

If $x \ge 0, y < 0$, $\begin{cases} 2x + y = 10 \\ x - 2y = 10 \end{cases}$, solving give $x = 6$, $y = -2$

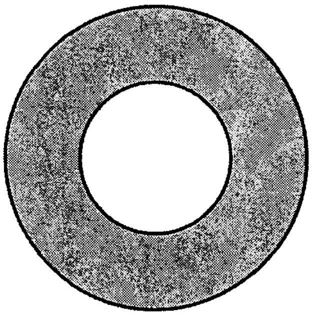
If $x < 0, y \ge 0$, $\begin{cases} y = 10 \\ x = 10 \end{cases}$, reject

If $x < 0, y < 0$; $\begin{cases} y = 10 \\ x - 2y = 10 \end{cases}$, solving give $x = 30$, $y = 10$ (reject)

$\therefore x = 6, y = -2$

$P = x + y = 6 - 2 = 4$

I4.2 In Figure 1, the shaded area is formed by two concentric circles and has area 96π cm^2. If the two radii differ by $2P$ cm and the large circle has area Q cm^2, find the value of Q. (Take $\pi = 3$)



Let the radii of the large and small circles be R and r respectively.

$\pi(R^2 - r^2) = 96\pi$ and $R - r = 8$ … (1)

$(R + r)(R - r) = 96 \Rightarrow (R + r)\times 8 = 96 \Rightarrow R + r = 12$ … (2)

(2) + (1): $2R = 20 \Rightarrow R = 10$

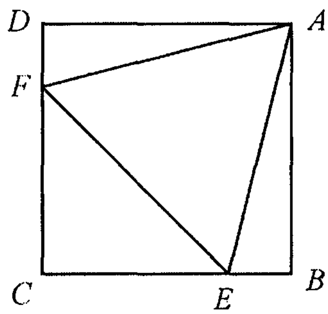
$\Rightarrow Q = \pi(10)^2 = 300$

I4.3 Let R be the largest integer such that $R^Q < 5^{200}$, find the value of R.

Reference: 1996 HI4, 1999 FG5.3, 2008 FG2.4

$R^{300} < 5^{200} \Rightarrow R^3 < 25$, the largest integer is 2.

I4.4 In Figure 2, there are a square $ABCD$ with side length $(R - 1)$ cm and an equilateral triangle AEF. (E and F are points on BC and CD respectively). If the area of ΔAEF is $(S - 3)$ cm^2, find the value of S.



Reference: 1995 HG7

Let $AF = x$ cm $= FE = AE$

$\angle FAE = 60°$, $\angle DAF = \angle BAE = 15°$

$AD = 1 = AF \cos 15° = x \cos 15° \Rightarrow x = \sec 15°$

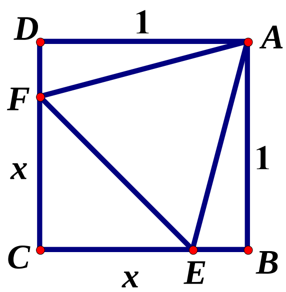
$$S - 3 = \frac{1}{2}x^2 \sin 60° = \frac{1}{2}\cdot\frac{1}{\cos^2 15°}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}\cdot\frac{1}{1+\cos 30°} = \frac{\sqrt{3}}{2}\cdot\frac{1}{1+\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}}{2}\cdot\frac{2}{2+\sqrt{3}} = 2\sqrt{3} - 3$$

$S = 2\sqrt{3}$

Method 2

Let $CE = CF = x$ cm

Then $BE = DF = (1 - x)$ cm



By Pythagoras' theorem,

$AF = FE \Rightarrow 1 + (1 - x)^2 = x^2 + x^2$

$2 - 2x + x^2 = 2x^2$

$x^2 + 2x - 2 = 0 \Rightarrow x^2 = 2 - 2x$

$x = -1 + \sqrt{3}$

Area of ΔAFE = Area of square – area of ΔCEF – 2 area of ΔADF

$$= 1 - \frac{x^2}{2} - 2\times\frac{1\times(1-x)}{2} = x - \frac{x^2}{2} = x - \frac{2-2x}{2} = 2x - 1$$

$$= 2(-1+\sqrt{3}) - 1 = 2\sqrt{3} - 3$$

$S - 3 = 2\sqrt{3} - 3$

$S = 2\sqrt{3}$

Individual Spare

IS.1 If all the positive factors of 28 are $d_1, d_2, \ldots, d_n$ and $a = \frac{1}{d_1}+\frac{1}{d_2}+\cdots+\frac{1}{d_n}$, find the value of a.

Positive factors of 28 are 1, 2, 4, 7, 14, 28. $a = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{7}+\frac{1}{14}+\frac{1}{28} = 2$

IS.2 Given that x is a negative real number that satisfy $\frac{1}{x+\frac{1}{x+2}} = a$. If $b = x+\frac{7}{2}$, find the value of b

$\frac{1}{x+\frac{1}{x+2}} = 2 \Rightarrow \frac{x+2}{x(x+2)+1} = 2$

$\Rightarrow x + 2 = 2x^2 + 4x + 2$

$\Rightarrow 2x^2 + 3x = 0$

$\Rightarrow x = -1.5$ or 0 (reject)

$b = -1.5 + 3.5 = 2$

IS.3 Let α and β be the two roots of the equation $x^2 + cx + b = 0$, where $c < 0$ and $\alpha - \beta = 1$. Find the value of c.

Reference: 2016 FI1.3

$\alpha\beta = b = 2$; $(\alpha - \beta)^2 = 1$

$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = 1$

$\Rightarrow (-c)^2 - 4\times2 = 1$

$c = -3$

IS.4 Let d be the remainder of $(196c)^{2008}$ divided by 97. Find the value of d.

$[196\times(-3)]^{2008} = 588^{2008} = (97\times6 + 6)^{2008}$

$= (97\times6)^{2008} + {}_{2008}C_1\cdot(97\times6)^{2007}\times6 + \ldots + 6^{2008}$

$= 97m + 6^{2008}$, where m is an integer.

Note that $2^5\times3 = 96 = 97 - 1 \equiv -1 \pmod{97}$;

$2\times3^5 = 486 = 97\times5 + 1 \equiv 1 \pmod{97}$;

$\therefore 6^6 = (2^5\times3)\times(2\times3^5) \equiv -1 \pmod{97}$

$6^{2008} = (6^6)^{334}\times6^4$

$\equiv (-1)^{334}\times1296$

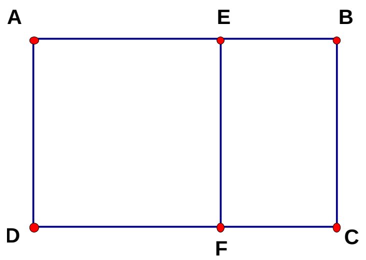
$\equiv 97\times13 + 35$

$\equiv 35 \pmod{97}$

$\therefore d = 35$

Sample Group Event (2007 Final Group Event 1)

SG.1 In Figure 1, *AEFD* is a unit square. The ratio of the length of the rectangle *ABCD* to its width is equal to the ratio of the length of the rectangle *BCFE* to its width. If the length of *AB* is *W* units, find the value of *W*.

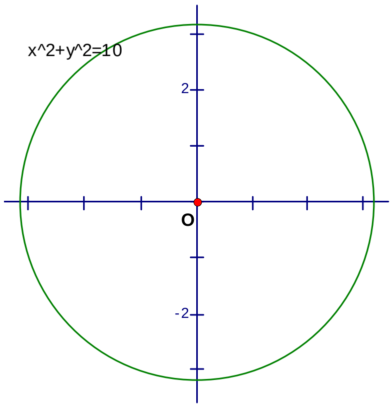


$\frac{W}{1} = \frac{1}{W-1}$

$W^2 - W - 1 = 0$

$\Rightarrow W = \frac{1+\sqrt{5}}{2}$

SG.2 On the coordinate plane, there are *T* points (*x*, *y*), where *x*, *y* are integers, satisfying $x^2+y^2 < 10$, find the value of *T*. **(Reference: 2002 FI4.3)**



T = number of integral points inside the circle $x^2 + y^2 = 10$.

We first count the number of integral points in the first quadrant:

$x = 1; y = 1, 2$

$x = 2; y = 1, 2$

Next, the number of integral points on the *x*-axis and *y*-axis

$= 3 + 3 + 3 + 3 + 1 = 13$

$T = 4\times 4 + 3 + 3 + 3 + 3 + 1$

$= 29$

SG.3 Let *P* and *P* + 2 be both prime numbers satisfying $P(P + 2) \le 2007$. If *S* represents the sum of such possible values of *P*, find the value of *S*.

$P^2 + 2P - 2007 \le 0$

$(P + 1)^2 - 2008 \le 0$

$(P + 1 + \sqrt{2008})(P + 1 - \sqrt{2008}) \le 0$

$(P + 1 + 2\sqrt{502})(P + 1 - 2\sqrt{502}) \le 0$

$-1 - 2\sqrt{502} \le P \le -1 + 2\sqrt{502}$

P is a prime $\Rightarrow 0 < P \le -1 + 2\sqrt{502}$

$22 = \sqrt{484} < \sqrt{502} < \sqrt{529} = 23$

$43 < -1 + 2\sqrt{502} < 45$

$\therefore (P, P + 2) = (3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43)$

$S = 3 + 5 + 11 + 17 + 29 + 41 = 106$

SG.4 It is known that $\log_{10}(2007^{2006} \times 2006^{2007}) = a\times 10^k$, where $1 \le a < 10$ and *k* is an integer. Find the value of *k* .

$a\times 10^k = 2006 \log 2007 + 2007 \log 2006 = 2006\times(\log 2007 + \log 2006) + \log 2006$

$2006\times(\log 2006 + \log 2006) + \log 2006 < a\times 10^k < 2006\times(\log 2007 + \log 2007) + \log 2007$

$4013 \log 2006 < a\times 10^k < 4013 \log 2007$

$4013 \log(2.006\times 10^3) < a\times 10^k < 4013 \log(2.007\times 10^3)$

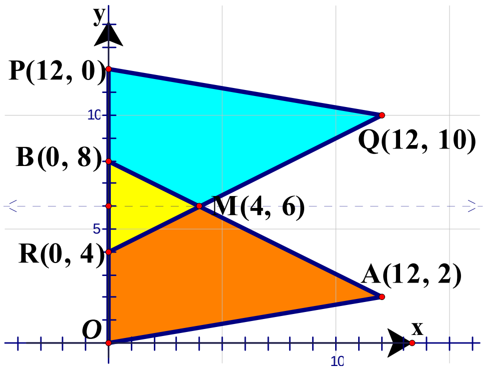
$4013 (\log 2.006 + 3) < a\times 10^k < 4013(\log 2.007 + 3)$

$4013 \log 2 + 4013\times 3 < a\times 10^k < 4013 \log 3 + 3$

$1.32429\times 10^4 = 4013\times 0.3 + 4013\times 3 < a\times 10^k < 4013\times 0.5 + 4013\times 3 = 1.40455\times 10^4$

$k = 4$

Group Event 1



G1.1 Given that there are three points on the coordinate plane: $O(0, 0)$, $A(12, 2)$ and $B(0, 8)$. A reflection of ΔOAB along the straight line $y = 6$ creates ΔPQR. If the overlapped area of ΔOAB and ΔPQR is m square units, find the value of m.

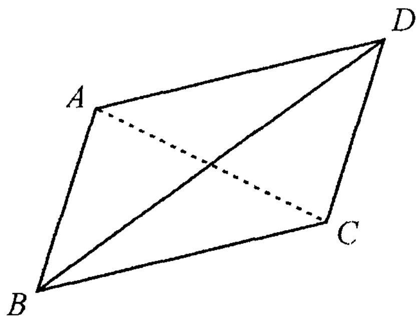
$P(12, 0)$, $Q(12, 10)$, $R(0, 4)$.

Suppose AB intersects QR at $M(x, 6)$.

Slope of MR = slope of QR

$\frac{6-4}{x}=\frac{10-4}{12} \Rightarrow x = 4$; $M(4, 6)$

Area of the overlap $\Delta BMR = \frac{1}{2}\cdot(8-4)\times 4 = 8$; $m = 8$



G1.2 In Figure 1, $ABCD$ is a parallelogram with $BA = 3$ cm, $BC = 4$ cm and $BD = \sqrt{37}$ cm. If $AC = h$ cm, find the value of h.

$CD = BA = 3$ cm (opp. sides, //-gram)

In ΔBCD, $\cos C = \frac{4^2+3^2-\sqrt{37}^2}{2\times 3\times 4} = -\frac{1}{2}$

$\cos B = \cos(180° - C) = -\cos C = \frac{1}{2}$ (int. $\angle$s AB // DC)

In ΔABC, $AC = \sqrt{3^2+4^2-2\cdot 3\cdot 4\cos B} = \sqrt{13}$; $h = \sqrt{13}$

Method 2

By Apollonius theorem,

$2BA^2 + 2BC^2 = AC^2 + BD^2$

$2\times 3^2 + 2\times 4^2 = h^2 + 37$

$h = \sqrt{13}$

G1.3 Given that x, y and z are positive integers and the fraction $\frac{151}{44}$ can be written in the form of

$3+\cfrac{1}{x+\cfrac{1}{y+\cfrac{1}{z}}}$. Find the value of $x + y + z$.

$$\frac{151}{44} = 3+\frac{19}{44} = 3+\frac{1}{\frac{44}{19}} = 3+\frac{1}{2+\frac{6}{19}} = 3+\cfrac{1}{2+\cfrac{1}{\frac{19}{6}}} = 3+\cfrac{1}{2+\cfrac{1}{3+\frac{1}{6}}}$$

$x = 2$, $y = 3$, $z = 6$; $x + y + z = 11$

G1.4 When 491 is divided by a two-digit integer, the remainder is 59. Find this two-digit integer.

Let the number be $10x + y$, where $0 < x \le 9$, $0 \le y \le 9$.

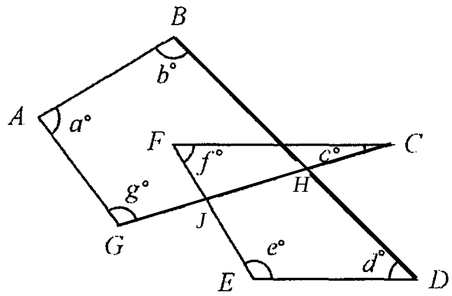
$491 = (10x + y)\cdot Q + 59$; $59 < 10x + y$

$491 - 59 = 432 = (10x + y)\cdot Q$; $432 = 72\times 6$; the number is 72.

Group Event 2

G2.1 In Figure 1, *BD*, *FC*, *GC* and *FE* are straight lines.

If $z = a + b + c + d + e + f + g$, find the value of z.



$a° + b° + g° + \angle BHG = 360°$ ($\angle$s sum of polygon *ABHG*)

$c° + f° = \angle CJE$ (ext. $\angle$ of ΔCFJ)

$c° + f° + e° + d° + \angle JHD = 360°$ ($\angle$s sum of polygon *JHDE*)

$a° + b° + g° + \angle BHG + c° + f° + e° + d° + \angle JHD = 720°$

$a° + b° + c° + d° + e° + f° + g° + 180° = 720°$

$z = 540$

G2.2 If R is the remainder of $1^6 + 2^6 + 3^6 + 4^6 + 5^6 + 6^6$ divided by 7, find the value of R.

$x^6 + y^6 = (x + y)(x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5) + 2y^6$

$6^6 + 1^6 = 7Q_1 + 2$; $5^6 + 2^6 = 7Q_2 + 2\times 2^6$; $4^6 + 3^6 = 7Q_3 + 2\times 3^6$

$2 + 2\times 2^6 + 2\times 3^6 = 2(1 + 64 + 729) = 1588 = 7\times 226 + 6$; $R = 6$

Method 2 $1^6 + 2^6 + 3^6 + 4^6 + 5^6 + 6^6 \equiv 1^6 + 2^6 + 3^6 + (-3)^6 + (-2)^6 + (-1)^6 \bmod 7$

$\equiv 2(1^6 + 2^6 + 3^6) \equiv 2(1 + 64 + 729) \bmod 7$

$\equiv 2(1 + 1 + 1) \bmod 7 \equiv 6 \bmod 7$

G2.3 If 14! is divisible by 6^k, where k is an integer, find the largest possible value of k.

We count the number of factors of 3 in 14!. They are 3, 6, 9, 12. So there are 5 factors of 3.

$k = 5$

G2.4 Let x, y and z be real numbers that satisfy $x + \frac{1}{y} = 4$, $y + \frac{1}{z} = 1$ and $z + \frac{1}{x} = \frac{7}{3}$.

Find the value of xyz. (**Reference 2010 FG2.2, 2017 FG1.1**)

Method 1

From (1), $x = 4 - \frac{1}{y} = \frac{4y-1}{y}$

$\Rightarrow \frac{1}{x} = \frac{y}{4y-1}$ …… (4)

Sub. (4) into (3): $z + \frac{y}{4y-1} = \frac{7}{3}$

$z = \frac{7}{3} - \frac{y}{4y-1}$ …… (5)

From (2): $\frac{1}{z} = 1 - y$

$z = \frac{1}{1-y}$ …… (6)

(5) = (6): $\frac{1}{1-y} = \frac{7}{3} - \frac{y}{4y-1}$

$\frac{1}{1-y} = \frac{28y - 7 - 3y}{3(4y-1)}$

$3(4y - 1) = (1 - y)(25y - 7)$

$12y - 3 = -25y^2 - 7 + 32y$

$25y^2 - 20y + 4 = 0$

$(5y - 2)^2 = 0 \Rightarrow y = \frac{2}{5}$

Sub. $y = \frac{2}{5}$ into (6): $z = \frac{1}{1-\frac{2}{5}} = \frac{5}{3}$

Sub. $y = \frac{2}{5}$ into (1): $x + \frac{5}{2} = 4 \Rightarrow x = \frac{3}{2}$

$xyz = \frac{2}{5}\times\frac{5}{3}\times\frac{3}{2} = 1$

Method 2

$$\begin{cases} x + \frac{1}{y} = 4 \cdots\cdots(1) \\ y + \frac{1}{z} = 1 \cdots\cdots(2) \\ z + \frac{1}{x} = \frac{7}{3} \cdots\cdots(3) \end{cases}$$

(1)×(2): $xy + 1 + \frac{x}{z} + \frac{1}{yz} = 4$

$x\left(y + \frac{1}{z}\right) + \frac{1}{yz} = 3$

Sub. (2) into the eqt.: $x + \frac{x}{xyz} = 3$

Let $a = xyz$, then $x + \frac{x}{a} = 3 \cdots\cdots(4)$

(2)×(3): $y\left(\frac{7}{3}\right) + \frac{y}{a} = \frac{4}{3} \Rightarrow y\left(\frac{7}{3} + \frac{1}{a}\right) = \frac{4}{3} \cdots\cdots(5)$

(1)×(3): $z(4) + \frac{z}{a} = \frac{25}{3} \Rightarrow z\left(4 + \frac{1}{a}\right) = \frac{25}{3} \cdots\cdots(6)$

(4)×(5)×(6): $a\left(1 + \frac{1}{a}\right)\left(\frac{7}{3} + \frac{1}{a}\right)\left(4 + \frac{1}{a}\right) = \frac{100}{3}$

$\frac{(a+1)(7a+3)(4a+1)}{3a^2} = \frac{100}{3}$

which reduces to $28a^3 - 53a^2 + 22a + 3 = 0$

$\Rightarrow (a - 1)^2(28a + 3) = 0$

$\therefore a = 1$

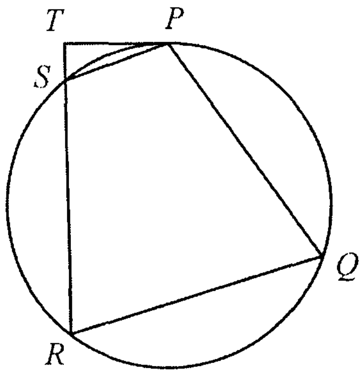
Method 3 (1)×(2)×(3) – (1) – (2) – (3):

$$xyz + x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xyz} - \left(x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = \frac{28}{3} - \frac{22}{3} \Rightarrow xyz + \frac{1}{xyz} = 2$$

$xyz = 1$

Group Event 3

G3.1 In Figure 1, *PQRS* is a cyclic quadrilateral, where *S* is on the straight line *RT* and *TP* is tangent to the circle. If *RS* = 8 cm, *RT* = 11 cm and *TP* = *k* cm, find the value of *k*.



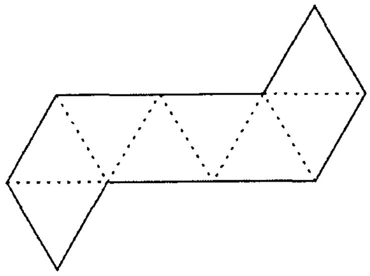
Join *PR*. $\angle SPT = \angle PRS$ ($\angle$ in alt. seg.)

$\angle STP = \angle PTR$ (common $\angle$)

$\Delta STP \sim \Delta PTR$ (equiangular)

$\frac{TP}{TR} = \frac{TS}{TP}$ (ratio of sides, ~ Δ) $\frac{k}{11} = \frac{11-8}{k}$; $k = \sqrt{33}$

G3.2 The layout in Figure 2 can be used to fold a polyhedron. If this polyhedron has *v* vertices, find the value of *v*.



There are 8 faces. $f = 8$.

There are 8 equilateral Δs, no of sides = 8×3 = 24

Each side is shared by 2 faces. Number of edge $e = 12$

By Euler formula, $v - e + f = 2$

$v - 12 + 8 = 2 \Rightarrow v = 6$

G3.3 For arbitrary real number *x*, define [*x*] to be the largest integer less than or equal to *x*. For instance, [2] = 2 and [3.4] = 3. Find the value of $[1.008^8 \times 100]$.

$1.008^8 \times 100 = (1 + 0.008)^8 \times 100 = 100(1 + 8\times0.008 + 28\times0.008^2 + \ldots) \approx 106.4$

The integral value = 106

G3.4 When choosing, without replacement, 4 out of 30 labelled balls that are marked from 1 to 30, there are *r* combinations. Find the value of *r*.

$r = C_4^{30} = \frac{30\times29\times28\times27}{1\times2\times3\times4} = 27405$

Remark: If the question is changed to: Choose 4 out of 30 labelled balls that are marked from 1 to 30 with repetition is allowed, there are *r* combinations. Find the value of *r*.

We shall divide into 5 different cases:

Case 1: All 4 balls are the same number, 30 combinations

Case 2: XXXY, where X, Y are different numbers, 30×29 = 870 combinations

Case 3 XXYY, where X, Y are different numbers, C_2^{30} = 435 combinations

Case 4 XXYZ, where X, Y, Z are different numbers, $30\times C_2^{29}$ = 12180 combinations

Case 5 XYZW, where X, Y, Z, W are different numbers, C_4^{30} = 27405 combinations

Total number of combinations = 30 + 870 + 435 + 12180 + 27405 = 40920

Group Event 4

G4.1 Regular tessellation is formed by identical regular *m*-polygons for some fixed *m*. Find the sum of all possible values of *m* .

Each interior angle = $\frac{180^\circ(m-2)}{m}$ (∠s sum of polygon)

Suppose *n* *m*-polygons tessellate the space.

$\frac{180^\circ(m-2)}{m}\cdot n = 360^\circ$ (∠s at a point)

$n(m-2) = 2m$

$n(m-2) - 2m + 4 = 4$

$(n-2)(m-2) = 4$

$m - 2 = 1, 2$ or 4

$m = 3, 4$ or 6

Sum of all possible $m = 3 + 4 + 6 = 13$

G4.2 Amongst the seven numbers 3624, 36024, 360924, 3609924, 36099924, 360999924 and 3609999924, there are *n* of them that are divisible by 38. Find the value of *n*.

$38 = 2\times19$, we need to investigate which number is divisible by 19.

$19^2 = 361$

$3624 = 3610 + 13$

$36024 = 36100 - 76 = 100(19^2) - 19\times4$

$360924 = 361000 - 76$

$3609924 = 3610000 - 76$

$36099924 = 36100000 - 76$

$360999924 = 361000000 - 76$

$3609999924 = 3610000000 - 76$

$n = 6$

G4.3 If $208208 = 8^5a + 8^4b + 8^3c + 8^2d + 8e + f$, where *a*, *b*, *c*, *d*, *e*, and *f* are integers and $0 \le a, b, c, d, e, f \le 7$, find the value of $a\times b\times c + d\times e\times f$.

Reference: 2011 FI1.2

8 | 2 0 8 2 0 8
8 | 2 6 0 2 6 ……0
8 | 3 2 5 3 ……2
8 | 4 0 6 ……5
8 | 5 0 ……6
6 ……2

$a = 6, b = 2, c = 6, d = 5, e = 2, f = 0$

$a\times b\times c + d\times e\times f = 72$

G4.4 In the coordinate plane, rotate point *A*(6, 8) about the origin *O*(0, 0) counter-clockwise for 20070° to point *B*(*p*, *q*). Find the value of *p* + *q*.

$20070^\circ = 360^\circ\times55 + 270^\circ$

$\therefore B(8, -6)$

$p + q = 2$

Group Spare

GS.1 Calculate the value of $\left(\sqrt{2008}+\sqrt{2007}\right)^{2007}\times\left(\sqrt{2007}-\sqrt{2008}\right)^{2007}$.

$$\left(\sqrt{2008}+\sqrt{2007}\right)^{2007}\times\left(\sqrt{2007}-\sqrt{2008}\right)^{2007}=(2007-2008)^{2007}=(-1)^{2007}=-1$$

GS.2 If $x-\frac{1}{x}=\sqrt{2007}$, find the value of $x^4+\frac{1}{x^4}$.

$$\left(x-\frac{1}{x}\right)^2=2007$$

$$x^2-2+\frac{1}{x^2}=2007$$

$$x^2+\frac{1}{x^2}=2009$$

$$\left(x^2+\frac{1}{x^2}\right)^2=2009^2=4036081$$

$$x^4+2+\frac{1}{x^4}=4036081$$

$$x^4+\frac{1}{x^4}=4036079$$

GS.3 Given that $\cos\alpha=-\frac{99}{101}$ and $180°<\alpha<270°$. Find the value of $\cot\alpha$.

$$\sec\alpha=-\frac{101}{99}$$

$$\tan^2\alpha=\sec^2\alpha-1$$

$$=\left(-\frac{101}{99}\right)^2-1$$

$$=\frac{101^2-99^2}{99^2}$$

$$=\frac{(101-99)\cdot(101+99)}{99^2}$$

$$=\frac{400}{99^2}$$

$$\tan\alpha=\frac{20}{99}$$

$$\cot\alpha=\frac{99}{20}\quad(=4.95)$$

GS.4 Calculate the value of $\frac{2008^3+4015^3}{2007^3+4015^3}$.

Let $x=2007.5$, then $2x=4015$

$$\frac{2008^3+4015^3}{2007^3+4015^3}=\frac{(x+0.5)^3+(2x)^3}{(x-0.5)^3+(2x)^3}=\frac{8\left(x+\frac{1}{2}\right)^3+8(2x)^3}{8\left(x-\frac{1}{2}\right)^3+8(2x)^3}=\frac{(2x+1)^3+(4x)^3}{(2x-1)+(4x)^3}$$

$$=\frac{(2x+1+4x)\left[(2x+1)^2-4x(2x+1)+(4x)^2\right]}{(2x-1+4x)\left[(2x-1)^2-4x(2x-1)+(4x)^2\right]}$$

$$=\frac{(6x+1)\left(4x^2+4x+1-8x^2-4x+16x^2\right)}{(6x-1)\left(4x^2-4x+1-8x^2+4x+16x^2\right)}$$

$$=\frac{(6x+1)\left(12x^2+1\right)}{(6x-1)\left(12x^2+1\right)}=\frac{6x+1}{6x-1}$$

$$=\frac{6023}{6022}$$

Individual Events

SI	*A*	15	I1	*R*	30	I2	*a*	16	I3	*m*	3	I4	*m*	3
	B	3		*S*	120		*b*	$\frac{3}{2}$		*n*	9		*n*	$\frac{9}{4}$
	C	4		*T*	11		*c*	36		*p*	2		*p*	9
	D	8		*U*	72		*d*	42		*q*	1141		*q*	8

Group Events

SG	*z*	540	G1	*q*	3	G2	*A*	$-\frac{17}{13}$	G3	*A*	5	G4	*P*	$\frac{3}{8}$
	R	6		*k*	1		*B*	13		*R*	4018		*R*	$\frac{1}{2}$
	k	5		*w*	25		*C*	46		*Q*	$\frac{4\sqrt{5}}{5}$		*S*	320
	xyz	1		*p*	$\frac{3}{2}$		*D*	30		*T*	$5-2\sqrt{3}$		*Q*	–1

Sample Individual Event (2008 Final Individual Event 1)

SI.1 Let $A = 15\times\tan 44°\times\tan 45°\times\tan 46°$, find the value of *A*.

Similar question 2012 G2.1

$$A = 15\times\tan 44^\circ\times 1\times\frac{1}{\tan 44^\circ} = 15$$

SI.2 Let *n* be a positive integer and $\overbrace{20082008\cdots2008}^{n\ 2008\text{'s}}15$ is divisible by *A*. If the least possible value of *n* is *B*, find the value of *B*.

The given number is divisible by 15. Therefore it is divisible by 3 and 5.

The last 2 digits of the given number is 15, which is divisible by 15.

The necessary condition is: $\overbrace{20082008\cdots2008}^{n\ 2008\text{'s}}$ must be divisible by 3.

2 + 0 + 0 + 8 = 10 which is not divisible by 3.

The least possible *n* is 3: 2+0+0+8+2+0+0+8+2+0+0+8 = 30 which is divisible by 3.

SI.3 Given that there are *C* integers that satisfy the equation $|x - 2| + |x + 1| = B$, find the value of *C*

Reference: 1994 HG1, 2001 HG9, 2004 FG4.2, 2008 HI8, 2008 FI1.3, 2010 HG6, 2012 FG2.3

$|x - 2| + |x + 1| = 3$

If $x < -1$, $2 - x - x - 1 = 3 \Rightarrow x = -1$ (rejected)

If $-1 \le x \le 2$, $2 - x + x + 1 = 3 \Rightarrow 3 = 3$, always true $\quad -1 \le x \le 2$

If $2 < x$, $x - 2 + x + 1 = 3 \Rightarrow x = 2$ (reject)

$\quad -1 \le x \le 2$ only

$\because$ *x* is an integer, $\quad x = -1, 0, 1, 2$; $C = 4$

SI.4 In the coordinate plane, the distance from the point $(-C, 0)$ to the straight line $y = x$ is $\sqrt{D}$, find the value of *D*.

The distance from $P(x_0, y_0)$ to the straight line $Ax + By + C = 0$ is $\left|\frac{Ax_0 + By_0 + C}{\sqrt{A^2 + B^2}}\right|$.

The distance from $(-4, 0)$ to $x - y = 0$ is $\sqrt{D} = \left|\frac{-4 - 0 + 0}{\sqrt{1^2 + (-1)^2}}\right| = \frac{4}{\sqrt{2}} = 2\sqrt{2} = \sqrt{8}$; $D = 8$

Individual Event 1

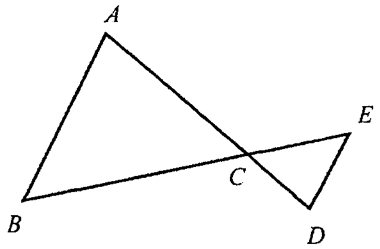
I1.1 Let a, b, c and d be the distinct roots of the equation $x^4 - 15x^2 + 56 = 0$.
If $R = a^2 + b^2 + c^2 + d^2$, find the value of R.

$x^4 - 15x^2 + 56 = 0 \Rightarrow (x^2 - 7)(x^2 - 8) = 0$

$a = \sqrt{7}$, $b = -\sqrt{7}$, $c = \sqrt{8}$, $d = -\sqrt{8}$

$R = a^2 + b^2 + c^2 + d^2 = 7 + 7 + 8 + 8 = 30$

I1.2 In Figure 1, AD and BE are straight lines with $AB = AC$ and $AB \parallel ED$. If $\angle ABC = R°$ and $\angle ADE = S°$, find the value of S.



$\angle ABC = 30° = \angle ACB$ (base $\angle$ isos. Δ)

$\angle BAC = 120°$ ($\angle$s sum of Δ)

$\angle ADE = 120°$ (alt. $\angle$s $AB \parallel ED$)

$S = 120$

I1.3 Let $F = 1 + 2 + 2^2 + 2^3 + \ldots + 2^S$ and $T = \sqrt{\frac{\log(1+F)}{\log 2}}$, find the value of T.

Reference: 2015 FI1.4, 2017 FI3.4

$F = 1 + 2 + 2^2 + 2^3 + \ldots + 2^{120}$

$= \frac{2^{121}-1}{2-1} = 2^{121} - 1$

$T = \sqrt{\frac{\log(1+F)}{\log 2}}$

$= \sqrt{\frac{\log 2^{121}}{\log 2}} = 11$

I1.4 Let f(x) be a function such that f(n) = (n – 1) f(n – 1) and f(n) ≠ 0 hold for all integers $n \geq 6$.

If $U = \frac{f(T)}{(T-1)f(T-3)}$, find the value of U.

Reference: 1999 FI5.4

$f(n) = (n-1)f(n-1) = (n-1)(n-2)f(n-2) = \ldots\ldots$

$U = \frac{f(11)}{(11-1)f(11-3)}$

$= \frac{10\times 9\times 8\times f(8)}{10\times f(8)}$

$= 8\times 9 = 72$

Individual Event 2

I2.1 Let $[x]$ be the largest integer not greater than x. If $a = \left[\left(\sqrt{3}-\sqrt{2}\right)^{2009}\right] + 16$, find the value of a.

$0 < \sqrt{3}-\sqrt{2} = \dfrac{1}{\sqrt{3}+\sqrt{2}} < 1$

$0 < \left(\sqrt{3}-\sqrt{2}\right)^{2009} < 1$

$\Rightarrow a = \left[\left(\sqrt{3}-\sqrt{2}\right)^{2009}\right] + 16$

$= 0 + 16 = 16$

I2.2 In the coordinate plane, if the area of the triangle formed by the x-axis, y-axis and the line $3x + ay = 12$ is b square units, find the value of b.

$3x + 16y = 12$

x-intercept = 4

y-intercept $= \dfrac{3}{4}$

Area $= b = \dfrac{1}{2}\cdot 4\cdot\dfrac{3}{4} = \dfrac{3}{2}$

I2.3 Given that $x - \dfrac{1}{x} = 2b$ and $x^3 - \dfrac{1}{x^3} = c$, find the value of c.

Reference: 1990 FI2.2

$x - \dfrac{1}{x} = 3$

$\Rightarrow x^2 - 2 + \dfrac{1}{x^2} = 9$

$\Rightarrow x^2 + \dfrac{1}{x^2} = 11$

$c = x^3 - \dfrac{1}{x^3}$

$= \left(x - \dfrac{1}{x}\right)\left(x^2 + 1 + \dfrac{1}{x^2}\right)$

$= 3\times(11 + 1) = 36$

I2.4 In Figure 1, $\alpha = c$, $\beta = 43$, $\gamma = 59$ and $\omega = d$, find the value of d.

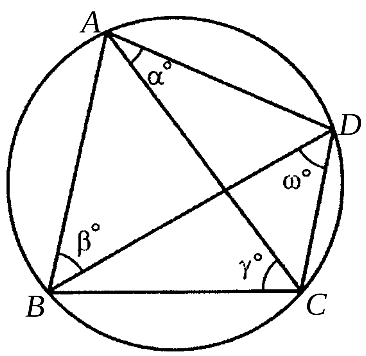
$\angle BAC = \omega°$ ($\angle$s in the same seg.)

$\angle ACD = \beta°$ ($\angle$s in the same seg.)

$\angle BAD + \angle BCD = 180°$ (opp. $\angle$s cyclic quad.)

$c + d + 43 + 59 = 180$

$d = 180 - 43 - 59 - 36 = 42$ ($\because$ $c = 36$)



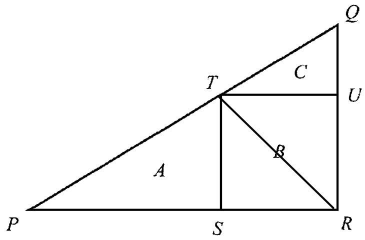
Individual Event 3

I3.1 Given that $\frac{4}{\sqrt{6}+\sqrt{2}}-\frac{1}{\sqrt{3}+\sqrt{2}}=\sqrt{a}-\sqrt{b}$. If $m = a - b$, find the value of m.

$$\frac{4}{\sqrt{6}+\sqrt{2}}-\frac{1}{\sqrt{3}+\sqrt{2}}=\frac{4}{\sqrt{6}+\sqrt{2}}\cdot\frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}-\sqrt{2}}-\frac{1}{\sqrt{3}+\sqrt{2}}\cdot\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}$$
$$=\frac{4\left(\sqrt{6}-\sqrt{2}\right)}{6-2}-\frac{\sqrt{3}-\sqrt{2}}{3-2}$$
$$=\sqrt{6}-\sqrt{2}-\left(\sqrt{3}-\sqrt{2}\right)$$
$$=\sqrt{6}-\sqrt{3}$$

$a = 6$, $b = 3$; $m = 6 - 3 = 3$

I3.2 In figure 1, PQR is a right-angled triangle and $RSTU$ is a rectangle. Let A, B and C be the areas of the corresponding regions. If $A : B = m : 2$ and $A : C = n : 1$, find the value of n.



$A : B = 3 : 2$, $A : C = n : 1 \Rightarrow A : B : C = 3n : 2n : 3$

Let $TS = UR = x$, $QU = y$

$\Delta PTS \sim \Delta TQU \sim \Delta PQR$ (equiangular)

$S_{\Delta PTS} : S_{\Delta TQU} : S_{\Delta PQR} = A : C : (A + B + C) = 3n : 3 : (5n + 3)$

$x^2 : y^2 : (x + y)^2 = 3n : 3 : (5n + 3)$

$\frac{y}{x}=\frac{1}{\sqrt{n}}$ … (1), $\frac{x+y}{y}=\frac{\sqrt{5n+3}}{\sqrt{3}}$ … (2)

From (2): $\frac{x}{y}+1=\frac{\sqrt{5n+3}}{\sqrt{3}}$ … (3)

Sub. (1) into (3): $\sqrt{n}+1=\frac{\sqrt{5n+3}}{\sqrt{3}}$

$\sqrt{3n}+\sqrt{3}=\sqrt{5n+3}$

$\left(\sqrt{3n}+\sqrt{3}\right)^2=5n+3$

$3n+6\sqrt{n}+3=5n+3$

$6\sqrt{n}=2n\Rightarrow n=9$

Method 2

Let $SR = x$, $PS = z$

Join TR which bisects the area of the rectangle.

$\frac{A}{B}=\frac{3}{2}\Rightarrow\frac{A}{\frac{B}{2}}=\frac{3}{1}$

$\frac{S_{\Delta TPS}}{S_{\Delta TSR}}=\frac{3}{1}\Rightarrow\frac{z}{x}=\frac{3}{1}$ … (4)

$\because$ $\Delta PTS \sim \Delta TQU$ (equiangular)

$\therefore$ $\frac{A}{C}=\frac{n}{1}\Rightarrow\left(\frac{z}{x}\right)^2=\frac{n}{1}$

$n = 3^2 = 9$ (by (4))

I3.3 Let x_1, x_2, x_3, x_4 be real numbers and $x_1 \neq x_2$. If $(x_1 + x_3)(x_1 + x_4) = (x_2 + x_3)(x_2 + x_4) = n - 10$ and $p = (x_1 + x_3)(x_2 + x_3) + (x_1 + x_4)(x_2 + x_4)$, find the value of p.

Reference: 2002 HI7, 2006HG6, 2014 HG7

$x_1^2+x_1x_3+x_1x_4+x_3x_4=x_2^2+x_2x_3+x_2x_4+x_3x_4=-1$

$x_1^2-x_2^2+x_1x_3-x_2x_3+x_1x_4-x_2x_4=0$

$(x_1 - x_2)(x_1 + x_2 + x_3 + x_4) = 0 \Rightarrow x_1 + x_2 + x_3 + x_4 = 0$ … (1)

$p = (x_1 + x_3)(x_2 + x_3) + (x_1 + x_4)(x_2 + x_4)$

$= -(x_2 + x_3)(x_2 + x_4) - (x_1 + x_3)(x_1 + x_4)$ (by (1), $x_1 + x_3 = -(x_2 + x_4)$, $x_2 + x_4 = -(x_1 + x_3)$)

$= 1 + 1 = 2$ (given $(x_1 + x_3)(x_1 + x_4) = (x_2 + x_3)(x_2 + x_4) = -1$)

I3.4 The total number of students in a school is a multiple of 7 and not less than 1000. Given that the same remainder 1 will be obtained when the number of students is divided by $p + 1$, $p + 2$ and $p + 3$. Let q be the least of the possible numbers of students in the school, find the value of q.

$p + 1 = 3$, $p + 2 = 4$, $p + 3 = 5$; HCF = 1, LCM = 60

$q = 60m + 1$, where m is an integer.

$\because$ $q \geq 1000$ and $q = 7k$, k is an integer.

$60m + 1 = 7k$

$7k - 60m = 1$

$k = 43$, $m = 5$ satisfies the equation

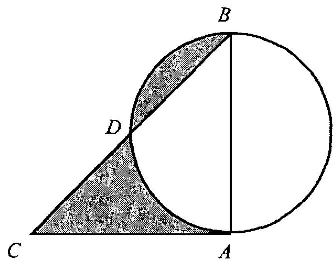
$k = 43 + 60t$; $7k \geq 1000 \Rightarrow 7(43 + 60t) \geq 1000 \Rightarrow t \geq 2 \Rightarrow$ Least $q = 7\times(43 + 60\times2) = 1141$

Individual Event 4

I4.1 Given that $x_0^2 + x_0 - 1 = 0$. If $m = x_0^3 + 2x_0^2 + 2$, find the value of m.

$m = x_0^3 + 2x_0^2 + 2 = x_0^3 + x_0^2 - x_0 + x_0^2 + x_0 - 1 + 3 = 3$

I4.2 In Figure 1, ΔBAC is a right-angled triangle, $AB = AC = m$ cm. Suppose that the circle with diameter AB intersects the line BC at D, and the total area of the shaded region is n cm^2. Find the value of n.



$AB = AC = 3$ cm; $\angle ADB = 90°$ ($\angle$ in semi-circle)

Shaded area = area of $\Delta ACD = \frac{1}{2}$ area of $\Delta ABC = \frac{1}{2}\cdot\frac{1}{2}\cdot 3\cdot 3\,\text{cm}^2$

$n = \frac{9}{4}$

I4.3 Given that $p = 4n\left(\frac{1}{2^{2009}}\right)^{\log(1)}$, find the value of p.

$p = 4n\left(\frac{1}{2^{2009}}\right)^{0} = 4\cdot\frac{9}{4} = 9$

I4.4 Let x and y be real numbers satisfying the equation $\left(x-\sqrt{p}\right)^2 + \left(y-\sqrt{p}\right)^2 = 1$.

If $k = \frac{y}{x-3}$ and q is the least possible values of k^2, find the value of q.

$(x - 3)^2 + (y - 3)^2 = 1$ … (1)

Sub. $y = k(x - 3)$ into (1): $(x - 3)^2 + [k(x - 3) - 3]^2 = 1$

$(x - 3)^2 + k^2(x - 3)^2 - 6k(x - 3) + 9 = 1$

$(1 + k^2)(x - 3)^2 - 6k(x - 3) + 8 = 0 \Rightarrow (1 + k^2)t^2 - 6kt + 8 = 0$; where $t = x - 3$

For real values of t: $\Delta = 4[3^2k^2 - 8(1 + k^2)] \ge 0$

$k^2 \ge 8$

The least possible value of $k^2 = q = 8$.

Method 2

The circle $(x - 3)^2 + (y - 3)^2 = 1$ intersects with the variable line $y = k(x - 3)$ which passes through a fixed point (3, 0), where k is the slope of the line.

From the graph, the extreme points when the variable line touches the circle at B and C.

$H(3, 3)$ is the centre of the circle.

In ΔABH, $\angle ABH = 90°$ ($\angle$ in semi-circle)

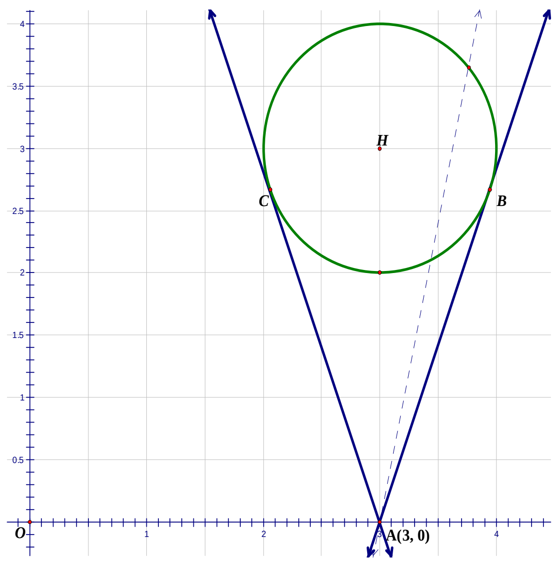
Let $\angle HAB = \theta$, $AH = 3$, $HB = 1$,

$AB = \sqrt{8} = 2\sqrt{2}$ (Pythagoras' theorem)

$\tan\theta = \frac{1}{\sqrt{8}}$

Slope of $AB = \tan(90° - \theta) = \sqrt{8}$

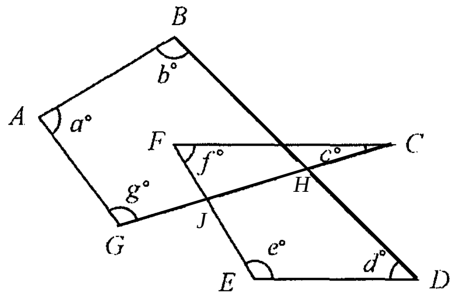
Least possible $k^2 = q = 8$



Sample Group Event (2008 Final Group Event 2)

SG.1 In Figure 1, *BD*, *FC*, *GC* and *FE* are straight lines.

If $z = a + b + c + d + e + f + g$, find the value of z.



$a° + b° + g° + \angle BHG = 360°$ ($\angle$s sum of polygon *ABHG*)

$c° + f° = \angle CJE$ (ext. $\angle$ of ΔCFJ)

$c° + f° + e° + d° + \angle JHD = 360°$ ($\angle$s sum of polygon *JHDE*)

$a° + b° + g° + \angle BHG + c° + f° + e° + d° + \angle JHD = 720°$

$a° + b° + c° + d° + e° + f° + g° + 180° = 720°$

$z = 540$

SG.2 If R is the remainder of $1^6 + 2^6 + 3^6 + 4^6 + 5^6 + 6^6$ divided by 7, find the value of R.

$x^6 + y^6 = (x + y)(x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5) + 2y^6$

$6^6 + 1^6 = 7Q_1 + 2$; $5^6 + 2^6 = 7Q_2 + 2\times 2^6$; $4^6 + 3^6 = 7Q_3 + 2\times 3^6$

$2 + 2\times 2^6 + 2\times 3^6 = 2(1 + 64 + 729) = 1588 = 7\times 226 + 6$; $R = 6$

Method 2 $1^6 + 2^6 + 3^6 + 4^6 + 5^6 + 6^6 \equiv 1^6 + 2^6 + 3^6 + (-3)^6 + (-2)^6 + (-1)^6 \bmod 7$

$\equiv 2(1^6 + 2^6 + 3^6) \equiv 2(1 + 64 + 729) \bmod 7$

$\equiv 2(1 + 1 + 1) \bmod 7 \equiv 6 \bmod 7$

SG.3 If 14! is divisible by 6^k, where k is an integer, find the largest possible value of k.

We count the number of factors of 3 in 14!. They are 3, 6, 9, 12. So there are 5 factors of 3.

$k = 5$

SG.4 Let x, y and z be real numbers that satisfy $x + \frac{1}{y} = 4$, $y + \frac{1}{z} = 1$ and $z + \frac{1}{x} = \frac{7}{3}$.

Find the value of xyz. (**Reference 2010 FG2.2**)

Method 1

From (1), $x = 4 - \frac{1}{y} = \frac{4y-1}{y}$

$\Rightarrow \frac{1}{x} = \frac{y}{4y-1}$ …… (4)

Sub. (4) into (3): $z + \frac{y}{4y-1} = \frac{7}{3}$

$z = \frac{7}{3} - \frac{y}{4y-1}$ …… (5)

From (2): $\frac{1}{z} = 1 - y$

$z = \frac{1}{1-y}$ …… (6)

(5) = (6): $\frac{1}{1-y} = \frac{7}{3} - \frac{y}{4y-1}$

$\frac{1}{1-y} = \frac{28y-7-3y}{3(4y-1)}$

$3(4y - 1) = (1 - y)(25y - 7)$

$12y - 3 = -25y^2 - 7 + 32y$

$25y^2 - 20y + 4 = 0$

$(5y - 2)^2 = 0$

$y = \frac{2}{5}$

Sub. $y = \frac{2}{5}$ into (6): $z = \frac{1}{1-\frac{2}{5}} = \frac{5}{3}$

Sub. $y = \frac{2}{5}$ into (1): $x + \frac{5}{2} = 4 \Rightarrow x = \frac{3}{2}$

$xyz = \frac{2}{5}\times\frac{5}{3}\times\frac{3}{2} = 1$

Method 2

$$\begin{cases} x + \frac{1}{y} = 4 \cdots\cdots(1) \\ y + \frac{1}{z} = 1 \cdots\cdots(2) \\ z + \frac{1}{x} = \frac{7}{3} \cdots\cdots(3) \end{cases}$$

(1)×(2): $xy + 1 + \frac{x}{z} + \frac{1}{yz} = 4$

$x\left(y + \frac{1}{z}\right) + \frac{1}{yz} = 3$

Sub. (2) into the eqt.: $x + \frac{x}{xyz} = 3$

Let $a = xyz$, then $x + \frac{x}{a} = 3 \cdots\cdots(4)$

(2)×(3): $y\left(\frac{7}{3}\right) + \frac{y}{a} = \frac{4}{3} \Rightarrow y\left(\frac{7}{3} + \frac{1}{a}\right) = \frac{4}{3} \cdots\cdots(5)$

(1)×(3): $z(4) + \frac{z}{a} = \frac{25}{3} \Rightarrow z\left(4 + \frac{1}{a}\right) = \frac{25}{3} \cdots\cdots(6)$

(4)×(5)×(6): $a\left(1 + \frac{1}{a}\right)\left(\frac{7}{3} + \frac{1}{a}\right)\left(4 + \frac{1}{a}\right) = \frac{100}{3}$

$\frac{(a+1)(7a+3)(4a+1)}{3a^2} = \frac{100}{3}$

which reduces to $28a^3 - 53a^2 + 22a + 3 = 0$

$\Rightarrow (a - 1)^2(28a + 3) = 0$

$\therefore a = 1$

Method 3 (1)×(2)×(3) – (1) – (2) – (3):

$$xyz + x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xyz} - \left(x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = \frac{28}{3} - \frac{22}{3} \Rightarrow xyz + \frac{1}{xyz} = 2$$

$xyz = 1$

Group Event 1

G1.1 Given some triangles with side lengths a cm, 2 cm and b cm, where a and b are integers and $a \le 2 \le b$. If there are q non-congruent classes of triangles satisfying the above conditions, find the value of q.

When $a = 1$, possible $b = 2$

When $a = 2$, possible $b = 2$ or 3

$\therefore q = 3$

G1.2 Given that the equation $|x| - \frac{4}{x} = \frac{3|x|}{x}$ has k distinct real root(s), find the value of k.

When $x > 0$: $x^2 - 4 = 3x \Rightarrow x^2 - 3x - 4 = 0 \Rightarrow (x + 1)(x - 4) = 0 \Rightarrow x = 4$

When $x < 0$: $-x^2 - 4 = -3x \Rightarrow x^2 - 3x + 4 = 0$; $D = 9 - 16 < 0 \Rightarrow$ no real roots.

$k = 1$ (There is only one real root.)

G1.3 Given that x and y are non-zero real numbers satisfying the equations $\frac{\sqrt{x}}{\sqrt{y}} - \frac{\sqrt{y}}{\sqrt{x}} = \frac{7}{12}$ and $x - y = 7$. If $w = x + y$, find the value of w.

The first equation is equivalent to $\frac{x-y}{\sqrt{xy}} = \frac{7}{12} \Rightarrow \sqrt{xy} = 12 \Rightarrow xy = 144$

Sub. $y = \frac{144}{x}$ into $x - y = 7$: $x - \frac{144}{x} = 7 \Rightarrow x^2 - 7x - 144 = 0 \Rightarrow (x + 9)(x - 16) = 0$

$x = -9$ or 16; when $x = -9$, $y = -16$ (rejected $\because \sqrt{x}$ is undefined); when $x = 16$; $y = 9$

$w = 16 + 9 = 25$

Method 2 The first equation is equivalent to $\frac{x-y}{\sqrt{xy}} = \frac{7}{12} \Rightarrow \sqrt{xy} = 12 \Rightarrow xy = 144$ …… (1)

$\because x - y = 7$ and $x + y = w$

$\therefore x = \frac{w+7}{2}$, $y = \frac{w-7}{2}$

Sub. these equations into (1): $\left(\frac{w+7}{2}\right)\left(\frac{w-7}{2}\right) = 144$

$w^2 - 49 = 576 \Rightarrow w = \pm 25$

$\because$ From the given equation $\frac{\sqrt{x}}{\sqrt{y}} - \frac{\sqrt{y}}{\sqrt{x}} = \frac{7}{12}$, we know that both $x > 0$ and $y > 0$

$\therefore w = x + y = 25$ only

G1.4 Given that x and y are real numbers and $\left|x - \frac{1}{2}\right| + \sqrt{y^2 - 1} = 0$.

Let $p = |x| + |y|$, find the value of p.

Reference: 2005 FI4.1, 2006 FI4.2, 2011 FI4.3, 2013 FI1.4, 2015 HG4, 2015 FI1.1

Both $\left|x - \frac{1}{2}\right|$ and $\sqrt{y^2 - 1}$ are non-negative numbers.

The sum of two non-negative numbers = 0 means each of them is zero

$x = \frac{1}{2}$, $y = \pm 1$; $p = \frac{1}{2} + 1 = \frac{3}{2}$

Group Event 2

G2.1 Given $\tan\theta = \frac{5}{12}$, where $180° \le \theta \le 270°$. If $A = \cos\theta + \sin\theta$, find the value of A.

$\cos\theta = -\frac{12}{13}$, $\sin\theta = -\frac{5}{13}$

$A = -\frac{12}{13} - \frac{5}{13} = -\frac{17}{13}$

G2.2 Let $[x]$ be the largest integer not greater than x.

If $B = \left[10 + \sqrt{10 + \sqrt{10 + \sqrt{10 + \cdots}}}\right]$, find the value of B.

Reference: 2007 FG2.2 … $x \ge 3 + \sqrt{3 + \sqrt{3 + \sqrt{3 + \sqrt{3 + \sqrt{3}}}}}$ …

Let $y = \sqrt{10 + \sqrt{10 + \sqrt{10 + \cdots}}}$

$y^2 = 10 + \sqrt{10 + \sqrt{10 + \sqrt{10 + \cdots}}} = 10 + y$

$y^2 - y - 10 = 0$

$y = \frac{1 + \sqrt{41}}{2}$ or $\frac{1 - \sqrt{41}}{2}$ (rejected)

$6 < \sqrt{41} < 7 \Rightarrow \frac{7}{2} < \frac{1 + \sqrt{41}}{2} < 4$

$13.5 < 10 + \sqrt{10 + \sqrt{10 + \sqrt{10 + \cdots}}} < 14$; $B = 13$

G2.3 Let $a \oplus b = ab + 10$. If $C = (1\oplus 2) \oplus 3$, find the value of C.

$1\oplus 2 = 2 + 10 = 12$; $C = 12\oplus 3 = 36 + 10 = 46$

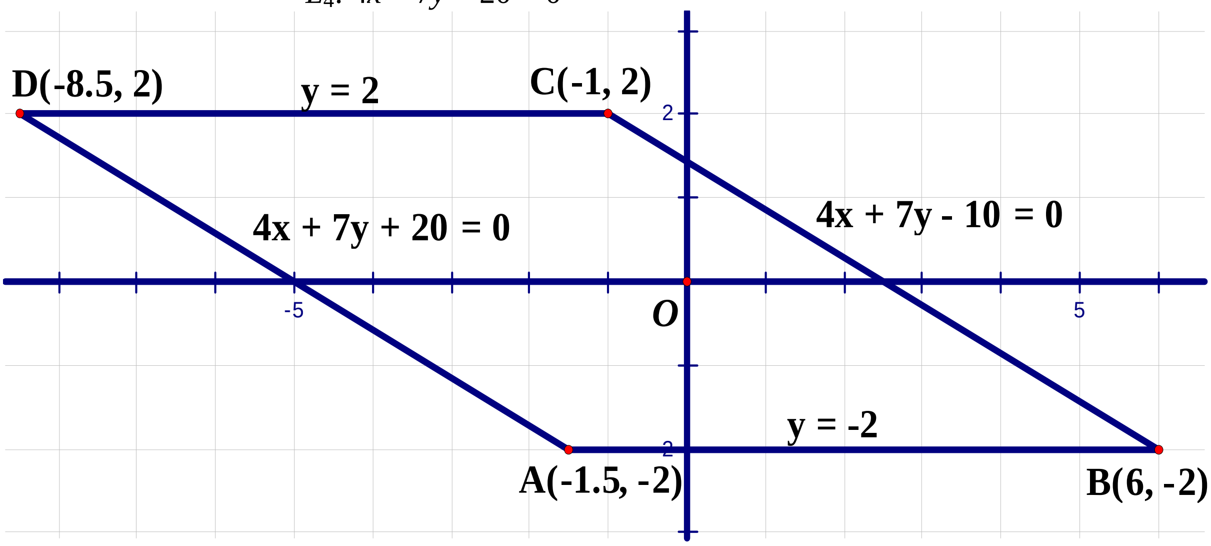
G2.4 In the coordinate plane, the area of the region bounded by the following lines is D square units, find the value of D.

L_1: $y - 2 = 0$

L_2: $y + 2 = 0$

L_3: $4x + 7y - 10 = 0$

L_4: $4x + 7y + 20 = 0$



It is easy to show that the bounded region is a parallelogram $ABCD$ with vertices $A(-1.5, -2)$, $B(6, -2)$, $C(-1, 2)$, $C(-8.5, 2)$.

The area $D = |6 - (-1.5)|\times|2 - (-2)| = 7.5\times 4 = 30$

Group Event 3

G3.1 Let $[x]$ be the largest integer not greater than x.

If $A=\left[\frac{2008\times 80+2009\times 130+2010\times 180}{2008\times 15+2009\times 25+2010\times 35}\right]$, find the value of A.

Reference: 2008 FGS.4 Calculate the value of $\frac{2008^3+4015^3}{2007^3+4015^3}$.

Let a = 2009, b = 130, c = 25

$$\frac{2008\times 80+2009\times 130+2010\times 180}{2008\times 15+2009\times 25+2010\times 35}=\frac{(a-1)(b-50)+ab+(a+1)(b+50)}{(a-1)(c-10)+ac+(a+1)(c+10)}$$

$$=\frac{ab-b-50a+50+ab+ab+b+50a+50}{ac-c-10a+10+ac+ac+c+10a+10}$$

$$=\frac{3ab+100}{3ac+20}=\frac{3\cdot 2009\cdot 130+100}{3\cdot 2009\cdot 25+20}=\frac{783610}{150695}=5+d$$

where $0 < d < 1$; $A = 5$

G3.2 There are R zeros at the end of $\underbrace{99\ldots9}_{2009\text{ of }9\text{'s}}\times\underbrace{99\ldots9}_{2009\text{ of }9\text{'s}}+1\underbrace{99\ldots9}_{2009\text{ of }9\text{'s}}$, find the value of R.

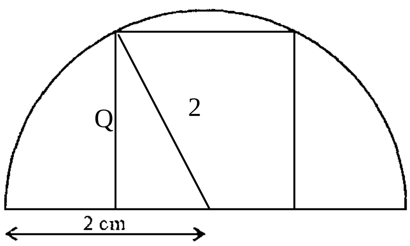
$$\underbrace{99\ldots9}_{2009\text{ of }9\text{'s}}\times\underbrace{99\ldots9}_{2009\text{ of }9\text{'s}}+1\underbrace{99\ldots9}_{2009\text{ of }9\text{'s}}=\left(1\underbrace{0\ldots0}_{2009\text{ of }0\text{'s}}-1\right)\times\left(1\underbrace{0\ldots0}_{2009\text{ of }0\text{'s}}-1\right)+\left(2\underbrace{0\ldots0}_{2009\text{ of }0\text{'s}}-1\right)$$

$$=(10^{2009}-1)(10^{2009}-1)+2\times 10^{2009}-1$$

$$=10^{4018}-2\times 10^{2009}+1+2\times 10^{2009}-1=10^{4018}$$

$R = 4018$

G3.3 In Figure 1, a square of side length Q cm is inscribed in a semi-circle of radius 2 cm. Find the value of Q.

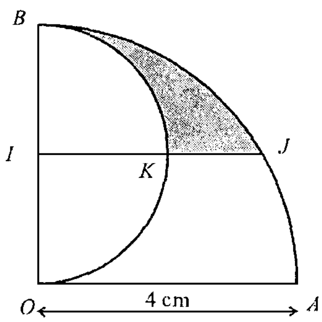


$$Q^2+\left(\frac{Q}{2}\right)^2=4 \quad \text{(Pythagoras' Theorem)}$$

$5Q^2 = 16$

$$Q=\frac{4}{\sqrt{5}}=\frac{4\sqrt{5}}{5}$$

G3.4 In Figure 2, the sector OAB has radius 4 cm and $\angle AOB$ is a right angle. Let the semi-circle with diameter OB be centred at I with IJ // OA, and IJ intersects the semi-circle at K. If the area of the shaded region is T cm^2, find the value of T. (Take $\pi = 3$)



OI = 2 cm, OJ = 4 cm

$$\cos\angle IOJ=\frac{OI}{OJ}=\frac{1}{2}$$

$\angle IOJ = 60°$

$S_{BIJ} = S_{\text{sector } OBJ} - S_{\Delta OIJ}$

$$=\left(\frac{1}{2}\cdot 4^2\cdot\frac{\pi}{3}-\frac{1}{2}\cdot 2\cdot 4\sin 60^\circ\right)\text{cm}^2$$

$$=\left(\frac{8\pi}{3}-2\sqrt{3}\right)\text{ cm}^2$$

Shaded area = $S_{BIJ} - S_{BIK}$

$$=\left(\frac{8\pi}{3}-2\sqrt{3}-\frac{1}{4}\pi\cdot 2^2\right)\text{cm}^2$$

$$=\left(\frac{5\pi}{3}-2\sqrt{3}\right)\text{cm}^2$$

$T=5-2\sqrt{3}$

Group Event 4

G4.1 Let P be a real number. If $\sqrt{3-2P}+\sqrt{1-2P}=2$, find the value of P.

$\left(\sqrt{3-2P}\right)^2=\left(2-\sqrt{1-2P}\right)^2$

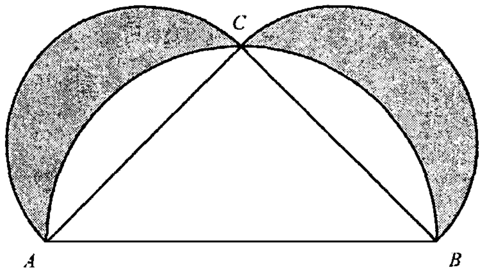
$3-2P=4-4\sqrt{1-2P}+1-2P$

$4\sqrt{1-2P}=2$

$4(1-2P)=1$

$P=\frac{3}{8}$

G4.2 In Figure 1, let AB, AC and BC be the diameters of the corresponding three semi-circles. If $AC = BC = 1$ cm and the area of the shaded region is R cm^2. Find the value of R.



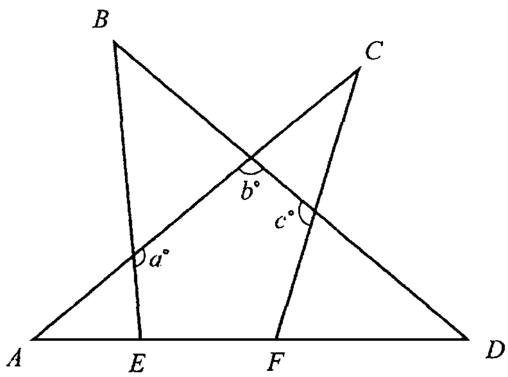
Reference: 1994 HI9

$AB=\sqrt{2}$

Shaded area = R cm^2 = $S_{\text{circle with diameter } AC} - 2\,S_{\text{segment } AC}$

$R=\pi\left(\frac{1}{2}\right)^2-\left[\frac{1}{2}\pi\cdot\left(\frac{\sqrt{2}}{2}\right)^2-\frac{1}{2}\cdot 1^2\right]=\frac{1}{2}$

G4.3 In Figure 2, AC, AD, BD, BE and CF are straight lines. If $\angle A+\angle B+\angle C+\angle D=140°$ and $a+b+c=S$, find the value of S.



$\angle CFD=\angle A+\angle C$ (ext. ∠ of Δ)

$\angle AEB=\angle B+\angle D$ (ext. ∠ of Δ)

$a°=\angle A+\angle AEB=\angle A+\angle B+\angle D$ (ext. ∠ of Δ)

$c°=\angle D+\angle CFD=\angle D+\angle A+\angle C$ (ext. ∠ of Δ)

$b°+\angle A+\angle D=180°$ (∠s sum of Δ)

$a°+b°+c°=\angle A+\angle B+\angle D+180°-(\angle A+\angle D)+\angle D+\angle A+\angle C$

$=\angle A+\angle B+\angle C+\angle D+180°$

$S=a+b+c=140+180=320$

G4.4 Let $Q=\log_{2+\sqrt{2^2-1}}\left(2-\sqrt{2^2-1}\right)$, find the value of Q.

$$Q=\frac{\log\left(2-\sqrt{3}\right)}{\log\left(2+\sqrt{3}\right)}$$

$$=\frac{\log\left(2-\sqrt{3}\right)}{\log\left(2+\sqrt{3}\right)\cdot\frac{\left(2-\sqrt{3}\right)}{\left(2-\sqrt{3}\right)}}$$

$$=\frac{\log\left(2-\sqrt{3}\right)}{\log\frac{1}{2-\sqrt{3}}}$$

$$=\frac{\log\left(2-\sqrt{3}\right)}{-\log\left(2-\sqrt{3}\right)}=-1$$

Method 2 $Q=\log_{2+\sqrt{3}}\left(2-\sqrt{3}\right)\cdot\frac{\left(2+\sqrt{3}\right)}{\left(2+\sqrt{3}\right)}$

$$=\log_{2+\sqrt{3}}\frac{1}{\left(2+\sqrt{3}\right)}$$

$$=\log_{2+\sqrt{3}}\left(2+\sqrt{3}\right)^{-1}=-1$$

Individual Events

SI	***a***	16	**I1**	***a***	72	**I2**	***A***	2	**I3**	***a***	3	**I4**	***a***	$*\frac{2}{3}$ see the remark	**IS**	***a***	20
	b	$\frac{3}{2}$		***b***	9		***B***	2		***b***	18		***b***	2		***b***	$-\frac{1}{3}$
	c	36		***c***	*31 see the remark		***C***	200		***c***	21		***c***	1		***c***	–4
	d	42		***d***	1984		***D***	100		***d***	$-\frac{1}{9}$		***d***	$\frac{3}{4}$		***d***	–24

Group Events

SG	***A***	$-\frac{17}{13}$	**G1**		44.5	**G2**	***p***	6	**G3**		2	**G4**	***m***	*3 see the remark	**GS**	***n***	4
	B	13		$\frac{x}{y}$	*2 see the remark		***m***	1		***n***	66		**Minimum *y***	$4\sqrt{2}$			86975
	C	46			$\frac{7}{2}$		***k***	15		***x***	$\frac{8}{\sqrt{3}}=\frac{8}{3}\sqrt{3}$		***A*+*B*+*C***	6			$*\frac{1}{3}$ see the remark
	D	30		***n***	10			478			8		***EC***	4			4

Sample Individual Event (2009 Final Individual Event 2)

SI.1 Let $[x]$ be the largest integer not greater than x. If $a=\left[\left(\sqrt{3}-\sqrt{2}\right)^{2009}\right]+16$, find the value of a.

$$0<\sqrt{3}-\sqrt{2}=\frac{1}{\sqrt{3}+\sqrt{2}}<1$$

$$0<\left(\sqrt{3}-\sqrt{2}\right)^{2009}<1$$

$$a=\left[\left(\sqrt{3}-\sqrt{2}\right)^{2009}\right]+16=0+16=16$$

SI.2 In the coordinate plane, if the area of the triangle formed by the x-axis, y-axis and the line $3x + ay = 12$ is b square units, find the value of b.

$3x + 16y = 12$

x-intercept = 4, y-intercept $=\frac{3}{4}$

Area $= b=\frac{1}{2}\cdot 4\cdot\frac{3}{4}=\frac{3}{2}$

SI.3 Given that $x-\frac{1}{x}=2b$ and $x^3-\frac{1}{x^3}=c$, find the value of c.

$$x-\frac{1}{x}=3\Rightarrow x^2-2+\frac{1}{x^2}=9\Rightarrow x^2+\frac{1}{x^2}=11$$

$$c=x^3-\frac{1}{x^3}=\left(x-\frac{1}{x}\right)\left(x^2+1+\frac{1}{x^2}\right)=3\times(11+1)=36$$

SI.4 In Figure 1, $\alpha = c$, $\beta = 43$, $\gamma = 59$ and $\omega = d$, find the value of d.

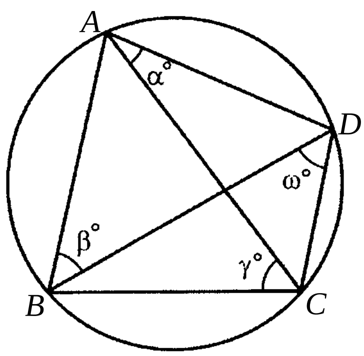
$\angle BAC = \omega°$ ($\angle$s in the same seg.)

$\angle ACD = \beta°$ ($\angle$s in the same seg.)

$\angle BAD + \angle BCD = 180°$ (opp. $\angle$s cyclic quad.)

$c + d + 43 + 59 = 180$

$d = 180 – 43 – 59 – 36 = 42$ ($\because$ $c = 36$)



Individual Event 1

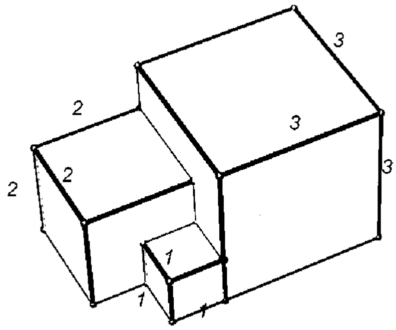
I1.1 Three cubes with volumes 1, 8, 27 are glued together at their faces. If a is the smallest possible surface area of the resulting polyhedron, find the value of a.

The lengths of the 3 cubes are 1, 2 and 3 with surface areas 6, 24 and 54 respectively.

As shown in the figure, if the three cubes are glued together, the faces stuck together are 2×2, 2×2, 1×1, 1×1, 1×1 and 1×1.

The smallest possible surface area is $6 + 24 + 54 - 4 - 4 - 4$.

$a = 72$



I1.2 Given that $f(x) = -x^2 + 10x + 9$, and $2 \le x \le \frac{a}{9}$. If b is the difference of the maximum and minimum values of f, find the value of b.

$f(x) = -x^2 + 10x + 9 = -(x - 5)^2 + 34$ for $2 \le x \le 8$

Maximum $= f(5) = 34$; minimum $= f(2) = f(8) = 25$

$b = 34 - 25 = 9$

I1.3 Given that p and q are real numbers with $pq = b$ and $p^2q + q^2p + p + q = 70$. If $c = p^2 + q^2$, find the value of c.

$pq = 9 \ldots (1)$, and $pq(p + q) + (p + q) = 70 \Rightarrow (pq + 1)(p + q) = 70 \ldots (2)$

Sub. (1) into (2): $10(p + q) = 70 \Rightarrow p + q = 7 \ldots (3)$

$c = p^2 + q^2 = (p + q)^2 - 2pq = 7^2 - 2\times9 = 31$

Remark: The original question is

Given that p and q are **integers** with $pq = b$ and $p^2q + q^2p + p + q = 70$.

However, $pq = 9$, $p + q = 7$, which give no integral solution.

I1.4 There are c rows in a concert hall and each succeeding row has two more seats than the previous row. If the middle row has 64 seats, how many seats (d) does the concert have ?

There are altogether 31 rows. The 16^{th} row is the middle row, which has 64 seats.

The 15^{th} row has $64 - 2 = 62$ seats.

The 14^{th} row has $64 - 2\times2 = 60$ seats.

....

The 1^{st} row has $64 - 2\times15 = 34$ seats.

Total number of seats $= \frac{n}{2}[2a + (n-1)d] = \frac{31}{2}[2\cdot34 + (31-1)\cdot2] = 1984$

Method 2 Total number of seats = (1^{st} row + 31^{st} row) + (2^{nd} row + 30^{th} row) + … + 16^{th} row

$= (64 + 64) + (64 + 64) + \ldots + 64$ (31 terms) $= 1984$

Individual Event 2

I2.1 If a, p, q are primes with $a < p$ and $a + p = q$, find the value of a.
'2' is the only prime number which is even. All other primes are odd numbers.
If both a and p are odd, then q must be even, which means that either q is not a prime or $q = 2$.
Both cases lead to contradiction.
$\therefore a = 2$

I2.2 If b and h are positive integers with $b < h$ and $b^2 + h^2 = b(a + h) + ah$, find the value of b.
Reference: 2000 FI5.2, 2001 FI2.1, 2011 FI3.1, 2013 HG1
$b < h$ and $b^2 + h^2 = b(2 + h) + 2h$
$b^2 + h^2 = 2b + bh + 2h$
$(b + h)^2 - 2(b + h) = 3bh < 3\left(\frac{b+h}{2}\right)^2$ (G.M. < A.M., given that $b < h$)
Let $t = b + h$, $t^2 - 2t < \frac{3t^2}{4} \Rightarrow t^2 - 8t < 0$, where t is a positive integer
$t - 8 < 0 \Rightarrow t < 8 \Rightarrow b + h < 8 \Rightarrow 2b < b + h < 8 \Rightarrow 2b < 8 \Rightarrow b < 4$
$b = 1, 2$ or 3
When $b = 1$, $1 + h^2 = 2 + h + 2h \Rightarrow h^2 - 3h - 1 = 0 \Rightarrow h$ is not an integer, rejected
When $b = 2$, $4 + h^2 = 4 + 2h + 2h \Rightarrow h^2 - 4h = 0 \Rightarrow h = 4$
When $b = 3$, $9 + h^2 = 6 + 3h + 2h \Rightarrow h^2 - 5h + 3 = 0 \Rightarrow h$ is not an integer, rejected
$\therefore b = 2$
Method 2 $h^2 - (b + 2)h + b^2 - 2b = 0$
$\Delta = (b + 2)^2 - 4(b^2 - 2b) = m^2$, where m is an integer
$-3b^2 + 12b + 4 = m^2$
$-3(b - 2)^2 + 16 = m^2$
$m^2 + 3(b - 2)^2 = 16$, both b and m are integers
$m = 0$, no integral solution for b
$m = 1$, no integral solution for b
$m = 2$, $b = 4$, $h^2 - 6h + 8 = 0 \Rightarrow h = 2$ or $h = 4$, contradicting $b < h$, reject
$m = 3$, no integral solution for b
$m = 4$, $b = 2$, $h = 4$ (accept)

I2.3 In a $(2b + 1)\times(2b + 1)$ checkerboard, two squares not lying in the same row are randomly chosen. If c is the number of combinations of different pairs of squares chosen, find the value of c.
There are 25 squares. First we count the number of ways of choosing two squares lying in the same column or the same row: ${}_5C_2\times5 + {}_5C_2\times5 = 100$
$\therefore c = {}_{25}C_2 - 100 = 200$
Method 2 Label the two squares as A, B.
For each chosen square A (out of 25 squares), B has 16 possible positions.
$\therefore$ There are $25\times16 = 400$ combinations.
However, A, B may be inter-changed. $\therefore$ We have double counted. $c = 200$.

I2.4 Given that $f(x) = c\left|\frac{1}{x} - \left\lfloor\frac{1}{x} + \frac{1}{2}\right\rfloor\right|$, where $\lfloor x\rfloor$ is the greatest integer less than or equal to the real number x. If d is the maximum value of $f(x)$, find the value of d.
Let $\frac{1}{x} + \frac{1}{2} = a + b$, where a is an integer and $0 \le b < 1$.
$\left\lfloor\frac{1}{x} + \frac{1}{2}\right\rfloor = a \Rightarrow -\left\lfloor\frac{1}{x} + \frac{1}{2}\right\rfloor = -a \Rightarrow \frac{1}{x} - \left\lfloor\frac{1}{x} + \frac{1}{2}\right\rfloor = a + b - \frac{1}{2} - a = b - \frac{1}{2}$
$0 \le b < 1 \Rightarrow -\frac{1}{2} \le b - \frac{1}{2} < \frac{1}{2} \Rightarrow \left|b - \frac{1}{2}\right| \le \frac{1}{2}$ (equality holds when $b = 0$)
$f(x) = 200\times\left|\frac{1}{x} - \left\lfloor\frac{1}{x} + \frac{1}{2}\right\rfloor\right| = 200\times\left|b - \frac{1}{2}\right| \le 200\times\frac{1}{2} = 100$
$d = 100$ (You may verify the result by putting $x = 2$.)

Individual Event 3

I3.1 If a is the number of distinct prime factors of 15147, find the value of a.

$15147 = 3^4 \times 11 \times 17$

$a = 3$

I3.2 If $x + \frac{1}{x} = a$ and $x^3 + \frac{1}{x^3} = b$, find the value of b.

Reference: 1983 FG7.3, 1996 FI1.2, 1998 FG5.2

$x + \frac{1}{x} = 3 \Rightarrow x^2 + 2 + \frac{1}{x^2} = 9 \Rightarrow x^2 + \frac{1}{x^2} = 7$

$b = x^3 + \frac{1}{x^3}$

$= \left(x + \frac{1}{x}\right)\left(x^2 - 1 + \frac{1}{x^2}\right)$

$= 3\times(7 - 1)$

$= 18$

I3.3 Let $f(x) = \begin{cases} x + 5 & \text{if } x \text{ is an odd integer} \\ \frac{x}{2} & \text{if } x \text{ is an even integer} \end{cases}$.

If c is an odd integer and $f(f(f(c))) = b$, find the least value of c.

$f(c) = c + 5$, which is even

$f(f(c)) = \frac{c+5}{2}$

If $\frac{c+5}{2}$ is odd, $f(f(f(c))) = 18$

$\Rightarrow \frac{c+5}{2} + 5 = 18$

$\Rightarrow c + 5 = 26$

$\Rightarrow c = 21$

If $\frac{c+5}{2}$ is even, $f(f(f(c))) = 18$

$\Rightarrow \frac{c+5}{4} = 18$

$\Rightarrow c + 5 = 72$

$\Rightarrow c = 67$

The least value of $c = 21$.

I3.4 Let $f\left(\frac{x}{3}\right) = x^2 + x + 1$. If d is the sum of all x for which $f(3x) = c$, find the value of d.

$f(x) = (3x)^2 + 3x + 1$

$= 9x^2 + 3x + 1$

$f(3x) = 81x^2 + 9x + 1$

$f(3x) = 21$

$\Rightarrow 81x^2 + 9x + 1 = 21$

$\Rightarrow 81x^2 + 9x - 20 = 0$

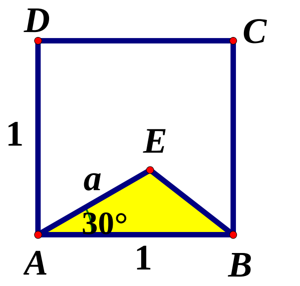
$\Rightarrow d$ = sum of roots

$= -\frac{9}{81}$

$= -\frac{1}{9}$

Individual Event 4

I4.1 In Figure 1, *ABCD* is a square, *E* is a point and $\angle EAB = 30°$. If the area of *ABCD* is six times that of ΔABE, then the ratio of $AE : AB = a : 1$. Find the value of *a*.



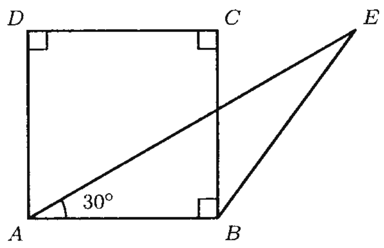
Let $AB = AD = 1$, $AE = a$, let the altitude of ΔABE from *E* to *AB* be *h*.

area of *ABCD* is six times that of $\Delta ABE \Leftrightarrow 1^2 = 6\times\frac{1}{2}\cdot 1\cdot h$

$h = \frac{1}{3}$, $a = \frac{h}{\sin 30^\circ} = 2h = \frac{2}{3}$

Remark: The original questions is

在圖一中，*ABCD* 為一正方形，*E* 為此正方形以外的一點及 $\angle EAB = 30°$。若 *ABCD* 的面積是 ΔABE 的面積的六倍，則 $AE : AB = a : 1$。求 *a* 的值。



In Figure 1, *ABCD* is a square, *E* is a point outside the square and $\angle EAB = 30°$. If the area of *ABCD* is six times that of ΔABE, then the ratio of $AE : AB = a : 1$. Find the value of *a*.

In fact, *E* must lie **inside** the square.

I4.2 Given that $b = \frac{\log 8^a + \log 27^a + \log 125^a}{\log 9 + \log 25 + \log 2 - \log 15}$, find the value of *b*.

$b = \frac{a\log 8 + a\log 27 + a\log 125}{2\log 3 + 2\log 5 + \log 2 - \log 3 - \log 5} = \frac{a(3\log 2 + 3\log 3 + 3\log 5)}{\log 3 + \log 5 + \log 2} = \frac{2}{3}\times 3 = 2$

I4.3 Let *c* be the remainder of $1^3 + 2^3 + \dots + 2009^3 + 2010^3$ divided by b^2, find the value of *c*.

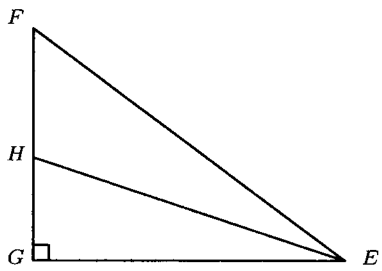
Use the formula $1^3 + 2^3 + \dots + n^3 = \frac{1}{4}n^2(n+1)^2$,

$1^3 + 2^3 + \dots + 2009^3 + 2010^3 = \frac{1}{4}\cdot 2010^2\cdot 2011^2 = 1005^2\cdot 2011^2 = (4\times 251 + 1)^2\cdot(4\times 502 + 3)^2$

$= (4p + 1)\cdot(4q + 1) = 4r + 1$, where *p*, *q*, *r* are positive integers.

$\therefore$ When it is divided by 2^2, the remainder is 1, $c = 1$.

I4.4 In Figure 2, *EFG* is a right-angled triangle. Given that *H* is a point on *FG*, such that $GH : HF = 4 : 5$ and $\angle GEH = \angle FEH$. If $EG = c$ and $FG = d$, find the value of *d*.



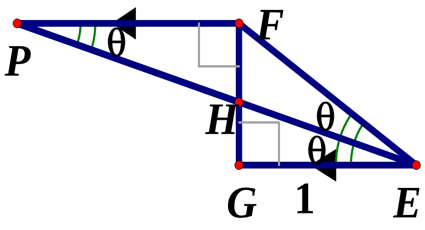
Let $\angle FEH = \theta = \angle FEH$, $GH = 4k$, $FH = 5k$, $EG = 1$

In ΔEGH, $\tan\theta = 4k$ …… (1)

In ΔEFG, $\tan 2\theta = 9k$ …… (2)

Sub. (1) into (2): $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \Leftrightarrow 9k = \frac{2\cdot 4k}{1-(4k)^2}$

$9(1 - 16k^2) = 8 \Leftrightarrow k = \frac{1}{12} \Leftrightarrow d = FG = 5k + 4k = 9k = \frac{3}{4}$



Method 2 Let $\angle GEH = \theta = \angle FEH$

From *P*, draw a line segment *FP* parallel to *GE*, which intersects with *EH* produced at *P*.

$\angle FPH = \theta$	(alt. ∠s, *PF* // *GE*)
$\Delta FPH \sim \Delta GEH$	(equiangular)
$\frac{GH}{HF} = \frac{GE}{PF} \Rightarrow \frac{4}{5} = \frac{1}{PF} \Rightarrow PF = 1.25$	(ratio of sides, ~Δ's)
$FE = PF = 1.25$	(sides opp. eq. ∠s)
$d = FG = \sqrt{1.25^2 - 1} = 0.75$	(Pythagoras' Theorem)

Individual Spare

IS.1 Given that $a=\sqrt{(19.19)^2+(39.19)^2-(38.38)(39.19)}$. Find the value of m.

Let $x = 19.19$, $y = 39.19$ then $x < y$ and $38.38 = 2x$

$a=\sqrt{x^2+y^2-2xy}=\sqrt{(y-x)^2}=y-x=39.19-19.19=20$

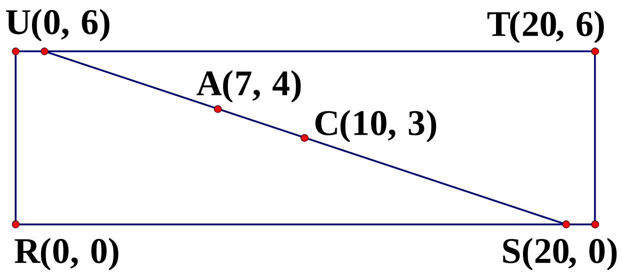
IS.2 Given four points $R(0, 0)$, $S(a, 0)$, $T(a, 6)$ and $U(0, 6)$. If the line $y = b(x - 7) + 4$ cuts the quadrilateral $RSTU$ into two halves of equal area, find the value of b.

$y = b(x - 7) + 4$ represents a family of straight lines with slope b which always pass through a fixed point $A(7, 4)$.

$R(0, 0)$, $S(20, 0)$, $T(20, 6)$ and $U(0, 6)$.

$RSTU$ is a rectangle whose base is parallel to x-axis with centre at $C(10, 3)$.

The line joining AC bisect the area of the rectangle. $b=\frac{3-4}{10-7}=-\frac{1}{3}$



IS.3 Given that c is the minimum value of $f(x)=\dfrac{x^2-2x+\frac{1}{b}}{2x^2+2x+1}$. Find the value of c.

Let $y=f(x)=\dfrac{x^2-2x-3}{2x^2+2x+1}$

$2yx^2 + 2yx + y = x^2 - 2x - 3$

$(2y - 1)x^2 + 2(y + 1)x + (y + 3) = 0$

For any values of x, the above quadratic equation has real solution.

$\therefore \Delta \ge 0$

$(y + 1)^2 - (2y - 1)(y + 3) \ge 0$

$y^2 + 2y + 1 - (2y^2 + 5y - 3) \ge 0$

$-y^2 - 3y + 4 \ge 0$

$y^2 + 3y - 4 \le 0$

$(y + 4)(y - 1) \le 0$

$-4 \le y \le 1$

c = the minimum of $y = -4$

IS.4 Given that $f(x) = px^6 + qx^4 + 3x - \sqrt{2}$, and p, q are non-zero real numbers.

If $d = f(c) - f(-c)$, find the value of d.

$d = (pc^6 + qc^4 + 3c - \sqrt{2}) - (pc^6 + qc^4 - 3c - \sqrt{2}) = 6c = 6(-4) = -24$

Sample Group Event (2009 Final Group Event 2)

SG.1 Given $\tan\theta = \frac{5}{12}$, where 180° ≤ θ ≤ 270°. If $A = \cos\theta + \sin\theta$, find the value of A.

$\cos\theta = -\frac{12}{13}$, $\sin\theta = -\frac{5}{13}$

$A = -\frac{12}{13} - \frac{5}{13} = -\frac{17}{13}$

SG.2 Let $[x]$ be the largest integer not greater than x. If $B = \left[10+\sqrt{10+\sqrt{10+\sqrt{10+\cdots}}}\right]$,

find the value of B .

Reference 2007 FG2.2 … $x \ge 3+\sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3+\sqrt{3}}}}}$ …

Let $y = \sqrt{10+\sqrt{10+\sqrt{10+\cdots}}}$

$y^2 = 10 + \sqrt{10+\sqrt{10+\sqrt{10+\cdots}}} = 10 + y$

$y^2 - y - 10 = 0$

$y = \frac{1+\sqrt{41}}{2}$ or $\frac{1-\sqrt{41}}{2}$ (rejected)

$6 < \sqrt{41} < 7 \Rightarrow \frac{7}{2} < \frac{1+\sqrt{41}}{2} < 4$

$13.5 < 10+\sqrt{10+\sqrt{10+\sqrt{10+\cdots}}} < 14$

$B = 13$

SG.3 Let $a \oplus b = ab + 10$. If $C = (1\oplus 2) \oplus 3$, find the value of C.

$1\oplus 2 = 2 + 10 = 12$; $C = 12\oplus 3 = 36 + 10 = 46$

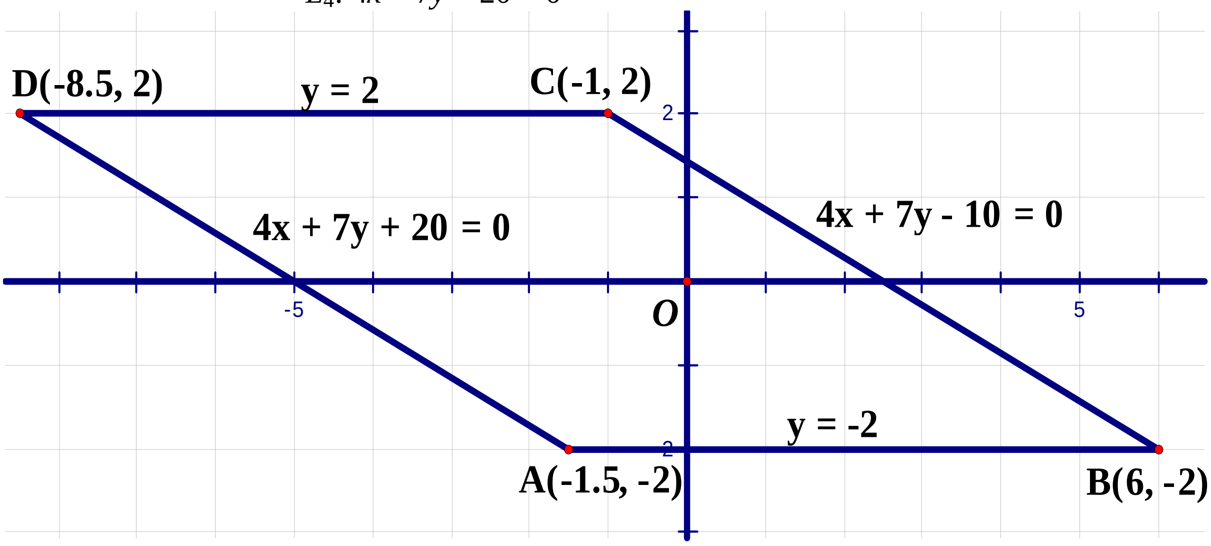
SG.4 In the coordinate plane, the area of the region bounded by the following lines is D square units, find the value of D .

L_1: $y - 2 = 0$

L_2: $y + 2 = 0$

L_3: $4x + 7y - 10 = 0$

L_4: $4x + 7y + 20 = 0$



It is easy to show that the bounded region is a parallelogram $ABCD$ with vertices $A(-1.5, -2)$, $B(6, -2)$, $C(-1, 2)$, $C(-8.5, 2)$.

The area $D = |6 - (-1.5)|\times|2 - (-2)| = 7.5\times 4 = 30$

Group Event 1

G1.1 Find the value of $\sin^2 1° + \sin^2 2° + \ldots + \sin^2 89°$.

Reference 2012 HG9

$\sin^2 1° + \sin^2 89° = \sin^2 1° + \cos^2 1° = 1$

$\sin^2 2° + \sin^2 88° = \sin^2 2° + \cos^2 2° = 1$

....

$\sin^2 44° + \sin^2 46° = \sin^2 44° + \cos^2 44° = 1$

$\sin^2 1° + \sin^2 2° + \ldots + \sin^2 89° = (\sin^2 1° + \sin^2 89°) + \ldots + (\sin^2 44° + \sin^2 46°) + \sin^2 45°$

$= 44.5$

G1.2 Let x, y and z be positive numbers. Given that $\frac{x+z}{2z-x} = \frac{z+2y}{2x-z} = \frac{x}{y}$. Find the value of $\frac{x}{y}$.

Reference 1998 FG1.2

It is equivalent to $\frac{x+z}{2z-x} = \frac{x}{y}$ … (1) and $\frac{z+2y}{2x-z} = \frac{x}{y}$ … (2)

From (2), $yz + 2y^2 = 2x^2 - xz \Rightarrow (x + y)z = 2(x^2 - y^2)$

$\Rightarrow x + y = 0$ (rejected, $x > 0$ and $y > 0$) or $z = 2(x - y)$ … (3)

From (1): $xy + yz = 2xz - x^2 \Rightarrow (2x - y)z = x^2 + xy$ … (4)

Sub. (4) into (5): $2(x - y)(2x - y) = x^2 + xy$

$2(2x^2 - 3xy + y^2) = x^2 + xy$

$3x^2 - 7xy + 2y^2 = 0$

$(3x - y)(x - 2y) = 0 \Rightarrow \frac{x}{y} = \frac{1}{3}$ or 2

When $y = 3x$, sub. into (3): $z = 2(x - 3x) = -4x$ (rejected, $x > 0$ and $z > 0$)

$\therefore \frac{x}{y} = 2$

Method 2 $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k \Rightarrow a = bk,\ c = dk,\ e = fk \Rightarrow \frac{a+c+e}{b+d+f} = \frac{bk+dk+fk}{b+d+f} = k$

$k = \frac{x}{y} = \frac{a+c+e}{b+d+f} = \frac{(x+z)+(z+2y)+x}{(2z-x)+(2x-z)+y} = \frac{2x+2y+2z}{x+y+z} = 2$ ($\because x+y+z>0$)

Remark: The original question is: Given that $\frac{x+z}{2z-x} = \frac{z+2y}{2x-z} = \frac{x}{y}$. Find the value of $\frac{x}{y}$.

The question has **more than one solution**.

G1.3 Find the sum of all real roots x of the equation $(2^x - 4)^3 + (4^x - 2)^3 = (4^x + 2^x - 6)^3$.

Let $a = 2^x - 4$, $b = 4^x - 2$, $a + b = 4^x + 2^x - 6$, the equation is equivalent to $a^3 + b^3 = (a + b)^3$

$(a + b)(a^2 - ab + b^2) = (a + b)^3$

$a^2 - ab + b^2 = a^2 + 2ab + b^2$ or $a + b = 0$

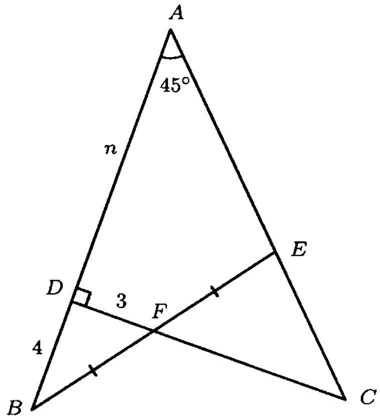
$3ab = 0$ or $4^x + 2^x - 6 = 0$

$2^x = 4$ or $4^x = 2$ or $(2^x - 2)(2^x + 3) = 0$

$x = 2$, $\frac{1}{2}$ or 1

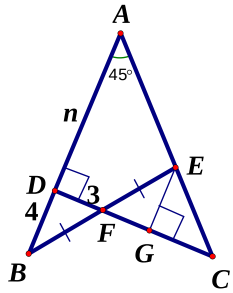
Sum of all real roots = 3.5

G1.4 In Figure 1, if $AB \perp CD$，F is the midpoint of BE, $\angle A = 45°$, $DF = 3$, $BD = 4$ and $AD = n$, find the value of n.



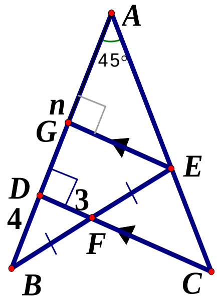
Let G be the foot of perpendicular drawn from E onto CF.

$\angle BFD = \angle CFE$ (vert. opp. ∠s)
$BF = 5$ (Pythagoras' Theorem)
$EF = 5$ (Given F is the midpoint)
$\angle BDF = 90° = \angle EGF$ (by construction)
$\Delta BDF \cong \Delta EGF$ (A.A.S.)
$\therefore FG = DF = 3$ (corr. sides, ≅ Δ's)
$EG = 4$ (corr. sides, ≅ Δ's)
$\angle ACD = 45°$ (∠s sum of ΔACD)
$\therefore \Delta ACD$ is a right-angled isosceles triangle.
$CD = AD = n$ (sides opp. equal angle)
$EG \perp CD \Rightarrow EG // AD \Rightarrow \Delta CEG \sim \Delta CAD$ (equiangular)
$\Rightarrow CG = EG = 4$ (ratio of sides, ~Δ's)
$n = AD = CD = 3 + 3 + 4 = 10$



Method 2

Draw $EG // CD$, which intersects AB at G.
$GD = BD = 4$ ($BF = FE$ and $FD // EG$, intercept theorem)
$GE = 2DF = 6$ (mid-points theorem)
$\angle AGE = 90°$ (corr. ∠s, $EG // CD$)
ΔAGE is a right-angled isosceles triangle.
$\therefore AG = GE = 6$ (sides opp. eq. ∠s)
$n = AG + GD = 6 + 4 = 10$



Group Event 2

G2.1 If $p = 2 - 2^2 - 2^3 - 2^4 - \ldots - 2^9 - 2^{10} + 2^{11}$, find the value of p.

$p = 2 + 2^{11} - (2^2 + 2^3 + 2^4 + \ldots + 2^9 + 2^{10}) = 2 + 2^{11} - \frac{2^2(2^9-1)}{2-1} = 2 + 2^{11} - 2^{11} + 2^2 = 6$

G2.2 Given that x, y, z are three distinct real numbers.

Reference: 2008 FG2.4, 2017 FG2.1

If $x+\frac{1}{y}=y+\frac{1}{z}=z+\frac{1}{x}$ and $m = x^2y^2z^2$, find the value of m.

Let $x+\frac{1}{y}=k$ … (1), $y+\frac{1}{z}=k$ … (2), $z+\frac{1}{x}=k$ … (3)

From (1), $x = k-\frac{1}{y}=\frac{ky-1}{y} \Rightarrow \frac{1}{x}=\frac{y}{ky-1}$

Sub. into (3): $z+\frac{y}{ky-1}=k \Rightarrow z=\frac{(k^2-1)y-k}{ky-1}$

Sub. into (2): $y+\frac{ky-1}{(k^2-1)y-k}=k \Rightarrow (k^2-1)y^2 - ky + ky - 1 = k(k^2-1)y - k^2$

$\Rightarrow (k^2-1)y^2 - k(k^2-1)y + (k^2-1) = 0$

$\Rightarrow k^2 - 1 = 0$ or $y^2 - ky + 1 = 0$

If $k^2 - 1 \neq 0$, then $y^2 - ky + 1 = 0 \Rightarrow y = \frac{k\pm\sqrt{k^2-4}}{2}$

$\because$ The system is symmetric for x, y, z

$\therefore$ x or $z = \frac{k\pm\sqrt{k^2-4}}{2}$, this contradict to the fact that x, y, z are distinct.

$\therefore y^2 - ky + 1 \neq 0$

$\Rightarrow k^2 = 1 \Rightarrow k = 1$ or -1

When $k = 1$, $x = k-\frac{1}{y}=1-\frac{1}{y}=\frac{y-1}{y}$; $z=\frac{(k^2-1)y-k}{ky-1}=\frac{-1}{y-1}$

$xyz=\frac{y-1}{y}\cdot y\cdot\frac{-1}{y-1}=-1$

When $k = -1$, $x = k-\frac{1}{y}=-1-\frac{1}{y}=-\frac{y+1}{y}$; $z=\frac{(k^2-1)y-k}{ky-1}=\frac{-1}{y+1}$

$xyz=-\frac{y+1}{y}\cdot y\cdot\frac{-1}{y+1}=1$

$\therefore m = x^2y^2z^2 = 1$

Method 2 $x+\frac{1}{y}=y+\frac{1}{z}\Leftrightarrow x-y=\frac{1}{z}-\frac{1}{y}\Leftrightarrow x-y=\frac{y-z}{yz}$ …… (1)

$y+\frac{1}{z}=z+\frac{1}{x}\Leftrightarrow y-z=\frac{1}{x}-\frac{1}{z}\Leftrightarrow y-z=\frac{z-x}{xz}$ …… (2)

$x+\frac{1}{y}=z+\frac{1}{x}\Leftrightarrow z-x=\frac{1}{y}-\frac{1}{x}\Leftrightarrow z-x=\frac{x-y}{xy}$ …… (3)

(1)×(2)×(3): $(x-y)(y-z)(z-x)=\frac{y-z}{yz}\cdot\frac{z-x}{xz}\cdot\frac{x-y}{xy}$

$\Leftrightarrow 1=\frac{1}{x^2y^2z^2}$

$\Leftrightarrow m = x^2y^2z^2 = 1$

G2.3 Given that x is a positive real number and $x \cdot 3^x = 3^{18}$. If k is a positive integer and $k < x < k + 1$, find the value of k.

The equation is equivalent to $3^{18-x} - x = 0$. Let $f(x) = 3^{18-x} - x$.

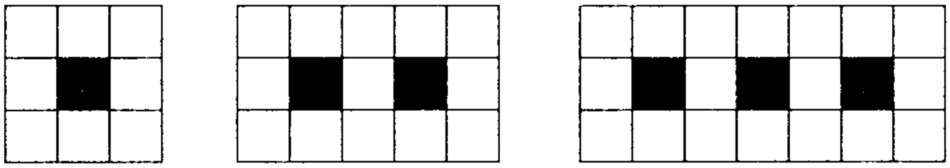
Clearly $f(x)$ is a continuous function.

$f(15) = 3^3 - 15 = 12 > 0$, $f(16) = 3^2 - 16 = -7 < 0$

By intermediate value theorem (or Bolzano's theorem), we can find a real root $15 < x < 16$.

$k = 15$

G2.4 Figure 1 shows the sequence of figures that are made of squares of white and black. Find the number of white squares in the 95th figure.



1st figure 2nd figure 3rd figure

1st figure = 8 white squares; 2nd figure = 13 squares; 3rd figure = 18 squares

$T(1) = 8$, $T(2) = 8 + 5$, $T(3) = 8 + 5\times2$, … , $T(95) = 8 + 5\times(95 - 1) = 478$

Group Event 3

G3.1 Find the smallest prime factor of $101^{303} + 301^{101}$.

Both 101^{303} and 301^{101} are odd integers

$\therefore\ 101^{303} + 301^{101}$ is even

The smallest prime factor is 2.

G3.2 Let n be the integral part of $\dfrac{1}{\frac{1}{1980}+\frac{1}{1981}+\cdots+\frac{1}{2009}}$. Find the value of n.

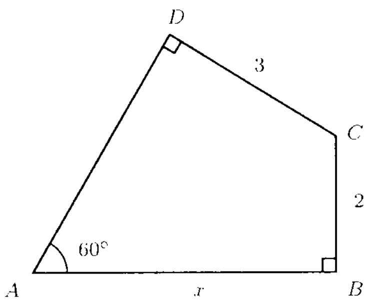
$$\frac{1}{2009}+\cdots+\frac{1}{2009}<\frac{1}{1980}+\frac{1}{1981}+\cdots+\frac{1}{2009}<\frac{1}{1980}+\cdots+\frac{1}{1980} \quad \text{(30 terms)}$$

$$\frac{30}{2009}<\frac{1}{1980}+\frac{1}{1981}+\cdots+\frac{1}{2009}<\frac{30}{1980}$$

$$66=\frac{1980}{30}<\frac{1}{\frac{1}{1980}+\frac{1}{1981}+\cdots+\frac{1}{2009}}<\frac{2009}{30}<\frac{2010}{30}=67$$

n = integral part = 66

G3.3 In Figure 1, $\angle A = 60°$, $\angle B = \angle D = 90°$. $BC = 2$, $CD = 3$ and $AB = x$, find the value of x.



$AC^2 = x^2 + 4$ (Pythagoras' Theorem on ΔABC)

$AD^2 = AC^2 - 3^2$ (Pythagoras' Theorem on ΔACD)

$= x^2 - 5$

$BD^2 = x^2 + (x^2 - 5) - 2x\sqrt{x^2-5}\cos 60°$ (cosine rule on ΔABD)

$BD^2 = 2^2 + 3^2 - 2\cdot 2\cdot 3\cos 120°$ (cosine rule on ΔBCD)

$\therefore\ 2x^2 - 5 - x\sqrt{x^2-5} = 13 + 6$

$x\sqrt{x^2-5} = 2x^2 - 24$

$x^2(x^2 - 5) = 4x^4 - 96x^2 + 576$

$3x^4 - 91x^2 + 576 = 0$

$(x^2 - 9)(3x^2 - 64) = 0$

$x = 3$ or $\dfrac{8}{\sqrt{3}}$

When $x = 3$, $AD = \sqrt{x^2-5} = 2$

$\tan\angle BAC = \dfrac{2}{3}$, $\tan\angle CAD = \dfrac{3}{2} = \tan(90° - \angle BAC)$

$\angle BAD = 90° \neq 60°$ $\therefore$ reject $x = 3$

When $x = \dfrac{8}{\sqrt{3}}$, $AD = \sqrt{x^2-5} = \dfrac{7}{\sqrt{3}}$

$\tan\angle BAC = \dfrac{\sqrt{3}}{4}$, $\tan\angle CAD = \dfrac{3\sqrt{3}}{7}$

$$\tan(\angle BAC + \angle CAD) = \frac{\frac{\sqrt{3}}{4}+\frac{3\sqrt{3}}{7}}{1-\frac{\sqrt{3}}{4}\cdot\frac{3\sqrt{3}}{7}} = \frac{19\sqrt{3}}{19} = \sqrt{3} = \tan 60°$$

$$\therefore\ x = \frac{8}{\sqrt{3}} = \frac{8}{3}\sqrt{3}$$

Method 2

$BD^2 = 2^2 + 3^2 - 2\cdot 2\cdot 3\cos 120°$ (cosine rule on ΔBCD)

$BD = \sqrt{19}$

$\angle ABC + \angle ADC = 180°$

A, B, C, D are concyclic (opp. $\angle$s supp.)

$AC = \sqrt{x^2+4}$ = diameter = $2R$ (converse, $\angle$ in semi-circle, R = radius)

$\dfrac{BD}{\sin 60°} = 2R$ (Sine rule on ΔABD)

$$\frac{\sqrt{19}}{\frac{\sqrt{3}}{2}} = \sqrt{x^2+4}$$

$76 = 3x^2 + 12$

$$x = \frac{8}{\sqrt{3}} = \frac{8}{3}\sqrt{3}$$

G3.4 Given that the function f satisfies $f(2 + x) = f(2 - x)$ for every real number x and that $f(x) = 0$ has exactly four distinct real roots. Find the sum of these four distinct real roots.

Reference: 1994 FI3.4

Let two of these distinct roots be $2 + \alpha$, $2 + \beta$, where $\alpha \neq \beta$ and $\alpha, \beta \geq 0$.

$f(2 + x) = f(2 - x) \Leftrightarrow f(2 + \alpha) = f(2 - \alpha) = 0$; $f(2 + \beta) = f(2 - \beta) = 0$

If $\alpha = 0$ and $\beta \neq 0 \Rightarrow$ there are only three real roots 2, $2 + \beta$, $2 - \beta$ contradiction, rejected.

$\therefore\ \alpha \neq 0$ and $\beta \neq 0 \Leftrightarrow$ The four roots are $2 + \alpha$, $2 - \alpha$, $2 + \beta$, $2 - \beta$.

Sum of roots $= 2 + \alpha + 2 - \alpha + 2 + \beta + 2 - \beta = 8$

Group Event 4

G4.1 Let a be an integer and $a \neq 1$. Given that the equation $(a - 1)x^2 - mx + a = 0$ has two roots which are positive integers. Find the value of m.

Let the 2 roots be α, β.

$\alpha\beta = \frac{a}{a-1} = \frac{1}{a-1} + 1$

α, β are positive integers $\Rightarrow \frac{1}{a-1}$ is a positive integer

$\Rightarrow a - 1 = 1$ or -1

$\Rightarrow a = 2$ or 0 (rejected)

Put $a = 2$ into the original equation: $x^2 - mx + 2 = 0$

$\alpha\beta = 2 \Rightarrow \alpha = 2, \beta = 1$ or $\alpha = 1, \beta = 2$

$m = \alpha + \beta = 3$

Remark: The original question is

Given that the equation $(a - 1)x^2 - mx + a = 0$ has two roots which are positive integers. Find the value of m.

If $a = 1$, then it is not a quadratic equation, it cannot have 2 positive integral roots.

If a is any real number $\neq 1$, the equality $\alpha\beta = \frac{1}{a-1} + 1$ could not implies $a = 2$

e.g. $a = 1.5 \Rightarrow \alpha\beta = \frac{1}{1.5-1} + 1 = 3 \Rightarrow \alpha = 1, \beta = 3 \Rightarrow \alpha + \beta = 4 = \frac{m}{1.5-1} \Rightarrow m = 2$

e.g. $a = 1.1 \Rightarrow \alpha\beta = \frac{1}{1.1-1} + 1 = 11 \Rightarrow \alpha = 1, \beta = 11 \Rightarrow \alpha + \beta = 12 = \frac{m}{1.1-1} \Rightarrow m = 1.2$

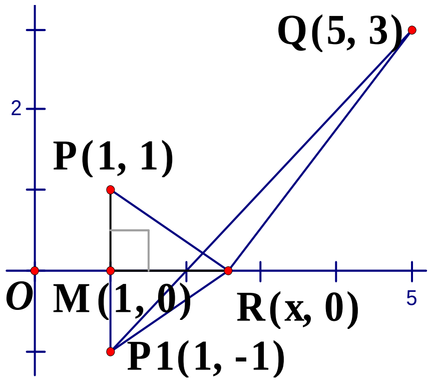
There are infinitely many possible values of m !!!

G4.2 Given that x is a real number and $y = \sqrt{x^2 - 2x + 2} + \sqrt{x^2 - 10x + 34}$. Find the minimum value of y. **Reference: 2015 HI9**

Consider the following problem:

Let $P(1, 1)$ and $Q(5, 3)$ be two points. $R(x, 0)$ is a variable point on x-axis.

To find the minimum sum of distances $PR + RQ$.



Let y = sum of distances $= \sqrt{(x-1)^2 + 1} + \sqrt{(x-5)^2 + 9}$

If we reflect $P(1, 1)$ along x-axis to $P_1(1, -1)$, $M(1, 0)$ is the foot of perpendicular,

then $\Delta PMR \cong \Delta P_1MR$ (S.A.S.)

$y = PR + RQ = P_1R + RQ \geq P_1Q$ (triangle inequality)

$y \geq \sqrt{(5-1)^2 + (3+1)^2} = 4\sqrt{2}$

G4.3 Given that A, B, C are positive integers with their greatest common divisor equal to 1. If A, B, C satisfy $A \log_{500} 5 + B \log_{500} 2 = C$, find the value of $A + B + C$.

$\log_{500} 5^A + \log_{500} 2^B = \log_{500} 500^C \Rightarrow \log_{500} 5^A \cdot 2^B = \log_{500} 500^C \Rightarrow 5^A \cdot 2^B = 5^{3C} \cdot 2^{2C}$

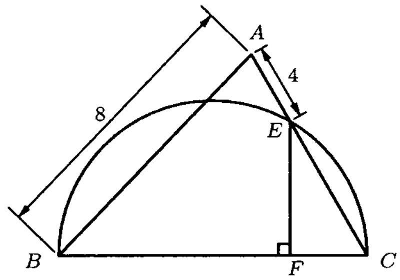
$A = 3C$, $B = 2C$ (unique factorization theorem)

$\because A, B, C$ are relatively prime.

$\therefore C = 1, A = 3, B = 2$

$A + B + C = 6$

G4.4 In figure 1, *BEC* is a semicircle and *F* is a point on the diameter *BC*. Given that *BF* : *FC* = 3 : 1, *AB* = 8 and *AE* = 4. Find the length of *EC*.



Join *BE*. It is easy to show that $\Delta BEF \sim \Delta ECF$ (equiangular)

Let $BF = 3k$, $CF = k$

$EF : 3k = k : EF$ (ratio of sides, ~Δ's)

$EF = \sqrt{3}\,k$

$BE^2 = BF^2 + EF^2 = 9k^2 + 3k^2$ (Pythagoras' Theorem on Δ*BEF*) $\Rightarrow BE = \sqrt{12}\,k$

$\angle BEC = 90°$ (∠ in semi-circle)

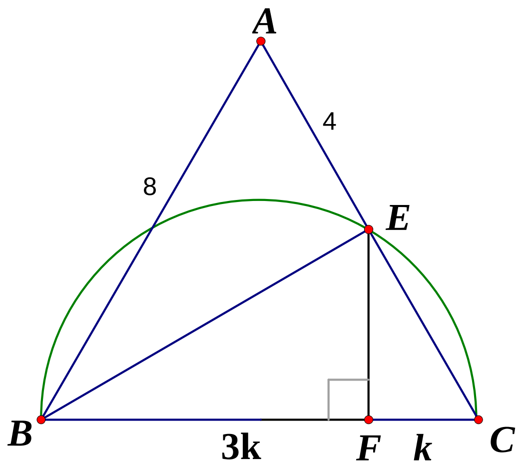
$BE^2 + AE^2 = AB^2$ (Pythagoras' Theorem on Δ*ABE*)

$12k^2 + 16 = 64 \Rightarrow k = 2$

$EC^2 = CF^2 + EF^2 = 2^2 + 3\times 2^2$ (Pythagoras' Theorem on Δ*CEF*)

$EC = 4$

Method 2



$\angle BEC = 90°$ (∠ in semi-circle)

$\angle BEA = 90°$ (adj. ∠s on st. line)

$\cos \angle BAE = \dfrac{AE}{AB} = \dfrac{4}{8} = \dfrac{1}{2}$

$\angle BAE = 60°$

Let $BF = 3k$, $CF = k$, $\angle ECB = \theta$.

$\angle CEF = 90° - \theta$ (∠s sum of Δ*CEF*)

$\angle CBE = 90° - \theta$ (∠s sum of Δ*BCE*)

$\angle BEF = \theta$ (∠s sum of Δ*BEF*)

$\Delta CEF \sim \Delta EBF$ (equiangular)

$\dfrac{CF}{EF} = \dfrac{EF}{BF}$ (corr. sides, ~Δs)

$EF^2 = k\cdot 3k$

$EF = \sqrt{3}\,k$

$\tan \angle ECF = \dfrac{EF}{CF} = \dfrac{\sqrt{3}k}{k} = \sqrt{3}$

$\angle ECF = 60°$

$\Delta ABE \cong \Delta CBE$ (A.A.S.)

$EC = AE = 4$ (corr. sides, ≅ Δs)

Group Spare

GS.1 Given that n is a positive integer. If $n^2 + 5n + 13$ is a perfect square, find the value of n.

$n^2 + 5n + 13 = n^2 + 5n + 2.5^2 - 2.5^2 + 13 = (n + 2.5)^2 + 6.75 = m^2$, where m is an integer

$m^2 - (n + 2.5)^2 = 6.75 \Rightarrow (m + n + 2.5)(m - n - 2.5) = 6.75 \Rightarrow (2m + 2n + 5)(2n - 2m - 5) = 27$

$$\begin{cases} 2m+2n+5=27 \\ 2m-2n-5=1 \end{cases} \text{ or } \begin{cases} 2m+2n+5=9 \\ 2m-2n-5=3 \end{cases}$$

$n = 4$ or $n = -1$ (rejected, $\because n > 0$)

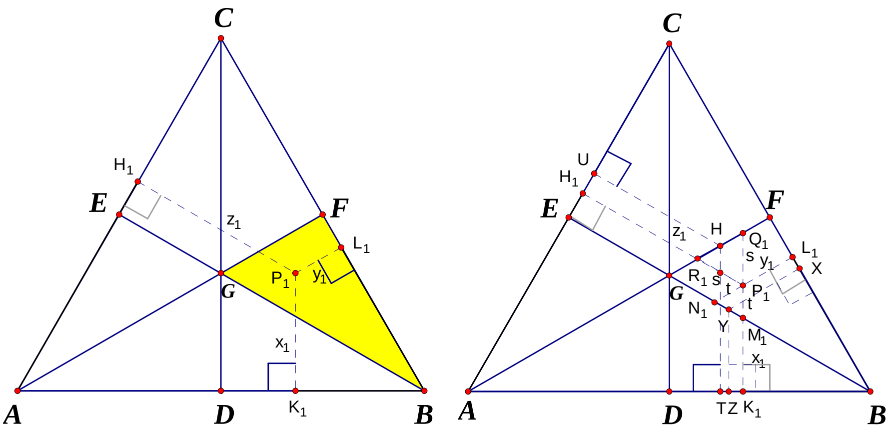
GS.2 Given that $1^3 + 2^3 + \ldots + k^3 = \left(\frac{k(k+1)}{2}\right)^2$. Find the value of $11^3 + 12^3 + \ldots + 24^3$.

$11^3 + 12^3 + \ldots + 24^3 = 1^3 + 2^3 + \ldots + 24^3 - (1^3 + 2^3 + \ldots + 10^3)$

$$= \frac{1}{4}\cdot 24^2 \cdot 25^2 - \frac{1}{4}\cdot 10^2 \cdot 11^2 = \frac{1}{4}\cdot 6^2 \cdot 100^2 - \frac{1}{4}\cdot 10^2 \cdot 121$$

$$= \frac{1}{4}\cdot(360000 - 12100) = \frac{1}{4}\cdot 347900 = 86975$$

GS.3 If P is an arbitrary point in the interior of the equilateral triangle ABC, find the probability that the area of ΔABP is greater than **each** of the areas of ΔACP and ΔBCP.



D, E, F be the mid-points of AB, AC and BC respectively.

The medians CD, BE and AF are concurrent at the centroid G.

It is easy to see that ΔCEG, ΔCFG, ΔAEG, ΔADG, ΔBDG, ΔBFG are congruent triangles having the same areas.

P is any point inside the triangle $\Rightarrow P$ lies on or inside one of these six congruent triangles.

As shown in the diagram, P_1 lies inside ΔBFG. Let the feet of perpendiculars from P_1 to AB, BC, CA be K_1, L_1, H_1 with lengths x_1, y_1 and z_1 respectively.

P_1H_1 and AF meet at R_1, P_1K_1 intersects BE at M_1, and AF at Q_1, L_1P_1 produced meet BE at N_1

By the properties on parallel lines, we can easily prove that $\Delta P_1M_1N_1$ and $\Delta P_1Q_1R_1$ are equilateral triangles. Let $P_1M_1 = P_1N_1 = N_1M_1 = t$, $P_1Q_1 = P_1R_1 = Q_1R_1 = s$

Let H and Y be the midpoints of Q_1R_1 and N_1M_1 respectively. $R_1H = 0.5s$, $YM_1 = 0.5t$

Let U and T be the feet of perpendiculars from H to AC and AB respectively.

Let X and Z be the feet of perpendiculars from Y to BC and AB respectively.

$UH = z_1 - s + 0.5s \cos 60° = z_1 - 0.75s$, $YZ = x_1 - t + 0.5t \cos 60° = x_1 - 0.75t$

$HT = x_1 + 0.75s$, $YX = y_1 + 0.75t$

It is easy to show that $\Delta AHU \cong \Delta AHT$, $\Delta BYX \cong \Delta BYZ$ (A.A.S.)

$UH = HT$ and $YZ = YX$ (corr. sides, $\cong \Delta$'s) $\Rightarrow z_1 - 0.75s = x_1 + 0.75s$, $x_1 - 0.75t = y_1 + 0.75t$

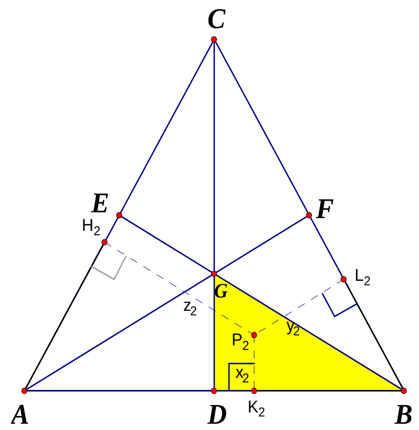
$z_1 = x_1 + 1.5s$, $x_1 = y_1 + 1.5t \Rightarrow z_1 > x_1 > y_1$

$\therefore \frac{1}{2}AC\cdot z_1 > \frac{1}{2}AB\cdot x_1 > \frac{1}{2}BC\cdot y_1 \Rightarrow$ area of ΔACP_1 > area of ΔABP_1 > area of ΔBCP_1

If P_2 lies inside ΔBDG, using a similar method, we can easily prove that

area of ΔACP_2 > area of ΔBCP_2 > area of ΔABP_2.

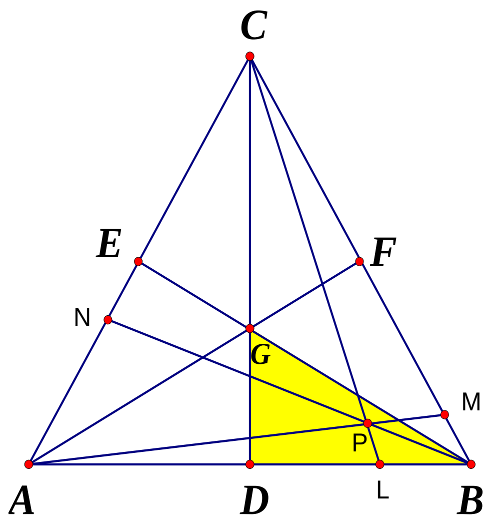
If P_3 lies inside ΔADG, then area of ΔBCP_3 > area of ΔACP_3 > area of ΔABP_3.



If P_4 lies inside ΔAEG, then
area of ΔBCP_4 > area of ΔABP_4 > area of ΔACP_4.
If P_5 lies inside ΔCEG, then
area of ΔABP_5 > area of ΔBCP_5 > area of ΔACP_5.
If P_6 lies inside ΔCFG, then
area of ΔABP_6 > area of ΔACP_6 > area of ΔBCP_6
In order that the area of ΔABP is greater than **each** of the areas of ΔACP and ΔBCP, P must lie inside ΔCEG or ΔCFG

Required probability

$$=\frac{\text{Area of }\Delta CEG+\text{area of }\Delta CFG}{\text{Area of }\Delta ABC}=\frac{2}{6}=\frac{1}{3}$$



Method 2 Suppose P lies inside ΔBDG.
Produce AP, BP, CP to intersect BC, CA, AB at L, M, N respectively.
Let $S_{\Delta XYZ}$ denotes the area of ΔXYZ.

$$\frac{S_{\Delta APB}}{S_{\Delta APC}}=\frac{S_{\Delta ABM}-S_{\Delta BPM}}{S_{\Delta ACM}-S_{\Delta CPM}}$$

$$=\frac{\frac{1}{2}BM\cdot AM\sin AMB-\frac{1}{2}BM\cdot PM\sin AMB}{\frac{1}{2}CM\cdot AM\sin AMC-\frac{1}{2}CM\cdot PM\sin AMC}$$

$$=\frac{\frac{1}{2}BM\sin AMB\cdot(AM-PM)}{\frac{1}{2}CM\sin AMC\cdot(AM-PM)}=\frac{BM\sin AMB}{CM\sin AMC}$$

$$=\frac{BM}{MC}\quad(\because \sin AMB=\sin(180°-AMC)=\sin AMC)$$

$$\frac{S_{\Delta APB}}{S_{\Delta BPC}}=\frac{S_{\Delta ABN}-S_{\Delta APN}}{S_{\Delta BCN}-S_{\Delta CPN}}$$

$$=\frac{\frac{1}{2}BN\cdot AN\sin ANB-\frac{1}{2}AN\cdot PN\sin ANB}{\frac{1}{2}BN\cdot NC\sin BNC-\frac{1}{2}NC\cdot PN\sin BNC}$$

$$=\frac{\frac{1}{2}AN\sin ANB\cdot(BN-PN)}{\frac{1}{2}NC\sin BNC\cdot(BN-PN)}=\frac{AN\sin ANB}{NC\sin BNC}$$

$$=\frac{AN}{NC}\quad(\because \sin ANB=\sin(180°-BNC)=\sin BNC)$$

In order that the area of ΔABP is greater than **each** of the areas of ΔACP and ΔBCP,
$BM > MC$ and $AN > NC$
$\therefore$ P must lie inside ΔCEG or ΔCFG

Required probability $=\dfrac{S_{\Delta CEG}+S_{\Delta CFG}}{S_{\Delta ABC}}=\dfrac{2}{6}=\dfrac{1}{3}$

Remark: The original question is
若 *P* 是等邊三角形 *ABC* 內部的隨意一點，求Δ*ABP* 的面積同時大於Δ*ACP* 及Δ*BCP* 的面積的概率。
If P is an arbitrary point in the interior of the equilateral triangle ABC, find the probability that the area of ΔABP is greater than **both** of the areas of ΔACP and ΔBCP.
There is a slight difference between the Chinese version and the English version.

GS.4 How many positive integers m are there for which the straight line passing through points $A(-m, 0)$ and $B(0, 2)$ and also passes through the point $P(7, k)$, where k is a positive integer?
Let the slope of the variable straight line be a. Then its equation is: $y = ax + 2$

It passes through $A(-m, 0)$ and $P(7, k)$: $\begin{cases}-am+2=0\cdots(1)\\ 7a+2=k\cdots(2)\end{cases}$

$7(1) + m(2)$: $14 + 2m = km \Rightarrow m(k-2) = 14$
$m = 1$, $k = 16$ or $m = 2$, $k = 9$ or $m = 7$, $k = 4$ or $m = 14$, $k = 3$
Number of positive integral values of m is 4.

Individual Events

SI			I1			I2			I3			I4			IS		
	R	30		***P***	20		***P***	3		***P***	7		***a***	2		***P***	95
	S	120		***Q***	36		***Q***	5		***Q***	13		***b***	1		***Q***	329
	T	11		***R***	8		***R***	6		***R***	5		***c***	2		****R*** see the remark	6
	U	72		****S*** see the remark	5040		***S***	$\frac{-95+3\sqrt{1505}}{10}$		***S***	$\sqrt{5}$		****d*** see the remark	2		***S***	198

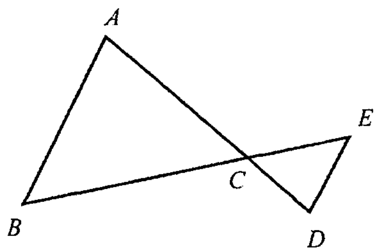
Group Events

SG			G1			G2			G3			G4			GS		
	q	3		***a***	2		***area***	40		***a***	1		***P***	20		****m*** see the remark	4
	k	1		***b***	3		****pairs*** see the remark	2550		***a+b+c***	1		$\frac{n}{m}$	$\frac{2}{3}$		***v***	6
	w	25		***c***	2		***x***	60		***y – x***	$\frac{1}{2}$		***r***	3		α	3
	p	$\frac{3}{2}$		***x***	3		***P***	–1		$\frac{P_1}{P_2}$	7		****BGHI*** see the remark	6		***F***	208

Sample Individual Event (2009 Final Individual Event 1)

S1.1 Let a, b, c and d be the distinct roots of the equation $x^4 - 15x^2 + 56 = 0$.
If $R = a^2 + b^2 + c^2 + d^2$, find the value of R.
$x^4 - 15x^2 + 56 = 0 \Rightarrow (x^2 - 7)(x^2 - 8) = 0$
$a = \sqrt{7}$, $b = -\sqrt{7}$, $c = \sqrt{8}$, $d = -\sqrt{8}$
$R = a^2 + b^2 + c^2 + d^2 = 7 + 7 + 8 + 8 = 30$

S1.2 In Figure 1, AD and BE are straight lines with $AB = AC$ and $AB // ED$.
If $\angle ABC = R°$ and $\angle ADE = S°$, find the value of S.



$\angle ABC = 30° = \angle ACB$ (base $\angle$ isos. Δ)
$\angle BAC = 120°$ ($\angle$s sum of Δ)
$\angle ADE = 120°$ (alt. $\angle$s $AB // ED$)
$S = 120$

S1.3 Let $F = 1 + 2 + 2^2 + 2^3 + \ldots + 2^S$ and $T = \sqrt{\frac{\log(1+F)}{\log 2}}$, find the value of T.

$F = 1 + 2 + 2^2 + 2^3 + \ldots + 2^{120} = \frac{2^{121}-1}{2-1} = 2^{121} - 1$

$T = \sqrt{\frac{\log(1+F)}{\log 2}} = \sqrt{\frac{\log 2^{121}}{\log 2}} = 11$

S1.4 Let $f(x)$ be a function such that $f(n) = (n - 1) f(n - 1)$ and $f(n) \neq 0$ hold for all integers $n \geq 6$.
If $U = \frac{f(T)}{(T-1)f(T-3)}$, find the value of U.
$f(n) = (n - 1) f(n - 1) = (n - 1)(n - 2)f(n - 2) = \ldots\ldots$
$U = \frac{f(11)}{(11-1)f(11-3)} = \frac{10\times 9\times 8\times f(8)}{10\times f(8)} = 8\times 9 = 72$

Individual Event 1

I1.1 If the average of a, b and c is 12, and the average of $2a + 1$, $2b + 2$, $2c + 3$ and 2 is P, find the value of P.

$a + b + c = 36$ …… (1)

$$P = \frac{2a+1+2b+2+2c+3+2}{4} = \frac{2(a+b+c)+8}{4} = \frac{2\times 36+8}{4} = 20$$

I1.2 Let $20112011 = aP^5 + bP^4 + cP^3 + dP^2 + eP + f$, where a, b, c, d, e and f are integers and $0 \le a, b, c, d, e, f < P$. If $Q = a + b + c + d + e + f$, find the value of Q.

Reference: 2008 FG4.3

$$\begin{array}{r|l l}
20 & 2\ 0\ 1\ 1\ 2\ 0\ 1\ 1 & \\ \hline
20 & 1\ 0\ 0\ 5\ 6\ 0\ 0 & \cdots\cdots\ 11 \\ \hline
20 & 5\ 0\ 2\ 8\ 0 & \cdots\cdots\ 0 \\ \hline
20 & 2\ 5\ 1\ 4 & \cdots\cdots\ 0 \\ \hline
20 & 1\ 2\ 5 & \cdots\cdots\ 14 \\ \hline
 & 6 & \cdots\cdots\ 5
\end{array}$$

$a = 6$, $b = 5$, $c = 14$, $d = 0$, $e = 0$, $f = 11$; $Q = 6 + 5 + 14 + 0 + 0 + 11 = 36$

I1.3 If R is the units digit of the value of $8^Q + 7^{10Q} + 6^{100Q} + 5^{1000Q}$, find the value of R.

$8^{36} \equiv 6 \pmod{10}$, $7^{360} \equiv 1 \pmod{10}$, $6^{360} \equiv 6 \pmod{10}$, $5^{36000} \equiv 5 \pmod{10}$

$8^{36} + 7^{360} + 6^{3600} + 5^{36000} \equiv 6 + 1 + 6 + 5 \equiv 8 \pmod{10}$

$R = 8$

I1.4 If S is the number of ways to arrange R persons in a circle, find the value of S.

Reference: 1998 FI5.3, 2000 FG4.4

First arrange the 8 persons in a row. Number of permutations = $P_8^8 = 8!$

Suppose the first and the last in the row are A and H respectively.

Now join the first and the last persons to form a ring.

A can be in any position of the ring. Each pattern is repeated 8 times.

$\therefore$ Number of permutations $= \frac{8!}{8} = 5040$

Remark: the original version was …"arrange R people" …

Note that the word "people" is an uncountable noun, whereas the word "persons" is a countable noun.

Individual Event 2

I2.1 If the solution of the system of equations $\begin{cases} x+y=P \\ 3x+5y=13 \end{cases}$ are positive integers, find the value of P.

5(1) – (2): $2x = 5P - 13 \Rightarrow x = \dfrac{5P-13}{2}$

(2) – 3(1): $2y = 13 - 3P \Rightarrow y = \dfrac{13-3P}{2}$

$\because$ x and y are positive integers $\therefore$ $\dfrac{5P-13}{2} > 0$ and $\dfrac{13-3P}{2} > 0$ and P is odd

$\dfrac{13}{5} < P < \dfrac{13}{3}$ and P is odd $\Rightarrow P = 3$

I2.2 If $x + y = P$, $x^2 + y^2 = Q$ and $x^3 + y^3 = P^2$, find the value of Q.

Reference: 2002 FG1.2

$x + y = 3$, $x^2 + y^2 = Q$ and $x^3 + y^3 = 9$

$(x + y)^2 = 3^2 \Rightarrow x^2 + y^2 + 2xy = 9 \Rightarrow Q + 2xy = 9$ …… (1)

$(x + y)(x^2 + y^2 - xy) = 9 \Rightarrow 3(Q - xy) = 9 \Rightarrow Q - xy = 3$ …… (2)

(1) + 2(2): $3Q = 15 \Rightarrow Q = 5$

I2.3 If a and b are distinct prime numbers and $a^2 - aQ + R = 0$ and $b^2 - bQ + R = 0$, find the value of R.

$a^2 - 5a + R = 0$ and $b^2 - 5b + R = 0$

a, b are the (prime numbers) roots of $x^2 - 5x + R = 0$

$a + b = 5$ …… (1), $ab = R$ …… (2)

$a = 2$, $b = 3 \Rightarrow R = 6$

I2.4 If $S > 0$ and $\dfrac{1}{S(S-1)} + \dfrac{1}{(S+1)S} + \cdots + \dfrac{1}{(S+20)(S+19)} = 1 - \dfrac{1}{R}$, find the value of S.

$$\left(\frac{1}{S-1} - \frac{1}{S}\right) + \left(\frac{1}{S} - \frac{1}{S+1}\right) + \cdots + \left(\frac{1}{S+19} - \frac{1}{S+20}\right) = \frac{5}{6}$$

$$\frac{1}{S-1} - \frac{1}{S+20} = \frac{5}{6}$$

$$\frac{21}{(S-1)(S+20)} = \frac{5}{6}$$

$5(S^2 + 19S - 20) = 126$

$5S^2 + 95S - 226 = 0$

$$S = \frac{-95+\sqrt{13545}}{10}$$

$$= \frac{-95+3\sqrt{1505}}{10}$$

Individual Event 3

I3.1 If P is a prime number and the roots of the equation $x^2 + 2(P+1)x + P^2 - P - 14 = 0$ are integers, find the least value of P.

Reference: 2000 FI5.2, 2001 FI2.1, 2010 FI2.2, 2013 HG1

$\Delta = 4(P + 1)^2 - 4(P^2 - P - 14) = m^2$

$$\left(\frac{m}{2}\right)^2 = P^2 + 2P + 1 - P^2 + P + 14 = 3P + 15$$

The possible square numbers are 16, 25, 36, …

$3P + 15 = 16$ (no solution); $3P + 15 = 25$ (not an integer); $3P + 15 = 36 \Rightarrow P = 7$

The least possible $P = 7$

I3.2 Given that $x^2 + ax + b$ is a common factor of $2x^3 + 5x^2 + 24x + 11$ and $x^3 + Px - 22$. If $Q = a + b$, find the value of Q.

Let $f(x) = 2x^3 + 5x^2 + 24x + 11$; $g(x) = x^3 + 7x - 22$

$g(2) = 8 + 14 - 22 = 0 \Rightarrow x - 2$ is a factor

By division $g(x) = (x - 2)(x^2 + 2x + 11)$; $f(x) = (2x + 1)(x^2 + 2x + 11)$

$a = 2$, $b = 11$; $Q = a + b = 13$

Method 2

Let $f(x) = 2x^3 + 5x^2 + 24x + 11 = (x^2 + ax + b)(cx + d)$

$g(x) = x^3 + 7x - 22 = (x^2 + ax + b)(px + q)$

$f(x) - 2g(x) = 2x^3 + 5x^2 + 24x + 11 - 2(x^3 + 7x - 22) \equiv (x^2 + ax + b)[(c - 2d)x + d - 2q]$

$5x^2 + 10x + 55 \equiv (x^2 + ax + b)[(c - 2d)x + d - 2q]$

By comparing coefficients of x^3 and x^2 on both sides:

$c = 2d$ and $d - 2q = 5$

$5x^2 + 10x + 55 \equiv 5(x^2 + ax + b)$

$a = 2$, $b = 11$

$Q = a + b = 13$

I3.3 If R is a positive integer and $R^3 + 4R^2 + (Q - 93)R + 14Q + 10$ is a prime number, find the value of R. **(Reference: 2004 FI4.2)**

Let $f(R) = R^3 + 4R^2 - 80R + 192$

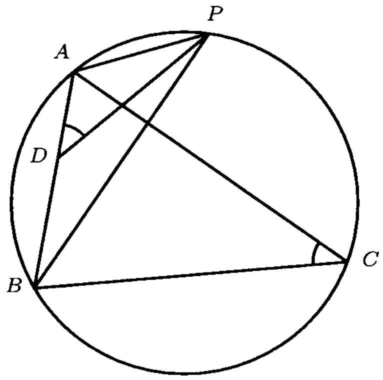
$f(4) = 64 + 64 - 320 + 192 = 0 \Rightarrow x - 4$ is a factor

By division, $f(R) = (R - 4)(R^2 + 8R - 48) = (R - 4)^2(R + 12)$

$\because f(R)$ is a prime number

$\therefore R - 4 = 1 \Rightarrow R = 5$ and $R + 12 = 17$, which is a prime.

I3.4 In Figure 1, AP, AB, PB, PD, AC and BC are line segments and D is a point on AB. If the length of AB is R times that of AD, $\angle ADP = \angle ACB$ and $S = \dfrac{PB}{PD}$, find the value of S.



Consider ΔADP and ΔABP.

$\angle ADP = \angle ACB = \angle APB$ (given, $\angle$s in the same segment AB)

$\angle DAP = \angle PAB$ (Common)

$\angle APD = \angle ABP$ ($\angle$s sum of Δ)

$\therefore \Delta ADP \sim \Delta APB$ (equiangular)

Let $AD = k$, $AB = 5k$, $AP = y$

$$\frac{PB}{PD} = \frac{AB}{AP} = \frac{AP}{AD} \quad \text{(Ratio of sides, }\sim\Delta\text{'s)}$$

$$\frac{PB}{PD} = \frac{5k}{y} = \frac{y}{k}$$

$$\therefore \left(\frac{y}{k}\right)^2 = 5 \Rightarrow \frac{y}{k} = \sqrt{5}$$

$$\frac{PB}{PD} = \sqrt{5}$$

Individual Event 4

I4.1 Consider the function $y = \sin x + \sqrt{3}\cos x$. Let a be the maximum value of y. Find the value of a

$y = \sin x + \sqrt{3}\cos x = 2\left(\sin x \cdot \frac{1}{2} + \cos x \cdot \frac{\sqrt{3}}{2}\right) = 2\left(\sin x \cdot \cos 60^\circ + \cos x \cdot \sin 60^\circ\right) = 2\sin(x + 60^\circ)$

a = maximum value of $y = 2$

I4.2 Find the value of b if b and y satisfy $|b - y| = b + y - a$ and $|b + y| = b + a$.

From the first equation:

$(b - y = b + y - 2$ or $y - b = b + y - 2)$ and $b + y - 2 \ge 0$

$(y = 1$ or $b = 1)$ and $b + y - 2 \ge 0$

When $y = 1 \Rightarrow b \ge 1$ …… (3)

When $b = 1 \Rightarrow y \ge 1$ …… (4)

From the second equation:

$(b + y = b + 2$ or $b + y = -b - 2)$ and $b + 2 \ge 0$

$(y = 2$ or $2b + y = -2)$ and $b \ge -2$

When $y = 2$ and $b \ge -2$ …… (5)

When $2b + y = -2$ and $b \ge -2 \Rightarrow (y \le 2$ and $b \ge -2$ and $2b + y = -2)$ …… (6)

(3) and (5): $y = 1, b \ge 1$ and $y = 2$ and $b \ge -2 \Rightarrow$ contradiction

(3) and (6): $y = 1, b \ge 1$ and $(y \le 2, b \ge -2, 2b + y = -2) \Rightarrow y = 1$ and $b = -1.5$ and $b \ge 1$!!!

(4) and (6): $(y \ge 1, b = 1)$ and $(y \le 2, b \ge -2, 2b + y = -2) \Rightarrow y = -4, b = 1$ and $y \ge 1$!!!

(4) and (5): $(b = 1, y \ge 1)$ and $(y = 2, b \ge -2) \Rightarrow b = 1$ and $y = 2$

$\therefore b = 1$

I4.3 Let x, y and z be positive integers. If $|x - y|^{2010} + |z - x|^{2011} = b$ and $c = |x - y| + |y - z| + |z - x|$, find the value of c.

Reference: 1996 FI2.3, 2005FI4.1, 2006 FI4.2, 2013 FI1.4, 2015 HG4, 2015 FI1.1

Clearly $|x - y|$ and $|z - x|$ are non-negative integers

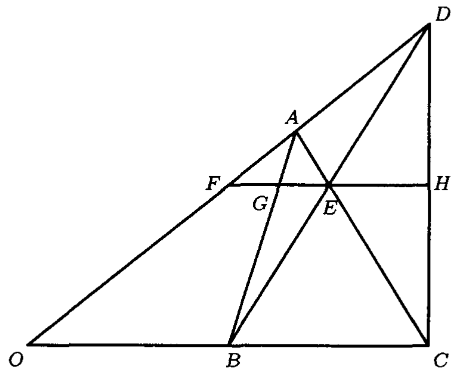
$|x - y|^{2010} + |z - x|^{2011} = 1 \Rightarrow (|x - y| = 0$ and $|z - x| = 1)$ or $(|x - y| = 1$ and $|z - x| = 0)$

When $x = y$ and $|z - x| = 1$, $c = 0 + |y - z| + |z - x| = 2|z - x| = 2$

When $|x - y| = 1$ and $|z - x| = 0$, $c = 1 + |y - z| + 0 = 1 + |y - x| = 1 + 1 = 2$

I4.4 In Figure 1, let ODC be a triangle. Given that FH, AB, AC and BD are line segments such that AB intersects FH at G, AC, BD and FH intersect at E, $GE = 1$, $EH = c$ and FH // OC. If $d = EF$, find the value of d.

Remark: there are some typing mistakes in the Chinese old version: … AC 及 AD 為綫段 … FH // BC …



$\Delta AGE \sim \Delta ABC$ (equiangular)

Let $\dfrac{CE}{AE} = k$, $AE = x$, $AG = t$.

$BC = k + 1$, $EC = kx$, $GB = kt$ (ratio of sides, ~Δ's)

$\Delta DEH \sim \Delta DBC$ (equiangular)

$\dfrac{BC}{EH} = \dfrac{k+1}{2} = \dfrac{DB}{DE}$ (corr. sides, ~Δ's)

Let $DE = 2y \Rightarrow DB = (k + 1)y$

$EB = DB - DE = (k - 1)y$

$\Delta AFG \sim \Delta AOB$ (equiangular)

$FG = d - 1$, $\dfrac{OB}{FG} = \dfrac{AB}{AG}$ (corr. sides, ~Δ's)

$OB = (d-1)\cdot\dfrac{(k+1)t}{t} = (d - 1)(k + 1)$

$\Delta DFE \sim \Delta DOB$ (equiangular)

$\dfrac{FE}{OB} = \dfrac{DE}{DB}$ (corr. sides, ~Δ's)

$\Rightarrow d = (d-1)(k+1)\cdot\dfrac{2y}{(k+1)y} \Rightarrow d = 2$

Method 2

$\Delta AFG \sim \Delta AOB$ and $\Delta AGE \sim \Delta ABC$

$\dfrac{d-1}{1} = \dfrac{OB}{BC}$ (corr. sides, ~Δ's)

$\Delta DFE \sim \Delta DOB$ and $\Delta DEH \sim \Delta DBC$

$\dfrac{d}{2} = \dfrac{OB}{BC}$ (corr. sides, ~Δ's)

Equating the two equations

$\dfrac{d-1}{1} = \dfrac{d}{2}$

$d = 2$

Individual Spare

IS.1 Let P be the number of triangles whose side lengths are integers less than or equal to 9. Find the value of P.

The sides must satisfy triangle inequality. i.e. $a + b > c$.

Possible order triples are (1, 1, 1), (2, 2, 2), … , (9, 9, 9),
(2, 2, 1), (2, 2, 3), (3, 3, 1), (3, 3, 2), (3, 3, 4), (3, 3, 5),
(4, 4, 1), (4, 4, 2), (4, 4, 3), (4, 4, 5), (4, 4, 6), (4, 4, 7),
(5, 5, 1), … , (5, 5, 4), (5, 5, 6), (5, 5, 7), (5, 5, 8), (5, 5, 9),
(6, 6, 1), … , (6, 6, 9) (except (6, 6, 6))
(7, 7, 1), … , (7, 7, 9) (except (7, 7, 7))
(8, 8, 1), … , (8, 8, 9) (except (8, 8, 8))
(9, 9, 1), … , (9, 9, 8)
(2, 3, 4), (2, 4, 5), (2, 5, 6), (2, 6, 7), (2, 7, 8), (2, 8, 9),
(3, 4, 5), (3, 4, 6), (3, 5, 6), (3, 5, 7), (3, 6, 7), (3, 6, 8), (3, 7, 8), (3, 7, 9), (3, 8, 9),
(4, 5, 6), (4, 5, 7), (4, 5, 8), (4, 6, 7), (4, 6, 8), (4, 6, 9), (4, 7, 8), (4, 7, 9), (4, 8, 9),
(5, 6, 7), (5, 6, 8), (5, 6, 9), (5, 7, 8), (5, 7, 9), (5, 8, 9), (6, 7, 8), (6, 7, 9), (6, 8, 9), (7, 8, 9).

Total number of triangles = $9 + 6 + 6 + 8\times5 + 6 + 9 + 9 + 6 + 4 = 95$

Method 2 First we find the number of order triples.

Case 1 All numbers are the same: (1, 1, 1), … , (9, 9, 9).

Case 2 Two of them are the same, the third is different: (1, 1, 2), … , (9, 9, 1)

There are $C_1^9 \times C_1^8 = 72$ possible triples.

Case 3 All numbers are different. There are $C_3^9 = 84$ possible triples.

∴ Total 9 + 72 + 84 = **165** possible triples.

Next we find the number of triples which **cannot form a triangle**, i.e. $a + b \le c$.

Possible triples are (1, 1, 2), … (1, 1, 9) (8 triples)
(1, 2, 3), … , (1, 2, 9) (7 triples)
(1, 3, 4), … , (1, 3, 9) (6 triples)
(1, 4, 5), … , (1, 4, 9) (5 triples)
(1, 5, 6), … , (1, 5, 9) (4 triples)
(1, 6, 7), (1, 6, 8), (1, 6, 9), (1, 7, 8), (1, 7, 9), (1, 8, 9),
(2, 2, 4), … , (2, 2, 9) (6 triples)
(2, 3, 5), … , (2, 3, 9) (5 triples)
(2, 4, 6), … , (2, 4, 9) (4 triples)
(2, 5, 7), (2, 5, 8), (2, 5, 9), (2, 6, 8), (2, 6, 9), (2, 7, 9),
(3, 3, 6), … , (3, 3, 9) (4 triples)
(3, 4, 7), (3, 4, 8), (3, 4, 9), (3, 5, 8), (3, 5, 9), (3, 6, 9), (4, 4, 8), (4, 4, 9), (4, 5, 9).

Total number of triples which cannot form a triangle
= (8 + 7 + … + 1) + (6 + 5 + … + 1) + (4 + 3 + 2 + 1) + (2 + 1) = 36 + 21 + 10 + 3 = **70**

∴ Number of triangles = **165 – 70** = **95**

IS.2 Let $Q = \log_{128} 2^3 + \log_{128} 2^5 + \log_{128} 2^7 + \ldots + \log_{128} 2^P$. Find the value of Q.

$Q = 3 \log_{128} 2 + 5 \log_{128} 2 + 7 \log_{128} 2 + \ldots + 95 \log_{128} 2$

$$= (3 + 5 + \ldots + 95) \log_{128} 2 = \frac{3+95}{2}\times 47\times \log_{128} 2 = \log_{128} 2^{2303} = \log_{128} (2^7)^{329} = 329$$

IS.3 Consider the line $12x – 4y + (Q – 305) = 0$. If the area of the triangle formed by the x-axis, the y-axis and this line is R square units, what is the value of R?

$12x – 4y + 24 = 0 \Rightarrow$ Height = 6, base = 2; area $R = \frac{1}{2}\cdot 6\cdot 2 = 6$

Remark: the original question is …$12x – 4y + Q = 0$….
The answer is very difficult to carry forward to next question.

IS.4 If $x + \frac{1}{x} = R$ and $x^3 + \frac{1}{x^3} = S$, find the value of S.

$$S = \left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right) = R\left[\left(x+\frac{1}{x}\right)^2-3\right] = R^3 – 3R = 216 – 3(6) = 198$$

Sample Group Event

Group Event 1 (2009 Final Group Event 1)

SG.1 Given some triangles with side lengths a cm, 2 cm and b cm, where a and b are integers and $a \le 2 \le b$. If there are q non-congruent classes of triangles satisfying the above conditions, find the value of q.

When $a = 1$, possible $b = 2$

When $a = 2$, possible $b = 2$ or 3

$\therefore q = 3$

SG.2 Given that the equation $|x| - \frac{4}{x} = \frac{3|x|}{x}$ has k distinct real root(s), find the value of k.

When $x > 0$: $x^2 - 4 = 3x \Rightarrow x^2 - 3x - 4 = 0 \Rightarrow (x + 1)(x - 4) = 0 \Rightarrow x = 4$

When $x < 0$: $-x^2 - 4 = -3x \Rightarrow x^2 - 3x + 4 = 0$; $D = 9 - 16 < 0 \Rightarrow$ no real roots.

$k = 1$ (There is only one real root.)

SG.3 Given that x and y are non-zero real numbers satisfying the equations $\frac{\sqrt{x}}{\sqrt{y}} - \frac{\sqrt{y}}{\sqrt{x}} = \frac{7}{12}$ and $x - y = 7$. If $w = x + y$, find the value of w.

The first equation is equivalent to $\frac{x-y}{\sqrt{xy}} = \frac{7}{12} \Rightarrow \sqrt{xy} = 12 \Rightarrow xy = 144$

Sub. $y = \frac{144}{x}$ into $x - y = 7$: $x - \frac{144}{x} = 7 \Rightarrow x^2 - 7x - 144 = 0 \Rightarrow (x + 9)(x - 16) = 0$

$x = -9$ or 16; when $x = -9$, $y = -16$ (rejected $\because \sqrt{x}$ is undefined); when $x = 16$; $y = 9$

$w = 16 + 9 = 25$

Method 2 The first equation is equivalent to $\frac{x-y}{\sqrt{xy}} = \frac{7}{12} \Rightarrow \sqrt{xy} = 12 \Rightarrow xy = 144$ …… (1)

$\because x - y = 7$ and $x + y = w$

$\therefore x = \frac{w+7}{2}, y = \frac{w-7}{2}$

Sub. these equations into (1): $\left(\frac{w+7}{2}\right)\left(\frac{w-7}{2}\right) = 144$

$w^2 - 49 = 576 \Rightarrow w = \pm 25$

$\because$ From the given equation $\frac{\sqrt{x}}{\sqrt{y}} - \frac{\sqrt{y}}{\sqrt{x}} = \frac{7}{12}$, we know that both $x > 0$ and $y > 0$

$\therefore w = x + y = 25$ only

SG.4 Given that x and y are real numbers and $\left|x - \frac{1}{2}\right| + \sqrt{y^2 - 1} = 0$.

Let $p = |x| + |y|$, find the value of p.

Reference: 2006 FI4.2 … $y^2 + 4y + 4 + \sqrt{x + y + k} = 0$. If $r = |xy|$, …

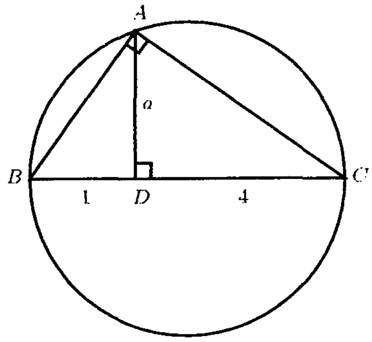
Both $\left|x - \frac{1}{2}\right|$ and $\sqrt{y^2 - 1}$ are non-negative numbers.

The sum of two non-negative numbers = 0 means each of them is zero

$x = \frac{1}{2}$, $y = \pm 1$; $p = \frac{1}{2} + 1 = \frac{3}{2}$

Group Event 1

G1.1 In Figure 1, *BC* is the diameter of the circle. *A* is a point on the circle, *AB* and *AC* are line segments and *AD* is a line segment perpendicular to *BC*. If *BD* = 1, *DC* = 4 and *AD* = *a*, find the value of *a*.



$\Delta ABD \sim \Delta CAD$ (equiangular)

$\frac{a}{1}=\frac{4}{a}$ (ratio of sides ~Δ's)

$a^2 = 1\times 4$

$a = 2$

G1.2 If $b = 1-\cfrac{1}{1-\cfrac{1}{1-\cfrac{1}{1-\cfrac{1}{-\frac{1}{2}}}}}$, find the value of *b*.

$1-\frac{1}{-\frac{1}{2}}=3$; $1-\cfrac{1}{1-\cfrac{1}{-\frac{1}{2}}}=\frac{2}{3}$; $1-\cfrac{1}{1-\cfrac{1}{1-\cfrac{1}{-\frac{1}{2}}}}=-\frac{1}{2}$; $b = 1 + 2 = 3$

G1.3 If *x*, *y* and *z* are real numbers, $xyz \neq 0$, $2xy = 3yz = 5xz$ and $c=\frac{x+3y-3z}{x+3y-6z}$, find the value of *c*.

$\frac{2xy}{xyz}=\frac{3yz}{xyz}=\frac{5xz}{xyz}\Rightarrow\frac{2}{z}=\frac{3}{x}=\frac{5}{y}\Rightarrow x:y:z=3:5:2$

Let $x = 3k$, $y = 5k$, $z = 2k$

$c=\frac{x+3y-3z}{x+3y-6z}=\frac{3k+15k-6k}{3k+15k-12k}=2$

G1.4 If *x* is an integer satisfying $\log_{1/4}(2x+1)<\log_{1/2}(x-1)$, find the maximum value of *x*.

$\frac{\log(2x+1)}{\log\frac{1}{4}}<\frac{\log(x-1)}{\log\frac{1}{2}}$

$\frac{\log(2x+1)}{-2\log 2}<\frac{\log(x-1)}{-\log 2}$

$\log(2x + 1) > 2\log(x - 1)$

$2x + 1 > (x - 1)^2$

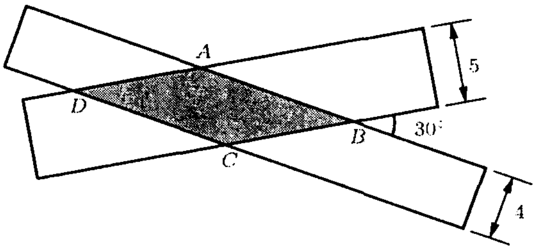
$x^2 - 4x < 0$

$0 < x < 4$

The maximum integral value of *x* is 3.

Group Event 2

G2.1 In Figure 1, two rectangles with widths 4 and 5 units cross each other at 30°. Find the area of the overlapped region.



Let $AB = x$, $BC = y$, $\angle ABC = 30°$

$x \sin 30° = 5 \Rightarrow x = 10$

$y \sin 30° = 4 \Rightarrow y = 8$

Area = $xy \sin 30° = 10 \times 8 \times 0.5 = 40$

G2.2 From 1 to 100, take a pair of integers (repetitions allowed) so that their sum is greater than 100. How many ways are there to pick such pairs?

Reference: 2002 FG3.3

(1, 100), (2, 100), … , (100, 100) (100 pairs)

(2, 99), … , (99, 99) (98 pairs)

(3, 98), … , (98, 98) (96 pairs)

....

(49, 52), (50, 52), (51, 52), (52, 52) (4 pairs)

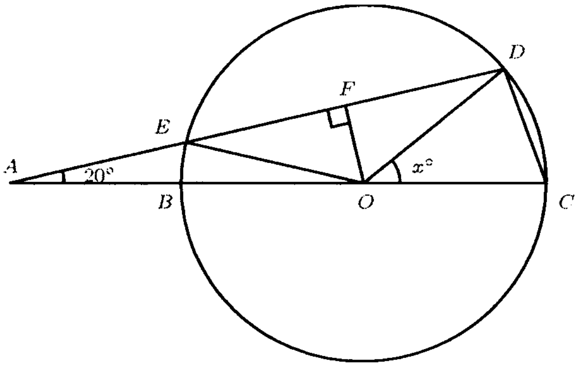
(50, 51), (51, 51) (2 pairs)

Total number of pairs = 2 + 4 + … + 100 = $\frac{2+100}{2} \times 50 = 2550$

Remark: the original version was …"take a pair of numbers"…從1到100選取兩數…

There are infinitely many ways if the numbers are not confined to be integers.

G2.3 In Figure 2, there is a circle with centre O and radius r. Triangle ACD intersects the circle at B, C, D and E. Line segment AE has the same length as the radius. If $\angle DAC = 20°$ and $\angle DOC = x°$, find the value of x.



$\angle AOE = 20°$ (Given $AE = OE$, base $\angle$s isos. Δ)

$\angle OED = 20° + 20° = 40°$ (ext. $\angle$ of ΔAOE)

$\angle ODE = \angle OED = 40°$ (base $\angle$s isos. Δ)

$x = 20 + 40 = 60$ (ext. $\angle$ of ΔAOD)

G2.4 Given that $\frac{1}{x} + \frac{2}{y} + \frac{3}{z} = 0$ and $\frac{1}{x} - \frac{6}{y} - \frac{5}{z} = 0$. If $P = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, find the value of P.

(1) – (2): $\frac{8}{y} + \frac{8}{z} = 0 \Rightarrow y = -z$

3(1) + (2): $\frac{4}{x} + \frac{4}{z} = 0 \Rightarrow x = -z$

$P = \frac{x}{y} + \frac{y}{z} + \frac{z}{x} = \frac{-z}{-z} + \frac{-z}{z} + \frac{z}{-z} = -1$

Group Event 3

G3.1 If a is a positive integer and $a^2 + 100a$ is a prime number, find the maximum value of a.

$a(a + 100)$ is a prime number.

$a = 1$, $a^2 + 100a = 101$ which is a prime number

G3.2 Let a, b and c be real numbers. If 1 is a root of $x^2 + ax + 2 = 0$ and a and b be roots of $x^2 + 5x + c = 0$, find the value of $a + b + c$.

$1 + a + 2 = 0 \Rightarrow a = -3$

$-3 + b = -5 \Rightarrow b = -2$

$c = -3b = 6$

$a + b + c = 1$

G3.3 Let x and y be positive real numbers with $x < y$. If $\sqrt{x}+\sqrt{y}=1$, $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{10}{3}$ and $x < y$, find the value of $y - x$.

$(1)^2$: $x + y + 2\sqrt{xy} = 1$

$\sqrt{xy}=\frac{1-(x+y)}{2}$ … (3)

(2): $\frac{x+y}{\sqrt{xy}}=\frac{10}{3}$ … (4)

Sub. (3) into (4): $\frac{x+y}{\frac{1-(x+y)}{2}}=\frac{10}{3}$

$6(x + y) = 10(1 - x - y)$

$16(x + y) = 10$

$x+y=\frac{5}{8}$

$\sqrt{xy}=\frac{1-(x+y)}{2}=\frac{1}{2}\left(1-\frac{5}{8}\right)=\frac{3}{16}$

$xy=\frac{9}{256}$

$(y-x)^2=(x+y)^2-4xy=\frac{25}{64}-\frac{9}{64}=\frac{1}{4}$

$y-x=\frac{1}{2}$

Method 2

Let $z=\sqrt{\frac{x}{y}}$, then $\frac{1}{z}=\sqrt{\frac{y}{x}}$

(2) becomes $z+\frac{1}{z}=\frac{10}{3}$

$3z^2 - 10z + 3 = 0$

$(3z - 1)(z - 3) = 0$

$z=\frac{1}{3}$ or 3

$\because x < y$

$\therefore z=\sqrt{\frac{x}{y}}<1\Rightarrow z=\frac{1}{3}$ only

$\frac{\sqrt{y}-\sqrt{x}}{\sqrt{y}+\sqrt{x}}=\frac{1-\sqrt{\frac{x}{y}}}{1+\sqrt{\frac{x}{y}}}=\frac{1-\frac{1}{3}}{1+\frac{1}{3}}=\frac{1}{2}$

$\because \sqrt{x}+\sqrt{y}=1$ $\therefore \sqrt{y}-\sqrt{x}=\frac{1}{2}$

$y-x=\left(\sqrt{y}-\sqrt{x}\right)\left(\sqrt{y}+\sqrt{x}\right)=\frac{1}{2}$

G3.4 Spilt the numbers 1, 2, … , 10 into two groups and let P_1 be the product of the first group and P_2 the product of the second group. If P_1 is a multiple of P_2, find the minimum value of $\frac{P_1}{P_2}$.

$P_1 = kP_2$, where k is a positive integer.

$\therefore$ All prime factors of P_2 can divide P_1.

$\frac{10}{5}=2$, 10 must be a factor of the numerator and 5 must a factor of the denominator

7 is a prime which must be a factor of the numerator.

Among the even numbers 2, 4, 6, 8, 10, there are 8 factors of 2.

4 factors of 2 should be put in the numerator and 4 factors should be put in the denominator.

Among the number 3, 6, 9, there are 4 factors of 3.

2 factors of 3 should be put in the numerator and 2 factors should be put in the denominator.

Minimum value of $\frac{P_1}{P_2}=\frac{8\times7\times9\times10}{1\times2\times3\times4\times5\times6}=7$

Group Event 4

G4.1 If $P = 2\sqrt[4]{2007 \cdot 2009 \cdot 2011 \cdot 2013 + 10 \cdot 2010 \cdot 2010 - 9} - 4000$, find the value of P.

Let $x = 2010$, $2007 = x - 3$, $2009 = x - 1$, $2011 = x + 1$, $2013 = x + 3$

$P = 2\sqrt[4]{(x-3)\cdot(x-1)\cdot(x+1)\cdot(x+3)+10x^2-9} - 4000 = 2\sqrt[4]{(x^2-9)\cdot(x^2-1)+10x^2-9} - 4000$

$= 2\sqrt[4]{x^4-10x^2+9+10x^2-9} - 4000 = 2x - 4000 = 20$

G4.2 If $9x^2 + nx + 1$ and $4y^2 + 12y + m$ are squares with $n > 0$, find the value of $\frac{n}{m}$.

$9x^2 + nx + 1 = (3x + 1)^2 \Rightarrow n = 6$

$4y^2 + 12y + m = (2y + 3)^2 \Rightarrow m = 9$

$\frac{n}{m} = \frac{2}{3}$

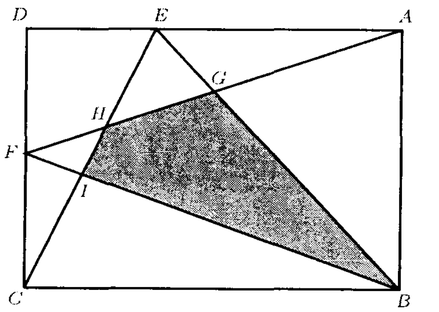
G4.3 Let n and $\frac{47}{5}\left(\frac{4}{47}+\frac{n}{141}\right)$ be positive integers. If r is the remainder of n divided by 15, find the value of r.

$\frac{47}{5}\left(\frac{4}{47}+\frac{n}{141}\right) = \frac{4}{5} + \frac{n}{15} = \frac{n+12}{15}$, which is an integer

$n + 12 = 15k$, where k is a positive integer

$r = 3$

G4.4 In figure 1, *ABCD* is a rectangle, and *E* and *F* are points on *AD* and *DC*, respectively. Also, *G* is the intersection of *AF* and *BE*, *H* is the intersection of *AF* and *CE*, and *I* is the intersection of *BF* and *CE*. If the areas of *AGE*, *DEHF* and *CIF* are 2, 3 and 1, respectively, find the area of the grey region *BGHI*. (**Reference: 2014 FI1.1)**



Let the area of $EGH = x$, area of $BCI = z$, area of $BGHI = w$

Area of $BCE = \frac{1}{2}$(area of $ABCD$) = area ADF + area BCF

$x + w + z = 3 + x + 2 + 1 + z$

$\Rightarrow w = 6$

$\therefore$ Area of the grey region $BGHI = 6$

Remark: there is a spelling mistake in the English version. old version: … gray region …

Group Spare

GS.1 Let α and β be the real roots of $y^2 - 6y + 5 = 0$. Let m be the minimum value of $|x - \alpha| + |x - \beta|$ over all real values of x. Find the value of m.

Reference: 1994 HG1, 2001 HG9, 2004 FG4.2, 2008 HI8, 2008 FI1.3, 2010 HG6, 2012 FG2.3

$\alpha = 1$, $\beta = 5$

If $x < 1$, $|x - \alpha| + |x - \beta| = 1 - x + 5 - x = 6 - 2x > 4$

If $1 \le x \le 5$, $|x - \alpha| + |x - \beta| = x - 1 + 5 - x = 4$

If $x > 5$, $|x - \alpha| + |x - \beta| = x - 1 + x - 5 = 2x - 6 > 4$

m = min. of $|x - \alpha| + |x - \beta| = 4$

Method 2 Using the triangle inequality: $|a| + |b| \ge |a + b|$

$|x - \alpha| + |x - \beta| \ge |x - 1 + 5 - x| = 4 \Rightarrow m = 4$

Remark: there is a typing mistake in the English version. … minimum value *a* of …

GS.2 Let α, β, γ be real numbers satisfying $\alpha + \beta + \gamma = 2$ and $\alpha\beta\gamma = 4$. Let v be the minimum value of $|\alpha| + |\beta| + |\gamma|$. Find the value of v.

If at least one of $\alpha, \beta, \gamma = 0$, then $\alpha\beta\gamma \ne 4 \Rightarrow \alpha, \beta, \gamma \ne 0$

If $\alpha, \beta, \gamma > 0$, then

$$\frac{\alpha+\beta+\gamma}{3} \ge \sqrt[3]{\alpha\beta\gamma} \quad \text{(A.M. } \ge \text{ G.M.)}$$

$$\frac{2}{3} \ge \sqrt[3]{4}$$

$2^3 \ge 27\times4 = 108$, which is a contradiction

If $\beta < 0$, in order that $\alpha\beta\gamma = 4 > 0$, WLOG let $\gamma < 0$, $\alpha > 0$

$\alpha = 2 - \beta - \gamma > 2$

$|\alpha| + |\beta| + |\gamma| = \alpha - (\beta + \gamma) = 2 + 2(-\beta - \gamma) \ge 2 + 4\sqrt{(-\beta)(-\gamma)}$, equality holds when $\beta = \gamma$

$4 = (2 - 2\beta)\beta^2$

$\beta^3 - \beta^2 + 2 = 0$

$(\beta + 1)(\beta^2 - 2\beta + 2) = 0$

$\beta = -1$ (For the 2nd equation, $\Delta = -4 < 0$, no real solution)

$\gamma = -1$, $\alpha = 4$

$|\alpha| + |\beta| + |\gamma| = 1 + 1 + 4 = 6$

v = min. of $|\alpha| + |\beta| + |\gamma| = 6$

GS.3 Let $y = |x + 1| - 2|x| + |x - 2|$ and $-1 \le x \le 2$. Let α be the maximum value of y. Find the value of α.

$y = x + 1 - 2|x| + 2 - x = 3 - 2|x|$

$0 \le |x| \le 2 \Rightarrow 3 \ge 3 - 2|x| \ge -1$

$\alpha = 3$

GS.4 Let F be the number of integral solutions of $x^2 + y^2 + z^2 + w^2 = 3(x + y + z + w)$.
Find the value of F.

$(x, y, z, w) = (0, 0, 0, 0)$ is a trivial solution.

$x^2 + y^2 + z^2 + w^2 - 3(x + y + z + w) = 0$

$$\left(x^2 - 3x + \frac{9}{4}\right) + \left(y^2 - 3y + \frac{9}{4}\right) + \left(z^2 - 3z + \frac{9}{4}\right) + \left(w^2 - 3w + \frac{9}{4}\right) = 9$$

$$\left(x - \frac{3}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 + \left(z - \frac{3}{2}\right)^2 + \left(w - \frac{3}{2}\right)^2 = 9$$

$(2x - 3)^2 + (2y - 3)^2 + (2z - 3)^2 + (2w - 3)^2 = 36$

Let $a = 2x - 3$, $b = 2y - 3$, $c = 2z - 3$, $d = 2w - 3$, the equation becomes $a^2 + b^2 + c^2 + d^2 = 36$

For integral solutions of (x, y, z, w), (a, d, c, d) must be odd integers.

In addition, the permutation of (a, b, c, d) is also a solution. (e.g. (b, d, c, a) is a solution)

$\because$ a, b, c, d are odd integers and $a^2 + b^2 + c^2 + d^2 \ge 0$

If one of the four unknowns, say, $a > 6$, then L.H.S. > 36, so L.H.S. ≠ R.H.S.

$\therefore$ a , b, c, $d = \pm 1, \pm 3, \pm 5$

When $a = \pm 5$, then $25 + b^2 + c^2 + d^2 = 36 \Rightarrow b^2 + c^2 + d^2 = 11$

The only integral solution to this equation is $b = \pm 3$, $c = \pm 1 = d$ or its permutations.

When the largest (in magnitude) of the 4 unknowns, say, a is ± 3, then $9 + b^2 + c^2 + d^2 = 36$
$\Rightarrow b^2 + c^2 + d^2 = 27$, the only solution is $b = \pm 3$, $c = \pm 3$, $d = \pm 3$ or its permutations.

$\therefore$ The integral solutions are $(a, b, c, d) = (5, 3, 1, 1)$ and its permutations … (1) $\times P_2^4 = 12$

$(3, 3, 3, 3)$ … (2) $\times 1$

If (a, b, c, d) is a solution, then $(\pm a, \pm b, \pm c, \pm d)$ are also solutions.

There are 16 solutions with different signs for $(\pm a, \pm b, \pm c, \pm d)$.

$\therefore$ $F = (12 + 1)\times 16$
$= 208$

Individual Events

SI	***P***	30	**I1**	***A***	12	**I2**	***P***	5	**I3**	α	45	**I4**	***A***	22	**IS**	***P***	95
	***Q** see the remark	120		***B***	108		***Q***	12		*β see the remark	56		***B***	12		***Q***	329
	R	11		***C** see the remark	280		***R***	3		γ	23		***C***	12		***R** see the remark	6
	***S** see the remark	72		***D***	69		***S***	17		δ	671		***D***	7		***S***	198

Group Events

SG	***q***	3	**G1**	tens digit	1	**G2**		2	**G3**	***z***	6	**G4**	$\frac{BD}{CE}$	2	**GS**	***m** see the remark	4
	k	1		***P** see the remark	1031		*K*	2		***r** see the remark	540		***Q***	1		***v***	6
	w	25		***k***	21		**l**	45		***D***	998		***R***	1		α	3
	p	$\frac{3}{2}$		*$S_{\Delta BCD}$ see the remark	32		see the remark	$\frac{1}{4}$		*$F_{2012}(7)$ see the remark	1		x_5	5		***F***	208

Sample Individual Event (2009 Final Individual Event 1)

SI.1 Let a, b, c and d be the roots of the equation $x^4 - 15x^2 + 56 = 0$.
If $P = a^2 + b^2 + c^2 + d^2$, find the value of P.

$x^4 - 15x^2 + 56 = 0 \Rightarrow (x^2 - 7)(x^2 - 8) = 0$

$a = \sqrt{7}$, $b = -\sqrt{7}$, $c = \sqrt{8}$, $d = -\sqrt{8}$

$P = a^2 + b^2 + c^2 + d^2 = 7 + 7 + 8 + 8 = 30$

SI.2 In Figure 1, $AB = AC$ and AB // ED.
If $\angle ABC = P°$ and $\angle ADE = Q°$, find the value of Q.

$\angle ABC = 30° = \angle ACB$ (base $\angle$s isos. Δ)

$\angle BAC = 120°$ ($\angle$s sum of Δ)

$\angle ADE = 120°$ (alt. $\angle$s, AB // ED)

$Q = 120$

Remark: Original question $\cdots$ AB // DE $\cdots$.
It is better for AB and ED to be oriented in the same direction.

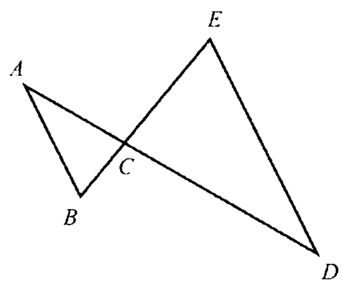


Figure 1

SI.3 Let $F = 1 + 2 + 2^2 + 2^3 + \cdots + 2^Q$ and $R = \sqrt{\frac{\log(1+F)}{\log 2}}$, find the value of R.

$F = 1 + 2 + 2^2 + 2^3 + \cdots + 2^{120} = \frac{2^{121}-1}{2-1} = 2^{121} - 1$

$R = \sqrt{\frac{\log(1+F)}{\log 2}} = \sqrt{\frac{\log 2^{121}}{\log 2}} = 11$

SI.4 Let f(x) be a function such that f(n) = (n – 1) f(n – 1) and f(1) ≠ 0.

If $S = \frac{\mathrm{f}(R)}{(R-1)\mathrm{f}(R-3)}$, find the value of S.

$\mathrm{f}(n) = (n-1)\,\mathrm{f}(n-1) = (n-1)(n-2)\mathrm{f}(n-2) = \cdots\cdots$

$S = \frac{\mathrm{f}(11)}{(11)\mathrm{f}(11-3)} = \frac{10\times 9\times 8\times \mathrm{f}(8)}{10\times \mathrm{f}(8)} = 9\times 8 = 72$

Remark: Original question:

Let f(x) be a function such that f(n) = (n – 1) f(n – 1). If $S = \frac{\mathrm{f}(R)}{(R-1)\mathrm{f}(R-3)}$, find the value of S.

Note that S is undefined when f(n) = 0 for some integers n.

Individual Event 1

I1.1 If A is the sum of the squares of the roots of $x^4 + 6x^3 + 12x^2 + 9x + 2$, find the value of A.

Let $f(x) = x^4 + 6x^3 + 12x^2 + 9x + 2$

$f(-1) = 1 - 6 + 12 - 9 + 2 = 0$

$f(-2) = 16 - 48 + 48 - 18 + 2 = 0$

$\therefore$ By factor theorem, $(x^2 + 3x + 2)$ is a factor of $f(x)$.

$x^4 + 6x^3 + 12x^2 + 9x + 2 = (x + 1)(x + 2)(x^2 + 3x + 1)$

The roots are -1, -2 and α, β; where $\alpha + \beta = -3$, $\alpha\beta = 1$

$A = (-1)^2 + (-2)^2 + \alpha^2 + \beta^2 = 5 + (\alpha + \beta)^2 - 2\alpha\beta$

$= 5 + 9 - 2$

$A = 12$

By division,

$$\begin{array}{r} x^2+3x+1 \\ x^2+3x+2\,\big)\,\overline{x^4+6x^3+12x^2+9x+2} \\ \underline{x^4-3x^3+\ 2x^2} \\ 3x^3+10x^2+9x \\ \underline{3x^3+\ 9x^2+6x} \\ x^2+3x+2 \\ \underline{x^2+3x+2} \end{array}$$

Method 2 By the change of subject, let $y = x^2$, then the equation becomes

$x^4 + 12x^2 + 2 = -x(6x^2 + 9) \Rightarrow y^2 + 12y + 2 = \mp\sqrt{y}(6y + 9)$

$(y^2 + 12y + 2)^2 - y(6y + 9)^2 = 0$

Coefficient of $y^4 = 1$, coefficient of $y^2 = 24 - 36 = -12$

If α, β, δ and γ are the roots of x, then α^2, β^2, δ^2 and γ^2 are the roots of y

$$\alpha^2 + \beta^2 + \delta^2 + \gamma^2 = -\frac{\text{coefficient of } y^3}{\text{coefficient of } y^4} = 12$$

Method 3

Let α, β, δ and γ are the roots of x, then by the relation between roots and coefficients,

$\alpha + \beta + \delta + \gamma = -6 \cdots\cdots (1)$

$\alpha\beta + \alpha\delta + \alpha\gamma + \beta\delta + \beta\gamma + \delta\gamma = 12 \cdots\cdots (2)$

$\alpha^2 + \beta^2 + \delta^2 + \gamma^2 = (\alpha + \beta + \delta + \gamma)^2 - 2(\alpha\beta + \alpha\delta + \alpha\gamma + \beta\delta + \beta\gamma + \delta\gamma)$

$= (-6)^2 - 2(12) = 12$

I1.2 Let x, y, z, w be four consecutive vertices of a regular A-gon. If the length of the line segment xy is 2 and the area of the quadrilateral $xyzw$ is $a + \sqrt{b}$, find the value of $B = 2^a \cdot 3^b$.

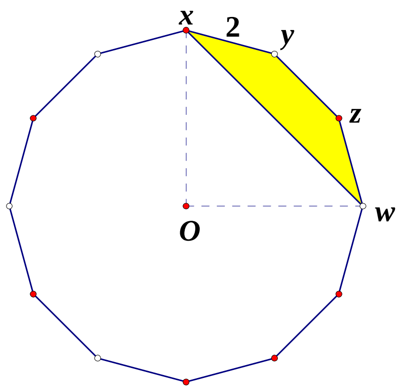
Let O be the centre of the regular dodecagon.

Let $Ox = r = Oy = Oz = Ow$

$$\angle xOy = \angle yOz = \angle zOw = \frac{360^\circ}{12} = 30^\circ \ (\angle\text{s at a point})$$

In ΔxOy, $r^2 + r^2 - 2r^2 \cos 30^\circ = 2^2$ (cosine rule)

$$(2 - \sqrt{3})r^2 = 4 \Rightarrow r^2 = \frac{4}{2 - \sqrt{3}} = 4\left(2 + \sqrt{3}\right)$$



Area of $xyzw$ = area of $Oxyzw$ – area of ΔOxw

$$= 3\times\frac{1}{2}\cdot r^2 \sin 30^\circ - \frac{1}{2}r^2 \sin 90^\circ = \left(\frac{3}{4} - \frac{1}{2}\right)\cdot 4\left(2 + \sqrt{3}\right) = 2 + \sqrt{3}$$

$a = 2$, $b = 3$, $B = 2^2 \cdot 3^3 = 4\times 27 = 108$

I1.3 If C is the sum of all positive factors of B, including 1 and B itself, find the value of C.

$108 = 2^2 \cdot 3^3$

$C = (1 + 2 + 2^2)\cdot(1 + 3 + 3^2 + 3^3) = 7\times 40 = 280$

Remark: Original version: 若C是B的所有因子之和… If C is the sum of all factors ⋯

Note that if negative factors are also included, then the answer will be different.

I1.4 If $C! = 10^D k$, where D and k are integers such that k is not divisible by 10, find the value of D.

Reference: 1990 HG6, 1994 FG7.1, 1996 HI3, 2004 FG1.1, 2011 HG7, 2012 FG1.3

Method 1

When each factor of 5 is multiplied by 2, a trailing zero will appear in $n!$.

The number of factors of 2 is clearly more than the number of factors of 5 in 280!

It is sufficient to find the number of factors of 5.

5, 10, 15, … , 280; altogether 56 numbers, each have at least one factor of 5.

25, 50, 75, … , 275; altogether 11 numbers, each have at least two factors of 5.

125, 250; altogether 2 numbers, each have at least three factors of 5.

$\therefore$ Total number of factors of 5 is 56 + 11 + 2 = 69

$D = 69$

Method 2

We can find the total number of factors of 5 by division as follow:

5	2 8 0		
5	5 6		
5	1 1	…	1
	2	…	1

$\therefore$ Total no. of factors of 5 is 56 + 11 + 2 = 69

$D = 69$

Individual Event 2

I2.1 If the product of the real roots of the equation $x^2 + 9x + 13 = 2\sqrt{x^2+9x+21}$ is P, find the value of P.

Let $y = x^2 + 9x$, then the equation becomes $y + 13 = 2\sqrt{y+21}$

$(y + 13)^2 = 4y + 84$

$y^2 + 26y + 169 - 4y - 84 = 0$

$y^2 + 22y + 85 = 0$

$(y + 17)(y + 5) = 0$

$y = -17$ or $y = -5$

Check put $y = -17$ into the original equation: $-17 + 13 = 2\sqrt{-17+21}$

LHS < 0, RHS > 0, rejected

Put $y = -5$ into the original equation: LHS = $-5 + 13 = 2\sqrt{-5+21}$ = RHS, accepted

$x^2 + 9x = -5$

$x^2 + 9x + 5 = 0$

Product of real roots = 5

Method 2

Let $y = \sqrt{x^2+9x+21} \ge 0$

Then the equation becomes $y^2 - 8 = 2y \Rightarrow y^2 - 2y - 8 = 0$

$(y - 4)(y + 2) = 0 \Rightarrow y = 4$ or -2 (rejected)

$\Rightarrow x^2 + 9x + 21 = 16$

$x^2 + 9x + 5 = 0$

$\Delta = 9^2 - 4(5) > 0$

Product of real roots = 5

I2.2 If $f(x) = \dfrac{25^x}{25^x + P}$ and $Q = f\left(\dfrac{1}{25}\right) + f\left(\dfrac{2}{25}\right) + \cdots + f\left(\dfrac{24}{25}\right)$, find the value of Q.

Reference: 2004 FG4.1, 2011 HG5

$$f(x) + f(1-x) = \frac{25^x}{25^x+5} + \frac{25^{1-x}}{25^{1-x}+5} = \frac{25+5\cdot 25^x + 25 + 5\cdot 25^{1-x}}{25+5\cdot 25^{1-x} + 5\cdot 25^x + 25} = 1$$

$$Q = f\left(\frac{1}{25}\right) + f\left(\frac{2}{25}\right) + \cdots + f\left(\frac{24}{25}\right)$$

$$= f\left(\frac{1}{25}\right) + f\left(\frac{24}{25}\right) + f\left(\frac{2}{25}\right) + f\left(\frac{23}{25}\right) + \cdots + f\left(\frac{12}{25}\right) + f\left(\frac{13}{25}\right) = 12$$

I2.3 If $X=\sqrt{(100)(102)(103)(105)+(Q-3)}$ is an integer and R is the units digit of X, find the value of R.

Reference: 1993 HG6, 1995 FI4.4, 1996 FG10.1, 2000 FG3.1, 2004 FG3.1

Let $y = 102.5$, then

$(100)(102)(103)(105) + (12 – 3)$

$= (y – 2.5)(y – 0.5)(y + 0.5)(y + 2.5) + 9$

$= (y^2 – 6.25)(y^2 – 0.25) + 9$

$= y^4 - 6.5y^2 + \frac{25}{16} + 9 = y^4 - 6.5y^2 + \frac{169}{16}$

$= \left(y^2 - \frac{13}{4}\right)^2 = \left(102.5^2 - \frac{13}{4}\right)^2 = \left(\frac{205^2}{4} - \frac{13}{4}\right)^2$

$X = \frac{42025^2 - 13}{4} = 10503$

R = the units digit of $X = 3$

Method 2 $X=\sqrt{(100)(102)(103)(105)+9}=\sqrt{(100)(100+5)(100+2)(100+3)+9}$

$=\sqrt{(100^2+500)(100^2+500+6)+9}=\sqrt{(100^2+500)^2+6(100^2+500)+9} = (100^2 + 500) + 3$

R = the units digit of $X = 3$

I2.4 If S is the sum of the last 3 digits (hundreds, tens, units) of the product of the positive roots of $\sqrt{2012}\cdot x^{\log_{2012} x} = x^R$, find the value of S.

$\log_{2012}\left(\sqrt{2012}\cdot x^{\log_{2012} x}\right) = \log_{2012} x^3$

$\frac{1}{2} + (\log_{2012} x)^2 = 3\log_{2012} x$

Let $y = \log_{2012} x$, then $2y^2 – 6y + 1 = 0$

$y = \log_{2012} x = \frac{3 \pm \sqrt{7}}{2}$

$\Rightarrow x = 2012^{\frac{3+\sqrt{7}}{2}}$ or $2012^{\frac{3-\sqrt{7}}{2}}$

Product of positive roots $= 2012^{\frac{3+\sqrt{7}}{2}} \times 2012^{\frac{3-\sqrt{7}}{2}}$

$= 2012^3$

$\equiv 12^3 \pmod{1000}$

$= 1728 \pmod{1000}$

S = sum of the last 3 digits = 7 + 2 + 8 = 17

Individual Event 3

I3.1 In Figure 1, a rectangle is sub-divided into 3 identical squares of side length 1 .
If $\alpha° = \angle ABD + \angle ACD$, find the value of α .

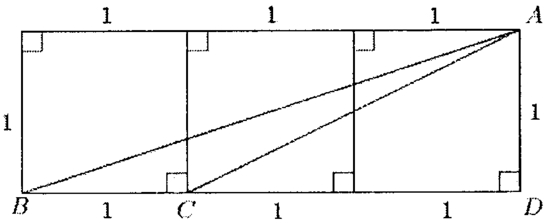


Figure 1

Method 1 (compound angle)

$\tan \angle ABD = \frac{1}{3}$, $\tan \angle ACD = \frac{1}{2}$

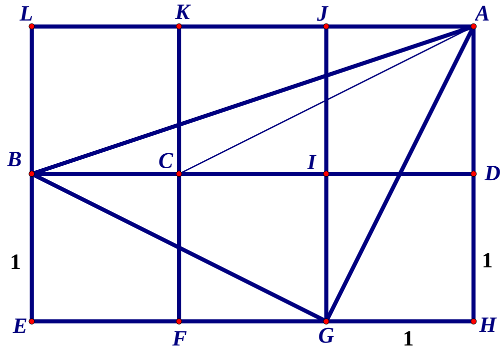
$0° < \angle ABD, \angle ACD < 45°$

$\therefore 0° < \angle ABD + \angle ACD < 90°$

$$\tan \alpha° = \frac{\tan \angle ABD + \tan \angle ACD}{1 - \tan \angle ABD \cdot \tan \angle ACD} = \frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\cdot\frac{1}{2}} = 1 > 0$$

$\alpha = 45$

Method 2 (congruent triangles)



Draw 3 more identical squares *BCFE*, *CIGH*, *IDHG* of length 1 as shown in the figure. *ALBD*, *DBEH* are identical rectangles. Join *BG*, *AG*.

$BE = GH = AD$ (sides of a square)

$EG = HA = CD$ (sides of 2 squares)

$\angle BEG = 90° = \angle GHA = \angle ADC$ (angle of a square)

$\Delta BEG \cong \Delta GHA \cong \Delta ADC$ (SAS)

Let $\angle BGE = \theta = \angle GAH$ (corr. $\angle$s $\cong \Delta$'s)

$\angle AGH = 90° - \theta$ ($\angle$s sum of Δ)

$\angle AGB = 180° - \angle AGH - \angle BGE$ (adj. $\angle$s on st. line)

$= 180° - \theta - (90° - \theta) = 90°$

$BG = AG$ (corr. sides $\cong \Delta$'s)

$\angle ABG = \angle BAG$ (base $\angle$s isos. Δ)

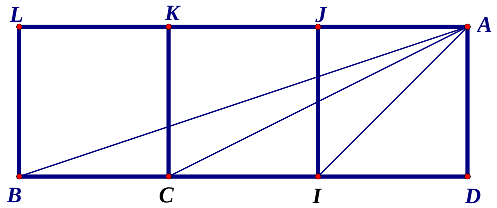
$= \frac{180° - 90°}{2} = 45°$ ($\angle$s sum of Δ)

$\alpha° = \angle ABD + \angle ACD$

$= \angle ABD + \angle GBI$ (corr. $\angle$s, $\cong\Delta$'s)

$= 45° \Rightarrow \alpha = 45$

Method 3 (similar triangles)



Join *AI*.

$AI = \sqrt{2}$ (Pythagoras' theorem)

$\frac{BI}{AI} = \frac{2}{\sqrt{2}} = \sqrt{2}$, $\frac{AI}{CI} = \frac{\sqrt{2}}{1} = \sqrt{2}$

$\angle AIB = \angle CIA$ (common $\angle$)

$\Delta AIB \sim \Delta CIA$ (2 sides proportional, included $\angle$)

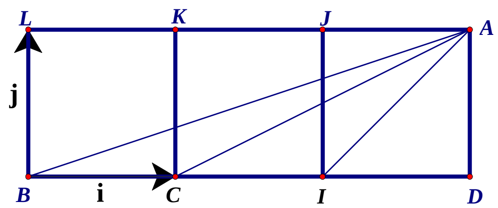
$\therefore \angle ACD = \angle BAI$ (corr. $\angle$s, ~Δ's)

$\alpha° = \angle ABD + \angle ACD = \angle ABI + \angle BAI$

$= \angle AID$ (ext. $\angle$ of Δ)

$= 45°$ (diagonal of a square) $\Rightarrow \alpha = 45$

Method 4 (vector dot product)



Define a rectangular system with $\overrightarrow{BC} = \mathbf{i}$, $\overrightarrow{BL} = \mathbf{j}$.

$\overrightarrow{AB} = -3\mathbf{i} - \mathbf{j}$, $\overrightarrow{AC} = -2\mathbf{i} - \mathbf{j}$, $\overrightarrow{AI} = -\mathbf{i} - \mathbf{j}$.

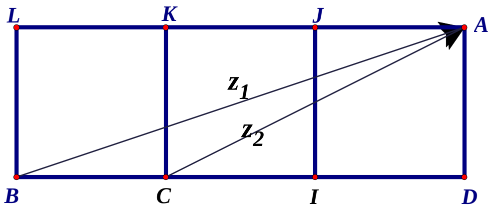
$\overrightarrow{AB} \cdot \overrightarrow{AI} = |\overrightarrow{AB}||\overrightarrow{AI}| \cos \angle BAI$

$$\cos\angle BAI = \frac{(-3)(-1)+(-1)(-1)}{\sqrt{(-3)^2+(-1)^2}\cdot\sqrt{(-1)^2+(-1)^2}} = \frac{2}{\sqrt{5}}$$

$\cos \angle ACD = \frac{2}{\sqrt{5}} \Rightarrow \angle BAI = \angle ACD$

$\alpha° = \angle ABD + \angle ACD = \angle ABI + \angle BAI = \angle AID$ (ext. $\angle$ of Δ) $\Rightarrow \alpha = 45$

Method 5 (complex number)



Define the Argand diagram with B as the origin, BD as the real axis, BL as the imaginary axis.

Let the complex numbers represented by AB and AC be z_1 and z_2 respectively.

$z_1 = 3 + i$, $z_2 = 2 + i$

$z_1 \cdot z_2 = (3 + i)(2 + i) = 6 - 1 + (2 + 3)i = 5 + 5i$

$\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$

$\alpha° = \angle ABD + \angle ACD = \tan^{-1}\frac{5}{5} = 45°$

$\alpha = 45$

I3.2 Let ABC be an acute-angled triangle. If $\sin A = \frac{36}{\alpha}$, $\sin B = \frac{12}{13}$ and $\sin C = \frac{\beta}{y}$, find the value of β, where β and y are in the lowest terms.

(Reference: 2003 FG2.4)

$\sin A = \frac{36}{45} = \frac{4}{5} \Rightarrow \cos A = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \frac{3}{5}$

$\sin B = \frac{12}{13} \Rightarrow \cos B = \sqrt{1 - \left(\frac{12}{13}\right)^2} = \frac{5}{13}$

$\sin C = \sin(180° - (A + B)) = \sin(A + B) = \sin A \cos B + \cos A \sin B$

$= \frac{4}{5} \cdot \frac{5}{13} + \frac{3}{5} \cdot \frac{12}{13} = \frac{56}{65}$

$\beta = 56$

Remark The original question is: Let ABC be a triangle. ……

Case 1 If all angles are acute, then $\beta = 56$ (done above)

Case 2 If $\angle A$ is obtuse, then $\cos A = -\frac{3}{5}$

$\cos B = \frac{5}{13}$, $\sin C = \frac{4}{5} \cdot \frac{5}{13} - \frac{3}{5} \cdot \frac{12}{13} = -\frac{16}{65} \Rightarrow C > 180°$ or $C < 0°$ (rejected)

Case 3 If $\angle B$ is obtuse, then $\cos B = -\frac{5}{13}$

$\cos A = \frac{3}{5}$, $\sin C = \frac{4}{5} \cdot \left(-\frac{5}{13}\right) + \frac{3}{5} \cdot \frac{12}{13} = \frac{16}{65}$

$\beta = 16$

There are two possible values of β, of which $\beta = 16$ could not be carried forward.

I3.3 In Figure 2, a circle at centre O has three points on its circumference, A, B and C. There are line segments OA, OB, AC and BC, where OA is parallel to BC. If D is the intersection of OB and AC with $\angle BDC = (2\beta - 1)°$ and $\angle ACB = \gamma°$, find the value of γ.

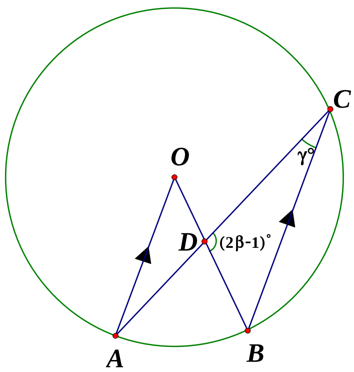


Figure 2

$\angle AOB = 2\gamma°$ ($\angle$ at centre twice $\angle$ at circumference)

$\angle OBC = 2\gamma°$ (alt. $\angle$, OA // CB)

$\gamma° + 2\gamma° + (2\beta - 1)° = 180°$ ($\angle$s sum of Δ)

$3\gamma + 111 = 180$

$\gamma = 23$

I3.4 In the expansion of $(ax+b)^{2012}$, where *a* and *b* are relatively prime positive integers. If the coefficients of x^{γ} and $x^{\gamma+1}$ are equal, find the value of $\delta = a + b$.

Coefficient of $x^{23} = C_{23}^{2012} \cdot a^{23} \cdot b^{1989}$; coefficient of $x^{24} = C_{24}^{2012} \cdot a^{24} \cdot b^{1988}$

$C_{23}^{2012} \cdot a^{23} \cdot b^{1989} = C_{24}^{2012} \cdot a^{24} \cdot b^{1988}$

$b = \frac{C_{24}^{2012}}{C_{23}^{2012}} \cdot a$

$b = \frac{2012-24+1}{24} \cdot a$

$24b = 1989a$

$8b = 663a$

$\because$ *a* and *b* are relatively prime integers

$\therefore a = 8, b = 663$

$\delta = 8 + 663 = 671$

Individual Event 4

I4.1 If A is a positive integer such that $\frac{1}{1\times 3}+\frac{1}{3\times 5}+\cdots+\frac{1}{(A+1)(A+3)}=\frac{12}{25}$, find the value of A.

$$\frac{1}{(n+1)(n+3)}=\frac{1}{2}\left(\frac{1}{n+1}-\frac{1}{n+3}\right) \text{ for } n\geq 0$$

$$\frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\cdots+\frac{1}{A+1}-\frac{1}{A+3}\right)=\frac{12}{25}$$

$$1-\frac{1}{A+3}=\frac{24}{25}$$

$$\frac{1}{A+3}=\frac{1}{25}$$

$A = 22$

I4.2 If x and y be positive integers such that $x > y > 1$ and $xy = x + y + A$.

Let $B=\frac{x}{y}$, find the value of B.

Reference: 1987 FG10.4, 2002 HG9

$xy = x + y + 22$

$xy - x - y + 1 = 23$

$(x - 1)(y - 1) = 23$

$\because$ 23 is a prime number and $x > y > 1$

$\therefore$ $x - 1 = 23$, $y - 1 = 1$

$x = 24$ and $y = 2$

$B=\frac{24}{2}=12$

I4.3 Let f be a function satisfying the following conditions:

(*i*) f(n) is an integer for every positive integer n;

(*ii*) f(2) = 2;

(*iii*) f(mn) = f(m)f(n) for all positive integers m and n and

(*iv*) f(m) > f(n) if $m > n$.

If C = f B), find the value of C.

Reference: 2003 HI1

$2 = f(2) > f(1) > 0 \Rightarrow f(1) = 1$

$f(4) = f(2\times 2) = f(2)f(2) = 4$

$4 = f(4) > f(3) > f(2) = 2$

$\Rightarrow f(3) = 3$

$C = f(12) = f(4\times 3) = f(4)f(3) = 4\times 3 = 12$

I4.4 Let D be the sum of the last three digits of 2401×7^C (in the denary system).

Find the value of D.

$2401\times 7^C = 7^4\times 7^{12} = 7^{16} = (7^2)^8 = 49^8 = (50 - 1)^8$

$= 50^8 - 8\times 50^7 + \ldots - 56\times 50^3 + 28\times 50^2 - 8\times 50 + 1$

$\equiv 28\times 2500 - 400 + 1 \pmod{1000}$

$\equiv -399 \equiv 601 \pmod{1000}$

$D = 6 + 0 + 1 = 7$

Method 2 $2401\times 7^C = 7^4\times 7^{12} = 7^{16}$

$7^4 = 2401$

$7^8 = (2400 + 1)^2 = 5760000 + 4800 + 1 \equiv 4801 \pmod{1000}$

$7^{16} \equiv (4800 + 1)^2 \equiv 48^2\times 10000 + 9600 + 1 \equiv 9601 \pmod{1000}$

$D = 6 + 0 + 1 = 7$

Individual Spare (2011 Final Group Spare Event)

IS.1 Let P be the number of triangles whose side lengths are integers less than or equal to 9. Find the value of P .

The sides must satisfy triangle inequality. i.e. $a + b > c$.

Possible order triples are (1, 1, 1), (2, 2, 2), … , (9, 9, 9) ,
(2, 2, 1), (2, 2, 3), (3, 3, 1), (3, 3, 2), (3, 3, 4), (3, 3, 5) ,
(4, 4, 1), (4, 4, 2), (4, 4, 3), (4, 4, 5), (4, 4, 6), (4, 4, 7) ,
(5, 5, 1), … , (5, 5, 4), (5, 5, 6), (5, 5, 7), (5, 5, 8), (5, 5, 9) ,
(6, 6, 1), … , (6, 6, 9) (except (6, 6, 6))
(7, 7, 1), … , (7, 7, 9) (except (7, 7, 7))
(8, 8, 1), … , (8, 8, 9) (except (8, 8, 8))
(9, 9, 1), … , (9, 9, 8)
(2, 3, 4), (2, 4, 5), (2, 5, 6), (2, 6, 7), (2, 7, 8), (2, 8, 9) ,
(3, 4, 5), (3, 4, 6), (3, 5, 6), (3, 5, 7), (3, 6, 7), (3, 6, 8), (3, 7, 8), (3, 7, 9), (3, 8, 9) ,
(4, 5, 6), (4, 5, 7), (4, 5, 8), (4, 6, 7), (4, 6, 8), (4, 6, 9), (4, 7, 8), (4, 7, 9), (4, 8, 9) ,
(5, 6, 7), (5, 6, 8), (5, 6, 9), (5, 7, 8), (5, 7, 9), (5, 8, 9), (6, 7, 8), (6, 7, 9), (6, 8, 9), (7, 8, 9) .

Total number of triangles = 9 + 6 + 6 + 8×5 + 6 + 9 + 9 + 6 + 4 = 95

Method 2 First we find the number of order triples.

Case 1 All numbers are the same: (1, 1, 1), … , (9, 9, 9).

Case 2 Two of them are the same, the third is different: (1, 1, 2), … , (9, 9, 1)

There are $C_1^9 \times C_1^8 = 72$ possible triples.

Case 3 All numbers are different. There are $C_3^9 = 84$ possible triples.

∴ Total 9 + 72 + 84 = **165** possible triples.

Next we find the number of triples which **cannot form a triangle**, i.e. $a + b \le c$.

Possible triples are (1, 1, 2), … (1, 1, 9) (8 triples)
(1, 2, 3), … , (1, 2, 9) (7 triples)
(1, 3, 4), … , (1, 3, 9) (6 triples)
(1, 4, 5), … , (1, 4, 9) (5 triples)
(1, 5, 6), … , (1, 5, 9) (4 triples)
(1, 6, 7), (1, 6, 8), (1, 6, 9), (1, 7, 8), (1, 7, 9), (1, 8, 9) ,
(2, 2, 4), … , (2, 2, 9) (6 triples)
(2, 3, 5), … , (2, 3, 9) (5 triples)
(2, 4, 6), … , (2, 4, 9) (4 triples)
(2, 5, 7), (2, 5, 8), (2, 5, 9), (2, 6, 8), (2, 6, 9), (2, 7, 9) ,
(3, 3, 6), … , (3, 3, 9) (4 triples)
(3, 4, 7), (3, 4, 8), (3, 4, 9), (3, 5, 8), (3, 5, 9), (3, 6, 9), (4, 4, 8), (4, 4, 9), (4, 5, 9) .

Total number of triples which cannot form a triangle

= (8 + 7 + ⋯ + 1) + (6 + 5 + ⋯ + 1) + (4 + 3 + 2 + 1) + (2 + 1) = 36 + 21 + 10 + 3 = **70**

∴ Number of triangles = **165 – 70** = **95**

IS.2 Let $Q = \log_{128} 2^3 + \log_{128} 2^5 + \log_{128} 2^7 + \cdots + \log_{128} 2^P$. Find the value of Q .

$Q = 3 \log_{128} 2 + 5 \log_{128} 2 + 7 \log_{128} 2 + \cdots + 95 \log_{128} 2$

$= (3 + 5 + \ldots + 95) \log_{128} 2 = \frac{3+95}{2} \times 47 \times \log_{128} 2 = \log_{128} 2^{2303} = \log_{128} (2^7)^{329} = 329$

IS.3 Consider the line $12x – 4y + (Q – 305) = 0$. If the area of the triangle formed by the x-axis, the y-axis and this line is R square units, what is the value of R ?

$12x – 4y + 24 = 0 \Rightarrow$ Height = 6, base = 2; area $R = \frac{1}{2} \cdot 6 \cdot 2 = 6$

Remark: the original question is … $12x – 4y + Q = 0$ ….
The answer is very difficult to carry forward to next question.

IS.4 If $x + \frac{1}{x} = R$ and $x^3 + \frac{1}{x^3} = S$, find the value of S.

$S = \left(x + \frac{1}{x}\right)\left(x^2 - 1 + \frac{1}{x^2}\right) = R\left[\left(x + \frac{1}{x}\right)^2 - 3\right] = R^3 – 3R = 216 – 3(6) = 198$

Sample Group Event (2009 Final Group Event 1)

SG.1 Given some triangles with side lengths a cm, 2 cm and b cm, where a and b are integers and $a \le 2 \le b$. If there are q non-congruent classes of triangles satisfying the above conditions, find the value of q.

When $a = 1$, possible $b = 2$

When $a = 2$, possible $b = 2$ or 3

$\therefore q = 3$

SG.2 Given that the equation $|x| - \frac{4}{x} = \frac{3|x|}{x}$ has k distinct real root(s), find the value of k.

When $x > 0$: $x^2 - 4 = 3x \Rightarrow x^2 - 3x - 4 = 0 \Rightarrow (x + 1)(x - 4) = 0 \Rightarrow x = 4$

When $x < 0$: $-x^2 - 4 = -3x \Rightarrow x^2 - 3x + 4 = 0$; $D = 9 - 16 < 0 \Rightarrow$ no real roots.

$k = 1$ (There is only one real root.)

SG.3 Given that x and y are non-zero real numbers satisfying the equations $\frac{\sqrt{x}}{\sqrt{y}} - \frac{\sqrt{y}}{\sqrt{x}} = \frac{7}{12}$ and $x - y = 7$. If $w = x + y$, find the value of w.

The first equation is equivalent to $\frac{x-y}{\sqrt{xy}} = \frac{7}{12} \Rightarrow \sqrt{xy} = 12 \Rightarrow xy = 144$

Sub. $y = \frac{144}{x}$ into $x - y = 7$: $x - \frac{144}{x} = 7 \Rightarrow x^2 - 7x - 144 = 0 \Rightarrow (x + 9)(x - 16) = 0$

$x = -9$ or 16; when $x = -9$, $y = -16$ (rejected $\because \sqrt{x}$ is undefined); when $x = 16$; $y = 9$

$w = 16 + 9 = 25$

Method 2 The first equation is equivalent to $\frac{x-y}{\sqrt{xy}} = \frac{7}{12} \Rightarrow \sqrt{xy} = 12 \Rightarrow xy = 144$ …… (1)

$\because x - y = 7$ and $x + y = w$

$\therefore x = \frac{w+7}{2}, y = \frac{w-7}{2}$

Sub. these equations into (1): $\left(\frac{w+7}{2}\right)\left(\frac{w-7}{2}\right) = 144$

$w^2 - 49 = 576 \Rightarrow w = \pm 25$

$\because$ From the given equation $\frac{\sqrt{x}}{\sqrt{y}} - \frac{\sqrt{y}}{\sqrt{x}} = \frac{7}{12}$, we know that both $x > 0$ and $y > 0$

$\therefore w = x + y = 25$ only

SG.4 Given that x and y are real numbers and $\left|x - \frac{1}{2}\right| + \sqrt{y^2 - 1} = 0$.

Let $p = |x| + |y|$, find the value of p.

Reference: 2006 FI4.2 $\cdots y^2 + 4y + 4 + \sqrt{x + y + k} = 0$. If $r = |xy|$, $\cdots$

Both $\left|x - \frac{1}{2}\right|$ and $\sqrt{y^2 - 1}$ are non-negative numbers.

The sum of two non-negative numbers = 0 means each of them is zero

$x = \frac{1}{2}$, $y = \pm 1$; $p = \frac{1}{2} + 1 = \frac{3}{2}$

Group Event 1

G1.1 Calculate the tens digit of 2011^{2011} .

$2011^{2011} \equiv (10 + 1)^{2011} \bmod 100$

$\equiv 10^{2011} + \cdots + 2011\times10 + 1$ (binomial theorem)

$\equiv 11 \bmod 100$

The tens digit is 1 .

G1.2 Let $a_1, a_2, a_3, \cdots$ be an arithmetic sequence with common difference 1 and $a_1 + a_2 + a_3 + \cdots + a_{100} = 2012$. If $P = a_2 + a_4 + a_6 + \cdots + a_{100}$, find the value of P .

Let $a_1 = a$ then $\frac{100(2a+99)}{2} = 2012$

$2a + 99 = \frac{1006}{25}$

$2a = \frac{1006-99\times25}{25} = -\frac{1469}{25}$

$P = a_2 + a_4 + a_6 + \cdots + a_{100} = \frac{50(a+1+a+99)}{2} = 25\times(2a + 100) = 25\times\left(100-\frac{1469}{25}\right)$

$P = 2500 - 1469 = 1031$

Method 2 $P = a_2 + a_4 + a_6 + \cdots + a_{100}$

$\underline{Q = a_1 + a_3 + a_3 + \cdots + a_{99}}$

$P - Q = 1 + 1 + 1 + \cdots + 1$ (50 terms) $= 50$

But since $P + Q = a_1 + a_2 + a_3 + \cdots + a_{100} = 2012$

$\therefore P = \frac{2012+50}{2} = 1031$

Remark: the original question …等差級數… , … arithmetic progression …

The phrases are changed to … 等差數列 … and … arithmetic sequence … according to the mathematics syllabus since 1999 .

G1.3 If 90! is divisible by 10^k, where k is a positive integer, find the greatest possible value of k .

Reference: 1990 HG6, 1994 FG7.1, 1996 HI3, 2004 FG1.1, 2011 HG7, 2012 FI1.4

Method 1

When each factor of 5 is multiplied by 2, a trailing zero will appear in $n!$.

The number of factors of 2 is clearly more than the number of factors of 5 in 280!

It is sufficient to find the number of factors of 5.

5, 10, 15, ⋯ , 90; altogether 18 numbers, each have at least one factor of 5.

25, 50, 75, altogether 3 numbers, each have at least two factors of 5.

$\therefore$ Total number of factors of 5 is 18 + 3 = 21

$k = 21$

Method 2

We can find the total number of factors of 5 by division as follow:

5 | 9 0

5 | 1 8

3 … 3

No. of factors of 5 is 18+3

$k = 21$

G1.4 In Figure 1, ΔABC is a right-angled triangle with $AB \perp BC$. If $AB = BC$, D is a point such that $AD \perp BD$ with $AD = 5$ and $BD = 8$, find the value of the area of ΔBCD.

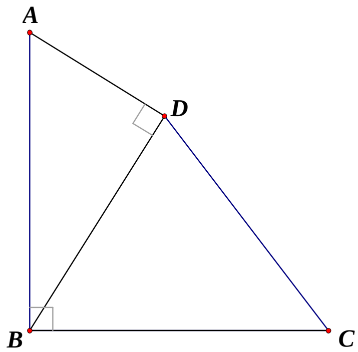


Figure 1

$AB = BC = \sqrt{5^2 + 8^2} = \sqrt{89}$ (Pythagoras' theorem)

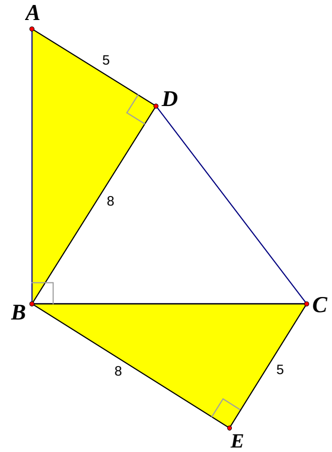
Let $\angle ABD = \theta$, then $\cos\theta = \dfrac{8}{\sqrt{89}}$

$\angle CBD = 90° - \theta$, $\sin\angle CBD = \dfrac{8}{\sqrt{89}}$

$S_{\Delta BCD} = \dfrac{1}{2} BD \cdot BC \sin\angle CBD = \dfrac{1}{2} \cdot 8 \cdot \sqrt{89} \cdot \dfrac{8}{\sqrt{89}} = 32$

Method 2

Rotate ΔABD about the centre at B in clockwise direction through 90° to ΔCBE.



Then $\Delta ABD \cong \Delta CBE$

$\angle CEB = \angle ADB = 90°$ (corr. $\angle$s $\cong \Delta$s)

$CE = 5$, $BE = 8$ (corr. sides $\cong \Delta$s)

$\angle DBE = \angle DBC + \angle CBE$

$= \angle DBC + \angle ABD$ (corr. $\angle$s $\cong \Delta$s)

$= 90°$

$\angle DBE + \angle CEB = 180°$

BD // CE (int. $\angle$s supp.)

$\therefore$ $BDCE$ is a right-angled trapezium.

Area of ΔBCD = area of trapezium $BDCE$ – area of ΔBCE

$= \dfrac{1}{2}(8 + 5)\times 8 - \dfrac{1}{2} \cdot 8\times 5$

$= 32$

Remark: the original question "⋯ right-angle triangle ⋯ "

It should be changed to right-angled triangle. Furthermore, the condition $AD \perp BD$ is not specified.

Group Event 2

G2.1 Find the value of $2\times\tan 1°\times\tan 2°\times \tan 3°\times \cdots \times \tan 87°\times \tan 88°\times \tan 89°$.

Similar question: 2008 FI1.1

$\tan \theta\times\tan(90° - \theta) = 1$ for $\theta = 1°, 2°, \cdots . 44°$ and $\tan 45° = 1$

$2\times\tan 1°\times\tan 2°\times \tan 3°\times \cdots \times \tan 87°\times \tan 88°\times \tan 89° = 2$

G2.2 If there are K integers that satisfy the equation $(x^2 - 3x + 2)^2 - 3(x^2 - 3x) - 4 = 0$, find the value of K.

$(x^2 - 3x + 2)^2 - 3(x^2 - 3x) - 4 = 0$

$(x^2 - 3x)^2 + 4(x^2 - 3x) - 3(x^2 - 3x) = 0$

$(x^2 - 3x)^2 + (x^2 - 3x) = 0$

$(x^2 - 3x)(x^2 - 3x + 1) = 0$

$x = 0, 3$ or $\dfrac{3\pm\sqrt{5}}{2}$

K = number of integral roots = 2

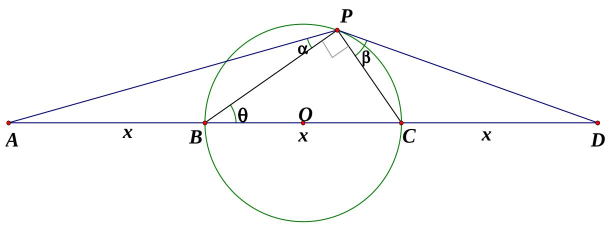
G2.3 If ℓ is the minimum value of $|x - 2| + |x - 47|$, find the value of ℓ.

Reference: 1994 HG1, 2001 HG9, 2004 FG4.2, 2008 HI8, 2008 FI1.3, 2010 HG6, 2011 FGS.1

Using the triangle inequality: $|a| + |b| \ge |a + b|$

$|x - 2| + |x - 47| = |x - 2| + |47 - x| \ge |x - 2 + 47 - x| = 45 \Rightarrow \ell = 45$

G2.4 In Figure 1, P, B and C are points on a circle with centre O and diameter BC. If A, B, C, D are collinear such that $AB = BC = CD$, $\alpha = \angle APB$ and $\beta = \angle CPD$, find the value of $(\tan \alpha)(\tan \beta)$.



Let $AB = x = BC = CD$, $\angle CBP = \theta$.

$\angle BPC = 90°$ ($\angle$ in semi circle), $\angle BCP = 90° - \theta$ ($\angle$s sum of Δ)

$BP = x \cos \theta$, $CP = x \sin \theta$

$\angle BAP = \theta - \alpha$, $\angle CDP = 90° - \theta - \beta$ (ext. $\angle$ of Δ)

$\dfrac{x}{\sin\alpha} = \dfrac{BP}{\sin\angle BAP}$ (sine rule on ΔABP); $\dfrac{x}{\sin\beta} = \dfrac{CP}{\sin\angle CDP}$ (sine rule on ΔCDP)

$\dfrac{x}{\sin\alpha} = \dfrac{x\cos\theta}{\sin(\theta-\alpha)}$; $\dfrac{x}{\sin\beta} = \dfrac{x\sin\theta}{\cos(\theta+\beta)}$

$\sin \theta \cos \alpha - \cos \theta \sin \alpha = \cos \theta \sin \alpha$; $\cos \theta \cos \beta - \sin \theta \sin \beta = \sin \theta \sin \beta$

$\sin \theta \cos \alpha = 2 \cos \theta \sin \alpha$; $\cos \theta \cos \beta = 2 \sin \theta \sin \beta$

$\tan \alpha = \dfrac{\tan\theta}{2}$; $\tan \beta = \dfrac{1}{2\tan\theta}$

$(\tan \alpha)(\tan \beta) = \dfrac{\tan\theta}{2}\cdot\dfrac{1}{2\tan\theta} = \dfrac{1}{4}$

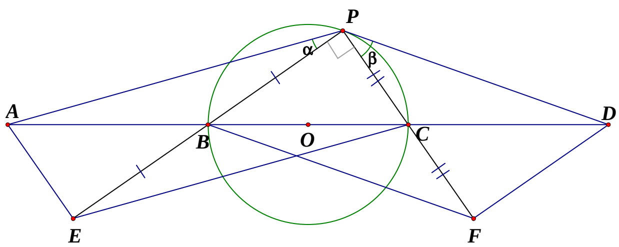
Method 2 $\angle BPC = 90°$ ($\angle$ in semi circle),

Produce PB to E so that $PB = BE$.

Produce PC to F so that $PC = CF$.

$\because$ $AB = BC = CD$ (given)

$\therefore$ $APCE$, $BPDF$ are //-grams (diagonals bisect each other)



$\angle PEC = \alpha$ (alt. $\angle$s, $AP//EC$)

$\angle PFB = \beta$ (alt. $\angle$s, $PD//BF$)

In ΔEPC, $\tan \alpha = \dfrac{PC}{PE} = \dfrac{PC}{2PB}$

In ΔBPF, $\tan \beta = \dfrac{PB}{PF} = \dfrac{PB}{2PC}$

$(\tan \alpha)(\tan \beta) = \dfrac{PC}{2PB}\cdot\dfrac{PB}{2PC} = \dfrac{1}{4}$

Method 3 Lemma A Given a triangle ABC. D is a point on BC such that $BD : DC = m : n$, $AD = t$.

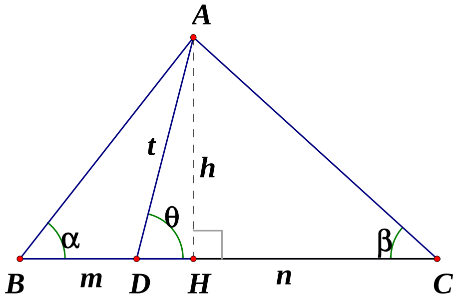
$\angle ABD = \alpha$, $\angle ADC = \theta < 90°$, $\angle ACD = \beta$.

Then $n \cot \alpha - m \cot \beta = (m + n) \cot \theta$

Proof: Let H be the projection of A on BC.

$AH = h$, $DH = h \cot \theta$.

$BH = \cot \alpha$, $CH = h \cot \beta$



$$\frac{m}{n} = \frac{BD}{DC} = \frac{BH - DH}{CH + HD} = \frac{h\cot\alpha - h\cot\theta}{h\cot\beta + h\cot\theta}$$

$m(\cot \beta + \cot \theta) = n(\cot \alpha - \cot \theta)$

$\therefore n \cot \alpha - m \cot \beta = (m + n) \cot \theta$

Lemma B Given a triangle ABC. D is a point on BC such that $BD : DC = m : n$, $AD = t$.

$\angle BAD = \alpha$, $\angle ADC = \theta < 90°$, $\angle CAD = \beta$.

Then $m \cot \alpha - n \cot \beta = (m + n) \cot \theta$

Proof: Draw the circumscribed circle ABC.

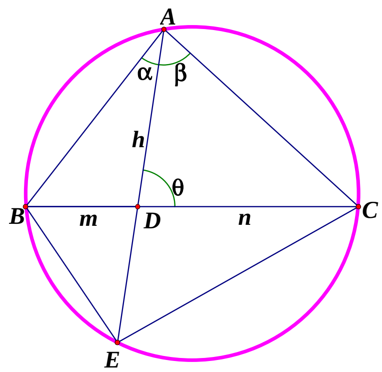
Produce AD to cut the circle again at E.

$\angle BCE = \alpha$, $\angle CBE = \beta$ ($\angle$s in the same seg.)

$\angle BDE = \theta < 90°$ (vert. opp. $\angle$s)

Apply **Lemma A** on ΔBEC.

$\therefore m \cot \alpha - n \cot \beta = (m + n) \cot \theta$



Now return to our original problem

$\angle BPC = 90°$ ($\angle$ in semi circle)

Apply **Lemma B** to ΔAPC:

$x \cot \alpha - x \cot 90° = (x + x) \cot \theta$

$\cot \alpha = 2 \cot \theta \cdots\cdots (1)$

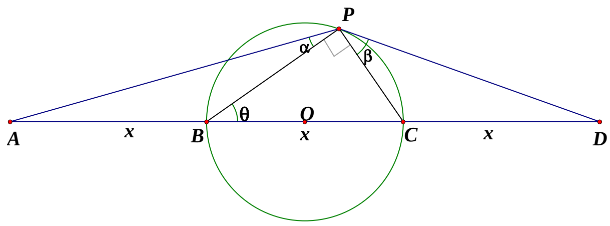
Apply **Lemma B** to ΔBPC, $\angle BPC = 90° - \theta$

$x \cot \beta - x \cot 90° = (x + x) \cot(90° - \theta)$

$\cot \beta = 2 \tan \theta \cdots\cdots (2)$

$(1)\times(2)$: $\cot \alpha \cot \beta = 2 \cot \theta \times 2 \tan \theta = 4$

$\therefore (\tan \alpha)(\tan \beta) = \frac{1}{4}$



Remark: the original question 圓有直徑 BC，圓心在 O，P、B 及 C 皆為圓周上的點。若 $AB = BC = CD$ 及 AD 為一綫段 ⋯ $AB = BC = CD$ and AD is a line segment ⋯

Both versions are not smooth and clear. The new version is as follow:

BC 是圓的直徑，圓心為 O，P、B 及 C 皆為圓周上的點。若 A、B、C 及 D 共綫且 $AB = BC = CD$ ⋯ If A, B, C, D are collinear such that $AB = BC = CD$ ⋯

Group Event 3

G3.1 Let $x=\dfrac{\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}}$, $y=\dfrac{\sqrt{7}-\sqrt{3}}{\sqrt{7}+\sqrt{3}}$ and $192z = x^4 + y^4 + (x + y)^4$, find the value of z.

$x+y=\dfrac{\left(\sqrt{7}+\sqrt{3}\right)^2+\left(\sqrt{7}-\sqrt{3}\right)^2}{7-3}=\dfrac{2(7+3)}{4}=5$; $xy = 1$

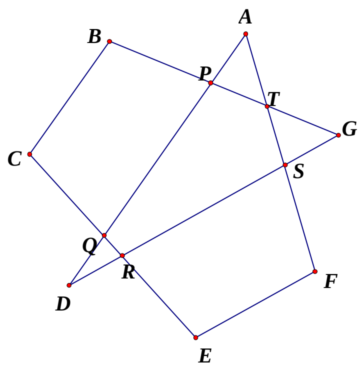
$x^2 + y^2 = (x + y)^2 - 2xy = 5^2 - 2 = 23$

$x^4 + y^4 = (x^2 + y^2)^2 - 2(xy)^2 = 23^2 - 2 = 527$

$192z = 527 + 5^4 = 527 + 625 = 1152$

$z = 6$

G3.2 In Figure 1, *AD*, *DG*, *GB*, *BC*, *CE*, *EF* and *FA* are line segments. If $\angle FAD + \angle GBC + \angle BCE + \angle ADG + \angle CEF + \angle AFE + \angle DGB = r°$, find the value of r.



Reference: 1992 HI13, 2000 HI5

In the figure, let *P*, *Q*, *R*, *S*, *T* be as shown.

$\angle ATP + \angle BPQ + \angle DQR + \angle ERS + \angle GST = 360°$ …… (1)

$\angle APT + \angle CQP + \angle DRQ + \angle FSR + \angle GTS = 360°$ …… (2)

(sum of ext. ∠ of polygon)

$\angle FAD = 180° - (\angle ATP + \angle APT)$ (∠s sum of Δ)

$\angle GBC + \angle BCE = 360° - (\angle BPQ + \angle CQP)$ (∠s sum of polygon)

$\angle ADG = 180° - (\angle DQR + \angle DRQ)$ (∠s sum of Δ)

$\angle CEF + \angle AFE = 360° - (\angle ERS + \angle FSR)$ (∠s sum of polygon)

$\angle DGB = 180° - (\angle GST + \angle GST)$ (∠s sum of Δ)

Add these 5 equations up and make use of equations (1) and (2):

$r° = 180°\times 7 - 2\times 360° \Rightarrow r = 540$

Remark: The original question $\angle FAD + \angle GBC + \angle BCE + \angle ADG + \angle CEF + \angle DGB = r°$
$\angle AFE$ is missing, the original question is wrong.

中文版題目：“…及 *FA* 都是綫段。…” is changed into “…及 *FA* 都是直綫綫段。…”

G3.3 Let k be positive integer and f(k) a function that if $\dfrac{k-1}{k}=0.k_1k_2k_3\ldots\ldots$, then $f(k)=\overline{k_1k_2k_3}$, for example, f(3) = 666 because $\dfrac{3-1}{3}=0.666\ldots\ldots$, find the value of D = f(f(f(f(f(112))))).

$0.99=1-\dfrac{1}{100}<\dfrac{112-1}{112}=1-\dfrac{1}{112}<1\Rightarrow f(112)=\overline{99k_3}$

$0.998=1-\dfrac{1}{500}<\dfrac{\overline{99k_3}-1}{\overline{99k_3}}=1-\dfrac{1}{\overline{99k_3}}<1-\dfrac{1}{1000}=0.999\Rightarrow f(f(112))=998$

$\Rightarrow$ f(f(f(112))) = 998 $\Rightarrow D$ = f(f(f(f(f(112))))) = 998

G3.4 If F_n is an integral valued function defined recursively by $F_n(k) = F_1(F_{n-1}(k))$ for $n \ge 2$ where $F_1(k)$ is the sum of squares of the digits of k, find the value of $F_{2012}(7)$.

$F_1(7) = 7^2 = 49$

$F_2(7) = F_1(F_1(7)) = F_1(49) = 4^2 + 9^2 = 97$

$F_3(7) = 9^2 + 7^2 = 130$

$F_4(7) = 1^2 + 3^2 + 0^2 = 10$

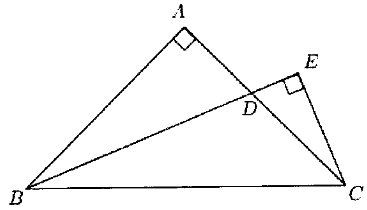
$F_5(7) = 1$

$F_{2012}(7) = 1$

Remark: the original question If f is an integer valued function …
The recursive function is defined for F_n, not f .

Group Event 4

G4.1 In figure 1, ABC and EBC are two right-angled triangles, $\angle BAC = \angle BEC = 90°$, $AB = AC$ and EDB is the angle bisector of $\angle ABC$. Find the value of $\frac{BD}{CE}$.



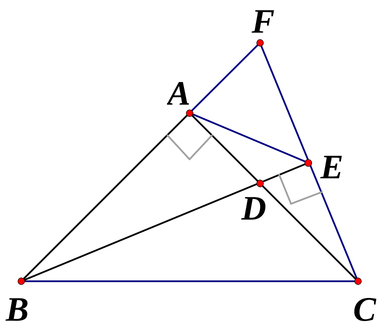
ΔABC is a right-angled isosceles triangle.
$\angle ABC = \angle ACB = 45°$ ($\angle$s sum of isos. Δ)
$\angle ABD = \angle CBD = 22.5°$ ($\angle$ bisector)
Let $BC = x$

$AB = x \cos 45° = \frac{\sqrt{2}x}{2}$; $CE = x \sin 22.5°$

$BD = AB \div \cos 22.5° = \frac{\sqrt{2}x}{2\cos 22.5^\circ}$

$\frac{BD}{CE} = \frac{\sqrt{2}x}{2\cos 22.5^\circ \cdot x\sin 22.5^\circ} = \frac{\sqrt{2}}{\cos 45^\circ} = 2$

Method 2 Produce CE and BA to meet at F.



$AB = AC \Rightarrow \angle ABC = \angle ACB = 45°$
$\angle BAC = \angle BEC = 90°$ (given)
$\Rightarrow ABCE$ is a cyclic quad. (converse, $\angle$s in the same seg.)
$\angle ABD = \angle CBD = 22.5°$ ($\angle$ bisector)
$\angle ACF = 22.5°$ ($\angle$s in the same seg.)
$\angle CAE = \angle CBE = 22.5°$ ($\angle$s in the same seg.)
$CE = AE$ … (1) (sides, opp. eq. $\angle$s)
$\angle EAF = 180° - 90° - 22.5° = 67.5°$ (adj. $\angle$s on st. line)
$\angle AFE = 180° - 90° - 22.5° = 67.5°$ ($\angle$s of ΔACF)
$AE = EF$ … (2) (sides, opp. eq. $\angle$s)
By (1) and (2), $CF = 2CE$ … (3)
$\Delta ACF \cong \Delta ABD$ (A.S.A.)
$BD = CF$ (corr. sides, $\cong \Delta$'s)

$\frac{BD}{CE} = \frac{CF}{CE} = \frac{2CE}{CE} = 2$ by (3)

G4.2 If $Q > 0$ and satisfies $\left|3Q - |1-2Q|\right| = 2$, find the value of Q.

Reference: 2002 FG4.3, 2005 FG4.2, 2009 HG9

$\left|3Q - |1-2Q|\right| = 2$

$3Q - |1 - 2Q| = 2$ or $3Q - |1 - 2Q| = -2$
$3Q - 2 = |1 - 2Q|$ or $3Q + 2 = |1 - 2Q|$
$3Q - 2 = 1 - 2Q$ or $3Q - 2 = 2Q - 1$ or $3Q + 2 = 1 - 2Q$ or $3Q + 2 = 2Q - 1$

$Q = \frac{3}{5}$ or 1 or $-\frac{1}{5}$ (rejected) or -3 (rejected)

Check: when $Q = \frac{3}{5}$, LHS $= \left|\frac{9}{5} - \left|1 - \frac{6}{5}\right|\right| = \frac{8}{5} \neq 2$, rejected

When $Q = 1$, LHS $= |3 - |1 - 2|| = 2 =$ RHS accepted
$\therefore Q = 1$

G4.3 Let $xyzt = 1$. If $R = \frac{1}{1+x+xy+xyz} + \frac{1}{1+y+yz+yzt} + \frac{1}{1+z+zt+ztx} + \frac{1}{1+t+tx+txy}$,

find the value of R.

$$\frac{1}{1+x+xy+xyz} = \frac{1}{1+x+xy+\frac{1}{t}} = \frac{t}{1+t+tx+txy}$$

$$\frac{1}{1+y+yz+yzt} = \frac{1}{1+y+\frac{1}{tx}+\frac{t}{tx}} = \frac{tx}{1+t+tx+txy}$$

$$\frac{1}{1+z+zt+ztx} = \frac{1}{1+\frac{1}{txy}+\frac{t}{txy}+\frac{tx}{txy}} = \frac{txy}{1+t+tx+txy}$$

$$R = \frac{t}{1+t+tx+txy} + \frac{tx}{1+t+tx+txy} + \frac{txy}{1+t+tx+txy} + \frac{1}{1+t+tx+txy} = 1$$

G4.4 If x_1, x_2, x_3, x_4 and x_5 are positive integers that satisfy $x_1 + x_2 + x_3 + x_4 + x_5 = x_1x_2x_3x_4x_5$, that is the sum is the product, find the maximum value of x_5.

The expression is symmetric. We may assume that $1 \le x_1 \le x_2 \le x_3 \le x_4 \le x_5$.

If $x_1 = x_2 = x_3 = x_4 = 1$, then $1 + 1 + 1 + 1 + x_5 = 1\times1\times1\times1\times x_5 \Rightarrow$ no solution

$\therefore x_1x_2x_3x_4 - 1 \ne 0$

$x_1 + x_2 + x_3 + x_4 = (x_1x_2x_3x_4 - 1)x_5$

$x_5 = \frac{x_1+x_2+x_3+x_4}{x_1x_2x_3x_4-1}$

When x_5 attains the maximum value, the denominator must be 1, i.e. $x_1x_2x_3x_4 = 2$

$\because 1 \le x_1 \le x_2 \le x_3 \le x_4$ $\therefore x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 2$, max. $x_5 = \frac{1+1+1+2}{2-1} = 5$

Method 2 We begin from the lowest integer.

Case 1 Let $x_1 = x_2 = x_3 = x_4 = 1$, then $1 + 1 + 1 + 1 + x_5 = 1\times1\times1\times1\times x_5 \Rightarrow$ no solution

Case 2 Let $x_1 = x_2 = x_3 = 1$ and $x_4 > 1$, then $3 + x_4 + x_5 = x_4x_5 \Rightarrow x_5 = \frac{x_4+3}{x_4-1}$

When $x_4 = 2$, $x_5 = 5$; when $x_4 = 3$, $x_5 = 3$

When $x_4 = 4$, no integral solution for x_5

When $x_4 = 5$, $x_5 = 2$, contradicting that $1 \le x_1 \le x_2 \le x_3 \le x_4 \le x_5$.

When $x_4 > 5$, then $x_5 < x_4$, which is a contradiction

Case 3 Let $x_1 = x_2 = 1$ and $x_3 > 1$, then $2 + x_3 + x_4 + x_5 = x_3x_4x_5$

When $x_3 = 2$, $4 + x_4 + x_5 = 2x_4x_5 \Rightarrow x_5 = \frac{x_4+4}{2x_4-1} > 1 \Rightarrow x_4 + 4 > 2x_4 - 1 \Rightarrow x_4 < 5$

When $x_4 = 2$, $x_5 = 2$

When $x_4 = 3, 4$, no integral solution for x_5

Case 4 $1 = x_1 < x_2 \le x_3 \le x_4 \le x_5$, then $x_5 = \frac{1+x_2+x_3+x_4}{x_2x_3x_4-1} < \frac{1+3x_4}{4x_4-1}$

When $x_2 = x_3 = x_4 = 2$, $x_5 = 1 < x_4$, contradiction

When $2 \le x_2 = x_3 < x_4$

$1 + 3x_4 < 4x_4 - 1$

$\frac{1+3x_4}{4x_4-1} < 1 \Rightarrow x_5 < 1$, contradiction $\therefore$ There is no integral solution for x_5.

Case 5 $2 \le x_1 \le x_2 \le x_3 \le x_4 \le x_5$, then $x_5 = \frac{x_1+x_2+x_3+x_4}{x_1x_2x_3x_4-1} < \frac{4x_4}{8x_4-1}$

$1 < 4x_4$

$4x_4 < 8x_4 - 1$

$\frac{4x_4}{8x_4-1} < 1 \Rightarrow x_5 < 1$, contradiction $\therefore$ There is no integral solution for x_5.

Conclusion: The solution set for $(x_1, x_2, x_3, x_4, x_5)$ is {(1, 1, 1, 2, 5), (1, 1, 1, 3, 3), (1,1,2,2,2)}.

Maximum for $x_5 = 5$

Group Spare (2011 Final Group Spare Event)

GS.1 Let α and β be the real roots of $y^2 - 6y + 5 = 0$. Let m be the minimum value of $|x - \alpha| + |x - \beta|$ over all real values of x . Find the value of m .

Remark: there is a typing mistake in the English version. ⋯ minimum value a of ⋯

Reference: 1994 HG1, 2001 HG9, 2004 FG4.2, 2008 HI8, 2008 FI1.3, 2010 HG6, 2012 FG2.3

$\alpha = 1, \beta = 5$

If $x < 1$, $|x - \alpha| + |x - \beta| = 1 - x + 5 - x = 6 - 2x > 4$

If $1 \le x \le 5$, $|x - \alpha| + |x - \beta| = x - 1 + 5 - x = 4$

If $x > 5$, $|x - \alpha| + |x - \beta| = x - 1 + x - 5 = 2x - 6 > 4$

m = min. of $|x - \alpha| + |x - \beta| = 4$

Method 2 Using the triangle inequality: $|a| + |b| \ge |a + b|$

$|x - \alpha| + |x - \beta| \ge |x - 1 + 5 - x| = 4 \Rightarrow m = 4$

GS.2 Let α, β, γ be real numbers satisfying $\alpha + \beta + \gamma = 2$ and $\alpha\beta\gamma = 4$.

Let v be the minimum value of $|\alpha| + |\beta| + |\gamma|$. Find the value of v .

If at least one of α, β, γ = 0, then $\alpha\beta\gamma \ne 4 \Rightarrow \alpha, \beta, \gamma \ne 0$

If α, β, γ > 0, then

$$\frac{\alpha+\beta+\gamma}{3} \ge \sqrt[3]{\alpha\beta\gamma} \quad \text{(A.M.} \ge \text{G.M.)}$$

$$\frac{2}{3} \ge \sqrt[3]{4}$$

$2^3 \ge 27\times 4 = 108$, which is a contradiction

If β < 0, in order that $\alpha\beta\gamma = 4 > 0$, WLOG let $\gamma < 0$, $\alpha > 0$

$\alpha = 2 - \beta - \gamma > 2$

$|\alpha| + |\beta| + |\gamma| = \alpha - (\beta + \gamma) = 2 + 2(-\beta - \gamma) \ge 2 + 4\sqrt{(-\beta)(-\gamma)}$, equality holds when $\beta = \gamma$

$4 = (2 - 2\beta)\beta^2$

$\beta^3 - \beta^2 + 2 = 0$

$(\beta + 1)(\beta^2 - 2\beta + 2) = 0$

$\beta = -1$ (For the 2nd equation, $\Delta = -4 < 0$, no real solution)

$\gamma = -1, \alpha = 4$

$|\alpha| + |\beta| + |\gamma| = 1 + 1 + 4 = 6$

v = min. of $|\alpha| + |\beta| + |\gamma| = 6$

GS.3 Let $y = |x + 1| - 2|x| + |x - 2|$ and $-1 \le x \le 2$. Let α be the maximum value of y .

Find the value of α .

$y = x + 1 - 2|x| + 2 - x = 3 - 2|x|$

$0 \le |x| \le 2 \Rightarrow 3 \ge 3 - 2|x| \ge -1$

$\alpha = 3$

GS.4 Let F be the number of integral solutions of $x^2 + y^2 + z^2 + w^2 = 3(x + y + z + w)$.

Find the value of F.

$(x, y, z, w) = (0, 0, 0, 0)$ is a trivial solution.

$x^2 + y^2 + z^2 + w^2 - 3(x + y + z + w) = 0$

$$\left(x^2 - 3x + \frac{9}{4}\right) + \left(y^2 - 3y + \frac{9}{4}\right) + \left(z^2 - 3z + \frac{9}{4}\right) + \left(w^2 - 3w + \frac{9}{4}\right) = 9$$

$$\left(x - \frac{3}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 + \left(z - \frac{3}{2}\right)^2 + \left(w - \frac{3}{2}\right)^2 = 9$$

$(2x - 3)^2 + (2y - 3)^2 + (2z - 3)^2 + (2w - 3)^2 = 36$

Let $a = 2x - 3$, $b = 2y - 3$, $c = 2z - 3$, $d = 2w - 3$, the equation becomes $a^2 + b^2 + c^2 + d^2 = 36$

For integral solutions of (x, y, z, w), (a, d, c, d) must be odd integers.

In addition, the permutation of (a, b, c, d) is also a solution. (e.g. (b, d, c, a) is a solution)

$\because$ a, b, c, d are odd integers and $a^2 + b^2 + c^2 + d^2 \ge 0$

If one of the four unknowns, say, $a > 6$, then L.H.S. > 36, so L.H.S. ≠ R.H.S.

$\therefore$ a , b, c, $d = \pm1, \pm3, \pm5$

When $a = \pm5$, then $25 + b^2 + c^2 + d^2 = 36 \Rightarrow b^2 + c^2 + d^2 = 11$

The only integral solution to this equation is $b = \pm3$, $c = \pm1 = d$ or its permutations.

When the largest (in magnitude) of the 4 unknowns, say, a is ±3, then $9 + b^2 + c^2 + d^2 = 36$

$\Rightarrow b^2 + c^2 + d^2 = 27$, the only solution is $b = \pm3$, $c = \pm3$, $d = \pm3$ or its permutations.

$\therefore$ The integral solutions are $(a, b, c, d) = (5, 3, 1, 1)$ and its permutations … (1) $\times P^4_2 = 12$

$(3, 3, 3, 3)$ … (2) $\times 1$

If (a, b, c, d) is a solution, then $(\pm a, \pm b, \pm c, \pm d)$ are also solutions.

There are 16 solutions with different signs for $(\pm a, \pm b, \pm c, \pm d)$.

$\therefore F = (12 + 1)\times16$

$= 208$

Individual Events

SI	***P***	30	**I1**	***a***	100	**I2**	***a***	3	**I3**	***a** see the remark	2	**I4**	***a***	1
	***Q** see the remark	120		***b***	5		***b***	600		***b***	7		***b***	7
	R	11		***c***	0		***c***	2		***c***	4		***c***	–61
	***S** see the remark	72		***d***	2		***d***	36		***d***	$4\frac{1}{3}$		***d***	69

Group Events

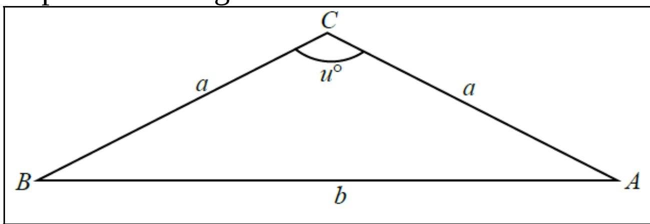
SG	***q***	3	**G1**	**unit digit**	5	**G2**	**minimum *r***	1	**G3**	$m^3 - n^3$	1387	**G4**	no. of digits	34
	k	1		**Integral part**	1		***s***	40		**Maximum**	31			2000
	w	25			24		***t***	–0.5		$a \cdot b$	$-\frac{1}{3}$			2519
	p	$\frac{3}{2}$		**Greatest *A***	3		***u***	120		***BC***	9		***A*+*B*+*C*+*D*+*E***	15

Errata

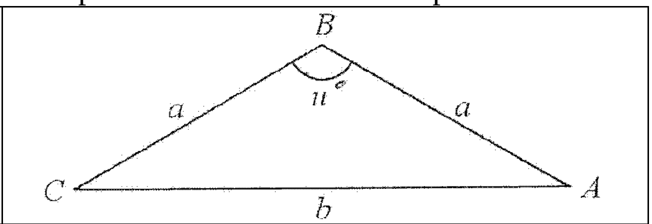
FI1.2 "the remainder <u>of</u> .... divided by" is changed into "the remainder <u>when</u> .... <u>is</u> divided by"
FI1.4 "Find the <u>maximum possible</u> value of" is changed into "Find the value of"
FI2.2"增加 $(2b - a)$ <u>cm^2</u>" is changed into "增加 $(2b - a)$ <u>cm^3</u>"
FI3.1 "integer" is deleted, 求 a 的整數值更改為求 a 的值。。
FI3.3 "The remainder <u>of</u> 392 divided by" is changed into "The remainder <u>when</u> 392 <u>is</u> divided by"

FI4.4 $\begin{cases} xy = 6 \\ x^2y + \underline{yx^2} + x + y + c = 2 \end{cases}$ is changed into $\begin{cases} xy = 6 \\ x^2y + \underline{xy^2} + x + y + c = 2 \end{cases}$

FG1.2 "integer" is changed into "integral"
FG1.3 "three-digit numbers how many" is changed into "three-digit numbers, how many".
FG2.4 wrong figure 1 on the internet
http://www.edb.gov.hk/attachment/tc/curriculum-development/kla/ma/res/sa/2012d.pdf



Original wrong diagram



New diagram

Sample Individual Event (2009 Final Individual Event 1)

SI.1 Let a, b, c and d be the roots of the equation $x^4 - 15x^2 + 56 = 0$.
If $P = a^2 + b^2 + c^2 + d^2$, find the value of P.
$x^4 - 15x^2 + 56 = 0 \Rightarrow (x^2 - 7)(x^2 - 8) = 0$
$a = \sqrt{7}$, $b = -\sqrt{7}$, $c = \sqrt{8}$, $d = -\sqrt{8}$
$P = a^2 + b^2 + c^2 + d^2 = 7 + 7 + 8 + 8 = 30$

SI.2 In Figure 1, $AB = AC$ and $AB \,//\, ED$.
If $\angle ABC = P°$ and $\angle ADE = Q°$, find the value of Q.
$\angle ABC = 30° = \angle ACB$ (base $\angle$s isos. Δ)
$\angle BAC = 120°$ ($\angle$s sum of Δ)
$\angle ADE = 120°$ (alt. $\angle$s, $AB \,//\, ED$)
$Q = 120$
Remark: Original question … $AB \,//\, DE$ ….
It is better for AB and ED to be oriented in the same direction.

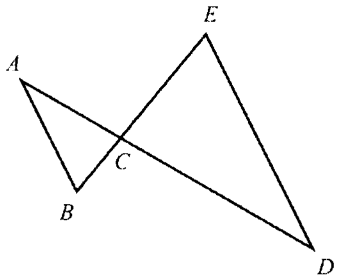


Figure 1

SI.3 Let $F = 1 + 2 + 2^2 + 2^3 + \ldots + 2^Q$ and $R = \sqrt{\frac{\log(1+F)}{\log 2}}$, find the value of R.

$F = 1 + 2 + 2^2 + 2^3 + \ldots + 2^{120} = \frac{2^{121}-1}{2-1} = 2^{121} - 1$

$R = \sqrt{\frac{\log(1+F)}{\log 2}} = \sqrt{\frac{\log 2^{121}}{\log 2}} = 11$

SI.4 Let $f(x)$ be a function such that $f(n) = (n - 1) f(n - 1)$ and $f(1) \neq 0$.
If $S = \frac{f(R)}{(R-1)f(R-3)}$, find the value of S.
$f(n) = (n - 1) f(n - 1) = (n - 1)(n - 2) f(n - 2) = \ldots\ldots$
$S = \frac{f(11)}{(11)f(11-3)} = \frac{10\times9\times8\times f(8)}{10\times f(8)} = 9\times8 = 72$

Remark: Original question:
Let $f(x)$ be a function such that $f(n) = (n - 1) f(n - 1)$. If $S = \frac{f(R)}{(R-1)f(R-3)}$, find the value of S.
Note that S is undefined when $f(n) = 0$ for some integers n.

Individual Event 1

I1.1 Figure 1 has *a* rectangles, find the value of *a*.

Reference: 1993HG9

$a = C_2^5 \times C_2^5 = 100$

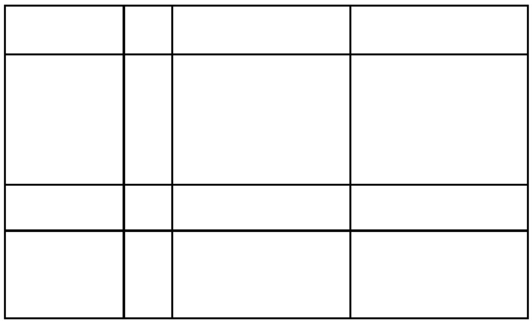


Figure 1

I1.2 Given that 7 divides 111111. If *b* is the remainder when $\underbrace{111111\ldots111111}_{a-times}$ is divided by 7, find the value of *b*.

$\underbrace{111111\ldots111111}_{100-times} = \underbrace{111111\ldots11}_{96-times}0000 + 1111$

$= 111111 \times \underbrace{1000001 \cdots 1000001}_{16\text{ '1's}} \times 10000 + 7 \times 158 + 5$

$= 7m + 5$, where *m* is an integer.

$b = 5$

I1.3 If *c* is the remainder of $\left[(b-2)^{4b^2} + (b-1)^{2b^2} + b^{b^2}\right]$ divided by 3, find the value of *c*.

$\left[(5-2)^{100} + (4)^{50} + 5^{25}\right] = 3^{100} + 4^{50} + 5^{25}$

$= 3^{100} + (3 + 1)^{50} + (3\times2 - 1)^{25}$

$= 3^{100} + 3m + 1 + 3n - 1$ (by binomial theorem, *n*, *m* are integers)

The remainder $c = 0$

I1.4 If $|x + 1| + |y - 1| + |z| = c$, find the value of $d = x^2 + y^2 + z^2$.

Reference: 2005 FI4.1, 2006 FI4.2, 2009 FG1.4, 2011 FI4.3, 2015 HG4, 2015 FI1.1

$|x + 1| + |y - 1| + |z| = 0$

$x = -1$, $y = 1$ and $z = 0$

$d = (-1)^2 + 1^2 + 0^2 = 2$

Individual Event 2

I2.1 Given that functions $f(x) = x^2 + rx + s$ and $g(x) = x^2 - 9x + 6$ have the properties that the sum of roots of $f(x)$ is the product of the roots of $g(x)$, and the product of roots of $f(x)$ is the sum of roots of $g(x)$. If $f(x)$ attains its minimum at $x = a$, find the value of a.

Let α, β be the roots of $f(x)$.

$\alpha + \beta = -r = 6$; $\alpha\beta = s = 9$

$\therefore f(x) = x^2 - 6x + 9 = (x - 3)^2 + 0$

$f(x)$ attains the minimum value at $x = a = 3$

I2.2 The surface area of a cube is b cm². If the length of each side is increased by 3 cm, its volume is increased by $(2b - a)$ cm³, find the value of b.

Let the original length of each side be x cm.

Old surface area b cm^2 = $6x^2$ cm^2

Original volume = x^3 cm^3

New length of side = $(x + 3)$ cm.

New volume = $(x + 3)^3$ cm^3

Increase in volume = $[(x + 3)^3 - x^3]$ cm^3 = $(2b - a)$ cm^3

$9x^2 + 27x + 27 = 2(6x^2) - 3$

$3x^2 - 27x - 30 = 0$

$x^2 - 9x - 10 = 0$

$(x - 10)(x + 1) = 0$

$x = 10$

$b = 6x^2 = 600$

I2.3 Let $f(1) = 3$, $f(2) = 5$ and $f(n + 2) = f(n + 1) + f(n)$ for positive integers n.

If c is the remainder of $f(b)$ divided by 3, find the value of c.

$f(1) = 3, f(2) = 5, f(3) = 8, f(4) = 13, f(5) = 21, f(6) = 34, f(7) = 55, f(8) = 89,$

$\equiv 0, \quad \equiv 2, \quad \equiv 2, \quad \equiv 1, \quad \equiv 0, \quad \equiv 1, \quad \equiv 1, \quad \equiv 2 \pmod 3$

$f(9) \equiv 0, f(10) \equiv 2, f(11) \equiv 2, f(12) \equiv 1, f(13) \equiv 0, f(14) \equiv 1, f(15) \equiv 1, f(16) \equiv 2 \pmod 3$

$\therefore$ When $f(n)$ is divided by 3, the pattern of the remainders repeats for every 8 integers.

$600 = 8\times75$

$c = 2$

I2.4 In Figure 2, the angles of triangle XYZ satisfy $\angle Z \le \angle Y \le \angle X$ and $c\cdot\angle X = 6\cdot\angle Z$.

If the maximum possible value of $\angle Z$ is $d°$, find the value of d.

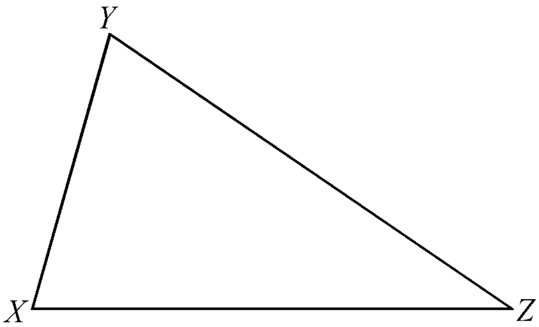


Figure 2

$2\cdot\angle X = 6\cdot\angle Z \Rightarrow \angle X = 3\angle Z$

Let $\angle Z = z°$, $\angle Y = y°$, $\angle X = 3z°$

$z + y + 3z = 180$ ($\angle$s sum of Δ)

$y = 180 - 4z$

$\because \angle Z \le \angle Y \le \angle X$

$\therefore z \le 180 - 4z \le 3z$

$\frac{180}{7} \le z$ and $z \le 36$

d = the maximum possible value of $z = 36$

Individual Event 3

I3.1 If $a=\dfrac{\left(7+4\sqrt{3}\right)^{\frac{1}{2}}-\left(7-4\sqrt{3}\right)^{\frac{1}{2}}}{\sqrt{3}}$, find the value of a.

Reference: 2015 FI4.2

$\left(7+4\sqrt{3}\right)^{\frac{1}{2}}=\sqrt{4+2\sqrt{4}\cdot\sqrt{3}+3}=\sqrt{\left(\sqrt{4}+\sqrt{3}\right)^2}=2+\sqrt{3}$

$\left(7-4\sqrt{3}\right)^{\frac{1}{2}}=\sqrt{4-2\sqrt{4}\cdot\sqrt{3}+3}=\sqrt{\left(\sqrt{4}-\sqrt{3}\right)^2}=2-\sqrt{3}$

$a=\dfrac{2+\sqrt{3}-\left(2-\sqrt{3}\right)}{\sqrt{3}}=2$

Remark: The original question is: ……, find the integer value of a. …… 求 a 的整數值。

As the value of a is exact, there is no need to emphasize the integer value of a.

I3.2 Suppose $f(x) = x - a$ and $F(x, y) = y^2 + x$. If $b = F(3, f(4))$, find the value of b.

Reference: 1985 FI3.3, 1990 HI3, 2015 FI4.3

$f(x) = x - 2$

$f(4) = 4 - 2 = 2$

$b = F(3, f(4))$

$= F(3, 2)$

$= 2^2 + 3 = 7$

I3.3 The remainder when 392 is divided by a 2-digit positive integer is b. If c is the number of such 2-digit positive integers, find the value of c.

$392 - 7 = 385 = 5\times77 = 5\times7\times11$

Possible 2-digit positive integer = 11, 35, 55 or 77

$c = 4$

I3.4 If x is a real number and d is the maximum value of the function $y=\dfrac{3x^2+3x+c}{x^2+x+1}$, find the value of d.

$(x^2 + x + 1)y = 3x^2 + 3x + 4$

$(3 - y)x^2 + (3 - y) + (4 - y) = 0$ …… (*), this is a quadratic equation in x.

For any real value of x, $y=\dfrac{3x^2+3x+c}{x^2+x+1}$ is a well-defined function

$\therefore$ (*) must have real roots in x.

$\Delta = (3 - y)^2 - 4(3 - y)(4 - y) \ge 0$

$(3 - y)(3 - y - 16 + 4y) \ge 0$

$(y - 3)(3y - 13) \le 0$

$3\le y\le\dfrac{13}{3}$

d = the maximum value of $y=\dfrac{13}{3}$

Method 2

$y=\dfrac{3x^2+3x+4}{x^2+x+1}=3+\dfrac{1}{x^2+x+1}=3+\dfrac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}$

$\left(x+\dfrac{1}{2}\right)^2+\dfrac{3}{4}\ge\dfrac{3}{4}$

$y\le3+\dfrac{4}{3}=\dfrac{13}{3}$

d = the maximum value of $y=\dfrac{13}{3}$

Individual Event 4

I4.1 Let $f(x)$ be a real value function that satisfies $f(xy) = f(x) f(y)$ for all real numbers x and y and $f(0) \neq 0$. Find the value of $a = f(1)$.

Reference: 2015 FI1.3

$f(0\times0) = f(0) f(0)$

$f(0) - [f(0)]^2 = 0$

$f(0)[1 - f(0)] = 0$

$\because f(0) \neq 0$

$\therefore f(0) = 1$

$f(0) = f(1\times0) = f(1) f(0)$

$1 = f(1) \Rightarrow a = f(1) = 1$

I4.2 Let $F(n)$ be a function with $F(1) = F(2) = F(3) = a$ and $F(n + 1) = \frac{F(n)\cdot F(n-1)+1}{F(n-2)}$ for positive integer $n \geq 3$, find the value of $b = F(6)$.

$F(4) = F(3 + 1) = \frac{F(3)\cdot F(3-1)+1}{F(3-2)} = \frac{F(3)\cdot F(2)+1}{F(1)} = \frac{1\cdot1+1}{1} = 2$

$F(5) = F(4 + 1) = \frac{F(4)\cdot F(4-1)+1}{F(4-2)} = \frac{F(4)\cdot F(3)+1}{F(2)} = \frac{2\cdot1+1}{1} = 3$

$F(6) = \frac{F(5)\cdot F(4)+1}{F(3)} = \frac{3\cdot2+1}{1} = 7$

I4.3 If $b - 6$, $b - 5$, $b - 4$ are three roots of the equation $x^4 + rx^2 + sx + t = 0$, find the value of $c = r + t$.

Reference: 2015 FI2.4

The three roots are 1, 2 and 3. Let the fourth root be α.

$\alpha + 1 + 2 + 3 = 0 \Rightarrow \alpha = -6$

$r = -6\times1 - 6\times2 - 6\times3 + 1\times2 + 1\times3 + 2\times3 = -25$

$t = -6\times1\times2\times3 = -36$

$c = r + t = -25 - 36 = -61$

I4.4 Suppose that (x_0, y_0) is a solution of the system: $\begin{cases} xy = 6 \\ x^2y + xy^2 + x + y + c = 2 \end{cases}$.

Find the value of $d = x_0^2 + y_0^2$.

Reference: 1993 HG8, 2010 FI1.3

From (2): $xy(x + y) + x + y - 61 = 2$

$6(x + y) + (x + y) - 63 = 0$

$x + y = 9$

$d = x^2 + y^2 = (x + y)^2 - 2xy = 9^2 - 2\times6 = 69$

Sample Group Event (2009 Final Group Event 1)

SG.1 Given some triangles with side lengths a cm, 2 cm and b cm, where a and b are integers and $a \le 2 \le b$. If there are q non-congruent classes of triangles satisfying the above conditions, find the value of q.

When $a = 1$, possible $b = 2$

When $a = 2$, possible $b = 2$ or 3

$\therefore q = 3$

SG.2 Given that the equation $|x| - \frac{4}{x} = \frac{3|x|}{x}$ has k distinct real root(s), find the value of k.

When $x > 0$: $x^2 - 4 = 3x \Rightarrow x^2 - 3x - 4 = 0 \Rightarrow (x + 1)(x - 4) = 0 \Rightarrow x = 4$

When $x < 0$: $-x^2 - 4 = -3x \Rightarrow x^2 - 3x + 4 = 0$; $D = 9 - 16 < 0 \Rightarrow$ no real roots.

$k = 1$ (There is only one real root.)

SG.3 Given that x and y are non-zero real numbers satisfying the equations $\frac{\sqrt{x}}{\sqrt{y}} - \frac{\sqrt{y}}{\sqrt{x}} = \frac{7}{12}$ and $x - y = 7$. If $w = x + y$, find the value of w.

The first equation is equivalent to $\frac{x-y}{\sqrt{xy}} = \frac{7}{12} \Rightarrow \sqrt{xy} = 12 \Rightarrow xy = 144$

Sub. $y = \frac{144}{x}$ into $x - y = 7$: $x - \frac{144}{x} = 7 \Rightarrow x^2 - 7x - 144 = 0 \Rightarrow (x + 9)(x - 16) = 0$

$x = -9$ or 16; when $x = -9$, $y = -16$ (rejected $\because \sqrt{x}$ is undefined); when $x = 16$; $y = 9$

$w = 16 + 9 = 25$

Method 2 The first equation is equivalent to $\frac{x-y}{\sqrt{xy}} = \frac{7}{12} \Rightarrow \sqrt{xy} = 12 \Rightarrow xy = 144$ …… (1)

$\because x - y = 7$ and $x + y = w$

$\therefore x = \frac{w+7}{2}, y = \frac{w-7}{2}$

Sub. these equations into (1): $\left(\frac{w+7}{2}\right)\left(\frac{w-7}{2}\right) = 144$

$w^2 - 49 = 576 \Rightarrow w = \pm 25$

$\because$ From the given equation $\frac{\sqrt{x}}{\sqrt{y}} - \frac{\sqrt{y}}{\sqrt{x}} = \frac{7}{12}$, we know that both $x > 0$ and $y > 0$

$\therefore w = x + y = 25$ only

SG.4 Given that x and y are real numbers and $\left|x - \frac{1}{2}\right| + \sqrt{y^2 - 1} = 0$. Let $p = |x| + |y|$, find the value of p.

Reference: 2006 FI4.2 … $y^2 + 4y + 4 + \sqrt{x + y + k} = 0$. If $r = |xy|$, …

Both $\left|x - \frac{1}{2}\right|$ and $\sqrt{y^2 - 1}$ are non-negative numbers.

The sum of two non-negative numbers = 0 means each of them is zero

$x = \frac{1}{2}$, $y = \pm 1$; $p = \frac{1}{2} + 1 = \frac{3}{2}$

Group Event 1

G1.1 Find the units digit of $(2^{13}+1)(2^{14}+1)(2^{15}+1)(2^{16}+1)$.

$2^{10}=1024$, $2^{11}=2048$, $2^{12}=4096$, $2^{13}=8192$, $2^{14}=16384$, $2^{15}=32768$, $2^{16}=65536$

$(2^{13}+1)(2^{14}+1)(2^{15}+1)(2^{16}+1)=8193\times16385\times32769\times65537\equiv 3\times5\times9\times7\equiv 5 \pmod{10}$

Units digit = 5

G1.2 Find the integral part of $16 \div (0.40 + 0.41 + 0.42 + \dots + 0.59)$.

$$0.40 + 0.41 + 0.42 + \dots + 0.59 = \frac{20}{2}\cdot(0.40+0.59) = 9.9$$

$1 = 9.9 \div 9.9 < 16 \div 9.9 < 18 \div 9 = 2$

Integral part = 1

G1.3 Choose three digits from 1, 2, 4, 6, 7 to construct three-digit numbers. Of these three-digit numbers, how many of them are divisible by 3?

126, 246, 147, 267 are divisible by 3.

The permutations of the digits of 126, 246, 147, 267 are also divisible by 3.

Total number of such integers = $3!\times4 = 24$

G1.4 Using numbers: 1, 2, 3, 4, 5, 6 to form a six-digit number: *ABCDEF* such that *A* is divisible by 1, *AB* is divisible by 2, *ABC* is divisible by 3, *ABCD* is divisible by 4, *ABCDE* is divisible by 5, *ABCDEF* is divisible by 6. Find the greatest value of *A*.

Reference: **http://www2.hkedcity.net/citizen_files/aa/gi/fh7878/public_html/Number_Theory/1234567890.pdf**

$\overline{ABCDE}$ is divisible by 5 $\Rightarrow E = 5$

$(A, C) = (1, 3)$ or $(3, 1)$

$\because$ $\overline{AB}$ is divisible by 2, $\overline{ABCD}$ is divisible by 4, $\overline{ABCDEF}$ is divisible by 6

$\therefore$ B, D, F are even.

$\overline{ABC}$ is divisible by 3 $\Rightarrow 1 + B + 3$ is divisible by 3 $\Rightarrow B = 2$

$\Rightarrow (D, F) = (4, 6)$ or $(6, 4)$

$\overline{ABCD}$ is divisible by 4 $\Rightarrow$ $\overline{CD}$ is divisible by 4

When $C = 1$, $D = 6$ … (1)

When $C = 3$, $D = 6$ … (2)

$\Rightarrow F = 4$

$\therefore$ $\overline{ABCDEF} = \overline{A2C654}$

Greatest value of $A = 3$

Group Event 2

G2.1 If $4^3 + 4^r + 4^4$ is a perfect square and r is a positive integer, find the minimum value of r.

$4^3 + 4^r + 4^4 = 2^2(4^2 + 4^{r-1} + 4^3) = 2^2(80 + 4^{r-1})$

The least perfect square just bigger than 80 is $81 = 9^2$.

$4^{r-1} = 1 \Rightarrow r = 1$

$\therefore$ The minimum value of r is 1.

G2.2 Three boys B_1, B_2, B_3 and three girls G_1, G_2, G_3 are to be seated in a row according to the following rules:

1) A boy will not sit next to another boy and a girl will not sit next to another girl,
2) Boy B_1 must sit next to girl G_1

If s is the number of different such seating arrangements, find the value of s.

First, arrange the three boys in a line, there are 3! permutations.

$B_1B_2B_3$, $B_1B_3B_2$, $B_2B_1B_3$, $B_2B_3B_1$, $B_3B_1B_2$, $B_3B_2B_1$.

If B_1 sits in the middle, there are two different cases. For instance, B_2 B_1 B_3. Then the possible seating arrangements are:

B_2 G_2 B_1 G_1 B_3 G_3, B_2 G_3 B_1 G_1 B_3 G_2, G_2 B_2 G_1 B_1 G_3 B_3 or G_3 B_2 G_1 B_1 G_2 B_3,

G_3 B_2 G_2 B_1 G_1 B_3, G_2 B_2 G_3 B_1 G_1 B_3, B_2 G_1 B_1 G_3 B_3 G_2 or B_2 G_1 B_1 G_2 B_3 G_3.

If B_1 sits in the left end or right end, there are four different cases. For instance, B_2 B_3 B_1, then the possible seating arrangements are:

B_2 G_2 B_3 G_3 B_1 G_1, B_2 G_3 B_3 G_2 B_1 G_1, G_2 B_2 G_3 B_3 G_1 B_1, G_3 B_2 G_2 B_3 G_1 B_1,

B_2 G_2 B_3 G_1 B_1 G_3, B_2 G_3 B_3 G_1 B_1 G_2.

$\therefore$ Total number of seating arrangements = $2\times8 + 4\times6 = 40$

Method 2 Label the 6 positions as

1	2	3	4	5	6

B_1 and G_1 sit next to each other, their positions can be 12, 23, 34, 45 or 56, altogether 5 ways.

B_1 and G_1 can interchange positions to G_1B_1, 2 different ways.

For the other positions, the two other boys and the two other girls can sit in $2\times2 = 4$ ways.

For instance, if B_1G_1 sit in the 2-3 positions,

1	B_1	G_1	4	5	6

then B_2, B_3, G_2, G_3 can sit in the following 4 ways:

G_2 B_1 G_1 B_2 G_3 B_3, G_3 B_1 G_1 B_2 G_1 B_3, G_2 B_1 G_1 B_3 G_3 B_2, G_3 B_1 G_1 B_3 G_2 B_2.

The total number of sitting arrangements = $5\times2\times4 = 40$ ways

G2.3 Let $f(x)=\dfrac{x+a}{x^2+\frac{1}{2}}$, where x is a real number and the maximum value of $f(x)$ is $\dfrac{1}{2}$ and the minimum value of $f(x)$ is -1. If $t = f(0)$, find the value of t.

Let $y=\dfrac{x+a}{x^2+\frac{1}{2}}=\dfrac{2x+2a}{2x^2+1}\Rightarrow 2yx^2+y=2x+2a\Rightarrow(2y)x^2-2x+(y-2a)=0$

For real values of x, $\Delta=(-2)^2-4(2y)(y-2a)\ge 0$

$1-(2y^2-4ay)\ge 0\Rightarrow 2y^2-4ay-1\le 0$ …… (*)

Given that $-1\le y\le\dfrac{1}{2}\Rightarrow(y+1)(2y-1)\le 0\Rightarrow 2y^2+y-1\le 0$ …… (**)

(*) is equivalent to (**) $\therefore a=-\dfrac{1}{4}$

$$f(x)=\frac{x-\frac{1}{4}}{x^2+\frac{1}{2}}$$

$$t=f(0)=\frac{-\frac{1}{4}}{\frac{1}{2}}$$

$$=-\frac{1}{2}$$

G2.4 In Figure 3, ABC is an isosceles triangle with $\angle ABC = u°$, $AB = BC = a$ and $AC = b$. If the quadratic equation $ax^2-\sqrt{2}\cdot bx+a=0$ has two real roots, whose absolute difference is $\sqrt{2}$, find the value of u.

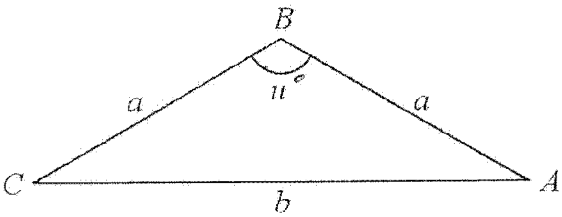


Figure 3

Let the roots be α, β.

$|\alpha-\beta|=\sqrt{2}$

$(\alpha-\beta)^2=2$

$(\alpha+\beta)^2-4\alpha\beta=2$

$\left(\dfrac{\sqrt{2}b}{a}\right)^2-4=2$

$b^2=3a^2$

$$\cos u°=\frac{a^2+a^2-b^2}{2a^2}$$

$$=\frac{2a^2-3a^2}{2a^2}$$

$$=-\frac{1}{2}$$

$u = 120$

Group Event 3

G3.1 If m and n are positive integers with $m^2 - n^2 = 43$, find the value of $m^3 - n^3$.

$(m + n)(m - n) = 43$, which is a prime number.

$\begin{cases} m+n=43 \\ m-n=1 \end{cases} \Rightarrow m = 22, n = 21$

$m^3 - n^3 = (m - n)(m^2 + mn + n^2) = 1\times[(m + n)^2 - mn] = 43^2 - 22\times21 = 1849 - 462 = 1387$

G3.2 Let $x_1, x_2, \dots, x_{10}$ be non-zero integers satisfying $-1 \le x_i \le 2$ for $i = 1, 2, \dots, 10$.

If $x_1 + x_2 + \dots + x_{10} = 11$, find the maximum possible value for $x_1^2 + x_2^2 + \cdots + x_{10}^2$.

In order to maximize $x_1^2 + x_2^2 + \cdots x_{10}^2$, the number of "2" appeared in $x_1, x_2, \dots, x_{10}$ must be as many as possible and the remaining numbers should be "–1".

Let the number of "2" be n and the number of "–1" be $10 - n$.

$2n - 1\times(10 - n) = 11$

$\Rightarrow n = 7$

Maximum $= 2^2 + 2^2 + 2^2 + 2^2 + 2^2 + 2^2 + 2^2 + 1 + 1 + 1 = 31$

G3.3 If $f(n) = a^n + b^n$, where n is a positive integer and $f(3) = [f(1)]^3 + f(1)$, find the value of $a\cdot b$.

$f(1) = a + b$

$f(3) = (a + b)^3 + a + b = a^3 + b^3$

$a^2 + 2ab + b^2 + 1 = a^2 - ab + b^2$

$3ab = -1$

$\Rightarrow ab = -\frac{1}{3}$

G3.4 In Figure 4, AD, BC and CD are tangents to the circle with centre at O and diameter $AB = 12$. If $AD = 4$, find the value of BC.

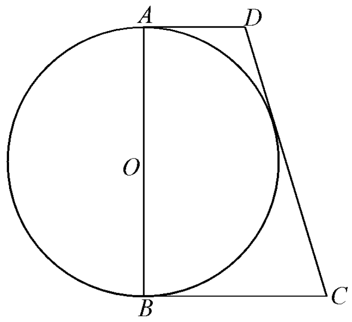


Figure 4

Suppose CD touches the circle at E. Let $BC = x$.

$DE = 4$ and $CE = x$ (tangent from ext. point)

From D, draw a line segment DF // AB, cutting BC at F.

$\angle DAB = \angle ABC = 90°$ (tangent $\perp$ radius)

$\angle DFC = 90°$ (corr. $\angle$s AB // DC)

$\therefore$ $ABFD$ is a rectangle.

$DF = 12$, $BF = 4$ (opp. sides of rectangle)

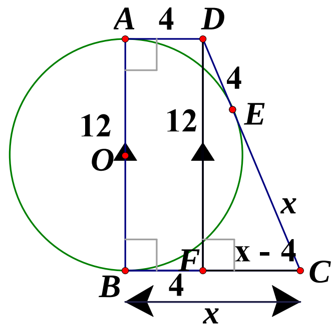
$CF = x - 4$

In ΔCDF, $(x - 4)^2 + 12^2 = (x + 4)^2$ (Pythagoras' theorem)

$x^2 - 8x + 16 + 144 = x^2 + 8x + 16$

$144 = 16x$

$BC = x = 9$



Group Event 4

G4.1 In P be the product of 3,659,893,456,789,325,678 and 342,973,489,379,256 , find the number of digits of P.

Reference: 2015 FG1.3

3,659,893,456,789,325,678 = 3.7×10^{18} (correct to 2 sig. fig.)

342,973,489,379,256 = 3.4×10^{14} (correct to 2 sig. fig.)

$P \approx 3.7\times10^{18}\times3.4\times10^{14} = 12.58\times10^{32} = 1.258\times10^{33}$

The number of digits is 34.

G4.2 If $\frac{1}{4}+4\left(\frac{1}{2013}+\frac{1}{x}\right)=\frac{7}{4}$, find the value of $1872+48\times\left(\frac{2013x}{x+2013}\right)$.

$$4\cdot\frac{x+2013}{2013x}=\frac{3}{2}$$

$$\frac{2013x}{x+2013}=\frac{8}{3}$$

$$1872+48\times\left(\frac{2013x}{x+2013}\right)=1872+48\times\frac{8}{3}= 1872 + 128 = 2000$$

G4.3 The remainders of an integer when divided by 10, 9, 8, … , 2 are 9, 8, 7, … , 1 respectively. Find the smallest such an integer.

Reference: 1985 FG7.2, 1990 HI13

Let the integer be N.

N + 1 is divisible by 10, 9, 8, 7, 6, 5, 4, 3, 2.

The L.C.M. of 2, 3, 4, 5, 6, 7, 8, 9, 10 is 2520.

$\therefore$ $N = 2520k - 1$, where k is an integer.

The least positive integral of $N = 2520 - 1 = 2519$

G4.4 In Figure 5, A, B, C, D, E represent different digits. Find the value of $A + B + C + D + E$.

$$\begin{array}{r} ABCDE \\ \times \quad\quad 9 \\ \hline 1AAA0E \end{array}$$

Figure 5

$9E \equiv E \pmod{10} \Rightarrow E = 0$ or 5

Consider the multiplication of ten thousands digit

$9A$ + carry digit = $10 + A \Rightarrow A = 1$ or 2

Possible products are 122205, 111105, 122200, 111100.

Of these 4 numbers, only 111105 is divisible by 9.

$\overline{ABCDE} = 111105\div9 = 12345$

$A + B + C + D + E = 1 + 2 + 3 + 4 + 5 = 15$

Individual Events

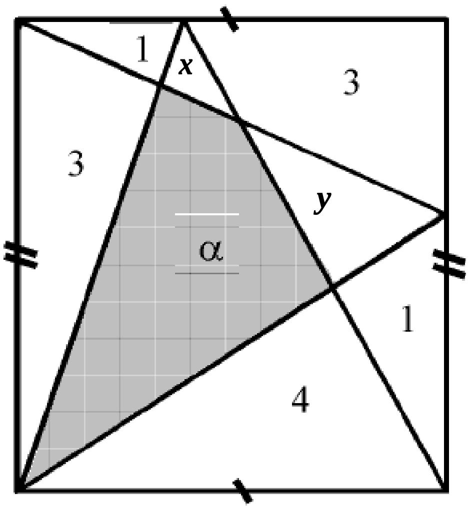
I1	α	5	**I2**	α	$\frac{4}{3}$	**I3**	α	11	**I4**	α	6
	β	55		β	24		β	45		β	5
	γ	6		γ	3		γ	45		γ	7.5
	δ	16		δ	7		δ	81		δ	$-\frac{33}{64}$

Group Events

G1	area	48	**G2**	Product	$\frac{1}{80}$	**G3**	Product	$\frac{11}{20}$	**G4**	***PZ***	1.6
	minimum	6		$S_{17}+S_{33}+S_{50}$	1		Sum	1		$x^3y+2x^2y^2+xy^3$	5
	remainder	0		Day	5		α	15		***d***	2
	a_{100}	$\frac{1}{10100}$		α	30		α	5		product	4

Individual Event 1

I1.1 Determine the area of the shaded region, α, in the figure. **(Reference: 2011 FG4.4)**



Label the unmarked regions by x and y respectively.

$3 + \alpha + y = \frac{1}{2}$ area of //-gram $= 4 + \alpha + x$

$\Rightarrow y = x + 1$ …… (1)

$1+x+3+3+\alpha+y+4+1$ = area of //-gram $= 2(4 + \alpha + x)$

$\Rightarrow 12 + x + y + \alpha = 8 + 2\alpha + 2x$ …… (2)

Sub. (1) into (2): $12 + x + x + 1 + \alpha = 8 + 2\alpha + 2x$

$\Rightarrow \alpha = 5$

I1.2 If the average of 10 distinct positive integers is 2α, what is the largest possible value of the largest integer, β, of the ten integers?

Let the 10 distinct positive integers be $0 < x_1 < x_2 < \dots < x_{10}$, in ascending order.

$$\frac{x_1 + x_2 + \cdots + x_{10}}{10} = 2\times 5 = 10$$

$x_1 + x_2 + \dots + x_9 + \beta = 100$

If β is the largest possible, then $x_1, x_2, \dots, x_9$ must be as small as possible.

The least possible $x_1, x_2, \dots, x_9$ are 1, 2, 3, … , 9.

The largest possible $\beta = 100 - (1 + 2 + \dots + 9) = 100 - 45 = 55$

I1.3 Given that 1, 3, 5, 7, … , β and 1, 6, 11, 16, … , β + 1 are two finite sequences of positive integers. Determine γ, the numbers of positive integers common to both sequences.

The two finite sequences are: 1, 3, 5, 7, … , 55 and 1, 6, 11, 16, … , 56.

The terms common to both sequences are 1, 11, 21, 31, 41, 51.

$\gamma = 6$

I1.4 If $\log_2 a + \log_2 b \geq \gamma$, determine the smallest positive value δ for $a + b$.

$\log_2 a + \log_2 b \geq 6$

$ab \geq 2^6 = 64$

$a + b = \left(\sqrt{a} - \sqrt{b}\right)^2 + 2\sqrt{ab} \geq 0 + 2\times\sqrt{64} = 16$

The smallest positive value of $\delta = 16$

Individual Event 2

I2.1 Determine the positive real root, α, of $\sqrt{(x+\sqrt{x})}-\sqrt{(x-\sqrt{x})}=\sqrt{x}$.

$\left[\sqrt{(x+\sqrt{x})}-\sqrt{(x-\sqrt{x})}\right]^2=x$

$x+\sqrt{x}-2\sqrt{x^2-x}+x-\sqrt{x}=x$

$x=2\sqrt{x^2-x}$

$x^2 = 4(x^2 - x)$

$3x^2 = 4x$

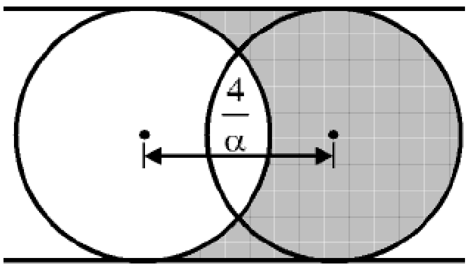
$x = 0$ (rejected) or $\frac{4}{3}$

Check: When $x=\frac{4}{3}$,

L.H.S. = $\sqrt{\left(\frac{4}{3}+\sqrt{\frac{4}{3}}\right)}-\sqrt{\left(\frac{4}{3}-\sqrt{\frac{4}{3}}\right)}=\sqrt{\frac{4+2\sqrt{3}}{3}}-\sqrt{\frac{4-2\sqrt{3}}{3}}=\frac{\sqrt{3}+1}{\sqrt{3}}-\frac{\sqrt{3}-1}{\sqrt{3}}=\sqrt{\frac{4}{3}}$ = R.H.S.

$\therefore \alpha=\frac{4}{3}$

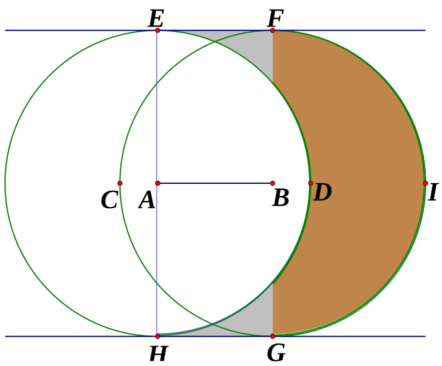
I2.2 In the figure, two circles of radii 4 with their centres placed apart by $\frac{4}{\alpha}$. Determine the area β, of the shaded region.



Let the centres of circles be ***A*** and ***B*** as shown. ***AB*** = 3

Suppose the two circles touches the two given line segments at ***E***, ***F***, ***G***, ***H*** as shown. Then ***EFGH*** is a rectangle with ***FE*** = ***AB*** = ***GH*** = 3, ***EH*** = ***FG*** = 8

β = Area of semi-circle ***FIG*** + area of rectangle ***EFGH***
– area of semi-circle ***EDH***
= Area of rectangle ***EFGH***
= 3×8 = 24



I2.3 Determine the smallest positive integer γ such that the equation $\sqrt{x}-\sqrt{\beta\gamma}=4\sqrt{2}$ has an integer solution in x.

$\sqrt{x}-\sqrt{24\gamma}=4\sqrt{2}$

$\sqrt{x}=2\sqrt{6\gamma}+4\sqrt{2}$

The smallest positive integer γ = 3

I2.4 Determine the units digit, δ, of $\left((\gamma^{\gamma})^{\gamma}\right)^{\gamma}$.

$\left((3^3)^3\right)^3=(3^9)^3=3^{27}$

The units digit of 3, 3^2, 3^3, 3^4 are 3, 9, 7, 1 respectively.

This pattern repeats for every multiples of 4.

27 = 6×4 + 3

δ = 7

Individual Event 3

I3.1 If the product of numbers in the sequence $10^{\frac{1}{11}}$, $10^{\frac{2}{11}}$, $10^{\frac{3}{11}}$, … , $10^{\frac{\alpha}{11}}$ is 1 000 000, determine the value of the positive integer α.

$10^{\frac{1}{11}}\times 10^{\frac{2}{11}}\times 10^{\frac{3}{11}}\times\ldots\times 10^{\frac{\alpha}{11}} = 10^6$

$\frac{1}{11}+\frac{2}{11}+\frac{3}{11}+\cdots+\frac{\alpha}{11}=6$

$\frac{1}{2}(1+\alpha)\alpha=66$

$\alpha^2 + \alpha - 132 = 0$

$(\alpha - 11)(\alpha + 12) = 0$

$\alpha = 11$

I3.2 Determine the value of β if $\frac{\beta}{1\times2\times3}+\frac{\beta}{2\times3\times4}+\cdots+\frac{\beta}{8\times9\times10}=\alpha$.

Reference: 2003 HG1

$\frac{1}{\boldsymbol{r}(\boldsymbol{r}+1)}-\frac{1}{(\boldsymbol{r}+1)(\boldsymbol{r}+2)}=\frac{(\boldsymbol{r}+2)-\boldsymbol{r}}{\boldsymbol{r}(\boldsymbol{r}+1)(\boldsymbol{r}+2)}=2\cdot\frac{1}{\boldsymbol{r}(\boldsymbol{r}+1)(\boldsymbol{r}+2)}$

Put $\boldsymbol{r} = 1$, $\frac{1}{1\times2}-\frac{1}{2\times3}=2\cdot\frac{1}{1\times2\times3}$

Put $\boldsymbol{r} = 2$, $\frac{1}{2\times3}-\frac{1}{3\times4}=2\cdot\frac{1}{2\times3\times4}$

……………………………………………………

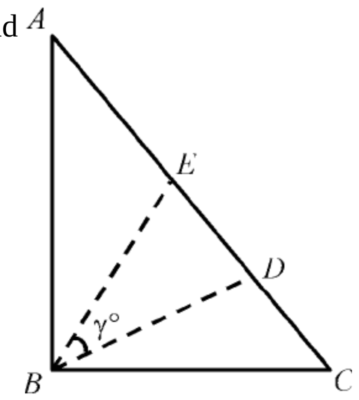
Put $\boldsymbol{r} = 8$, $\frac{1}{8\times9}-\frac{1}{9\times10}=2\cdot\frac{1}{8\times9\times10}$

Add these equations together and multiply both sides by β and divide by 2:

$\frac{\beta}{2}\left[\frac{1}{2}-\frac{1}{9\times10}\right]=\frac{\beta}{1\times2\times3}+\frac{\beta}{2\times3\times4}+\cdots+\frac{\beta}{8\times9\times10}=\alpha=11$

$\beta = 45$

I3.3 In the figure, triangle ***ABC*** has $\angle ABC = 2\beta°$, ***AB*** = ***AD*** and ***CB*** = ***CE***. If γ° = ∠***DBE***, determine the value of γ.



Let ∠***ABE*** = ***x***

∠***ABC*** = 90°

∠***CBE*** = 90° – ***x***

∠***ADB*** = ***x*** + γ° (base ∠s isos. Δ)

∠***CEB*** = ∠***CBE*** = 90° – ***x*** (base ∠s isos. Δ)

In Δ***BDE***, γ° + ***x*** + γ° + 90° – ***x*** = 180° (∠s sum of Δ)

γ = 45

I3.4 For the sequence 1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 2, 1, 2, … , determine the sum δ of the first γ terms.

δ = 1 + 2 + 1 + 2 + 2 + 1 + 2 + 2 + 2 + 1 + 2 + 2 + 2 + 2 + 1 + 2 +…+ 2 + (5 terms) + 1 + 2 + … + 2 (6 terms) + 1 + 2 +…+ 2 (7 terms) + 1 + 2 + … + 2 (8 terms) + 1

= 9 + 2×36 = 81

Individual Event 4

I4.1 If $\frac{6\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}=3\sqrt{\alpha}-6$, determine the value of α.

Reference: 1989 FG10.1

$$\frac{6\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}\cdot\frac{3\sqrt{2}-2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}=3\sqrt{\alpha}-6$$

$$6\sqrt{3}\cdot\frac{3\sqrt{2}-2\sqrt{3}}{18-12}=3\sqrt{\alpha}-6$$

$$3\sqrt{6}-6=3\sqrt{\alpha}-6$$

$\alpha = 6$

I4.2 Consider fractions of the form $\frac{\boldsymbol{n}}{\boldsymbol{n}+1}$, where $\boldsymbol{n}$ is a positive integer. If 1 is subtracted from both the numerator and the denominator, and the resultant fraction remains positive and is strictly less than $\frac{\alpha}{7}$, determine, β, the number of these fractions.

$$0<\frac{\boldsymbol{n}-1}{\boldsymbol{n}}<\frac{6}{7}$$

$7\boldsymbol{n} - 7 < 6\boldsymbol{n}$ and $\boldsymbol{n} > 1$

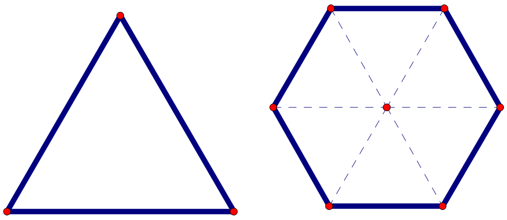
$1 < \boldsymbol{n} < 7$

Possible $\boldsymbol{n}$ = 2, 3, 4, 5, 6

$\beta = 5$

I4.3 The perimeters of an equilateral triangle and a regular hexagon are equal. If the area of the triangle is β square units, determine the area, γ, of the hexagon in square units.

Reference: 1996 FI1.1, 2016 FI2.1



Let the length of the equilateral triangle be $\boldsymbol{x}$, and that of the regular hexagon be $\boldsymbol{y}$.

Since they have equal perimeter, $3\boldsymbol{x} = 6\boldsymbol{y}$

$\therefore\ \boldsymbol{x} = 2\boldsymbol{y}$

The hexagon can be divided into 6 identical equilateral triangles.

Ratio of areas $=\frac{1}{2}\boldsymbol{x}^2\sin 60^\circ : 6\times\frac{1}{2}\boldsymbol{y}^2\sin 60^\circ = 2 : 3$

$\beta = 5$

$\gamma = 5\times\frac{3}{2} = 7.5$

I4.4 Determine the value of $\delta=\frac{3}{2}+\frac{5}{4}+\frac{9}{8}+\frac{17}{16}+\frac{33}{32}+\frac{65}{64}-\gamma$.

$$\delta=\frac{3}{2}+\frac{5}{4}+\frac{9}{8}+\frac{17}{16}+\frac{33}{32}+\frac{65}{64}-7\frac{1}{2}$$

$$=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}-1\frac{1}{2}$$

$$=-\frac{33}{64}$$

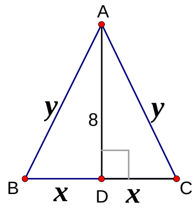
Group Event 1

G1.1 If an isosceles triangle has height 8 from the base, not the legs, and perimeters 32, determine the area of the triangle.

Let the isosceles triangle be $\boldsymbol{ABC}$ with $\boldsymbol{AB} = \boldsymbol{AC} = \boldsymbol{y}$

$\boldsymbol{D}$ is the midpoint of the base $\boldsymbol{BC}$, $\boldsymbol{AD} \perp \boldsymbol{BC}$, $\boldsymbol{AD} = 8$

Let $\boldsymbol{BD} = \boldsymbol{DC} = \boldsymbol{x}$



Perimeter $= 2x + 2y = 32 \Rightarrow y = 16 - x$ …… (1)

$x^2 + 8^2 = y^2$ …… (2)

Sub. (1) into (2): $x^2 + 64 = 256 - 32x + x^2$

$x = 6$, $y = 10$

Area of the triangle = 48

G1.2 If $f(x) = \dfrac{\left(x+\dfrac{1}{x}\right)^6 - \left(x^6+\dfrac{1}{x^6}\right) - 2}{\left(x+\dfrac{1}{x}\right)^3 + \left(x^3+\dfrac{1}{x^3}\right)}$ where x is a positive real number, determine the minimum value of f(x).

Reference: 1979 American High School Mathematics Examination Q29

$$f(x) = \frac{\left[\left(x+\frac{1}{x}\right)^3\right]^2 - \left(x^3+\frac{1}{x^3}\right)^2}{\left(x+\frac{1}{x}\right)^3 + \left(x^3+\frac{1}{x^3}\right)}$$

$$= \left(x+\frac{1}{x}\right)^3 - \left(x^3+\frac{1}{x^3}\right)$$

$$= 3\left(x+\frac{1}{x}\right)$$

$$f(x) = 3\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 + 6 \geq 6$$

G1.3 Determine the remainder of the 81-digit integer $\overline{111\cdots1}$ divided by 81.

$$\underbrace{\overline{11\cdots1}}_{81\text{ digits}} = 10^{80} + 10^{79} + \cdots + 10 + 1$$

$$= (10^{80} - 1) + (10^{79} - 1) + \cdots + (10 - 1) + 81$$

$$= \underbrace{\overline{99\cdots9}}_{80\text{ digits}} + \underbrace{\overline{99\cdots9}}_{79\text{ digits}} + \cdots + 9 + 81$$

$$= 9\times\left(\underbrace{\overline{11\cdots1}}_{80\text{ digits}} + \underbrace{\overline{11\cdots1}}_{79\text{ digits}} + \cdots + 1\right) + 81$$

Let $x = \underbrace{\overline{11\cdots1}}_{80\text{ digits}} + \underbrace{\overline{11\cdots1}}_{79\text{ digits}} + \cdots + 1 \equiv 80 + 79 + \cdots + 1 \pmod 9$

$x \equiv \dfrac{81}{2}\cdot 80 \equiv 9\times 40 \equiv 0 \bmod 9$

$x = 9m$ for some integer m

$\underbrace{\overline{11\cdots1}}_{81\text{ digits}} = 9\times 9m + 81 \equiv 0 \pmod{81}$

The remainder = 0

G1.4 Given a sequence of real numbers $a_1, a_2, a_3, \ldots$ that satisfy

1) $a_1 = \frac{1}{2}$, and

2) $a_1 + a_2 + \ldots + a_k = k^2 a_k$, for $k \ge 2$.

Determine the value of a_{100}.

Reference: 2013 HG10

$\frac{1}{2} + a_2 = 2^2 a_2 \Rightarrow a_2 = \frac{1}{2\times 3} = \frac{1}{6}$

$\frac{1}{2} + \frac{1}{6} + a_3 = 3^2 a_3 \Rightarrow a_3 = \frac{1}{3\times 4} = \frac{1}{12}$

Claim: $a_n = \frac{1}{n\times(n+1)}$ for $n \ge 1$

Pf: By M.I., n = 1, 2, 3, proved already.

Suppose $a_k = \frac{1}{k\times(k+1)}$ for some $k \ge 1$

$\frac{1}{2} + \frac{1}{6} + \ldots + \frac{1}{k\times(k+1)} + a_{k+1} = (k+1)^2 a_{k+1}$

$\left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \cdots + \left(\frac{1}{k}-\frac{1}{k+1}\right) = (k^2+2k)a_{k+1}$

$a_{k+1} = \frac{1}{k(k+2)}\cdot\left(1-\frac{1}{k+1}\right) = \frac{1}{(k+1)(k+2)}$ ∴ The statement is true for all $n \ge 1$

$a_{100} = \frac{1}{100\times 101} = \frac{1}{10100}$

Group Event 2

G2.1 By removing certain terms from the sum, $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}$, we can get 1. What is the product of the removed term(s)?

$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{12}=\frac{3}{4}+\frac{1}{4}=1$

The removed terms are $\frac{1}{8};\frac{1}{10}$.

Product $=\frac{1}{80}$

G2.2 If $\boldsymbol{S_n} = 1 - 2 + 3 - 4 + \ldots + (-1)^{n-1}\boldsymbol{n}$, where $\boldsymbol{n}$ is a positive integer, determine the value of $\boldsymbol{S}_{17} + \boldsymbol{S}_{33} + \boldsymbol{S}_{50}$.

If $\boldsymbol{n} = 2\boldsymbol{m}$, where $\boldsymbol{m}$ is a positive integer,

$\boldsymbol{S}_{2m} = (1 - 2) + (3 - 4) + \ldots + (2\boldsymbol{m} - 1 - 2\boldsymbol{m}) = -\boldsymbol{m}$

$\boldsymbol{S}_{2m+1} = -\boldsymbol{m} + 2\boldsymbol{m} + 1 = \boldsymbol{m} + 1$

$\boldsymbol{S}_{17} + \boldsymbol{S}_{33} + \boldsymbol{S}_{50} = 9 + 17 - 25 = 1$

G2.3 Six persons ***A***, ***B***, ***C***, ***D***, ***E*** and ***F*** are to rotate for night shifts in alphabetical order with ***A*** serving on the first Sunday, ***B*** on the first Monday and so on. In the fiftieth week, which day does ***A*** serve on? (Represent Sunday by 0, Monday by 1, … , Saturday by 6 in your answer.)

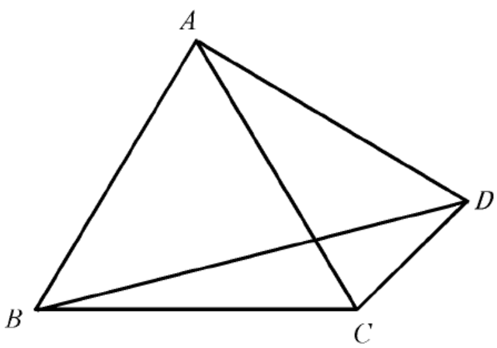
$50\times7 = 350 = 6\times58 + 2$

B serves on Saturday in the fiftieth week.

A serves on Friday in the fiftieth week.

Answer 5.

G2.4 In the figure, vertices of equilateral triangle ***ABC*** are connected to ***D*** in straight line segments with ***AB*** = ***AD***. If $\angle \boldsymbol{BDC} = \alpha°$, determine the value of α.



Reference: 2003 HG8, 2011 HG9

Use ***A*** as centre, ***AB*** as radius to draw a circle to pass through ***B***, ***C***, ***D***.

$\angle \boldsymbol{BAC} = 2\angle \boldsymbol{BDC}$ (∠ at centre twice ∠ at ⊙[ce])

$60° = 2\alpha°$

$\alpha = 30$

Group Event 3

G3.1 Determine the value of the product $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{10^2}\right)$.

Reference: 1986 FG10.4, 1999 FIS.4

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{10^2}\right)=\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{10}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{10}\right)$$

$$=\left(\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{9}{10}\right)\cdot\left(\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{11}{10}\right)=\frac{1}{10}\times\frac{11}{2}=\frac{11}{20}$$

G3.2 Determine the value of the sum $\frac{1}{\log_2 100!}+\frac{1}{\log_3 100!}+\frac{1}{\log_4 100!}+\cdots+\frac{1}{\log_{100} 100!}$,

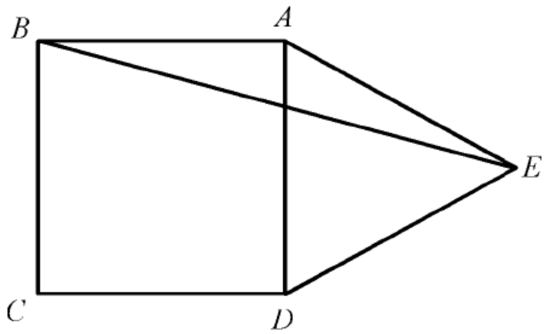
where 100! = 100×99×98×…×3×2×1.

$$\frac{1}{\log_2 100!}+\frac{1}{\log_3 100!}+\frac{1}{\log_4 100!}+\cdots+\frac{1}{\log_{100} 100!}$$

$$=\frac{\log 2}{\log 100!}+\frac{\log 3}{\log 100!}+\frac{\log 4}{\log 100!}+\cdots+\frac{\log 100}{\log 100!}$$

$$=\frac{\log 100!}{\log 100!}=1$$

G3.3 In the figure, ***ABCD*** is a square, ***ADE*** is an equilateral triangle and ***E*** is a point outside of the square ***ABCD***. If $\angle$***AEB*** = α°, determine the value of α. (**Reference: 1991 FI1.1)**



$\alpha° = \frac{180^\circ - 90^\circ - 60^\circ}{2}$ ($\angle$s sum of isos. Δ)

α = 15

G3.4 Fill the white squares in the figure with distinct non-zero digits so that the arithmetical expressions, read both horizontally and vertically, are correct. What is the value of α?

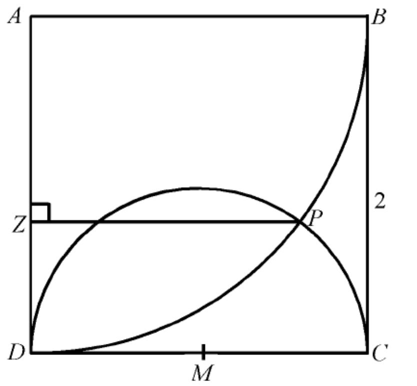
α = 5

	÷		=	
+		×		
	+		=	*α*
=		=		

6	÷	2	=	3
+		×		
1	+	4	=	5
=		=		
7		8		

Group Event 4

G4.1 In the figure below, $ABCD$ is a square of side length 2. A circular arc with centre at A is drawn from B to D. A semicircle with centre at M, the midpoint of CD, is drawn from C to D and sits inside the square. Determine the shortest distance from P, the intersection of the two arcs, to side AD, that is, the length of PZ.



Join AP, DP, MP.

Let F be the foot of perpendicular from P to CD.

Let $AZ = x$. Then $DZ = 2 - x = PF$, $DM = MP = 1$

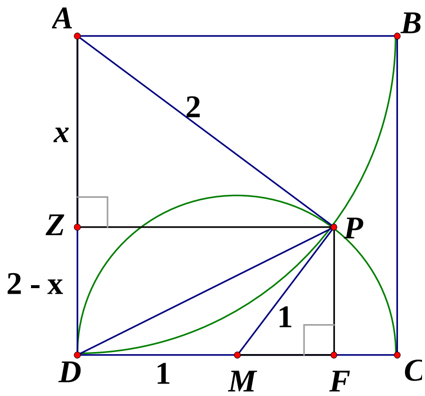
In ΔAZP, $AZ^2 + ZP^2 = AP^2$ (Pythagoras' theorem)

$ZP^2 = 4 - x^2$ …… (1)

In ΔPMF, $MF^2 + PF^2 = PM^2$ (Pythagoras' theorem)

$MF^2 = 1 - (2 - x)^2 = 4x - x^2 - 3$ …… (2)

$PZ = DF = 1 + MF$



$4 - x^2 = (1 + \sqrt{4x - x^2 - 3})^2$

$4 - x^2 = 1 + 2\sqrt{4x - x^2 - 3} + 4x - x^2 - 3$

$3 - 2x = \sqrt{4x - x^2 - 3}$

$9 - 12x + 4x^2 = 4x - x^2 - 3$

$5x^2 - 16x + 12 = 0$

$(5x - 6)(x - 2) = 0$

$x = \frac{6}{5}$ or 2 (rejected)

$PZ = \sqrt{4 - x^2} = \sqrt{4 - 1.2^2} = \sqrt{2.56} = 1.6$

Method 2 Let D be the origin, DC be the x-axis, DA be the y-axis.

Equation of circle DPC: $(x - 1)^2 + y^2 = 1 \Rightarrow x^2 - 2x + y^2 = 0$ …… (1)

Equation of circle BPD: $x^2 + (y - 2)^2 = 2^2 \Rightarrow x^2 + y^2 - 4y = 0$ …… (2)

$(1) - (2) \Rightarrow y = \frac{x}{2}$ …… (3)

Sub. (3) into (1): $x^2 - 2x + \frac{x^2}{4} = 0 \Rightarrow x = 0$ (rejected) or $1.6 \Rightarrow PZ = 1.6$

G4.2 If $x = \frac{\sqrt{5}+1}{2}$ and $y = \frac{\sqrt{5}-1}{2}$, determine the value of $x^3y + 2x^2y^2 + xy^3$.

$xy = \frac{5-1}{4} = 1$

$x^3y + 2x^2y^2 + xy^3 = xy(x^2 + 2xy + y^2) = x^2 + y^2 + 2 = \frac{1}{4}(5+1+5+1)+2 = 5$

G4.3 If a, b, c and d are distinct digits and $\begin{array}{r} abcd \\ -\,daabc \\ \hline 2014d \end{array}$, determine the value of d.

Consider the unit digit subtraction, $c = 0$ and there is no borrow digit in the tens digit.

Consider the tens digit, $10 + 0 - b = 4 \Rightarrow b = 6$ and there is a borrow digit in the hundreds.

Consider the hundreds digit, $5 - a = 1 \Rightarrow a = 4$ and there is no borrow digit in the thousands.

Consider the ten thousands digit, $4 - d = 2 \Rightarrow d = 2$; Check: $\begin{array}{r} 44602 \\ -24460 \\ \hline 20142 \end{array}$

G4.4 Determine the product of all real roots of the equation $x^4 + (x - 4)^4 = 32$. **Reference: 2017 FG3.3**

Let $t = x - 2$, then the equation becomes $(t + 2)^4 + (t - 2)^4 = 32$

$2(t^4 + 24t^2 + 16) = 32$

$t^4 + 24t^2 = 0$

$t^2 = 0$ or -24 (rejected)

$t = 0 \Rightarrow x = 2$ (repeated root)

Product of all real roots = $2\times2 = 4$

Individual Events

I1	α	10	**I2**	α	7	**I3**	α	*686 see the remark	**I4**	α	3
	β	90		β	5		β	236328		β	2
	γ	10		γ	2		γ	*15 see the remark		γ	7
	δ	2047		δ	–56		δ	$\frac{15}{4}$		δ	1

Group Events

G1		$\frac{3}{5}$	**G2**		417	**G3**		$\sqrt{10}$	**G4**		1
		15			23			0			625
		34			–3			$\frac{1+\sqrt{5}}{2}$			1
		15			$12\sqrt{3}$			10			$\frac{3-\sqrt{5}}{2}$

Individual Event 1

I1.1 If $\left|x+\sqrt{5}\right|+\left|y-\sqrt{5}\right|+\left|z\right|=0$, determine $\alpha = x^2 + y^2 + z^2$.

Reference: 2005 FI4.1, 2006 FI4.2, 2009 FG1.4, 2013 FI1.4, 2015 HG4, 2015 FI1.1

Sum of non-negative terms = 0 ⇒ each term = 0 at the same time

$x+\sqrt{5}=0$ and $y-\sqrt{5}=0$ and $z = 0$

$x=-\sqrt{5}$ and $y=\sqrt{5}$ and $z = 0$

$\alpha = x^2 + y^2 + z^2 = 5 + 5 + 0 = 10$

I1.2 If β is the sum of all digits of the product $\underbrace{11111\cdots11}_{\alpha\ 1\text{'s}}\times\underbrace{99999\cdots99}_{\alpha\ 9\text{'s}}$, determine the value of β.

Reference: 2000 FI4.4

Observe the patterns $11 \times 99 = 1089$; $111\times999 = 110889$.

Deductively, $\underbrace{11111\cdots11}_{10\ 1\text{'s}}\times\underbrace{99999\cdots99}_{10\ 9\text{'s}} = \underbrace{11111\cdots11}_{9\ 1\text{'s}}0\underbrace{88888\cdots88}_{9\ 9\text{'s}}9$

β = the sum of all digits = 9 + 9×8 + 9 = 90

I1.3 Suppose that the real function $f(x)$ satisfies $f(xy) = f(x)\,f(y)$ for all real numbers x and y, and $f(1) < 1$. Determine the value of $\gamma = f(\beta) + 100 - \beta$.

Reference: 2013 FI4.1

$f(1) = f(1)\,f(1)$

$\Rightarrow f(1)[f(1) - 1] = 0$

$\Rightarrow f(1) = 0$ or 1 (rejected)

$\because f(1) = 0$

$f(x) = f(1\times x) = f(1)f(x) = 0$ for all real values of x.

$\gamma = f(\beta) + 100 - \beta = 0 + 100 - 90 = 10$

I1.4 If n is a positive integer and $f(n) = 2^n + 2^{n-1} + 2^{n-2} + \ldots + 2^2 + 2^1 + 1$, determine the value of $\delta = f(\gamma)$.

Reference: 2009 FI1.3, 2017 FI3.4

$f(n) = 2^{n+1} - 1$ (sum to n terms of a G.S. $a = 1$, $r = 2$, no. of terms = $n + 1$)

$\delta = f(10) = 2^{11} - 1 = 2047$

Individual Event 2

I2.1 If x_0, y_0, z_0 is a solution to the simultaneous equations below, $\begin{cases} x - y - z = -1 \\ y - x - z = -2 \\ z - x - y = -4 \end{cases}$

determine the value of $\alpha = x_0 + y_0 + z_0$.

(1) + (2) + (3): $-(x + y + z) = -7$

$\alpha = 7$

I2.2 If β is the reminder of $\underbrace{111\cdots111}_{100\ 1's} \div \alpha$, determine the value of β.

$111111 \div 7 = 15873$

$\underbrace{111\cdots111}_{100\ 1's} = \underbrace{111\cdots111}_{96\ 1's}0000 + 1111$

$= 7m + 7\times158 + 5$, where m is an integer

$\beta = 5$

I2.3 If γ is the remainder of $[(\beta - 2)^{100} + \beta^{50} + (\beta + 2)^{25}] \div 3$, determine the value of γ.

$3^{100} + 5^{50} + 7^{25} = 3^{100} + (6 - 1)^{50} + (6 + 1)^{25}$

$= 3^{100} + 6^n + 1 + 6^m + 1$, where m and n are integers

$\gamma = 2$

I2.4 If the equation $x^4 + ax^2 + bx + \delta = 0$ has four real roots with three of them being 1, γ and γ^2, determine the value of δ.

Reference: 2013 FI4.3

Let the fourth root be t.

$1 + 2 + 2^2 + t = \text{sum of roots} = -\dfrac{\text{coefficient of } x^3}{\text{coefficient of } x^4} = 0$

$t = -7$

$1\times2\times2^2\times(-7) = \text{product of roots} = \dfrac{\text{constant term}}{\text{coefficient of } x^4} = \delta$

$\delta = -56$

Individual Event 3

I3.1 Of the positive integers from 1 to 1000, including 1 and 1000, there are α of them that are not divisible by 5 or 7. Determine the value of α.

Reference: 1993 FG8.3-4, 1994 FG8.1-2, 1998 HI6

Numbers divisible by 5: 5, 10, 15, $\cdots$, 1000, there are 200 numbers

Numbers divisible by 7: 7, 14, 21, $\cdots$, 994, there are 142 numbers

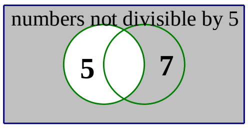
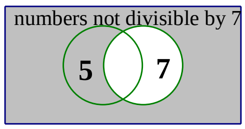
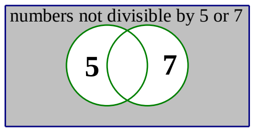
Numbers divisible by 35: 35, 70, $\cdots$, 980, there are 28 numbers

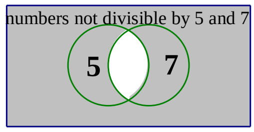
Numbers divisible by 5 or 7 = 200 + 142 – 28 = 314

Numbers that are not divisible by 5 or 7 = 1000 – 314 = 686

Remark: The original question is:

Of the positive integers from 1 to 1000, including 1 and 1000, there are α of them that are not divisible by 5 or not divisible by 7. Determine the value of α.

<table>
<tr>
<td>
</td>
<td>
</td>
<td>

'numbers neither divisible by 5 nor 7'
'numbers not divisible by 5 and numbers not divisible by 7'</td>
<td>

'numbers not divisible by 5 or not divisible by 7'</td>
</tr>
</table>

I3.2 Determine the value of $\beta = 1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^\alpha (\alpha + 1)^2$.

Reference: 1997 HI5, 2002 FG2.3, 2004 HI1, 2015 FG4.1

$1^2 - 2^2 + 3^2 - 4^2 + \cdots + 685^2 - 686^2 + 687^2$

$= 1 + (3^2 - 2^2) + (5^2 - 4^2) + \cdots + (687^2 - 686^2)$

$= 1 + (3 + 2)(3 - 2) + (5 + 4)(5 - 4) + \cdots + (687 + 686)(687 - 686)$

$= 1 + 5 + 9 + \cdots + 1373$ (sum of 344 terms of an A.S., $\boldsymbol{a} = 1$, $\boldsymbol{d} = 4$)

$= \frac{1+1373}{2} \times 344$

$= 236328$

I3.3 If γ is the remainder of β divided by the 1993rd term of the following sequence:

1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, $\cdots$. Determine the value of γ.

$1 + 2 + 3 + \cdots + 62 = \frac{1+62}{2} \times 62 = 1953$ and $1993 - 1953 = 40 < 63$

The 1993rd term of the sequence is 63.

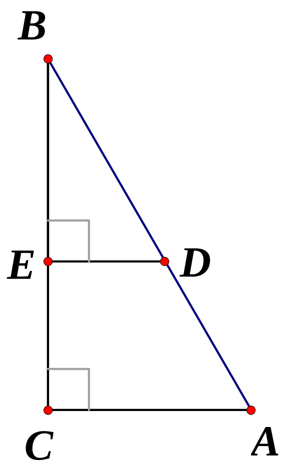
236328 ÷ 63, by division, the remainder is 15.

Remark: The original question is:

Determine the remainder of β divided by the 1993rd term of the following sequence:

1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, $\cdots$. γ is not mentioned.

I3.4 In the figure below, $BE = AC$, $BD = \frac{1}{2}$ and $DE + BC = 1$.



If δ is γ times the length of ED, determine the value of δ.

Let $DE = x$, $BE = y$

Then $AC = y$, $BC = 1 - x$

It is easy to show that $\Delta BED \sim \Delta BCA$ (equiangular)

$\frac{DE}{BE} = \frac{AC}{BC}$ (cor. sides, ~Δ' s) $\Rightarrow \frac{x}{y} = \frac{y}{1-x} \Rightarrow y^2 = x(1 - x)$

$BE^2 + DE^2 = BD^2$ (Pythagoras' theorem)

$y^2 + x^2 = \frac{1}{4} \Rightarrow x(1 - x) + x^2 = \frac{1}{4} \Rightarrow x = \frac{1}{4}$

$\delta = \frac{15}{4}$

Individual Event 4

I4.1 Let α be the remainder of 2^{1000} divided by 13, determine the value of α.

Reference: 1972 American High School Mathematics Examination Q31, 2011 HI1

$13\times5 = 64 + 1 \Rightarrow 2^6 = 13\times5 - 1$

$2^{1000} = 2^4\cdot2^{996} = 16\cdot(2^6)^{166} = (13 + 3)\cdot(13\times5 - 1)^{166}$

$= (13 + 3)\cdot(13\boldsymbol{m} + 1)$, by using binomial theorem

$= 13\boldsymbol{n} + 3$, where $\boldsymbol{n}$ and $\boldsymbol{m}$ are integers

$\alpha = 3$

I4.2 Determine the value of $\beta = \dfrac{\left(7+4\sqrt{\alpha}\right)^{\frac{1}{2}} - \left(7-4\sqrt{\alpha}\right)^{\frac{1}{2}}}{\sqrt{\alpha}}$.

Reference: 2013 FI3.1

$\sqrt{7+4\sqrt{3}} = \sqrt{7+2\sqrt{12}} = \sqrt{4+3+2\sqrt{4\times3}} = \sqrt{4}+\sqrt{3} = 2+\sqrt{3}$

$\sqrt{7-4\sqrt{3}} = 2-\sqrt{3}$

$\beta = \dfrac{\left(7+4\sqrt{3}\right)^{\frac{1}{2}} - \left(7-4\sqrt{3}\right)^{\frac{1}{2}}}{\sqrt{3}} = \dfrac{2+\sqrt{3}-2+\sqrt{3}}{\sqrt{3}} = 2$

I4.3 If $\boldsymbol{f}(\boldsymbol{a}) = \boldsymbol{a} - \beta$ and $\boldsymbol{F}(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{b}^2 + \boldsymbol{a}$, determine the value of $\gamma = \boldsymbol{F}(3, \boldsymbol{f}(4))$.

Reference: 1985 FI3.3, 1990 HI3, 2013 FI3.2

$\boldsymbol{f}(4) = 4 - 2 = 2$

$\gamma = \boldsymbol{F}(3, \boldsymbol{f}(4)) = \boldsymbol{F}(3, 2) = 2^2 + 3 = 7$

I4.4 If δ is the product of all real roots of $\boldsymbol{x}^{\log_\gamma \boldsymbol{x}} = 10$, determine the value of δ.

$\boldsymbol{x}^{\log_7 \boldsymbol{x}} = 10$

$\log_7 \boldsymbol{x} \log \boldsymbol{x} = \log 10$

$\dfrac{(\log \boldsymbol{x})^2}{\log 7} = 1$

$\log \boldsymbol{x} = \pm\sqrt{\log 7}$

$\boldsymbol{x} = 10^{\sqrt{\log 7}}$ or $10^{-\sqrt{\log 7}}$

Product of roots $= 10^{\sqrt{\log 7}} \times 10^{-\sqrt{\log 7}} = 1$

Group Event 1

G1.1 Simplify $\left(\frac{1\times3\times9+2\times6\times18+\cdots+\boldsymbol{n}\times3\boldsymbol{n}\times9\boldsymbol{n}}{1\times5\times25+2\times10\times50+\cdots+\boldsymbol{n}\times5\boldsymbol{n}\times25\boldsymbol{n}}\right)^{\frac{1}{3}}$.

Reference: 2000 FI5.1

$$\left(\frac{1\times3\times9+2\times6\times18+\cdots+\boldsymbol{n}\times3\boldsymbol{n}\times9\boldsymbol{n}}{1\times5\times25+2\times10\times50+\cdots+\boldsymbol{n}\times5\boldsymbol{n}\times25\boldsymbol{n}}\right)^{\frac{1}{3}}$$

$$=\left[\frac{3^3\left(1^3+2^3+\cdots+\boldsymbol{n}^3\right)}{5^3\left(1^3+2^3+\cdots+\boldsymbol{n}^3\right)}\right]^{\frac{1}{3}}$$

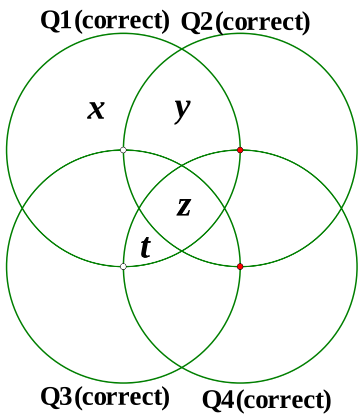
$$=\frac{3}{5}$$

G1.2 Among 50 school teams joining the HKMO, no team answered all four questions correctly in the paper of a group event. If the first question was solved by 45 teams, the second by 40 teams, the third by 35 teams and the fourth by 30 teams. How many teams solved both the third and the fourth questions?

$\because$ No team answered all four questions correctly

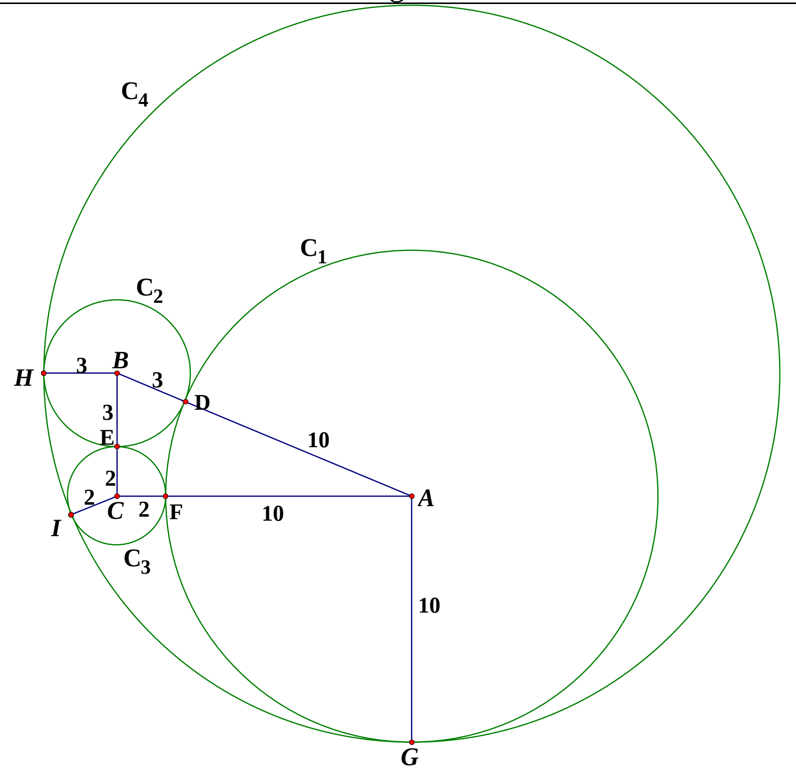
$\therefore$ Each team can solve at most three questions.
The maximum number of solved questions = 50×3 = 150
The actual number of solved questions = 45 + 40 + 35 + 30 = 150
$\therefore$ Each team can solve exactly three questions.
Number of teams that cannot solve the first question = (50 – 45) teams = 5 teams
$\Rightarrow$ These 5 teams can solve Q2, Q3 and Q4 but not Q1.
Number of teams that cannot solve the second question = (50 – 40) teams =10 teams
$\Rightarrow$ These 10 teams can solve Q1, Q3 and Q4 but not Q2.
Number of teams that cannot solve the third question = (50 – 35) teams = 15 teams
$\Rightarrow$ These 15 teams can solve Q1, Q2 and Q4 but not Q3.
Number of teams that cannot solve the fourth question = (50 – 30) teams = 20 teams
$\Rightarrow$ These 20 teams can solve Q1, Q2 and Q3 but not Q4.
Number of school teams solved both the third and the fourth questions = 5 + 10 = 15

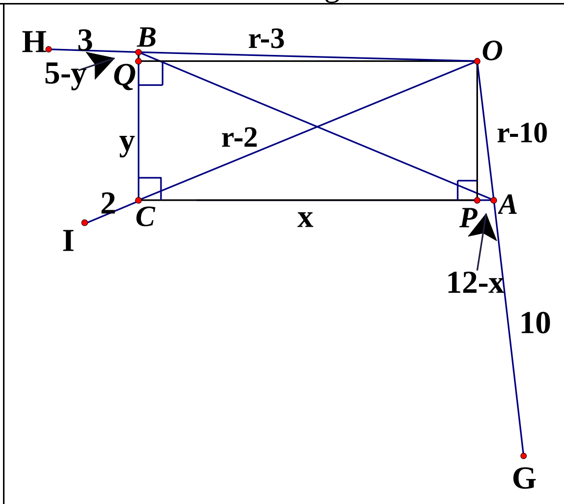
Remark We cannot use the Venn diagram on the right with explanation below:
$\boldsymbol{x}$ = school teams that can solve Q1 but not Q2, Q3 nor Q4.
$\boldsymbol{y}$ = school teams that can solve Q1, Q2 but not Q3 nor Q4.
$\boldsymbol{z}$ = school teams that can solve all four questions.
$\boldsymbol{t}$ = school teams that can solve Q1, Q3 and Q4 but not Q2
However, we could not find any part in Venn diagram representing school teams that can solve Q1, Q4 but not Q2 nor Q3 !!!



G1.3 Let $\boldsymbol{n}$ be the product 3659893456789325678 and 342973489379256. Determine the number of digits of $\boldsymbol{n}$. **(Reference: 2013 FG4.1)**
Let $\boldsymbol{x}$ = 3 659 893 456 789 325 678, $\boldsymbol{y}$ = 342 973 489 379 256
$\boldsymbol{x} = 3.7\times10^{18}$, $\boldsymbol{y} = 3.4\times10^{14}$ (correct to 2 sig. fig.)
$\boldsymbol{n} = \boldsymbol{xy} = 3.7\times10^{18}\times3.4\times10^{14} = 12.58\times10^{32} = 1.258\times10^{33}$
The number of digits of $\boldsymbol{n}$ is 34.

G1.4 Three circles of radii 2, 3 and 10 units are placed inside another big circle in such a way that all circles are touching one another. Determine the value of the radius of the big circle.





Let ***A*** be the centre of circle $\boldsymbol{C}_1$ with radius 10, ***B*** be the centre of circle $\boldsymbol{C}_2$ with radius 3, ***C*** be the centre of circle $\boldsymbol{C}_3$ with radius 2. Join ***AB***, ***BC***, ***AC***.

Suppose $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ touch each other at ***D***, $\boldsymbol{C}_2$ and $\boldsymbol{C}_3$ touch each other at ***E***, $\boldsymbol{C}_3$ and $\boldsymbol{C}_1$ touch each other at ***F***. Then ***A***, ***D***, ***B*** are collinear, ***B***, ***E***, ***C*** are collinear, ***C***, ***F***, ***A*** are collinear.

$AB = 10 + 3 = 13$, $BC = 3 + 2 = 5$, $AC = 10 + 2 = 12$

$BC^2 + AC^2 = 5^2 + 12^2 = 25 + 144 = 169 = 13^2 = AB^2$

$\therefore \angle ACB = 90°$ (converse, Pythagoras' theorem)

Let ***O*** be the centre of circle $\boldsymbol{C}_4$ with radius ***r*** circumscribing all three circles $\boldsymbol{C}_1$, $\boldsymbol{C}_2$, $\boldsymbol{C}_3$ at ***G***, ***H*** and ***I*** respectively. Then ***O***, ***A***, ***G*** are collinear, ***O***, ***B***, ***H*** are collinear, ***O***, ***C***, ***I*** are collinear.

$AG = 10$, $BH = 3$, $CI = 2$, $OA = r - 10$, $OB = r - 3$, $OC = r - 2$.

Let ***P*** and ***Q*** be the feet of perpendiculars drawn from ***O*** onto ***AC*** and ***AB*** respectively.

Then ***OPCQ*** is a rectangle.

Let $CP = x = QO$ (opp. sides of rectangle), $CQ = y = PO$ (opp. sides of rectangle)

$AP = 12 - x$, $BQ = 5 - y$.

In ΔOCP, $x^2 + y^2 = (r - 2)^2$ ...... (1) (Pythagoras' theorem)

In ΔOAP, $(12 - x)^2 + y^2 = (r - 10)^2$ ...... (2) (Pythagoras' theorem)

In ΔOBQ, $x^2 + (5 - y)^2 = (r - 3)^2$ ...... (3) (Pythagoras' theorem)

(1) – (2): $24x - 144 = 16r - 96 \Rightarrow x = \dfrac{2r+6}{3}$ ...... (4)

(1) – (3): $10y - 25 = 2r - 5 \Rightarrow y = \dfrac{r+10}{5}$ ...... (5)

Sub. (4) and (5) into (1): $\left(\dfrac{2r+6}{3}\right)^2 + \left(\dfrac{r+10}{5}\right)^2 = (r-2)^2$

$25(4r^2 + 24r + 36) + 9(r^2 + 20r + 100) = 225(r^2 - 4r + 4)$

$116r^2 - 1680r - 900 = 0 \Rightarrow 29r^2 - 420r - 225 = 0$

$(r - 15)(29r + 15) = 0 \Rightarrow r = 15$, the radius of the big circle is 15.

Group Event 2

G2.1 On a 3×3 grid of 9 squares, each squares is to be painted with either Red or Blue. If α is the total number of possible colouring in which no 2×2 grid consists of only Red squares, determine the value of α.

If there is no restriction, number of possible colouring = $2^9 = 512$

(1)If all 9 squares are painted as red, number of colouring = 1

(2)If there are exactly three 2×2 grid consists of only Red squares, possible pattern may be:

R	R	B
R	R	R
R	R	R

90° rotation gives another possible pattern
Number of colouring = 4

(3)If there are exactly two 2×2 grid consists of only Red squares, possible pattern may be:

B	R	R
R	R	R
R	R	B

R	R	B
R	R	R
B	R	R

Number of colouring = 2

(4)

R	R	R
R	R	B
R	R	R

R	R	R
R	R	B
R	R	B

R	R	B
R	R	R
R	R	B

R	R	B
R	R	B
R	R	R

R	R	B
R	R	B
R	R	B

90° rotation gives another possible pattern
Number of colouring
= 4×5 = 20

(5)If there is exactly one 2×2 grid consists of only Red squares, possible pattern may be:

B	B	B
R	R	B
R	R	B

90° rotation gives another possible pattern
Number of colouring = 4

(6)

R	B	B
R	R	B
R	R	B

B	R	B
R	R	B
R	R	B

B	B	R
R	R	B
R	R	B

B	B	B
R	R	R
R	R	B

B	B	B
R	R	B
R	R	R

90° rotation gives another possible pattern
Number of colouring
= 4×5 = 20

(7)

R	B	R
R	R	B
R	R	B

R	B	B
R	R	R
R	R	B

R	B	B
R	R	B
R	R	R

B	R	R
R	R	B
R	R	B

B	R	B
R	R	R
R	R	B

90° rotation gives another possible pattern
Number of colouring
= 4×5 = 20

(8)

B	R	B
R	R	B
R	R	R

B	B	R
R	R	R
R	R	B

B	B	R
R	R	B
R	R	R

90° rotation gives another possible pattern
Number of colouring
= 4×3 = 12

(9)

B	R	R
R	R	B
R	R	R

R	B	R
R	R	B
R	R	R

R	B	R
R	R	R
R	R	B

90° rotation gives another possible pattern
Number of colouring
= 4×3 = 12

∴ Total number of possible colouring in which no 2×2 grid consists of only Red squares
= No restriction – all 9 red squares – exactly 3 2×2 red grid – exactly 2 2×2 red grid – exactly 1 2×2 red grid
= 512 – 1 – 4 – 2 – 20 – 4 – 20 – 20 – 12 – 12 = 417

Method 2 (a)All 9 blue squares = 1 pattern. (b)8 blue squares + 1 red squares = 9 patterns.
(c)7B+2R = $\boldsymbol{C}_2^9$ = 36 patterns, (d)6B+3R = $\boldsymbol{C}_3^9$ = 84 patterns, (e)5B+4R = $\boldsymbol{C}_4^9 - 4$ = 122 patterns
(f)4B+5R = $\boldsymbol{C}_5^9 - 4\times 5$ = 106 patterns, (g)3B+6R = $\boldsymbol{C}_6^9 - 4\times\boldsymbol{C}_2^5 + 4$ = 48 patterns
(h)2B+7R = 8 + 2 = 10 patterns, (i)1B+8R = 1 pattern
Total number of different patterns = 1 + 9 + 36 + 84 + 122 + 106 + 48 + 10 + 1 = 417

G2.2 If the sum of 25 consecutive positive integers is the product of 3 prime numbers, what is the minimum sum of these 3 prime numbers?

Let the smallest positive integer be $\boldsymbol{x}$. We use the formula: $S(n) = \frac{n}{2}\left[2a + (n-1)d\right]$.

$\frac{25}{2}(2x + 24\times 1) = 25(x + 12) = 5\times 5\times(x + 12)$ = product of 3 prime numbers

The minimum prime for $x + 12$ is 13. The minimum sum of these 3 prime numbers is 23.

G2.3 Determine the sum of all real roots of the following equation $|x + 3| - |x - 1| = x + 1$.

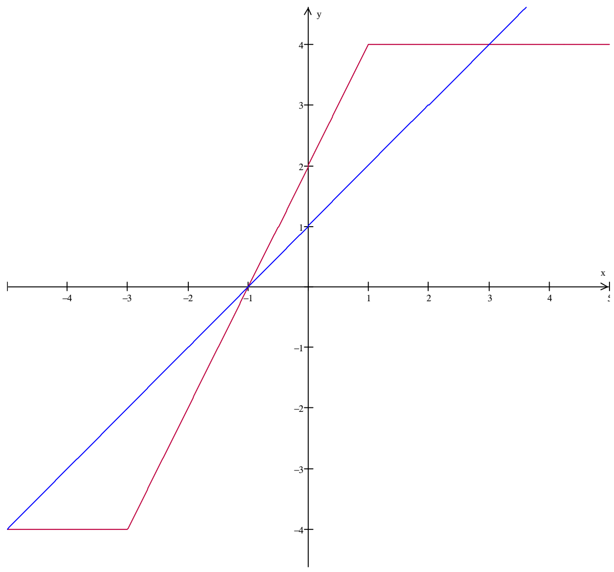
When $x \le -3$, the equation becomes $-x - 3 - (1 - x) = x + 1 \Rightarrow x = -5$

When $-3 < x \le 1$, the equation becomes $x + 3 - (1 - x) = x + 1 \Rightarrow x = -1$

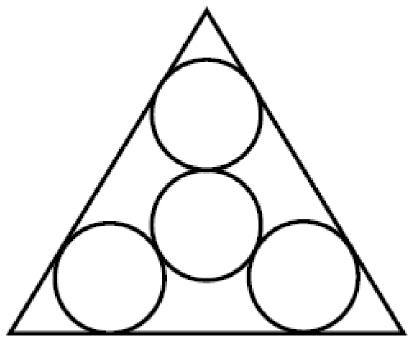
When $1 < x$, the equation becomes $x + 3 - (x - 1) = x + 1 \Rightarrow x = 3$

$\therefore$ Sum of all real roots $= -5 + (-1) + 3 = -3$

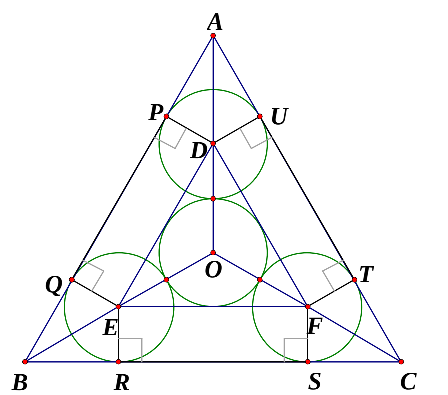
A graph is given below:



G2.4 In the figure below, there are 4 identical circles placed inside an equilateral triangle. If the radii of the circles are 1 unit, what is the value of the area of the triangle?



Let the triangle be ABC, O is the centre of the middle circle, D, E, F are the centres of the other 3 circles respectively. Let P, Q, R, S, T, U be the points of contact as shown.



$DP \perp AB$, $EQ \perp AB$, $ER \perp BC$, $FS \perp BC$, $FT \perp AC$, $DU \perp AC$ (tangent $\perp$ radius)

$DP = EQ = ER = FS = FT = DU = 1$ (radii)

$OD = OE = OF = 2$ (radii 1 + radii 1)

$\Delta ODE \cong \Delta OEF \cong \Delta OFD$ (S.S.S.)

$\angle DOE = \angle EOF = \angle FOD$ (corr. $\angle$s $\cong \Delta$s)

$\angle DOE + \angle EOF + \angle FOD = 360°$ ($\angle$s at a point)

$\therefore \angle DOE = \angle EOF = \angle FOD = 120°$

$DPQE$, $ERSF$, $FTUD$ are rectangles (opp. sides are eq. and //)

$DE = EF = FD = 2\times 2 \sin 60° = 2\sqrt{3} = PQ = RS = TU$

In ΔADU, $\angle DAU = 30°$, $DU = 1$, $DU \perp AU$,

$AU = 1 \tan 60° = \sqrt{3}$

$\therefore AB = BC = CA = 2\sqrt{3} + 2\sqrt{3} = 4\sqrt{3}$

Area of $\Delta ABC = \frac{1}{2}\cdot\left(4\sqrt{3}\right)^2 \sin 60° = 12\sqrt{3}$

Group Event 3

G3.1 Simplify $\sqrt{3+\sqrt{5}}+\sqrt{3-\sqrt{5}}$.

Reference: 1993 FI1.4, 1999 HG3, 2001 FG2.1, 2011 HI7, 2015 FI4.2

$$\sqrt{3+\sqrt{5}}=\sqrt{\frac{6+2\sqrt{5}}{2}}=\frac{1}{\sqrt{2}}\cdot\sqrt{5+2\sqrt{5}+1}=\frac{1}{\sqrt{2}}\cdot\left(\sqrt{5}+1\right)$$

$$\sqrt{3-\sqrt{5}}=\sqrt{\frac{6-2\sqrt{5}}{2}}=\frac{1}{\sqrt{2}}\cdot\sqrt{5-2\sqrt{5}+1}=\frac{1}{\sqrt{2}}\cdot\left(\sqrt{5}-1\right)$$

$$\sqrt{3+\sqrt{5}}+\sqrt{3-\sqrt{5}}=\frac{1}{\sqrt{2}}\cdot\left(\sqrt{5}+1\right)+\frac{1}{\sqrt{2}}\cdot\left(\sqrt{5}-1\right)=\frac{1}{\sqrt{2}}\cdot\left(2\sqrt{5}\right)=\sqrt{10}$$

G3.2 Let $\boldsymbol{p}$ be a prime and $\boldsymbol{m}$ be an integer. If $\boldsymbol{p}(\boldsymbol{p}+\boldsymbol{m})+2\boldsymbol{p}=(\boldsymbol{m}+2)^3$, find the greatest possible value of $\boldsymbol{m}$.

$\boldsymbol{p}(\boldsymbol{p}+\boldsymbol{m}+2)=(\boldsymbol{m}+2)^3$

If $\boldsymbol{m}$ is even and $\boldsymbol{p}$ is odd, then odd×(odd + even + 2) = (even + 2)^3 ⇒ LHS ≠ RHS !!!

If $\boldsymbol{m}$ is odd and $\boldsymbol{p}$ is odd, then odd×(odd + odd + 2) = (odd + 2)^3 ⇒ LHS ≠ RHS !!!

In all cases, $\boldsymbol{p}$ must be even.

∵ the only even prime is 2 ∴ $\boldsymbol{p}=2$

$2(\boldsymbol{m}+4)=(\boldsymbol{m}+2)^3$

LHS is even ⇒ $(\boldsymbol{m}+2)^3$ is even ⇒ $\boldsymbol{m}+2$ is even ⇒ RHS is divisible by 8 ⇒ LHS is divisible by 8

⇒ $\boldsymbol{m}+4=4\boldsymbol{n}$, where $\boldsymbol{n}$ is an integer ⇒ $\boldsymbol{m}+2=4\boldsymbol{n}-2$

Put $\boldsymbol{m}+2=4\boldsymbol{n}-2$ into the equation: $2(4\boldsymbol{n})=(4\boldsymbol{n}-2)^3$

$\boldsymbol{n}=(2\boldsymbol{n}-1)^3$

⇒ $\boldsymbol{n}=1$, $\boldsymbol{m}=0$ (This is the only solution, $\boldsymbol{n}<(2\boldsymbol{n}-1)^3$ for $\boldsymbol{n}>1$ and $\boldsymbol{n}>(2\boldsymbol{n}-1)^3$ for $\boldsymbol{n}<1$)

G3.3 Determine a root to $\boldsymbol{x}=\left(\boldsymbol{x}-\frac{1}{\boldsymbol{x}}\right)^{\frac{1}{2}}+\left(1-\frac{1}{\boldsymbol{x}}\right)^{\frac{1}{2}}$.

$$\boldsymbol{x}-\sqrt{1-\frac{1}{\boldsymbol{x}}}=\sqrt{\boldsymbol{x}-\frac{1}{\boldsymbol{x}}}\Rightarrow\left(\boldsymbol{x}-\sqrt{1-\frac{1}{\boldsymbol{x}}}\right)^2=\left(\sqrt{\boldsymbol{x}-\frac{1}{\boldsymbol{x}}}\right)^2$$

$$\boldsymbol{x}^2-2\boldsymbol{x}\sqrt{1-\frac{1}{\boldsymbol{x}}}+1-\frac{1}{\boldsymbol{x}}=\boldsymbol{x}-\frac{1}{\boldsymbol{x}}$$

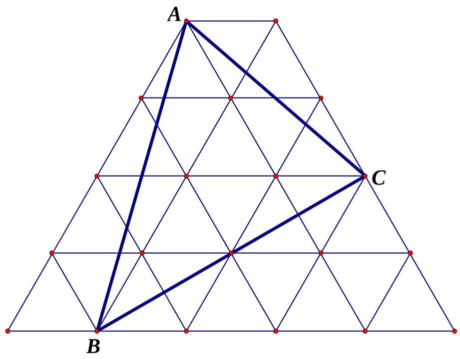
$$\boldsymbol{x}^2-\boldsymbol{x}+1=2\sqrt{\boldsymbol{x}^2-\boldsymbol{x}}\Rightarrow\left(\boldsymbol{x}^2-\boldsymbol{x}\right)-2\sqrt{\boldsymbol{x}^2-\boldsymbol{x}}+1=0$$

$$\left(\sqrt{\boldsymbol{x}^2-\boldsymbol{x}}-1\right)^2=0$$

$$\sqrt{\boldsymbol{x}^2-\boldsymbol{x}}=1\Rightarrow\boldsymbol{x}^2-\boldsymbol{x}-1=0$$

$$\boldsymbol{x}=\frac{1+\sqrt{5}}{2}\quad\text{or}\quad\frac{1-\sqrt{5}}{2}\quad(\text{rejected as } \boldsymbol{x}>0)$$

G3.4 In the figure below, the area of each small triangle is 1. Determine the value of the area of the triangle ***ABC***.



Total number of equilateral triangles = 24

Area of ***ABC***

$= 24 - \frac{1}{2}\cdot 4 - \frac{1}{2}\cdot 6 - 1 - \frac{1}{2}\cdot 4 - 6$

= 10

Group Event 4

G4.1 Let $\boldsymbol{b} = 1^2 - 2^2 + 3^2 - 4^2 + 5^2 - \ldots - 2012^2 + 2013^2$.

Reference: 1997 HI5, 2002 FG2.3, 2004 HI1, 2015 FI3.2

Determine the remainder of ***b*** divided by 2015.

$\boldsymbol{b} = 1 + (3-2)(3+2) + (5-4)(5+4) + \ldots + (2013-2012)(2012+2013)$

$\boldsymbol{b} = 1 + 5 + 9 + \ldots + 4025$

This is an arithmetic series with $\boldsymbol{a} = 1$, $\boldsymbol{d} = 4$.

$1 + (\boldsymbol{n} - 1)\times 4 = 4025$

$\Rightarrow \boldsymbol{n} = 1007$

$\boldsymbol{b} = \frac{1007}{2}(1 + 4025)$

$= 1007\times 2013$

$= 1007\times(2015 - 2)$

$= 1007\times 2015 - 2014$

$= 1006\times 2015 + 1$

Remainder = 1

G4.2 There are positive integers with leading digits being 6 and upon removing this leading digit, the resulting integer is $\frac{1}{25}$ of the original value. Determine the least of such positive integers.

Let the original number be ***x***.

$\boldsymbol{x} = 6\times 10^{\boldsymbol{n}} + \boldsymbol{y}$, where $\boldsymbol{y} < 10^{\boldsymbol{n}}$ and $\boldsymbol{y} = \frac{1}{25}\boldsymbol{x}$

$\boldsymbol{x} = 6\times 10^{\boldsymbol{n}} + \frac{1}{25}\boldsymbol{x}$

$24\boldsymbol{x} = 150\times 10^{\boldsymbol{n}}$

$4\boldsymbol{x} = 25\times 10^{\boldsymbol{n}}$

4 is not a factor of 25, so 4 must be a factor of $10^{\boldsymbol{n}}$

Least possible $\boldsymbol{n} = 2$

The least positive ***x*** is $25\times 10^2 \div 4 = 625$

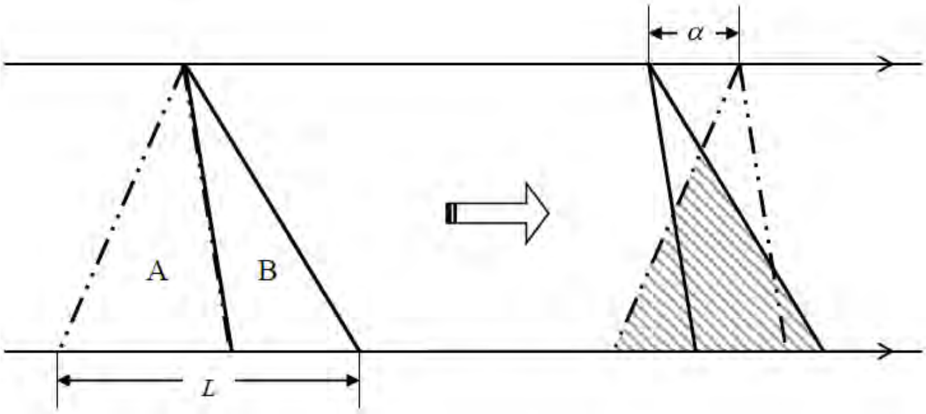
G4.3 If $\boldsymbol{x} + \frac{1}{\boldsymbol{x}} = 1$, determine the value of $\boldsymbol{x}^5 + \frac{1}{\boldsymbol{x}^5}$.

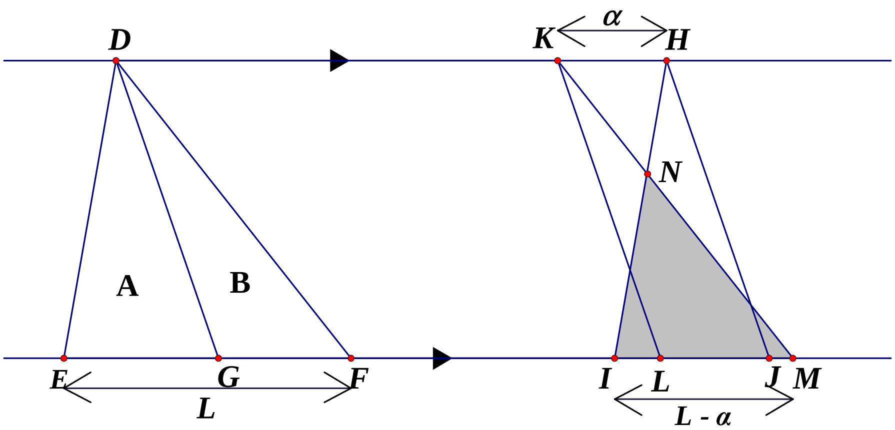
$\left(\boldsymbol{x} + \frac{1}{\boldsymbol{x}}\right)^2 = 1 \Rightarrow \boldsymbol{x}^2 + 2 + \frac{1}{\boldsymbol{x}^2} = 1 \Rightarrow \boldsymbol{x}^2 + \frac{1}{\boldsymbol{x}^2} = -1$

$\left(\boldsymbol{x} + \frac{1}{\boldsymbol{x}}\right)\left(\boldsymbol{x}^2 + \frac{1}{\boldsymbol{x}^2}\right) = 1\times(-1) = -1 \Rightarrow \boldsymbol{x}^3 + \frac{1}{\boldsymbol{x}^3} + \boldsymbol{x} + \frac{1}{\boldsymbol{x}} = -1 \Rightarrow \boldsymbol{x}^3 + \frac{1}{\boldsymbol{x}^3} = -2$

$\left(\boldsymbol{x}^2 + \frac{1}{\boldsymbol{x}^2}\right)\left(\boldsymbol{x}^3 + \frac{1}{\boldsymbol{x}^3}\right) = (-1)\times(-2) = 2 \Rightarrow \boldsymbol{x}^5 + \frac{1}{\boldsymbol{x}^5} + \boldsymbol{x} + \frac{1}{\boldsymbol{x}} = 2 \Rightarrow \boldsymbol{x}^5 + \frac{1}{\boldsymbol{x}^5} = 1$

G4.4 In the figure below, when triangle ***A*** shifts α units to the right, the area of shaded region is $\frac{\alpha}{L}$ times of the total area of the triangles ***A*** and ***B***. Determine the value of $\frac{\alpha}{L}$.





Let the original triangle be ***DEF***. ***G*** is a point on ***EF*** with ***EF*** = ***L***.

Δ***DEG*** is translated to Δ***HIJ*** by α units, Δ***DEG*** ≅ Δ***HIJ***, Δ***DGF*** ≅ Δ***KLM***, ***EF*** = ***L***, ***HK*** = α.

Let ***HI*** intersects ***KM*** at ***N***, ***IM*** = ***L*** – α

Consider Δ***DEF*** in the left figure and Δ***NIM*** in the right figure.

$\angle DEF = \angle NIM$ (corr. ∠s, ***DE*** // ***HI***)

$\angle DFE = \angle NMI$ (corr. ∠s, ***DF*** // ***KM***)

∴ $\Delta DEF \sim \Delta NIM$ (equiangular)

$\frac{S_{\Delta NIM}}{S_{\Delta DEF}} = \left(\frac{L-\alpha}{L}\right)^2$ (ratio of areas of ~Δs)

$\frac{\alpha}{L} = \left(1-\frac{\alpha}{L}\right)^2$ (Given $\frac{S_{\Delta NIM}}{S_{\Delta DEF}} = \frac{\alpha}{L}$)

$\left(\frac{\alpha}{L}\right)^2 - 3\left(\frac{\alpha}{L}\right)^2 + 1 = 0$, this is a quadratic equation in $\frac{\alpha}{L}$.

$\frac{\alpha}{L} = \frac{3+\sqrt{5}}{2}$ or $\frac{3-\sqrt{5}}{2}$

From the figure, $\frac{\alpha}{L} < 1$ and $\frac{3+\sqrt{5}}{2} > 1$

∴ $\frac{\alpha}{L} = \frac{3-\sqrt{5}}{2}$ only

Individual Events

I1			I2			I3			I4		
	a	15		***a***	3		***a***	5		***a***	0
	b	30		***b***	$\frac{1}{2}$		***b***	0		***b***	*4 see the remark
	c	11		***c***	4		***c***	–1		***c***	*2 see the remark
	d	979		***d***	24		***d***	4		***d***	0

Group Events

G1			G2			G3			G4		
	P	12		***A***	9		***K***	4		***d***	*72 see the remark
	Q	60		***B***	5		***L***	3		***u***	6
	n	11		***c***	5		***x***	52		***c***	$\frac{7}{13}$
	T	$*\frac{1}{2}\left(3^{2048}-1\right)$ see the remark		***d***	30		***y***	$\frac{13}{3}$		***x***	$\frac{1+\sqrt{5}}{2}$

Individual Event 1

I1.1 解方程 $\log_5 a + \log_3 a = \log_5 a \cdot \log_3 a$，其中 $a > 1$ 為實數。

Solve the equation $\log_5 a + \log_3 a = \log_5 a \cdot \log_3 a$ for real number $a > 1$.

$$\frac{\log a}{\log 5}+\frac{\log a}{\log 3}=\frac{\log a}{\log 5}\cdot\frac{\log a}{\log 3}$$

Multiple both sides by log3·log 5 and divide both sides by log a ($\neq 0$).

$\log 3 + \log 5 = \log a$

$a = 15$

I1.2 若 $\sqrt{b}=\sqrt{8+\sqrt{a}}+\sqrt{8-\sqrt{a}}$ ，求 b 的實數值。

If $\sqrt{b}=\sqrt{8+\sqrt{a}}+\sqrt{8-\sqrt{a}}$, determine the real value of b.

Reference: 2007 FI1.1

$$b=\left(\sqrt{8+\sqrt{15}}+\sqrt{8-\sqrt{15}}\right)^2$$

$$=8+\sqrt{15}+2\left(\sqrt{8+\sqrt{15}}\cdot\sqrt{8-\sqrt{15}}\right)+8-\sqrt{15}$$

$$=16+2\sqrt{64-15}$$

$$=16+2\sqrt{49}$$

$= 16 + 2\times 7 = 30$

I1.3 若方程 $x^2 - cx + b = 0$ 有兩個實數根及兩根之差為 1，求兩根之和的最大可能值 c。

If the equation $x^2 - cx + b = 0$ has two distinct real roots and their difference is 1, determine the greatest possible value of the sum of the roots, c.

Reference: 2008 FIS.3

Let the roots be α, β.

$\alpha + \beta = c$, $\alpha\beta = b = 30$, $\alpha - \beta = 1$

$(\alpha - \beta)^2 = 1$

$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = 1$

$c^2 - 4\times 30 = 1$

$\Rightarrow c = 11$ or -11

The greatest possible value of $c = 11$

I1.4 設 $d=\overline{xyz}$ 為一不能被 10 整除的三位數。若 $\overline{xyz}$ 與 $\overline{zyx}$ 之和可被 c 整除，求此整數的最大可能值 d。

Let $d=\overline{xyz}$ be a three-digit integer that is **not** divisible by 10. If the sum of integers $\overline{xyz}$ and $\overline{zyx}$ is divisible by c, determine the greatest possible value of such an integer d.

$100x + 10y + z + 100z + 10y + x$

$= 100(x + z) + 20y + x + z$

$= 101(x + z) + 20y$

$= 99(x + z) + 22y + 2(x + z - y)$, which is divisible by 11

$x + z - y$ is a multiple of 11

$x + z - y = 0$ or 11

To maximize d, x should be as large as possible

$x + z = 11 + y$

$x = 9$, $z = 9$, $y = 7$

The greatest possible value of $d = 979$

Check: $\overline{xyz}+\overline{zyx} = 979 + 979 = 1958 = 11\times178$, which is divisible by 11.

Individual Event 2

I2.1 一個等邊三角形及一個正六邊形的周長比率為 1 : 1。若三角形與六邊形的面積比率為 2 : a，求 a 的值。

Let the ratio of perimeter of an equilateral triangle to the perimeter of a regular hexagon be 1 : 1. If the ratio of the area of the triangle to the area of the hexagon is 2 : a, determine the value of a.

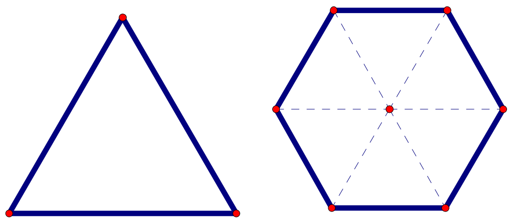
Reference: 1996 FI1.1, 2014 FI4.3

Let the length of the equilateral triangle be x, and that of the regular hexagon be y.

Since they have equal perimeter, $3x = 6y$

$\therefore x = 2y$

The hexagon can be divided into 6 identical equilateral triangles.



Ratio of areas $= \frac{1}{2}x^2 \sin 60^\circ : 6\times\frac{1}{2}y^2\sin 60^\circ = 2 : a$

$x^2 : 6y^2 = 2 : a$

$(2y)^2 : 6y^2 = 2 : a$

$\Rightarrow a = 3$

I2.2 求 $b = \left[\log_2(a^2) + \log_4\left(\frac{1}{a^2}\right)\right]\times\left[\log_a 2 + \log_{a^2}\left(\frac{1}{2}\right)\right]$ 的值。

Determine the value of $b = \left[\log_2(a^2) + \log_4\left(\frac{1}{a^2}\right)\right]\times\left[\log_a 2 + \log_{a^2}\left(\frac{1}{2}\right)\right]$.

$$b = \left[\log_2(3^2) + \log_4\left(\frac{1}{3^2}\right)\right]\times\left[\log_3 2 + \log_{3^2}\left(\frac{1}{2}\right)\right]$$

$$= \left[\log_2 9 + \frac{\log_2\left(\frac{1}{9}\right)}{\log_2 4}\right]\times\left[\log_3 2 + \frac{\log_3\left(\frac{1}{2}\right)}{\log_3 9}\right]$$

$$= \left[\log_2 9 + \frac{\log_2\left(\frac{1}{9}\right)}{2}\right]\times\left[\log_3 2 + \frac{\log_3\left(\frac{1}{2}\right)}{2}\right]$$

$$= \left[\log_2 9 + \log_2\left(\frac{1}{9}\right)^{\frac{1}{2}}\right]\times\left[\log_3 2 + \log_3\left(\frac{1}{2}\right)^{\frac{1}{2}}\right]$$

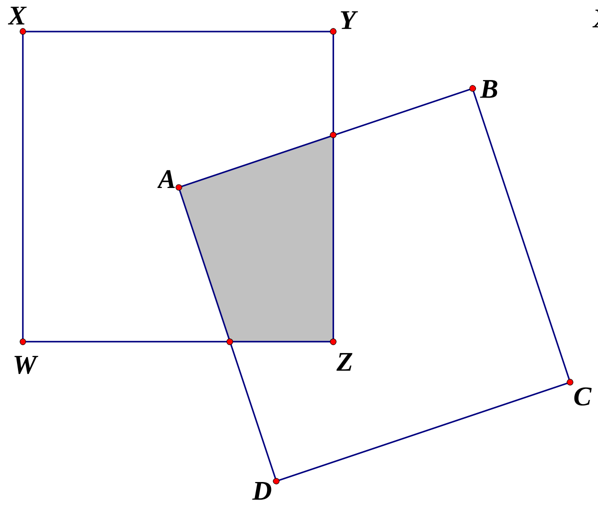
$$= \left[\log_2\left(9\times\frac{1}{\sqrt{9}}\right)\right]\times\left[\log_3\left(2\times\frac{1}{\sqrt{2}}\right)\right]$$

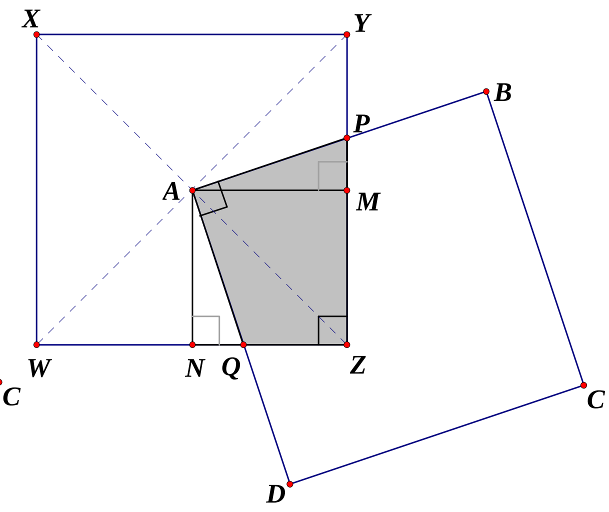
$$= [\log_2 3]\times\left[\log_3\left(2^{\frac{1}{2}}\right)\right]$$

$$= \frac{\log 3}{\log 2}\times\frac{\frac{1}{2}\log 2}{\log 3} = \frac{1}{2}$$

I2.3 在下圖中，正方形 ***ABCD*** 及 ***XYZW*** 相等而且互相交疊使得頂點 ***A*** 位在 ***XYZW*** 的中心及綫段 ***AB*** 將綫段 ***YZ*** 邊分為 1 : 2。若 ***XYZW*** 的面積與交疊部分的面積比率為 ***c*** : 1，求 ***c*** 的值。

In the figure below, identical squares ***ABCD*** and ***XYZW*** overlap each other in such a way that the vertex ***A*** is at the centre of ***XYZW*** and the line segment ***AB*** cuts line segment ***YZ*** into 1 : 2. If the ratio of the area of ***XYZW*** to the overlapped region is ***c*** : 1, determine the value of ***c***.





Reference: 2009 HI7

A is the intersection of the diagonals ***XZ*** and ***YW***. Suppose ***AB*** cuts ***YZ*** at ***P*** and ***AD*** cuts ***WZ*** at ***Q***. By the property of squares, $\angle PAQ = \angle PZQ = 90°$

$\therefore \angle PAQ + \angle PZQ = 90° + 90° = 180°$

$\Rightarrow$ ***A***, ***P***, ***Z***, ***Q*** are concyclic (opp. ∠s supp.)

$\angle APM = \angle AQN$ (ext. ∠s cyclic quad.)

Let ***M***, ***N*** be the mid-points of ***YZ*** and ***WZ*** respectively.

It is easy to show that $AM = AN$ and $\angle AMP = \angle ANQ = 90°$

$\therefore \Delta APM \cong \Delta ANQ$ (A.A.S.)

Area of shaded region = area of ***APM*** + area of ***AMZQ***

= area of ***ANQ*** + area of ***AMZQ***

= area of ***AMZN***

$= \frac{1}{4} \times$ area of ***XYZW***

$\therefore c = 4$

I2.4 若 76 與 ***d*** 的最小公倍數(L.C.M.)為 456 及 76 與 ***d*** 的最大公因數(H.C.F.)為 ***c***，求正整數 ***d*** 的值。

If the least common multiples (L.C.M.) of 76 and ***d*** is 456 and the highest common factor (H.C.F.) of 76 and ***d*** is ***c***, determine the value of the positive integer ***d***.

Reference: 2005 FI1.2

$76 \times d$ = L.C.M. × H.C.F. = 456×4

$d = 24$

Individual Event 3

I3.1 若 $f(x) = x^4 + x^3 + x^2 + x + 1$，求 $f(x^5)$ 除以 $f(x)$ 的餘值 a。

If $f(x) = x^4 + x^3 + x^2 + x + 1$, determine the remainder a of $f(x^5)$ divided by $f(x)$.

Reference: 1996 FG10.2

Clearly $f(1) = 5$.

By division algorithm, $f(x) = (x - 1)Q(x) + 5$, where $Q(x)$ is a polynomial

$$f(x^5) = (x^5 - 1)Q(x^5) + 5$$
$$= (x - 1)(x^4 + x^3 + x^2 + x + 1)Q(x^5) + 5$$
$$= f(x)(x - 1)Q(x^{10}) + 5$$

The remainder is $a = 5$.

I3.2 設 n 為整數。求 $n^a - n$ 除以 30 的餘值 b。

Let n be an integer. Determine the remainder b of $n^a - n$ divided by 30.

$n^5 - n = n(n^4 - 1) = n(n^2 + 1)(n^2 - 1) = (n - 1)n(n + 1)(n^2 + 1)$

$n - 1$, n and $n + 1$ are three consecutive integers, the product of which must be divisible by 6.

If any one of $n - 1$, n or $n + 1$ is divisible by 5, then the product is divisible by 30.

Otherwise, let $n - 1 = 5k + 1$, $n = 5k + 2$, $n + 1 = 5k + 3$, $n^2 + 1 = (5k + 2)^2 + 1 = 25k^2 + 20k + 5$ which is a multiple of 5, the product is divisible by 30.

If $n - 1 = 5k + 2$, $n = 5k + 3$, $n + 1 = 5k + 4$, $n^2 + 1 = (5k + 3)^2 + 1 = 25k^2 + 30k + 10$ which is a multiple of 5, the product is divisible by 30.

In all cases, $n^5 - n$ is divisible by 30. The remainder when $n^5 - n$ divided by 30 is 0.

I3.3 若 $0 < x < 1$，求 $c = \left(\frac{\sqrt{1+x}}{\sqrt{1+x}-\sqrt{1-x}} + \frac{1-x}{\sqrt{1-x^2}+x-1}\right) \times \left(\sqrt{\frac{1}{x^2-b^2}-1} - \frac{1}{x-b}\right)$ 的值。

If $0 < x < 1$, determine the value of

$$c = \left(\frac{\sqrt{1+x}}{\sqrt{1+x}-\sqrt{1-x}} + \frac{1-x}{\sqrt{1-x^2}+x-1}\right) \times \left(\sqrt{\frac{1}{x^2-b^2}-1} - \frac{1}{x-b}\right).$$

Reference: 2017 FG3.2

$$c = \left(\frac{\sqrt{1+x}}{\sqrt{1+x}-\sqrt{1-x}} + \frac{1-x}{\sqrt{1-x^2}+x-1}\right) \times \left(\sqrt{\frac{1}{x^2}-1} - \frac{1}{x}\right)$$

$$= \left\{\frac{\sqrt{1+x}\cdot\left(\sqrt{1+x}+\sqrt{1-x}\right)}{(1+x)-(1-x)} + \frac{(1-x)\cdot\left[\sqrt{1-x^2}-(x-1)\right]}{\left(1-x^2\right)-(x-1)^2}\right\} \times \left(\sqrt{\frac{1-x^2}{x^2}} - \frac{1}{x}\right)$$

$$= \left\{\frac{1+x+\sqrt{1-x^2}}{2x} + \frac{(1-x)\cdot\left[\sqrt{1-x^2}+(1-x)\right]}{\left(1-x^2\right)-\left(1-2x+x^2\right)}\right\} \times \left(\frac{\sqrt{1-x^2}}{x} - \frac{1}{x}\right)$$

$$= \left\{\frac{1+x+\sqrt{1-x^2}}{2x} + \frac{(1-x)\cdot\left[\sqrt{1-x^2}+(1-x)\right]}{2x(1-x)}\right\} \times \left(\frac{\sqrt{1-x^2}-1}{x}\right)$$

$$= \left[\frac{1+x+\sqrt{1-x^2}}{2x} + \frac{\sqrt{1-x^2}+(1-x)}{2x}\right] \times \left(\frac{\sqrt{1-x^2}-1}{x}\right)$$

$$= \left(\frac{2+2\sqrt{1-x^2}}{2x}\right) \times \left(\frac{\sqrt{1-x^2}-1}{x}\right) = \left(\frac{1+\sqrt{1-x^2}}{x}\right) \times \left(\frac{\sqrt{1-x^2}-1}{x}\right)$$

$$= \frac{\left(1-x^2\right)-1}{x^2} = -1$$

Remark: You may substitute $x = 0.5$ directly to find the value of c.

I3.4 若實數 x 及 y 滿足方程 $2\log_{10}(x+2cy)=\log_{10}x+\log_{10}y$，求 $d=\dfrac{x}{y}$ 的值。

If real numbers x and y satisfy the equation $2\log_{10}(x+2cy)=\log_{10}x+\log_{10}y$, determine the value of $d=\dfrac{x}{y}$.

$2\log_{10}(x-2y)=\log_{10}x+\log_{10}y$

$(x-2y)^2=xy$

$x^2-5xy+4y^2=0$

$(x-y)(x-4y)=0$

$d=\dfrac{x}{y}=1$ or 4

Check: When $x=y$, L.H.S. $=2\log_{10}(y-2y)=2\log_{10}(-y)$, R.H.S. $=\log_{10}y+\log_{10}y$

When $y>0$. L.H.S. is undefined and R.H.S. is well defined, rejected

When $y=0$, L.H.S. is undefined and R.H.S. is undefined, rejected

When $y<0$, L.H.S. is well defined, R.H.S. is undefined, rejected

When $x=4y$, L.H.S. $=2\log_{10}(4y-2y)=\log_{10}4+2\log_{10}y=\log_{10}4y+\log_{10}y=$ R.H.S.

When $y>0$. L.H.S. is well defined and R.H.S. is well defined, accepted

$\therefore d=4$ only

Individual Event 4

I4.1 若 m 和 n 為正整數及 $a=\log_2\left[\left(\frac{m^4n^{-4}}{m^{-1}n}\right)^{-3}\div\left(\frac{m^{-2}n^2}{mn^{-1}}\right)^5\right]$，求 a 的值。

If m and n are positive integers and $a=\log_2\left[\left(\frac{m^4n^{-4}}{m^{-1}n}\right)^{-3}\div\left(\frac{m^{-2}n^2}{mn^{-1}}\right)^5\right]$, determine the value of a.

$$a=\log_2\left[\left(\frac{m^5}{n^5}\right)^{-3}\div\left(\frac{n^3}{m^3}\right)^5\right]$$

$$=\log_2\left[\left(\frac{n^{15}}{m^{15}}\right)\times\left(\frac{m^{15}}{n^{15}}\right)\right]=\log_2 1=0$$

I4.2 當整數 1108 + a、1453、1844 + 2a 及 2281 除以正整數 n (>1)都得相同餘數 b，求 b 的值。

When the integers 1108 + a, 1453, 1844 + 2a and 2281 divided by some positive integer n (> 1), they all get the same remainder b. Determine the value of b.

Reference: 2000 FG1.1

$1108 = pn + b$ …… (1)

$1453 = qn + b$ …… (2)

$1844 = rn + b$ …… (3)

$2281 = sn + b$ …… (4)

p, q, r, s are non-negative integers and $0 \le b < n$.

(2) – (1): $345 = (q-p)n$ …… (5)

(3) – (2): $391 = (r-q)n$ …… (6)

(4) – (3): $437 = (s-r)n$ …… (7)

$\therefore$ n is the common factor of 345, 391 and 437.

$345 = 3\times5\times23$, $391 = 17\times23$, $437 = 19\times23$

$\therefore$ n = 1 or 23

When n = 1, b = 0. (rejected)

When n = 23, sub. n = 23 into (1): $1108 = 23\times48 + 4$

$b = 4$

Remark: original question: ……除以正整數n都得相同餘數b, ……divided by some positive integer n, they all get the same remainder b.

There are two possible answers for b: 0 or 4.

I4.3 若 $\frac{6}{b}<x<\frac{10}{b}$，求 $c=\sqrt{x^2-2x+1}+\sqrt{x^2-6x+9}$ 的值。

If $\frac{6}{b}<x<\frac{10}{b}$, determine the value of $c=\sqrt{x^2-2x+1}+\sqrt{x^2-6x+9}$.

Reference: 2005 HI2, 2018 FI1.2

$1.5 < x < 2.5$, $c=\sqrt{(x-1)^2}+\sqrt{(x-3)^2}=x-1+3-x=2$

Remark: original question: ……最大可能值 …… the greatest possible value of c.

The value of c is a constant.

I4.4 求 $1 + 3^c + (3^c)^2 + (3^c)^3 + (3^c)^4$ 除以 $1 + 3 + 3^2 + 3^3 + 3^4$ 的餘值 d。

Determine the remainder d when $1 + 3^c + (3^c)^2 + (3^c)^3 + (3^c)^4$ is divided by $1 + 3 + 3^2 + 3^3 + 3^4$.

$1 + 3 + 3^2 + 3^3 + 3^4 = \frac{3^5 - 1}{3 - 1}$

$1 + 3^2 + (3^2)^2 + (3^2)^3 + (3^2)^4 = \frac{(3^2)^5 - 1}{3^2 - 1} = \frac{3^5 + 1}{3 + 1} \cdot \frac{3^5 - 1}{3 - 1}$

$$\frac{1 + 3^2 + (3^2)^2 + (3^2)^3 + (3^2)^4}{1 + 3 + 3^2 + 3^3 + 3^4} = \frac{3^5 + 1}{3 + 1}$$
$$= \frac{(2+1)^5 + 1}{4}$$
$$= \frac{2^5 + C_1^5 2^4 + C_2^5 2^3 + C_3^5 2^2 + C_4^5 2 + 1 + 1}{4}$$
$$= 2^3 + C_1^5 2^2 + C_2^5 2 + C_3^5 + 3, \text{ which is an integer}$$

The remainder $d = 0$

Method 2

$1 + 3 + 3^2 + 3^3 + 3^4 = \frac{3^5 - 1}{3 - 1}$

$1 + 3^2 + (3^2)^2 + (3^2)^3 + (3^2)^4 = \frac{(3^2)^5 - 1}{3^2 - 1} = \frac{3^5 + 1}{3 + 1} \cdot \frac{3^5 - 1}{3 - 1}$

$$\frac{1 + 3^2 + (3^2)^2 + (3^2)^3 + (3^2)^4}{1 + 3 + 3^2 + 3^3 + 3^4} = \frac{3^5 + 1}{3 + 1} = \frac{244}{4} = 61, \text{ which is an integer}$$

The remainder $d = 0$

Method 3

$3 \equiv -1 \pmod 4$

$3^5 \equiv (-1)^5 \equiv -1 \pmod 4$

$3^5 + 1 \equiv 0 \pmod 4$

$\frac{3^5 + 1}{3 + 1}$ is an integer

$\frac{3^5 + 1}{3 + 1} \cdot \frac{3^5 - 1}{3 - 1}$ is an integral multiple of $\frac{3^5 - 1}{3 - 1}$.

$1 + 3^2 + (3^2)^2 + (3^2)^3 + (3^2)^4 = \frac{(3^2)^5 - 1}{3^2 - 1} = \frac{3^5 + 1}{3 + 1} \cdot \frac{3^5 - 1}{3 - 1}$ is an integral multiple of $1 + 3 + 3^2 + 3^3 + 3^4$

The remainder $d = 0$

Group Event 1

G1.1 一項工程包括三個項目：A、B 和 C。若項目 A 開始三天後，項目 B 才可開始進行。項目 C 亦必須在項目 B 開始四天後才可開始進行。若完成項目 A、B 和 C 分別需要四天、六天和五天，求最少天數 (P) 完成全項工程。

A project comprises of three tasks, A, B and C. Suppose task B must begin 3 days later than task A begins, and task C must begin 4 days later than task B begins. If the numbers of days to complete tasks A, B and C are 4, 6 and 5, respectively, determine the least number of days (P) to complete the project.

$P = 3 + 4 + 5 = 12$

G1.2 指示牌牌上掛有紅、黃、綠閃燈。紅、黃、綠閃燈分別每隔 3 秒、4 秒、8 秒閃爍一次。當 0 秒時，紅、黃、綠閃燈同時閃爍。若當 Q 秒時，第三次出現只有紅及黃閃燈同時閃爍，求 Q 的值。

There are 3 blinking lights, red, yellow and green, on a panel. Red, yellow and green lights blink at every 3, 4 and 8 seconds, respectively. Suppose each light blinks at the time $t = 0$. At time Q (in seconds), there is the third time at which only red and yellow lights blink, determine the value of Q.

The L.C.M. of 3, 4 and 8 is 24. i.e. The lights blink patterns repeat for every 24 seconds.

If only red and yellow lights blink, but not the green light, then the first time it happens is 12 s, the second time it happens is 12 + 24 = 36 s, in the third time it happens is 36 + 24 = 60 s.

$Q = 60$

G1.3 設 $f_{n+1} = \begin{cases} f_n + 3 & 若\ n\ 是雙數 \\ f_n - 2 & 若\ n\ 是單數 \end{cases}$。

若 $f_1 = 60$，求 n 的最少可能值，令當 $m \ge n$ 時，滿足 $f_m \ge 63$。

Let $f_{n+1} = \begin{cases} f_n + 3 & \text{if } n \text{ is even} \\ f_n - 2 & \text{if } n \text{ is odd} \end{cases}$.

If $f_1 = 60$, determine the smallest possible value of n satisfying $f_m \ge 63$ for all $m \ge n$.

$f_2 = 58$, $f_3 = 61$, $f_4 = 59$, $f_5 = 62$, $f_6 = 60$, $f_7 = 63$, $f_8 = 61$, $f_9 = 64$, $f_{10} = 62$, $f_{11} = 65$, $f_{12} = 63\cdots\cdots$

Now $f_m \ge 63$

The smallest possible value of n is 11.

G1.4 求 $T = \left(3^{2^0}+1\right)\times\left(3^{2^1}+1\right)\times\left(3^{2^2}+1\right)\times\cdots\times\left(3^{2^{10}}+1\right)$ 的值。(答案以指數表示。)

Determine the value of $T = \left(3^{2^0}+1\right)\times\left(3^{2^1}+1\right)\times\left(3^{2^2}+1\right)\times\cdots\times\left(3^{2^{10}}+1\right)$. (Leave your answer in index form.)

Reference: 1994 FG6.2

$T = (3-1)(3+1)(3^2+1)(3^4+1)(3^8+1)\times\cdots\times\left(3^{2^{10}}+1\right)\div 2$

$= (3^2-1)(3^2+1)(3^4+1)(3^8+1)\times\cdots\times\left(3^{2^{10}}+1\right)\div 2$

$= (3^4-1)(3^4+1)(3^8+1)\times\cdots\times\left(3^{2^{10}}+1\right)\div 2$

$= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$= \frac{1}{2}\left(3^{2^{11}}-1\right)$

$= \frac{1}{2}\left(3^{2048}-1\right)$

Remark: The original question: 求 $T = \left(3^{2^0}+1\right)\times\left(3^{2^1}+1\right)\times\left(3^{2^2}+1\right)\times\cdots\times\left(3^{2^{10}}+1\right)$ 的值。

Determine the value of $T = \left(3^{2^0}+1\right)\times\left(3^{2^1}+1\right)\times\left(3^{2^2}+1\right)\times\cdots\times\left(3^{2^{10}}+1\right)$.

It is difficult to find the exact value of the expression.

Group Event 2

G2.1 一個盒子有五個球，球面上分別印上號碼 3、4、6、9 或 10。由盒中同時隨機取出 2 個球，並得出其號碼的總和。若 $\boldsymbol{A}$ 為不同總和的數量，求 $\boldsymbol{A}$ 的值。

A box contains five distinctly marked balls with number markings being 3, 4, 6, 9 or 10. Two balls are randomly drawn without replacement from the box. If $\boldsymbol{A}$ is the number of possible distinct sums of the selected numbers, determine the value of $\boldsymbol{A}$.

3 + 4 = 7, 3 + 6 = 9, 3 + 9 = 12, 3 + 10 = 13, 4 + 6 = 10, 4 + 9 = 13, 4 + 10 = 14, 6 + 9 = 15, 6 + 10 = 16, 9 + 10 = 19

The distinct sums are 7, 9, 10, 12, 13, 14, 15, 16, 19. $\boldsymbol{A} = 9$

G2.2 設 $f_1 = 9$ 及 $f_n = \begin{cases} f_{n-1} + 3 & \text{若 } n \text{ 是3的倍數} \\ f_{n-1} - 1 & \text{若 } n \text{ 不是3的倍數} \end{cases}$。

若 $\boldsymbol{B}$ 為 $\boldsymbol{k}$ 的值的可能數量，使得 $f_k < 11$，求 $\boldsymbol{B}$ 的值。

Let $f_1 = 9$ and $f_n = \begin{cases} f_{n-1} + 3 & \text{if } n \text{ is a multiple of 3} \\ f_{n-1} - 1 & \text{if } n \text{ is not a multiple of 3} \end{cases}$.

If $\boldsymbol{B}$ is the number of possible values of $\boldsymbol{k}$ such that $f_k < 11$, determine the value of $\boldsymbol{B}$.

$f_1 = 9, f_2 = 8, f_3 = 11, f_4 = 10, f_5 = 9, f_6 = 12, f_7 = 11, f_8 = 10, f_9 = 13, \cdots$

There are 5 values of $\boldsymbol{k}$ such that $f_k < 11$, $\boldsymbol{B} = 5$

Remark: 中文版 …其中 $\boldsymbol{k}$ 滿足 $f_k < 11$… 改為 …使得 $f_k < 11$…

G2.3 設 a_1、a_2、a_3、a_4、a_5、a_6 為非負整數，並滿足 $\begin{cases} a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 + 6a_6 = 26 \\ a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 5 \end{cases}$。

若 $\boldsymbol{c}$ 為方程系統的解的數量，求 $\boldsymbol{c}$ 的值。

Let $a_1, a_2, a_3, a_4, a_5, a_6$ be non-negative integers and satisfy

$$\begin{cases} a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 + 6a_6 = 26 \\ a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 5 \end{cases}.$$

If $\boldsymbol{c}$ is the number of solutions to the system of equations, determine the value of $\boldsymbol{c}$.

Let $(a_1, a_2, a_3, a_4, a_5, a_6)$ be a solution, then $0 \le a_1, a_2, a_3, a_4, a_5, a_6 \le 5$

The solutions are:

(0, 1, 0, 0, 0, 4), (0, 0, 1, 0, 1, 3), (0, 0, 0, 2, 0, 3), (0, 0, 0, 1, 2, 2), (0, 0, 0, 0, 4, 1).

$\boldsymbol{c} = 5$

G2.4 設 $\boldsymbol{d}$ 及 $\boldsymbol{f}$ 為正整數及 $a_1 = 0.9$。若 $a_{i+1} = a_i^2$ 及 $\prod_{i=1}^{4} a_i = \frac{3^d}{f}$，求 $\boldsymbol{d}$ 的最小可能值。

Let $\boldsymbol{d}$ and $\boldsymbol{f}$ be positive integers and $a_1 = 0.9$. If $a_{i+1} = a_i^2$ and $\prod_{i=1}^{4} a_i = \frac{3^d}{f}$, determine the smallest possible value of $\boldsymbol{d}$.

$0.9 \times 0.9^2 \times 0.9^4 \times 0.9^8 = 0.9^{15} = \frac{9^{15}}{10^{15}} = \frac{3^{30}}{10^{15}}$

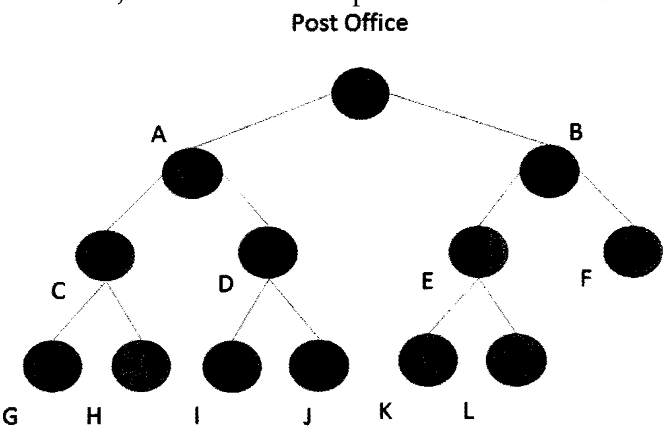
$\boldsymbol{d} = 30$

Remark 最少可能值改為最小可能值

Group Event 3

G3.1 下圖是郵差的送信路線圖：從郵局開始，到達十二個地點送信，最後返回郵局。若郵差從一地點步行到另一地點需要十分鐘及 K 為郵差需要的時數來完成整天路線，求 K 的最小可能值。

The figure below represents routes of a postman. Starting at the post office, the postman walks through all the 12 points and finally returns to the post office. If he takes 10 minutes from a point to another adjacent point by walk and K is the number of hours required for the postman to finish the routes, find the smallest possible value of K.



Post Office →A→C→G→C→H→C→A→D→I→D→J→D→A→Post Office

→B→E→K→E→L→E→B→F→B→Post Office

Total time = 240 minutes = 4 hours

$K = 4$

Remark 最少可能值改為最小可能值

G3.2 若 n 為正整數，$a_1 = 0.8$ 及 $a_{n+1} = a_n^2$，求 L 的最小值，滿足 $a_1 \times a_2 \times \cdots \times a_L < 0.3$。

If $a_1 = 0.8$ and $a_{n+1} = a_n^2$ for positive integers n, determine the least value of L satisfying

$a_1 \times a_2 \times \cdots \times a_L < 0.3$.

$0.8 \times 0.8^2 \times 0.8^4 \times \cdots \times 0.8^{\left(2^{L-1}\right)} < 0.3$

$0.8^{\left(1+2+4+\cdots+2^{L-1}\right)} < 0.3$

$0.8^{\left(2^L - 1\right)} < 0.3$

$(2^L - 1) \log 0.8 < \log 0.3$

$2^L - 1 > \frac{\log 0.3}{\log 0.8} = \frac{1 - \log 3}{1 - \log 8} \approx \frac{1 - 0.48}{1 - 3 \times 0.30} = \frac{0.52}{0.10} = 5.2$

$2^L > 6.2 \Rightarrow$ The least value of $L = 3$

Remark 最少值 改為 最小值

G3.3 若方程 $\sqrt[3]{5+\sqrt{x}}+\sqrt[3]{5-\sqrt{x}}=1$，求實數根 x。

Solve $\sqrt[3]{5+\sqrt{x}}+\sqrt[3]{5-\sqrt{x}}=1$ for real number x.

Reference: 1999 FI3.2, 2005 FI2.2, 2019 HI10

$\sqrt[3]{5+\sqrt{x}}+\sqrt[3]{5-\sqrt{x}}=1$

$\left(\sqrt[3]{5+\sqrt{x}}+\sqrt[3]{5-\sqrt{x}}\right)^3=1$

$5+\sqrt{x}+3\left(5+\sqrt{x}\right)^{2/3}\left(5-\sqrt{x}\right)^{1/3}+3\left(5+\sqrt{x}\right)^{1/3}\left(5-\sqrt{x}\right)^{2/3}+5-\sqrt{x}=1$

$10+3\left(25-x\right)^{1/3}\left(5+\sqrt{x}\right)^{1/3}+3\left(25-x\right)^{1/3}\left(5-\sqrt{x}\right)^{1/3}=1$

$9+3\left(25-x\right)^{1/3}\left[\left(5+\sqrt{x}\right)^{1/3}+\left(5-\sqrt{x}\right)^{1/3}\right]=0$

$3+\left(25-x\right)^{1/3}=0$

$\left(25-x\right)^{1/3}=-3$

$25-x=-27$

$x=52$

G3.4 若 a、b 及 y 為實數，並滿足 $\begin{cases}a+b+y=5\\ab+by+ay=3\end{cases}$，求 y 的最大值。

If a, b and y are real numbers and satisfy $\begin{cases}a+b+y=5\\ab+by+ay=3\end{cases}$,

determine the greatest possible value of y.

From (1): $a=5-(b+y)$ …… (3)

Sub. (3) into (2): $[5-(b+y)]b+by+[5-(b+y)]y=3$

$b^2+(y-5)b+(y^2-5y+3)=0$, this is a quadratic equation in b.

For real values of b, $\Delta=(y-5)^2-4(y^2-5y+3)\ge 0$

$-3y^2+10y+13\ge 0$

$3y^2-10y-13\le 0$

$(3y-13)(y+1)\le 0$

$-1\le y\le\frac{13}{3}$

The maximum value of $y=\frac{13}{3}$.

Group Event 4

G4.1 若 a 及 b 為整數，且 a^2 與 b^2 相差 144，求 $d = a + b$ 的最大值。

Let a and b are two integers and the difference between a^2 and b^2 is 144, determine the largest possible value of $d = a + b$.

$a^2 - b^2 = 144$

$(a + b)(a - b) = 144 = 144\times1 = 72\times2 = \cdots\cdots$

When $a + b = 144$, $a - b = 1$, then $a = 72.5$, $b = 71.5$, which are not integers.

When $a + b = 72$, $a - b = 2$, then $a = 37$, $b = 35$

$\therefore$ The largest possible value of $a + b = 72$.

Remark: The original question is

若 a^2 及 b^2 為整數，且相差 144，求 $d = a + b$ 的最大值。

Let a^2 and b^2 are two integers that differ by 144, determine the largest possible value of $d = a + b$.

The original question is wrong because d can be any positive number.

e.g. $a^2 = 100000144$, $b^2 = 100000000$, then $a^2 - b^2 = 144$ and $d = a + b \approx 200000$

e.g. $a^2 = 10^{100} + 144$, $b^2 = 10^{100}$, then $a^2 - b^2 = 144$ and $d = a + b \approx 2\times10^{50}$

G4.2 若 n 為整數，n^2 的個位及 10 位分別為 u 及 7，求 u 的值。

If n is an integer, and the units and tens digits of n^2 are u and 7, respectively, determine the value of u.

$0^2 = 0$, $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$, $6^2 = 36$, $7^2 = 49$, $8^2 = 64$, $9^2 = 81$

Let $n = 10a + b$, $n^2 = (10a + b)^2 = 100a + 20ab + b^2 \equiv 70 + u \pmod{100}$

$u = 0, 1, 4, 5, 6$ or 9

$20ab + b^2 \equiv 70 + u \pmod{100}$

$b(20a + b) \equiv 70 + u \pmod{100}$

If $b = 0$, L.H.S. $\not\equiv$ R.H.S.

If $b = 1$, $20a + 1 \equiv 71 \Rightarrow 20a \equiv 70$. No solution

If $b = 2$, $40a + 4 \equiv 74 \Rightarrow 40a \equiv 70$. No solution

If $b = 3$, $60a + 9 \equiv 79 \Rightarrow 60a \equiv 70$. No solution

If $b = 4$, $80a + 16 \equiv 76 \pmod{100}$

$80a \equiv 60 \pmod{100}$

$8a \equiv 6 \pmod{10}$

$a = 2$ or 7 (e.g. $24^2 = 576$ and $74^2 = 5476$)

If $b = 5$, $100a + 25 \equiv 75 \Rightarrow 100a \equiv 50$. No solution

If $b = 6$, $120a + 36 \equiv 76 \pmod{100}$

$20a \equiv 40 \pmod{100}$

$a = 2$ or 7 (e.g. $26^2 = 676$ and $76^2 = 5776$)

If $b = 7$, $140a + 49 \equiv 79 \Rightarrow 140a \equiv 30$. No solution

If $b = 8$, $160a + 64 \equiv 74 \Rightarrow 160a \equiv 10$. No solution

If $b = 9$, $180a + 81 \equiv 71 \Rightarrow 180a \equiv 90$. No solution

Conclusion $u = 6$

G4.3 求實數 $c = \dfrac{\left(4+\sqrt{15}\right)^{\frac{3}{2}} + \left(4-\sqrt{15}\right)^{\frac{3}{2}}}{\left(6+\sqrt{35}\right)^{\frac{3}{2}} - \left(6-\sqrt{35}\right)^{\frac{3}{2}}}$ 的值。

Determine the value of real number $c = \dfrac{\left(4+\sqrt{15}\right)^{\frac{3}{2}} + \left(4-\sqrt{15}\right)^{\frac{3}{2}}}{\left(6+\sqrt{35}\right)^{\frac{3}{2}} - \left(6-\sqrt{35}\right)^{\frac{3}{2}}}$.

$\sqrt{8+2\sqrt{15}} = \sqrt{5+2\sqrt{3\times5}+3} = \sqrt{5}+\sqrt{3}$, $\sqrt{8-2\sqrt{15}} = \sqrt{5-2\sqrt{3\times5}+3} = \sqrt{5}-\sqrt{3}$

$\sqrt{12+2\sqrt{35}} = \sqrt{7+2\sqrt{7\times5}+5} = \sqrt{7}+\sqrt{5}$, $\sqrt{12-2\sqrt{35}} = \sqrt{7-2\sqrt{7\times5}+5} = \sqrt{7}-\sqrt{5}$

$$c=\frac{\left(4+\sqrt{15}\right)^{\frac{3}{2}}+\left(4-\sqrt{15}\right)^{\frac{3}{2}}}{\left(6+\sqrt{35}\right)^{\frac{3}{2}}-\left(6-\sqrt{35}\right)^{\frac{3}{2}}}$$

$$=\frac{2^{\frac{3}{2}}\left[\left(4+\sqrt{15}\right)^{\frac{3}{2}}+\left(4-\sqrt{15}\right)^{\frac{3}{2}}\right]}{2^{\frac{3}{2}}\left[\left(6+\sqrt{35}\right)^{\frac{3}{2}}-\left(6-\sqrt{35}\right)^{\frac{3}{2}}\right]}$$

$$=\frac{\left(8+2\sqrt{15}\right)^{\frac{3}{2}}+\left(8-2\sqrt{15}\right)^{\frac{3}{2}}}{\left(12+2\sqrt{35}\right)^{\frac{3}{2}}-\left(12-2\sqrt{35}\right)^{\frac{3}{2}}}$$

$$=\frac{\left(\sqrt{5}+\sqrt{3}\right)^{3}+\left(\sqrt{5}-\sqrt{3}\right)^{3}}{\left(\sqrt{7}+\sqrt{5}\right)^{3}-\left(\sqrt{7}-\sqrt{5}\right)^{3}}$$

$$=\frac{\left(\sqrt{5}\right)^{3}+3\left(\sqrt{5}\right)\left(\sqrt{3}\right)^{2}}{3\left(\sqrt{7}\right)^{2}\left(\sqrt{5}\right)+\left(\sqrt{5}\right)^{3}}=\frac{5\left(\sqrt{5}\right)+9\left(\sqrt{5}\right)}{21\left(\sqrt{5}\right)+5\left(\sqrt{5}\right)}=\frac{7}{13}$$

G4.4 求下列方程 $x=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}}}$ 的正實數解。

Determine the positive real root of the following equation: $x=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}}}$.

$$\frac{1}{x-1}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}}\Rightarrow\frac{1}{x-1}-1=\cfrac{1}{1+\cfrac{1}{\frac{x+1}{x}}}$$

$$\frac{2-x}{x-1}=\cfrac{1}{1+\cfrac{x}{x+1}}$$

$$\frac{2-x}{x-1}=\frac{1}{\frac{2x+1}{x+1}}$$

$$\frac{2-x}{x-1}=\frac{x+1}{2x+1}$$

$(2-x)(2x+1)=(x-1)(x+1)$

$-2x^2+3x+2=x^2-1$

$3x^2-3x-3=0$

$x^2-x-1=0$

$x=\frac{1+\sqrt{5}}{2}$ or $\frac{1-\sqrt{5}}{2}$ (< 0, rejected)

Remark: "求實數……的正數值。" is changed into "求下列方程……的正實數解。"

"Determine the positive value of the real number ……" is changed into "Determine the positive real root of the following equation ……"

Individual Events

I1	*a*	0	**I2**	*a*	8	**I3**	*a*	8	**I4**	*a*	2
	b	–1		*b*	64		*b*	2		*b*	1
	c	7		*c*	15936		*c*	10		*c*	$\frac{3}{44}$
	d	18		*d*	*5312 see the remark		*d*	1023		*d*	$\frac{1945}{3872}$

Group Events

G1	*a*	–1	**G2**	*A*	2	**G3**	*R*	0	**G4**	*P*	*18 see the remark
	b	3		*B*	28		*S*	–1		*Q*	$\frac{63}{512}$
	c	$\frac{1}{9}$		*C*	300		*T*	4		*R*	377
	d	393		*D*	11		*U*	$-2\sqrt{3}$		*S*	5

Individual Event 1

I1.1 若 a 為 $\frac{1}{(x+2)(x+3)}=\frac{1}{(x+1)(x+4)}$ 的實數解的數量，求 a 的值。

If a is the number of real roots of $\frac{1}{(x+2)(x+3)}=\frac{1}{(x+1)(x+4)}$, determine the value of a.

$(x+1)(x+4)=(x+2)(x+3)$ $x^2+5x+4=x^2+5x+6$ $0=2$ 無解，$a=0$	$(x+1)(x+4)=(x+2)(x+3)$ $x^2+5x+4=x^2+5x+6$ $0=2$ No solution, $a=0$

I1.2 若 x 為實數及 b 為 $-|x-a-9|-|10-x|$ 的最大值，求 b 的值。

If x is a real number and b is the maximum value of $-|x-a-9|-|10-x|$, determine the value of b. **(Reference: 2008, HI8, 2010 HG6, 2011 FGS.1, 2012 FG2.3, 2016 FI4.3)**

我們利用三角不等式 $\lvert p\rvert+\lvert q\rvert\ge\lvert p+q\rvert$ $-\lvert x-9\rvert-\lvert 10-x\rvert=-(\lvert x-9\rvert+\lvert 10-x\rvert)$ $\le-\lvert x-9+10-x\rvert=-1$ $\therefore b=-1$	We use the triangle inequality $\lvert p\rvert+\lvert q\rvert\ge\lvert p+q\rvert$ $-\lvert x-9\rvert-\lvert 10-x\rvert=-(\lvert x-9\rvert+\lvert 10-x\rvert)$ $\le-\lvert x-9+10-x\rvert=-1$ $\therefore b=-1$

I1.3 若實數 x 及 y 滿足 $4x^2+4y^2+9xy=-119b$，求 xy 的最大值 c。

If real numbers x and y satisfy $4x^2+4y^2+9xy=-119b$, determine c, the maximum value of xy.

$4x^2+4y^2+9xy=119$

$119\ge2\sqrt{(2x)^2\cdot(2y)^2}+9xy$ (A.M. ≥ G.M.)

$119\ge17xy$

$7\ge xy$

$c=7$

I1.4 若正實數 x 滿足方程 $x^2+\frac{1}{x^2}=c$，求 $d=x^3+\frac{1}{x^3}$ 。

If a positive real number x satisfies $x^2+\frac{1}{x^2}=c$, determine the value of $d=x^3+\frac{1}{x^3}$.

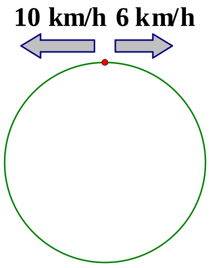
Reference: 1985 FI1.2, 1990 HI12

$x^2+\frac{1}{x^2}=7$ $\Rightarrow\left(x+\frac{1}{x}\right)^2=9$ $\Rightarrow x+\frac{1}{x}=3$ or -3 (捨去，$\because x>0$) $d=x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right)$ $=3\times(7-1)=18$	$x^2+\frac{1}{x^2}=7$ $\Rightarrow\left(x+\frac{1}{x}\right)^2=9$ $\Rightarrow x+\frac{1}{x}=3$ or -3 (rejected, $\because x>0$) $d=x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right)$ $=3\times(7-1)=18$

Individual Event 2

I2.1 兩個學生於長 1-km 的圓形跑道的起點開始分別以10 km/h 及 6 km/h 的速率跑沿相反方向跑步。當他們於起點再相遇時便停止跑步。若 a 為他們開始及停止前相互經過的次數，求 a 的值。

Two students run in opposite directions from a starting point of a 1-km circular track at speeds of 10 km/h and 6 km/h, respectively. They stop running when they meet each other at the starting point again. If a is number of times they cross each other after they start and before they stop, determine the value of a.



在半小時內，他們分別經過起點 5 次和 3 次。 總跑步距離= (5 + 3)km = 8 km 總相遇次數= a = 8 他們相遇在 $\frac{5}{8}, \frac{10}{8}, \frac{15}{8}, \frac{20}{8}, \frac{25}{8}, \frac{30}{8}, \frac{35}{8}, \frac{40}{8}$ 。	In half an hour, they will pass the starting point 5 times and 3 times respectively. Total distance travelled = (5 + 3)km = 8 km Number of times they meet = a = 8 They meet at $\frac{5}{8}, \frac{10}{8}, \frac{15}{8}, \frac{20}{8}, \frac{25}{8}, \frac{30}{8}, \frac{35}{8}, \frac{40}{8}$.

I2.2 袋中有若干粒紅色及藍色的彈珠，紅色彈珠與藍色彈珠的比例為 3：1。若加入 a 粒藍色彈珠，紅色彈珠與藍色彈珠的比例則為 2：1。求彈珠的總數 b。

There is a set of red marbles and blue marbles. When a red marbles are added to the set, the ratio of red marbles to the blue marbles is 3 : 1. When a blue marbles are added, the ratio of red marbles to blue marbles becomes 2 : 1. Determine the total number of marbles, b.

假設原本有 $3k$ 粒紅色彈珠及 k 粒藍色彈珠。 $3k : (k + 8) = 2 : 1$ $3k = 2k + 16$ $k = 16$ 彈珠的總數= $b = 4k = 64$	Let the original number of red marbles and blue marbles be $3k$ and k respectively. $3k : (k + 8) = 2 : 1$ $3k = 2k + 16$ $k = 16$ The total number of marbles = $b = 4k = 64$

I2.3 若 c 為 1 000 000 與一個平方數之最小的相差，其中此平方數為 b 的倍數，求 c 的值。

If c is the smallest difference between 1 000 000 and a square, where the square is a multiple of b, determine the value of c.

假設該平方數為 $64n^2$。 $1000 = 8\times125$ $1\,000\,000 = 64\times125^2$ $1\,000\,000 - 64n^2 = 64\times(125^2 - n^2)$ $64\times(125^2 - 124^2)$ 或 $64\times(126^2 - 125^2)$ $64\times(125 + 124)$ 或 $64\times(126 + 125)$ 最小值= $c = 64\times249 = 15936$	Let the square be $64n^2$. $1000 = 8\times125$ $1\,000\,000 = 64\times125^2$ $1\,000\,000 - 64n^2 = 64\times(125^2 - n^2)$ $64\times(125^2 - 124^2)$ or $64\times(126^2 - 125^2)$ $64\times(125 + 124)$ or $64\times(126 + 125)$ Minimum value = $c = 64\times249 = 15936$

I2.4 於一個月的時間完成建築一個水庫需要 d 個技工或 y 個勞工，當中 $d + y = c$。若挑選 d 個勞工去建築一個同樣的水庫，所需要的時間是挑選 y 個技工的 4 倍，求 d 的值。
The building of a reservoir takes d technicians, or alternatively y labours to complete in a month, where $d + y = c$. If d labours are employed to build the same reservoir, the time taken is 4 times as much as the time taken when y technicians are employed. Determine the value of d.

$d + y = 15936$ 每名技工每天工作量 $=\frac{1}{30d}$. 每名勞工每天工作量 $=\frac{1}{30y}$. d 名勞工完成工程所需日數 $=1\div\frac{d}{30y}=\frac{30y}{d}$ y 名技工完成工程所需日數 $=1\div\frac{y}{30d}=\frac{30d}{15936-d}$ $\frac{30(15936-d)}{d}=\frac{4\times30d}{15936-d}$ $(15936 - d)^2 = 4d^2$ $15936 - d = 2d$ $d = 5312$	$d + y = 15936$ Amount of work for one technician per day $=\frac{1}{30d}$. Amount of work for one labour per day $=\frac{1}{30y}$. Days for d labours to finish the job $=1\div\frac{d}{30y}=\frac{30y}{d}$ Days for y technicians to finish the job $=1\div\frac{y}{30d}=\frac{30d}{15936-d}$ $\frac{30(15936-d)}{d}=\frac{4\times30d}{15936-d}$ $(15936 - d)^2 = 4d^2$ $15936 - d = 2d$ $d = 5312$

Remark: The Chinese version and the English version have different meaning.
Original version: …所需要的時間較挑選 y 個技工的多4倍… the time taken is 4 times as much as …
New version: …所需要的時間是挑選 y 個技工的4倍… the time taken is 4 times as much as …

Individual Event 3

I3.1 若 $\{x_0, y_0, z_0\}$ 為以下方程組的解，求 $a = x_0 + y_0 + z_0$ 的值。

If $\{x_0, y_0, z_0\}$ is a solution to the set of simultaneous equations below, determine the value of $a = x_0 + y_0 + z_0$.

$$\begin{cases} 2x - 2y + z = -15 \\ x + 2y + 2z = 18 \\ 2x - y + 2z = -5 \end{cases}$$

$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & -15 \\ 1 & 2 & 2 & 18 \\ 2 & -1 & 2 & -5 \end{array}\right] \begin{array}{c} R_2 \to R_1 \\ R_3 - R_1 \\ 2R_2 - R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 18 \\ 0 & 1 & 1 & 10 \\ 0 & 5 & 2 & 41 \end{array}\right] \begin{array}{c} R_1 - 2R_2 \\ \\ 5R_2 - R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 1 & 10 \\ 0 & 0 & 3 & 9 \end{array}\right] \begin{array}{c} \\ R_2 - \frac{1}{3}R_3 \\ \frac{1}{3}R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$x_0 = -2$, $y_0 = 7$, $z_0 = 3$

$a = -2 + 7 + 3 = 8$

I3.2 求 $b = \dfrac{\sqrt{6+2\sqrt{a}} + \sqrt{6-2\sqrt{a}}}{2}$ 的值。

Determine the value of $b = \dfrac{\sqrt{6+2\sqrt{a}} + \sqrt{6-2\sqrt{a}}}{2}$.

Reference: 2011 HI7, 2013 HI1, 2013 FI3.1, 2015 FG3.1, 2016 FG4.3

$$b = \frac{\sqrt{6+2\sqrt{8}} + \sqrt{6-2\sqrt{8}}}{2}$$

$$= \frac{\sqrt{\left(\sqrt{4}\right)^2 + 2\sqrt{4}\cdot\sqrt{2} + \left(\sqrt{2}\right)^2} + \sqrt{\left(\sqrt{4}\right)^2 - 2\sqrt{4}\cdot\sqrt{2} + \left(\sqrt{2}\right)^2}}{2}$$

$$= \frac{\sqrt{\left(\sqrt{4}+\sqrt{2}\right)^2} + \sqrt{\left(\sqrt{4}-\sqrt{2}\right)^2}}{2}$$

$$= \frac{\sqrt{4}+\sqrt{2}+\sqrt{4}-\sqrt{2}}{2} = 2$$

I3.3 若 x 是正整數且 $\log_{10} b^x > 3$，求 x 的最小值 c。

If x is a positive integer and $\log_{10} b^x > 3$, determine c, the minimum value of x.

$\log_{10} 2^x > 3 = \log_{10} 1000$

$2^9 = 512 < 1000 < 1024 = 2^{10}$

$c = 10$

I3.4 若 $f(x) = 2^0 + 2^1 + 2^2 + \cdots + 2^{x-2} + 2^{x-1}$，求 $d = f(c)$ 的值。

If $f(x) = 2^0 + 2^1 + 2^2 + \cdots + 2^{x-2} + 2^{x-1}$, determine the value of $d = f(c)$.

Reference: 2009 FI1.3, 2015 FI1.4

$d = f(10) = 2^0 + 2^1 + 2^2 + \cdots + 2^8 + 2^9$

$= 2^{10} - 1$

$= 1023$

Individual Event 4

I4.1 若 a 為正整數，求 a 的最大值使得 $ax^2 - (a-3)x + (a-2) = 0$ 有實根。

If a is a positive integer, determine the greatest value of a such that $ax^2 - (a-3)x + (a-2) = 0$ has real root(s).

$\Delta = (a-3)^2 - 4a(a-2) \geq 0$ $a^2 - 6a + 9 - 4a^2 + 8a \geq 0$ $3a^2 - 2a - 9 \leq 0$ Let $3a^2 - 2a - 9 = 0$ $a = \frac{2 \pm \sqrt{2^2 + 4(3)(9)}}{2 \cdot 3}$ $a = \frac{1 \pm \sqrt{28}}{3}$ $a \approx \frac{1 \pm 5.3}{3} = 2.1$ or -1.4 $-1.4 \leq a \leq 2.1$ a 的最大整數值= 2。	$\Delta = (a-3)^2 - 4a(a-2) \geq 0$ $a^2 - 6a + 9 - 4a^2 + 8a \geq 0$ $3a^2 - 2a - 9 \leq 0$ Let $3a^2 - 2a - 9 = 0$ $a = \frac{2 \pm \sqrt{2^2 + 4(3)(9)}}{2 \cdot 3}$ $a = \frac{1 \pm \sqrt{28}}{3}$ $a \approx \frac{1 \pm 5.3}{3} = 2.1$ 或 -1.4 $-1.4 \leq a \leq 2.1$ The largest integral value of $a = 2$.

I4.2 若 x 及 y 為實數且 $1 < y < x$ 及 $\log_x y + 3\log_y x = \frac{13}{a}$，求 $b = \frac{x + y^4}{x^2 + y^2}$ 的值。

If x and y are real numbers with $1 < y < x$ and $\log_x y + 3\log_y x = \frac{13}{a}$,

determine the value of $b = \frac{x + y^4}{x^2 + y^2}$.

設 $t = \log_x y$, 則 $\log_y x = \frac{1}{t}$ 原式變成： $t + \frac{3}{t} = \frac{13}{2}$ $2t^2 - 13t + 6 = 0$ $(2t-1)(t-6) = 0$ $t = \frac{1}{2}$ 或 $t = 6$ $\because 1 < y < x \therefore \log_x y < 1$ $\log_x y = \frac{1}{2}$ $y = \sqrt{x}$ $b = \frac{x + y^4}{x^2 + y^2} = \frac{x + x^2}{x^2 + x} = 1$	Let $t = \log_x y$, then $\log_y x = \frac{1}{t}$ The equation becomes: $t + \frac{3}{t} = \frac{13}{2}$ $2t^2 - 13t + 6 = 0$ $(2t-1)(t-6) = 0$ $t = \frac{1}{2}$ or $t = 6$ $\because 1 < y < x \therefore \log_x y < 1$ $\log_x y = \frac{1}{2}$ $y = \sqrt{x}$ $b = \frac{x + y^4}{x^2 + y^2} = \frac{x + x^2}{x^2 + x} = 1$

I4.3 一個袋中有紅球 $b + 2$ 個，白球 $b + 3$ 個及藍球 $b + 4$ 個，從袋中隨機抽出 3 個並不重新放進袋中。求三個抽出的球都是相同顏色的概率 c 的值。

A bag contains $b + 2$ red balls, $b + 3$ white balls and $b + 4$ blue balls. Three balls are randomly drawn from the bag without replacement. Determine the value of the probability, c, that the 3 balls are of the same colours.

紅球 = 3、白球 = 4、藍球 = 5 P(同色) $= \frac{3\times2\times1 + 4\times3\times2 + 5\times4\times3}{12\times11\times10}$ $= \frac{3}{44}$	Red balls = 3, White balls = 4, Blue balls = 5 P(same colour) $= \frac{3\times2\times1 + 4\times3\times2 + 5\times4\times3}{12\times11\times10}$ $= \frac{3}{44}$

I4.4 若 $\cos 2\theta = c$，求 $d = \sin^4\theta + \cos^4\theta$ 的值。

If $\cos 2\theta = c$, determine the value of $d = \sin^4\theta + \cos^4\theta$.

$d = (\cos^2\theta + \sin^2\theta)^2 - 2\sin^2\theta\cos^2\theta$

$d = 1 - 0.5\sin^2 2\theta$

$d = 1 - \frac{1}{2}\left(1 - \cos^2 2\theta\right)$

$d = \frac{1}{2}\left(1 + \frac{3^2}{44^2}\right)$

$d = \frac{1936 + 9}{2 \cdot 1936}$

$d = \frac{1945}{3872}$

Group Event 1

G1.1 若實數 x、y 及 z 滿足 $x+\frac{1}{y}=-1$，$y+\frac{1}{z}=-2$ 及 $z+\frac{1}{x}=-5$。求 $a=\frac{1}{xyz}$ 的值。

If real numbers x, y and z satisfy $x+\frac{1}{y}=-1$, $y+\frac{1}{z}=-2$ and $z+\frac{1}{x}=-5$. Determine the value of $a=\frac{1}{xyz}$. **(Reference: 2008 FG2.4, 2010 FG2.2)**

(1)×(2)×(3) – (1) – (2) – (3):

$$xyz+x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{xyz}-\left(x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=-10+8 \Rightarrow xyz+\frac{1}{xyz}=-2$$

$(xyz+1)^2=0$

$a=\frac{1}{xyz}=-1$

G1.2 若 $\left|x-|2x-1|\right|=\frac{1}{2}$ 為實數方程，求實根數量 b 的值。

If $\left|x-|2x-1|\right|=\frac{1}{2}$ is a real equation, determine the value of b, the number of real solutions of the equation.

$x-\|2x-1\|=\frac{1}{2}$ 或 $x-\|2x-1\|=-\frac{1}{2}$	$x-\|2x-1\|=\frac{1}{2}$ or $x-\|2x-1\|=-\frac{1}{2}$
$x-\frac{1}{2}=\|2x-1\|$ 或 $x+\frac{1}{2}=\|2x-1\|$	$x-\frac{1}{2}=\|2x-1\|$ or $x+\frac{1}{2}=\|2x-1\|$
$2x-1=\pm\left(x-\frac{1}{2}\right)$ 或 $2x-1=\pm\left(x+\frac{1}{2}\right)$	$2x-1=\pm\left(x-\frac{1}{2}\right)$ or $2x-1=\pm\left(x+\frac{1}{2}\right)$
$x=\frac{1}{2}$ 或 $x=\frac{3}{2}$, $\frac{1}{6}$	$x=\frac{1}{2}$ or $x=\frac{3}{2}$, $\frac{1}{6}$
b =實根數量= 3	b = number of real solution = 3

G1.3 若實數 x 及 y 滿足 $xy>0$ 及 $x+y=3$，求 $\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)$ 的最大值 c。

If real numbers x and y satisfy $xy > 0$ and $x + y = 3$,

find c, the maximum value of $\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)$.

$\because xy>0$, $x+y=3$	$\because xy>0$, $x+y=3$
$\therefore x>0$, $y>0$	$\therefore x>0$, $y>0$
$0<xy\le\left(\frac{x+y}{2}\right)^2=\frac{9}{4}$ (A.M. ≥ G.M.)	$0<xy\le\left(\frac{x+y}{2}\right)^2=\frac{9}{4}$ (A.M. ≥ G.M.)
$0<xy\le 1$ 或 $1\le xy\le\frac{9}{4}$	$0<xy\le 1$ or $1\le xy\le\frac{9}{4}$
$\frac{1}{xy}\ge 1$ 或 $\frac{4}{9}\le\frac{1}{xy}\le 1 \Rightarrow \frac{4}{9}\le\frac{1}{xy}$	$\frac{1}{xy}\ge 1$ or $\frac{4}{9}\le\frac{1}{xy}\le 1 \Rightarrow \frac{4}{9}\le\frac{1}{xy}$
$-\frac{4}{9}\ge-\frac{1}{xy}$	$-\frac{4}{9}\ge-\frac{1}{xy}$
$\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)=\frac{xy-(x+y)+1}{xy}=1-\frac{2}{xy}\le\frac{1}{9}=c$	$\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)=\frac{xy-(x+y)+1}{xy}=1-\frac{2}{xy}\le\frac{1}{9}=c$

G1.4 若實數 x 滿足 $x-\frac{1}{x}=3$，求 $d=x^5-\frac{1}{x^5}$ 的值。

If a real number x satisfies $x-\frac{1}{x}=3$, determine the value of $d=x^5-\frac{1}{x^5}$.

$\left(x-\frac{1}{x}\right)^2=3^2=9$

$x^2+\frac{1}{x^2}=11$

$\left(x^2+\frac{1}{x^2}\right)^2=11^2=121$

$x^4+\frac{1}{x^4}=119$

$d=x^5-\frac{1}{x^5}=\left(x-\frac{1}{x}\right)\left(x^4+x^2+1+\frac{1}{x^2}+\frac{1}{x^4}\right)$

$d = 3\times(119 + 11 + 1)$

$d = 393$

Group Event 2

G2.1 在六進制中，若 A 為 $12345_6 \div 13_6$ 的餘數，求 A 的值。

In base-6 system, if $12345_6 \div 13_6$ has remainder A, determine the value of A.

$12345_6 = 6^4 + 2\times6^3 + 3\times6^2 + 4\times6 + 5 = 1296 + 2\times216 + 108 + 24 + 5 = 1865_{10}$

$13_6 = 9_{10}$

$1865 \div 9$, $A = 2$

G2.2 立方體的任意兩個頂點可相連成一線段。若 B 為最多所能夠相連成的直線的數量，求 B 的值。

Any two vertices in a cube can form a line segment. If B is the greatest number of line segments thus formed, determine the value of B.

立方體有 8 個頂點。 從中選取兩點形成一線段。 $B = C^8_2 = 28$	There are 8 vertices in a cube. Select any two vertices to form a line segment. $B = C^8_2 = 28$

G2.3 若實數 x、y 及 z 滿足 $(x + y + z) = 30$ 及 $C = x^2 + y^2 + z^2$，求 C 的最小值。

If real numbers x, y and z satisfy $(x + y + z) = 30$ and $C = x^2 + y^2 + z^2$, determine the least value of C.

考慮 $t^2 - 2xt + x^2 = (t - x)^2 \cdots (1)$ $t^2 - 2yt + y^2 = (t - y)^2 \cdots (2)$ $t^2 - 2zt + z^2 = (t - z)^2 \cdots (3)$ (1) + (2) + (3): L.H.S. $= 3t^2 - 2(x + y + z)t + (x^2 + y^2 + z^2)$ 此函數必為非負 $\Delta = 4(x + y + z)^2 - 4(3)(x^2 + y^2 + z^2) \le 0$ $(1^2 + 1^2 + 1^2)(x^2 + y^2 + z^2) \ge (x + y + z)^2$ $3C \ge 30^2$ $C \ge 300$ C 的最小值 $= 300$	Consider $t^2 - 2xt + x^2 = (t - x)^2 \cdots (1)$ $t^2 - 2yt + y^2 = (t - y)^2 \cdots (2)$ $t^2 - 2zt + z^2 = (t - z)^2 \cdots (3)$ (1) + (2) + (3): L.H.S. $= 3t^2 - 2(x + y + z)t + (x^2 + y^2 + z^2)$ The function is always non-negative $\Delta = 4(x + y + z)^2 - 4(3)(x^2 + y^2 + z^2) \le 0$ $(1^2 + 1^2 + 1^2)(x^2 + y^2 + z^2) \ge (x + y + z)^2$ $3C \ge 30^2$ $C \ge 300$ The minimum value of $C = 300$

G2.4 已知 $D = (x - 1)^3 + 3$。當 $-3 \le x \le 3$，求 D 的最大值。

Given that $D = (x - 1)^3 + 3$. Determine the greatest value of D for $-3 \le x \le 3$.

$-4 \le x - 1 \le 2$ $-64 \le (x - 1)^3 \le 8$ $-61 \le (x - 1)^3 + 3 \le 11$ D 的最大值 $= 11$	$-4 \le x - 1 \le 2$ $-64 \le (x - 1)^3 \le 8$ $-61 \le (x - 1)^3 + 3 \le 11$ The greatest value of $D = 11$

Group Event 3

G3.1 設 a、b 及 c 為整數且 $1 < a < b < c$。若 $(ab-1)(bc-1)(ac-1)$可被 abc 整除，求 $ab+bc+ac-1$ 除以 abc 所得之餘數 R 的值。

Let a, b and c be integers with $1 < a < b < c$. If $(ab-1)(bc-1)(ac-1)$ is divisible by abc, determine the value of the remainder R when $ab+bc+ac-1$ is divided by abc.

$(ab-1)(bc-1)(ac-1)$ $=(abc)^2-abc(a+b+c)+(ab+bc+ca)-1$ 它可被abc 整除。 $\therefore ab+bc+ca-1=mabc$，$m$為整數。 餘數$R=0$	$(ab-1)(bc-1)(ac-1)$ $=(abc)^2-abc(a+b+c)+(ab+bc+ca)-1$ It is divisible by abc. $\therefore ab+bc+ca-1=mabc$, m is an integer The remainder $R=0$

G3.2 若 $0<x<1$，求 $S=\left(\frac{\sqrt{1+x}}{\sqrt{1+x}-\sqrt{1-x}}+\frac{1-x}{\sqrt{1-x^2}+x-1}\right)\cdot\left(\sqrt{\frac{1}{x^2}-1}-\frac{1}{x}\right)$ 的值。

If $0<x<1$, determine the value of $S=\left(\frac{\sqrt{1+x}}{\sqrt{1+x}-\sqrt{1-x}}+\frac{1-x}{\sqrt{1-x^2}+x-1}\right)\cdot\left(\sqrt{\frac{1}{x^2}-1}-\frac{1}{x}\right)$.

Reference: 2016 FI3.3

$$S=\left(\frac{\sqrt{1+x}}{\sqrt{1+x}-\sqrt{1-x}}+\frac{1-x}{\sqrt{1-x^2}+x-1}\right)\times\left(\sqrt{\frac{1}{x^2}-1}-\frac{1}{x}\right)$$

$$=\left\{\frac{\sqrt{1+x}\cdot\left(\sqrt{1+x}+\sqrt{1-x}\right)}{(1+x)-(1-x)}+\frac{(1-x)\cdot\left[\sqrt{1-x^2}-(x-1)\right]}{\left(1-x^2\right)-(x-1)^2}\right\}\times\left(\sqrt{\frac{1-x^2}{x^2}}-\frac{1}{x}\right)$$

$$=\left\{\frac{1+x+\sqrt{1-x^2}}{2x}+\frac{(1-x)\cdot\left[\sqrt{1-x^2}+(1-x)\right]}{\left(1-x^2\right)-\left(1-2x+x^2\right)}\right\}\times\left(\frac{\sqrt{1-x^2}}{x}-\frac{1}{x}\right)$$

$$=\left\{\frac{1+x+\sqrt{1-x^2}}{2x}+\frac{(1-x)\cdot\left[\sqrt{1-x^2}+(1-x)\right]}{2x(1-x)}\right\}\times\left(\frac{\sqrt{1-x^2}-1}{x}\right)$$

$$=\left[\frac{1+x+\sqrt{1-x^2}}{2x}+\frac{\sqrt{1-x^2}+(1-x)}{2x}\right]\times\left(\frac{\sqrt{1-x^2}-1}{x}\right)$$

$$=\left(\frac{2+2\sqrt{1-x^2}}{2x}\right)\times\left(\frac{\sqrt{1-x^2}-1}{x}\right)=\left(\frac{1+\sqrt{1-x^2}}{x}\right)\times\left(\frac{\sqrt{1-x^2}-1}{x}\right)$$

$$=\frac{\left(1-x^2\right)-1}{x^2}=-1$$

Remark: You may substitute $x = 0.5$ directly to find the value of c.

G3.3 求方程 $x^4+(x-4)^4=544$ 的實根之和 T 的值。

Determine the value of T, the sum of real roots of $x^4+(x-4)^4=544$.

Reference: 2014 FG4.4

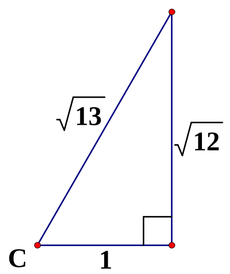
設 $t=2-x$，則 $x=t+2$，$x-4=t-2$ 方程變成：$(t+2)^4+(t-2)^4=544$ $2[t^4+6(2)^2t^2+2^4]=544$ $t^4+24t^2-256=0$ $(t^2+32)(t^2-8)=0$ $t^2=-32$ (捨去) or $t^2=8$ $x=2\pm2\sqrt{2}$ T =實根之和 $=4$	Let $t=2-x$, then $x=t+2$, $x-4=t-2$ The equation becomes: $(t+2)^4+(t-2)^4=544$ $2[t^4+6(2)^2t^2+2^4]=544$ $t^4+24t^2-256=0$ $(t^2+32)(t^2-8)=0$ $t^2=-32$ (rejected) or $t^2=8$ $x=2\pm2\sqrt{2}$ T = Sum of real roots = 4

G3.4 在三角形 ABC 中，$BC = a$，$\angle ABC = \frac{\pi}{3}$ 及面積為 $\sqrt{3}a^2$。求 $U = \tan(\angle ACB)$的值。

In triangle ABC, $BC = a$, $\angle ABC = \frac{\pi}{3}$ and its area is $\sqrt{3}a^2$.

Determine the value of $U = \tan(\angle ACB)$.

設 $AB = c$ $\frac{1}{2}ac\sin\frac{\pi}{3} = \sqrt{3}a^2$ $c = 4a$ $BC^2 = a^2 + (4a)^2 - 2a\cdot 4a\cos\frac{\pi}{3}$ $BC = \sqrt{13}a$ $\cos\angle ACB = \frac{a^2 + \left(\sqrt{13}a\right)^2 - (4a)^2}{2\cdot a\cdot\sqrt{13}a} = -\frac{1}{\sqrt{13}}$ $U = \tan\angle ACB = -\frac{\sqrt{13-1}}{1} = -2\sqrt{3}$	Let $AB = c$ $\frac{1}{2}ac\sin\frac{\pi}{3} = \sqrt{3}a^2$ $c = 4a$ $BC^2 = a^2 + (4a)^2 - 2a\cdot 4a\cos\frac{\pi}{3}$ $BC = \sqrt{13}a$ $\cos\angle ACB = \frac{a^2 + \left(\sqrt{13}a\right)^2 - (4a)^2}{2\cdot a\cdot\sqrt{13}a} = -\frac{1}{\sqrt{13}}$ $U = \tan\angle ACB = -\frac{\sqrt{13-1}}{1} = -2\sqrt{3}$



Group Event 4

G4.1 製作某玩具，需要先倒模，後上色。甲先生每日可以為 3 件玩具倒模，或為 15 件玩具上色；乙先生每日則可以為 5 件玩具倒模，或為 15 件玩具上色。各人每日只能倒模或上色，而不能同做兩事。若甲先生和乙先生合作，求最小多少日 P 才可以製作 120 件玩具。

To make a specific toy, it must be first moulded and then painted. Mr. A can mould 3 pieces of toys or paint 15 pieces of toys in one day, whereas Mr. B can mould 5 pieces or paint 15 pieces of toys in one day. Each of them can either mould or paint toys in one day, but not both. If Mr. A and Mr. B work together, determine the least number of days P to make 120 toys.

A倒模的速度比B慢，而A上色的速度和B一樣。因此，為了要使得製作120件玩具的日數最短，B所有時間皆被指派去倒模。假設A花了x日倒模，y日上色。 $3x + 5(x + y) = 120 \Rightarrow 8x + 5y = 120$ …… (1) $15y = 120 \Rightarrow y = 8$ …… (2) 代 (2) 入 (1): $8x + 40 = 120$ $x = 10$ 最小日數$P = 18$	The speed of making moulds for A is slower than B while the speed of painting for A is the same as B. So, in order to minimize the number of days to make 120 toys, all time of B is allocated for moulding. Let A uses x days for moulding and y days for painting. $3x + 5(x + y) = 120 \Rightarrow 8x + 5y = 120$ …… (1) $15y = 120 \Rightarrow y = 8$ …… (2) Sub. (2) into (1): $8x + 40 = 120$ $x = 10$ The least number of days $P = 18$

Remark: The following sentence is missing in the Chinese version:
各人每日只能倒模或上色，而不能同做兩事。

G4.2 在一個射鴨子遊戲中一男孩射了 10 發子彈，該男孩每發子彈射中鴨子的概率為 0.5。求他於最後一發子彈射中第六隻鴨子的概率 Q。

In a duck shooting game, a boy fires 10 shots. The probability of him shooting down a duck with a shot is 0.5. Determine the probability Q of him shooting down the 6th duck at the last shot.

P(最後一發子彈射中第六隻鴨子) = P(頭9發射中5隻，第10發射中1隻) $=\frac{C_5^9}{2^9}\cdot\frac{1}{2}=\frac{63}{512}$	P(shoot down the 6th duck at the last shot) = P(1-9 shots 5 ducks, last shot 6th duck) $=\frac{C_5^9}{2^9}\cdot\frac{1}{2}=\frac{63}{512}$

G4.3 如圖 1，求按箭咀方向由 A 往 B 的路線總數 R。

As shown in Figure 1 below, determine the number of ways R getting from point A to B with the direction indicated by the arrows.

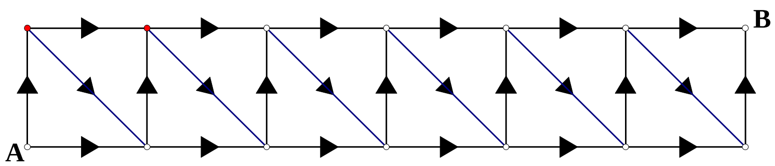
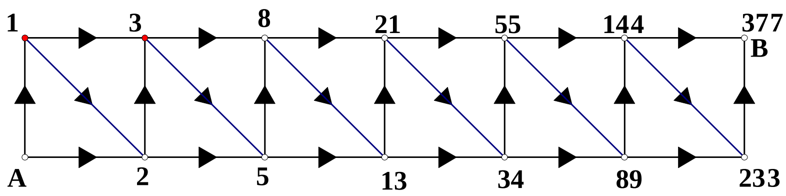


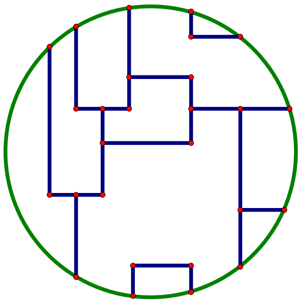
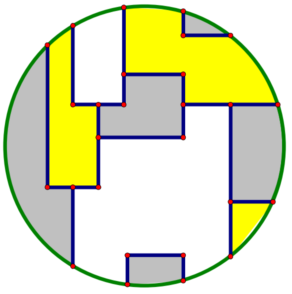
Figure 1/圖 1



$R = 377$

G4.4 如果用 3 款顏料替下圖中所有區域著色，並且相鄰的區域不可用相同顏料。求同一款顏料最多可用作上色的區域數目 S。

To shade all the regions inside the following circular map using 3 colours, for which adjacent regions must not be in the same colour. Determine the maximum number S of regions being shaded by the same colour.

$S = 5$

Individual Events

I1	A	4	I2	a	40	I3	A	6	I4	a	0
	B	*1 see the remark		b	9		B	48		b	9
	C	8		c	1		C	2		c	24
	D	488		d	2		D	3		d	1344

Group Events

G1	s	5	G2	u	$\frac{1}{2}$	G3	α	$2\sqrt{3}$	G4	A	4068289
	w	8		v	122.5		β	594		B	$-\frac{1}{24}$
	q	4		n	9		φ	$-\frac{\sqrt{3}}{2}$		C	$\frac{1}{128}$
	v	0		m	15		γ	9		D	2

Individual Event 1

I1.1 已知 $x^2 = y^2 - 4y$，其中 x 及 y 為整數。求 $A = x + y$ 的最大值。

Given that $x^2 = y^2 - 4y$, where x and y are integers. Determine the largest value of $A = x + y$.

$x^2 = y^2 - 4y + 4 - 4$

$4 = (y-2)^2 - x^2 = (y + x - 2)(y - x - 2)$

$y + x - 2$	$y - x - 2$	x	y	$x + y$
4	1	非整數	捨去	
2	2	0	4	4
1	4	非整數	捨去	
–1	–4	非整數	捨去	
–2	–2	0	0	0
–4	–1	非整數	捨去	

A的最大值 = 4

$x^2 = y^2 - 4y + 4 - 4$

$4 = (y-2)^2 - x^2 = (y + x - 2)(y - x - 2)$

$y + x - 2$	$y - x - 2$	x	y	$x + y$
4	1	non-integer	rejected	
2	2	0	4	4
1	4	non-integer	rejected	
–1	–4	non-integer	rejected	
–2	–2	0	0	0
–4	–1	non-integer	rejected	

The largest value of A = 4

I1.2 已知 $y=\sqrt{9A^2-12A+4}\pm\sqrt{A^2-4A+4}\pm\sqrt{A^2+6A+9}$，且 B 是 y 的最小值，求 B 的值。

Given that $y=\sqrt{9A^2-12A+4}\pm\sqrt{A^2-4A+4}\pm\sqrt{A^2+6A+9}$, and B is the least value of y, determine the value of B. (**Reference: 2016 FI4.3**)

$y=\sqrt{(3A-2)^2}\pm\sqrt{(A-2)^2}\pm\sqrt{(A+3)^2}$

$=\sqrt{(3\times4-2)^2}\pm\sqrt{(4-2)^2}\pm\sqrt{(4+3)^2}$

$= 10 \pm 2 \pm 7$

y 的最小值 = $B = 10 - 2 - 7 = 1$

$y=\sqrt{(3A-2)^2}\pm\sqrt{(A-2)^2}\pm\sqrt{(A+3)^2}$

$=\sqrt{(3\times4-2)^2}\pm\sqrt{(4-2)^2}\pm\sqrt{(4+3)^2}$

$= 10 \pm 2 \pm 7$

The least value of $y = B = 10 - 2 - 7 = 1$

Remark: Original version …y 的最小正數值…the least positive value of y

The value of y must be positive, there is no need to emphasize the word "positive".

I1.3 設 C 為正整數。已知 $144 + (B + 1)^C$ 為平方數，求 C 的值。

Let C be a positive integer. Given that $144 + (B + 1)^C$ is a perfect square, determine the value of C.

$12^2 + 2^C = m^2$，其中 m 為整數

$2^4\cdot(3^2 + 2^{C-4}) = m^2$

$3^2 + 2^{C-4} = n^2$，其中 n 為整數

$2^{C-4} = (n + 3)(n - 3)$

$n + 3 = 2^a$，$n - 3 = 2^b$，$a + b = C - 4$

$6 = 2^a - 2^b = 2^b(2^{a-b} - 1)$

$b = 1$，$2^{a-1} - 1 = 3$，$a = 3$

$C = 8$

$12^2 + 2^C = m^2$, where m is an integer

$2^4\cdot(3^2 + 2^{C-4}) = m^2$

$3^2 + 2^{C-4} = n^2$, where n is an integer

$2^{C-4} = (n + 3)(n - 3)$

$n + 3 = 2^a$, $n - 3 = 2^b$, $a + b = C - 4$

$6 = 2^a - 2^b = 2^b(2^{a-b} - 1)$

$b = 1$, $2^{a-1} - 1 = 3$, $a = 3$

$C = 8$

I1.4 已知 $x+\frac{1}{x}=C$，求 $D=x^3+\frac{1}{x^3}$ 的值。

Given that $x+\frac{1}{x}=C$, determine the value of $D=x^3+\frac{1}{x^3}$.

Reference 1991 HI3

$$x+\frac{1}{x}=8$$

$$\left(x+\frac{1}{x}\right)^2=64$$

$$\Rightarrow x^2+\frac{1}{x^2}=62$$

$$x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right)$$

$$= 8\times(62-1) = 488$$

Individual Event 2

I2.1 $7778^2 - 2223^2$ 之值的所有數字之和是 $\boldsymbol{a}$，求 $\boldsymbol{a}$ 的值。

Determine the value of $\boldsymbol{a}$, where $\boldsymbol{a}$ is the sum of all digits of $7778^2 - 2223^2$.

$7778^2 - 2223^2 = (7778 + 2223)(7778 - 2223)$ $= 10001 \times 5555$ $= 55555555$ 數位之和 $= \boldsymbol{a} = 5\times 8 = 40$	$7778^2 - 2223^2 = (7778 + 2223)(7778 - 2223)$ $= 10001 \times 5555$ $= 55555555$ Sum of all digits $= \boldsymbol{a} = 5\times 8 = 40$

I2.2 若 $\boldsymbol{b}$ 是乘積 $\boldsymbol{a}\times(\boldsymbol{a}-1)\times(\boldsymbol{a}-2)\times\cdots\times 2\times 1$ 的尾隨零的數量。求 $\boldsymbol{b}$ 的值。

$\boldsymbol{a}\times(\boldsymbol{a}-1)\times(\boldsymbol{a}-2)\times\cdots\times 2\times 1 = \overline{\cdots * \underbrace{00\cdots 0}_{\text{"0" 的數量是}\, b}}$， * 代表非零數字。

If the number of trailing zeros of the product $\boldsymbol{a}\times(\boldsymbol{a}-1)\times(\boldsymbol{a}-2)\times\cdots\times 2\times 1$ is $\boldsymbol{b}$, determine the value of $\boldsymbol{b}$.

$\boldsymbol{a}\times(\boldsymbol{a}-1)\times(\boldsymbol{a}-2)\times\cdots\times 2\times 1 = \overline{\cdots * \underbrace{00\cdots 0}_{\text{The number of "0" is } b}}$, * represents a non-zero digit.

Reference: 1994 FG7.1, 1996 HI3, 2011 HG7, 2012 FI1.4, 2012 FG1.3

方法一 當每一個因子 5 乘以 2 時，乘積 40!.的末位便出現一個 '0'。 40! 當中 2 的因子很明顯比 5 的因子多。 我們只需數一數 5 的因子。 5, 10, 15, … , 40; 共有 8 個數，每個數有最小一個 '5' 的因子。 25 這個數有兩個 5 因子。 5 的因子合共有 8 + 1 = 9 個 $\boldsymbol{b} = 9$	**Method 1** When each factor of 5 is multiplied by 2, a trailing zero will appear in 40!. The number of factors of 2 is clearly more than the number of factors of '5' in 40! It is sufficient to find the number of factors of 5. 5, 10, 15, ⋯ , 40; altogether 8 numbers, each have at least one factor of 5. The number "25" has two factors of 5. Total number of factors of 5 is 8 + 1= 9 $\boldsymbol{b} = 9$
方法二 我們可以用以下的連除式找出5的因子的數量: 5 | 4 0 5 | 8 ⋯ 0 1 ⋯ 3 ∴ 5 的因子合共有 8 + 1 = 9 個 $\boldsymbol{b} = 9$	**Method 2** We can find the total number of factors of 5 by division as follow: 5 | 4 0 5 | 8 ⋯ 1 1 ⋯ 3 ∴Total no. of factors of 5 is 8 + 1 = 9 $\boldsymbol{b} = 9$

I2.3 若 $\boldsymbol{c}$ 是 $2^{10} - 2^8 + 2^6 - 2^4 + 2^2$ 除以 $\boldsymbol{b}$ 的餘數，求 $\boldsymbol{c}$ 的值。

If $\boldsymbol{c}$ is the remainder when $2^{10} - 2^8 + 2^6 - 2^4 + 2^2$ is divided by $\boldsymbol{b}$, determine the value of $\boldsymbol{c}$.

$2^{10} - 2^8 + 2^6 - 2^4 + 2^2 = 1024 - 256 + 64 - 16 + 4$

$= 820 = 9\times 91 + 1$

$\boldsymbol{c} = 1$

I2.4 求整數 d，使得對於任何實數 x，$x^{13}+cx+90$ 可被 x^2-x+d 整除。

Determine the integral value of d, so that $x^{13}+cx+90$ is divisible by x^2-x+d for any real number x. (**Reference 24th Putnam competition 1963 B1**)

$x^{13}+x+90$ 可被 x^2-x+d 整除。
若 $d=0$ 及 $x^{13}+x+90$ 可被 x^2-x 整除。
$\Rightarrow x^{13}+x+90$ 可被 x 整除，不可能。
若 $d<0$，則 x^2-x+d 的判別式$=1-4d>0$
$\Rightarrow x^2-x+d$ 有兩個實數根
但是，$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{13}+x+90\right)=13x^{12}+1>0\ \forall x$
$\therefore x^{13}+x+90$ 絕對遞增 $\forall x$
$\Rightarrow x^{13}+x+90$ 只有一個實數根，矛盾!
$\therefore d>0$
代 $x=0$，d 整除 90
代 $x=1$，d 整除 92
$\therefore d$ 是 90 和 92 的公因數
d 的可能值$=1$ 或 2
代 $x=2$，$d+2$ 整除 $2^{13}+92=8284$
若 $d=1$，則 3 整除 8284，錯，捨去
$\therefore d=2$
事實上，利用短除
$x^{13}+x+90=(x^2-x+2)\times(x^{11}+x^{10}-x^9-3x^8-x^7+5x^6+7x^5-3x^4-17x^3-11x^2+23x+45)$

$x^{13}+x+90$ is divisible by x^2-x+d.
If $d=0$ and $x^{13}+x+90$ is divisible by x^2-x
$\Rightarrow x^{13}+x+90$ is divisible by x, impossible.
If $d<0$, then Δ of x^2-x+d is $1-4d>0$
$\Rightarrow x^2-x+d$ has two real roots
However, $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{13}+x+90\right)=13x^{12}+1>0\ \forall x$
$\therefore x^{13}+x+90$ is strictly increasing $\forall x$
$x^{13}+x+90$ has only one real root !!!
$\therefore d>0$
Put $x=0$, d divides 90
Put $x=1$, d divides 92
$\therefore d$ is the common factor of 90 and 92
Possible $d=1$ or 2
Put $x=2$, $d+2$ divides $2^{13}+92=8284$
If $d=1$, then 3 divides 8284, false, rejected
$\therefore d=2$
In fact, by using synthetic division,
$x^{13}+x+90=(x^2-x+2)\times(x^{11}+x^{10}-x^9-3x^8-x^7+5x^6+7x^5-3x^4-17x^3-11x^2+23x+45)$

	x^{13}	x^{12}	x^{11}	x^{10}	x^9	x^8	x^7	x^6	x^5	x^4	x^3	x	x	1
	1	0	0	0	0	0	0	0	0	0	0	0	1	90
1		1	1	–1	–3	–1	5	7	–3	–17	–11	23	45	
–2			–2	–2	2	6	2	–10	–14	6	34	22	–46	–90
	1	1	–1	–3	–1	5	7	–3	–17	–11	23	45	0	0
	x^{11}	x^{10}	x^9	x^8	x^7	x^6	x^5	x^4	x^3	x^2	x	1	x	1
							商 Quotient						餘數 remainder	

Individual Event 3

I3.1 已知 $\boldsymbol{a}$、$\boldsymbol{b}$、$\boldsymbol{c}$ 為實數，且 $\boldsymbol{A} = (3\boldsymbol{a} - \boldsymbol{X})^2 + (3\boldsymbol{b} - \boldsymbol{X})^2 + (3\boldsymbol{c} - \boldsymbol{X})^2 + 6$.
若 $\boldsymbol{X} = \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$ 及 $\boldsymbol{X}^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2 + \boldsymbol{c}^2$，求 $\boldsymbol{A}$ 的最小值。
Given that $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are real numbers and $\boldsymbol{A} = (3\boldsymbol{a} - \boldsymbol{X})^2 + (3\boldsymbol{b} - \boldsymbol{X})^2 + (3\boldsymbol{c} - \boldsymbol{X})^2 + 6$.
If $\boldsymbol{X} = \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$ 及 $\boldsymbol{X}^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2 + \boldsymbol{c}^2$, determine the least value of $\boldsymbol{A}$.

$\boldsymbol{A} = (3\boldsymbol{a} - \boldsymbol{X})^2 + (3\boldsymbol{b} - \boldsymbol{X})^2 + (3\boldsymbol{c} - \boldsymbol{X})^2 + 6$ $= 3\boldsymbol{X}^2 - (6\boldsymbol{a} + 6\boldsymbol{b} + 6\boldsymbol{c})\boldsymbol{X} + (9\boldsymbol{a}^2 + 9\boldsymbol{b}^2 + 9\boldsymbol{c}^2) + 6$ $= 3\boldsymbol{X}^2 - 6\boldsymbol{X}^2 + 9\boldsymbol{X}^2 + 6$ $= 6(\boldsymbol{X}^2 + 1) \ge 6$ 當 $\boldsymbol{X}^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2 + \boldsymbol{c}^2 = 0$ 時，等式成立。 即 $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{c} = 0$ $\boldsymbol{A}$ 的最小值= 6	$\boldsymbol{A} = (3\boldsymbol{a} - \boldsymbol{X})^2 + (3\boldsymbol{b} - \boldsymbol{X})^2 + (3\boldsymbol{c} - \boldsymbol{X})^2 + 6$ $= 3\boldsymbol{X}^2 - (6\boldsymbol{a} + 6\boldsymbol{b} + 6\boldsymbol{c})\boldsymbol{X} + (9\boldsymbol{a}^2 + 9\boldsymbol{b}^2 + 9\boldsymbol{c}^2) + 6$ $= 3\boldsymbol{X}^2 - 6\boldsymbol{X}^2 + 9\boldsymbol{X}^2 + 6$ $= 6(\boldsymbol{X}^2 + 1) \ge 6$ Equality holds when $\boldsymbol{X}^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2 + \boldsymbol{c}^2 = 0$ i.e. $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{c} = 0$ The least value of $\boldsymbol{A} = 6$

I3.2 假設,班中有 $\boldsymbol{A}$ 名男同學及 $30 - \boldsymbol{A}$ 名女同學。若男同學的平均體重為 60 kg，女同學的平均體重為 45 kg 及全班同學的平均體重為 $\boldsymbol{B}$ kg，求 $\boldsymbol{B}$ 的值。
Suppose that there are $\boldsymbol{A}$ boys and $30 - \boldsymbol{A}$ girls in a class. If the average weight of the boys is 60 kg, the average weight of the girls is 45 kg, and the average weight of the students in the class is $\boldsymbol{B}$ kg, determine the value of $\boldsymbol{B}$.

班中有 6 名男同學及 24 名女同學。 平均體重$= \dfrac{60\times6+45\times24}{30} = 48$ kg $\boldsymbol{B} = 48$	There are 6 boys and 24 girls. Average weight $= \dfrac{60\times6+45\times24}{30} = 48$ kg $\boldsymbol{B} = 48$

I3.3 若 $\boldsymbol{n}$ 是正整數、$\boldsymbol{a}_1 = \boldsymbol{B}$ 及 $\boldsymbol{a}_{\boldsymbol{n}+1} = \begin{cases} \dfrac{\boldsymbol{a}_n}{2} & \text{若 } \boldsymbol{a}_n \text{ 是偶數；} \\ 3\boldsymbol{a}_n + 1 & \text{若 } \boldsymbol{a}_n \text{ 是奇數。} \end{cases}$ 求 $\boldsymbol{C} = \boldsymbol{a}_{2018}$ 的最值。

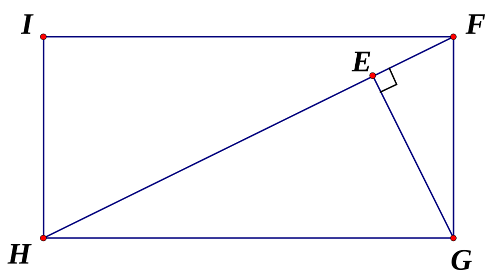
If $\boldsymbol{n}$ is a positive integer $\boldsymbol{a}_1 = \boldsymbol{B}$ and $\boldsymbol{a}_{\boldsymbol{n}+1} = \begin{cases} \dfrac{\boldsymbol{a}_n}{2} & \text{if } \boldsymbol{a}_n \text{ is even;} \\ 3\boldsymbol{a}_n + 1 & \text{if } \boldsymbol{a}_n \text{ is odd.} \end{cases}$

determine the value of $\boldsymbol{C} = \boldsymbol{a}_{2018}$.

$\boldsymbol{a}_1 = 48,\ \boldsymbol{a}_2 = 24,\ \boldsymbol{a}_3 = 12,\ \boldsymbol{a}_4 = 6,\ \boldsymbol{a}_5 = 3$ $\boldsymbol{a}_6 = 3\times3 + 1 = 10,\ \boldsymbol{a}_7 = 5,\ \boldsymbol{a}_8 = 3\times5 + 1 = 16$ $\boldsymbol{a}_9 = 8,\ \boldsymbol{a}_{10} = 4,\ \boldsymbol{a}_{11} = 2,\ \boldsymbol{a}_{12} = 1,$ $\boldsymbol{a}_{13} = 3\times1 + 1 = 4,\ \boldsymbol{a}_{14} = 2,\ \boldsymbol{a}_{15} = 1, \cdots$ $\boldsymbol{a}_{3k} = 1$ 由 $\boldsymbol{k} = 4, 5, 6, \cdots$ $\boldsymbol{a}_{3k+1} = 4$，$\boldsymbol{a}_{3k+2} = 2$ 由 $\boldsymbol{k} = 4, 5, 6, \cdots$ $2018 = 3\times672 + 2$ $\boldsymbol{a}_{2018} = 2$	$\boldsymbol{a}_1 = 48,\ \boldsymbol{a}_2 = 24,\ \boldsymbol{a}_3 = 12,\ \boldsymbol{a}_4 = 6,\ \boldsymbol{a}_5 = 3$ $\boldsymbol{a}_6 = 3\times3 + 1 = 10,\ \boldsymbol{a}_7 = 5,\ \boldsymbol{a}_8 = 3\times5 + 1 = 16$ $\boldsymbol{a}_9 = 8,\ \boldsymbol{a}_{10} = 4,\ \boldsymbol{a}_{11} = 2,\ \boldsymbol{a}_{12} = 1,$ $\boldsymbol{a}_{13} = 3\times1 + 1 = 4,\ \boldsymbol{a}_{14} = 2,\ \boldsymbol{a}_{15} = 1, \cdots$ $\boldsymbol{a}_{3k} = 1$ for $\boldsymbol{k} = 4, 5, 6, \cdots$ $\boldsymbol{a}_{3k+1} = 4$，$\boldsymbol{a}_{3k+2} = 2$ for $\boldsymbol{k} = 4, 5, 6, \cdots$ $2018 = 3\times672 + 2$ $\boldsymbol{a}_{2018} = 2$

I3.4 長方形 ***FGHI*** 被直綫 ***FH*** 分為兩個直角三角形。三角形 Δ***FGH*** 被直綫 ***EG*** 分為另外兩個直角三角形。若 ***FH*** ∶ ***FG*** = ***C*** ∶ 1 及三角形 Δ***EGH*** 與三角形 Δ***FEG*** 的面積比為 ***D*** ∶ 1，求 ***D*** 的值。

Suppose that a rectangle ***FGHI*** is divided into two right-angled triangles by line ***FH***. The triangle Δ***FGH*** is then divided into two right-angled triangles by line ***EG***. If the ratio of lengths ***FH*** : ***FG*** is ***C*** : 1 and the ratio of the areas of Δ***EGH*** to Δ***FEG*** is ***D*** : 1, determine the value of ***D***.



設∠***GFH*** = θ $FH : FG = 2 : 1 \Rightarrow \cos\theta = \frac{1}{2}$ θ = 60° ∠***GHE*** = 30° (三角形內角和) $EF = EG \div \tan 60° = \frac{EG}{\sqrt{3}}$ $EH = EG \div \tan 30° = \sqrt{3}EG$ $S_{\Delta EGH} : S_{\Delta FEG} = \frac{1}{2}EG\times EH : \frac{1}{2}EG\times EF$ $= EH : EF$ $= \sqrt{3}EG : \frac{EG}{\sqrt{3}} = 3 : 1$ ***D*** = 3	Let ∠***GFH*** = θ $FH : FG = 2 : 1 \Rightarrow \cos\theta = \frac{1}{2}$ θ = 60° ∠***GHE*** = 30° (∠ sum of Δ) $EF = EG \div \tan 60° = \frac{EG}{\sqrt{3}}$ $EH = EG \div \tan 30° = \sqrt{3}EG$ $S_{\Delta EGH} : S_{\Delta FEG} = \frac{1}{2}EG\times EH : \frac{1}{2}EG\times EF$ $= EH : EF$ $= \sqrt{3}EG : \frac{EG}{\sqrt{3}} = 3 : 1$ ***D*** = 3

Individual Event 4

I4.1 若 a 為 $(1^{2018}+2^{2018}+3^{2018}+4^{2018})\div 5$ 的餘數，求 a 的值。

If a is the remainder of $(1^{2018}+2^{2018}+3^{2018}+4^{2018})\div 5$, determine the value of a .

$1^{2018}\equiv 1 \pmod{10}$ $2^1=2,\ 2^2=4,\ 2^3=8,\ 2^4=16,\ 2^5=32$ $2018=4\times504+2,\ 2^{2018}\equiv 4 \pmod{10}$ $3^1=3,\ 3^2=9,\ 3^3=27,\ 3^4=81,\ 3^5=243$ $3^{2018}\equiv 9 \pmod{10}$ $4^1=4,\ 4^2=16,\ 4^3=64$ $2018=2\times1009$ $4^{2018}\equiv 6 \pmod{10}$ $1^{2018}+2^{2018}+3^{2018}+4^{2018}\equiv 1+4+9+6\equiv 0$ 餘數 = $a=0$	$1^{2018}\equiv 1 \pmod{10}$ $2^1=2,\ 2^2=4,\ 2^3=8,\ 2^4=16,\ 2^5=32$ $2018=4\times504+2,\ 2^{2018}\equiv 4 \pmod{10}$ $3^1=3,\ 3^2=9,\ 3^3=27,\ 3^4=81,\ 3^5=243$ $3^{2018}\equiv 9 \pmod{10}$ $4^1=4,\ 4^2=16,\ 4^3=64$ $2018=2\times1009$ $4^{2018}\equiv 6 \pmod{10}$ $1^{2018}+2^{2018}+3^{2018}+4^{2018}\equiv 1+4+9+6\equiv 0$ The remainder = $a=0$

I4.2 若 x、y 為正整數及 b 為 x、y 組合的數量使得它們的乘積 $x\times y=\overline{1aa}$，求 b 的值。

If x , y are positive integers numbers and b is the number of groups of x, y such that the product $x\times y=\overline{1aa}$, determine the value of b .

$xy=100$

$(x, y)=(1, 100), (2, 50), (4, 25), (5, 20), (10, 10), (20, 5), (25, 4), (50, 2), (100, 1)$

$b=9$

I4.3 若對於正整數 $x>y>z$， $xyz+xy+xz+yz+x+y+z+1=30b+87$。求 $c=x+y+z$ 的值。

If $xyz+xy+xz+yz+x+y+z+1=30b+87$ for positive integers $x>y>z$, determine the value of $c=x+y+z$.

Reference: 2004 HG6

$(x+1)(y+1)(z+1)=30\times9+87=357$

$(x+1)(y+1)(z+1)=17\times7\times3$

$x=16,\ y=6,\ z=2$

$c=16+6+2=24$

I4.4 若某長方形的面積為 d cm^2，它能被邊長為 $\frac{c}{3}$ cm 的正方形階磚密鋪，若該長方形亦能被闊度為 $\frac{c}{2}$ cm、長度為 7 cm 的長方形階磚密鋪，求 d 的最小值。

Let d cm^2 be the area of a rectangle that can be tessellated by square tiles with sides length of $\frac{c}{3}$ cm. If the rectangle can also be tessellated by rectangular tiles with width of $\frac{c}{2}$ cm and length of 7 cm, determine the least value of d .

$\frac{c}{3}=8,\ \frac{c}{2}=12$ 假設長方形的長、度為 $8p$ cm×$8q$ cm，其中 p 及 q 為正整數。 $8p=12r\cdots(1),\ 8q=7s\cdots(2)$ 其中 r 及 s 為正整數。 由(1)式， $2p=3r$ 最小值為 $s=8$、$q=7$、$r=2$、$p=3$ $d=8p\times8q=64\times3\times7=1344$	$\frac{c}{3}=8,\ \frac{c}{2}=12$ Let the dimensions of the rectangle be $8p$ cm×$8q$ cm , where p and q are positive integers. $8p=12r\cdots(1),\ 8q=7s\cdots(2)$ where r and s are positive integers From (1), $2p=3r$ For minimum values $s=8,\ q=7,\ r=2,\ p=3$ $d=8p\times8q=64\times3\times7=1344$

Remark: original question …求 d 的最小正數值。 …the least positive value of d .

d must be positive, there is no need to emphasize the word "positive".

Group Event 1

G1.1 瑪莉和小明在中文科、英文科及數學科獲得的分數為 $\boldsymbol{s}$ 或 $\boldsymbol{t}$，及 $\boldsymbol{s} > \boldsymbol{t} > 0$。若瑪莉於中文科的分數比小明的高以及小明於英文的分數比瑪莉的高，而瑪莉和小明的總分分別為 12 分和 9 分。求 $\boldsymbol{s}$ 的值。

Suppose that Mary and Ming obtained a score of either $\boldsymbol{s}$ or $\boldsymbol{t}$ in each of the subjects: Chinese, English and Mathematics, where $\boldsymbol{s} > \boldsymbol{t} > 0$. It is known that Mary did better in Chinese but Ming did better in English. Mary's and Ming's total scores are 12 and 9 respectively. Determine the value of $\boldsymbol{s}$.

根據已知資料，

	中文科	英文科	數學科	總分
瑪莉	$\boldsymbol{s}$	$\boldsymbol{t}$	$\boldsymbol{s}$	12
小明	$\boldsymbol{t}$	$\boldsymbol{s}$	$\boldsymbol{t}$	9

$2\boldsymbol{s} + \boldsymbol{t} = 12$ …… (1)

$2\boldsymbol{t} + \boldsymbol{s} = 9$ …… (2)

2(1) – (2): $3\boldsymbol{s} = 15 \Rightarrow \boldsymbol{s} = 5$

According to the given information,

	Chinese	English	Mathematics	Total
Mary	$\boldsymbol{s}$	$\boldsymbol{t}$	$\boldsymbol{s}$	12
Ming	$\boldsymbol{t}$	$\boldsymbol{s}$	$\boldsymbol{t}$	9

$2\boldsymbol{s} + \boldsymbol{t} = 12$ …… (1)

$2\boldsymbol{t} + \boldsymbol{s} = 9$ …… (2)

2(1) – (2): $3\boldsymbol{s} = 15 \Rightarrow \boldsymbol{s} = 5$

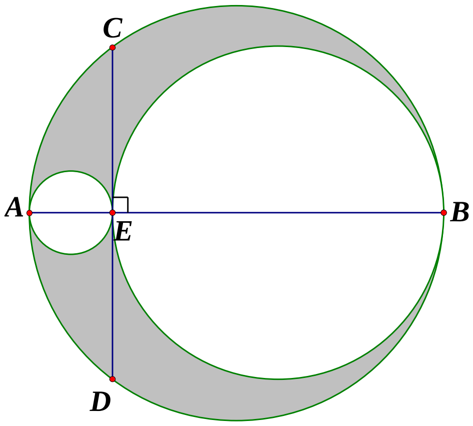
Remark: Original question: … 分數為 $\boldsymbol{s}$ 或 $\boldsymbol{t}$ 的整數… an integral score of either $\boldsymbol{s}$ or $\boldsymbol{t}$

$\boldsymbol{s}$ and $\boldsymbol{t}$ can be solved without the information of integral value.

G1.2 已知兩圓的直徑為 ***AE*** 及 ***BE***，內接於直徑為 ***AB*** 的圓中。若 $CE \perp AB$，$AB = 10$，$CE = 4$ 及陰影部份總面積為 $w\pi$，求 w 的值。

Given that the two circles, one with diameter ***AE*** and the other with diameter ***BE***, are inscribed by a larger circle with diameter ***AB***. If $CE \perp AB$ with $AB = 10$ and $CE = 4$, and the total area of the shaded regions is $w\pi$, determine the value of w.

Reference: 1990 HG10



將 ***CE*** 延長交圓形於 ***D***。
假設三個以 ***AE***、***BE*** 及 ***AB*** 為直徑凡圓形的半徑分別為 $\boldsymbol{a}$、$\boldsymbol{b}$ 及 $\boldsymbol{c}$。

$2\boldsymbol{a} + 2\boldsymbol{b} = 2\boldsymbol{c} = 10$

$\boldsymbol{c} = 5$ 及 $\boldsymbol{a} + \boldsymbol{b} = 5$ …… (1)

利用相交弦定理，$2\boldsymbol{a}\times 2\boldsymbol{b} = 4^2$

$\boldsymbol{ab} = 4$ …… (2)

陰影面積 $= \pi c^2 - \pi a^2 - \pi b^2$
$= \pi[5^2 - (a^2 + b^2)]$
$= \pi[5^2 - (a + b)^2 + 2ab]$
$= \pi(5^2 - 5^2 + 2\times 4)$ 由(1)及(2)所得
$= 8\pi$

$\boldsymbol{w} = 8$

Produce ***CE*** to meet the circle again at ***D***.
Let the radii of the 3 circles with diameters ***AE***, ***BE*** and ***AB*** are $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively.

$2\boldsymbol{a} + 2\boldsymbol{b} = 2\boldsymbol{c} = 10$

$\boldsymbol{c} = 5$ and $\boldsymbol{a} + \boldsymbol{b} = 5$ …… (1)

By intersecting chords theorem, $2\boldsymbol{a}\times 2\boldsymbol{b} = 4^2$

$\boldsymbol{ab} = 4$ …… (2)

Shaded area $= \pi c^2 - \pi a^2 - \pi b^2$
$= \pi[5^2 - (a^2 + b^2)]$
$= \pi[5^2 - (a + b)^2 + 2ab]$
$= \pi(5^2 - 5^2 + 2\times 4)$ by (1) and (2)
$= 8\pi$

$\boldsymbol{w} = 8$

G1.3 設 m 及 r 為非負整數。若 $f(7m + r) = r$，求 $q = f(2^{2018})$ 的值。

Let m and r be non-negative integers.

If $f(7m + r) = r$, determine the value of $q = f(2^{2018})$.

我們找出 $2^{2018} \div 7$ 的餘數。 $2 \div 7 \cdots\cdots 2$ $2^2 \div 7 \cdots\cdots 4$ $2^3 \div 7 \cdots\cdots 1$ $2^4 \div 7 \cdots\cdots 2$ 餘數出現的規律就是每隔 3 的倍數重複一次。 $2018 = 3\times672 + 2$，$2^{2018} \div 7$ 的餘數是 4。 $q = f(2^{2018}) = f(7m + 4) = 4$	We find the remainder of $2^{2018} \div 7$. $2 \div 7 \cdots\cdots 2$ $2^2 \div 7 \cdots\cdots 4$ $2^3 \div 7 \cdots\cdots 1$ $2^4 \div 7 \cdots\cdots 2$ The pattern of the remainders repeats for every multiple of 3. $2018 = 3\times672 + 2$, the remainder of $2^{2018} \div 7$ is 4 . $q = f(2^{2018}) = f(7m + 4) = 4$

G1.4 在五進制中，若 v 為 $234234_5 \div 234_5$ 的餘數，求 v 的值。

In base 5 system, if v is the remainder of $234234_5 \div 234_5$, determine the value of v .

$234234_5 = (234\times1000 + 234)_5$ $= (234\times1001)_5$ 餘數 $v = 0$	$234234_5 = (234\times1000 + 234)_5$ $= (234\times1001)_5$ The remainder is $v = 0$

Group Event 2

G2.1 已知 $\frac{1-2^{-1/u}}{2^{-1/u}-2^{-2/u}}=4$，求 $\boldsymbol{u}$ 的值。Given that $\frac{1-2^{-1/u}}{2^{-1/u}-2^{-2/u}}=4$, determine the value of $\boldsymbol{u}$.

Reference: 2018 HI5

$$\frac{\left(1-\frac{1}{2^{\frac{1}{u}}}\right)}{\left(\frac{1}{2^{\frac{1}{u}}}-\frac{1}{2^{\frac{2}{u}}}\right)}\cdot\frac{2^{\frac{2}{u}}}{2^{\frac{2}{u}}}=4$$

$$\frac{2^{\frac{1}{u}}\left(2^{\frac{1}{u}}-1\right)}{2^{\frac{1}{u}}-1}=4$$

$$2^{\frac{1}{u}}=2^2$$

$$\boldsymbol{u}=\frac{1}{2}$$

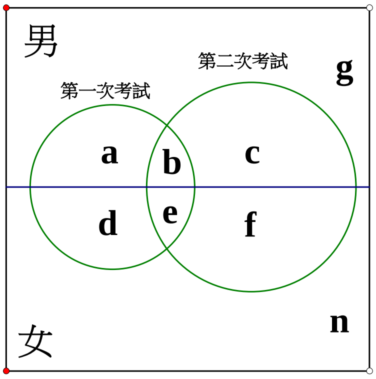
G2.2 已知 $\boldsymbol{b}\geq 1$、$\boldsymbol{a}-12\boldsymbol{b}=15$ 及 $\boldsymbol{x}$ 是實數，求 $\boldsymbol{v}=\frac{(\boldsymbol{x}-\boldsymbol{a})^2}{2\boldsymbol{b}}+5\boldsymbol{x}$ 的最小值。

Given that $\boldsymbol{b}\geq 1$, $\boldsymbol{a}-12\boldsymbol{b}=15$ and $\boldsymbol{x}$ is a real number, determine the least value of $\boldsymbol{v}=\frac{(\boldsymbol{x}-\boldsymbol{a})^2}{2\boldsymbol{b}}+5\boldsymbol{x}$.

$\boldsymbol{v}=\frac{(\boldsymbol{x}-\boldsymbol{a})^2}{2\boldsymbol{b}}+5\boldsymbol{x}$ $=\frac{(\boldsymbol{x}-12\boldsymbol{b}-15)^2}{2\boldsymbol{b}}+5\boldsymbol{x}$ $=\frac{\boldsymbol{x}^2-2(12\boldsymbol{b}+15)\boldsymbol{x}+(12\boldsymbol{b}+15)^2+10\boldsymbol{bx}}{2\boldsymbol{b}}$ $=\frac{\boldsymbol{x}^2-2(7\boldsymbol{b}+15)\boldsymbol{x}+(12\boldsymbol{b}+15)^2}{2\boldsymbol{b}}$ $=\frac{[\boldsymbol{x}-(7\boldsymbol{b}+15)]^2+(12\boldsymbol{b}+15)^2-(7\boldsymbol{b}+15)^2}{2\boldsymbol{b}}$ $=\frac{[\boldsymbol{x}-(7\boldsymbol{b}+15)]^2+5\boldsymbol{b}(19\boldsymbol{b}+30)}{2\boldsymbol{b}}$ $=\frac{[\boldsymbol{x}-(7\boldsymbol{b}+15)]^2}{2\boldsymbol{b}}+\frac{5}{2}(19\boldsymbol{b}+30)$ $\geq 0+\frac{5}{2}(19\times 1+30)=\frac{245}{2}=122.5$ $\boldsymbol{v}$ 的最小值= 122.5	$\boldsymbol{v}=\frac{(\boldsymbol{x}-\boldsymbol{a})^2}{2\boldsymbol{b}}+5\boldsymbol{x}$ $=\frac{(\boldsymbol{x}-12\boldsymbol{b}-15)^2}{2\boldsymbol{b}}+5\boldsymbol{x}$ $=\frac{\boldsymbol{x}^2-2(12\boldsymbol{b}+15)\boldsymbol{x}+(12\boldsymbol{b}+15)^2+10\boldsymbol{bx}}{2\boldsymbol{b}}$ $=\frac{\boldsymbol{x}^2-2(7\boldsymbol{b}+15)\boldsymbol{x}+(12\boldsymbol{b}+15)^2}{2\boldsymbol{b}}$ $=\frac{[\boldsymbol{x}-(7\boldsymbol{b}+15)]^2+(12\boldsymbol{b}+15)^2-(7\boldsymbol{b}+15)^2}{2\boldsymbol{b}}$ $=\frac{[\boldsymbol{x}-(7\boldsymbol{b}+15)]^2+5\boldsymbol{b}(19\boldsymbol{b}+30)}{2\boldsymbol{b}}$ $=\frac{[\boldsymbol{x}-(7\boldsymbol{b}+15)]^2}{2\boldsymbol{b}}+\frac{5}{2}(19\boldsymbol{b}+30)$ $\geq 0+\frac{5}{2}(19\times 1+30)=\frac{245}{2}=122.5$ The least value of $\boldsymbol{v}$ = 122.5

G2.3 若班中有 20 位男同學及 15 位女同學參加兩次考試。已知 8 位同學在第一次考試中不合格，12 位同學在第二次考試中不合格，及 6 位同學於兩次考試均不合格。若 5 位男同學在第一次考試中不合格，7 位男同學在第二次考試中不合格，4 位男同學兩次考試均不合格及 n 位女同學兩次考試均合格，求 n 的值。

Suppose that there were 20 boys and 15 girls in a class taking two examinations. Given that 8 students failed in the first examinations, 12 students failed in the second examinations, and 6 students failed in both examinations. If 5 boys failed in the first examinations, 7 boys failed in the second examinations, 4 boys failed in both examinations, and n girls passed in both examinations, determine the value of n .



如上溫氏圖所示，左圓表示在第一次考試中不合格的同學。右圓表示在第二次考試中不合格的同學。兩圓重疊部分(b 及 e)表示在兩次考試中皆不合格的同學。g 及 n 在兩圓以外，表示在兩次考試中皆合格的同學。上半部份 (a, b, c, g) 表示男同學。下半部份 (d, e, f, n) 表示女同學。

根據已給資料，

$b + e = 6 \cdots (1)$

$a + d + (b + e) = 8 \Rightarrow a + d = 2 \cdots (2)$

$c + f + (b + e) = 12 \Rightarrow c + f = 6 \cdots (3)$

$b = 4 \cdots (4)$

$a + b = 5 \Rightarrow a = 1 \cdots (5)$

$b + c = 7 \Rightarrow c = 3 \cdots (6)$

代 (5) 入 (2): $1 + d = 2 \Rightarrow d = 1 \cdots (7)$

代 (4) 入 (1): $4 + e = 6 \Rightarrow e = 2 \cdots (8)$

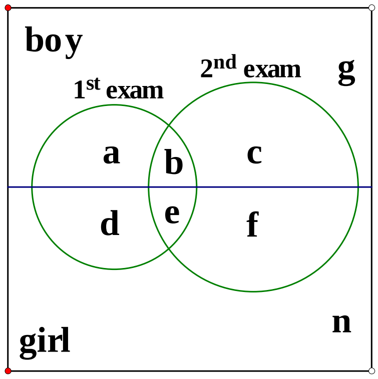
代 (6) 入 (3): $3 + f = 6 \Rightarrow f = 3 \quad \cdots (9)$

女同學數目 $d + e + f + n = 15 \cdots (10)$

代 (7)、(8)、(9) 入 (10): $1 + 2 + 3 + n = 15$

$n = 9$

9名女同學在兩次考試中皆合格。



As shown in the Venn diagram, the left circle represents the students failed in the first examination. The right circle represents the students failed in the second examination. The overlapping part of the two circles (b and e) represents the students failing in both examinations. g and n outside the circles represent students passed in both examinations. The upper part (a, b, c, g) represents boys. The lower part (d, e, f, n) represents girls.

According to the given information,

$b + e = 6 \cdots (1)$

$a + d + (b + e) = 8 \Rightarrow a + d = 2 \cdots (2)$

$c + f + (b + e) = 12 \Rightarrow c + f = 6 \cdots (3)$

$b = 4 \cdots (4)$

$a + b = 5 \Rightarrow a = 1 \cdots (5)$

$b + c = 7 \Rightarrow c = 3 \cdots (6)$

Sub. (5) into (2): $1 + d = 2 \Rightarrow d = 1 \cdots (7)$

Sub. (4) into (1): $4 + e = 6 \Rightarrow e = 2 \cdots (8)$

Sub. (6) into (3): $3 + f = 6 \Rightarrow f = 3 \quad \cdots (9)$

Number of girls: $d + e + f + n = 15 \cdots (10)$

Sub. (7), (8), (9) into (10): $1 + 2 + 3 + n = 15$

$n = 9$

9 girls passed in both examinations.

G2.4 求最小正整數 $\boldsymbol{m}$，使得 $\boldsymbol{m}^{200} > 6^{300}$。

Determine the least positive integer $\boldsymbol{m}$ such that $\boldsymbol{m}^{200} > 6^{300}$.

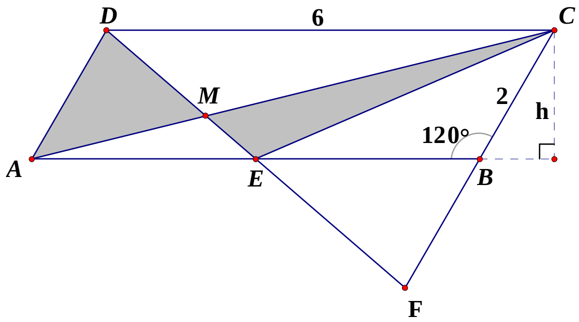
Reference: 1996 HI4, 1999 FG5.3, 2008 FI4.3

$(\boldsymbol{m}^2)^{100} > (6^3)^{100}$ $\boldsymbol{m}^2 > 216$ $\boldsymbol{m} > \sqrt{216} > \sqrt{196} = 14$ $\boldsymbol{m}$的最小正整數= 15	$(\boldsymbol{m}^2)^{100} > (6^3)^{100}$ $\boldsymbol{m}^2 > 216$ $\boldsymbol{m} > \sqrt{216} > \sqrt{196} = 14$ The least positive integer $\boldsymbol{m}$ = 15

Group Event 3

G3.1 AC 是平行四邊形 $ABCD$ 的對角綫，$CD = 6$，$BC = 2$ 及 $\angle ABC = 120°$。若 E 是 AB 的中點，AC 與 DE 相交於 M 及陰影部分的總面積是α，求 α 的值。

$ABCD$ is a parallelogram with diagonal AC, $CD = 6$, $BC = 2$, and $\angle ABC = 120°$. If E is the midpoint of AB, AC and DE intersect at M, and the total area of the shaded regions in α, determine the value of α.



Reference 1998 HG5, 2016 HI14

$AB = 6$，$AE = 3$，假設平行四邊形的高為 h。 $S_{ABCD} = 6\times2 \sin 120° = 6\sqrt{3}$ $h = 2\sin 120° = \sqrt{3}$ $S_{\Delta ACD} = \frac{1}{2}\cdot 6h = 3\sqrt{3}$ $S_{\Delta ACE} = \frac{1}{2}\cdot 3h = \frac{3\sqrt{3}}{2}$ 將 DE 及 CB 延長並相交於 F。 $AE = EB$ (E 是 AB 的中點) $\angle AED = \angle BEF$ (對頂角) $\angle EAD = \angle EBF$ (交錯角，AD // CF) $\Delta ADE \cong \Delta BFE$ (A.S.A.) $BF = AD = 2$ (全等三角形的對應角) $CF = CB + BF = 2 + 2 = 4$ $\angle AMD = \angle CMF$ (對頂角) $\angle ADM = \angle CFM$ (交錯角，AD // CF) $\angle DAM = \angle FCM$ (交錯角，AD // CF) $\Delta ADM \cong \Delta FCM$ (等角) $AM : MC = AD : CF$ (相似三角形的對應邊) $AM : MC = 2 : 4 = 1 : 2$ ΔADM 與 ΔCDM 同高不同底 $S_{\Delta ADM} = \frac{1}{1+2}\cdot S_{\Delta ACD} = \frac{1}{3}\cdot 3\sqrt{3} = \sqrt{3}$ ΔAEM 與 ΔCEM 同高不同底 $S_{\Delta CEM} = \frac{2}{1+2}\cdot S_{\Delta ACE} = \frac{2}{3}\cdot\frac{3\sqrt{3}}{2} = \sqrt{3}$ 陰影面積 $\alpha = \sqrt{3} + \sqrt{3} = 2\sqrt{3}$	$AB = 6$, $AE = 3$, let the height of the //-gram be h. $S_{ABCD} = 6\times2 \sin 120° = 6\sqrt{3}$ $h = 2\sin 120° = \sqrt{3}$ $S_{\Delta ACD} = \frac{1}{2}\cdot 6h = 3\sqrt{3}$ $S_{\Delta ACE} = \frac{1}{2}\cdot 3h = \frac{3\sqrt{3}}{2}$ Produce DE and CB to meet at F. $AE = EB$ (E is the mid-point of AB) $\angle AED = \angle BEF$ (vert. opp. ∠s) $\angle EAD = \angle EBF$ (alt. ∠s, AD // CF) $\Delta ADE \cong \Delta BFE$ (A.S.A.) $BF = AD = 2$ (corr. ∠s, ≅ Δ's) $CF = CB + BF = 2 + 2 = 4$ $\angle AMD = \angle CMF$ (vert. opp. ∠s) $\angle ADM = \angle CFM$ (alt. ∠s, AD // CF) $\angle DAM = \angle FCM$ (alt. ∠s, AD // CF) $\Delta ADM \cong \Delta FCM$ (equiangular) $AM : MC = AD : CF$ (corr. sides, ~Δs) $AM : MC = 2 : 4 = 1 : 2$ ΔADM and ΔCDM have the same height but different bases $S_{\Delta ADM} = \frac{1}{1+2}\cdot S_{\Delta ACD} = \frac{1}{3}\cdot 3\sqrt{3} = \sqrt{3}$ ΔAEM and ΔCEM have the same height but different bases $S_{\Delta CEM} = \frac{2}{1+2}\cdot S_{\Delta ACE} = \frac{2}{3}\cdot\frac{3\sqrt{3}}{2} = \sqrt{3}$ Shaded area $\alpha = \sqrt{3} + \sqrt{3} = 2\sqrt{3}$

G3.2 設 β為三位正整數且能被 11 整除，且其商相等於其值的各數字之和的三倍，求 β 的值。

If β is a 3-digit positive integer that is divisible by 11 and whose quotient when divided by 11 is 3 times the sum of its digits, determine the value of β .

假設該數 $\beta = 100\boldsymbol{a} + 10\boldsymbol{b} + \boldsymbol{c}$，其中$\boldsymbol{a}$、$\boldsymbol{b}$ 及 $\boldsymbol{c}$ 為 0 至 9 之間的整數及 $\boldsymbol{a} \neq 0$。 它能被11整除$\Rightarrow \boldsymbol{a} + \boldsymbol{c} - \boldsymbol{b} = 11\boldsymbol{k}$，$\boldsymbol{k} = 0$或1 $\boldsymbol{b} = \boldsymbol{a} + \boldsymbol{c} - 11\boldsymbol{k}$ …… (1) 代 (1) 入 $\beta = 100\boldsymbol{a} + 10(\boldsymbol{a} + \boldsymbol{c} - 11\boldsymbol{k}) + \boldsymbol{c}$ $= 110\boldsymbol{a} + 11\boldsymbol{c} - 110\boldsymbol{k}$ $= 11(10\boldsymbol{a} + \boldsymbol{c} - 10\boldsymbol{k})$ $\boldsymbol{Q}$ = 商 $= 10\boldsymbol{a} + \boldsymbol{c} - 10\boldsymbol{k}$ …… (2) 由已知資料， $\boldsymbol{Q} = 3(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$ …… (3) 代 (1) 及 (2) 入 (3): $10\boldsymbol{a} + \boldsymbol{c} - 10\boldsymbol{k} = 3(2\boldsymbol{a} + 2\boldsymbol{c} - 11\boldsymbol{k})$ $4\boldsymbol{a} + 23\boldsymbol{k} = 5\boldsymbol{c}$ 當$\boldsymbol{k} = 0$，$4\boldsymbol{a} = 5\boldsymbol{c} \Rightarrow \boldsymbol{a} = 5$，$\boldsymbol{c} = 4$，$\boldsymbol{b} = \boldsymbol{a} + \boldsymbol{c} = 9$ $\beta = 594$ 當 $\boldsymbol{k} = 1$，$4\boldsymbol{a} + 23 = 5\boldsymbol{c}$ 當 $\boldsymbol{c} = 5$、6、8 或 9，對 $\boldsymbol{a}$ 沒有整數解 當 $\boldsymbol{c} = 7$，$\boldsymbol{a} = 3$，$\boldsymbol{b} = \boldsymbol{a} + \boldsymbol{c} - 11\boldsymbol{k} < 0$，捨去	Let the integer be $\beta = 100\boldsymbol{a} + 10\boldsymbol{b} + \boldsymbol{c}$, where $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are integers from 0, 1, ⋯ , 9 and $\boldsymbol{a} \neq 0$. It is divisible by 11 $\Rightarrow \boldsymbol{a} + \boldsymbol{c} - \boldsymbol{b} = 11\boldsymbol{k}$, $\boldsymbol{k} = 0$ or 1 $\boldsymbol{b} = \boldsymbol{a} + \boldsymbol{c} - 11\boldsymbol{k}$ …… (1) Sub. (1) into $\beta = 100\boldsymbol{a} + 10(\boldsymbol{a} + \boldsymbol{c} - 11\boldsymbol{k}) + \boldsymbol{c}$ $= 110\boldsymbol{a} + 11\boldsymbol{c} - 110\boldsymbol{k}$ $= 11(10\boldsymbol{a} + \boldsymbol{c} - 10\boldsymbol{k})$ $\boldsymbol{Q}$ = quotient $= 10\boldsymbol{a} + \boldsymbol{c} - 10\boldsymbol{k}$ …… (2) According to the given information, $\boldsymbol{Q} = 3(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$ …… (3) Sub. (1) and (2) into (3): $10\boldsymbol{a} + \boldsymbol{c} - 10\boldsymbol{k} = 3(2\boldsymbol{a} + 2\boldsymbol{c} - 11\boldsymbol{k})$ $4\boldsymbol{a} + 23\boldsymbol{k} = 5\boldsymbol{c}$ When $\boldsymbol{k} = 0$, $4\boldsymbol{a} = 5\boldsymbol{c} \Rightarrow \boldsymbol{a} = 5$, $\boldsymbol{c} = 4$, $\boldsymbol{b} = \boldsymbol{a} + \boldsymbol{c} = 9$ $\beta = 594$ When $\boldsymbol{k} = 1$, $4\boldsymbol{a} + 23 = 5\boldsymbol{c}$ When $\boldsymbol{c} = 5$, 6, 8 or 9, no integral solution for $\boldsymbol{a}$ When $\boldsymbol{c} = 7$, $\boldsymbol{a} = 3$, $\boldsymbol{b} = \boldsymbol{a} + \boldsymbol{c} - 11\boldsymbol{k} < 0$, rejected

Remark: original question: ⋯ β 的最大值 ⋯ largest value of β

β is uniquely found. There is no need to emphasize the word "largest".

G3.3 求 φ 的最大實數值，使不等式 $\sqrt{1-\varphi}-\sqrt{1+\varphi} \geq 1$ 成立。

Determine the largest real value of φ such that the inequality $\sqrt{1-\varphi}-\sqrt{1+\varphi} \geq 1$ holds.

為使表達式成立，$1 - \varphi \geq 0$ 及 $1 + \varphi \geq 0$ $\Rightarrow -1 \leq \varphi \leq 1$ $\sqrt{1-\varphi} \geq 1+\sqrt{1+\varphi} > 0$ 當 $\varphi \geq 0$，則 $1 > \sqrt{1-\varphi} \geq 1+\sqrt{1+\varphi} > 1$ 矛盾 $\therefore -1 \leq \varphi < 0$ $\left(\sqrt{1-\varphi}\right)^2 \geq \left(1+\sqrt{1+\varphi}\right)^2$ $1-\varphi \geq 1+2\sqrt{1+\varphi}+1+\varphi$ $-2\varphi-1 \geq 2\sqrt{1+\varphi} \geq 0$ $(-2\varphi-1)^2 \geq \left(2\sqrt{1+\varphi}\right)^2$ $4\varphi^2 + 4\varphi + 1 \geq 4(1 + \varphi)$ $4\varphi^2 \geq 3$ $-1 \leq \varphi \leq -\frac{\sqrt{3}}{2}$ φ 的最大實數值$= -\frac{\sqrt{3}}{2}$	In order that the expression is well defined, $1 - \varphi \geq 0$ and $1 + \varphi \geq 0 \Rightarrow -1 \leq \varphi \leq 1$ $\sqrt{1-\varphi} \geq 1+\sqrt{1+\varphi} > 0$ If $\varphi \geq 0$, then $1 > \sqrt{1-\varphi} \geq 1+\sqrt{1+\varphi} > 1$!!! $\therefore -1 \leq \varphi < 0$ $\left(\sqrt{1-\varphi}\right)^2 \geq \left(1+\sqrt{1+\varphi}\right)^2$ $1-\varphi \geq 1+2\sqrt{1+\varphi}+1+\varphi$ $-2\varphi-1 \geq 2\sqrt{1+\varphi} \geq 0$ $(-2\varphi-1)^2 \geq \left(2\sqrt{1+\varphi}\right)^2$ $4\varphi^2 + 4\varphi + 1 \geq 4(1 + \varphi)$ $4\varphi^2 \geq 3$ $-1 \leq \varphi \leq -\frac{\sqrt{3}}{2}$ The largest value of $\varphi = -\frac{\sqrt{3}}{2}$.

G3.4 設 θ 及 γ 為正整數，當中 $\theta < \gamma$。若 $\frac{\theta+\gamma}{2} : \sqrt{\theta\gamma} = 13 : 12$，求 γ 的最小值。

Suppose that θ and γ are positive integers, where $\theta < \gamma$.

If $\frac{\theta+\gamma}{2} : \sqrt{\theta\gamma} = 13 : 12$, determine the least value of γ.

$6(\theta+\gamma) = 13\sqrt{\theta\gamma}$	$6(\theta+\gamma) = 13\sqrt{\theta\gamma}$
$36(\theta + \gamma)^2 = 169\theta\gamma$	$36(\theta + \gamma)^2 = 169\theta\gamma$
$36\theta^2 - 97\theta\gamma + 36\gamma^2 = 0$	$36\theta^2 - 97\theta\gamma + 36\gamma^2 = 0$
$(4\theta - 9\gamma)(9\theta - 4\gamma) = 0$	$(4\theta - 9\gamma)(9\theta - 4\gamma) = 0$
$\theta : \gamma = 9 : 4$ or $4 : 9$	$\theta : \gamma = 9 : 4$ or $4 : 9$
∵ θ 及 γ 為正整數及 $\theta < \gamma$	∵ θ and γ are positive integers and $\theta < \gamma$
∴ $\theta : \gamma = 4 : 9$	∴ $\theta : \gamma = 4 : 9$
γ 的最小值= 9 (當 $\theta = 4$)	The least value of $\gamma = 9$ (when $\theta = 4$)

Group Event 4

G4.1 設 $\boldsymbol{X}=\sqrt{2018-\sqrt{\boldsymbol{A}}}$ 是正整數，求 $\boldsymbol{A}$ 的最大值。

Let $\boldsymbol{X}=\sqrt{2018-\sqrt{\boldsymbol{A}}}$ be a positive integer. Determine the largest value of $\boldsymbol{A}$.

Reference 2016 HI3

$45=\sqrt{2025}>\sqrt{2018-\sqrt{\boldsymbol{A}}}$	$45=\sqrt{2025}>\sqrt{2018-\sqrt{\boldsymbol{A}}}$
$\boldsymbol{X}=\sqrt{2018-\sqrt{\boldsymbol{A}}}=1$、2、⋯、43 或 44	$\boldsymbol{X}=\sqrt{2018-\sqrt{\boldsymbol{A}}}=1, 2, \cdots, 43$ or 44
$2018-\sqrt{\boldsymbol{A}}=1^2$、$2^2$、⋯ 、$43^2$ 或 44^2	$2018-\sqrt{\boldsymbol{A}}=1^2, 2^2, \cdots, 43^2$ or 44^2
$\sqrt{\boldsymbol{A}}=2018-1^2$、$2018-2^2$、⋯、或$2018-44^2$	$\sqrt{\boldsymbol{A}}=2018-1^2, 2018-2^2, \cdots$, or $2018-44^2$
$\boldsymbol{A}=(2018-1^2)^2$、$(2018-2^2)^2$、⋯、或$(2018-44^2)^2$	$\boldsymbol{A}=(2018-1^2)^2, (2018-2^2)^2, \cdots$, or $(2018-44^2)^2$
$\boldsymbol{A}$ 的最大值$=2017^2=4068289$	The largest value of $\boldsymbol{A}=2017^2=4068289$

G4.2 求方程 $(12\boldsymbol{x}-1)(6\boldsymbol{x}-1)(4\boldsymbol{x}-1)(3\boldsymbol{x}-1)=5$ 的所有實根之乘積 $\boldsymbol{B}$ 的值。

Determine the value of $\boldsymbol{B}$, the product of all real roots of $(12\boldsymbol{x}-1)(6\boldsymbol{x}-1)(4\boldsymbol{x}-1)(3\boldsymbol{x}-1)=5$

方法一 方程式兩邊乘以24: $(12\boldsymbol{x}-1)(12\boldsymbol{x}-2)(12\boldsymbol{x}-3)(12\boldsymbol{x}-4)=120$ 設 $\boldsymbol{a}=12\boldsymbol{x}-1$，則方程式可寫成： $\boldsymbol{a}(\boldsymbol{a}-1)(\boldsymbol{a}-2)(\boldsymbol{a}-3)=120$ $(\boldsymbol{a}^2-3\boldsymbol{a})(\boldsymbol{a}^2-3\boldsymbol{a}+2)=120$ $(\boldsymbol{a}^2-3\boldsymbol{a})^2+2(\boldsymbol{a}^2-3\boldsymbol{a})-120=0$ $(\boldsymbol{a}^2-3\boldsymbol{a}-10)(\boldsymbol{a}^2-3\boldsymbol{a}+12)=0$ $\boldsymbol{a}=-2$ 或 5 或 沒有解 $12\boldsymbol{x}-1=-2$ 或 $12\boldsymbol{x}-1=5$ $\boldsymbol{x}=-\frac{1}{12}$ 或 $\frac{1}{2}$ 所有實根之乘積$=\frac{1}{2}\times\left(-\frac{1}{12}\right)=-\frac{1}{24}$	**Method 1** Multiply both sides by 24 : $(12\boldsymbol{x}-1)(12\boldsymbol{x}-2)(12\boldsymbol{x}-3)(12\boldsymbol{x}-4)=120$ Let $\boldsymbol{a}=12\boldsymbol{x}-1$, then the equation becomes: $\boldsymbol{a}(\boldsymbol{a}-1)(\boldsymbol{a}-2)(\boldsymbol{a}-3)=120$ $(\boldsymbol{a}^2-3\boldsymbol{a})(\boldsymbol{a}^2-3\boldsymbol{a}+2)=120$ $(\boldsymbol{a}^2-3\boldsymbol{a})^2+2(\boldsymbol{a}^2-3\boldsymbol{a})-120=0$ $(\boldsymbol{a}^2-3\boldsymbol{a}-10)(\boldsymbol{a}^2-3\boldsymbol{a}+12)=0$ $\boldsymbol{a}=-2$ or 5 or no solution $12\boldsymbol{x}-1=-2$ or $12\boldsymbol{x}-1=5$ $\boldsymbol{x}=-\frac{1}{12}$ or $\frac{1}{2}$ Product of roots $=\frac{1}{2}\times\left(-\frac{1}{12}\right)=-\frac{1}{24}$
方法二 將方程重新排列: $(12\boldsymbol{x}-1)(3\boldsymbol{x}-1)(6\boldsymbol{x}-1)(4\boldsymbol{x}-1)=5$ $(36\boldsymbol{x}^2-15\boldsymbol{x}+1)(24\boldsymbol{x}^2-10\boldsymbol{x}+1)=5$ 設 $\boldsymbol{t}=12\boldsymbol{x}^2-5\boldsymbol{x}$，則方程式可寫成： $(3\boldsymbol{t}+1)(2\boldsymbol{t}+1)=5$ $6\boldsymbol{t}^2+5\boldsymbol{t}-4=0$ $(2\boldsymbol{t}-1)(3\boldsymbol{t}+4)=0$ $(24\boldsymbol{x}^2-10\boldsymbol{x}-1)(36\boldsymbol{x}^2-15\boldsymbol{x}+4)=0$ $(2\boldsymbol{x}-1)(12\boldsymbol{x}+1)=0$ 或 沒有解 $\boldsymbol{x}=\frac{1}{2}$ 或$-\frac{1}{12}$ 所有實根之乘積$=\frac{1}{2}\times\left(-\frac{1}{12}\right)=-\frac{1}{24}$	**Method 2** Rearrange the equation as $(12\boldsymbol{x}-1)(3\boldsymbol{x}-1)(6\boldsymbol{x}-1)(4\boldsymbol{x}-1)=5$ $(36\boldsymbol{x}^2-15\boldsymbol{x}+1)(24\boldsymbol{x}^2-10\boldsymbol{x}+1)=5$ Let $\boldsymbol{t}=12\boldsymbol{x}^2-5\boldsymbol{x}$, then the equation becomes: $(3\boldsymbol{t}+1)(2\boldsymbol{t}+1)=5$ $6\boldsymbol{t}^2+5\boldsymbol{t}-4=0$ $(2\boldsymbol{t}-1)(3\boldsymbol{t}+4)=0$ $(24\boldsymbol{x}^2-10\boldsymbol{x}-1)(36\boldsymbol{x}^2-15\boldsymbol{x}+4)=0$ $(2\boldsymbol{x}-1)(12\boldsymbol{x}+1)=0$ or no solution $\boldsymbol{x}=\frac{1}{2}$ or $-\frac{1}{12}$ Product of roots $=\frac{1}{2}\times\left(-\frac{1}{12}\right)=-\frac{1}{24}$

方法三

該方程式可寫成：

$$\left(x-\frac{1}{12}\right)\left(x-\frac{1}{6}\right)\left(x-\frac{1}{4}\right)\left(x-\frac{1}{3}\right)=\frac{5}{12\times6\times4\times3}$$

設 $y=\frac{1}{4}\left(x-\frac{1}{12}+x-\frac{1}{6}+x-\frac{1}{4}+x-\frac{1}{3}\right)$

所以，$y=x-\frac{5}{24}\Rightarrow x=y+\frac{5}{24}$

該方程式可寫成：

$$\left(y+\frac{5}{24}-\frac{1}{12}\right)\left(y+\frac{5}{24}-\frac{1}{6}\right)\left(y+\frac{5}{24}-\frac{1}{4}\right)\left(y+\frac{5}{24}-\frac{1}{3}\right)=\frac{5}{864}$$

$$\left(y+\frac{3}{24}\right)\left(y+\frac{1}{24}\right)\left(y-\frac{1}{24}\right)\left(y-\frac{3}{24}\right)=\frac{5}{864}$$

$$\left(y^2-\frac{1}{24^2}\right)\left(y^2-\frac{3^2}{24^2}\right)=\frac{5}{864}$$

設 $t=y^2-\frac{5}{24^2}\Rightarrow y^2=t+\frac{5}{24^2}$

該方程式可寫成：

$$\left(t+\frac{4}{24^2}\right)\left(t-\frac{4}{24^2}\right)=\frac{5}{864}\Rightarrow\ t^2-\frac{16}{24^4}=\frac{5}{864}$$

$$t^2=\frac{1}{12^4}+\frac{5}{12^2\times6}\times\frac{2}{2}\times\frac{12}{12}=\frac{121}{144^2}$$

$t=\frac{11}{144}$ 或 $-\frac{11}{144}$

$y^2=\frac{11}{144}+\frac{5}{24^2}$ 或 $-\frac{11}{144}+\frac{5}{24^2}=\frac{49}{24^2}$ 或 $-\frac{39}{24^2}$

$y=\frac{7}{24}$ 或 $-\frac{7}{24}$

$x=\frac{7}{24}+\frac{5}{24}$ 或 $-\frac{7}{24}+\frac{5}{24}=\frac{1}{2}$ 或 $-\frac{1}{12}$

B =所有實根之乘積$=\frac{1}{2}\times\left(-\frac{1}{12}\right)=-\frac{1}{24}$

Method 3

The equation can be written as:

$$\left(x-\frac{1}{12}\right)\left(x-\frac{1}{6}\right)\left(x-\frac{1}{4}\right)\left(x-\frac{1}{3}\right)=\frac{5}{12\times6\times4\times3}$$

Let $y=\frac{1}{4}\left(x-\frac{1}{12}+x-\frac{1}{6}+x-\frac{1}{4}+x-\frac{1}{3}\right)$

So, $y=x-\frac{5}{24}\Rightarrow x=y+\frac{5}{24}$

Then the equation becomes:

$$\left(y+\frac{5}{24}-\frac{1}{12}\right)\left(y+\frac{5}{24}-\frac{1}{6}\right)\left(y+\frac{5}{24}-\frac{1}{4}\right)\left(y+\frac{5}{24}-\frac{1}{3}\right)=\frac{5}{864}$$

$$\left(y+\frac{3}{24}\right)\left(y+\frac{1}{24}\right)\left(y-\frac{1}{24}\right)\left(y-\frac{3}{24}\right)=\frac{5}{864}$$

$$\left(y^2-\frac{1}{24^2}\right)\left(y^2-\frac{3^2}{24^2}\right)=\frac{5}{864}$$

Let $t=y^2-\frac{5}{24^2}\Rightarrow y^2=t+\frac{5}{24^2}$

The equation becomes:

$$\left(t+\frac{4}{24^2}\right)\left(t-\frac{4}{24^2}\right)=\frac{5}{864}\Rightarrow\ t^2-\frac{16}{24^4}=\frac{5}{864}$$

$$t^2=\frac{1}{12^4}+\frac{5}{12^2\times6}\times\frac{2}{2}\times\frac{12}{12}=\frac{121}{144^2}$$

$t=\frac{11}{144}$ or $-\frac{11}{144}$

$y^2=\frac{11}{144}+\frac{5}{24^2}$ or $-\frac{11}{144}+\frac{5}{24^2}=\frac{49}{24^2}$ or $-\frac{39}{24^2}$

$y=\frac{7}{24}$ or $-\frac{7}{24}$

$x=\frac{7}{24}+\frac{5}{24}$ or $-\frac{7}{24}+\frac{5}{24}=\frac{1}{2}$ or $-\frac{1}{12}$

B = product of all real roots $=\frac{1}{2}\times\left(-\frac{1}{12}\right)=-\frac{1}{24}$

G4.3 求 $\boldsymbol{C}=\cos\frac{\pi}{15}\times\cos\frac{2\pi}{15}\times\cos\frac{3\pi}{15}\times\cos\frac{4\pi}{15}\times\cos\frac{5\pi}{15}\times\cos\frac{6\pi}{15}\times\cos\frac{7\pi}{15}$ 的值。

Determine the value of $\boldsymbol{C}=\cos\frac{\pi}{15}\times\cos\frac{2\pi}{15}\times\cos\frac{3\pi}{15}\times\cos\frac{4\pi}{15}\times\cos\frac{5\pi}{15}\times\cos\frac{6\pi}{15}\times\cos\frac{7\pi}{15}$.

設 Let $\alpha=\frac{\pi}{5}$, $\beta=\frac{\pi}{15}$

$\cos\frac{7\pi}{15}=-\cos\left(\pi-\frac{7\pi}{15}\right)=-\cos\frac{8\pi}{15}=-\cos 8\beta$

$\cos\frac{3\pi}{15}=\cos\frac{\pi}{5}=\cos\alpha$

$\cos\frac{6\pi}{15}=\cos\frac{2\pi}{5}=\cos 2\alpha$

$\cos\frac{5\pi}{15}=\cos\frac{\pi}{3}=\frac{1}{2}$

$\cos\frac{\pi}{15}\times\cos\frac{2\pi}{15}\times\cos\frac{3\pi}{15}\times\cos\frac{4\pi}{15}\times\cos\frac{5\pi}{15}\times\cos\frac{6\pi}{15}\times\cos\frac{7\pi}{15}$

$=\cos\beta\times\cos 2\beta\times\cos\alpha\times\cos 4\beta\times\frac{1}{2}\times\cos 2\alpha\times(-\cos 8\beta)$

$=-\frac{1}{2}\times\cos\alpha\times\cos 2\alpha\times\cos\beta\times\cos 2\beta\times\cos 4\beta\times\cos 8\beta$

$=-\frac{1}{2}\times\frac{2\sin\alpha\cos\alpha\times\cos 2\alpha}{2\sin\alpha}\times\frac{2\sin\beta\cos\beta\times\cos 2\beta\times\cos 4\beta\times\cos 8\beta}{2\sin\beta}$

$=-\frac{1}{2}\times\frac{2\sin 2\alpha\times\cos 2\alpha}{4\sin\alpha}\times\frac{2\sin 2\beta\times\cos 2\beta\times\cos 4\beta\times\cos 8\beta}{4\sin\beta}$

$=-\frac{1}{2}\times\frac{\sin 4\alpha}{4\sin\alpha}\times\frac{2\sin 4\beta\times\cos 4\beta\times\cos 8\beta}{8\sin\beta}$

$=-\frac{1}{2}\times\frac{\sin\frac{4\pi}{5}}{4\sin\frac{\pi}{5}}\times\frac{2\sin 8\beta\times\cos 8\beta}{16\sin\beta}$

$=-\frac{1}{2}\times\frac{\sin\left(\pi-\frac{\pi}{5}\right)}{4\sin\frac{\pi}{5}}\times\frac{\sin 16\beta}{16\sin\beta}=-\frac{1}{2}\times\frac{\sin\frac{\pi}{5}}{4\sin\frac{\pi}{5}}\times\frac{\sin\frac{16\pi}{15}}{16\sin\frac{\pi}{15}}$

$=-\frac{1}{128}\times\frac{\sin\left(\pi+\frac{\pi}{15}\right)}{\sin\frac{\pi}{15}}=-\frac{1}{128}\times\frac{-\sin\frac{\pi}{15}}{\sin\frac{\pi}{15}}=\frac{1}{128}$

$\boldsymbol{C}=\frac{1}{128}$

G4.4 設 r、s 及 t 是正實數，且 $r^2 + s^2 + t^2 = rs + st + rt$。若 $r = 1$，求 $D = s + t$ 的值。

Let r, s and t be positive real numbers with $r^2 + s^2 + t^2 = rs + st + rt$.

If $r = 1$, determine the value of $D = s + t$.

Reference: 2005 FI4.1

$r^2 + s^2 + t^2 = rs + st + rt$ $2r^2 + 2s^2 + 2t^2 = 2rs + 2st + 2rt$ $(r^2 - 2rs + s^2) + (s^2 - 2st + t^2) + (t^2 - 2tr + r^2) = 0$ $(r - s)^2 + (s - t)^2 + (t - r)^2 = 0$ 3個非負數之和= 0 每個非負數= 0 $r = s$，$s = t$ 及 $t = r$ 當 $r = 1$，$s = t = 1$ $D = 1 + 1 = 2$	$r^2 + s^2 + t^2 = rs + st + rt$ $2r^2 + 2s^2 + 2t^2 = 2rs + 2st + 2rt$ $(r^2 - 2rs + s^2) + (s^2 - 2st + t^2) + (t^2 - 2tr + r^2) = 0$ $(r - s)^2 + (s - t)^2 + (t - r)^2 = 0$ sum of 3 non-negative numbers = 0 Each non-negative number = 0 $r = s$, $s = t$ and $t = r$ When $r = 1$, $s = t = 1$ $D = 1 + 1 = 2$