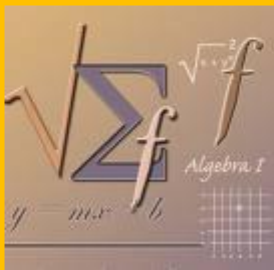
ព្រះរាជាណាចក្រកម្ពុជា

ជាតិ សាសនា ព្រះមហាក្សត្រ

Mathematical Intelligence



Series 01

វិញ្ញាសាគណិតវិទ្យាសម្រាប់ស្រាវជ្រាវ

ថ្នាក់ទី៖ ១០ ១១ និង ១២

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** ត្រៀមប្រលងអាហារូបករណ៍បរទេស*

ដកស្រង់ចេញពីឯកសារ អង់គ្លេស បារាំង និង វៀតណាម

 ២០១. ផុល កើន

អារម្ភកថា

សៀវភៅ **វិញ្ញាសា គណិតវិទ្យាសម្រាប់ស្រាវជ្រាវ** គឺជាវណ្ណកម្មទី១នៃ “Mathematical Intelligence Series” ត្រូវបានរៀបចំឡើងក្នុងគោលបំនង ជួយពង្រឹងសមត្ថភាព ប្អូនៗ ដែលកំពុងសិក្សា នៅកម្រិតវិទ្យាល័យ។ ជាពិសេសប្អូនៗ ដែលមានបំណងប្រលងអាហារូបករណ៍ទៅសិក្សានៅបរទេស។

សៀវភៅនេះត្រូវបានចែកជាពីរផ្នែកដែល ផ្នែកទី១ រៀបចំឡើងជាភាសាខ្មែរ និងផ្នែកទី២ ជាភាសាអង់គ្លេស។ ដោយសារការប្រលងអាហារូបករណ៍ទៅសិក្សានៅបរទេស ជាពិសេសប្រទេសជឿនលឿន ដូចជា ជប៉ុន សឹង្ហបូរី ត្រូវបានជម្រុះ ជាច្រើនដំណាក់កាល គឺ ជម្រុះជាភាសាខ្មែរ រួចទើបជម្រុះជាភាសាអង់គ្លេស ទើបសៀវភៅនេះត្រូវបានបញ្ចូលលំហាត់ ជាភាសាអង់គ្លេស ក្នុងគោលបំណងសម្រួលដល់ប្អូនៗ នៅតាមបណ្តាខេត្តដែលពិបាករកឯកសារជាភាសាអង់គ្លេស ។ លំហាត់មួយចំនួនធំនៅក្នុងសៀវភៅនេះ សុទ្ធសឹងតែជាលំហាត់ដែលដកស្រង់ពីវិញ្ញាសាធ្លាប់ប្រលងនៅបរទេស។

ទោះបីជាសៀវភៅនេះត្រូវបានពិនិត្យជាច្រើនសារក៏ដោយ ក៏មិនអាចជៀសវាងនូវកំហុសឆ្គងដោយអចេតនាបានឡើយ។ យើងខ្ញុំនឹងរងចាំទទួលនូវមតិរិះគន់ ស្ថាបនាដោយក្តីរីករាយ ដើម្បីអោយសៀវភៅនេះកាន់តែមានភាពសុក្រិត។

វិញ្ញាសាទី១

១. គេអោយ x, y ជាចំនួនពិតដែលផ្ទៀងផ្ទាត់លក្ខខណ្ឌ៖
$(x+5)^2+(y-12)^2=14^2$ ។ ចូរគណនាតំលៃអប្បបរមានៃ
x^2+y^2 ។

២. គេអោយអនុគមន៍ $f(x)$ កំនត់លើ $\mathbb{R}$ ហើយផ្ទៀងផ្ទាត់
$f(1)=1$ ។ចំពោះគ្រប់តំលៃ $x\in\mathbb{R}$; $f(x+5)\geq f(x)+5$ និង
$f(x+1)\leq f(x)+1$ ។ បើ $g(x)=f(x)+1-x$ ។ គណនា
$g(2011)$ ។

៣. តាង $\log_4(x+2y)+\log_4(x-2y)=1$ ។ គណនាតំលៃអប្បបរមានៃ $|x|-|y|$ ។

៤. គណនាអាំងតេក្រាល៖

ក. $I=\int\frac{dx}{x^2+2x-3}$ ខ. $J=\int\frac{dx}{x^2-6x+10}$

គ. $K=\int\frac{dx}{2x^2+8x+2}$ ។

៥. កំនត់តំលៃ $a<0$ ដើម្បីអោយវិសមភាព $(*)$ ផ្ទៀងផ្ទាត់ចំ ពោះគ្រប់តំលៃ $x\in\mathbb{R}$

$$\sin^2 x+a\cos x+a^2\geq 1+\cos x \qquad (*)$$

៦. គេមានត្រីកោណ ABC ។ បង្ហាញថា
$A=2B \Leftrightarrow a^2=b(b+c)$

សម្រាយបញ្ជាក់

១. តាង $x+5=14\cdot\cos\theta$ និង $y-12=14\cdot\sin\theta$ ចំពោះ
$\theta\in[0,2\pi[$

គេបាន $x^2+y^2=(14\cdot\cos\theta-5)^2+(14\cdot\sin\theta+12)^2$

$=196\cdot\cos^2\theta-140\cdot\cos\theta+25+196\cdot\sin^2\theta+336\cdot\sin\theta+144$

$=196\cdot(\cos^2\theta+\sin^2\theta)+28(12\cdot\sin\theta-5\cdot\cos\theta)+169$

$=365+28(12\cdot\sin\theta-5\cdot\cos\theta)$

$$=365+28\cdot 13\left(\frac{12}{13}\cdot\sin\theta-\frac{5}{13}\cdot\cos\theta\right)$$

ដោយ $\left(\frac{12}{13}\right)^2+\left(\frac{5}{13}\right)^2=1$ តាង $\cos\varphi=\frac{12}{13};\ \sin\varphi=\frac{5}{13}$

$\Rightarrow x^2+y^2=365+364(\sin\theta\cos\varphi-\sin\varphi\cos\theta)$

$=365+364\cdot\sin(\theta-\varphi)$

ដោយ $-1\le\sin(\theta-\varphi)\le 1$

នាំអោយ $x^2+y^2\ge 365-364\cdot 1=1$

យើងបាន x^2+y^2 មានតំលៃអប្បបរមាស្មើ 1 ពេលដែល

$$\sin(\theta-\varphi)=-1\Leftrightarrow\theta-\varphi=\frac{3\pi}{2}\Leftrightarrow\theta=\frac{3\pi}{2}+\arctan\frac{5}{12}$$

<u>ដូចនេះ</u> x^2+y^2 មានតំលៃអប្បបរមាស្មើ 1

២. យើងមាន $f(x)+5\le f(x+5)\le f(x+4)+1\le$
$f(x+3)+2\le f(x+2)+3\le f(x+1)+4\le f(x)+5$
នាំអោយ វិសមភាព $f(x+5)\ge f(x)+5$ និង
$f(x+1)\le f(x)+1$ក្លាយជាសមភាពចំពោះគ្រប់តំលៃ $x\in\mathbb{R}$
យើងបាន $f(x+1)=f(x)+1$
ដោយ $f(1)=1,\quad f(2)=f(1)+1=1+1=2,$
$f(3)=f(2)+1=2+1=3,\dots,$
$f(2011)=f(2010)+1=2010+1=2011$
នាំអោយ
$g(2011)=f(2011)+1-2011=2011+1-2011=1$
<u>ដូចនេះ</u> $\quad g(2011)=1$

៣. យើងមាន $\log_4(x+2y)+\log_4(x-2y)=1$
$\Leftrightarrow \log_4\left[(x+2y)(x-2y)\right]=1$

នាំអោយ $\begin{cases} x+2y>0 \\ x-2y>0 \\ (x-2y)(x+2y)=4 \end{cases}$

យើងបាន

$$\begin{cases} x>2|y|\ge 0 \\ x^2-4y^2=4 \end{cases}$$

តាមរយៈភាពស៊ីមេទ្រីនៃតំលៃ y ធៀបនឹងសូន្យ (មានន័យថា y អាចជាចំនួនវិជ្ជមាន ឬ អវិជ្ជមានដែលមាន 0 ជាចំនុចឆ្លុះ) នោះ

យើងសិក្សាតែក្នុងករណី $y \geq 0$ គឺគ្រប់គ្រាន់ហើយ។
ម្យ៉ាងទៀត $x > 0$
នាំអោយ $|x| - |y| = x - y$ ជាចំនួនពិតវិជ្ជមាន
យក $u = x - y \Rightarrow x = u + y$ ជំនួសក្នុង $x^2 - 4y^2 = 4$

$$\Rightarrow (u+y)^2 - 4y^2 = 4$$

$$\Rightarrow u^2 + 2uy + y^2 - 4y^2 = 4$$

$$\Rightarrow 3y^2 - 2uy + 4 - u^2 = 0 \qquad y \text{ ជាអថេរពិត}$$

$$\Delta' = u^2 - 12 + 3u^2 = 4u^2 - 12 \geq 0$$

នាំអោយ $u \geq \sqrt{3}$ ឬ $u \leq -\sqrt{3}$ តែ u វិជ្ជមាន
នាំអោយ $u \geq \sqrt{3}$

សញ្ញាស្មើកើតមានពេលដែល $x = \dfrac{4\sqrt{3}}{3}; y = \dfrac{\sqrt{3}}{3}$

<u>ដូចនេះ</u> តំលៃតូចបំផុតនៃ $|x| - |y|$ គឺ $\sqrt{3}$

៤. គណនាអាំងតេក្រាលខាងក្រោម៖

ក. $\displaystyle\int \frac{dx}{x^2 + 2x - 3} = \int \frac{dx}{(x-1)(x+3)}$

យក $\displaystyle\frac{1}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3} = \frac{Ax + 3A + Bx - B}{(x-1)(x+3)}$

$$= \frac{(A+B)x + (3A-B)}{(x-1)(x+3)}$$

ដើម្បីអោយសមភាពផ្ទៀងផ្ទាត់ចំពោះគ្រប់តំលៃ $x \in \mathbb{R}$

លុះត្រាតែ $A+B=0$ និង $3A-B=1$

នាំអោយ $A=\frac{1}{4}$ និង $B=-\frac{1}{4}$

$$\Rightarrow \int \frac{dx}{x^2+2x-3} = \int \left(\frac{1/4}{x-1} + \frac{-1/4}{x+3} \right) dx = \frac{1}{4}\int \frac{dx}{x-1} - \frac{1}{4}\int \frac{dx}{x+3}$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+3| + C$$ C ជាចំនួនពិតថេរ

<u>ដូចនេះ</u> $\int \frac{dx}{x^2+2x-3} = \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+3| + C,\ C\in\mathbb{R}$

ខ. $\int \frac{dx}{x^2-6x+10} = \int \frac{dx}{(x^2-6x+9)+1} = \int \frac{dx}{(x-3)^2+1}$

តាង $u=x-3 \Rightarrow du=dx$

$$\Rightarrow \int \frac{dx}{x^2-6x+10} = \int \frac{du}{u^2+1^2} = \arctan u + C$$

$$= \arctan(x-3) + C$$ C ជាចំនួនពិតថេរ

<u>ដូចនេះ</u> $\int \frac{dx}{x^2-6x+10} = \arctan(x-3) + C$

គ. $\int \frac{dx}{2x^2+8x+2} = \int \frac{dx}{2(x^2+4x+1)} = \frac{1}{2}\int \frac{dx}{(x+2)^2-\sqrt{3}^2}$

តាង $u=x+2 \Rightarrow du=dx$

$$\Rightarrow \int \frac{dx}{2x^2+8x+2} = \frac{1}{2}\int \frac{du}{u^2-\sqrt{3}^2} = \frac{1}{4\sqrt{3}}\ln\left|\frac{u-\sqrt{3}}{u+\sqrt{3}}\right| + C$$

$$=\frac{1}{4\sqrt{3}}\ln\left|\frac{x+2-\sqrt{3}}{x+2+\sqrt{3}}\right|+C \qquad C \text{ ជាចំនួនពិតថេរ}$$

<u>ដូចនេះ</u> $\int\frac{dx}{2x^2+8x+2}=\frac{1}{4\sqrt{3}}\ln\left|\frac{x+2-\sqrt{3}}{x+2+\sqrt{3}}\right|+C$ ដែល $C\in\mathbb{R}$

៥. កំនត់តំលៃអវិជ្ជមាន a

យើងមាន $\sin^2 x+a\cos x+a^2\geq 1+\cos x$

ករណី $x=0 \quad \Rightarrow a^2+a\geq 2$

$$\Rightarrow a^2+a-2\geq 0$$
$$\Rightarrow (a-1)(a+2)\geq 0$$
$$\Rightarrow a\in\left]-\infty,-2\right[\cup\left]1,+\infty\right[$$

ដោយ $a<0 \quad \Rightarrow a\in\left]-\infty,-2\right[$

ចំពោះ $a\in\left]-\infty,-2\right[$ យើងបាន

$$a^2+a\cos x\geq a^2+a\geq 2\geq\cos^2 x+\cos x=1+\cos x-\sin^2 x$$

$\Rightarrow\sin^2 x+a\cos x+a^2\geq 1+\cos x$ ចំពោះគ្រប់តំលៃ $x\in\mathbb{R}$

<u>ដូចនេះ</u> $a\in\left]-\infty,-2\right[$

៦. បង្ហាញថា $A=2B\Leftrightarrow a^2=b(b+c)$

- ករណីទី១. បង្ហាញថា $A=2B\Rightarrow a^2=b(b+c)$

យើងមាន $A=2B$ និង $A+B+C=\pi$

$\Rightarrow C=\pi-3B \quad \Rightarrow\sin C=\sin(\pi-3B)=\sin 3B$

តាមទ្រឹស្តីបទស៊ីនុស យើងបាន

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=\frac{b+c}{\sin B+\sin C}$$

$$\Rightarrow \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{b+c}{\sin B+\sin 3B}$$

$$\Rightarrow \frac{a^2}{\sin^2 A}=\frac{b(b+c)}{\sin B(\sin B+\sin 3B)} \qquad (*)$$

ដោយ $\sin B(\sin B+\sin 3B)=\sin B(2\sin 2B\cos B)$

$$=2\sin B\cos B\sin 2B$$
$$=(\sin 2B)(\sin 2B)=\sin^2 2B$$

តាម $(*)$ យើងបាន $\frac{a^2}{\sin^2 2B}=\frac{b(b+c)}{\sin^2 2B}$

នាំអោយ $a^2=b(b+c)$ ពិត

- ករណីទី២. បង្ហាញថា $a^2=b(b+c)\Rightarrow A=2B$

តាមទ្រឹស្តីបទស៊ីនុស $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$

$$\Rightarrow \frac{a^2}{\sin^2 A}=\frac{b^2}{\sin^2 B}=\frac{bc}{\sin B\sin C}=\frac{b^2+bc}{\sin^2 B+\sin B\sin C}$$

$$\Rightarrow \frac{a^2}{\sin^2 A}=\frac{b(b+c)}{\sin^2 B+\sin B\sin C}$$

ដោយ $a^2=b(b+c)$

យើងបាន $\sin^2 A=\sin^2 B+\sin B\sin C$

$$\Rightarrow \sin^2 A-\sin^2 B=\sin B\sin(A+B)$$

$$\Rightarrow (\sin A-\sin B)(\sin A+\sin B)=\sin B\sin(A+B)$$

$$\Rightarrow 4\sin\frac{A-B}{2}\cos\frac{A+B}{2}\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$=2\sin B\sin\frac{A+B}{2}\cos\frac{A+B}{2}$$

$$\Rightarrow 2\cos\frac{A-B}{2}\sin\frac{A-B}{2}=\sin B$$

$\Rightarrow \sin(A-B)=\sin B$ ដោយ $A-B<\pi$

យើងបាន $A-B=B$ ឬ $A-B=\pi-B$

នាំអោយ $\left[\begin{array}{l} A=2B \\ A=\pi \end{array}\right.$ យក / មិនយក ព្រោះ $A<\pi$

នាំអោយ $A=2B$

ដូចនេះ $A=2B\Leftrightarrow a^2=b(b+c)$

វិញ្ញាសាទី២

១. គណនាអាំងតេក្រាលខាងក្រោម៖

ក. $\int \frac{(2x+3)dx}{x^2+3x+4}$ ខ. $\int \frac{dx}{\sqrt{x^2-4x+3}}$

២. គេអោយពហុធាដឺក្រេទី 2 $f(x)=ax^2+bx+c$ ដែល $(a,b,c\in\mathbb{R};a\neq 0)$ ផ្ទៀងផ្ទាត់លក្ខខណ្ឌខាងក្រោម៖

$i.$ ចំពោះ $x\in\mathbb{R},\quad f(x-4)=f(2-x)$ និង $f(x)\geq x$

$ii.$ ចំពោះ $x\in]0,2[;\quad f(x)\leq\left(\frac{x+1}{2}\right)^2$

$iii.$ តំលៃតូចបំផុតនៃ f លើ $\mathbb{R}$ ស្មើ 0

កំនត់តំលៃធំបំផុតនៃ m ដែល $(m>1)$ ដើម្បីអោយ $\exists t\in\mathbb{R};$ $f(x+t)\leq x$ ចំពោះ $x\in[1,m]$

៣. កំនត់ចំនួនគត់ធម្មជាតិ n ដើម្បីអោយ $20n+2$ ចែកដាច់ $2003n+2002$

៤. រកគ្រប់ចំនួនពិត k ដែលនាំអោយវិសមភាព $a^3+b^3+c^3+d^3+1\geq k(a+b+c+d)$ ពិតចំពោះគ្រប់តំលៃ $a,b,c,d\in[-1,+\infty[$

៥. គេអោយ $\{a_n\}$ ជាស្វ៊ីតចំនួនពិត ដែល $a_1=2$ និង

$a_{n+1} = {a_n}^2 - a_n + 1$ ចំពោះ $n \in \mathbb{N}^*$ ។

បង្ហាញថា $\dfrac{1}{a_1} + \dfrac{1}{a_2} + \dfrac{1}{a_3} + \cdots + \dfrac{1}{a_{2011}} < 1$

៦. រកបណ្តាចំនួនគត់ធម្មជាតិ n ដើម្បីអោយ $n^4 - 4n^3 + 22n^2 - 36n + 18$ ជាការេប្រាកដ ។

<u>សម្រាយបញ្ជាក់</u>

១. គណនាអាំងតេក្រាល៖

ក. $\displaystyle\int \frac{(2x+3)dx}{x^2+3x+4} = \int \frac{(x^2+3x+4)'}{x^2+3x+4}dx = \ln\left|x^2+3x+4\right| + C$

ដែល C ជាចំនួនពិតថេរ

<u>ដូចនេះ</u> $\displaystyle\int \frac{2x+3}{x^2+3x+4}dx = \ln\left|x^2+3x+4\right| + C$

ខ. $\displaystyle\int \frac{dx}{\sqrt{x^2-4x+3}} = \int \frac{dx}{\sqrt{(x-2)^2-1}}$

តាង $u = x-2 \quad \Rightarrow du = dx$

នាំអោយ $\displaystyle\int \frac{dx}{\sqrt{x^2-4x+3}} = \int \frac{du}{\sqrt{u^2-1}} = \ln\left|u+\sqrt{u^2-1}\right| + C$

$= \ln\left|x-2+\sqrt{x^2-4x+3}\right| + C$ ដែល C ជាចំនួនពិតថេរ

<u>ដូចនេះ</u> $\displaystyle\int \frac{dx}{\sqrt{x^2-4x+3}} = \ln\left|x-2+\sqrt{x^2-4x+3}\right| + C$

២. កំនត់តំលៃអតិបរមានៃ m

ដោយ $f(x-4)=f(2-x)$ ចំពោះគ្រប់តំលៃ $x\in\mathbb{R}$

យើងបាន បន្ទាត់ $x=-1$ ជាអ័ក្សឆ្លុះនៃអនុគមន៍ f

តាម (iii) នោះ f បែរភាពផតឡើងលើ $\Rightarrow a>0$

នាំអោយ $f(x)=a(x+1)^2$ ដែល $a>0$ $(*)$

តាម (i) $\Rightarrow f(1)\geq 1$

(ii) $\Rightarrow f(1)\leq\left(\dfrac{1+1}{2}\right)^2=1$

យើងបាន $f(1)=1$

តាម $(*)$ $\Rightarrow 1=a(1+1)^2 \Rightarrow a=\dfrac{1}{4}$

នាំអោយ $f(x)=\dfrac{1}{4}(x+1)^2$

ដោយក្រាបនៃអនុគមន៍ $f(x)=\dfrac{1}{4}(x+1)^2$ បែរភាពផតឡើងលើ និង ក្រាបនៃ $y=f(x+t)$ បានមកដោយការបំលែងក្រាបនៃ $f(x)$ ចំនួន t ឯកតា។ ប្រសិនបើយើងចង់បានក្រាបនៃអនុគមន៍ $f(x+t)$ ស្ថិតនៅខាងក្រោមបន្ទាត់ពុះទីមួយ $y=x$ ចំពោះតំលៃ $x\in[1,m]$ ហើយ m អតិបរមា លុះត្រាតែ 1 និង m ជាឫសពីរនៃសមីការអញ្ញាតិ x

$$\frac{1}{4}(x+t+1)^2=x \qquad (**)$$

ជំនួស $x=1$ ចូលក្នុងសមីការ$(**)$ យើងបាន $t=0$ ឬ $t=-4$

- ករណី $t=0$ នោះ $(**)$មានឬស $x_1=x_2=1$ មិនយក$m>1$
- ករណី $t=-4$ នោះ $(**)$ មានឬស $x_1=1$ ឬ $x_2=9$

នាំអោយ $m=9$

ម្យ៉ាងទៀត ពេល $t=-4$ ចំពោះ $x\in[1,9]$

$$\Leftrightarrow(x-1)(x-9)\le 0$$

$$\Leftrightarrow x^2-10x+9\le 0$$

$$\Leftrightarrow x^2-9x+9\le x$$

$$\Leftrightarrow \frac{1}{4}(x+1-4)^2\le x$$

$$\Leftrightarrow f(x-4)\le x \qquad \text{ពិត}$$

<u>ដូចនេះ</u> តំលៃអតិបរមានៃm គឺ $m=9$

៣. កំនត់ចំនួនគត់ធម្មជាតិ n

ឧបមាថា $20n+2$ ចែកដាច់ $2003n+2002$

យើងបាន n ត្រូវតែជាចំនួនគូ ព្រោះ $20n+2$ ជាចំនួនគូចំពោះគ្រប់តំលៃ $n\in\mathbb{N}$ ។ ប៉ុន្តែ $2003n+2002$ គូតែក្នុងករណី n ជាចំនួនគត់គូតែប៉ុណ្ណោះ។

តាង $n=2m$ ចំពោះ $m\in\mathbb{N}^*$

គេបាន $20\times 2m+2$ ចែកដាច់ $2003\times 2m+2002$

$\Rightarrow 40m+2$ ចែកដាច់ $4006m+2002$

$\Rightarrow 20m+1$ ចែកដាច់ $2003m+1001$

ដោយ $2003m+1001=100(20m+1)+3m+901$

នាំអោយ $20m+1$ ចែកដាច់ $3m+901$

យើងបាន $k=\dfrac{3m+901}{20m+1}\in\mathbb{N}$

ករណី $k=1 \quad \Rightarrow 3m+901=20m+1 \quad \Rightarrow m=\dfrac{900}{17}\notin\mathbb{N}$

ករណី $k=2 \quad \Rightarrow 3m+901=40m+2 \quad \Rightarrow m=\dfrac{899}{37}\notin\mathbb{N}$

ករណី $k=3 \quad \Rightarrow 3m+901=60m+3 \quad \Rightarrow m=\dfrac{898}{57}\notin\mathbb{N}$

ករណី $k=4 \quad \Rightarrow 3m+901=80m+4 \quad \Rightarrow m=\dfrac{897}{77}\notin\mathbb{N}$

នាំអោយ $k\geq 5 \quad \Leftrightarrow \dfrac{3m+901}{20m+1}\geq 5$

$$\Leftrightarrow 3m+901\geq 100m+5$$
$$\Leftrightarrow 97m\leq 896$$
$$\Leftrightarrow m\leq\frac{896}{97}<10$$

យើងបាន $m=\{1,2,3,4,5,6,7,8,9\}$

ករណី $m=1 \quad \Rightarrow n=2 \quad \Rightarrow 42$ ចែកដាច់ 6008 មិនពិត

ករណី $m=2 \quad \Rightarrow n=4 \quad \Rightarrow 82$ ចែកដាច់ 10014 មិនពិត

ករណី $m=3 \quad \Rightarrow n=6 \quad \Rightarrow 122$ ចែកដាច់ 14020 មិនពិត

ករណី $m=4 \quad \Rightarrow n=8 \quad \Rightarrow 162$ ចែកដាច់ 18026 មិនពិត

ករណី $m=5 \quad \Rightarrow n=10 \quad \Rightarrow 202$ ចែកដាច់ 22032 មិនពិត

ករណី $m=6 \quad \Rightarrow n=12 \quad \Rightarrow 242$ ចែកដាច់ 26038 មិនពិត

ករណី $m=7 \quad \Rightarrow n=14 \quad \Rightarrow 282$ ចែកដាច់ 30044 មិនពិត

ករណី $m=8 \Rightarrow n=16 \Rightarrow 322$ចែកដាច់ 34050 មិនពិត

ករណី $m=9 \Rightarrow n=18 \Rightarrow 362$ចែកដាច់ 38056មិនពិត

ដូចនេះ គ្មានចំនួនគត់ធម្មជាតិ n ទេ

៩. រកគ្រប់ចំនួនពិត k

យើងមាន $a^3+b^3+c^3+d^3+1\geq k(a+b+c+d)$

ករណី $a=b=c=d=-1 \Rightarrow -3\geq k(-4) \Rightarrow k\geq \frac{3}{4}$

ករណី $a=b=c=d=\frac{1}{2} \Rightarrow 4\cdot\frac{1}{8}+1\geq 2k \Rightarrow k\leq\frac{3}{4}$

យើងបាន $k=\frac{3}{4}$

យើងនឹងបង្ហាញថា $a^3+b^3+c^3+d^3+1\geq \frac{3}{4}(a+b+c+d)$

ផ្ទៀងផ្ទាត់ចំពោះគ្រប់តំលៃ $a,b,c,d\in[-1,+\infty[$

យើងត្រូវបង្ហាញថា$4x^3+1\geq 3x$ ចំពោះគ្រប់តំលៃ $x\in[-1,+\infty)$

ឧបមាថា $4x^3+1\geq 3x \Leftrightarrow 4x^3-3x+1\geq 0$

$\Leftrightarrow 4x^3+4x^2-4x^2-4x+x+1\geq 0$

$\Leftrightarrow (x+1)(4x^2-4x+1)\geq 0$

$\Leftrightarrow (x+1)(2x-1)^2\geq 0$ ពិត ចំពោះគ្រប់តំលៃ $x\in[-1,+\infty)$

នាំអោយ ចំពោះគ្រប់តំលៃ $a,b,c,d\in[-1,+\infty[$

យើងបាន $4a^3+1\geq 3a,\ 4b^3+1\geq 3b,\ 4c^3+1\geq 3c$ និង

$4d^3+1\geq 3d$

ដោយធ្វើប្រមាណវិធីបូកអង្គនឹងអង្គនៃវិសមភាពទាំងបួន

យើងបាន $4\left(a^3+b^3+c^3+d^3+1\right)\geq 3\left(a+b+c+d\right)$

$\Rightarrow a^3+b^3+c^3+d^3+1\geq \frac{3}{4}\left(a+b+c+d\right)$ ពិត

ដូចនេះ: $k=\frac{3}{4}$

៥. បង្ហាញថា $\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_{2011}}<1$

យើងមាន $a_{n+1}=a_n{}^2-a_n+1 \Rightarrow a_{n+1}-1=a_n\left(a_n-1\right)$

$$\Rightarrow \frac{1}{a_{n+1}-1}=\frac{1}{a_n\left(a_n-1\right)}=\frac{1}{a_n-1}-\frac{1}{a_n}$$

បើ $n=1 \quad \Rightarrow \frac{1}{a_2-1}=\frac{1}{a_1-1}-\frac{1}{a_1} \qquad (1)$

បើ $n=2 \quad \Rightarrow \frac{1}{a_3-1}=\frac{1}{a_2-1}-\frac{1}{a_2} \qquad (2)$

....

....

បើ $n=2011 \Rightarrow \frac{1}{a_{2012}-1}=\frac{1}{a_{2011}-1}-\frac{1}{a_{2011}} \qquad (2011)$

យក $(1)+(2)+\cdots+(2011)$ យើងបាន

$$\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_{2011}}=\frac{1}{a_1-1}-\frac{1}{a_{2012}-1}=1-\frac{1}{a_{2012}-1}$$

ដោយ $a_{n+1}-a_n=a_n^2-2a_n+1=(a_n-1)^2\geq 0$ នោះ $\{a_n\}$ ជាស្វីតម៉ូណូតូនកើន ហើយ $a_1=2$ នាំអោយ $a_{2012}>1$

<u>ដូចនេះ</u> $\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_{2011}}<1$

៦. កំនត់ចំនួនគត់ធម្មជាតិ n

យើងមាន $A=n^4-4n^3+22n^2-36n+18$

$=n^4-4n^3+4n^2+18n^2-36n+18=(n^2-2n)^2+18(n^2-2n)+18$

$=\left[(n^2-2n)+9\right]^2-63$

តាង $x=n^2-2n;\quad A=y^2$ ដែល y ជាចំនួនគត់មិនអវិជ្ជមាន

យើងបាន $(x+9)^2-y^2=63$

$\Rightarrow(x-y+9)(x+y+9)=63=1\times 63=3\times 21=7\times 9$

$\Rightarrow((x-y+9),(x+y+9))=\{(1,63),(3,21),(7,9)\}$

- ករណីទី១. $\begin{cases}x-y+9=1\\x+y+9=63\end{cases}$

យើងបាន $x=23$ និង $y=31$

បើ $x=23$ នាំអោយ $n^2-2n=23\Leftrightarrow n^2-2n-23=0$

$\Delta'=1+23=24\quad\Rightarrow n=1\pm\sqrt{24}$ មិនយក ព្រោះ n ជាចំនួនគត់ធម្មជាតិ

- ករណីទី២. $\begin{cases}x-y+9=3\\x+y+9=21\end{cases}$

នាំអោយ $x=3$ និង $y=9$

បើ $x=3$ $\Rightarrow n^2-2n-3=0$

យើងបាន $\left[\begin{array}{ll} n=-1 & \text{មិនយក} \\ n=3 & \text{យក} \end{array}\right.$

- ករណីទី៣. $\begin{cases} x-y+9=7 \\ x+y+9=9 \end{cases}$

នាំអោយ $x=-1$ និង $y=1$

បើ $x=-1$ $\Rightarrow n^2-2n+1=0$ $\Rightarrow n=1$

ដូចនេះ $n=1$ ឬ $n=3$

វិញ្ញាសាទី៣

១. កំនត់ចំនួនគត់ធម្មជាតិ n ដែលបណ្តាលអោយ A ជាការេប្រាកដ។ $A = n^4 + 6n^3 + 11n^2 + 3n + 31$

២. គេអោយ x, y ជាចំនួនពិតវិជ្ជមានផ្ទៀងផ្ទាត់ $x^3 + y^3 = x - y$ ។ បង្ហាញថា $x^2 + 4y^2 < 1$

៣. ចំពោះ u, v, w ជាចំនួនពិតវិជ្ជមានផ្ទៀងផ្ទាត់លក្ខខណ្ឌ៖ $u\sqrt{vw} + v\sqrt{wu} + w\sqrt{uv} \geq 1$ ។ កំនត់តំលៃអប្បបរមានៃចំនួនពិតថេរ λ ដើម្បីអោយ $u + v + w \geq \lambda$ ។

៤. គណនាអាំងតេក្រាលខាងក្រោម៖

ក. $I = \int \frac{4x^2 + 8x + 7}{x^2 + 6x + 10} dx$

ខ. $J = \int \frac{2x^3 - 5x^2 + 6x + 2}{x^2 - x - 1} dx$

៥. គេអោយ a, b, c ជាចំនួនពិតវិជ្ជមាន។ កំនត់តំលៃអប្បបរមានៃ

$$A = \frac{a + 3c}{a + 2b + c} + \frac{4b}{a + b + 2c} - \frac{8c}{a + b + 3c}$$ ។

៦. គេអោយ $f : \mathbb{R} \to \mathbb{R}$ ជាអនុគមន៍ដែល $f(0) = 1$ ។ ចំពោះគ្រប់តំលៃ $x, y \in \mathbb{R}$; $f(xy + 1) = f(x) \cdot f(y) - f(y) - x + 2$

កំនត់អនុគមន៍ $f(x)$

សម្រាយបញ្ជាក់

១. កំនត់ចំនួនគត់ធម្មជាតិ n

ឧបមាថា $A=n^4+6n^3+11n^2+3n+31$ ជាការេប្រាកដ

$\Rightarrow A=\left(n^2+3n+1\right)^2-3(n-10)$ ជាការេប្រាកដ

• បើ $n>10\Rightarrow A<\left(n^2+3n+1\right)^2\Rightarrow A\le\left(n^2+3n\right)^2$

$\Rightarrow\left(n^2+3n+1\right)^2-3(n-10)\le\left(n^2+3n\right)^2$

$\Rightarrow\left(n^2+3n+1\right)^2-\left(n^2+3n\right)^2\le 3n-30$

$\Rightarrow\left(n^2+3n+1+n^2+3n\right)\left(n^2+3n+1-n^2-3n\right)\le 3n-30$

$\Rightarrow 2n^2+6n+1\le 3n-30$

$\Rightarrow 2n^2+3n+31\le 0$ មិនពិតព្រោះ n ជាចំនួនគត់ធម្មជាតិ

• បើ $n=10\Rightarrow A=\left(10^2+3\cdot 10+1\right)^2=131^2$ ជាការេប្រាកដ

• បើ $n<10\Rightarrow A>\left(n^2+3n+1\right)^2$ និង $n^2+3n\ge 0$

$\Rightarrow A\ge\left(n^2+3n+2\right)^2$

$\Rightarrow\left(n^2+3n+1\right)^2-3(n-10)\ge\left(n^2+3n+2\right)^2$

$\Rightarrow 2n^2+9n-27\le 0$

$$\Rightarrow\frac{-3\left(\sqrt{33}+3\right)}{4}\le n\le\frac{3\left(\sqrt{33}-3\right)}{4}<3$$

$\Rightarrow n=\{0,1,2\}$ ហើយចំពោះ n ទាំងអស់នេះ យើងបានតំលៃ

$A=\{31,52,145\}$ ដែលមិនមែនជាការប្រាកដទេ

ដូចនេះ $n=10$

២. បង្ហាញថា $x^2+4y^2<1$

តាមសម្មតិកម្ម $x,y\in\mathbb{R}_+$ យើងបាន $x^3+y^3>0$

ដោយ $x^3+y^3=x-y$ នោះដើម្បីបង្ហាញថា $x^2+4y^2<1$

យើងគ្រាន់តែបង្ហាញថា $x^3+y^3>(x-y)(x^2+4y^2)$

$\Leftrightarrow x^3+y^3>x^3+4xy^2-x^2y-4y^3$

$\Leftrightarrow 5y^3+x^2y>4xy^2$ ពិត ព្រោះតាមវិសមភាព កូស៊ី យើងបាន

$5y^3+x^2y\geq 2\sqrt{5y^3x^2y}=2\sqrt{5}xy^2>4xy^2$

ដូចនេះ $x^2+4y^2<1$

៣. កំនត់តំលៃអប្បបរមានៃចំនួនថេរ λ

របៀបទី១. តាមវិសមភាព កូស៊ី ចំពោះ $u,v,w\in\mathbb{R}_+$

យើងបាន $\frac{u+v}{2}\geq\sqrt{uv};\quad \frac{v+w}{2}\geq\sqrt{vw};\quad \frac{u+w}{2}\geq\sqrt{uw}$

$\Rightarrow u\cdot\frac{v+w}{2}+v\cdot\frac{u+w}{2}+w\cdot\frac{u+v}{2}\geq u\sqrt{vw}+v\sqrt{wu}+w\sqrt{uv}\geq 1$

$\Rightarrow \frac{uv}{2}+\frac{uw}{2}+\frac{uv}{2}+\frac{vw}{2}+\frac{uw}{2}+\frac{vw}{2}\geq 1$

$\Rightarrow uv+vw+uw\geq 1$

ម្យ៉ាងទៀត $(u-v)^2+(v-w)^2+(u-w)^2\geq 0$

$\Rightarrow 2(u^2+v^2+w^2)-2(uv+vw+uw)\geq 0$

$\Rightarrow u^2+v^2+w^2\geq uv+vw+uw\geq 1$

ដោយ $(u+v+w)^2=\left(u^2+v^2+w^2\right)+2(uv+vw+uw)$

$\Rightarrow (u+v+w)^2 \geq 1+2\times 1=3$

$\Rightarrow u+v+w \geq \sqrt{3}$

<u>ដូចនេះ</u> $\lambda=\sqrt{3}$

<u>របៀបទី២.</u>

យើងមាន $\dfrac{(u+v+w)^4}{9}=\left(\dfrac{u+v+w}{3}\right)^3\cdot 3(u+v+w)$

តាមវិសមភាព កូស៊ី យើងបាន

$\dfrac{u+v+w}{3} \geq \sqrt[3]{uvw} \quad \Rightarrow \left(\dfrac{u+v+w}{3}\right)^3 \geq uvw$

$\Rightarrow \dfrac{(u+v+w)^4}{9} \geq 3uvw(u+v+w)$

$\Rightarrow \dfrac{(u+v+w)^4}{9} \geq (uvw+uvw+uvw)(u+v+w)$

តាមវិសមភាព កូស៊ី ស្វាត យើងបាន

$$(uvw+uvw+uvw)(u+v+w) \geq \left(\sqrt{uvwu}+\sqrt{uvwv}+\sqrt{uvww}\right)^2$$

$$\geq \left(u\sqrt{vw}+v\sqrt{uw}+w\sqrt{uv}\right)^2$$

$$\geq 1$$

នាំអោយ $\dfrac{(u+v+w)^4}{9} \geq 1 \quad \Rightarrow (u+v+w)^4 \geq 9$

$\Rightarrow u+v+w\geq\sqrt{3}$

ដូចនេះ $\lambda=\sqrt{3}$

របៀបទី៣.

តាង $x=\sqrt{uv};\quad y=\sqrt{vw};\quad z=\sqrt{uw}$

យើងបាន $xy+yz+zx=\sqrt{uvvw}+\sqrt{vwwu}+\sqrt{wuuv}$

$$=v\sqrt{uw}+w\sqrt{uv}+u\sqrt{vw}$$

$$=u\sqrt{vw}+v\sqrt{uw}+w\sqrt{uv}\geq 1$$

នាំអោយ $xy+yz+zx\geq 1$

និង $u=\frac{xz}{y};\quad v=\frac{xy}{z};\quad w=\frac{yz}{x}$

$$\Rightarrow 2(u+v+w)=\left(\frac{xz}{y}+\frac{xy}{z}\right)+\left(\frac{xy}{z}+\frac{yz}{x}\right)+\left(\frac{yz}{x}+\frac{xz}{y}\right)$$

$$\geq 2\sqrt{\frac{xz}{y}\cdot\frac{xy}{z}}+2\sqrt{\frac{xy}{z}\cdot\frac{yz}{x}}+2\sqrt{\frac{yz}{x}\cdot\frac{xz}{y}}$$

$\Rightarrow 2(u+v+w)\geq 2(x+y+z)$

$\Rightarrow u+v+w\geq x+y+z \qquad (*)$

ម្យ៉ាងទៀត $(x-y)^2+(y-z)^2+(z-x)^2\geq 0$

$\Rightarrow 2\left(x^2+y^2+z^2\right)-2(xy+yz+zx)\geq 0$

$\Rightarrow x^2+y^2+z^2\geq xy+yz+zx\geq 1$

ដោយ $(x+y+z)^2=\left(x^2+y^2+z^2\right)+2(xy+yz+zx)$

$\Rightarrow (x+y+z)^2\geq 1+2\cdot 1=3$

$\Rightarrow x+y+z\geq\sqrt{3}$

តាម $(*)$ យើងបាន $u+v+w\geq\sqrt{3}$

ដូចនេះ $\lambda=\sqrt{3}$

៩. គណនាអាំងតេក្រាល៖

ក. $I=\int\frac{4x^2+8x+7}{x^2+6x+10}dx=\int\frac{4x^2+24x+40-16x-33}{x^2+6x+10}dx$

$=\int\frac{4\left(x^2+6x+10\right)-16x-33}{x^2+6x+10}dx=\int 4dx-\int\frac{16x+33}{x^2+6x+10}dx$

$=4x-\int\frac{16x+33}{x^2+6x+10}dx$

ដោយ $\int\frac{16x+33}{x^2+6x+10}dx=\int\frac{8(2x+6)-15}{x^2+6x+10}dx$

$=8\int\frac{\left(x^2+6x+10\right)'}{x^2+6x+10}dx-15\int\frac{1}{x^2+6x+10}dx$

$=8\ln\left|x^2+6x+10\right|-15\int\frac{1}{(x+3)^2+1}dx$

$=8\ln\left|x^2+6x+10\right|-15\arctan(x+3)+C$ ដែល $C\in\mathbb{R}$

ដូច្នេះ $I=4x-8\ln\left|x^2+6x+10\right|+15\arctan(x+3)+C$

ខ. $J=\int\frac{2x^3-5x^2+6x+2}{x^2-x-1}dx$

$$=\int\frac{2x^3-2x^2-2x-3x^2+3x+3+5x-1}{x^2-x-1}dx$$
$$=\int\frac{2x\left(x^2-x-1\right)-3\left(x^2-x-1\right)+5x-1}{x^2-x-1}dx$$
$$=\int 2xdx+\int(-3)dx+\int\frac{5x-1}{x^2-x-1}dx=x^2-3x+\int\frac{5x-1}{x^2-x-1}dx$$

ដោយ $\int\frac{5x-1}{x^2-x-1}dx=\int\frac{\frac{5}{2}(2x-1)+\frac{5}{2}-1}{x^2-x-1}dx$

$$=\frac{5}{2}\int\frac{\left(x^2-x-1\right)'}{x^2-x-1}dx+\frac{3}{2}\int\frac{1}{x^2-x-1}dx$$
$$=\frac{5}{2}\ln\left|x^2-x-1\right|+\frac{3}{2}\int\frac{dx}{\left(x-\frac{1}{2}\right)^2-\frac{5}{4}}$$
$$=\frac{5}{2}\ln\left|x^2-x-1\right|+\frac{3}{2}\cdot\frac{1}{2\sqrt{\frac{5}{4}}}\ln\left|\frac{x-\frac{1}{2}-\frac{\sqrt{5}}{2}}{x-\frac{1}{2}+\frac{\sqrt{5}}{2}}\right|+C$$ ដែល $C\in\mathbb{R}$

<u>ដូច្នេះ</u>

$$J=x^2-3x+\frac{5}{2}\ln\left|x^2-x-1\right|+\frac{3}{2\sqrt{5}}\ln\left|\frac{2x-1-\sqrt{5}}{2x-1+\sqrt{5}}\right|+C$$

៥. កំនត់តំលៃអប្បបរមានៃ A

តាង $x=a+2b+c;\ \ y=a+b+2c$ និង $z=a+b+3c$

យើងបាន $z-y=c$ និង $x-y=b-c$

$\Rightarrow x-y=b-(z-y)$

$\Rightarrow b=x+z-2y$ និង $a+3c=2y-x$

នាំអោយ $A=\dfrac{a+3c}{a+2b+c}+\dfrac{4b}{a+b+2c}-\dfrac{8c}{a+b+3c}$

$$=\frac{2y-x}{x}+\frac{4(x+z-2y)}{y}-\frac{8(z-y)}{z}$$

$$=2\cdot\frac{y}{x}-1+4\cdot\frac{x}{y}+4\cdot\frac{z}{y}-8-8+8\cdot\frac{y}{z}$$

$$=-17+2\cdot\frac{y}{x}+4\cdot\frac{x}{y}+4\cdot\frac{z}{y}+8\cdot\frac{y}{z}$$

តាមវិសមភាព កូស៊ី

$$2\cdot\frac{y}{x}+4\cdot\frac{x}{y}\geq 2\cdot\sqrt{2\cdot\frac{y}{x}\cdot 4\cdot\frac{x}{y}}=2\sqrt{8}=4\sqrt{2}$$

$$4\cdot\frac{z}{y}+8\cdot\frac{y}{z}\geq 2\cdot\sqrt{4\cdot\frac{z}{y}\cdot 8\cdot\frac{y}{z}}=2\sqrt{32}=8\sqrt{2}$$

នាំអោយ $A\geq -17+4\sqrt{2}+8\sqrt{2}=-17+12\sqrt{2}$

សញ្ញាស្មើកើតមាននៅពេលដែល $\dfrac{2y}{x}=\dfrac{4x}{y}$ និង $\dfrac{4z}{y}=\dfrac{8y}{z}$

ឬ $4x^2=2y^2=z^2$

យើងបាន ប្រព័ន្ធសមីការ

$$\begin{cases} a+b+2c=\sqrt{2}(a+2b+c) \\ a+b+3c=2(a+2b+c) \end{cases}$$

ដោយដោះស្រាយសមីការអញ្ញត្តិ b និង c ហើយយក a ជាប៉ារ៉ាម៉ែត្រ

នាំអោយ $b=\left(1+\sqrt{2}\right)a$ និង $c=\left(4+3\sqrt{2}\right)a$

ដូច្នេះ តំលៃអប្បបរមានៃ A គឺ $\min(A)=-17+12\sqrt{2}$

៦. កំនត់អនុគមន៍ $f(x)$

យើងមាន $f(xy+1)=f(x)\cdot f(y)-f(y)-x+2$ ចំពោះគ្រប់តំលៃ $x,y\in\mathbb{R}$

យើងបាន $f(yx+1)=f(y)\cdot f(x)-f(x)-y+2$

ដោយ $f(xy+1)=f(yx+1)$

$\Rightarrow f(x)\cdot f(y)-f(y)-x+2=f(y)\cdot f(x)-f(x)-y+2$

$\Rightarrow f(x)+y=f(y)+x$

ជំនួស $x=0 \Rightarrow f(0)+y=f(y)+0$ ដោយ $f(0)=1$

យើងបាន $f(y)=1+y$

ដូច្នេះ $f(x)=x+1$ ចំពោះគ្រប់តំលៃ $x\in\mathbb{R}$

វិញ្ញាសាទី៨

១. ចំពោះតំលៃ $n \in \mathbb{N}^*$ បង្ហាញថា $1+x^2+x^4+\cdots+x^{2n-2}$ ចែកដាច់នឹង $1+x+x^2+\cdots+x^{n-1}$ ។

២. គណនា $x \in \mathbb{N}^*$ ដើម្បីអោយ $\overline{xxx(x-1)} = (x-1)^{x-2}$ ។

៣. គេអោយ x_1, x_2, x_3, x_4, x_5 ជាចំនួនពិតមិនអវិជ្ជមាន ហើយផ្ទៀងផ្ទាត់ $\frac{1}{1+x_1}+\frac{1}{1+x_2}+\frac{1}{1+x_3}+\frac{1}{1+x_4}+\frac{1}{1+x_5}=1$ ។

បង្ហាញថា $\frac{x_1}{4+x_1^2}+\frac{x_2}{4+x_2^2}+\frac{x_3}{4+x_3^2}+\frac{x_4}{4+x_4^2}+\frac{x_5}{4+x_5^2}\le 1$

៤. បង្ហាញថា $\tan x + 2\tan 2x + 4\tan 4x + 8\cot 8x = \cot x$ ចំពោះគ្រប់តំលៃ $x \in \mathbb{R}$ ។

៥. កំនត់គ្រប់អនុគមន៍ $f:\mathbb{R}\to\mathbb{R}$ ដែលផ្ទៀងផ្ទាត់លក្ខខណ្ឌ៖ $f(x+y)+f(y+z)+f(z+x)\ge 3f(x+2y+3z)$ ចំពោះគ្រប់តំលៃ x, y, z ។

៦. ឧបមាថា $a+b$ និង $a-b$ ជាចំនួនបឋមរវាងគ្នា។ រកតួចែករួមធំបំផុតនៃ $2a+(1+2a)(a^2-b^2)$ និង $2a(a^2+2a-b^2)(a^2-b^2)$ ។

សម្រាយបញ្ជាក់

១. យើងមាន $\left(x^2-1\right)\left(1+x^2+x^4+\cdots+x^{2n-2}\right)=x^{2n}-1$

$\Rightarrow(x-1)(x+1)\left(1+x^2+x^4+\cdots+x^{2n-2}\right)=\left(x^n-1\right)\left(x^n+1\right)$

ដោយ $x^n-1=(x-1)\left(x^{n-1}+x^{n-2}+\cdots+x+1\right)$

$x^n+1=(x+1)\left(x^{n-1}-x^{n-2}+\cdots+(-1)^{n-2}x+(-1)^{n-1}\right)$

នាំអោយ $1+x^2+x^4+\cdots+x^{2n-2}=K\left(1+x+x^2+\cdots+x^{n-1}\right)$

ដែល $K=x^{n-1}-x^{n-2}+\cdots+(-1)^{n-2}x+(-1)^{n-1}\in\mathbb{Z}$

ដូច្នេះ $1+x^2+x^4+\cdots+x^{2n-2}$ ចែកដាច់នឹង $1+x+x^2+\cdots+x^{n-1}$

២. គណនា $x\in\mathbb{N}^*$

យើងមាន $\overline{xxx(x-1)}=(x-1)^{x-2}$

ដោយ $1000\le\overline{xxx(x-1)}<10000$ ហើយ $x\in\mathbb{N}^*$

នាំអោយ $1\le x\le 9$

$\Rightarrow 1000\le(x-1)^{x-2}<10000$

ដោយ $5^4=625<1000$ និង $7^6=343243>10000$

យើងបាន $5^4<(x-1)^{x-2}<7^6$

$\Rightarrow(x-1)^{x-2}=6^5 \qquad \Rightarrow x=7$

ដូច្នេះ $x=7$

៣.បង្ហាញថា $\sum_{i=1}^{5}\frac{x_i}{4+x_i^2}\le 1$

តាង $y_i=\frac{1}{1+x_i}$ ចំពោះ $i=\overline{1,5}$

$\Rightarrow x_i=\frac{1-y_i}{y_i}$ ហើយ $\sum_{i=1}^{5}y_i=1$

ឧបមាថា$\sum_{i=1}^{5}\frac{x_i}{4+x_i^2}\le 1 \Leftrightarrow \sum_{i=1}^{5}\frac{\frac{1-y_i}{y_i}}{4+\left(\frac{1-y_i}{y_i}\right)^2}\le 1$

$$\Leftrightarrow \sum_{i=1}^{5}\frac{y_i-y_i^2}{4y_i^2+1-2y_i+y_i^2}\le 1$$

$$\Leftrightarrow \sum_{i=1}^{5}\frac{y_i-y_i^2}{5y_i^2-2y_i+1}\le 1$$

$$\Leftrightarrow \sum_{i=1}^{5}\frac{5y_i-5y_i^2}{5y_i^2-2y_i+1}\le 5$$

$$\Leftrightarrow \sum_{i=1}^{5}\frac{-\left(5y_i^2-2y_i+1\right)+3y_i+1}{5y_i^2-2y_i+1}\le 5$$

$$\Leftrightarrow \sum_{i=1}^{5}\left(-1+\frac{3y_i+1}{5y_i^2-2y_i+1}\right)\le 5$$

$$\Leftrightarrow \sum_{i=1}^{5}\frac{3y_i+1}{5y_i^2-2y_i+1}\le 10$$

$$\Leftrightarrow \sum_{i=1}^{5} \frac{3y_i+1}{5\left(y_i^2 - 2\cdot\frac{1}{5}y_i + \frac{1}{25}\right) + \frac{4}{5}} \le 10 \qquad (*)$$

$$\Leftrightarrow \sum_{i=1}^{5} \frac{3y_i+1}{5\left(y_i - \frac{1}{5}\right)^2 + \frac{4}{5}} \le 10$$

ម្យ៉ាងទៀត $\sum_{i=1}^{5} \frac{3y_i+1}{5\left(y_i - \frac{1}{5}\right)^2 + \frac{4}{5}} \le \sum_{i=1}^{5} \frac{3y_i+1}{\frac{4}{5}} = \frac{5}{4}\sum_{i=1}^{5}(3y_i+1)$

$$\Rightarrow \sum_{i=1}^{5} \frac{3y_i+1}{5\left(y_i - \frac{1}{5}\right)^2 + \frac{4}{5}} \le \frac{5}{4}\left(3\sum_{i=1}^{5} y_i + \sum_{i=1}^{5} 1\right) = \frac{5}{4}(3\times 1 + 5) = 10$$

យើងបាន $(*)$ ពិត នាំអោយ ការឧបមាខាងលើពិត

ដូច្នេះ $\sum_{i=1}^{5} \frac{x_i}{4+x_i^2} \le 1$

៩. បង្ហាញថា $\tan x + 2\tan 2x + 4\tan 4x + 8\cot 8x = \cot x$

តាង $f(x) = \tan x - \cot x + 2\tan 2x + 4\tan 4x + 8\cot 8x$

$$\Rightarrow f'(x) = \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} + \frac{4}{\cos^2 2x} + \frac{16}{\cos^2 4x} - \frac{64}{\sin^2 8x}$$

$$= \frac{\sin^2 x + \cos^2 x}{\sin^2 x\cos^2 x} + \frac{4}{\cos^2 2x} + \frac{16}{\cos^2 4x} - \frac{64}{\sin^2 8x}$$

$$= \frac{4}{\sin^2 2x} + \frac{4}{\cos^2 2x} + \frac{16}{\cos^2 4x} - \frac{64}{\sin^2 8x}$$

$$= \frac{16}{\sin^2 4x} + \frac{16}{\cos^2 4x} - \frac{64}{\sin^2 8x}$$
$$= \frac{64}{\sin^2 8x} - \frac{64}{\sin^2 8x} = 0$$

នាំអោយ f ជាអនុគមន៍ថេរ ចំពោះគ្រប់តំលៃ x

ដោយ $f\left(\frac{\pi}{6}\right) = \tan\frac{\pi}{6} - \cot\frac{\pi}{6} + 2\tan\frac{\pi}{3} + 4\tan\frac{2\pi}{3} + 8\cot\frac{4\pi}{3}$

$$= \frac{\sqrt{3}}{3} - \sqrt{3} + 2\sqrt{3} + 4\left(-\sqrt{3}\right) + 8\left(\frac{\sqrt{3}}{3}\right) = 0$$

នាំអោយ $f(x) = 0$ ចំពោះ គ្រប់តំលៃ $x \in \mathbb{R}$

$\Rightarrow \tan x - \cot x + 2\tan 2x + 4\tan 4x + 8\cot 8x = 0$

$\Rightarrow \tan x + 2\tan 2x + 4\tan 4x + 8\cot 8x = \cot x$

<u>ដូចេ្នះ</u> $\tan x + 2\tan 2x + 4\tan 4x + 8\cot 8x = \cot x$

៥. កំនត់អនុគមន៍ f

យក $x = a$ និង $y = z = 0$

យើងបាន $f(a) + f(0) + f(a) \geq 3f(a)$

$\Rightarrow f(a) \leq f(0)$ $\quad (*)$

យក $x = \frac{a}{2}$; $y = \frac{a}{2}$ និង $z = -\frac{a}{2}$

យើងបាន $f(a) + f(0) + f(0) \geq 3f(0)$

$\Rightarrow f(a) \geq f(0)$ $\quad (**)$

តាម $(*)$ និង $(**)$

នាំអោយ $f(a) = f(0)$ $\Rightarrow f$ ជាអនុគមន៍ថេរ

ដោយ គ្រប់អនុគមន៍ថេរ សុទ្ធតែផ្ទៀងផ្ទាត់លក្ខខណ្ឌខាងលើ

យើងបាន $f(x)=c$ ចំពោះគ្រប់តំលៃ $c\in\mathbb{R}$

ដូច្នេះ $f(x)=c$ ចំពោះគ្រប់តំលៃ $c\in\mathbb{R}$

៦. តាង (r,s) ជាតួចែករួមធំបំផុតនៃ r និង s

ប្រសិនបើ $(r,s)=1$ នាំអោយមាន ចំនួនបឋម p ចែកដាច់ rs

នាំអោយ p ចែកដាច់ r ឬ p ចែកដាច់ s តែ p មិនអាចចែកដាច់ r និង s ព្រមគ្នាបានទេ

$\Rightarrow p$ ចែកមិនដាច់ $r+s$

$\Rightarrow (rs,r+s)=1$

តាង $x=a+b,\quad y=a-b$

យើងបាន $2a+(1+2a)(a^2-b^2)=x+y+(1+x+y)xy$

$$=(x+y+xy)+(x+y)xy$$

និង $2a(a^2+2a-b^2)(a^2-b^2)=(x+y)(xy+x+y)xy$

បើ $(x,y)=1\quad\Rightarrow(x+y,xy)=1$

$(x+y,xy)=1\ \Rightarrow(x+y+xy,(x+y)xy)=1$

ហើយ $(x+y+xy,(x+y)xy)=1$

នាំអោយ

$(x+y+xy+(x+y)xy,(x+y+xy)(x+y)xy)=1$

ដូច្នេះ តួរចែករួមធំបំផុតនៃ $2a+(1+2a)(a^2-b^2)$ និង $2a(a^2+2a-b^2)(a^2-b^2)$ គឺ 1

វិញ្ញាសាទី៤

១. បង្ហាញថា $\left(\sqrt{3}+\sqrt{2}\right)^{\frac{1}{n}}+\left(\sqrt{3}-\sqrt{2}\right)^{\frac{1}{n}}$ ជាចំនួនអសនិទាន ចំពោះគ្រប់ចំនួនគត់វិជ្ជមាន n ។

២. ដោះស្រាយប្រព័ន្ធសមីការ៖

$$\begin{cases} \log_2 x+\log_4 y+\log_4 z=2 & (1) \\ \log_3 y+\log_9 z+\log_9 x=2 & (2) \\ \log_4 z+\log_{16} x+\log_{16} y=2 & (3) \end{cases}$$

៣. គេអោយ $a=\tan\alpha$ ។ បង្ហាញថា $\tan 5\alpha=\dfrac{5a-10a^3+a^5}{1-10a^2+5a^4}$

៤. គេអោយ $A=\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$ ។ បង្ហាញថា $A^n=\begin{pmatrix} 1 & 0 \\ 1-2^n & 2^n \end{pmatrix}$ ចំពោះគ្រប់ចំនួនគត់វិជ្ជមាន n ។

៥. គណនាតំលៃអតិបរមានិងអប្បបរមានៃ

$$y=\frac{2+\cos x}{\sin x+\cos x-2}$$

៦. គេអោយ $f_1(x)=\dfrac{x}{\sqrt[3]{1+x^3}}$ និង $f_n(x)=f\left[f_{n-1}(x)\right]$ ។

គណនា $f_n(x)$

សម្រាយបញ្ជាក់

១. តាង $x=\left(\sqrt{3}+\sqrt{2}\right)^{\frac{1}{n}} \quad \Rightarrow x^{-1}=\left(\sqrt{3}-\sqrt{2}\right)^{\frac{1}{n}}$

ឧបមាថា $x+x^{-1}$ ជាចំនួនសនិទាន

យើងបាន $x^2+x^{-2}=\left(x+x^{-1}\right)^2-2$ ជាចំនួនសនិទាន

ឧបមាថា ពិតដល់ n គឺ x^n+x^{-n} ជាចំនួនសនិទាន

យើងនឹងបង្ហាញថា ពិតដល់ $n+1$

ដោយ $x^{n+1}+x^{-(n+1)}=\left(x+x^{-1}\right)\left(x^n+x^{-n}\right)-\left(x^{n-1}+x^{-(n-1)}\right)$

យើងបាន $x^{n+1}+x^{-(n+1)}$ ជាចំនួនសនិទាន

នាំអោយ x^n+x^{-n} ជាចំនួនសនិទាន

ប៉ុន្តែ $x^n+x^{-n}=\sqrt{3}+\sqrt{2}+\sqrt{3}-\sqrt{2}=2\sqrt{3}$ ជាចំនួន
អសនិទាន (សំនើរ ផ្ទុយ)

នាំអោយ $x+x^{-1}$ ជាចំនួនអសនិទាន

ដូច្នេះ $\left(\sqrt{3}+\sqrt{2}\right)^{\frac{1}{n}}+\left(\sqrt{3}-\sqrt{2}\right)^{\frac{1}{n}}$ ជាចំនួនអសនិទាន

២. ដោះស្រាយប្រព័ន្ធសមីការ៖

លក្ខខណ្ឌ $x,y,z>0$

តាម (1) $\Rightarrow \log_2 x+\frac{1}{2}\log_2 y+\frac{1}{2}\log_2 z=2$

$$\Rightarrow \log_2 x+\log_2\sqrt{y}+\log_2\sqrt{z}=2$$

$$\Rightarrow \log_2\left(x\sqrt{yz}\right)=2 \quad \Rightarrow x\sqrt{yz}=2^2$$

$$\Rightarrow x^2yz=2^4 \qquad (4)$$

តាម (2) $\Rightarrow \log_3 y+\frac{1}{2}\log_3 z+\frac{1}{2}\log_3 x=2$

$$\Rightarrow \log_3 y+\log_3\sqrt{z}+\log_3\sqrt{x}=2$$

$$\Rightarrow \log_3 y\sqrt{xz}=2 \quad \Rightarrow y\sqrt{zx}=3^2$$

$$\Rightarrow y^2zx=3^4 \qquad (5)$$

តាម (3) $\Rightarrow \log_4 z+\frac{1}{2}\log_4 x+\frac{1}{2}\log_4 y=2$

$$\Rightarrow \log_4 z+\log_4\sqrt{x}+\log_4\sqrt{y}=2$$

$$\Rightarrow \log_4 z\sqrt{xy}=2 \quad \Rightarrow z\sqrt{xy}=4^2$$

$$\Rightarrow z^2xy=4^4 \qquad (6)$$

យក (4) គុណ (5) គុណ (6)

យើងបាន $(xyz)^4=24^4 \quad \Rightarrow xyz=24$

តាម (4) $\Rightarrow x=\frac{2^4}{xyz}=\frac{2^4}{24}=\frac{2}{3}$

តាម (5) $\Rightarrow y=\frac{3^4}{xyz}=\frac{3^4}{24}=\frac{27}{8}$

តាម (6) $\Rightarrow z=\frac{4^4}{xyz}=\frac{4^4}{24}=\frac{32}{3}$

ដូច្នេះ $\{x,y,z\}=\left\{\frac{2}{3},\frac{27}{8},\frac{32}{3}\right\}$

៣. តាមទ្រឹស្តីបទ ដឺម័រ

យើងបាន $(\cos\alpha + i\sin\alpha)^5 = \cos 5\alpha + i\sin 5\alpha \qquad (*)$

ម្យ៉ាងទៀត $(\cos\alpha + i\sin\alpha)^5 = \cos^5\alpha + 5\cos^4\alpha(i\sin\alpha) +$

$10\cos^3\alpha(i\sin\alpha)^2 + 10\cos^2\alpha(i\sin\alpha)^3 + 5\cos\alpha(i\sin\alpha)^4 + (i\sin\alpha)^5$

$= \cos^5\alpha + 5i\cos^4\alpha\sin\alpha - 10\cos^3\alpha\sin^2\alpha - 10i\cos^2\alpha\sin^3\alpha +$

$5\cos\alpha\sin^4\alpha + i\sin^5\alpha$

$= \left(\cos^5\alpha - 10\cos^3\alpha\sin^2\alpha + 5\cos\alpha\sin^4\alpha\right)$

$+ i\left(5\cos^4\alpha\sin\alpha - 10\cos^2\alpha\sin^3\alpha + \sin^5\alpha\right) \qquad (**)$

តាម $(*)$ និង $(**)$ យើងបាន

$$\begin{cases} \cos 5\alpha = \cos^5\alpha - 10\cos^3\alpha\sin^2\alpha + 5\cos\alpha\sin^4\alpha \\ \sin 5\alpha = 5\cos^4\alpha\sin\alpha - 10\cos^2\alpha\sin^3\alpha + \sin^5\alpha \end{cases}$$

$$\Rightarrow \tan 5\alpha = \frac{\sin 5\alpha}{\cos 5\alpha} = \frac{5\cos^4\alpha\sin\alpha - 10\cos^2\alpha\sin^3\alpha + \sin^5\alpha}{\cos^5\alpha - 10\cos^3\alpha\sin^2\alpha + 5\cos\alpha\sin^4\alpha}$$

$$\Rightarrow \tan 5\alpha = \frac{\dfrac{5\cos^4\alpha\sin\alpha - 10\cos^2\alpha\sin^3\alpha + \sin^5\alpha}{\cos^5\alpha}}{\dfrac{\cos^5\alpha - 10\cos^3\alpha\sin^2\alpha + 5\cos\alpha\sin^4\alpha}{\cos^5\alpha}}$$

$$\Rightarrow \tan 5\alpha = \frac{5\tan\alpha - 10\tan^3\alpha + \tan^5\alpha}{1 - 10\tan^2\alpha + 5\tan^4\alpha}$$ យក $a = \tan\alpha$

យើងបាន $\tan 5\alpha = \dfrac{5a - 10a^3 + a^5}{1 - 10a^2 + 5a^4}$

<u>ដូច្នេះ</u> $\tan 5\alpha = \dfrac{5a - 10a^3 + a^5}{1 - 10a^2 + 5a^4}$ ដែល $a = \tan\alpha$

៤. យើងមាន $A=\begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}=\begin{pmatrix}1 & 0\\ 1-2^1 & 2^1\end{pmatrix}$

$$A^2=A\cdot A=\begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}=\begin{pmatrix}1\cdot 1+0(-1) & 1\cdot 0+0\cdot 2\\ (-1)\cdot 1+2(-1) & (-1)\cdot 0+2\cdot 2\end{pmatrix}$$

$$=\begin{pmatrix}1 & 0\\ -3 & 4\end{pmatrix}=\begin{pmatrix}1 & 0\\ 1-2^2 & 2^2\end{pmatrix}$$

ឧបមាថា ពិតដល់ n គឺ $A^n=\begin{pmatrix}1 & 0\\ 1-2^n & 2^n\end{pmatrix}$

យើងនឹង បង្ហាញថាពិតដល់ $n+1$

យើងបាន $A^{n+1}=A\cdot A^n=\begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 1-2^n & 2^n\end{pmatrix}$

$$=\begin{pmatrix}1\cdot 1+0(1-2^n) & 1\cdot 0+0\cdot 2^n\\ (-1)\cdot 1+2(1-2^n) & (-1)\cdot 0+2\cdot 2^n\end{pmatrix}=\begin{pmatrix}1 & 0\\ 1-2^{n+1} & 2^{n+1}\end{pmatrix}$$

ពិត

<u>ដូច្នេះ</u> $A^n=\begin{pmatrix}1 & 0\\ 1-2^n & 2^n\end{pmatrix}$

៥. យើងមាន $y=\dfrac{2+\cos x}{\sin x+\cos x-2}$

$\Leftrightarrow y(\sin x+\cos x-2)=2+\cos x$

$\Leftrightarrow y\sin x+(y-1)\cos x=2(y+1)$

$\Leftrightarrow \left[y\sin x+(y-1)\cos x\right]^2=\left[2(y+1)\right]^2 \qquad (*)$

តាមវិសមភាព កូស៊ី ស្វាត

$$\left[y\sin x+(y-1)\cos x\right]^2\le\left[y^2+(y-1)^2\right](\sin^2 x+\cos^2 x)$$

$$\le y^2+y^2-2y+1$$

តាម $(*)$ $\Rightarrow\left[2(y+1)\right]^2\le 2y^2-2y+1$

$$\Rightarrow 4y^2+8y+4\le 2y^2-2y+1$$

$$\Rightarrow 2y^2+10y+3\le 0$$

$$\Rightarrow \frac{-5-\sqrt{19}}{2}\le y\le\frac{-5+\sqrt{19}}{2}$$

<u>ដូច្នេះ</u> $\min(y)=\dfrac{-5-\sqrt{19}}{2}$ និង $\max(y)=\dfrac{-5+\sqrt{19}}{2}$

៦. គណនា $f_n(x)$

យើងមាន $f_1(x)=\dfrac{x}{\sqrt[3]{1+x^3}}$

$$\Rightarrow f_2(x)=f\left[f_1(x)\right]=\frac{f_1(x)}{\sqrt[3]{1+f_1^{\,3}(x)}}=\frac{\dfrac{x}{\sqrt[3]{1+x^3}}}{\sqrt[3]{1+\left(\dfrac{x}{\sqrt[3]{1+x^3}}\right)^3}}$$

$$=\frac{\dfrac{x}{\sqrt[3]{1+x^3}}}{\sqrt[3]{1+\dfrac{x^3}{1+x^3}}}=\frac{x}{\sqrt[3]{1+x^3}\cdot\dfrac{\sqrt[3]{1+2x^3}}{\sqrt[3]{1+x^3}}}=\frac{x}{\sqrt[3]{1+2x^3}}$$

ឧបមាថា វាពិតដល់ n គឺ៖ $f_n(x)=\dfrac{x}{\sqrt[3]{1+nx^3}}$

យើងនឹងបង្ហាញថាពិតដល់ $n+1$

$$f_{n+1}(x)=f\left[f_n(x)\right]=\frac{f_n(x)}{\sqrt[3]{1+f_n^{\,3}(x)}}=\frac{\dfrac{x}{\sqrt[3]{1+nx^3}}}{\sqrt[3]{1+\left(\dfrac{x}{\sqrt[3]{1+nx^3}}\right)^3}}$$

$$=\frac{\dfrac{x}{\sqrt[3]{1+nx^3}}}{\sqrt[3]{1+\dfrac{x^3}{1+nx^3}}}=\frac{x}{\sqrt[3]{1+nx^3}\cdot\dfrac{\sqrt[3]{1+(n+1)x^3}}{\sqrt[3]{1+nx^3}}}=\frac{x}{\sqrt[3]{1+(n+1)x^3}}$$

ពិត

ដូច្នេះ $\quad f_n(x)=\dfrac{x}{\sqrt[3]{1+nx^3}}$

វិញ្ញាសាទី៦

១. គេអោយត្រីកោណ ABC ។

បង្ហាញថា $\cot A\cot B+\cot B\cot C+\cot A\cot C=1$ ។

រួចទាញបញ្ជាក់ថា $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

២. គណនាបណ្តាចំនួនគត់ធម្មជាតិ x,y,z ដែលផ្ទៀងផ្ទាត់ $x^z y=x+y^z$ ។

៣. ចំពោះគ្រប់ x,y,z ជាចំនួនពិត ដែល $x^2+y^2+z^2=9$ ។

បង្ហាញថា $2(x+y+z)-xyz\leq 10$ ។ តើសញ្ញាស្មើកើតមាននៅពេលណា?

៤. គេអោយ u,v,w ជាចំនួនកុំផ្លិច ដែលមាន ម៉ូឌុលស្មើ 1 ។

បង្ហាញថា $|uv+vw+uw|=|u+v+w|$

៥. គណនាតួទូទៅនៃស្វ៊ីត $(a_n)_{n\in\mathbb{N}}$ និង $(b_n)_{n\in\mathbb{N}}$ ដែលផ្ទៀងផ្ទាត់៖

ក. $\{a_n\}=\{1,3,10,22,39,...\}$

ខ. $\{b_n\}=\{1,4,13,40,121,...\}$

៦. គេអោយ $I_n=\int_0^1 x^n\cos\pi x dx$ ចំពោះ $n\in\mathbb{N}$ ។

បង្ហាញថា $\pi^2 I_n+n(n-1)I_{n-2}+n=0$ ចំពោះ $n\geq 2$ ។

រួចទាញបង្ហាញថា $\int_0^1 x^4\cos\pi x dx = \frac{4(6-\pi^2)}{\pi^4}$

សម្រាយបញ្ជាក់

១. យើងមាន $A+B+C=\pi \Rightarrow A+B=\pi-C$

យើងបាន $\cot(A+B)=\cot(\pi-C)=-\cot C \qquad (*)$

ដោយ $\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$

$$\Rightarrow \cot(A+B)=\frac{1-\tan A\tan B}{\tan A+\tan B}=\frac{1-\frac{1}{\cot A\cot B}}{\frac{1}{\cot A}+\frac{1}{\cot B}}=\frac{\cot A\cot B-1}{\cot A+\cot B}$$

តាម $(*)$ $\Rightarrow \frac{\cot A\cot B-1}{\cot A+\cot B}=-\cot C$

$\Rightarrow \cot A\cot B-1=-\cot A\cot C-\cot B\cot C$

$\Rightarrow \cot A\cot B+\cot B\cot C+\cot A\cot C=1$

ដូច្នេះ $\cot A\cot B+\cot B\cot C+\cot A\cot C=1$

ទាញបង្ហាញថា $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

យើងមាន $\cot A\cot B+\cot B\cot C+\cot A\cot C=1$

$$\Rightarrow \frac{1}{\tan A\tan B}+\frac{1}{\tan B\tan C}+\frac{1}{\tan A\tan C}=1$$

$$\Rightarrow \frac{\tan A+\tan B+\tan C}{\tan A\tan B\tan C}=1$$

ដូច្នេះ $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

២. យើងមាន $x^z y = x + y^z$ (1)

$\Leftrightarrow x = x^z y - y^z$ (2)

តាម (1) x ចែកដាច់នឹង y ព្រោះ $x, y, z \in \mathbb{N}^*$

$\Rightarrow x^z$ ចែកដាច់នឹង y^z

តាម (2) $\Rightarrow x$ ចែកដាច់នឹង y^z $\Leftrightarrow x = ty^z$ ដែល $t \in \mathbb{N}^*$

តាម (2) $\Rightarrow t \cdot y^z = t^z \cdot y^{z^2} \cdot y - y^z$

$\Rightarrow t = t^z \cdot y^{z^2 - z + 1} - 1$

$\Rightarrow t + 1 = t^z \cdot y^{z^2 - z + 1}$ (3)

តាម (3) $\Rightarrow 1$ ចែកដាច់នឹង t $\Rightarrow t = 1$

$\Rightarrow y^{z^2 - z + 1} = 2$

ដោយ $y, z \in \mathbb{N}^*$ យើងបាន $y = 2$ និង $z^2 - z + 1 = 1$

នាំអោយ $y = 2,\ z = 1$

តាម (1) $\Rightarrow x = 2$

ដូច្នេះ $\{x, y, z\} = \{2, 2, 1\}$

៣. ឧបមា $x^2 \geq y^2 \geq z^2$ នោះ $x^2 \geq 3$ និង $6 \geq y^2 + z^2 \geq 2yz$

តាម វិសមភាព កូស៊ី ស្វាត

យើងបាន $\left[2(x+y+z) - xyz\right]^2 = \left[2(y+z) + x(2-yz)\right]^2$

$\leq \left[(y+z)^2 + x^2\right]\left[4 + (2-yz)^2\right] = (2yz+9)(y^2z^2 - 4yz + 8)$

តាង $a = yz$ នោះយើងគ្រាន់តែបង្ហាញថា

$(2a+9)(a^2 - 4a + 8) \leq 100$

$\Leftrightarrow 100-(2a+9)(a^2-4a+8)\ge 0$

$\Leftrightarrow 100-2a^3+8a^2-16a-9a^2+36a-72\ge 0$

$\Leftrightarrow -2a^3-a^2+20a+28\ge 0$

$\Leftrightarrow -2a^3+7a^2-8a^2+28a-8a+28\ge 0$

$\Leftrightarrow a^2(7-2a)+4a(7-2a)+4(7-2a)\ge 0$

$\Leftrightarrow (7-2a)(a^2+4a+4)\ge 0$

$\Leftrightarrow (a+2)^2(7-2a)\ge 0$ ពិត ព្រោះ $2a=2yz\le 6$

សមភាពកើតមានពេល $a=-2$

$\Rightarrow yz=-2$ និង $x^2=4(y+z)^2=20-4x^2$

$\Rightarrow x^2=4$ និង $y+z=\pm 1$ នាំអោយ $x=y=2$ និង $z=-1$

ដូច្នេះ $2(x+y+z)-xyz\le 10$

សញ្ញាស្មើកើតមានពេល $x=y=2$ និង $z=-1$

៤. ដោយ u,v,w ជាចំនួនកុំផ្លិច ដែលមាន ម៉ូឌុលស្មើ 1

តាង $u=e^{i\alpha};\ v=e^{i\beta}$ និង $w=e^{i\gamma}$

នាំអោយ $uv+vw+uw=e^{i(\alpha+\beta)}+e^{i(\beta+\gamma)}+e^{i(\alpha+\gamma)}$

$$=e^{i(\alpha+\beta+\gamma)}\left(e^{-i\gamma}+e^{-i\alpha}+e^{-i\beta}\right)$$

$$=e^{i(\alpha+\beta+\gamma)}\left(\overline{w}+\overline{u}+\overline{v}\right)$$

$$=e^{i(\alpha+\beta+\gamma)}\left(\overline{u+v+w}\right)$$

$\Rightarrow |uv+vw+uw|=\left|e^{i(\alpha+\beta+\gamma)}\left(\overline{u+v+w}\right)\right|=\left|e^{i(\alpha+\beta+\gamma)}\right|\left|\overline{u+v+w}\right|$

ដោយ $\left|e^{i(\alpha+\beta+\gamma)}\right|=1$ និង $\left|\overline{u+v+w}\right|=|u+v+w|$

ដូច្នេះ $|uv+vw+uw|=|u+v+w|$

៥. គណនា តួទូទៅនៃស្ទីត

ក. $\{a_n\}=\{1,3,10,22,39,...\}$

តាង $c_n=a_n-a_{n-1}$ ដែល $n\in\mathbb{N}^*$

យើងបាន $\sum_{i=1}^{n}c_i=a_n-a_0 \quad \Rightarrow a_n=\sum_{i=1}^{n}c_i+a_0$

ម្យ៉ាងទៀត យើងបាន $\{c_n\}=\{2,7,12,17,...\}$ ជាស្ទីតនព្វន្តដែល មានតួទីមួយ $c_1=2$ និង ផលសងរួម $d=5$

$$\Rightarrow\sum_{i=1}^{n}c_i=\frac{n\left(2c_1+(n-1)d\right)}{2}=\frac{n\left(4+(n-1)5\right)}{2}=\frac{5n^2-n}{2}$$

នាំអោយ $a_n=\frac{5n^2-n}{2}+1=\frac{5n^2-n+2}{2}$

ដូច្នេះ $a_n=\frac{5n^2-n+2}{2}$

ខ. $\{b_n\}=\{1,4,13,40,121,...\}$

តាង $d_n=b_n-b_{n-1}$ ដែល $n\in\mathbb{N}^*$

យើងបាន $\sum_{i=1}^{n}d_i=b_n-b_0 \quad \Rightarrow b_n=\sum_{i=1}^{n}b_i+b_0$

ម្យ៉ាងទៀត $\{d_n\}=\{3,9,27,81,...\}$ ជាស្ទីតធរណីមាត្រ ដែល មាន តួទីមួយ $d_1=3$ និង រេសុង $r=3$

$$\Rightarrow \sum_{i=1}^{n} d_i = d_1 \cdot \frac{1-r^n}{1-r} = 3\cdot\frac{1-3^n}{1-3} = \frac{3^{n+1}-3}{2}$$

នាំអោយ $b_n = \frac{3^{n+1}-3}{2} + 1 = \frac{3^{n+1}-1}{2}$

<u>ដូច្នេះ</u> $b_n = \frac{3^{n+1}-1}{2}$

៦. យើងមាន $I_n = \int_0^1 x^n \cos \pi x dx$

តាង $u = x^n \quad \Rightarrow du = nx^{n-1}dx$

$$dv = \cos \pi x dx \quad \Rightarrow v = \int \cos \pi x dx = \frac{1}{\pi}\sin \pi x$$

យើងបាន $I_n = \left(\frac{x^n}{\pi}\sin \pi x\right)\Bigg|_0^1 - \frac{n}{\pi}\int_0^1 x^{n-1}\sin \pi x dx$

$$\Rightarrow I_n = -\frac{n}{\pi}\int_0^1 x^{n-1}\sin \pi x dx \qquad (*)$$

ចំពោះ $\int_0^1 x^{n-1}\sin \pi x dx$

តាង $a = x^{n-1} \quad \Rightarrow da = (n-1)x^{n-2}dx$

$$db = \sin \pi x dx \quad \Rightarrow b = \int \sin \pi x dx = -\frac{1}{\pi}\cos \pi x$$

$$\Rightarrow \int_0^1 x^{n-1}\sin \pi x dx = \left(-\frac{x^{n-1}}{\pi}\cos \pi x\right)\Bigg|_0^1 - \int_0^1 \left(-\frac{1}{\pi}\cos \pi x\right)(n-1)x^{n-2}dx$$

$$\Rightarrow \int_0^1 x^{n-1}\sin \pi x dx = \frac{1}{\pi} + \frac{n-1}{\pi}\int_0^1 \left(x^{n-2}\cos \pi x\right)dx = \frac{1}{\pi} + \frac{n-1}{\pi} I_{n-2}$$

ជំនួសចូលក្នុង $(*)$ នាំអោយ

$$I_n = -\frac{n}{\pi}\left(\frac{1}{\pi} + \frac{n-1}{\pi} I_{n-2}\right) = -\frac{n}{\pi^2} - \frac{n(n-1)}{\pi^2} I_{n-2}$$

$$\Rightarrow \pi^2 I_n + n(n-1) I_{n-2} + n = 0$$

<u>ដូច្នេះ</u> $\pi^2 I_n + n(n-1) I_{n-2} + n = 0$

ទាញបង្ហាញថា $\int_0^1 x^4 \cos \pi x dx = \frac{4(6-\pi^2)}{\pi^4}$

តាមសម្រាយខាងលើ យើងបាន $\pi^2 I_4 + 4(4-1) I_2 + 4 = 0$

$$\Rightarrow I_4 = \frac{-12 I_2 - 4}{\pi^2}$$

ម្យ៉ាងទៀត $\pi^2 I_2 + 2(2-1) I_0 + 2 = 0$

$$\Rightarrow I_2 = \frac{-2 I_0 - 2}{\pi^2}$$ ដោយ $I_0 = \int_0^1 \cos \pi x dx \pi = \frac{1}{\pi}\sin \pi x\Big|_0^1 = 0$

$$\Rightarrow I_2 = -\frac{2}{\pi^2}$$

$$\Rightarrow I_4 = \int_0^1 x^4 \cos \pi x dx = \frac{-12\left(\frac{-2}{\pi^2}\right) - 4}{\pi^2} = \frac{-4\pi^2 + 24}{\pi^4} = \frac{4(6-\pi^2)}{\pi^4}$$

<u>ដូច្នេះ</u> $\int_0^1 x^4 \cos \pi x dx = \frac{4(6-\pi^2)}{\pi^4}$

វិញ្ញាសាទី៧

១. គេអោយស្វ៊ីត $0 \le a_1 \le a_2 \le \cdots \le a_{2011} < 1$ ។
ចូរប្រៀបធៀប $\log_{1/2010}\left(a_1 + a_2 + \cdots + a_{2010}\right)$ និង
$\log_{1/2011}\left(a_1 + a_2 + \cdots + a_{2011}\right)$

២. គណនាតំលៃអតិបរមា និង អប្បបរមានៃអនុគមន៍

$$f\left(x\right) = \frac{3 + 8x^2 + 12x^4}{\left(1 + 2x^2\right)^2}$$

៣. ដោះស្រាយសមីការៈ

ក. $P_x A_x^2 + 72 = 6\left(A_x^2 + 2P_x\right)$

ខ. $C_{n-1}^4 - C_{n-1}^3 - \frac{5}{4}A_{n-2}^2 = 0$

៤. បង្ហាញថា $A = \frac{1}{\cos^n \frac{\pi}{7}} + \frac{1}{\cos^n \frac{3\pi}{7}} + \frac{1}{\cos^n \frac{5\pi}{7}}$ ជាចំនួនសនិទាន ចំពោះគ្រប់ $n \in \mathbb{N}$ ។

៥. គេអោយ a,b,c ជាប្រវែងជ្រុងទាំងបីនៃត្រីកោណ និង p ជាកន្លះបរិមាត្រ។
បង្ហាញថា $\sqrt{p} < \sqrt{p-a} + \sqrt{p-b} + \sqrt{p-c} \le \sqrt{3p}$

៦. គណនាតួទី n នៃស្វីតដែលគោរពលក្ខខណ្ឌៈ $U_0 = \sqrt{2}$; $U_1 = \sqrt{3}$ និង $U_{n+2} - 5U_{n+1} + 6U_n = 0$ ។

សម្រាយបញ្ជាក់

១. យើងមាន $0 \le a_1 \le a_2 \le \cdots \le a_{2011} < 1$

$$\Rightarrow a_1 + a_2 + \cdots + a_{2010} \le 2010a_{2010} \le 2010a_{2011}$$

បូកអង្គទាំងពីរនៃវិសមភាពនឹង $2010(a_1 + a_2 + \cdots + a_{2010})$

$$\Rightarrow 2011(a_1 + a_2 + \cdots + a_{2010}) \le 2010(a_1 + a_2 + \cdots + a_{2011})$$

$$\Rightarrow 0 < \frac{a_1 + a_2 + \cdots + a_{2010}}{2010} \le \frac{a_1 + a_2 + \cdots + a_{2011}}{2011} < 1$$

$$\Rightarrow \log_{1/2010}\left(\frac{a_1 + a_2 + \cdots + a_{2010}}{2010}\right) \ge \log_{1/2010}\left(\frac{a_1 + a_2 + \cdots + a_{2011}}{2011}\right) \quad (1)$$

ម្យ៉ាងទៀត $0 < \frac{1}{2011} < \frac{1}{2010} < 1$

$$\Rightarrow \log_{1/2011}\left(\frac{a_1 + a_2 + \cdots + a_{2011}}{2011}\right) < \log_{1/2010}\left(\frac{a_1 + a_2 + \cdots + a_{2011}}{2011}\right) \quad (2)$$

តាម (1) និង (2)

$$\Rightarrow \log_{1/2010}\left(\frac{a_1 + a_2 + \cdots + a_{2010}}{2010}\right) > \log_{1/2011}\left(\frac{a_1 + a_2 + \cdots + a_{2011}}{2011}\right)$$

$$\Rightarrow \log_{1/2010}(a_1 + a_2 + \cdots + a_{2010}) + 1 > \log_{1/2011}(a_1 + a_2 + \cdots + a_{2011}) + 1$$

$$\Rightarrow \log_{1/2010}(a_1 + a_2 + \cdots + a_{2010}) > \log_{1/2011}(a_1 + a_2 + \cdots + a_{2011})$$

ដូច្នេះ
$$\log_{1/2010}\left(a_1+a_2+\cdots+a_{2010}\right)>\log_{1/2011}\left(a_1+a_2+\cdots+a_{2011}\right)$$

២. តាង $\tan u = x\sqrt{2}$ ដែល $-\frac{\pi}{2}<u<\frac{\pi}{2}$

$$\Rightarrow f(x)=g(u)=\frac{3+4\tan^2 u+3\tan^4 u}{\left(1+\tan^2 u\right)^2}$$
$$=\frac{3\cos^4 u+4\sin^2 u\cos^2 u+3\sin^4 u}{\left(\sin^2 u+\cos^2 u\right)^2}$$
$$=3\left(\sin^2 u+\cos^2 u\right)^2-2\left(\sin u\cos u\right)^2$$
$$=3-\frac{\sin^2 2u}{2}$$

ដោយ $0\le\sin^2 2u\le 1$ $\quad\Rightarrow\frac{5}{2}\le g(u)\le 3$

$\Rightarrow Max\left(f(x)\right)=3$ ពេល $x=0$

$\Rightarrow \min\left(f(x)\right)=\frac{5}{2}$ ពេល $x=\frac{1}{\sqrt{2}}$

ដូច្នេះ $Max\left(f(x)\right)=3$ និង $\min\left(f(x)\right)=\frac{5}{2}$

៣. ដោះស្រាយសមីការ៖

ក. $P_x A_x^2+72=6\left(A_x^2+2P_x\right)$

លក្ខខណ្ឌ $x\in\mathbb{N}, x\ge 2$

យើងបានសមីការអាចសរសេរ

$$x!\frac{x!}{(x-2)!}+72=6\left(\frac{x!}{(x-2)!}+2(x!)\right)$$

$$\Leftrightarrow x!\frac{x(x-1)(x-2)!}{(x-2)!}+72=\frac{x(x-1)\cdot(x-2)!6}{(x-2)!}+12(x!)$$

$$\Leftrightarrow x(x-1)x!+72=6x(x-1)+12(x!)$$

$$\Leftrightarrow (x^2-x-12)x!-6(x^2-x-12)=0$$

$$\Leftrightarrow (x^2-x-12)(x!-6)=0$$

យើងបាន $x^2-x-12=0$ ឬ $x!-6=0$

នាំអោយ $x=-3$ ឬ $x=4$ ឬ $x=3$ ប៉ុន្តែ $x\geq 2$

ដូច្នេះ $x=4$ ឬ $x=3$

ខ. $C_{n-1}^{4}-C_{n-1}^{3}-\frac{5}{4}A_{n-2}^{2}=0$

លក្ខខណ្ឌ $n\geq 5$ និង $n\in\mathbb{N}$

យើងបាន សមីការ អាចសរសេរ

$$\frac{(n-1)!}{4!(n-5)!}-\frac{(n-1)!}{3!(n-4)!}-\frac{5}{4}\cdot\frac{(n-2)!}{(n-4)!}=0$$

$$\Leftrightarrow \frac{(n-1)(n-2)(n-3)(n-4)}{24}-\frac{(n-1)(n-2)(n-3)}{6}$$

$$-\frac{5(n-2)(n-3)}{4}=0$$

$\Leftrightarrow \frac{(n-1)(n-4)}{24}-\frac{n-1}{6}-\frac{5}{4}=0$

$\Leftrightarrow n^2-5n+4-4n+4-30=0$

$\Leftrightarrow n^2-9n-22=0$ នាំអោយ $\Delta=81+88=169=13^2$

យើងបាន $n=\frac{9-13}{2}=-2$ ឬ $n=\frac{9+13}{2}=11$ប៉ុន្តែ $n\geq 5$

ដូច្នេះ $n=11$

៩. យើងមាន $\frac{\pi}{7};\frac{3\pi}{7};\frac{5\pi}{7}$ ជាឫសនៃសមីការ

$\cos 3x+\cos 4x=0$

ដោយ $\cos 3x=\cos(2x+x)=\cos 2x\cos x-\sin 2x\sin x$

$=(2\cos^2 x-1)\cos x-2\sin x\cos x\sin x$

$=2\cos^3 x-\cos x-2(1-\cos^2 x)\cos x$

$=4\cos^3 x-3\cos x$

ហើយ $\cos 4x=(2\cos^2 2x-1)=\left[2(2\cos^2 x-1)^2-1\right]$

$=8\cos^4 x-8\cos^2 x+1$

យើងបាន $\cos 3x+\cos 4x=0$

$\Leftrightarrow 4\cos^3 x-3\cos x+8\cos^4 x-8\cos^2 x+1=0$

$\Leftrightarrow(\cos x+1)(8\cos^3 x-4\cos^2 x-4\cos x+1)=0$

$\Leftrightarrow 8\cos^3 x-4\cos^2 x-4\cos x+1=0$ ព្រោះ $\cos x+1>0$

នាំអោយ $\cos\frac{\pi}{7};\cos\frac{3\pi}{7};\cos\frac{5\pi}{7}$ ជាឫសនៃសមីការ

$8t^3 - 4t^2 - 4t + 1 = 0 \qquad (*)$

តាង $S_n = x_1^{\ n} + x_2^{\ n} + x_3^{\ n}$ ដែល $x_1 = \cos\frac{\pi}{7}; x_2 = \cos\frac{3\pi}{7}$

និង $x_3 = \cos\frac{5\pi}{7}$

នាំអោយ $(*) \Leftrightarrow 8S_{n+3} - 4S_{n+2} - 4S_{n+1} + S_n = 0 \qquad (**)$

ដែល $S_0 = 3,\ \ S_1 = x_1 + x_2 + x_3 = \frac{1}{2}$

$S_2 = x_1^{\ 2} + x_2^{\ 2} + x_3^{\ 2} = (x_1 + x_2 + x_3)^2 - 2(x_1x_2 + x_2x_3 + x_1x_3)$

$\quad = \frac{1}{4} - 2\left(-\frac{1}{2}\right) = \frac{5}{4}$

ដោយ $S_0; S_1; S_2 \in \mathbb{Q}$ តាម $(**)$ $\Rightarrow S_n \in \mathbb{Q}$ ដែល $\forall n \in \mathbb{N}$

ម្យ៉ាងទៀត

$$A = \left(\frac{1}{x_1}\right)^n + \left(\frac{1}{x_2}\right)^n + \left(\frac{1}{x_3}\right)^n = \frac{(x_1x_2)^n + (x_2x_3)^n + (x_1x_3)^n}{(x_1x_2x_3)^n}$$

$$= \frac{1}{2} \cdot \frac{S_n^{\ 2} - S_{2n}}{\left(-\frac{1}{8}\right)^n} \in \mathbb{Q}$$

<u>ដូច្នេះ</u> A ជាចំនួនសនិទាន

៥. • បង្ហាញថា $\sqrt{p-a} + \sqrt{p-b} + \sqrt{p-c} > \sqrt{p}$

យើងមាន $(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2xz$

ដោយ $x, y, z > 0 \quad \Rightarrow (x+y+z)^2 > x^2 + y^2 + z^2$

ដោយ $\sqrt{p-a};\sqrt{p-b};\sqrt{p-b}>0$ នាំអោយ

$$\left(\sqrt{p-a}+\sqrt{p-b}+\sqrt{p-c}\right)^2>\left(\sqrt{p-a}\right)^2+\left(\sqrt{p-b}\right)^2+\left(\sqrt{p-c}\right)^2$$

$$\Rightarrow\left(\sqrt{p-a}+\sqrt{p-b}+\sqrt{p-c}\right)^2>p-a+p-b+p-c$$

$$\Rightarrow\left(\sqrt{p-a}+\sqrt{p-b}+\sqrt{p-c}\right)^2>p$$

$$\Rightarrow\sqrt{p-a}+\sqrt{p-b}+\sqrt{p-c}>\sqrt{p} \qquad (*)$$

- បង្ហាញថា $\sqrt{p-a}+\sqrt{p-b}+\sqrt{p-c}\le\sqrt{3p}$

តាម វិសមភាព កូស៊ី ស្វាត

$$\left(\sqrt{p-a}+\sqrt{p-b}+\sqrt{p-c}\right)^2\le(1+1+1)(p-a+p-b+p-c)$$

$$\Rightarrow\left(\sqrt{p-a}+\sqrt{p-b}+\sqrt{p-c}\right)^2\le 3p$$

$$\Rightarrow\sqrt{p-a}+\sqrt{p-b}+\sqrt{p-c}\le\sqrt{3p} \qquad (**)$$

តាម $(*)$ និង $(**)$ វិសមភាពត្រូវបានស្រាយបញ្ជាក់

ដូច្នេះ $\sqrt{p}<\sqrt{p-a}+\sqrt{p-b}+\sqrt{p-c}\le\sqrt{3p}$

៦. រកតួទី n នៃស៊ីត

យើងមាន $U_{n+2}-5U_{n+1}+6U_n=0$

សមីការសំគាល់ $r^2-5r+6=0$ មានឬស $r_1=2$ ឬ $r_2=3$

យើងបាន $U_n=\alpha 2^n+\beta 3^n$ ចំពោះ $\alpha;\beta\in\mathbb{R}$

ដោយ $U_0=\sqrt{2}$ និង $U_1=\sqrt{3}$

នាំអោយ $\begin{cases}\alpha+\beta=\sqrt{2}\\ 2\alpha+3\beta=\sqrt{3}\end{cases}$

យើងបាន $\alpha = 3\sqrt{2} - \sqrt{3}$ និង $\beta = \sqrt{3} - 2\sqrt{2}$

ដូច្នេះ $U_n = \left(3\sqrt{2} - \sqrt{3}\right)2^n + \left(\sqrt{3} - 2\sqrt{2}\right)3^n$

វិញ្ញាសាទី៨

១. បង្ហាញថា $C_n^k+3C_n^{k-1}+3C_n^{k-2}+C_n^{k-3}=C_{n+3}^k$ ចំពោះ $3\le k\le n$ ។

២. ដោះស្រាយសមីការ៖

$$2010\cdot 2011^{2\left(\log_{2010}x-1\right)}=x^{1+\log_{2010}2011}-x^2$$

៣. គណនា $\lim\limits_{n\to+\infty}\left(\frac{\pi}{2n}\sum_{i=0}^{n-1}\cos\frac{i}{2n}\pi\right)$

៤. គេអោយ a,b,c ជាប្រវែងជ្រុងទាំងបីនៃត្រីកោណដែលមានក្រលាផ្ទៃ ស្មើ 1។ បង្ហាញថា $a^4+b^4+c^4\ge 16$

៥. គណនា $\arctan\frac{1}{2}+\arctan\frac{1}{5}+\arctan\frac{1}{8}$

៦. គណនាតួទី n នៃស៊ីតដែលផ្ទៀងផ្ទាត់លក្ខខណ្ឌ $U_0=\sqrt{2};$ $U_1=-\frac{\sqrt{10}+3\sqrt{5}}{2}$ និង $U_{n+2}+\sqrt{5}U_{n+1}+5U_n=0$ ។

សម្រាយបញ្ជាក់

១. បង្ហាញថា $C_n^k+3C_n^{k-1}+3C_n^{k-2}+C_n^{k-3}=C_{n+3}^k$

យើងមាន $C_n^k+C_n^{k-1}=C_{n+1}^k$

យើងបាន $C_n^k+3C_n^{k-1}+3C_n^{k-2}+C_n^{k-3}$

$=\left(C_n^k+C_n^{k-1}\right)+2\left(C_n^{k-1}+C_n^{k-2}\right)+\left(C_n^{k-2}+C_n^{k-3}\right)$

$=C_{n+1}^k+2C_{n+1}^{k-1}+C_{n+1}^{k-2}=\left(C_{n+1}^k+C_{n+1}^{k-1}\right)+\left(C_{n+1}^{k-1}+C_{n+1}^{k-2}\right)$

$=C_{n+2}^k+C_{n+2}^{k-1}=C_{n+3}^k$

ដូច្នេះ $C_n^k+3C_n^{k-1}+3C_n^{k-2}+C_n^{k-3}=C_{n+3}^k$

២. ដោះស្រាយសមីការ៖

តាង $a=2010$ និង $b=2011$

នោះសមីការអាចសរសេរ

$$a\cdot b^{2(\log_a x-1)}=x^{1+\log_a b}-x^2 \qquad (1)$$

- ចំពោះ $x=1$ យើងបាន សមីការមិនផ្ទៀងផ្ទាត់
- ចំពោះ $x\neq 1$ សមីការ (1) សមមូលនឹង

$$a\cdot b^{2(\log_a x-1)}=x^{(\log_a x)(\log_x ab)}-a^{\log_a x^2}$$

$$\Leftrightarrow \frac{a}{b^2}\cdot\left(b^2\right)^{\log_a x}=\left(x^{\log_x ab}\right)^{\log_a x}-\left(a^2\right)^{\log_a x}$$

$$\Leftrightarrow \frac{a}{b^2}\cdot\left(b^2\right)^{\log_a x}=(ab)^{\log_a x}-\left(a^2\right)^{\log_a x}$$

ចែកអង្គទាំងពីរនៃសមីការនឹង $\left(a^2\right)^{\log_a x}$ យើងបាន

$$\frac{a}{b^2}\cdot\left(\frac{b^2}{a^2}\right)^{\log_a x}=\left(\frac{b}{a}\right)^{\log_a x}-1$$

$$\Leftrightarrow \frac{a}{b^2}\cdot\left[\left(\frac{b}{a}\right)^{\log_a x}\right]^2-\left(\frac{b}{a}\right)^{\log_a x}+1=0$$

តាង $t=\left(\frac{b}{a}\right)^{\log_a x}$

យើងបាន $\frac{a}{b^2}t^2-t+1=0$

$\Leftrightarrow at^2-b^2t+b^2=0$

$\Leftrightarrow at^2-abt+abt-ab^2-b^2t+ab^2+b^2=0$

$\Leftrightarrow at(t-b)+ab(t-b)-b^2(t-(a+1))=0$

ដោយ $a+1=b$

$\Leftrightarrow (t-b)\left(at+ab-b^2\right)=0$

យើងបាន $t-b=0$ ឬ $at+b(a-b)=0$ ដោយ $a-b=-1$

នាំអោយ $t=b$ ឬ $t=\frac{b}{a}$

* ករណី $t=b \quad \Rightarrow \left(\frac{b}{a}\right)^{\log_a x}=b$

$\Rightarrow \log_a x=\log_{\frac{b}{a}} b=\frac{1}{1-\log_b a}$

$\Rightarrow x=a^{\frac{1}{1-\log_b a}}=2010^{\frac{1}{1-\log_{2011}2010}}$

* ករណី $t=\frac{b}{a} \quad \Rightarrow \left(\frac{b}{a}\right)^{\log_a x}=\frac{b}{a}$

$\Rightarrow \log_a x=1 \qquad \Rightarrow x=a=2010$

<u>ដូច្នេះ</u> $x=2010$ ឬ $x=2010^{\frac{1}{1-\log_{2011}2010}}$

៣.គណនា $\lim\limits_{n\to+\infty}\left(\frac{\pi}{2n}\sum\limits_{i=0}^{n-1}\cos\frac{i}{2n}\pi\right)$

យើងមាន $\lim\limits_{n\to+\infty}\frac{\pi}{2n}\sum\limits_{i=0}^{n-1}\cos\frac{i}{2n}\pi=\frac{\pi}{2}\lim\limits_{n\to+\infty}\frac{1}{n}\sum\limits_{i=0}^{n-1}\cos\left(\frac{\pi}{2}\cdot\frac{i}{n}\right)$

តាង $f(x)=\cos\frac{\pi}{2}x$ ជាប់លើចន្លោះ $[0,1]$

ដោយចែកចន្លោះ $[0,1]$ ជា n ចន្លោះតូចៗដោយចំនុច $x_i=\frac{i}{n}$

ដែល $i=\overline{0,n}$

នាំអោយ $\frac{1}{n}\sum\limits_{i=0}^{n-1}\cos\left(\frac{\pi}{2}\cdot\frac{i}{n}\right)=\frac{1}{n}\sum\limits_{i=0}^{n}f(x_i)$

$$\Rightarrow\lim_{n\to+\infty}\frac{1}{n}\sum_{i=0}^{n-1}\cos\left(\frac{\pi}{2}\cdot\frac{i}{n}\right)=\lim_{n\to+\infty}\frac{1}{n}\sum_{i=0}^{n}f(x_i)=\int_0^1 f(x)dx$$

$$=\int_0^1\cos\frac{\pi}{2}xdx=\frac{2}{\pi}\sin\frac{\pi}{2}x\Big|_0^1=\frac{2}{\pi}\left(\sin\frac{\pi}{2}-\sin 0\right)=\frac{2}{\pi}$$

យើងបាន $\lim\limits_{n\to+\infty}\left(\frac{\pi}{2n}\sum\limits_{i=0}^{n-1}\cos\frac{i}{2n}\pi\right)=\frac{\pi}{2}\cdot\frac{2}{\pi}=1$

<u>ដូច្នេះ</u> $\lim\limits_{n\to+\infty}\left(\frac{\pi}{2n}\sum\limits_{i=0}^{n-1}\cos\frac{i}{2n}\pi\right)=1$

៤. បង្ហាញថា $a^4+b^4+c^4\geq 16$

យើងមាន $\left(a^2-b^2\right)^2\geq 0\quad\Rightarrow a^4+b^4\geq 2a^2b^2$

$$\left(b^2-c^2\right)^2\geq 0 \qquad \Rightarrow b^4+c^4\geq 2b^2c^2$$

យើងបាន

$$a^4+b^4+c^4\geq 2a^2b^2+2b^2c^2-b^4=b^2\left(2a^2+2c^2-b^2\right)$$

ដោយ $a^2+c^2\geq 2ac$ និង $b^2\geq b^2-(a-c)^2$

$$\Rightarrow a^4+b^4+c^4\geq\left(b^2-(a-c)^2\right)\left(a^2+c^2+2ac-b^2\right)$$

$$\Rightarrow a^4+b^4+c^4\geq\left(b^2-(a-c)^2\right)\left((a+c)^2-b^2\right)$$

$$\geq(b-a+c)(b+a-c)(a+c-b)(a+c+b)$$

$$\geq 16\left[\left(\frac{b+c-a}{2}\right)\left(\frac{a+b-c}{2}\right)\left(\frac{a+c-b}{2}\right)\left(\frac{a+c+b}{2}\right)\right]$$

$$\geq 16p(p-a)(p-b)(p-c)$$

ដោយ $S=\sqrt{p(p-a)(p-b)(p-c)}=1$

$$\Rightarrow p(p-a)(p-b)(p-c)=1$$

យើងបាន $a^4+b^4+c^4\geq 16$

ដូច្នេះ $a^4+b^4+c^4\geq 16$

៥. គណនា $\arctan\frac{1}{2}+\arctan\frac{1}{5}+\arctan\frac{1}{8}$

យើងមាន $Z=x+iy=|Z|e^{i\arctan\frac{y}{x}}$

ចំពោះស្ទីត $Z_1=2+i,\ \ Z_2=5+i$ និង $Z_3=8+i$

$$\Rightarrow \arg(Z_1)=\arctan\frac{1}{2},\arg(Z_2)=\arctan\frac{1}{5},\arg(Z_3)=\arctan\frac{1}{8}$$

ដោយ $\arg\left(Z_1Z_2Z_3\right)=\arg\left(Z_1\right)+\arg\left(Z_2\right)+\arg\left(Z_3\right)$

$\Rightarrow \arctan\frac{1}{2}+\arctan\frac{1}{5}+\arctan\frac{1}{8}=\arg\left(Z_1Z_2Z_3\right)$

ដោយ $Z_1Z_2Z_3=(2+i)(5+i)(8+i)=65(1+i)=65\sqrt{2}e^{i\frac{\pi}{4}}$

$\Rightarrow \arg\left(Z_1Z_2Z_3\right)=\frac{\pi}{4}+k\pi$ ចំពោះ $k\in\mathbb{Z}$

ប៉ុន្តែ $0\le\arctan x\le x$ ចំពោះគ្រប់តំលៃ $x\ge 0$

$\Rightarrow 0\le\arctan\frac{1}{2}+\arctan\frac{1}{5}+\arctan\frac{1}{8}\le\frac{1}{2}+\frac{1}{5}+\frac{1}{8}<\pi$

$\Rightarrow \arctan\frac{1}{2}+\arctan\frac{1}{5}+\arctan\frac{1}{8}=\frac{\pi}{4}$

ដូចេ្នះ $\arctan\frac{1}{2}+\arctan\frac{1}{5}+\arctan\frac{1}{8}=\frac{\pi}{4}$

៦. កំនត់តួទី n នៃស្វ៊ីត (U_n)

យើងមាន $U_{n+2}+\sqrt{5}U_{n+1}+5U_n=0$

សមីការសំគាល់ $r^2+\sqrt{5}r+5=0$

មានឫស $r=-\frac{\sqrt{5}}{2}\pm\frac{\sqrt{15}}{2}i=\sqrt{5}\left(-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}\right)$

$$=\sqrt{5}\left(\cos\frac{2\pi}{3}\pm i\sin\frac{2\pi}{3}\right)$$

នាំអោយ $U_n=\sqrt{5}^n\left(\alpha\cos\frac{2n\pi}{3}+\beta\sin\frac{2n\pi}{3}\right)$

ដោយ $U_0=\sqrt{2}$ និង $U_1=-\frac{\sqrt{10}+3\sqrt{5}}{2}$

យើងបាន $\begin{cases}\sqrt{5}^0(\alpha\cos 0+\beta\sin 0)=\sqrt{2}\\ \sqrt{5}^1\left(\alpha\cos\frac{2\pi}{3}+\beta\sin\frac{2\pi}{3}\right)=-\frac{\sqrt{10}+3\sqrt{5}}{2}\end{cases}$

$$\begin{cases}\alpha=\sqrt{2}\\ \sqrt{5}\left(-\frac{1}{2}\alpha+\frac{\sqrt{3}}{2}\beta\right)=-\frac{\sqrt{10}+3\sqrt{5}}{2}\end{cases}$$

នាំអោយ $\alpha=\sqrt{2}$ និង $\beta=-\sqrt{3}$

ដូច្នេះ $U_n=\sqrt{5}^n\left(\sqrt{2}\cos\frac{2n\pi}{3}-\sqrt{3}\sin\frac{2n\pi}{3}\right)$

វិញ្ញាសាទី៩

១. បង្ហាញថា គ្រប់ចំនួនគត់ធម្មជាតិ n៖

$$\frac{1}{A(2,2)}+\frac{1}{A(3,2)}+\frac{1}{A(4,2)}+\cdots+\frac{1}{A(n,2)}=\frac{n-1}{n}$$

២. គេអោយត្រីកោណ ABC ។ បង្ហាញថា គ្រប់តំលៃ $x\in\mathbb{R}$

គេបាន $1+\frac{1}{2}x^2\geq\cos A+(\cos B+\cos C)x$

៣. ដោះស្រាយសមីការ

$4^{\cos 2x}+4^{\cos^2 x}=3$ ចំពោះ $x\in\left[-\frac{3}{4},\frac{5}{2}\right]$

៤. គេអោយត្រីកោណ ABC ដែលផ្ទៀងផ្ទាត់ ៖

$$\begin{cases}\sin B+\sin C=2\sin A\\ \cos B+\cos C=2\cos A\end{cases}$$

រកប្រភេទនៃត្រីកោណ ABC

៥. គេអោយ $A=\begin{pmatrix}\frac{1}{2} & 0\\ a & a\end{pmatrix}$ ។

ក. គណនា A^3 ជាអនុគមន៍នៃ a

ខ. គណនា A^n ជាអនុគមន៍នៃ a និង n ដែល n ជាចំនួនគត់ធម្មជាតិខុសពីសូន្យ។

៦. គេអោយ $f(x)=x^3+ax^2+bx+c$ ផ្ទៀងផ្ទាត់ $f(-2)=-10$ និងមានតំលៃបរមាស្មើ $\frac{50}{27}$ ត្រង់ $x=\frac{2}{3}$ ។ កំនត់តំលៃ a,b,c

សម្រាយបញ្ជាក់

១. យើងមាន $\frac{1}{A(2,2)}+\frac{1}{A(3,2)}+\frac{1}{A(4,2)}+\cdots+\frac{1}{A(n,2)}$

$$=\frac{1}{\frac{2!}{(2-2)!}}+\frac{1}{\frac{3!}{(3-2)!}}+\frac{1}{\frac{4!}{(4-2)!}}+\cdots+\frac{1}{\frac{n!}{(n-2)!}}$$

$$=\frac{1}{2!}+\frac{1}{3!}+\frac{2!}{4!}+\cdots+\frac{(n-2)!}{n!}$$

$$=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(n-1)n}$$

$$=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n-1}-\frac{1}{n}\right)$$

$$=1-\frac{1}{n}=\frac{n-1}{n}$$

ដូច្នេះ $\frac{1}{A(2,2)}+\frac{1}{A(3,2)}+\frac{1}{A(4,2)}+\cdots+\frac{1}{A(n,2)}=\frac{n-1}{n}$

២. បង្ហាញថា $1+\frac{1}{2}x^2 \geq \cos A+(\cos B+\cos C)x$

ឧបមាថា $1+\frac{1}{2}x^2 \geq \cos A+(\cos B+\cos C)x$

$$\Leftrightarrow x^2-2(\cos B+\cos C)x-2(\cos A-1)\geq 0$$

តាង $f(x)=x^2-2(\cos B+\cos C)x-2(\cos A-1)$

ដើម្បីបង្ហាញថា $f(x)\geq 0$ យើងគ្រាន់តែបង្ហាញថា $\Delta'\leq 0$

ព្រោះ $a=1>0$

បើ $f(x)=0\Leftrightarrow x^2-2(\cos B+\cos C)x-2(\cos A-1)=0$

យើងបាន $\Delta'=(\cos B+\cos C)^2+2(\cos A-1)$

$$=\left(2\cos\frac{B+C}{2}\cos\frac{B-C}{2}\right)^2-2\left(2\sin^2\frac{A}{2}\right)$$

$$=4\left[\cos\left(\frac{\pi}{2}-\frac{A}{2}\right)\cos\frac{B-C}{2}\right]^2-4\sin^2\frac{A}{2}$$

$$=4\sin^2\frac{A}{2}\cos^2\frac{B-C}{2}-4\sin^2\frac{A}{2}$$

$$=4\sin^2\frac{A}{2}\left(\cos^2\frac{B-C}{2}-1\right)$$

$$=-4\sin^2\frac{A}{2}\sin^2\frac{B-C}{2}\leq 0$$

នាំអោយ $f(x)\geq 0$

ដូច្នេះ $1+\frac{1}{2}x^2\geq \cos A+(\cos B+\cos C)x$ ចំពោះគ្រប់ $x\in\mathbb{R}$

៣. ដោះស្រាយសមីការ

$$4^{\cos 2x}+4^{\cos^2 x}=3$$

$$\Leftrightarrow 4^{2\cos^2 x-1}+4^{\cos^2 x}=3$$

$$\Leftrightarrow \frac{1}{4}\cdot\left(4^{\cos^2 x}\right)^2+4^{\cos^2 x}-3=0$$

តាង $a=4^{\cos^2 x}$ ដែល $1\le a\le 4$ ព្រោះ $0\le\cos^2 x\le 1$

យើងបានសមីការអាចសរសេរ $\frac{1}{4}a^2+a-3=0$

$$\Leftrightarrow a^2+4a-12=0$$

មានឫស $a=2$ ឬ $a=-6$ $(a=-6$ មិនយក$)$

បើ $a=2$ $\Rightarrow 4^{\cos^2 x}=2$ $\Rightarrow \cos^2 x=\frac{1}{2}$ $\Rightarrow \frac{1+\cos 2x}{2}=\frac{1}{2}$

$$\Rightarrow \cos 2x=0$$

$$\Rightarrow 2x=\frac{\pi}{2}+k\pi \quad \text{ដែល } k\in\mathbb{Z}$$

$$\Rightarrow x=\frac{\pi}{4}+\frac{k\pi}{2}$$

ដោយ $-\frac{3}{4}\le x\le\frac{5}{2}$

$$\Rightarrow -\frac{3}{4}\le\frac{\pi}{4}+\frac{k\pi}{2}\le\frac{5}{2} \quad \Rightarrow \frac{-3-\pi}{2\pi}\le k\le\frac{10-\pi}{2\pi}$$

$\Rightarrow -1<k<2$ ដោយ $k\in\mathbb{Z}$ $\Rightarrow k=0$ ឬ $k=1$

ករណី $k=0$ $\Rightarrow x=\frac{\pi}{4}$

ករណី $k=1 \quad \Rightarrow x=\frac{3\pi}{4}$

ដូច្នេះ $x=\frac{\pi}{4}$ ឬ $x=\frac{3\pi}{4}$

៤. រកប្រភេទនៃត្រីកោណ ABC

យើងមាន $\begin{cases} \sin B+\sin C=2\sin A & (*) \\ \cos B+\cos C=2\cos A & (**) \end{cases}$

លើកអង្គសងខាងនៃសមភាពទាំងពីរជាការេ

យើងបាន $\begin{cases} \sin^2 B+2\sin B\sin C+\sin^2 C=4\sin^2 A & (1) \\ \cos^2 B+2\cos B\cos C+\cos^2 C=4\cos^2 A & (2) \end{cases}$

យក (1) បូក (2)

នាំអោយ $2+2(\sin B\sin C+\cos B\cos C)=4$

$\Rightarrow \cos(B-C)=1=\cos 0$

$\Rightarrow B=C$ ជំនួសចូល $(**)$

យើងបាន $\cos B+\cos B=2\cos A \Rightarrow \cos A=\cos B$

$\Rightarrow A=B$

នាំអោយ $A=B=C$

ដូច្នេះ ត្រីកោណ ABC ជាត្រីកោណសម័ង្ស

៥. ក. គណនា A^3 ជាអនុគមន៍នៃ a

យើងមាន $A=\begin{pmatrix}\frac{1}{2} & 0\\ a & a\end{pmatrix}$

$$\Rightarrow A^2=A\cdot A=\begin{pmatrix}\frac{1}{2} & 0\\ a & a\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 0\\ a & a\end{pmatrix}=\begin{pmatrix}\frac{1}{4}+0 & 0+0\\ \frac{1}{2}a+a^2 & 0+a^2\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{2^2} & 0\\ \frac{1}{2}a+a^2 & a^2\end{pmatrix}$$

$$\Rightarrow A^3=A^2\cdot A=\begin{pmatrix}\frac{1}{2^2} & 0\\ \frac{1}{2}a+a^2 & a^2\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 0\\ a & a\end{pmatrix}=\begin{pmatrix}\frac{1}{2^3} & 0\\ \frac{1}{2^2}a+\frac{1}{2}a^2+a^3 & a^3\end{pmatrix}$$

ដូច្នេះ $A^3=\begin{pmatrix}\frac{1}{2^3} & 0\\ \frac{1}{2^2}a+\frac{1}{2}a^2+a^3 & a^3\end{pmatrix}$

ខ. គណនា A^n ជាអនុគមន៍នៃ a និង n

តាមរយៈសំនួរ ក.

យើងឧបមាថា $A^n=\begin{pmatrix}\dfrac{1}{2^n} & 0\\ \dfrac{1}{2^{n-1}}a+\dfrac{1}{2^{n-2}}a^2+\cdots+a^n & a^n\end{pmatrix}$

យើងនឹងបង្ហាញថា ពិតដល់ $n+1$

$$A^{n+1}=A^n\cdot A=\begin{pmatrix}\dfrac{1}{2^n} & 0\\ \dfrac{1}{2^{n-1}}a+\dfrac{1}{2^{n-2}}a^2+\cdots+a^n & a^n\end{pmatrix}\begin{pmatrix}\dfrac{1}{2} & 0\\ a & a\end{pmatrix}$$

$$=\begin{pmatrix}\dfrac{1}{2^{n+1}} & 0\\ \dfrac{1}{2^n}a+\dfrac{1}{2^{n-1}}a^2+\cdots+a^{n+1} & a^{n+1}\end{pmatrix}$$ ពិត

<u>ដូច្នេះ</u> $A^n=\begin{pmatrix}\dfrac{1}{2^n} & 0\\ \dfrac{1}{2^{n-1}}a+\dfrac{1}{2^{n-2}}a^2+\cdots+a^n & a^n\end{pmatrix}$

៦. កំនត់ $a,\ b$ និង c

យើងមាន $f(x)=x^3+ax^2+bx+c$

ដោយ $f(-2)=-10 \Rightarrow (-2)^3+a(-2)^2+b(-2)+c=-10$

$\Rightarrow 4a-2b+c=-2 \qquad (1)$

ម្យ៉ាងទៀត f មានតំលៃបរមាស្មើ $\dfrac{50}{27}$ ត្រង់ $x=\dfrac{2}{3}$

យើងបាន $f\left(\frac{2}{3}\right)=\frac{50}{27}$ និង $f'\left(\frac{2}{3}\right)=0$

ដោយ $f'(x)=3x^2+2ax+b$

យើងបាន

$$\begin{cases} \left(\frac{2}{3}\right)^3+a\left(\frac{2}{3}\right)^2+b\left(\frac{2}{3}\right)+c=\frac{50}{27} \\ 3\left(\frac{2}{3}\right)^2+2a\left(\frac{2}{3}\right)+b=0 \end{cases}$$

នាំអោយ $\begin{cases} 12a+18b+27c=42 & (2) \\ 4a+3b=-2 & (3) \end{cases}$

តាម (1), (2) និង (3) យើងបាន ប្រព័ន្ធសមីការ

$$\begin{cases} 4a-2b+c=-2 \\ 12a+18b+27c=42 \\ 4a+3b=-2 \end{cases}$$

ដោយដោះស្រាយប្រព័ន្ធសមីការនេះ យើងទទួលបានតំលៃ

$a=-\frac{3}{4}$; $b=\frac{1}{3}$ និង $c=\frac{5}{3}$

ដូច្នេះ $a=-\frac{3}{4}$; $b=\frac{1}{3}$ និង $c=\frac{5}{3}$

វិញ្ញាសាទី១០

១. បង្ហាញថា បើ $ab+cd$ ចែកដាច់នឹង $a-c$ នោះ $ad+bc$ ចែកដាច់នឹង $a-c$ ។

២. ក. គេអោយសមីការ $x^4+bx^3+cx^2+bx+1=0$ មានឫស។ បង្ហាញថា $b^2+(c-2)^2>3$

ខ. ដោះស្រាយសមីការ $x^3+3x-3=0$

៣. គណនាលីមីតខាងក្រោម៖

ក. $\lim\limits_{n\to+\infty} 2^n\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}$ n រ៉ាឌីកាល់

ខ. $\lim\limits_{x\to1}\left(\dfrac{m}{1-x^m}-\dfrac{n}{1-x^n}\right)$

៤. តាង $f(x)=2x^4+ax^2+bx-60$ ។ បើ f ចែកនឹង $(x-1)$ នោះសល់សំនល់ -94 និង $(x-3)$ ជាកត្តាមួយនៃ $f(x)$ ។ កំនត់តំលៃចំនួនថេរ a និង b ។

៥. បង្ហាញថា $1<\dfrac{1}{n+1}+\dfrac{1}{n+2}+\cdots+\dfrac{1}{3n+1}<2$

៦. កំនត់តំលៃ $k\in\mathbb{N}$ ដើម្បីអោយ $C_{14}^{k}+C_{14}^{k+2}=2C_{14}^{k+1}$ ។

សម្រាយបញ្ជាក់

១. ដើម្បីបង្ហាញនូវសំនើរខាងលើ យើងគ្រាន់តែបង្ហាញថា $(ab+cd)-(ad+bc)$ ចែកដាច់នឹង $(a-c)$

ដោយ $(ab+cd)-(ad+bc)=ab-bc+cd-ad$

$$=b(a-c)-d(a-c)$$

$$=(a-c)(b-d)$$

យើងបាន $(ab+cd)-(ad+bc)$ ចែកដាច់នឹង $(a-c)$

<u>ដូច្នេះ</u> បើ $(ab+cd)\vdots(a-c)\Rightarrow(ad+bc)\vdots(a-c)$

២. ក. បង្ហាញថា $b^2+(c-2)^2>3$

យើងមានសមីការ $x^4+bx^3+cx^2+bx+1=0$

$$\Leftrightarrow x^2+bx+c+\frac{b}{x}+\frac{1}{x^2}=0$$

$$\Leftrightarrow\left(x^2+\frac{1}{x^2}\right)+b\left(x+\frac{1}{x}\right)+c=0 \qquad (*)$$

តាង $t=x+\frac{1}{x}\quad\Rightarrow x^2+\frac{1}{x^2}=t^2-2$

$(*)\Leftrightarrow t^2-2+bt+c=0\quad\Leftrightarrow t^2+bt+c-2=0$

$\Leftrightarrow t^2=(2-c)-bt$

តាមវិសមភាព កូស៊ី ស្វាត

យើងបាន $t^4=\left[(2-c)\cdot 1-bt\right]^2\le\left[(2-c)^2+b^2\right](1+t^2)$

$$\Leftrightarrow \frac{t^4}{1+t^2} \leq (2-c)^2 + b^2 \qquad (1)$$

ម្យ៉ាងទៀត $\left(\frac{1}{x} - x\right)^2 \geq 0 \Rightarrow x^2 + \frac{1}{x^2} \geq 2$

$\Rightarrow t^2 \geq 4 \qquad \Rightarrow t^4 \geq 16$

នាំអោយ $t^4 - 3(t^2+1) = t^4 - 3t^2 - 3 \geq 16 - 12 - 3 > 0$

$$\Rightarrow \frac{t^4}{1+t^2} > 3 \qquad (2)$$

តាម(1) និង (2) $\quad \Rightarrow b^2 + (2-c)^2 > 3$

ដូច្នេះ $\quad b^2 + (2-c)^2 > 3$

ខ. ដោះស្រាយសមីការ $x^3 + 3x - 3 = 0$

តាង $x = y - \frac{1}{y}$

សមីការអាចសរសេរ$\left(y - \frac{1}{y}\right)^3 + 3\left(y - \frac{1}{y}\right) - 3 = 0$

$\Leftrightarrow y^3 - \frac{1}{y^3} - 3 = 0 \qquad$ តាង $t = y^3$

យើងបាន $t - \frac{1}{t} - 3 = 0 \quad \Rightarrow t^2 - 3t - 1 = 0$

មានឫស $\quad t = \frac{1}{2}\left(3 \pm \sqrt{13}\right)$

នាំអោយ $y=\sqrt[3]{\frac{3\pm\sqrt{13}}{2}} \Rightarrow x=\sqrt[3]{\frac{3\pm\sqrt{13}}{2}}-\sqrt[3]{\frac{2}{3\pm\sqrt{13}}}$

ដូច្នេះ $x=\sqrt[3]{\frac{3\pm\sqrt{13}}{2}}-\sqrt[3]{\frac{2}{3\pm\sqrt{13}}}$

៣. គណនាលីមីត៖

ក. $\lim\limits_{n\to+\infty} 2^n\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}$

យើងមាន $\sqrt{2}=2\frac{\sqrt{2}}{2}=2\cos\frac{\pi}{4}=2\cos\frac{\pi}{2^2}$

$$\sqrt{2+\sqrt{2}}=\sqrt{2+2\cos\frac{\pi}{2^2}}=\sqrt{2\left(1+\cos\frac{\pi}{2^2}\right)}=\sqrt{4\cos^2\frac{\pi}{2^3}}$$

$$=2\cos\frac{\pi}{2^3}$$

តាមវិចារដោយកំនើន យើងបាន

$$\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{(n-1)radikal}=2\cos\frac{\pi}{2^n}$$

នាំអោយ $\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}=\sqrt{2-2\cos\frac{\pi}{2^n}}$

$$=\sqrt{2\left(1-\cos\frac{\pi}{2^n}\right)}=\sqrt{2\cdot 2\sin^2\frac{\pi}{2^{n+1}}}=2\sin\frac{\pi}{2^{n+1}}$$

$$\Rightarrow \lim_{n\to+\infty} 2^n\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}} = \lim_{n\to+\infty} 2^n\left(2\sin\frac{\pi}{2^{n+1}}\right)$$

$$= \lim_{n\to+\infty} 2^{n+1}\sin\frac{\pi}{2^{n+1}} = \lim_{n\to+\infty}\frac{\sin\frac{\pi}{2^{n+1}}}{\frac{1}{2^{n+1}}} = \pi\lim_{n\to+\infty}\frac{\sin\frac{\pi}{2^{n+1}}}{\frac{\pi}{2^{n+1}}}$$

តាង $u=\frac{\pi}{2^{n+1}}$ បើ $n\to+\infty$ នាំអោយ $u\to 0$

$$\Rightarrow \lim_{n\to+\infty} 2^n\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}} = \pi\lim_{u\to0}\frac{\sin u}{u}=\pi$$

ដូច្នេះ $\lim_{n\to+\infty} 2^n\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}} = \pi$

ខ. $\lim_{x\to1}\left(\frac{m}{1-x^m}-\frac{n}{1-x^n}\right)$

$$=\lim_{x\to1}\left[\left(\frac{m}{1-x^m}-\frac{1}{1-x}\right)-\left(\frac{n}{1-x^n}-\frac{1}{1-x}\right)\right]$$

$$=\lim_{x\to1}\left(\frac{m}{1-x^m}-\frac{1}{1-x}\right)-\lim_{x\to1}\left(\frac{n}{1-x^n}-\frac{1}{1-x}\right)$$

ដោយ $\lim_{x\to1}\left(\frac{m}{1-x^m}-\frac{1}{1-x}\right)$

$$=\lim_{x\to1}\left(\frac{m-\left(1+x+x^2+\cdots+x^{m-1}\right)}{(1-x)\left(1+x+x^2+\cdots+x^{m-1}\right)}\right)$$

$$=\lim_{x\to1}\left(\frac{(1-1)+(1-x)+\left(1-x^2\right)+\cdots+\left(1-x^{m-1}\right)}{(1-x)\left(1+x+x^2+\cdots+x^{m-1}\right)}\right)$$

$$=\lim_{x\to1}\left(\frac{(1-x)\left(1+(1+x)+\cdots+\left(1+x+\cdots+x^{m-2}\right)\right)}{(1-x)\left(1+x+x^2+\cdots+x^{m-1}\right)}\right)$$

$$=\lim_{x\to1}\frac{1+(1+x)+\cdots+\left(1+x+\cdots+x^{m-2}\right)}{1+x+x^2+\cdots+x^{m-1}}$$

$$=\frac{1+2+3+\cdots+(m-1)}{m}=\frac{\frac{m(m-1)}{2}}{m}=\frac{m-1}{2}$$

ដូចគ្នាដែរយើងបាន $\lim_{x\to1}\left(\frac{n}{1-x^n}-\frac{1}{1-x}\right)=\frac{n-1}{2}$

យើងបាន $\lim_{x\to1}\left(\frac{m}{1-x^m}-\frac{n}{1-x^n}\right)=\frac{m-1}{2}-\frac{n-1}{2}=\frac{m-n}{2}$

<u>ដូច្នេះ</u> $\lim_{x\to1}\left(\frac{m}{1-x^m}-\frac{n}{1-x^n}\right)=\frac{m-n}{2}$

៤. កំនត់តំលៃ a និង b

យើងមាន ពហុធា $f(x)=2x^4+ax^2+bx-60$

ដោយ សំនល់នៃ $f(x)$ ពេលចែកនឹង $(x-1)$ ស្មើនឹង -94

$\Rightarrow f(1)=-94 \qquad \Rightarrow 2+a+b-60=-94$

$\Rightarrow a+b=-36 \qquad (1)$

ម្យ៉ាងទៀត $(x-3)$ ជាកត្តាមួយនៃ $f(x)$ មានន័យថា $f(3)=0$

$\Rightarrow 2\cdot 3^4 + a\cdot 3^2 + 3b - 60 = 0$

$\Rightarrow 9a + 3b = 6 \qquad (2)$

តាម (1) និង (2) យើងបានប្រព័ន្ធសមីការ

$$\begin{cases} a + b = -36 \\ 9a + 3b = 6 \end{cases}$$

នាំអោយ $a = 19$ និង $b = -55$

ដូច្នេះ $a = 19$ និង $b = -55$

៥. បង្ហាញថា $1 < \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} < 2$

* បង្ហាញថា $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} < 2$

តាង $f(n) = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1}$

$\Rightarrow f(n) < \frac{1}{n+1} + \frac{1}{n+1} + \cdots + \frac{1}{n+1} = \frac{2n+1}{n+1} < \frac{2n+2}{n+1} = 2$

នាំអោយ $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} < 2 \qquad (i)$

** បង្ហាញថា $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} > 1$

យើងមាន $f(n) = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1}$

ករណី $n = 1 \Rightarrow f(1) = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12} > 1$ ពិត

ឧបមាថា ពិតដល់ $n = k$ គឺ $f(k) > 1$

យើងនឹងបង្ហាញថា ពិតដល់ $n = k+1$ គឺ $f(k+1) > 1$

$$f\left(k+1\right)=\frac{1}{k+2}+\cdots+\frac{1}{3k+1}+\frac{1}{3k+2}+\frac{1}{3k+3}+\frac{1}{3k+4}$$

$$\Rightarrow f\left(k+1\right)=f\left(k\right)-\frac{1}{k+1}+\frac{1}{3k+2}+\frac{1}{3k+3}+\frac{1}{3k+4}$$

$$\Rightarrow f\left(k+1\right)-f\left(k\right)=-\frac{1}{k+1}+\frac{1}{3k+2}+\frac{1}{3k+3}+\frac{1}{3k+4}$$

ដោយ $\frac{1}{3k+2}+\frac{1}{3k+4}=\frac{6k+6}{\left(3k+2\right)\left(3k+4\right)}$

ម្យ៉ាងទៀត

$$\left(3k+2\right)\left(3k+4\right)\leq\left(\frac{3k+2+3k+4}{2}\right)^2=\left(3k+3\right)^2$$

$$\Rightarrow\frac{1}{3k+2}+\frac{1}{3k+4}\geq\frac{6k+6}{\left(3k+3\right)^2}=\frac{2}{3k+3}$$

យើងបាន $f\left(k+1\right)-f\left(k\right)\geq-\frac{1}{k+1}+\frac{1}{3k+3}+\frac{2}{3k+3}=0$

នាំអោយ $f\left(k+1\right)\geq f\left(k\right)>1$ ពិត

យើងបាន $\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n+1}>1 \qquad (ii)$

តាម (i) និង (ii) នាំអោយ វិសមភាពត្រូវបានស្រាយបញ្ជាក់

<u>ដូច្នេះ</u> $1<\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n+1}<2$

៦. កំនត់តំលៃ $k\in\mathbb{N}$

យើងមាន $C_{14}^{k}+C_{14}^{k+2}=2C_{14}^{k+1} \qquad (*)$

លក្ខខណ្ឌ $k \in \mathbb{N}$ និង $0 \le k \le 12$

$$(*) \Leftrightarrow \frac{14!}{k!(14-k)!} + \frac{14!}{(k+2)!(12-k)!} = 2 \cdot \frac{14!}{(k+1)!(13-k)!}$$

$$\Leftrightarrow \frac{1}{k!(14-k)!} + \frac{1}{(k+2)!(12-k)!} = \frac{2}{(k+1)!(13-k)!}$$

$$\Leftrightarrow \frac{1}{(13-k)(14-k)} + \frac{1}{(k+1)(k+2)} = \frac{2}{(k+1)(13-k)}$$

$$\Leftrightarrow (k+1)(k+2) + (13-k)(14-k) - 2(k+2)(14-k) = 0$$

$$\Leftrightarrow k^2 + 3k + 2 + 182 - 27k + k^2 + 2k^2 - 24k - 56 = 0$$

$$\Leftrightarrow 4k^2 - 48k + 128 = 0$$

$\Leftrightarrow k^2 - 12k + 32 = 0$ មានឫស $k = 4$ ឬ $k = 8$

ដូច្នេះ $k = 4$ ឬ $k = 8$

វិញ្ញាសាទី១១

១. គេអោយ a,b,c ជាចំនួនពិតវិជ្ជមាន ដែល $0\le a,b,c\le 1$ ។

បង្ហាញថា $\dfrac{a}{1+bc}+\dfrac{b}{1+ac}+\dfrac{c}{1+ab}\le 2$

២. ដោះស្រាយសមីការក្នុង $\mathbb{N}^3$

$10x+15y+6z=133$

៣. ដោយដឹងថា ឬសនៃសមីការ $x^3-26x^2+156x+p=0$ ជាស្វីតធរណីមាត្រកើន ។ កំនត់តំលៃ p

៤. គេអោយត្រីកោណ ABC ផ្ទៀងផ្ទាត់ $\tan\dfrac{A}{2}\tan\dfrac{B}{2}=\dfrac{1}{2}$ ។

បង្ហាញថា ត្រីកោណ ABC ជាត្រីកោណកែងត្រង់ A សមមូលនឹង $\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}=\dfrac{1}{10}$ ។

៥. រកតួចែករួមធំបំផុតនៃ $\left(2^{2007}-1\right)$ និង $\left(2^{2^{2007}}+1\right)$ ។

៦.បង្ហាញថា $\lim\limits_{n\to+\infty}\left(\dfrac{1}{\sqrt{n^3}}\sum_{k=1}^{n}\sqrt{k}\right)=\dfrac{2}{3}$

សម្រាយបញ្ជាក់

១. បង្ហាញថា $\frac{a}{1+bc}+\frac{b}{1+ac}+\frac{c}{1+ab}\le 2$

ឧបមាថា $0\le a\le b\le c\le 1$

យើងបាន $(1-a)(1-b)\ge 0$

$$\Rightarrow 1+ab-a-b\ge 0$$

$$\Rightarrow a+b\le 1+ab$$

$$\Rightarrow a+b\le 1+2ab$$

ម្យ៉ាងទៀត $1+bc\ge 1+ab$ និង $1+ac\ge 1+ab$

នាំអោយ $\frac{1}{1+bc}\le\frac{1}{1+ab}$ និង $\frac{1}{1+ac}\le\frac{1}{1+ab}$

$\Rightarrow \frac{a}{1+bc}\le\frac{a}{1+ab}$ និង $\frac{b}{1+ac}\le\frac{b}{1+ab}$

យើងបាន $\frac{a}{1+bc}+\frac{b}{1+ac}+\frac{c}{1+ab}\le\frac{a}{1+ab}+\frac{b}{1+ab}+\frac{c}{1+ab}$

$$\Rightarrow \frac{a}{1+bc}+\frac{b}{1+ac}+\frac{c}{1+ab}\le\frac{a+b+c}{1+ab}$$

ដោយ $a+b+c\le a+b+1\le 2+2ab=2(1+ab)$

នាំអោយ $\frac{a}{1+bc}+\frac{b}{1+ac}+\frac{c}{1+ab}\le\frac{2(1+ab)}{1+ab}=2$

ដូច្នេះ $\frac{a}{1+bc}+\frac{b}{1+ac}+\frac{c}{1+ab}\le 2$

២. ដោះស្រាយសមីការក្នុង $\mathbb{N}^3$

យើងមាន $10x+15y+6z=133$ (1)

យក (x,y,z) ជាឫសនៃសមីការ (1) $\Rightarrow y$ ជាចំនួនសេស

តាង $y=2Y+1$ ដែល $Y\in\mathbb{N}$

នោះសមីការ អាចសរសេរ

$$10x+30Y+15+6z=133$$

$$\Rightarrow 5x+15Y+3z=59 \qquad (2)$$

យក (x,Y,z) ជាឫសនៃសមីការ (2)

$\Rightarrow 2x-2\vdots 3$ និង $x-1\vdots 3$

តាង $x=3X+1$ ដែល $X\in\mathbb{N}$

នោះសមីការ (2) អាចសរសេរ

$$15X+5+15Y+3z=59$$

$$\Rightarrow 5X+5Y+z=18 \qquad (3)$$

យក (X,Y,z) ជាឫសនៃសមីការ (3)

$\Rightarrow z-3\vdots 5$

តាង $z=5Z+3$ ដែល $Z\in\mathbb{N}$

នោះសមីការ (3) អាចសរសេរ

$$5X+5Y+5Z+3=18$$

$$\Rightarrow X+Y+Z=3 \qquad (4)$$

ដោយ $X,Y,Z\in\mathbb{N}$ យើងបានឫសនៃសមីការ(4) គឺ៖

$$(X,Y,Z)=\{(1,1,1);(1,2,0);(1,0,2);(2,1,0);(2,0,1);(0,2,1);$$
$$(0,1,2);(0,0,3);(0,3,0);(3,0,0)\}$$

ដោយគណនាតំលៃនៃ(x,y,z) តាមករណីនីមួយៗខាងលើ

យើងបាន $(x,y,z)=\{(4,3,8);(4,5,3);(4,1,13);(7,3,3);$
$(7,1,8);(1,5,8);(1,3,13);(1,1,18);(1,7,3);(10,1,3)\}$

ដូច្នេះ $(x,y,z)=\{(4,3,8);(4,5,3);(4,1,13);(7,3,3);$
$(7,1,8);(1,5,8);(1,3,13);(1,1,18);(1,7,3);(10,1,3)\}$

៣. កំនត់តំលៃ p

យើងមាន សមីការ $x^3-26x^2+156x+p=0$

តាង a,b,c ជាឬសនៃសមីការនេះ

តាម ទ្រឹស្តីបទវៀែតគេបាន

$$\begin{cases} a+b+c=26 & (1) \\ ab+bc+ca=-156 & (2) \\ abc=-p & (3) \end{cases}$$

ម្យ៉ាងទៀត a,b,c ជាស្វីតធរណីមាត្រកើន

យើងបាន $ac=b^2$ ជំនួសចូលក្នុង(3)

យើងបាន $p=-b^3$

យក $ac=b^2$ ជំនួសចូលក្នុង(2)

យើងបាន $ab+bc+b^2=-156$

$$\Rightarrow b(a+c+b)=-156$$

$$\Rightarrow b\times 26=-156 \Rightarrow b=-6$$

នាំអោយ $p=-b^3=-(-6)^3=216$

ដូច្នេះ $p=216$

៩. បង្ហាញថា ត្រីកោណ ABC កែងត្រង់ A សមមូលនឹង

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{1}{10}$$

* បង្ហាញថា បើ $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{1}{10} \Rightarrow$ ត្រីកោណ ABC កែងត្រង់ A

តាមរូបមន្ត ក្រលាផ្ទៃនៃត្រីកោណ

$$S=p(p-a)\tan\frac{A}{2}=p(p-b)\tan\frac{B}{2}$$

$$S=\sqrt{p(p-a)(p-b)(p-c)}$$

$$\Rightarrow p^2(p-a)(p-b)\tan\frac{A}{2}\tan\frac{B}{2}=p(p-a)(p-b)(p-c)$$

$$\Rightarrow \tan\frac{A}{2}\tan\frac{B}{2}=\frac{p-c}{p}=\frac{a+b-c}{a+b+c}$$

ដោយ $\tan\frac{A}{2}\tan\frac{B}{2}=\frac{1}{2}$

$$\Rightarrow \frac{a+b-c}{a+b+c}=\frac{1}{2} \qquad \Rightarrow a+b=3c$$

នាំអោយ $p=\frac{a+b+c}{2}=\frac{4c}{2}=2c \qquad (1)$

ម្យ៉ាងទៀត $S=pr=\frac{abc}{4R}$

$$\Rightarrow pabc\frac{r}{4R}=p(p-a)(p-b)(p-c)$$

$$\Rightarrow abc\frac{r}{4R}=(p-a)(p-b)(2c-c)=c(p-a)(p-b)$$

$$\Rightarrow ab\frac{r}{4R}=(p-a)(p-b)=p^2-(a+b)p+ab$$

$$\Rightarrow ab\left(1-\frac{r}{4R}\right)=(a+b)p-p^2=3c\cdot 2c-(2c)^2=2c^2 \quad (2)$$

តាមសម្មតិកម្ម $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{1}{10}$

ដោយ $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{r}{4R}$

យើងបាន $\frac{r}{4R}=\frac{1}{10}$ ជំនួសចូលក្នុង (2)

នាំអោយ $ab\left(1-\frac{1}{10}\right)=2c^2 \quad \Rightarrow ab=\frac{20}{9}c^2 \qquad (3)$

តាម (1) និង (3) យើងបាន a និង b ជាឬសពីរនៃសមីការ

$t^2-3ct+\frac{20}{9}c^2=0$ មានឬស $t=\frac{4}{3}c$ ឬ $t=\frac{5}{3}c$

យក $a>b \quad \Rightarrow a=\frac{5}{3}c$ និង $b=\frac{4}{3}c$

$$\Rightarrow b^2+c^2=\left(\frac{4}{3}c\right)^2+c^2=\left(\frac{5}{3}c\right)^2=a^2$$

ផ្ទៀងផ្ទាត់ទ្រឹស្តីបទ ពីតាករ

យើងបាន ត្រីកោណ ABC ជាត្រីកោណកែងត្រង់ A

* បង្ហាញថា បើត្រីកោណ ABC ជាត្រីកោណកែងត្រង់ A

$$\Rightarrow \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{1}{10}$$

បើត្រីកោណ ABC ជាត្រីកោណកែងត្រង់ A

យើងបាន $\begin{cases} a = 2R \\ a^2 = b^2 + c^2 \\ S = \dfrac{1}{2}bc = pr \end{cases}$

តាម $(1) \Leftrightarrow a + b = 3c \quad \Rightarrow a = 3c - b$

នាំអោយ $\begin{cases} a = 2R \\ (3c - b)^2 = b^2 + c^2 \\ \dfrac{1}{2}b \cdot \dfrac{1}{2}p = pr \end{cases}$

នាំអោយ $\begin{cases} a = 2R \\ c = \dfrac{6b}{8} = 3r \\ b = 4r \end{cases}$

$\Rightarrow a = \sqrt{(3r)^2 + (4r)^2} = 5r$

$\Rightarrow \dfrac{5r}{2R} = \dfrac{a}{a} = 1 \qquad \Rightarrow \dfrac{r}{4R} = \dfrac{1}{10}$

យើងបាន $\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2} = \dfrac{1}{10}$

<u>ដូច្នេះ</u> ត្រីកោណ ABC កែងត្រង់ $A \Leftrightarrow \sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2} = \dfrac{1}{10}$

៥. រកតួចែករួមធំបំផុតនៃ $\left(2^{2007} - 1\right)$ និង $\left(2^{2^{2007}} + 1\right)$

តាង d ជាតួចែករួមធំបំផុតនៃ $\left(2^{2007} - 1\right)$ និង $\left(2^{2^{2007}} + 1\right)$

ដែល d ជាចំនួនគត់វិជ្ជមានសេស

យើងបាន $2^{2007}-1=md$ ដែល $m\in\mathbb{N}^*$

$2^{2^{2007}}+1=nd$ ដែល $n\in\mathbb{N}^*$

$(m,n)=1$

$\Rightarrow 2^{2007}=md+1$ និង $2^{2^{2007}}=nd-1$

$$\Rightarrow\left(2^{2007}\right)^{2^{2007}}=(md+1)^{2^{2007}}=kd+1 \text{ ដែល} k\in\mathbb{N}^* \qquad (1)$$

$$\text{និង } \left(2^{2^{2007}}\right)^{2007}=(nd-1)^{2007}=pd-1 \text{ ដែល } p\in\mathbb{N}^* \qquad (2)$$

តាម (1) និង $(2)\Rightarrow kd+1=pd-1$

$\Rightarrow d(p-k)=2$ ដែល $(p-k)\in\mathbb{N}^*$

នាំអោយ d ចែកដាច់ 2

ដោយ d ជាចំនួនគត់វិជ្ជមានសេស$\Rightarrow d=1$

ដូច្នេះ $d=\gcd\left(\left(2^{2007}-1\right),\left(2^{2^{2007}}+1\right)\right)=1$

៦. បង្ហាញថា $\lim_{n\to+\infty}\left(\frac{1}{\sqrt{n^3}}\sum_{k=1}^{n}\sqrt{k}\right)=\frac{2}{3}$

យើងមាន $\int_0^n\sqrt{x}dx=\int_0^1\sqrt{x}dx+\int_1^2\sqrt{x}dx+\cdots+\int_{n-1}^n\sqrt{x}dx$

$$\Rightarrow\int_0^n\sqrt{x}dx<\int_0^1\sqrt{1}dx+\int_1^2\sqrt{2}dx+\cdots+\int_{n-1}^n\sqrt{n}dx=\sum_{k=1}^{n}\sqrt{k}$$

ម្យ៉ាងទៀត $\int_1^{n+1}\sqrt{x}dx=\int_1^2\sqrt{x}dx+\int_2^3\sqrt{x}dx+\cdots+\int_n^{n+1}\sqrt{x}dx$

$$\Rightarrow \int_1^{n+1}\sqrt{x}dx>\int_1^2\sqrt{1}dx+\int_2^3\sqrt{2}dx+\cdots+\int_n^{n+1}\sqrt{n}dx=\sum_{k=1}^{n}\sqrt{k}$$

យើងបាន $\int_0^n\sqrt{x}dx<\sum_{k=1}^{n}\sqrt{k}<\int_1^{n+1}\sqrt{x}dx \qquad (*)$

ដោយ $\int_0^n\sqrt{x}dx=\frac{2}{3}\sqrt{x^3}\Big|_0^n=\frac{2}{3}\sqrt{n^3}$

$$\int_1^{n+1}\sqrt{x}dx=\frac{2}{3}\sqrt{x^3}\Big|_1^{n+1}=\frac{2}{3}\sqrt{(n+1)^3}-\frac{2}{3}$$

តាម $(*)\Rightarrow\frac{2}{3}\sqrt{n^3}<\sum_{k=1}^{n}\sqrt{k}<\frac{2}{3}\sqrt{(n+1)^3}-\frac{2}{3}$

$$\Rightarrow\frac{2}{3}<\frac{1}{\sqrt{n^3}}\sum_{k=1}^{n}\sqrt{k}<\frac{2}{3}\sqrt{\left(\frac{n+1}{n}\right)^3}-\frac{2}{3\sqrt{n^3}}$$

$$\Rightarrow\lim_{n\to+\infty}\frac{2}{3}<\lim_{n\to+\infty}\frac{1}{\sqrt{n^3}}\sum_{k=1}^{n}\sqrt{k}<\lim_{n\to+\infty}\left(\frac{2}{3}\sqrt{\left(\frac{n+1}{n}\right)^3}-\frac{2}{3\sqrt{n^3}}\right)$$

$$\Rightarrow\frac{2}{3}<\lim_{n\to+\infty}\frac{1}{\sqrt{n^3}}\sum_{k=1}^{n}\sqrt{k}<\frac{2}{3}\Rightarrow\lim_{n\to+\infty}\frac{1}{\sqrt{n^3}}\sum_{k=1}^{n}\sqrt{k}=\frac{2}{3}$$

<u>ដូច្នេះ</u> $\lim_{n\to+\infty}\frac{1}{\sqrt{n^3}}\sum_{k=1}^{n}\sqrt{k}=\frac{2}{3}$

វិញ្ញាសាទី១២

១. គេអោយ x,y ជាចំនួនពិត ។ បង្ហាញថា

$$-\frac{1}{2}\le\frac{(x+y)(1-xy)}{(x^2+1)(y^2+1)}\le\frac{1}{2}$$

២. គេអោយ $I_n=\int_0^1 x^2(1-x^2)^n dx$ ។ គណនា $\lim_{n\to+\infty}\frac{I_{n+1}}{I_n}$

៣. ដោយដឹងថា $\sin x+\sin y+\sin z=0$ ។ ចូរគណនាតំលៃអតិបរមានិង អប្បបរមានៃ $P=\sin^2 x+\sin^4 y+\sin^6 z$ ។

៤. កំនត់ អនុគមន៍ f និង g ដែលផ្ទៀងផ្ទាត់លក្ខខណ្ឌ៖

$$\begin{cases} f(3x-1)+g(6x-1)=3x \\ f(x+1)+xg(2x+3)=2x^2+x \end{cases}$$ ចំពោះគ្រប់ $x\neq-1$

៥. បង្ហាញថា $\frac{n^7}{7}+\frac{n^5}{5}+\frac{23n}{35}$ ជាចំនួនគត់ ចំពោះគ្រប់ $n\in\mathbb{Z}$ ។

៦. គណនា $A=\int_0^{\frac{\pi}{2}}\left(\cos^2(\cos x)+\sin^2(\sin x)\right)dx$ ។

សម្រាយបញ្ជាក់

១. បង្ហាញថា $-\frac{1}{2}\le\frac{(x+y)(1-xy)}{(x^2+1)(y^2+1)}\le\frac{1}{2}$

តាង $\vec{a}=\left(\frac{2x}{1+x^2},\frac{1-x^2}{1+x^2}\right)$ និង $\vec{b}=\left(\frac{1-y^2}{1+y^2},\frac{2y}{1+y^2}\right)$

$$\Rightarrow\left\|\vec{a}\right\|=\left|\vec{a}\right|=\sqrt{\frac{4x^2}{(1+x^2)^2}+\frac{(1-x^2)^2}{(1+x^2)^2}}=\sqrt{\frac{(1+x^2)^2}{(1+x^2)^2}}=1$$

និង $\left\|\vec{b}\right\|=\left|\vec{b}\right|=\sqrt{\frac{(1-y^2)^2}{(1+y^2)^2}+\frac{4y^2}{(1+y^2)^2}}+=\sqrt{\frac{(1+y^2)^2}{(1+y^2)^2}}=1$

នាំអោយ $\left\|\vec{a}\right\|\cdot\left\|\vec{b}\right\|=\left|\vec{a}\right|\cdot\left|\vec{b}\right|=1$

ម្យ៉ាងទៀត

$$\left|\vec{a}\cdot\vec{b}\right|=\left|2\cdot\frac{x(1-y^2)+y(1-x^2)}{(1+x^2)(1+y^2)}\right|=2\left|\frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)}\right|$$

ប៉ុន្តែ $\left|\vec{a}\cdot\vec{b}\right|\le\left|\vec{a}\right|\cdot\left|\vec{b}\right|$ $\quad\Rightarrow 2\left|\frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)}\right|\le 1$

$$\Rightarrow\left|\frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)}\right|\le\frac{1}{2}\quad\Rightarrow-\frac{1}{2}\le\frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)}\le\frac{1}{2}$$

ដូច្នេះ $-\dfrac{1}{2}\leq\dfrac{(x+y)(1-xy)}{(1+x^2)(1+y^2)}\leq\dfrac{1}{2}$

២. គណនា $\lim\limits_{n\to+\infty}\dfrac{I_{n+1}}{I_n}$

យើងមាន $I_n=\int\limits_0^1 x^2\left(1-x^2\right)^n dx$

$$\Rightarrow I_{n+1}=\int\limits_0^1 x^2\left(1-x^2\right)^{n+1}dx=\int\limits_0^1 x^2\left(1-x^2\right)^n\left(1-x^2\right)dx$$

$$=\int\limits_0^1 x^2\left(1-x^2\right)^n dx-\int\limits_0^1 x^4\left(1-x^2\right)^n dx$$

$$=I_n-\int\limits_0^1 x^4\left(1-x^2\right)^n dx \qquad (1)$$

ចំពោះ $I_{n+1}=\int\limits_0^1 x^2\left(1-x^2\right)^{n+1}dx$

តាង $u=\left(1-x^2\right)^{n+1} \quad \Rightarrow du=-2(n+1)x\left(1-x^2\right)^n dx$

$dv=x^2dx \quad \Rightarrow v=\dfrac{1}{3}x^3$

$$\Rightarrow I_{n+1}=\frac{1}{3}x^3\left(1-x^2\right)^{n+1}\Big|_0^1-\int\limits_0^1\left[\frac{1}{3}x^3\left(-2(n+1)x\left(1-x^2\right)^n\right)\right]dx$$

$$\Rightarrow I_{n+1}=\frac{2(n+1)}{3}\int_0^1 x^4\left(1-x^2\right)^n dx$$

$$\Rightarrow \int_0^1 x^4\left(1-x^2\right)^n dx=\frac{3}{2n+2}I_{n+1} \qquad (2)$$

តាម (1) និង (2) យើងបាន $I_{n+1}=I_n-\frac{3}{2n+2}I_{n+1}$

$$\Rightarrow I_{n+1}+\frac{3}{2n+2}I_{n+1}=I_n \quad \Rightarrow \frac{I_{n+1}}{I_n}=\frac{2n+2}{2n+5}$$

យើងបាន $\lim\limits_{n\to+\infty}\frac{I_{n+1}}{I_n}=\lim\limits_{n\to+\infty}\frac{2n+2}{2n+5}=1$

ដូច្នេះ $\lim\limits_{n\to+\infty}\frac{I_{n+1}}{I_n}=1$

៣. កំនត់ $\min(P)$ និង $\max(P)$

យើងមាន $P=\sin^2 x+\sin^4 y+\sin^6 z$

ដោយ $-1\le \sin x,\sin y,\sin z\le 1$

$\Rightarrow 0\le \sin^2 x,\sin^4 y,\sin^6 z\le 1$

យើងបាន $P\ge 0$

នាំអោយ $\min(P)=0$ ពេល $\sin x=\sin y=\sin z=0$

$\Rightarrow x=y=z=k\pi$ ដែល $k\in\mathbb{Z}$

ម្យ៉ាងទៀត $\sin^2 x\le|\sin x|;\sin^4 y\le|\sin y|;\sin^6 z\le|\sin z|$

$\Rightarrow P\le|\sin x|+|\sin y|+|\sin z|$

ឧបមាថា $\sin x \le \sin y \le \sin z$

ដោយ $\sin x + \sin y + \sin z = 0$ $\Rightarrow \sin x \le 0$ និង $\sin z \ge 0$

ករណី $\sin y \ge 0$ $\Rightarrow P \le -\sin x + \sin y + \sin z$

$$\Rightarrow P \le -2\sin x + (\sin x + \sin y + \sin z)$$

$$\Rightarrow P \le -2\sin x \le 2$$

ករណី $\sin y \le 0$ $\Rightarrow P \le -\sin x - \sin y + \sin z$

$$\Rightarrow P \le -(\sin x + \sin y + \sin z) + 2\sin z$$

$$\Rightarrow P \le 2\sin z \le 2$$

យើងបាន $\max(P) = 2$ ពេល $\sin x = -1; \sin y = 0; \sin z = 1$

ដូច្នេះ $\min(P) = 0$ និង $\max(P) = 2$

៤. កំនត់អនុគមន៍ $f(x)$ និង $g(x)$

យើងមាន $\begin{cases} f(3x-1) + g(6x-1) = 3x & (1) \\ f(x+1) + xg(2x+3) = 2x^2 + x & (2) \end{cases}$

តាង $u = 3x - 1$ $\Rightarrow x = \dfrac{u+1}{3}$ ជំនួសចូលក្នុង (1)

យើងបាន $f(u) + g\left(6\dfrac{u+1}{3} - 1\right) = 3\dfrac{u+1}{3}$

$$\Leftrightarrow f(u) + g(2u+1) = u + 1 \qquad (*)$$

តាង $u = x + 1$ $\Rightarrow x = u - 1$ ជំនួសចូលក្នុង (2)

យើងបាន $f(u) + (u-1)g(2u-2+3) = 2(u-1)^2 + (u-1)$

$$\Leftrightarrow f(u) + (u-1)g(2u+1) = 2u^2 - 3u + 1 \qquad (**)$$

យក $(**)$ ដក $(*)$

នាំអោយ $(u-1)g(2u+1)-g(2u+1)=2u^2-3u+1-u-1$

$\Leftrightarrow g(2u+1)=\dfrac{2u^2-4u}{u-2}=2u$ តាង $y=2u+1\Rightarrow u=\dfrac{y-1}{2}$

$\Rightarrow g(y)=2\left(\dfrac{y-1}{2}\right)=y-1$

យើងបាន $g(x)=x-1$

យក $(*)$ គុណនឹង $(u-1)$ រួចដក $(**)$

នាំអោយ $(u-1)f(u)-f(u)=u^2-1-(2u^2-3u+1)$

$\Leftrightarrow (u-2)f(u)=-u^2+3u-2=-(u-1)(u-2)$

$\Leftrightarrow f(u)=-(u-1)$

យើងបាន $f(x)=-(x-1)$

<u>ដូច្នេះ</u> $f(x)=-(x-1)$ និង $g(x)=x-1$

៥. បង្ហាញថា $\dfrac{n^7}{7}+\dfrac{n^5}{5}+\dfrac{23n}{35}$ ជាចំនួនគត់

យើងមាន $\dfrac{n^7}{7}+\dfrac{n^5}{5}+\dfrac{23n}{35}=\dfrac{5n^7+7n^5+23n}{35}$

ដើម្បីបង្ហាញថា $\dfrac{n^7}{7}+\dfrac{n^5}{5}+\dfrac{23n}{35}$ ជាចំនួនគត់ យើងគ្រាន់តែបង្ហាញថា $5n^7+7n^5+23n$ ចែកដាច់នឹង 35 គឺ បង្ហាញថា វាចែកដាច់នឹង 5 រួចចែកដាច់នឹង 7 ព្រោះ 5 និង 7 ជាចំនួនបឋមរវាងគ្នា ។

$*$ បង្ហាញថា $5n^7+7n^5+23n$ ចែកដាច់នឹង 5

តាមទ្រឹស្តីបទ *Fermat*

យើងបាន $n^5 \equiv n \quad (\bmod 5)$

$\Rightarrow 7n^5 \equiv 7n \quad (\bmod 5)$

$\Rightarrow 7n^5+23n \equiv 7n+23n \equiv 0 \quad (\bmod 5)$

$\Rightarrow 5n^7+7n^5+23n \equiv 0 \quad (\bmod 5)$

នាំអោយ $5n^7+7n^5+23n$ ចែកដាច់នឹង 5

$*$ បង្ហាញថា $5n^7+7n^5+23n$ ចែកដាច់នឹង 7

តាមទ្រឹស្តីបទ *Fermat*

យើងបាន $n^7 \equiv n \quad (\bmod 7)$

$\Rightarrow 5n^7 \equiv 5n \quad (\bmod 7)$

$\Rightarrow 5n^7+23n \equiv 5n+23n \equiv 0 \quad (\bmod 7)$

$\Rightarrow 5n^7+7n^5+23n \equiv 0 \quad (\bmod 7)$

នាំអោយ $5n^7+7n^5+23n$ ចែកដាច់នឹង 7

ដូច្នេះ $\frac{n^7}{7}+\frac{n^5}{5}+\frac{23n}{35}$ ជាចំនួនគត់

៦. គណនា $A=\int_0^{\frac{\pi}{2}}\left(\cos^2(\cos x)+\sin^2(\sin x)\right)dx \qquad (*)$

តាង $y=\frac{\pi}{2}-x \quad \Rightarrow x=\frac{\pi}{2}-y \quad \Rightarrow dx=-dy$

បើ $x=0 \Rightarrow y=\frac{\pi}{2}$ និង $x=\frac{\pi}{2} \Rightarrow y=0$

$$\Rightarrow A=\int_{\frac{\pi}{2}}^{0}\left[\cos^2\left(\cos\left(\frac{\pi}{2}-y\right)\right)+\sin^2\left(\sin\left(\frac{\pi}{2}-y\right)\right)\right](-dy)$$

$$\Rightarrow A=\int_{0}^{\frac{\pi}{2}}\left(\cos^2(\sin y)+\sin^2(\cos y)\right)dy$$

$$\Rightarrow A=\int_{0}^{\frac{\pi}{2}}\left(\cos^2(\sin x)+\sin^2(\cos x)\right)dx \qquad (**)$$

យក $(*)$ បូក $(**)$

យើងបាន $2A=\int_{0}^{\frac{\pi}{2}}\left(\cos^2(\cos x)+\sin^2(\sin x)\right)dx$

$$+\int_{0}^{\frac{\pi}{2}}\left(\cos^2(\sin x)+\sin^2(\cos x)\right)dx$$

$$2A=\int_{0}^{\frac{\pi}{2}}\left(\cos^2(\cos x)+\sin^2(\cos x)+\cos^2(\sin x)+\sin^2(\sin x)\right)dx$$

$$\Rightarrow 2A = \int_0^{\frac{\pi}{2}} 2dx \quad \Rightarrow 2A = 2x\Big|_0^{\frac{\pi}{2}} = 2\left(\frac{\pi}{2} - 0\right) = \pi$$

$$\Rightarrow A = \frac{\pi}{2}$$

ដូចេ្នះ $A = \int_0^{\frac{\pi}{2}} \left(\cos^2(\cos x) + \sin^2(\sin x)\right) dx = \frac{\pi}{2}$

Mathematics Preparation Test

1 Multiple Choice Questions

1. Sam-the-Super-Snail is climbing a vertical stone 1 meter high. He climbs at a steady speed of 30cm per hour, but each time the church clock strikes the shock causes him to slip down 1cm. The clock strikes the hour, so at 1 o’clock he would slip back 1cm, at 2 o’clock he would slip back 2cm and so on. If he starts to climb just after the clock strikes 3:00 pm, when will he reach the top?

(a). 3:50 pm (b). 6:20 pm (c). 6:50 pm (d). 7:20 pm

(e). None of the above

2. The floor of a room is a 10m by 10m square and the room is 4m high. A spider is in one of the corners of the floor and sees a fly across the room at the diagonally opposite corner on the ceiling. If the fly does not move, what is the shortest distance (in meters) the spider must travel to catch the fly?

(a). $\sqrt{216}$ (b). $4+\sqrt{200}$ (c). 24 (d). $\sqrt{296}$

(e). None of the above

3. A cathedral is located at the top of a hill. When the top of the spire is viewed from the base of the hill, the angle of the elevation is 60°. When the spire is viewed from a distance of 200 feet from the base of the hill, the angle of the elevation is 45°. If the hill

rises at an angle of 30°, then the height, in feet, of the cathedral is

(a). $200-\frac{100\sqrt{3}}{6}$ (b). $100+\frac{200\sqrt{3}}{6}$

(c). $100+\frac{100\sqrt{3}}{6}$ (d). $200+\frac{200\sqrt{3}}{3}$

(e). None of the above

4. Two points are randomly and simultaneously selected from the 4×5 grid of 20 lattice points {$(m,n):1\leq m\leq 5, 1\leq n\leq 4$, with m and n are integers}. What is the probability that the distance between them is a rational number?

(a). 7/19 (b). 36/95 (c). 1/2 (d). 10/19

(e). None of the above

5. The sum of the roots of the equation $x^3+4x^2-7x-10=0$ is -4. What is the sum of the roots of the equation $(x-3)^3+4(x-3)^2-7(x-3)-10=0$?

(a). -13 (b). -7 (c). -1 (d). 5

(e). None of the above

6. At noon, a car and a van are 120 miles apart on a straight road and driving toward each other at a speed of 40 mi/hr. A fly starts from the front bumper of the van at noon and files to the bumper

of the car, then immediately back to the bumper of the van, back to the car, and so on, until the car and the van meet. If the fly files at a speed of 100mi/hr, what is the total distance it travels?

(a). 150 miles (b). 120 miles (c). 100 miles

(d). 200 miles (e). None of the above

7. An amoeba propagates by simple division; each split into two takes 3 minutes to complete. When such an amoeba is put into a glass container with a nutrient fluid, the container will be full of amoebas in one hour. How long would it take for the container to be filled if we start with not one amoeba, but two?

(a). 30 min (b). 45 min (c). 51 min

(d). 57 min (e). None of the above

8. A square of perimeter 20 is inscribed in a square of perimeter 28 (inscribed means the vertices of the smaller square are on the sides of the larger square). What is the greatest distance between a vertex of the inner square and a vertex of the outer square?

(a). $5\sqrt{3}$ (b). $\sqrt{58}$ (c). $\frac{7\sqrt{5}}{2}$

(d). $\sqrt{65}$ (e). None of the above

9. The Beatles (Paul, John, George, and Ringo) took a math exam. Paul got correct half of the questions plus 7 questions, John got correct one third of the questions plus 17 questions, George got correct one fourth of the questions plus 22 questions, and Ringo

got correct one fifth of the questions plus 25 questions. There were between one and 100 questions on the exam and each Beatle got an integer number of questions correct. Which Beatle got the most questions correct?

(a). Only Paul (b). Only John (c). Only George

(d). Only Ringo (e). At least two Beatles got the most questions correct

10. If L is the line whose equation is $ax+by=c$. Let M be the reflection of L through the y-axis, and let N be the reflection of L through the x-axis. Which of the following must be true about M and N for all choices of a, b, and c?

(a). The x-intercepts of M and N are equal

(b). The y-intercepts of M and N are equal

(c). The slopes of M and N are equal

(d). The slopes of M and N are negative reciprocals of each another

(e). None of the preceding

11. Let $f(x)$ be a cubic polynomial in one variable x such that $f(0)\neq 0$ and with no repeated roots. Suppose that $f(x)$ and $f(-x)$ share a common root. Which of the following polynomials could be the least common multiple (LCM) of

$f(x)$ and $f(-x)$? (The numbers $\pm a$, $\pm b$, and $\pm c$ below are all distinct)

(a). $(x^2-a^2)^2(x-b)(x-c)$

(b). $(x^2-a)^2(x^2-b)$

(c). $(x^2-a^2)(x-b)(x-c)$

(d). $(x^2-a^2)(x^2-b^2)$

(e). None of the preceding could be the LCM

12. 10% of a high school senior class participate in Math Olympics. 95% of the seniors that participate in Math Olympics get into the college of their choice. Only 50% of the seniors who don't participate in Math Olympics get into the college of their choice. If a senior from that high school got into the college of his or her choice, what is the probability that this senior participated in Math Olympics?

(a). 0.095 (b). 0.545 (c). 19/109 (d). 0.95

(e). None of the above

13. A middle school has 100 lockers numbered 1 to 100, and 100 students. The first student goes down the row of lockers and opens every locker. Then the second student goes down the row of lockers and closes every locker that is numbered with a multiple of 2. Then the third student goes down the row of lockers, and for

every locker that is numbered with a multiple of 3, if it is open, she closes it, but if it is already closed, she opens it again. The fourth student then does the same thing for the lockers numbered with multiple of 4, and so on, down to the hundredth student. In the end, how many lockers are still open?

(a). 1

(b). All of the lockers that are not numbered with prime numbers

(c). 10 (d). 15 (e). None of the above

14. The line $3x+y=b$ and the equation $2x^2+y^2=1$ are graphed on an $(x,y)-$rectangular coordinate system. For what values of b will the graph of the line $3x+y=b$ be tangent to the graph of the equation $2x^2+y^2=1$?

(a). The graph of the line $3x+y=b$ is not tangent to the graph of $2x^2+y^2=1$ for any b.

(b). The graph of the line $3x+y=b$ is tangent to the graph of $2x^2+y^2=1$ for all b.

(c). $b=\pm 1$ (d). $b=\pm\frac{\sqrt{22}}{2}$ (e). None of the above

15. Suppose that the digits 1, 0, and 8 are written so that they look the same when they are upside down as they do when they are right-side up. Also, the digits 6 and 9 are written so that by turning

(rotating) a 6 upside down, we get a 9. Two 5-digit numbers are "flips" of each other if one is obtained by rotating the other upside down. For example, 61891 and 16819 are "flips" of each other. If all 5-digit numbers (including those starting with 0 like 00027) are being printed on slips of paper such that two numbers are printed on the same slip if and only if they are flips of one another, how many slips of paper are needed?

(a). 100,000 (b). 98475 (c). 96875

(d). $10^5 - \frac{5^5}{2}$ (e). None of the above

16. In a local election, 3000 votes were cast to elect one of the three candidates A, B, and C. Of the first 2000 votes, A received 45 %, B received 35%, and the rest went to C. Of the remaining 1000 votes, 60% went to C, 25% went to B, and the rest went to A. Which of the following is true?

(a). A places first with 60% of the votes

(b). C wins the election

(c). A and C tie for first place

(d). C received 50 votes more than B

(e). None of the above

17. Mary's top book shelf holds five books with the following widths, in centimeters: 6, 1/2, 1, 2.5, and 10. What is the average book width, in centimeters?

(a). 1 (b). 2 (c). 3 (d). 4 (e). 5

18. Tyrone had 97 marbles and Eric had 11 marbles. Tyrone then gave some of his marbles to Eric so that Tyrone ended with twice as many marbles as Eric. How many marbles did Tyrone give to Eric?

(a). 3 (b). 13 (c). 18 (d). 25 (e). 29

19. A book that is to be recorded onto compact discs takes 412 minutes to read aloud. Each disc can hold up to 56 minutes of reading. Assume that the smallest possible number of discs is used and that each disc contains the same length of reading. How many minutes of reading will each disc contain?

(a). 50.2 (b). 51.5 (c). 52.4 (d). 53.8 (e). 55.2

20. For positive numbers x and y the operation $*(x, y)$ is defined as $*(x, y) = x - \frac{1}{y}$. What is $*(2, *(2, 2))$?

(a). 2/3 (b). 1 (c). 4/3 (d). 5/3 (e). 2

21. The length of the interval of solutions of the inequality $a \le 2x + 3 \le b$ is 10. What is $b - a$?

(a). 6 (b). 10 (c). 15 (d). 20 (e). 30

22. A new sequence is obtained from the sequence of the positive integers {1, 2, 3, …} by deleting all the perfect squares. Then the 2003rd term of the new sequence is (………).

(a). 2046 (b). 2047 (c). 2048 (d). 2049 (e).2050

23. Let O be an interior point of ΔABC such that $\overrightarrow{OA}+2\overrightarrow{OB}+3\overrightarrow{OC}=0$. Then the ratio of the area of ΔABC to the area of ΔAOC is (……….).

(a). 2 (b). 3/2 (c).3 (d). 5/3 (e).5/4

24. Let $T=\{0,1,2,3,4,5,6\}$ and $M=\left\{\frac{a_1}{7},\frac{a_2}{7^2},\frac{a_3}{7^3},\frac{a_4}{7^4};\ a_i\in T, i=1,2,3,4\right\}$. Arrange the numbers in M in the descending order. Then the 2005th number is (………)

(a). $\frac{5}{7}+\frac{5}{7^2}+\frac{6}{7^3}+\frac{3}{7^4}$ (b). $\frac{5}{7}+\frac{5}{7^2}+\frac{6}{7^3}+\frac{2}{7^4}$

(c). $\frac{1}{7}+\frac{1}{7^2}+\frac{0}{7^3}+\frac{4}{7^4}$ (d). $\frac{1}{7}+\frac{1}{7^2}+\frac{0}{7^3}+\frac{3}{7^4}$

25. Assume that $M=\{(x,y)/x^2+2y^2=3\}$ and $N=\{(x,y)/y=mx+b\}$. If $M\cap N\neq\varnothing$ for all $m\in\mathbb{R}$, then b takes values from (…………)

$(a).\left[-\frac{\sqrt{6}}{2},\frac{\sqrt{6}}{2}\right]$ $(b).\left(-\frac{\sqrt{6}}{2},\frac{\sqrt{6}}{2}\right)$

$(c).\left(-\frac{\sqrt{6}}{2},\frac{\sqrt{6}}{2}\right]$ $(d).\left[-\frac{2\sqrt{3}}{3},\frac{2\sqrt{3}}{3}\right]$

2. Mathematical Question:

1. Daniel likes two things—kittens and prime numbers. One day, he decides to fill up his whiteboard with kittens and prime numbers. He begins by drawing five cute kittens side by side on the board, and then writes the following numbers under the kittens: 110437, 124297, 138157, 152017. Before he writes a number under the last kitten, however, he suddenly realizes that he forgot to check whether the numbers he wrote down are prime. Feeling adventurous, he decides to test their primality by writing down all the numbers up to 152017 on the back of his whiteboard and crossing out every second number after 2, then every third number after 3, and so on until only the primes are left. Satisfied that the four numbers he wrote down are prime, and also pleased at the size of his whiteboard and the amount time he just wasted, Daniel returns to the front of his whiteboard and writes down a number under the last kitten. What number does he write down?

2. Justin is reading a tourist guide listing some of the straight-line distances between five towns: Aberystwyth, Botcherby, Caarnduncan, Damnaglaur, and Evercreech. They are:
Aberystwyth to Botcherby: 102km
Botcherby to Caarnduncan: 79km
Caarnduncan to Damnaglaur: 231km
Damnaglaur to Evercreech: 16km
Evercreech to Aberystwyth: 34km
If Justin is currently in Botcherby, how far is he from Evercreech?

3. The letters a, b, c, d, e, f, and g each represent a distinct digit (number from 0 to 9 inclusive). The three products $a \times b \times c$, $c \times d \times e$, and $e \times f \times g$ are equal. What 3 digits are not used?

4. A large building has 64 rooms arranged in a $4 \times 4 \times 4$ lattice. Each room has doors to each of the adjacent rooms on the same floor, and stairs to the corresponding rooms in adjacent floors. If Thara has just entered the building and is in one corner of the building on the bottom floor, and wants to get to his bedroom in the opposite corner of the building on the top floor, in how many ways can he get there in the minimum number of moves?

5. Tom, Dick, and Harry attend the same gym. One attends every 2 days, one every 3 days, and one every 7 days. In a particular month of gym attendance, Tom's first visit was on a Monday, Dick's on a Wednesday, and Harry's on a Friday. Also, on one day of this month, all three attended the gym together. Which day of the month was it?

6. John is a judge with a keen interest in numerology. He uses the code A=1, B=2, and C=3, etc to encode his wife's name. He notices that the product of the letters of her name is the same as for his profession, JUDGE. Given that it has no letter in common with JUDGE, and that its letters are in alphabetical order if you swap two of them. What is her name?

7. If S is a set of points in the plane, a point x is said to belong to the environment of S if there exist points y and z in S with the property that the distance between x and y is less than the distance between y and z. If S is an isosceles right-angled triangle of area 1; what is the area of its environment?

8. Given that $x+y+z=1,\ xy+yz+xz=2,$ and $x^5+y^5+z^5=5$. What is xyz ?

9. Three sets of twin girls are married to three sets of twin boys such that each girl's sister is married to her husband's brother. One day, the six couples decide to play mixed doubles tennis. However, no girl wants to be paired with her husband, or her husband's brother. How many possible pairings are there?

10. A car travels from A to B at the rate of 20km/h and then returns from B to A at the rate of 80km/h, without stopping. What is the average speed for the round trip, in Kilometers per hour?

11. Let $M(-1,2)$ and $N(1,4)$ be two points in a plane rectangular coordinate system xOy. P is a moving point on the $x-axis$. When $\angle MPN$ takes its maximum value, the $x-coordinate$ of point P is ………………….. .

12. In a planar rectangular coordinate system xOy, the area enclosed by the graph of function $f(x) = a\sin ax + \cos ax$ (for all $a > 0$) defined on an interval with the least positive period and by the graph of function $g(x) = \sqrt{a^2 + 1}$ is …………… .

13. It is given that complex numbers z_1 and z_2 satisfy $|z_1| = 2$ and $|z_2| = 3$. If the included angle of their corresponding vectors is 60°, then $\left|\frac{z_1 + z_2}{z_1 - z_2}\right| =$……………… .

14. The roots of the equation $x^3 + px^2 + qx + 30 = 0$ are in the ratios $2:3:5$. Find the values of p and q.

15. In MUMSland coins are shaped like regular polygons. All coins have the same side length but they vary in the number of sides they have. There are some coins that are shaped like squares, and others that are shaped like hexagons. There is another type of coin, so that the square, hexagon and the third type of coin fit together snugly at a vertex, with no hole between coins. How many sides does the third type of coin have?

រៀនដោះស្រាយ លំហាត់គណិតវិទ្យា ជាភាសាអង់គ្លេស

Exercise 01. Prove by induction that $9^n - 1$ is a multiple of 8 for all positive integer values of n.

Solution

Let P_n be the statement

'For all positive integer values of n, $9^n - 1$ is a multiple of 8'.

If the formula is valid when $n = k$

i.e. if $9^k - 1 = 8a$, say,

then $9^{k+1} - 1 = (9^k)(9) - 1$

$$= (9^k)(9) - 9 + 8$$

$$= 9(9^k - 1) + 8$$

$$= 9(8a) + 8 = 8(9a + 1)$$ which is the multiple of 8

i.e. If $9^k - 1$ is a multiple of 8 then so is $9^{k+1} - 1$.

But if $n = 1$, $9^1 - 1 = 8$, so P_1 is true .

So P_2 is true, P_3 is true, … and so on for all positive integer values of n.

Therefore by induction, P_n is correct for $n \in \mathbb{N}$

Exercise 02. Prove that if n is a natural number such that n^2 is even then n is even.

Solution

The contrapositive of the statement

'n is a natural number such that n^2 is even $\Rightarrow n$ is even'

Is 'n is an odd integer $\Rightarrow n^2$ is an odd integer'

Now n is an odd integer $\Rightarrow n = 2k+1$ where k is an integer

$$\Rightarrow n^2 = (2k+1)^2$$

$$\Rightarrow n^2 = 4k^2 + 4k + 1$$

$$\Rightarrow n^2 = 2\left(2k^2 + 2k\right) + 1$$

$$\Rightarrow n^2 \text{ is an odd integer}$$

Hence if n is an odd integer then n^2 is an odd integer.

Hence if n is a natural number such that n^2 is an even integer then n is an even integer.

Note: **(Indirect Proof)** If we want to prove that the proposition $p \Rightarrow q$ is true, thus we need to prove that $\neg q \Rightarrow \neg p$ is true.

In short, $(p \Rightarrow q)$ true (or Valid) $\Leftrightarrow (\neg q \Rightarrow \neg p)$ true (or Valid)

Exercise 03. In MUMSland money comes in the currency of proofs. Proofs are divided up into smaller denominations called propositions and lemmas. 1 proof is worth 10 propositions. 1 proposition is worth 10 lemmas. There are coins in circulation worth 1 lemma, 2 lemmas, 5 lemmas, 1 proposition, 2 propositions, 5 propositions, 1 proof, and 2 proofs.

Lori is about to travel on a hot air balloon to her economics tutorial but she needs to pay the fare, and must pay exactly. She knows the fare is more than 1 proof and less than 3 proofs. At least how many coins must she carry to be sure of carrying the exact correct fare?

Solution

For this question, it is much easier to obtain the answer than to prove it rigorously!

Note that the currency is identical to a dollars-cents currency , with 1c, 2c, 5c, 10c, 20c, 50c, $1, $2 coins. We will use the language of dollars and cents for this solution as it is simpler.

Consider how many coins we need to obtain the amounts of $1.88 and $2.99.

Note that, if we have a collection of coins worth $1.88, then there must be a sub-collection of coins worth exactly 8c. (This is not entirely obvious—the same argument would not apply for $1.11= $1+5c+2c+2c+2c). Of the remaining $1.80 in coins, similarly there must be a sub-collection of coins worth exactly $1. Thus we can divide the collection into three parts worth $1, 80c and 8c respectively.

We can do the same for a $2.99 collection into parts worth $2, 90c, and 9c.

For the 8c and 9c sub-collections, taken together, at least 4 coins are required, since at least 3 coins are required for 8c, at least 3 coins are required for 9c, and the sets of three coins are not identical, using at least 4 distinct coins in total. These 4 or more coins must all be 5c or less.

Similarly for the 80c and 90c sub-collections, at least four coins are required. If these sub-collections include 5c or smaller denominations, then at least 5 coins are required. In either case all the coins are 50c or smaller.

And obviously, for the $1 and $2 sub-collections, at least 2 coins are required. If these sub-collections include 50c or smaller denominations, then at least 3 coins are required.

So, these sub-collections in total required 4+4+2 coins. It is easy to verify that there cannot be an overlap between sub-collections (eg. 90c and 8c sub-collections) allowing a smaller number of coins, since then there must be 50c or 5c coins requiring larger sub-collections. So at least 10 coins are required.

The following 10 coin collection can easily be verified to allow every possible fare to be achieved.

<u>Exercise 04</u>. If $I_n = \int \cos^n x dx$. Show that

$I_n = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n} I_{n-2}$. Hence find $\int \cos^5 xdx$

<u>Solution</u>

To use integration by parts, $\cos^n x$ is written in the product form

$\cos x \cos^{n-1} x$

So that $I_n = \int \cos x \cos^{n-1} xdx$

If $v = \cos^{n-1} x$ and $\frac{du}{dx} = \cos x$

Then $\frac{dv}{dx}=(n-1)\cos^{n-2}x(-\sin x)$ and $u=\sin x$

So $I_n=\sin x\cos^{n-1}x+\int(n-1)\cos^{n-2}x\sin^2 xdx$

$$=\sin x\cos^{n-1}x+(n-1)\int\cos^{n-2}x\left(1-\cos^2 x\right)dx$$

$$=\sin x\cos^{n-1}x+(n-1)\int\cos^{n-2}xdx-(n-1)\int\cos^n xdx$$

Hence $I_n=\sin x\cos^{n-1}x+(n-1)I_{n-2}-(n-1)I_n$

Or $nI_n=\sin x\cos^{n-1}x+(n-1)I_{n-2}$

So $I_n=\frac{1}{n}\sin x\cos^{n-1}x+\frac{n-1}{n}I_{n-2}$

Now we can apply this reduction formula to find $\int\cos^5 xdx$

First using $n=5$

i.e. $I_5=\frac{1}{5}\sin x\cos^4 x+\frac{4}{5}I_3 \qquad (1)$

Further using $n=3$, we have, $I_3=\frac{1}{3}\sin x\cos^2 x+\frac{2}{3}I_1 \quad (2)$

Now $I_1=\int\cos xdx=\sin x+K \quad (3)$

Combining (1), (2), and (3) gives,

$$I_5=\frac{1}{5}\sin x\cos^4 x+\frac{4}{5}\left(\frac{1}{3}\sin x\cos^2 x+\frac{2}{3}\sin x\right)+K$$

Where K is constant

Exercise 05. How many pairs of natural numbers (u,v) are there such that $1 \le u \le v$, with the property that the least common multiple of u and v is 2000?

Solution

We will first drop the consideration that $u \le v$, and just find pairs of natural numbers (u,v) with least common multiple 2000.

Let $u = 2^a 5^b$, $v = 2^c 5^d$. So the least common multiple is $2^{\max(a,c)} 5^{\max(b,d)} = 2000 = 2^4 5^3$. We can restate the problem now as: find all sets of four non-negative integers $\{a,b,c,d\}$ with $\max(a,c) = 4$ and $\max(b,d) = 3$.

Consider a and c. If $a = c$ then there is only one possibility: $a = c = 4$. If $a \ne c$ then one of them must be 4 -- there are 2 choices as to which one. The other one can then be 0, 1, 2,or 3—there are 4 choices. It is clear that we have then counted all possible $\{a,c\}$ exactly once, so there are $1 + 2 \times 4 = 9$ possibilities.

Now consider b and d with a similar argument. If $b = d$ then $b = d = 3$. If $b \ne d$ then one of them is 3 and the other is one of 0, 1, 2. This gives $1 + 2 \times 3 = 7$ possibilities.

Thus there are $9 \times 7 = 63$ sets of four non-negative integers $\{a,b,c,d\}$ with $\max(a,c) = 4$ and $\max(b,d) = 3$. So there are 63 pairs of natural numbers (u,v) with least common multiple 2000.

If now $u \le v$, then we have counted every pair twice, except the one where $u = v$, in which case $u = v = 2000$. So there are 31 pairs we have counted twice and 1 pair we have counted once –a total of 32 pairs.

Exercise 06. Find all positive integers n such that $2^{200} + 2^4 + 2^n - 2^{103}$ is a perfect square.

Solution

To find an answer, we might note that $2^{200} = \left(2^{100}\right)^2$,

$2^4 = \left(2^2\right)^2$, and that

$$\left(2^{100} + 2^2\right)^2 = 2^{200} + 2^4 + 2 \times 2^{102} = 2^{200} + 2^4 + 2^{103}$$

Is a perfect square, which is quite close to the expression given. In fact, if we note that $2^{104} - 2^{103} = 2^{103}$, then letting $n = 104$ gives us exactly this perfect square!

While this would have been sufficient for the purposes of the competition, we'll show that this is the only answer. First we note that we can factorize $2^{200} + 2^4 - 2^{103} = \left(2^{100} - 2^2\right)^2$.

This gives $\left(2^{100} - 2^2\right)^2 + 2^n = x^2$ for some positive integer x.

Now we can rewrite and factorize as follows:

$$2^n = x^2 - \left(2^{100} - 2^2\right)^2$$

$$=\left(x-2^{100}+2^{2}\right)\left(x+2^{100}-2^{2}\right)$$

Since both the expressions in brackets are integers, they must be powers of 2. Then we have

(1) $$x-2^{100}+2^{2}=2^{a}$$

(2) $$x+2^{100}-2^{2}=2^{b}$$

Where $a+b=n$

Subtracting (1) from (2) now gives

$$2^{b}-2^{a}=2\times 2^{100}-2\times 2^{2}$$
$$2^{a}\left(2^{b-a}-1\right)=2^{101}-2^{3}$$
$$2^{a}\left(2^{b-a}-1\right)=2^{3}\left(2^{98}-1\right)$$

Since 2^{a} and 2^{3} are powers of 2, while $2^{b-a}-1$ and $2^{98}-1$ are odd, we must have $a=3$ and $b-a=98$, so $b=101$.

This gives $n=a+b=104$

So $n=104$

<u>Exercise 07</u>. If the equation $x^{3}+px^{2}+qx+r=0$ has roots that are in arithmetic progression;

Show that $2p^{3}-9pq+27r=0$

<u>Solution</u>

If the roots from an arithmetic progression with common difference λ, and α is the middle root, then the three roots are: $\alpha-\lambda,\alpha,\alpha+\lambda$.

From the given equation we see that

$$a = 1, b = p, c = q, d = r$$

So $$\sum \alpha = (\alpha - \lambda) + \alpha + (\alpha + \lambda) = -p$$

$\Rightarrow$ $$3\alpha = -p \qquad (1)$$

But α satisfies the given equation so we have

$$\alpha^3 + p\alpha^2 + q\alpha + r = 0$$

$$(1) \Rightarrow \left(-\frac{p}{3}\right)^3 + p\left(-\frac{p}{3}\right)^2 + q\left(-\frac{p}{3}\right) + r = 0$$

$$\Leftrightarrow -p^3 + 3p^3 - 9pq + 27r = 0$$

$$\Leftrightarrow 2p^3 - 9pq + 27r = 0$$

Exercise 08. $P(x, y)$ is the point on an Argand diagram representing $z = x + yi$.

If $|z+2| = 3|z - 2 - 4i|$ show that the Cartesian equation of the locus of P is $x^2 + y^2 - 5x - 9y + 22 = 0$

Solution

To use the geometric approach in this problem we would first write $z + 2$ in the form $z - z_1$ so that it corresponds to a line joining two points, i.e. $|z - (-2 + 0i)| = 3|z - (2 + 4i)|$

Then, using $A(-2, 0)$; $B(2, 4)$ and $P(x, y)$ we find that $AP = 3BP$. It is not obvious from this property however, where

P can lie, so the method based on the algebraic definition of a modulus is used instead.

Writing $x+yi$ for z we have

$$z+2=x+yi+2=(x+2)+yi$$

$$\Rightarrow |z+2|=\sqrt{(x+2)^2+y^2}$$

And $\quad z-2-4i=x+yi-2-4i=(x-2)+(y-4)i$

$$\Rightarrow |z-2-4i|=\sqrt{(x-2)^2+(y-4)^2}$$

Thus the given condition can be given

$$\sqrt{(x+2)^2+y^2}=3\sqrt{(x-2)^2+(y-4)^2}$$

i.e. $\quad x^2+4x+4+y^2=9\left(x^2-4x+4+y^2-8y+16\right)$

$$\Rightarrow 8x^2+8y^2-40x-72y+176=0$$

$$\Rightarrow x^2+y^2-5x-9y+22=0$$

Because this is the condition which the coordinates of P must satisfy, it is the equation to the locus of P, which is seen to be a circle.

<u>Exercise 09</u>. If $z=x+yi$ and $w=u+vi$ represent points $P(x,y)$ and $Q(u,v)$ and $w=\dfrac{z+1}{z-1}$. Find the locus of Q when P describes

a. The circle $x^2+y^2=4$

b. The parabola $y^2=4x$

Solution

Find the locus of Q when P describes:

a. The circle $x^2 + y^2 = 4$

Although $x^2 + y^2 = 4$ can be expressed as $|z| = 2$, this form cannot be used directly from the relationship $w = \frac{z+1}{z-1}$

Rearranging this, however, gives $z = \frac{w+1}{w-1}$

Then $|z| = 2 \quad \Rightarrow \left|\frac{w+1}{w-1}\right| = 2$

i.e. $|w+1| = 2|w-1|$

$$\Rightarrow \quad (u+1)^2 + v^2 = 2\left[(u-1)^2 + v^2\right]$$

$$\Rightarrow \quad 3u^2 + 3v^2 - 10u + 3 = 0$$

Therefore the image of the circle $x^2 + y^2 = 4$ is another circle with equation $3u^2 + 3v^2 - 10u + 3 = 0$

b. The parabola $y^2 = 4x$

To find the image of $y^2 = 4x$ we need x and y separately in terms of u and v.

$$z = \frac{w+1}{w-1} \quad \Rightarrow x + yi = \frac{u+vi+1}{u+vi-1} = \frac{\left(u^2+v^2-1\right) - 2vi}{(u-1)^2 + v^2}$$

$$\Rightarrow x=\frac{u^2+v^2-1}{(u-1)^2+v^2} \quad \text{and} \quad y=-\frac{2v}{(u-1)^2+v^2}$$

So $y^2=4x$ maps to $\dfrac{4v^2}{\left[(u-1)^2+v^2\right]^2}=\dfrac{4\left(u^2+v^{2-1}\right)}{(u-1)^2+v^2}$

$\Rightarrow \quad v^2=\left[(u-1)^2+v^2\right]\left(u^2+v^2-1\right)$ provided that $u^2+v^2-1\neq 0$

Therefore the image of $y^2=4x$ is the curve $\left(u^2-2u+1+v^2\right)\left(u^2+v^2-1\right)=v^2$

<u>Exercise 10</u>. (IMO 1999). Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that $f\left(x-f(y)\right)=f\left(f(y)\right)+xf(y)+f(x)-1$ for all $x,y\in\mathbb{R}$.

<u>Solution</u>

let $c=f(0)$ and A be the image of $f(\mathbb{R})$. If a is in A, then it is straightforward to find $f(a)$: putting $a=f(y)$ and $x=a$, we get $f(a-a)=f(a)+a^2+f(a)-1$

$$\Rightarrow f(a)=\frac{1+c}{2}-\frac{a^2}{2} \qquad (*)$$

The next step is to show that $A-A=\mathbb{R}$. Note first that c cannot be zero, for if it were, then putting $y=0$, we get: $f(x-c)=f(c)+xc+f(x)-1 \qquad (**)$

and hence $f(0)=f(c)=1$, Contradiction. But $(**)$ also shows that $f(x-c)-f(x)=xc+(f(c)-1)$. Here x is free to vary over $\mathbb{R}$, so $xc+(f(c)-1)$ can take any value in $\mathbb{R}$.

Thus given any x in $\mathbb{R}$, we may find $a,b\in A$ such that $x=a-b$. Hence $f(x)=f(a-b)=f(b)+ab+f(a)-1.$

So, using $(*)$: $f(x)=c-\dfrac{b^2}{2}+ab-\dfrac{a^2}{2}=c-\dfrac{x^2}{2}$.

In particular, this is true for $x\in A$. Comparing with $(*)$ we deduce that $c=1$. So for all $x\in\mathbb{R}$ we must have $f(x)=1-\dfrac{x^2}{2}$

Finally, it is easy to check that this satisfies the original relation and hence is the unique solution.

Exercise 11. (Balkan 2000). Find all real-valued functions on the reals which satisfy $f(xf(x)+f(y))=f(x)^2+y$ for all x,y.

Solution

Put $x=0$, then $f(f(y))=f(0)^2+y$. Put $y=-f(0)^2$ and $k=f(y)$, then $f(k)=0$. Now put $x=y=k$, then $f(0)=0+k$, so $k=f(0)$. Put $y=k,x=0$, then

$f(0)=f(0)^2+k$, so $k=0$. Hence $f(0)=0$.

Put $x=0$, $f(f(y))=y \quad (*)$

Put $y=0$, $f(xf(x))=f(x)^2 \qquad (**)$

Put $x=f(z)$ in $(**)$, then using $f(z)=x$,

we have $f(zf(z))=z^2$. Hence $z^2=f(z)^2$ for all $z \quad (***)$

In particular, $f(1)=1$ or -1.

Suppose $f(1)=1$, then putting $x=1$ in the original relation we get $f(1+f(y))=1+y$, hence $(1+f(y))^2=(1+y)^2$. Hence $f(y)=y$ for all y.

Similarly if $f(1)=-1$, then putting $x=1$ in the original relation we get $f(-1+f(y))=1+y$. Hence $(-1+f(y))^2=(1+y)^2$

So $f(y)=-y$ for all y.

Finally, it is easy to check that $f(x)=x$ does indeed satisfy the original relation, as does $f(x)=-x$.

<u>Exercise 12</u>. Let $a, b,$ and c denote the angles of elevation of a tower measured at horizontal distances of 100, 200, and 300 meters from the tower, respectively . If $a+b+c$ is a right angle, find the height of the tower in meters.

<u>Solution</u>

Let the height of the tower in meters be h. So $\tan a = \frac{h}{100}$, $\tan b = \frac{h}{200}$, and $\tan c = \frac{h}{300}$. Since $a+b+c=90^\circ$, then $\tan c \tan(a+b)=1$. Expanding using the addition formula for tan gives, we have

$$\frac{\tan c(\tan a+\tan b)}{1-\tan a\tan b}=1$$

$$\Rightarrow \tan c(\tan a+\tan b)=1-\tan a\tan b$$

$$\Rightarrow \frac{h}{300}\left(\frac{h}{100}+\frac{h}{200}\right)=1-\frac{h}{100}\times\frac{h}{200}$$

$$\Rightarrow h^2=10000$$

$$\Rightarrow h=100$$

So the tower is 100 meters high.

Supplementary Exercises

Problem 01. Calculate $\lim_{x\to 0}\left(\frac{98}{83}\cdot\frac{1-\cos 3x\cos 5x\cos 7x}{\sin^2 7x}\right)$

Problem 02. Let $f(x)=\sqrt[3]{\frac{x}{2}+\sqrt{\frac{x^2}{4}-1}}+\sqrt[3]{\frac{x}{2}-\sqrt{\frac{x^2}{4}-1}}$

and $g(x)=x^4-4x^2+2$. Prove that $f(g(x))=g(f(x))$ for any $x\ge 2$.

Problem 03. Given that $I_n=\int_0^1 x^n\sqrt{(1-x)}dx$. Prove that

$$I_n<\frac{1}{(n+1)\sqrt{n+1}}.$$

Problem 04. Calculate the following limit:

$$\lim_{n\to+\infty}\left[\frac{3}{n}\left(1+\sqrt{\frac{n}{n+3}}+\sqrt{\frac{n}{n+6}}+\cdots+\sqrt{\frac{n}{n+3(n-1)}}\right)\right]$$

Problem 05. Let $f(x)$ and $g(x)$ are the continuous function on $[a,b]$. $0<a\le f(x)\le A$ and $0<b\le g(x)\le B$ for all $x\in[a,b]$. Prove that

$$\frac{(ab+AB)^2}{4aAbB} \ge \frac{\int_a^b f^2(x)\,dx \int_a^b g^2(x)\,dx}{\left(\int_a^b \left(f(x)g(x)\right)dx\right)^2} \ge \frac{4aAbB}{(ab+AB)^2}$$

Problem 06. (IMO 1995) a,b,c are positive real number with $abc=1$. Prove that $\frac{1}{a^3(b+c)}+\frac{1}{b^3(c+a)}+\frac{1}{c^3(a+b)} \ge \frac{3}{2}$.

Problem 07. (Russia 2000) Find all functions $f:\mathbb{R}\to\mathbb{R}$ which satisfy $f(x+y)+f(y+z)+f(z+x) \ge 3f(x+2y+3z)$ for all x,y,z.

Problem 08. (MOP 1997) Let f be a real-valued function which satisfies:

a) For all real numbers x,y; $f(x+y)+f(x-y)=2f(x)f(y)$

b) There exists a real number x_0 such that $f(x_0)=-1$

Prove that f is periodic

Problem 09. (IMO 1994) Let S be the set of all real numbers greater than -1. Find all functions $f:S\to S$ such that $f\left(x+f\left(y\right)+xf\left(y\right)\right)=y+f\left(x\right)+yf\left(x\right)$ for all x and y, and $\frac{f\left(x\right)}{x}$ is strictly increasing on each of the intervals $-1<x<0$ and $0<x$.

Problem 10. (Hungarian Olympiad 1981) Show that there is only one natural number n such that $2^8+2^{11}+2^n$ is a perfect square.

Problem 11. (IMO 2004) Find all polynomials $P\left(x\right)$ with real coefficients, which satisfy the equation:
$P\left(a-b\right)+P\left(b-c\right)+P\left(c-a\right)=2P\left(a+b+c\right)$ for all real numbers a,b,c such that $ab+bc+ca=0$.

Problem 12. Show that for all $n\in\mathbb{N}$:

a) $n^7-n \vdots 42$

b) $n^{13}-n \vdots 2730$

Problem 13. (BMO 1996) Let a,b,c be positive real numbers.

Prove that: a) $4\left(a^3+b^3\right)\geq\left(a+b\right)^3$

b) $9\left(a^3+b^3+c^3\right)\geq\left(a+b+c\right)^3$

Problem 14. Given the triangle ABC. Prove that:

a) $a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0$

b) $b\cos B+c\cos C=a\cos(B-C)$

c) $a^2\sin 2B+b^2\sin 2A=2ab\sin C$

Problem 15. Prove that for all $n\in\mathbb{N}$ such that $z+\frac{1}{z}=2\cos\alpha$ then $z^n+\frac{1}{z^n}=2\cos n\alpha$.

Problem 16. Solve the following system of equation:

$$\begin{cases}\log_5 x=\log_3\left(\sqrt{y}+4\right)\\ \log_5 y=\log_3\left(\sqrt{z}+4\right)\\ \log_5 z=\log_3\left(\sqrt{x}+4\right)\end{cases}$$

ANSWER

1. Multiple Choice Questions

1. (c). 6:50min
2. (d). $\sqrt{296}$
3. (d). $200+\frac{200\sqrt{3}}{3}$
4. (b). $\frac{36}{95}$
5. (d). 5
6. (a). 150miles
7. (d). 57min
8. (d). $\sqrt{65}$
9. (e). At least two Beatles got the most questions correct
10. (c). the slops of M and N are equal
11. (d). $\left(x^2-a\right)\left(x^2-b\right)$
12. (c). $\frac{19}{109}$
13. (c). 10
14. (d). $b=\pm\frac{\sqrt{2}}{2}$
15. (b). 98475

16. (d). C received 50 votes more than B

17. (d). 4cm

18. (d). 25

19. (b). 51.5

20. (c). $\frac{4}{3}$

21. (d). 20

22. (c). 2048

23. (c). 3

24. (c). $\frac{1}{7}+\frac{1}{7^2}+\frac{0}{7^3}+\frac{4}{7^4}$

25. (a). $\left[-\frac{\sqrt{6}}{2},\frac{\sqrt{6}}{2}\right]$

2. Mathematical Question

1. 165877
2. 136km
3. 0, 5, 7
4. 1680
5. 27
6. FANNY
7. $3+\sqrt{3}+\frac{31}{6}\pi$

8. $\frac{6}{5}$
9. 80
10. 32
11. 1
12. $\frac{2\pi}{a}\sqrt{a^2+1}$
13. $\frac{\sqrt{133}}{7}$
14. $p=10$ and $q=31$
15. 12 sides

Supplementary Exercises

1. 1
5. 2
7. $f(x)=k$ for all $k \in \mathbb{R}$
9. $f(x)=-\frac{x}{1+x}$
11. $P(x)=\alpha x^4+\beta x^2$ for all $\alpha,\beta \in \mathbb{R}$
16. $x=y=z=5$

Glossary

1. Multiple Choice Question:

1. Steady Speed: ល្បឿនថេរ

   Slip Down=Slip Back: រអិលចុះក្រោម

   Strike: ស្រែករ៉ូ

   Vertical: ឈរត្រង់

2. Diagonally: តាមអង្កត់ទ្រូង

3. Angle: មុំ

   Cathedral: វិហារនៃគ្រិស្តសាសនា

   Elevation: កំពស់

   Foot: ខ្នាតនៃរង្វាស់ប្រវែង(ស្មើ 30.48cm) [ពហុវចនៈ Feet]

   Hill: កំពូលភ្នំតូចៗ

   Spire: the spire of a building such as a church is a tall point structure on the top.

4. Grid: ក្រឡា

   Integer: ចំនួនគត់

   Lattice: is a partern or structure made of stripes of wood or another material which cross over each other diagonally leaving holes in between.

   Probability: ប្រូបាបបីលីតេ

   Randomly: ដែលព្រាវ ឬ ដោយចៃដន្យ

Rational Number: ចំនួនសនិទាន

Simultaneously: ដំណាលគ្នា

5. Root: ឫស (នៃសមីការ)

Sum: ផលបូក

6. Apart: ឃ្លាតពីគ្នា

Bumper: បាំងដែលនៅខាងមុខរថយន្ត (នាម)

Straight Road: ផ្លូវត្រង់

Van: ឡាន

7. Amoeba: អាមីប (សត្វដែលមានកោសិកាតែមួយ)

Container: អ្នកផ្ទុកឬ វត្ថុសម្រាប់ផ្ទុក

Nutrient Fluid: វត្ថុរាវដែលធ្វើអោយធំធាត់

Propagate: ដុះដាល ឬ បង្កើនចំនួន

Simple Division: ការបំបែកយ៉ាងសមញ្ញ

Split: បំបែក

8. Inner Square: ការេដែលនៅខាងក្នុង

Outer Square: ការេដែលនៅខាងក្រៅ

Perimeter: បរិមាត្រ

Square: ការេ

Vertex: កំពូល (ពហុវចនៈvertices ឬ vertexes)

9. At Least: តិចបំផុត

Beat: If you beat someone in a competition or election, you defeat them.

One third: មួយភាគបី

Plus: បូក

Take an Exam: ប្រលង

10. Equal: ស្មើ

Equation: សមីការ

Reciprocal: បញ្ច្រាស់ ឬ ផ្ទុយ

Reflection: ភាពឆ្លុះ ឬ រូបភាពឆ្លុះ

Slope: មេគុណប្រាប់ទិស

X-axis: អ័ក្ស x (អ័ក្ស អាប់ស៊ីស)

Y-axis: អ័ក្សY (អ័ក្ស អរដោនេ)

X-intercept: អង្កត់ពីចំនុចមួយទៅអ័ក្ស x

Y-intercept: អង្កត់ពីចំនុចមួយទៅអ័ក្សY

11. Cubic Polynomial: ពហុធាដឺក្រេទី ៣

Distinct: ផ្សេងគ្នា

LCM (Least Common Multiple): ពហុគុណរួមតូចបំផុត

Share: តែមួយ ឬ ដូចគ្នា

Variable: អថេរ

12. Participate: ចូលរួមប្រកួត

Senior Class: ថ្នាក់ទី១១ ឬ ១២

13. And So On: ដែលនៅមានតទៀត (តាមលំនាំដែលបានរៀបរាប់)

Multiple of: ពហុគុណនៃ

14. Graph: ក្រាប

Tangent: បន្ទាត់ប៉ះ (កិរិ). ប៉ះ

15. Digit: លេខ តួលេខ ឬ ខ្ទង់

If and only if: លុះត្រាតែ

Rotate: វិល

Slip of paper: ផ្នែកតូចនៃក្រដាស់ (Small piece of paper)

16. Candidate: បេក្ខជន

Cast: បោះឆ្នោត

Local Election: ការបោះឆ្នោតតាមតំបន់

Remaining: ដែលនៅសល់

17. Average: មធ្យម

Centimeter: សង់ទីម៉ែត្រ

Hold: ដាក់

Top book shelf: ធ្នើសៀវភៅជាន់លើបំផុត

Width: កំពស់ ឬ ទទឹង

18. As many as: មានច្រើនស្មើនឹង

Marble: ថ្មើមថ្ម

Twice: ពីរដង

19. Aloud: ដែលលឺសំលេង

Compact disc: ឌីសមួយប្រភេទដែលមានគុណភាពខ្ពស់

Contain: មាន ឬ ផ្ទុក

Record: ថតសំលេង

20. Defined as: កំនត់ដោយ

Operation: ប្រមាណវិធី

Positive number: ចំនួនពិតវិជ្ជមាន

21. Inequality: វិសមភាព

Interval: ចន្លោះ

22. Obtained: ដែលផ្ទុក

Perfect square: ការេប្រាកដ

Sequence: ស្វ៊ីត

Term: តួនៃស្វ៊ីត

23. Area: ផ្ទៃក្រលា

Interior point: ចំនុចក្នុង

Ratio: ផលធៀប

24. Arrange: តំរៀប

Descending: ដែលចុះ

25. Assume that: ឧបមាថា

Value: តម្លៃ

2. Mathematical Questions

1. Adventurous: ដែលផ្សងព្រេង

Crossing out: គូសចោល

Cute: ដែលទាក់ទាញអារម្មណ៍

Fill up: បំពេញ

Kitten: កូនឆ្មា

Prime number: ចំនួនបឋម

Realise: ដឹង

Satisfy: ផ្ទៀងផ្ទាត់

Side by side: If two people or things are side by side, they are next to each other.

2. Currently: នៅពេលនេះ

Listing: ដែលបានកត់ជាតារាង

Tourist guide: ការណែនាំសំរាប់អ្នកទេសចរណ៍

3. Digit: តួលេខ, ខ្ទង់

Distinct: ផ្សេងគ្នា

Inclusive: ដែលរួមទាំងខាងដើម និង ខាងចុង

Represent: តំណាង

Product: គុណ

4. Adjacent: ដែលជាប់គ្នា

Corresponding room: បន្ទប់ដែលនៅ ពីលើឬពីក្រោម

Minimum: អប្បបរមា

5. Attend: ចូលរួមក្នុងកម្មវិធីអ្វីមួយ

6. Alphabetical order: ដែលរៀបតាមអក្ខរក្រម

Encode: ដាក់កូដ (អក្សរសម្ងាត់)

Keen interest: ដែលចាប់អារម្មណ៍ខ្លាំង

Numerology: លេខវិទ្យា

Profession: អាជីព

Swap: ប្តូរទីតាំង ឬ ប្តូរយកអ្វីថ្មី

7. Environment of S: បរិវេណនៃសំនុំ S

Isosceles: សមបាត

Property: លក្ខណះ

Set of points: សំនុំចំនុច

9. Mixed double tennis: តេនីសដែលលេងគ្នាច្រើន

Paired: ដៃគូរ

Possible: ដែលអាចកើតឡើង

Twin boy: បងប្អូនភ្លោះប្រុស

Twin girl: បងប្អូនភ្លោះស្រី

10. Rate: អត្រា ឬ អត្រាកម្រិត

11. Maximum: អតិបរមា

Rectangular coordinate system: កូអរដោនេនៃប្រព័ន្ធចតុកោណកែង

X-coordinate: តម្លៃនៅលើអ័ក្សអាប់ស៊ីស(អ័ក្សx) នៃចំនុច

12. Enclosed: ដែលព័ទ្ធអ្វីមួយជិត

Function: អនុគមន៍

Graph: ក្រាប

Least positive period: ខួបចំនួនពិតវិជ្ជមានតូចបំផុត

Planar rectangular coordinate system: កូអរដោនេនៃប្រព័ន្ធចតុកោណកែង

13. Complex number: ចំនួនកុំផ្លិច

Corresponding vector: វិចទ័រដែលទាក់ទងគ្នា (បង្កើតបានមុំ)

Included angle: មុំដែលកើតឡើងពីវិចទ័រពីរកាត់គ្នា

14. Arithmetic progression: កំនើននព្វន្ត ឬ ស្វីតនព្វន្ត

Coefficient: មេគុណ

Decreasing: ដែលថយចុះ

Expansion: ពង្រីក ឬ ពន្លាត

Integer: ចំនួនគត់

Power: ស្វ័យគុណ ឬ អានុភាព

15. Coin: កាក់

Hexagon: ឆកោណ

Regular polygon: ពហុកោណនិយ័ត

Shaped: ដែលមានទម្រង់

Snugly: យ៉ាងល្អ

Vary: ខុសគ្នា

Vertex: កំពូល

Exercises

1. Induction: វិធីកំនើន ឬ វិចារកំនើន

Statement: សំនើរ ឬ សេចក្តីអះអាង

Valid: ពិត ឬ សមស្រប

2. Contrapositive: ផ្ទុយពីសម្មតិកម្ម

Even=Even number: ចំនួនគូ

Natural number: ចំនួនគត់ធម្មជាតិ

Odd integer: ចំនួនគត់សេស

Proposition: សំនើរ

3. Circulation: ការវង់ ឬ ការវង់លើខ្សែកោង

Currency: រូបិវត្ថុ

Denomination: ការបែងចែក

Entirely obvious: ច្បាស់លាស់ទាំងស្រុង

Verify=Satisfy: ផ្ទៀងផ្ទាត់

Worth: តម្លៃ

4. Constant: ចំនួនថេរ

Integration by part: ការគណនាអាំងតេក្រាលដោយផ្នែក

Reduction: ការបង្រួម ឬ ការបន្ថយ

5. Argument: អំណះអំណាង

Consideration: ការពិចារណា

Counted: ដែលបានរាប់

Non-negative integer: ចំនួនគត់មិនអវិជ្ជមាន

Pairs of natural number: គូនៃចំនួនពិតធម្មជាតិ

Restate: បញ្ជាក់ម្តងទៀត

6. Factorize: ដាក់ជាផលគុណកត្តា

Subtract: ដក

8. Argand Diagram: ដ្យាក្រាម Argand

Cartesian coordinate: កូអរដោនេកាតេសង់

Locus: សំនុំចំនុច

9. Image: រូបភាព (នៃអនុគមន៍)

Rearrange: រៀបឡើងវិញ

10. Contradiction: ភាពផ្ទុយ ឬ សំនើរផ្ទុយ