

**វិភាគលេខ**

Numerical Analysis

**ភាគ១**

Part I

**រៀបរៀងដោយ អេ មណីចក្រ**

**ឆ្នាំសិក្សា ២០១៨-២០១៩**

# មាតិកា

## Contents

|                                                                          |    |
|--------------------------------------------------------------------------|----|
| មាតិកា .....                                                             | i  |
| មេរៀនទី១ .....                                                           | 1  |
| I. រំលឹកការគណនា .....                                                    | 1  |
| 1. លីមីត (Limits) .....                                                  | 1  |
| 2. ភាពជាប់ (Continuity) .....                                            | 1  |
| 3. ទម្រង់ក្រាប .....                                                     | 1  |
| 4. ដេរីវេ (Derivatives) .....                                            | 3  |
| 5. អាំងតេក្រាល (Integration) .....                                       | 3  |
| II. ទ្រឹស្តីបទរ៉ូល (Rolle's Theorem) .....                               | 3  |
| III. ទ្រឹស្តីបទតម្លៃមធ្យម (Mean Value Theorem) .....                     | 5  |
| IV. ទ្រឹស្តីបទតម្លៃចុង (Extreme Value Theorem) .....                     | 6  |
| V. អាំងតេក្រាលរីម៉ាន (The Riemann integral) .....                        | 6  |
| VI. ទ្រឹស្តីបទតម្លៃកណ្តាល .....                                          | 7  |
| VII. ទ្រឹស្តីបទ Taylor (Taylor's Theorem) .....                          | 9  |
| មេរៀនទី២ .....                                                           | 12 |
| លេខល្អៀង និង នព្វន្ត .....                                               | 12 |
| I. ការតាងចំនួន (Representation of Numbers) .....                         | 12 |
| II. ប្រព័ន្ធរបាប .....                                                   | 12 |
| III. ល្អៀងដាច់ខាត និង ល្អៀងធៀប (Absolute Error and Relative Error) ..... | 15 |
| IV. ល្អៀងក្នុងការគណនា (Errors in Computation) .....                      | 16 |
| 1. ការបង្អត់ចំនួន (Rounding Number) .....                                | 16 |

|                                                                |     |
|----------------------------------------------------------------|-----|
| 2. ខ្ទង់សំខាន់ (Significant digits) .....                      | 16  |
| V. នព្វន្តកុំព្យូទ័រ (Computer Arithmetic) .....               | 17  |
| 1. ប្រមាណវិធី (Operation).....                                 | 17  |
| 2. ប្រព័ន្ធចំនួន (Radix Number Systems).....                   | 18  |
| 3. ការបម្លែងចំនួន (Conversion from One Radix to Another) ..... | 18  |
| មេរៀនទី៣ សមីការ $f(x) = 0$ (The Equation $f(x) = 0$ ).....     | 127 |
| មេរៀនទី៤ ឌីផេរ៉ង់ស្យែលលេខ (Numerical Differentiation).....     | 174 |
| ឯកសារយោង (Reference).....                                      |     |

មេរៀនទី១

សេចក្តីផ្តើមគណិតវិទ្យា

Mathematic Prelimery

I. រំលឹកការគណនា

1. លីមីត (Limits)

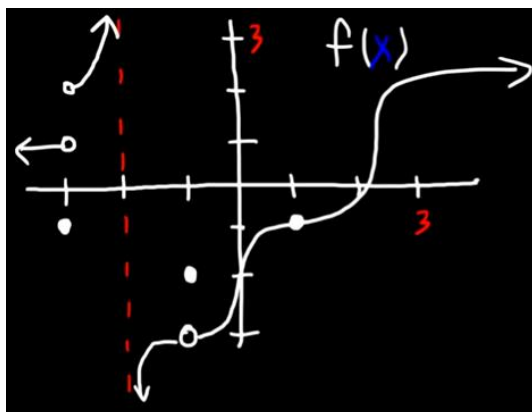
ឧទាហរណ៍៖

១. គេមាន  $f(x) = \frac{x^2 - 4}{x - 2}$  បើ  $x \rightarrow 2$  គណនា  $f(x)$  ។

២. គណនាលីមីត៖    ក.  $\lim_{x \rightarrow 3} \frac{x^2 - 8x + 15}{x - 3}$     ខ.  $\lim_{x \rightarrow 4} \left( \frac{1}{x} - \frac{1}{4} \right)$     គ.  $\lim_{x \rightarrow 9} \frac{\sqrt{x} - 3}{x - 9}$

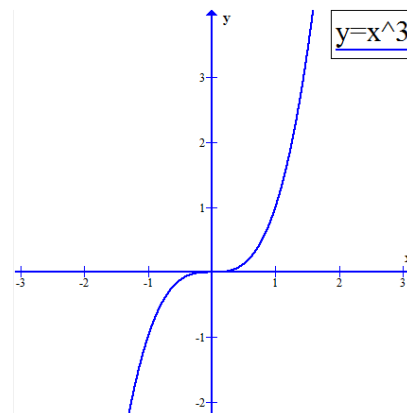
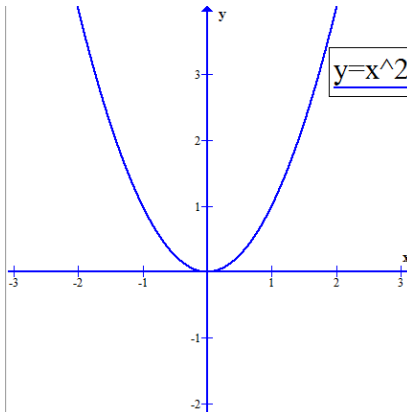
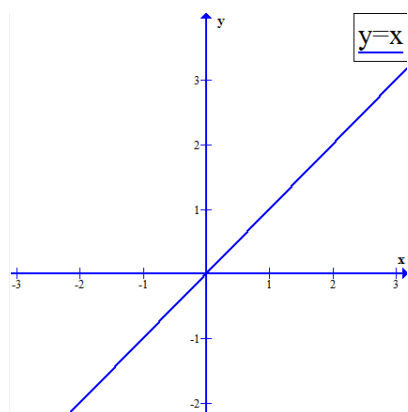
2. ភាពជាប់ (Continuity)

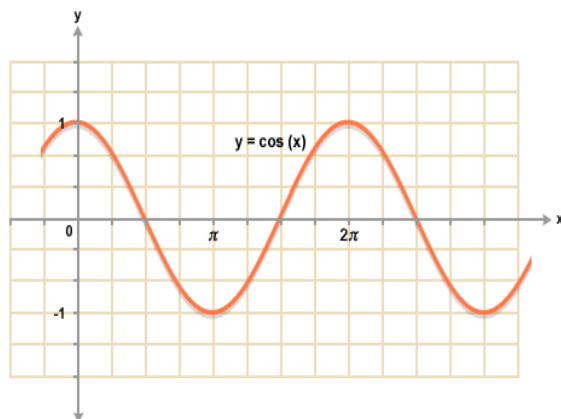
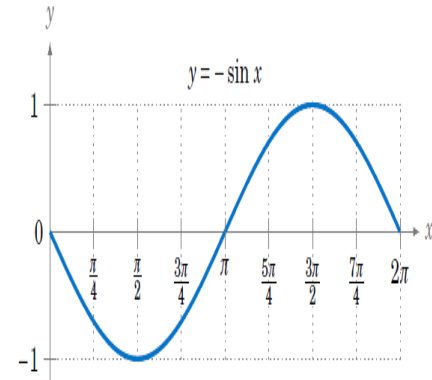
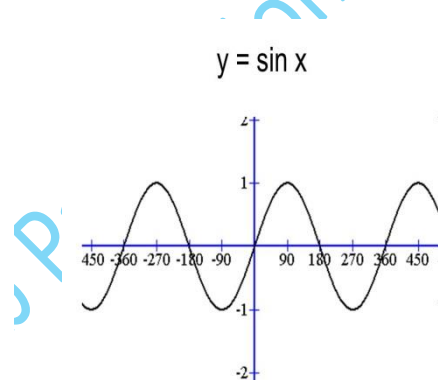
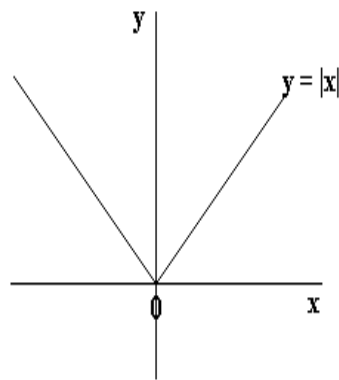
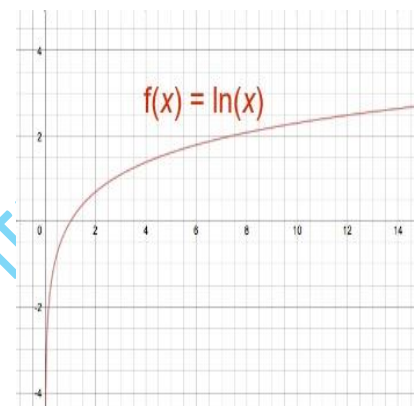
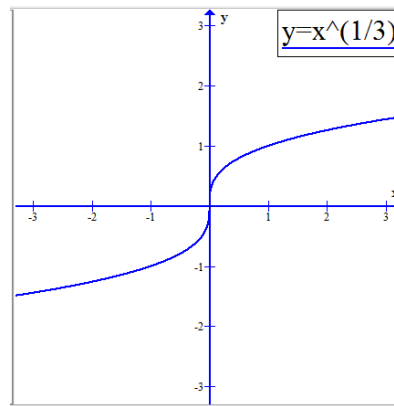
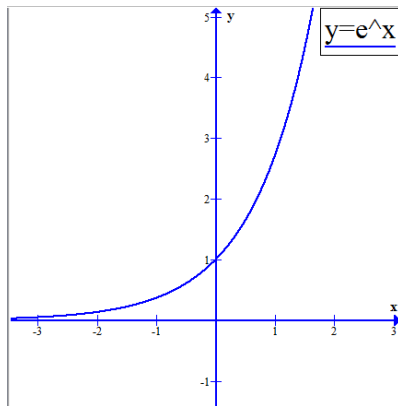
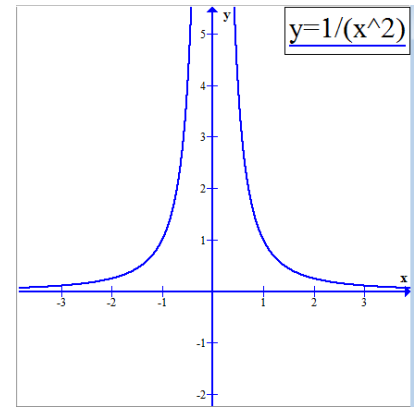
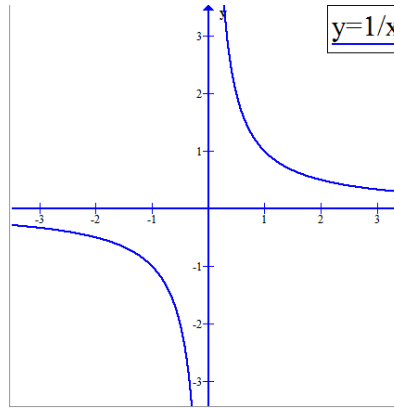
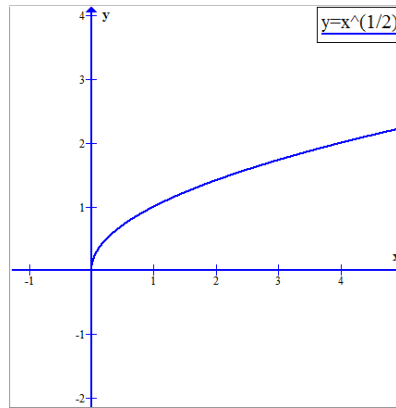
ឧទាហរណ៍១៖ គេមានអនុគមន៍  $f(x)$  (ដូចរូប)។ ចូររកភាពជាប់នៃអនុគមន៍ត្រង់  $x = -3, x = -1, x = -2$  និង  $x = 1$  ។



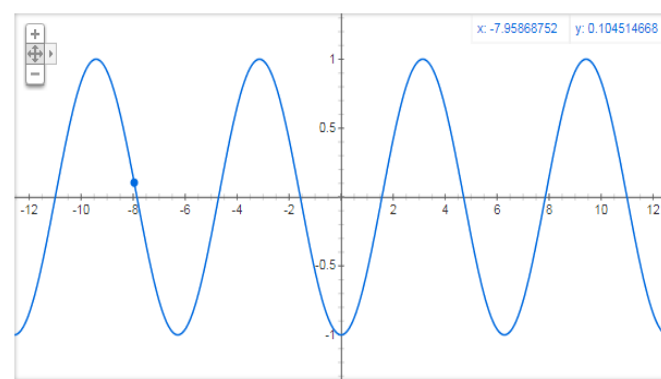
ឧទាហរណ៍២៖ សិក្សាភាពជាប់នៃអនុគមន៍៖  $f(x) = \begin{cases} 5x + 3 & , x < 1 \\ x^2 + 7 & , 1 \leq x < 2 \\ 3x - 1 & , x \geq 2 \end{cases}$

3. ទម្រង់ក្រាប





Graph for  $-\cos(x)$



#### 4. ដេរីវេ (Derivatives)

$$\begin{array}{ll} x^6 \rightarrow 6x^5 & 8x^5 \rightarrow 8(5x^4) = 40x^4 \\ x^7 \rightarrow 7x^6 & 9x^4 \rightarrow 9(4x^3) = 36x^3 \\ x^1 \rightarrow 1x^0 = 1 & 32x^2 \rightarrow 32(2x) = 64x \\ 5 \rightarrow 0 & 6x \rightarrow 6(1) = 6 \end{array}$$

ឧទាហរណ៍៖ គណនាដេរីវេនៃអនុគមន៍  $y = 4x^3 + 6x^2 - 5x + 8$

គេបាន  $y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx}[4x^3 + 6x^2 - 5x + 8] = 12x^2 + 12x - 5$

#### 5. អាំងតេក្រាល (Integration)

ឧទាហរណ៍៖ គេមាន  $\frac{d}{dx}[x^3] = 3x^2$  នោះ  $\int 3x^2 dx = x^3 + c$

ឧទាហរណ៍៖ គណនាអាំងតេក្រាលនៃអនុគមន៍៖  $I = \int x^4 dx$      $J = \int 8x^3 dx$      $K = \int 5x dx$      $L = \int 7 dx$

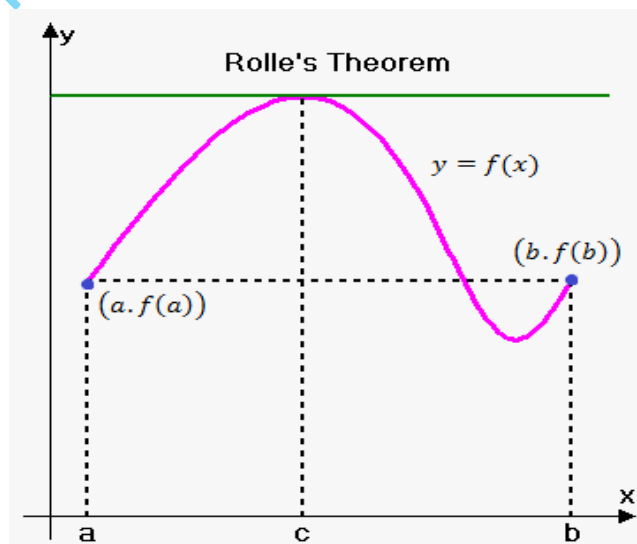
#### Basic introduction

គេមាន  $f(x) = x^2$  នោះ  $f'(x) = 2x$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= 2x \end{aligned}$$

#### II. ទ្រឹស្តីបទរ៉ូល (Rolle's Theorem)

ឧបមាថាអនុគមន៍  $f$  ជាប់លើចន្លោះបិទ  $C[a, b]$  ហើយមានដេរីវេលើចន្លោះបើក  $(a, b)$  និង  $f(a) = f(b)$  នោះមានចំនួនពិត  $c$  មួយយ៉ាងតិចនៅក្នុងចន្លោះបើក  $(a, b)$  ដែល  $f'(c) = 0$  ។



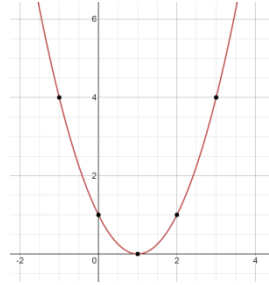
**ឧទាហរណ៍១៖** គេឱ្យ  $f(x) = x^2 - 2x + 1$  ។ បង្ហាញថាមានចំនួន  $c$  នៅលើចន្លោះ  $[-1, 3]$  ដែល  $f'(c) = 0$   
ចម្លើយ៖

$$f(-1) = 4 \quad (-1, 4)$$

$$f(3) = 4 \quad (3, 4)$$

$$f'(c) = 2c - 2 = 0$$

$$\Rightarrow c = 1$$



**ឧទាហរណ៍២៖** គេឱ្យ  $f(x) = x^2 - 5x + 3$  ។ បង្ហាញថាមានចំនួន  $c$  នៅលើចន្លោះ  $[0, 5]$  ដែល  $f'(c) = 0$

ចម្លើយ៖  $c = \frac{5}{2}$

**ឧទាហរណ៍៣៖** គេឱ្យ  $f(x) = \sin x$  ។ បង្ហាញថាមានចំនួន  $c$  នៅលើចន្លោះ  $[0, 2\pi]$  ដែល  $f'(c) = 0$  ។

ចម្លើយ៖  $c = \frac{\pi}{2}, c = \frac{3\pi}{2}$

**ឧទាហរណ៍៤៖** គេឱ្យ  $f$  ជាអនុគមន៍កំណត់ដោយ  $f(x) = x^2 - 3x + 2$  ។

ក. រកអាប់ស៊ីសនៃចំណុចប្រសព្វតាងដោយ  $x_1, x_2$  នៃក្រាបតាងអនុគមន៍ និងអ័ក្សអាប់ស៊ីស។

ខ. បង្ហាញថា មានចំនួនពិត  $c$  មួយនៅចន្លោះ  $(x_1, x_2)$  ដែល  $f'(c) = 0$  រួចគណនាតម្លៃ  $c$  នោះ។

ចម្លើយ៖

ក. រកអាប់ស៊ីសនៃចំណុចប្រសព្វតាងដោយ  $x_1, x_2$  នៃក្រាបតាងអនុគមន៍ និងអ័ក្សអាប់ស៊ីស

ឱ្យ  $f(x) = 0$  យើងបាន  $x^2 - 3x + 2 = 0$

មានមេគុណរាង  $a + b + c = 0$  នោះ  $x_1 = 1, x_2 = \frac{c}{a} = 2$

ដូចនេះ  $x_1 = 1, x_2 = 2$

ខ. បង្ហាញថា មានចំនួនពិត  $c$  មួយនៅចន្លោះ  $(x_1, x_2)$  ដែល  $f'(c) = 0$

ដោយ  $f(x_1) = f(1) = 0 = f(2) = f(x_2)$

តាមទ្រឹស្តីបទរ៉ូល មាន  $c$  មួយនៅចន្លោះ  $(x_1, x_2)$  ដែល  $f'(c) = 0$

- គណនាតម្លៃ  $c$

ដោយ  $f(x) = x^2 - 3x + 2$

គេបាន  $f'(x) = 2x - 3$

បើ  $f'(x) = 0 \Rightarrow x = \frac{3}{2} \in (1, 2)$

ដូចនេះ  $c = \frac{3}{2}$

**ឧទាហរណ៍៥៖**

ក. គេឱ្យ  $f(x) = x^2 - 2x$  ។ បញ្ជាក់អត្ថិភាព និង រកគ្រប់  $c \in [0, 2]$  ដែល  $f'(c) = 0$  ។

ខ. គេឱ្យ  $f(x) = (x-3)(x+1)^2$  ។ បញ្ជាក់អត្ថិភាព និង រកគ្រប់  $c \in [-1, 3]$  ដែល  $f'(c) = 0$  ។

ចម្លើយ៖

ក. បញ្ជាក់អត្ថិភាព និង រកគ្រប់  $c \in [0, 2]$  ដែល  $f'(c) = 0$

យើងមាន  $f(x) = x^2 - 2x \Rightarrow f(0) = 0$  និង  $f(2) = 2^2 - 2(2) = 0$

តាមទ្រឹស្តីបទរ៉ូល មាន  $c \in [0, 2]$  ដែល  $f'(c) = 0$

ហើយ  $f'(x) = 2x - 2$

$$f'(x) = 0 \Rightarrow x = 1 \in [0, 2]$$

ដូចនេះ  $c = 2$

$$ខ. c = -1, c = \frac{2}{3}$$

III. ទ្រឹស្តីបទតម្លៃមធ្យម (Mean Value Theorem)

បើ  $f$  ជាអនុគមន៍ជាប់លើចន្លោះបិទ  $[a, b]$  និងមានដេរីវេលើចន្លោះបើក  $(a, b)$  នោះមានចំនួនពិត

$$c \in (a, b) \text{ មួយយ៉ាងតិចដែល } f'(c) = \frac{f(b) - f(a)}{b - a} \text{ ។}$$

ឧទាហរណ៍១៖ គេឱ្យ  $f(x) = x^2 - 4x + 1$  ។ បង្ហាញថាមានចំនួន  $c$  មួយយ៉ាងតិចនៅលើចន្លោះ  $[1, 5]$  ។

ចម្លើយ៖

$$f(x) = x^2 - 4x + 1 = (x-2)^2 - 3$$

$$f'(x) = 2x - 4 = \frac{f(5) - f(1)}{5 - 1} = \frac{6 + 1}{4} = 2 \Rightarrow x = 2$$

ដូចនេះ  $c = 2$

ឧទាហរណ៍២៖ គេឱ្យ  $f(x) = x^{\frac{2}{3}}$  ។ បង្ហាញថាមានចំនួន  $c$  មួយយ៉ាងតិចនៅលើចន្លោះ  $[0, 1]$  ។

ចម្លើយ៖

$$f(x) = x^{\frac{2}{3}}$$

$$f'(x) = \frac{2}{3} x^{-\frac{1}{3}} = \frac{f(1) - f(0)}{1 - 0} = 1$$

$$\Rightarrow \sqrt[3]{x} = \frac{2}{3} \Rightarrow x = \frac{8}{27} \approx 0.296$$

$$\text{ដូចនេះ } c = \frac{8}{27} \approx 0.296$$

ឧទាហរណ៍៣៖ គេឱ្យ  $f(x) = \sqrt{x-4}$  ។ បង្ហាញថាមានចំនួន  $c$  មួយ នៅលើចន្លោះ  $[4, 8]$  ។

ចម្លើយ៖  $c = 5$

IV. ទ្រឹស្តីបទតម្លៃបំផុត (Extreme Value Theorem)

បើ  $f$  ជាអនុគមន៍ជាប់លើចន្លោះបិទ  $[a, b]$  មាន  $c_1, c_2 \in [a, b]$  នោះ  $f(c_1) \leq f(x) \leq f(c_2)$  គ្រប់  $x$  លើសពីនេះ បើ  $f$  មានដេរីវេ លើចន្លោះបើក  $(a, b)$  នោះចំនួនពិត  $c_1$  និង  $c_2$  កើតឡើងនៅ ខាងចុង នៃ  $x \in [a, b]$  ឬនៅត្រង់  $f'(x) = 0$  ។

ឧទាហរណ៍១៖ រកបរមាណៃអនុគមន៍  $f(x) = 3x^2 - 12x + 5, [1, 4]$  ។

ចម្លើយ៖

គេមាន  $f(x) = 3x^2 - 12x + 5, [1, 4]$

នោះ  $f'(x) = 6x - 12$

បើ  $6x - 12 = 0 \Rightarrow x = 3$

គេបាន

$$f(1) = -4$$

$$f(4) = 5$$

$$f(3) = -7$$

នោះ

$$f(1) = -4$$

$$f(4) = 5 \Rightarrow \max$$

$$f(3) = -7 \Rightarrow \min$$

ឧទាហរណ៍២៖ កំណត់បរមាណៃអនុគមន៍  $f(x) = 3x^4 - 4x^3, [-1, 2]$  ។

ឧទាហរណ៍៣៖ កំណត់បរមាណៃអនុគមន៍  $f(x) = 2x^3 - 3x^2 - 12x + 1; [-2, 3]$  ។

ឧទាហរណ៍៤៖ កំណត់បរមាណៃអនុគមន៍  $f(x) = \sin x + \cos x; \left[0, \frac{\pi}{3}\right]$  ។

ចម្លើយ  $f'(x) = 0 \Rightarrow \cos x - \sin x = 0 \Rightarrow \cos x = \sin x \Rightarrow x = \frac{\pi}{4}$

V. អាំងតេក្រាលរីម៉ាន (The Riemann integral)

បើអនុគមន៍  $f$  ជាប់នៅលើចន្លោះ  $[a, b]$  មានលីមីតដូចខាងក្រោម៖  $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$  ។

របៀបដោះស្រាយ៖

$$1. \text{ គណនា } \Delta x = \frac{b-a}{n}$$

$$2. \text{ រក } x_i = a + i\Delta x$$

$$3. \text{ ជំនួសរក } f(x_i)$$

$$4. \text{ រក } \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

ឧទាហរណ៍៖ ដោយប្រើអាំងតេក្រាលរីម៉ាន គណនាអាំងតេក្រាល៖

ក.  $\int_0^2 x dx$       ខ.  $\int_1^4 (x^2 - 4x + 2) dx$

ចម្លើយ

ក.  $\int_0^2 x dx$

1.  $\Delta x = \frac{2-0}{n} = \frac{2}{n}$

2.  $x_i = 0 + i \frac{2}{n}$

3.  $f(x) = x \Rightarrow f(x_i) = i \left( \frac{2}{n} \right)$

$\Rightarrow \int_0^2 x dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$

$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{2}{n} i \times \frac{2}{n}$

$= \lim_{n \rightarrow \infty} \frac{4}{n^2} \sum_{i=1}^n i$

$= \lim_{n \rightarrow \infty} \frac{4}{n^2} \times \frac{n(n+1)}{2}$

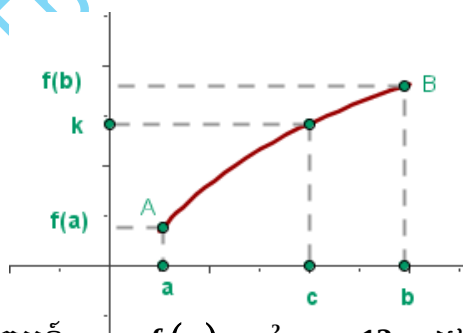
$= \lim_{n \rightarrow \infty} \frac{2(n+1)}{n} = 2$

ខ.  $\int_1^4 (x^2 - 4x + 2) dx$

ចម្លើយ -3

#### VI. ទ្រឹស្តីបទតម្លៃកណ្តាល

បើអនុគមន៍  $f$  ជាប់លើចន្លោះបិទ  $[a, b]$  និង  $k$  ជាចំនួនមួយនៅចន្លោះ  $f(a)$  និង  $f(b)$  នោះមានចំនួនពិត  $c$  មួយយ៉ាងតិចក្នុងចន្លោះបិទ  $[a, b]$  ដែល  $f(c) = k$  ។



ឧទាហរណ៍១៖ គេឱ្យអនុគមន៍  $f(x) = x^2 - x - 12$  ដោយប្រើទ្រឹស្តីបទតម្លៃកណ្តាលរកចំនួនពិត  $c$  នៅក្នុងចន្លោះ  $[3, 5]$  ដែល  $f(c) = 0$  ។

ចម្លើយ៖

$$f(3) = -6$$

$$f(5) = 8$$

$$f(c) = 0 \Leftrightarrow 0 = x^2 - x - 12$$

$$\Rightarrow x = 4, x = -3$$

តែ  $x = -3$  មិនយក

ដូចនេះ  $c = 4$

ឧទាហរណ៍២៖ គេឱ្យអនុគមន៍  $f(x) = 2x^2 + 3$  : ដោយប្រើទ្រឹស្តីបទតម្លៃកណ្តាលរកចំនួនពិត  $c$  នៅក្នុងចន្លោះ  $[1, 4]$  ដែល  $f(c) = 19$  ។

ចម្លើយ  $c = 2$

សម្គាល់៖ ទ្រឹស្តីបទតម្លៃកណ្តាលអាចប្រើបានចំពោះ  $k = 0$  ជាពិសេសបើអនុគមន៍  $f$  ជាប់លើចន្លោះបិទ  $[a, b]$  និង  $f(a)$  និង  $f(b)$  មានសញ្ញាផ្ទុយគ្នា នោះ  $f(a) \cdot f(b) < 0$  មានចំនួន  $c$  មួយយ៉ាងតិច ក្នុងចន្លោះបិទ  $[a, b]$  ដែល  $f(c) = 0$  ។

ឧទាហរណ៍១៖ ដោយប្រើទ្រឹស្តីបទតម្លៃកណ្តាល ចូរបង្ហាញថាចំពោះអនុគមន៍ពហុធា  $x^3 - 3x + 1 = 0$  មានចំនួនពិត  $c$  នៅក្នុងចន្លោះ  $[0, 1]$  ដែល  $f(c) = 0$  ។

ចម្លើយ៖

$$\text{ដោយ } f(0) = 1, f(1) = -1 \Rightarrow f(0) \cdot f(1) = (1)(-1) = -1 < 0$$

តាមទ្រឹស្តីបទតម្លៃកណ្តាលដោយ  $f(0) \cdot f(1) < 0$  មានចំនួនពិត  $c$  នៅក្នុងចន្លោះ  $[0, 1]$  ដែល  $f(c) = 0$

ឧទាហរណ៍២៖ ដោយប្រើទ្រឹស្តីបទតម្លៃកណ្តាល ចូរបង្ហាញថាចំពោះអនុគមន៍ពហុធា  $f(x) = x^3 + 4x - 2$  មានចំនួនពិត  $c$  នៅក្នុងចន្លោះ  $[0, 2]$  ដែល  $f(c) = 0$  ។

ឧទាហរណ៍៣៖ ដោយប្រើទ្រឹស្តីបទតម្លៃកណ្តាលចូររកចំនួនពិត  $c$  ចំពោះអនុគមន៍ពហុធា  $f(x) = x^2 + x - 1$  នៅក្នុងចន្លោះ  $[0, 5]$  ដែល  $f(c) = 11$  ។

ចម្លើយ៖

$$f(0) = -1$$

$$f(5) = 29$$

$$-1 < 11 < 29$$

$$-1 < 11 < 29$$

$$\text{នោះ: } 11 = x^2 + x - 1$$

$$x^2 + x - 12 = 0 \Rightarrow x = -4, x = 3$$

តែ  $x = -4$  មិនយក

ដូចនេះ  $c = 3, f(3) = 11$

ឧទាហរណ៍៤៖ ដោយប្រើទ្រឹស្តីបទតម្លៃកណ្តាល ចូររកចំនួនពិត  $c$  ចំពោះអនុគមន៍ពហុធា

$$f(x) = 2x^2 - 3x + 7 \text{ នៅក្នុងចន្លោះ } [0, 3] \text{ ដែល } f(c) = 9$$

ចម្លើយ៖  $c = 2$

VII. ទ្រឹស្តីបទ Taylor (Taylor's Theorem)

បើអនុគមន៍  $f$  ជាប់លើចន្លោះបិទ  $[a, b]$  ដែល  $f^{(n+1)}$  មានដេរីវេលើចន្លោះ  $[a, b]$  ហើយមាន  $x_0 \in [a, b]$  ។

គ្រប់តម្លៃ  $x \in [a, b]$  មានចំនួន  $\xi(x)$  មួយយ៉ាងតិចរវាង  $x_0$  និង  $x$  ដែល  $f(x) = P_n(x) + R_n(x)$

$$\begin{aligned} \text{ដែល} \quad P_n(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \\ &= \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \quad \text{ហៅថា ពហុធា Taylor} \end{aligned}$$

$$\text{ហើយ} \quad R_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)^{n+1} \quad \text{ហៅថា ល្អៀងខ្លី}$$

(called the remainder term or truncation error)

$$\text{បើ } N \rightarrow \infty \text{ នោះ: } f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n, \quad \text{ព្រោះ: } \lim_{n \rightarrow \infty} R_n = 0$$

ឧទាហរណ៍១៖ រកពហុធា Taylor ដឺក្រេទី៤ នៃអនុគមន៍  $f(x) = \ln x$  គ្រប់ចំនុច  $c = 1$  រួចទាញរកតម្លៃប្រហែលនៃ  $\ln(1.1)$  ។

ចម្លើយ៖

$$\begin{aligned} f(x) &= \ln x & \Rightarrow f(1) &= 0 \\ f'(x) &= \frac{1}{x} & \Rightarrow f'(1) &= 1 \\ f''(x) &= -\frac{1}{x^2} & \Rightarrow f''(1) &= -1 \\ f'''(x) &= \frac{2}{x^3} & \Rightarrow f'''(1) &= 2 \\ f^{(4)}(x) &= -\frac{6}{x^4} & \Rightarrow f^{(4)}(1) &= -6 \end{aligned}$$

គេបាន

$$\begin{aligned} P_4(x) &= 0 + 1(x-1) - \frac{1(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{6(x-1)^4}{4!} \\ P_4(x) &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 \\ P_4(1.1) &= 0.1 - \frac{1}{2}(0.1)^2 + \frac{1}{3}(0.1)^3 - \frac{1}{4}(0.1)^4 = 0.095308333 \approx 0.09531 \\ \ln(1.1) &= 0.0953101798 \end{aligned}$$

ឧទាហរណ៍២៖ រកពហុធា Maclaurin ដឺក្រេទី៤ នៃអនុគមន៍  $f(x) = e^x$  រួចទាញរកតម្លៃប្រហែលនៃ  $e^{0.2}$  ។

ចម្លើយ

$$P_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$P_4(0.2) = 1 + 0.2 + \frac{(0.2)^2}{2!} + \frac{(0.2)^3}{3!} + \frac{(0.2)^4}{4!} \approx \underline{1.2214}$$

$$e^{0.2} \approx 1.221402758$$

**ឧទាហរណ៍៣៖** រកស៊េរី Taylor នៃអនុគមន៍  $f(x) = \ln(x)$  ត្រង់  $c = 1$  ។

ចម្លើយ៖

$$f(x) = \ln x \quad \Rightarrow f(1) = 0$$

$$f'(x) = \frac{1}{x} \quad \Rightarrow f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2} \quad \Rightarrow f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3} \quad \Rightarrow f'''(1) = 2$$

$$f^{(4)}(x) = -\frac{6}{x^4} \quad \Rightarrow f^{(4)}(1) = -6$$

$$f(x) = f(c) + f'(c)(x-c)^1 + \frac{f''(c)(x-c)^2}{2!} + \dots$$

$$\ln x = 0 + 1(x-1)^1 + \frac{(-1)(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} + \frac{(-6)(x-1)^4}{4!} + \dots$$

$$\ln x = \frac{(x-1)^1}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$

$$\ln x = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1}$$

**ឧទាហរណ៍៤៖** រកស៊េរី Taylor នៃអនុគមន៍  $f(x) = e^x$  ត្រង់  $c = 3$  ។

ចម្លើយ៖  $e^x = \sum_{n=0}^{\infty} \frac{e^3 (x-3)^n}{n!}$

**ឧទាហរណ៍៥៖** រកស៊េរី Maclaurin នៃអនុគមន៍  $f(x) = \sin(x)$  ។

ចម្លើយ៖  $\sin(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1} (-1)^n}{(2n+1)!}$

**ឧទាហរណ៍៦៖** រកស៊េរី Maclaurin នៃអនុគមន៍  $f(x) = \cos(x)$  ដោយប្រើស៊េរី Maclaurin នៃអនុគមន៍  $\sin x$

ចម្លើយ៖

$$\begin{aligned}\frac{d}{dx}[\sin x] &= \frac{d}{dx}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right] \\ \cos x &= 1 - \frac{3x^2}{3 \cdot 2!} + \frac{5x^4}{5 \cdot 4!} - \frac{7x^6}{7 \cdot 6!} + \dots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ \cos x &= \sum_{n=0}^{\infty} \frac{x^{2n}(-1)^n}{(2n)!}\end{aligned}$$

ប្រើវិធីម្យ៉ាងទៀត

$$\begin{aligned}\frac{d}{dx}[\sin x] &= \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{x^{2n+1}(-1)^n}{(2n+1)!}\right] \\ \cos x &= \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}(-1)^n}{(2n+1) \times (2n)!} \\ &= \sum_{n=0}^{\infty} \frac{x^{2n}(-1)^n}{(2n)!}\end{aligned}$$

**ឧទាហរណ៍៧៖** រកស៊េរី Maclaurin នៃអនុគមន៍  $\cos(x^2)$  ។

ចម្លើយ៖ ដោយ 
$$\cos x = \sum_{n=0}^{\infty} \frac{x^{2n}(-1)^n}{(2n)!}$$

នោះ៖ 
$$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(x^2)^{2n}(-1)^n}{(2n)!} = \sum_{n=0}^{\infty} \frac{x^{4n}(-1)^n}{(2n)!}$$

**ឧទាហរណ៍៨៖** រកស៊េរី Maclaurin នៃអនុគមន៍  $x \cos(x)$  ។

ចម្លើយ៖ 
$$x \cos x = x \sum_{n=0}^{\infty} \frac{x^{2n}(-1)^n}{(2n)!} = \sum_{n=0}^{\infty} \frac{x^{2n+1}(-1)^n}{(2n)!}$$

**ឧទាហរណ៍៩៖** រកស៊េរី Maclaurin នៃអនុគមន៍  $\cos^2(x)$  ។

ចម្លើយ៖ 
$$\cos^2(x) = \frac{1}{2}[1 + \cos 2x] = \frac{1}{2}\left[1 + \sum_{n=0}^{\infty} \frac{x^{2n}(-1)^n}{(2n)!}\right]$$

មេរៀនទី២

លេខរៀង និង ទព្វន្ត  
(Numbers Errors and Arithmetic)

I. ការតាងចំនួន (Representation of Numbers)

យើងពិនិត្យមើលប្រភេទពីរនៃចំនួន គឺ ចំនួនគត់ និងចំនួនពិត។

**និយមន័យ៖** ចំនួនគត់  $N$  មានសញ្ញា  $+$  ឬ  $-$  ភ្ជាប់ពីក្រោយដោយខ្សែនៃខ្ទង់ទសភាគ

$N = \pm d_1 d_2 d_3 \dots d_m$  ,  $N = \pm d_1 d_2 d_3 \dots d_m$  គឺជាប្រវែងពាក្យ (Word Length) នៃ  $N$  ហើយស្មើនឹងចំនួននៃខ្ទង់នៅក្នុងខ្សែនោះ។

តាមការសន្មត ខ្ទង់ទីមួយ  $N = \pm d_1 d_2 d_3 \dots d_m$  ខុសពីសូន្យ។ ដូចជា យើងមិនសរសេរ  $N = \pm d_1 d_2 d_3 \dots d_m$  ដើម្បីមានន័យថា សាមសិបបួនទេ គឺយើងសរសេរ  $N = \pm d_1 d_2 d_3 \dots d_m$  ។ ដូចនេះ  $N = \pm d_1 d_2 d_3 \dots d_m$  ។

ខ្ទង់ដែលនៅសល់  $N = \pm d_1 d_2 d_3 \dots d_m$  អាចមានតម្លៃស្មើ  $N = \pm d_1 d_2 d_3 \dots d_m$  គឺថា  $N = \pm d_1 d_2 d_3 \dots d_m$  ។

ឧទាហរណ៍៖ 1205; -100 និង +34 ជាចំនួនគត់មានន័យចំនែក 009 និង -12,1 មិនមែនជាចំនួនគត់ទេ។

II. ប្រព័ន្ធបាម

1. ប្រព័ន្ធបាមគោលដប់

ជាទូទៅ  $n = abcd_{10} = a \times 10^3 + b \times 10^2 + c \times 10 + d$

ឧទាហរណ៍:  $1254 = 1000 + 200 + 50 + 4 = 10^3 + 2 \times 10^2 + 5 \times 10 + 4$  (ជាទម្រង់ពន្លាតនៃចំនួន 1254)។

**ប្រតិបត្តិ៖** សរសេរទម្រង់ពន្លាតខាងក្រោមនេះជាចំនួនក្នុងប្រព័ន្ធបាមគោល 10:  $3 \times 10^5 + 5 \times 10^4 + 7 \times 10$

2. ប្រព័ន្ធបាមគោល 2

ជាទូទៅ  $n = abcd_2 = a \times 2^3 + b \times 2^2 + c \times 2 + d$

ឧទាហរណ៍: ចូរបំប្លែងចំនួន  $100110_2$  ខាងក្រោមជាប្រព័ន្ធបាមគោល 10

គេបាន  $100110_2 = 1 \times 2^5 + 0 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 1 \times 2 + 0 = 32 + 0 + 0 + 4 + 2 = 38$

**ប្រតិបត្តិ៖** ចូរបំប្លែងចំនួនខាងក្រោមទៅជាចំនួនក្នុងប្រព័ន្ធបាមគោល 10

ក-100001101<sub>2</sub> ខ-11000101<sub>2</sub> គ-101010101<sub>2</sub>

**ឧទាហរណ៍៖** ចូរបំប្លែងចំនួនខាងក្រោមពីគោល 10 ទៅគោល 2

ក. 48                      ខ. 129                      គ. 432

ចម្លើយ

ក. 48

48 = 2 × 24 + 0  
24 = 2 × 12 + 0  
12 = 2 × 6 + 0  
6 = 2 × 3 + 0  
3 = 2 × 1 + 0

ដូចនេះ: 48 = 100000<sub>2</sub>

i. បម្លែងចំនួនទសភាគក្នុងប្រព័ន្ធគោល

១. បម្លែងចំនួនទសភាគខាងក្រោមពីប្រព័ន្ធគោល២ ទៅប្រព័ន្ធគោល ១០៖

ក.  $101.1101_2$

ខ.  $1101.0111_2$

គ.  $111.111_2$

ឃ.  $101.01011_2$

ដំណោះស្រាយ

១. បម្លែងចំនួនទសភាគខាងក្រោមពីប្រព័ន្ធគោល២ ទៅប្រព័ន្ធគោល ១០៖

ក.  $101.1101_2$

$$\begin{aligned} 101.1101_2 &= (1 \times 2^2) + (0 \times 2^1) + (1 \times 2^0) + (1 \times 2^{-1}) + (1 \times 2^{-2}) + (0 \times 2^{-3}) + (1 \times 2^{-4}) \\ &= 4 + 0 + 1 + \frac{1}{2} + \frac{1}{2^2} + 0 + \frac{1}{2^4} \\ &= 5 + 0.5 + 0.25 + 0.0625 \\ &= 5.8125_{10} \end{aligned}$$

ខ.  $1101.0111_2$

$$\begin{aligned} 1101.0111_2 &= (1 \times 2^3) + (1 \times 2^2) + (0 \times 2) + (1 \times 2^0) + (0 \times 2^{-1}) + (1 \times 2^{-2}) + (1 \times 2^{-3}) + (1 \times 2^{-4}) \\ &= 8 + 4 + 0 + 1 + 0 + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \\ &= 13 + 0.25 + 0.125 + 0.0625 = 13.4375_{10} \end{aligned}$$

គ.  $111.111_2$  និង ឃ.  $101.01011_2$  (អនុវត្តខ្លួនឯង)

២. បម្លែងចំនួនខាងក្រោមពីប្រព័ន្ធគោល១០ ទៅប្រព័ន្ធគោល ២៖

ក.  $13.6875_{10}$

ខ.  $123.15_{10}$

គ.  $32.45_{10}$

ឃ.  $58.3125_{10}$

ង.  $7.0202_{10}$

ច.  $28.555_{10}$

ដំណោះស្រាយ

២. បម្លែងចំនួនខាងក្រោមពីប្រព័ន្ធគោល១០ ទៅប្រព័ន្ធគោល ២៖

ក.  $13.6875_{10}$

$$\begin{aligned} 13 &= 2 \times 6 + 1 \\ 6 &= 2 \times 3 + 0 \\ 3 &= 2 \times 1 + 1 \\ 13_{10} &= 1101_2 \end{aligned}$$

$$\begin{aligned} 0.6875 \times 2 &= 1.375 \\ 0.375 \times 2 &= 0.75 \\ 0.75 \times 2 &= 1.50 \\ 0.50 \times 2 &= 1.00 \\ 0.6875_{10} &= 0.1011_2 \end{aligned}$$

ដូចនេះ  $13.6875_{10} = 1101.1011_2$

ខ.  $123.15_{10}$

$$\begin{aligned} 123 &= 2 \times 61 + 1 \\ 61 &= 2 \times 30 + 1 \\ 30 &= 2 \times 15 + 0 \\ 15 &= 2 \times 7 + 1 \\ 7 &= 2 \times 3 + 1 \\ 3 &= 2 \times 1 + 1 \end{aligned}$$

$$123_{10} = 1111011_2$$

$$\begin{aligned} 0.15 \times 2 &= 0.30 \\ 0.30 \times 2 &= 0.6 \\ 0.6 \times 2 &= 1.2 \\ 0.2 \times 2 &= 0.4 \\ 0.4 \times 2 &= 0.8 \\ 0.8 \times 2 &= 1.6 \\ 0.6 \times 2 &= 1.2 \\ 0.2 \times 2 &= 0.4 \\ 0.4 \times 2 &= 0.8 \end{aligned}$$

$$0.15_{10} = 0.00100110011..._2$$

ដូចនេះ:  $123.15_{10} = 1111011.00100110011..._2$

គ.  $32.45_{10}$

$$\begin{aligned} 32 &= 2 \times 16 + 0 \\ 16 &= 2 \times 8 + 0 \\ 8 &= 2 \times 4 + 0 \\ 4 &= 2 \times 2 + 0 \\ 2 &= 2 \times 1 + 0 \end{aligned}$$

$$32_{10} = 100000_2$$

$$\begin{aligned} 0.45 \times 2 &= 0.9 \\ 0.9 \times 2 &= 1.8 \\ 0.8 \times 2 &= 1.6 \\ 0.6 \times 2 &= 1.2 \\ 0.2 \times 2 &= 0.4 \\ 0.4 \times 2 &= 0.8 \\ 0.8 \times 2 &= 1.6 \\ 0.6 \times 2 &= 1.2 \end{aligned}$$

$$0.45_{10} = 0.0111001100110011..._2$$

ដូចនេះ:  $32.45_{10} = 100000.0111001100110011..._2$

ឃ.  $58.3125_{10}$

$$\begin{aligned} 58 &= 2 \times 29 + 0 \\ 29 &= 2 \times 14 + 1 \\ 14 &= 2 \times 7 + 0 \\ 7 &= 2 \times 3 + 1 \\ 3 &= 2 \times 1 + 1 \end{aligned}$$

$$\begin{aligned} 0.3125 \times 2 &= 0.625 \\ 0.625 \times 2 &= 1.25 \\ 0.25 \times 2 &= 0.5 \\ 0.5 \times 2 &= 1.00 \end{aligned}$$

ដូចនេះ:  $58.3125_{10} = 111010.0101_2$

ង.  $7.0202_{10}$  ច.  $28.555_{10}$  (អនុវត្តខ្លួនឯង)

ឧទាហរណ៍៖ ឧទាហរណ៍អំពីចំនួនពិត ចំនុចនឹងមានន័យគឺ

| ចំនួន   | ទំហំ              | ចំនួន | ទំហំ              |
|---------|-------------------|-------|-------------------|
| 31,02   | ខ្ទង់ដប់          | 0,93  | ខ្ទង់មួយចែកនឹងដប់ |
| -2,6147 | ឯកតា              | 429,  | ខ្ទង់រយ           |
| 0,97    | ខ្ទង់មួយចែកនឹងដប់ | 0,003 | ខ្ទង់1 ចែកនឹង1000 |

ឧទាហរណ៍អំពីចំនួនពិត ចំនុចនឹងដែលគ្មានន័យគឺ

| ចំនួន  | ហេតុផល              | ទម្រង់មានន័យ |
|--------|---------------------|--------------|
| 031,02 | សូន្យនាំមុខផ្នែកគត់ | 31,02        |
| 429    | គ្មានចំនុចទសភាគ     | 429,         |

**សង្កេត៖** ផ្នែកគត់ ឬផ្នែកប្រភាគ(តែមិនទាំងពីរ) អាចលប់ចោលបាន (ដូចជា 0,93 និង 429,)។ តែទោះជាយ៉ាងណាក៏ដោយ 429,មិនចាំបាច់ដូច 429,0 ហើយផ្នែកប្រភាគអាចមានសូន្យនាំមុខ (ដូចជា 0,003)។

- **និយមន័យ៖** ចំនួនពិតចំនុចអណ្តែត (Floating-point real number)  $x$  មានសញ្ញា  $+$  ឬ  $-$  , ចំនុចទសភាគ និងចំនួនគត់គ្មានសញ្ញាគុណដោយស្វ័យគុណនៃគោល  $x = \pm d_1 d_2 d_3 \dots d_m \times 10^n$  ។

ចំនួនគត់  $n$  មានសញ្ញាគឺជានិទស្សន្ត ហើយ  $d_i \in \{0,1,2,3,4,5,6,7,8,9\}, i=1, \dots, m$  ដោយ  $d_1 \neq 0$  ។ ឥឡូវនេះវាជាតម្លៃនៃនិទស្សន្ត ដែលធ្វើឱ្យទំហំនៃចំនួនមិនប្រែប្រួល ហើយទម្រង់នេះ (នៅក្នុងគោលពីរ) ត្រូវប្រើដោយកុំព្យូទ័រដើម្បីតាងចំនួនពិត  $x = \pm b_1 b_2 b_3 \dots b_m \times 2^n$  ។

ឧទាហរណ៍៖ ឧទាហរណ៍ អំពីចំនួនពិត ចំនុចអណ្តែតមានន័យគឺ

| ចំនួន                 | ទំហំ              | ចំនួន               | ទំហំ                |
|-----------------------|-------------------|---------------------|---------------------|
| $.3102 \times 10^2$   | ខ្ទង់ដប់          | $.93 \times 10^0$   | ខ្ទង់ 1 ចែកនឹង 100  |
| $-.26147 \times 10^1$ | ខ្ទង់ឯកតា         | $.429 \times 10^3$  | ខ្ទង់រយ             |
| $.93 \times 10^0$     | ខ្ទង់ 1 ចែកនឹង 10 | $.3 \times 10^{-2}$ | ខ្ទង់ 1 ចែកនឹង 1000 |

ឧទាហរណ៍អំពីចំនួនពិតចំនុចអណ្តែតគ្មានន័យ គឺ

| ចំនួន                 | ហេតុផល                               | ទម្រង់មានន័យ        |
|-----------------------|--------------------------------------|---------------------|
| $0.03102 \times 10^3$ | សូន្យនាំមុខ                          | $.3102 \times 10^2$ |
| $42,9 \times 10^1$    | ចំនុចទសភាគមិនស្ថិតនៅក្នុងទីតាំងទីមួយ | $.429 \times 10^3$  |

### III. រង្វង់ដាច់ខាត និង រង្វង់ធៀប (Absolute Error and Relative Error)

រង្វង់ដាច់ខាតនៃតម្លៃ  $y = y \pm \Delta y$  ឬ  $y \pm \delta y$

$$\text{រង្វង់ធៀប} = \frac{\Delta y}{y} \times 100\%$$

ឧទាហរណ៍៖ ចំពោះ  $0.94 \pm 0.2s$  គេបាន  $0.2s$  ជារង្វង់ដាច់ខាត

ឧទាហរណ៍២៖ គណនាល្បឿនដាច់ខាតនៃ  $0.0055kg \pm 2\%$  ។

ចម្លើយ៖

$$\text{ដោយ } 0.0055kg \times 0.02 = \pm 0.00011kg$$

$$\text{គេបានល្បឿនដាច់ខាត} = 0.0055 \pm 0.00011kg$$

ឧទាហរណ៍៣៖ គណនាល្បឿនធៀបនៃ  $0.94 \pm 0.2s$

$$\text{ចម្លើយ៖ ល្បឿនធៀបគឺ } \frac{0.2s}{0.94s} = 0.21 \times 100\% = 21\%$$

ឧទាហរណ៍៤៖ គណនាល្បឿនធៀបនៃ  $100.0 \pm 0.6cm$  ។

$$\text{ចម្លើយ៖ ល្បឿនធៀប} = \frac{0.6cm}{100.0cm} = 0.006 \times 100\% = 0.6\%$$

#### IV. កំហុសក្នុងការគណនា (Errors in Computation)

##### 1. ការបង្កប់ចំនួន (Rounding Number)

ឧទាហរណ៍៖ ចូរបង្កប់ចំនួនខាងក្រោម៖

១. 53,711 ទៅជា1ខ្ទង់មានន័យ

២. 4,642,069 ទៅជា1ខ្ទង់មានន័យ

៣. 5.47651 ទៅជា1ខ្ទង់មានន័យ

៤. 0.0045902 ទៅជា3ខ្ទង់មានន័យ

៥. 30 895 ទៅជា2ខ្ទង់មានន័យ

##### 2. ខ្ទង់សំខាន់ (Significant digits)

ឧទាហរណ៍១៖ ចំនួនទសភាគខាងក្រោមតើមានខ្ទង់សំខាន់ប៉ុន្មាន?

|         |   |              |                       |   |   |
|---------|---|--------------|-----------------------|---|---|
| 1.23    | → | ៤ខ្ទង់សំខាន់ | 800.                  | → | 3 |
| 0.00545 | → | ៣ខ្ទង់       | 800.0                 | → | 4 |
| 404     | → | 3            | 50060.                | → | 5 |
| 5006    | → | 4            | 50060.00              | → | 7 |
| 70,080  | → | 4            | 0.0153                | → | 3 |
| 8000    | → | 1            | 0.001008              | → | 4 |
| 800     | → | 1            | 0.00010700            | → | 5 |
|         |   |              | $2.53 \times 10^4$    | → | 3 |
|         |   |              | $3.06 \times 10^{-5}$ | → | 3 |
|         |   |              | $1.000 \times 10^8$   | → | 4 |
|         |   |              | $4.20 \times 10^{-7}$ | → | 3 |

ការបង្កប់ចំនួន

|          |          |                      |          |          |                     |
|----------|----------|----------------------|----------|----------|---------------------|
| 4257.16  | <u>1</u> | 4000                 | 158.1054 | <u>1</u> | 200                 |
|          | <u>2</u> | 4300                 |          | <u>2</u> | 160                 |
|          | <u>3</u> | 4260                 |          | <u>3</u> | 158                 |
|          | <u>4</u> | 4257                 |          | <u>4</u> | 158.1               |
|          | <u>5</u> | 4257.2               |          | <u>5</u> | 158.11              |
| 0.035047 | <u>1</u> | 0.04                 | 50,000   | <u>1</u> | $5 \times 10^4$     |
|          | <u>2</u> | 0.035                |          | <u>2</u> | $5.0 \times 10^4$   |
|          | <u>3</u> | 0.0350               |          | <u>3</u> | $5.00 \times 10^4$  |
|          | <u>4</u> | 0.03505              |          | <u>4</u> | $5.000 \times 10^4$ |
| 100      | →        | $1.00 \times 10^2$   |          |          |                     |
|          | <u>2</u> | $1.0 \times 10^2$    |          |          |                     |
| 3650000  | <u>1</u> | 4000000              |          |          |                     |
|          | <u>2</u> | 3700000              |          |          |                     |
|          | <u>3</u> | 3650000              |          |          |                     |
|          | <u>4</u> | $3.650 \times 10^6$  |          |          |                     |
|          | <u>5</u> | $3.6500 \times 10^6$ |          |          |                     |

V. ទព្វកុំព្យូទ័រ (Computer Arithmetic)

1. ប្រមាណវិធី (Operation)

ក. គណនាផលបូក

$$\begin{array}{r} 2.314 \\ + 5.23 \\ \hline 7.544 \approx 7.54 \end{array}$$

- a)  $3.45 + 5.3 \approx 8.8$
- b)  $3.65 + 14.1 + 8.136 \approx 25.9$
- c)  $530. + 4.63 \approx 535$
- d)  $8.2354 + 9.35 \approx 17.59$
- e)  $36500 + 143.56 \approx 36600$
- f)  $350000 + 57000.1 + 42.68 \approx 410000$
- g)  $72000 + 4300 + 160000 \approx 240000$

ខ. គណនាផលដក

- a)  $4.671 - 2.1 \approx 2.6$
- b)  $7.463 - 3.58 \approx 3.88$
- c)  $200 - 28.14 \approx 200$
- d)  $5000 - 62.413 \approx 5000 = 5 \times 10^3$

គ. គណនាផលគុណ

- a)  $9.6 \times 7 = 67.2 \approx 70 = 7 \times 10^1$   
 b)  $325 \times 75 = 24,375 \approx 24000 = 2.4 \times 10^4$   
 c)  $4.38 \times 2.3 = 10.074 \approx 10. = 1.0 \times 10^1$   
 d)  $8.315 \times 17.368 = 144.41492 \approx 144.4$   
 e)  $4000 \times 8.17 = 32680 \approx 30,00 = 3 \times 10^4$

ឃ. គណនាផលធៀប

- a)  $34.7 / 3.1 = 11.193548 \approx 11.$   
 b)  $536.3172 / 13.2 = 40.63009091 \approx 40.6$   
 c)  $250,000 / 3.176 = 78715.365 \approx 79000 = 7.9 \times 10^4$

ង. ប្រមាណវិធីចម្រុះ

- a)  $4.3 \times 5.231 + 6.814 = 22.4933 + 6.814 = 29.3073$   
 b)  $7.35 \times 4.265 + 7.34 = 31.34775 + 7.34 = 38.68775 \approx 38.7$   
 c)  $(8.431 + 9.25 + 12.6) / 4.7 \approx 6.4$   
 d)  $(5.312 \times 2.86) / (19.3 - 17.2) \approx 7.2$

## 2. ប្រព័ន្ធចំនួន (Radix Number Systems)

ឧទាហរណ៍៖ ប្រព័ន្ធចំនួន 2001 ជាចំនួនក្នុងប្រព័ន្ធគោល 2 គោល 8 គោល 10 និង គោល 16 ។

## 3. ការប្តូរចំនួន (Conversion from One Radix to Another)

| Decimal | Binary | Octal | Hex |
|---------|--------|-------|-----|
| 0       | 0      | 0     | 0   |
| 1       | 1      | 1     | 1   |
| 2       | 10     | 2     | 2   |
| 3       | 11     | 3     | 3   |
| 4       | 100    | 4     | 4   |
| 5       | 101    | 5     | 5   |
| 6       | 110    | 6     | 6   |
| 7       | 111    | 7     | 7   |
| 8       | 1000   | 10    | 8   |
| 9       | 1001   | 11    | 9   |
| 10      | 1010   | 12    | A   |
| 11      | 1011   | 13    | B   |
| 12      | 1100   | 14    | C   |
| 13      | 1101   | 15    | D   |
| 14      | 1110   | 16    | E   |

ឧទាហរណ៍

|         |        |       |     |
|---------|--------|-------|-----|
| Decimal | Binary | Octal | Hex |
|---------|--------|-------|-----|

|         |              |       |     |
|---------|--------------|-------|-----|
| 15      | 1111         | 17    | F   |
| 16      | 10000        | 20    | 10  |
| 20      | 10100        | 24    | 14  |
| 30      | 11110        | 36    | 1E  |
| 40      | 101000       | 50    | 28  |
| 50      | 110010       | 62    | 32  |
| 60      | 111100       | 74    | 3C  |
| 70      | 1000110      | 106   | 46  |
| 80      | 1010000      | 120   | 50  |
| Decimal | Binary       | Octal | Hex |
| 90      | 1011010      | 132   | 5A  |
| 100     | 11001000     | 144   | 64  |
| 1000    | 1111101000   | 1750  | 3E8 |
| 2989    | 101110101101 | 5655  | BAD |

ឧទាហរណ៍២៖ បម្លែងចំនួនក្នុងប្រព័ន្ធគោល៨ទៅគោល២និងគោល១៦៖

ក. Binary 0111101110100011.101111000100

Hexadecimal 7BA3.BC4

Octal 75643.704

ខ. Binary 0001100101001000.101101100

Hexadecimal 1948.B6

Octal 14510.554

ឧទាហរណ៍៣៖ បម្លែងចំនួន  $1492_{10}$  ទៅប្រព័ន្ធគោល២។

ឧទាហរណ៍៤៖ គណនា

a)  $(10111010)_2 + (110111)_2$

b)  $(101100)_2 \times (1011)_2$

c)  $(1010)_2 - (1)_2$

d)  $(110010)_2 - (101)_2$

e)  $(1101)_2 - (100101)_2$

f)  $(100101)_2 / (101)_2$

## How to Use the Bisection Algorithm

### How to Use the Bisection Algorithm

#### Quick Overview

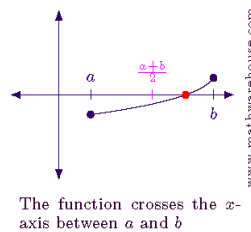
- The bisection algorithm approximates the location of an  $x$ -intercept (a root) of a continuous function.
- The bisection algorithm depends on the Intermediate Value Theorem.
- The algorithm is *iterative*. This means that the result from using it once will help us get a better result when we use the algorithm a second time.

#### Basic Idea

Suppose  $f(x)$  is continuous over  $[a, b]$ , and  $f(a)$  and  $f(b)$  have opposite signs (see the image below). Then, the Intermediate Value Theorem tells us that the function will achieve every value between  $f(a)$  and  $f(b)$  at least once somewhere in  $[a, b]$ .

Since  $f(a)$  and  $f(b)$  have opposite signs, then we know  $0$  is somewhere in-between. So the IVT guarantees that somewhere in  $[a, b]$  the function will equal  $0$  (again, see the image below).

We approximate the location of the root by finding the midpoint of the interval at  $x = \frac{a+b}{2}$  (see image below).



#### The Algorithm

Suppose  $f(x)$  is continuous over  $[a, b]$  and the function values at the end-points have different signs.

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1. Find the midpoint of  $[a, b]$ . Call it  $x_1$ .
  - (a) If  $f(x_1) = 0$ , we're done.
  - (b) If not, then  $x_1$  is our first approximation to the root of the function. It has a maximum error of  $\frac{1}{2}$  the length of the interval.
2. Find a smaller interval where  $f(x)$  has opposite signs at the endpoints. This new interval will either be  $[a, x_1]$ , or  $[x_1, b]$ .
3. Using this smaller interval, repeat Steps 1 and 2 until the error is small enough.

#### Example 1

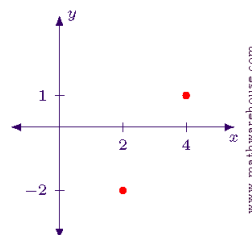
Using the Bisection Algorithm, find three approximations of the root of  $f(x) = \frac{1}{4}x^2 - 3$ . Determine the maximum error possible in using each approximation.

#### Solution

Step 1) Verify the Bisection Algorithm can be used.

We first note that the function is continuous everywhere on its domain.

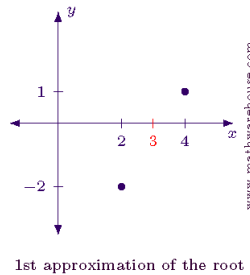
Next, we pick an interval to work with. If we pick  $x = 2$ , we see that  $f(2) = -2 < 0$  and if we pick  $x = 4$  we see  $f(4) = 1 > 0$ . So we can start with the interval  $[2, 4]$ .



1st interval for Example 1

Step 2) Find the first approximation to the root and its associated error.

The first approximation to the root is the midpoint of our starting interval. In this case, the midpoint of  $[2, 4]$  is at  $x = 3$ .

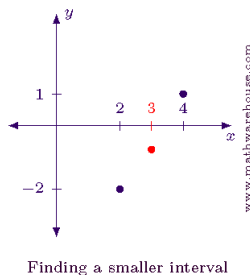


Maximum Error: Since the root has to be between  $x = 2$  and  $x = 4$ , using  $x = 3$  as an approximation for the root means the farthest away the root could possibly be is a distance of  $\pm 1$  unit (the plus/minus is because our approximation could be too big or too small).

In general, the maximum error in using a particular approximation is half the interval length.

Step 3) Use the midpoint to find a smaller interval so we can improve our approximation.

Notice that  $f(3) = \frac{1}{4}(3)^2 - 3 = -\frac{3}{4} < 0$ . Updating our graph, we now have three points on it.



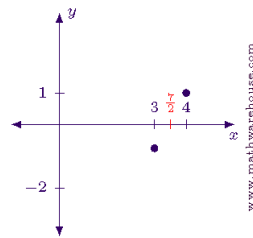
Examining this graph, we see that the root must lie between  $x = 3$  and  $x = 4$ . Consequently, this is our new interval.

Step 4) Find the second approximation and its associated error.

The midpoint of the interval  $[3, 4]$  is at  $x = \frac{7}{2}$ . This is the second approximation.

Max Error: Our interval has a length of 1 unit, so the maximum possible error in using  $x = \frac{7}{2}$  as an approximation will be  $\pm 1/2$  of a unit.

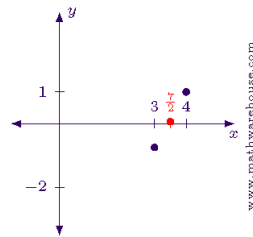
Our graph now looks like this:



The second approximation

Step 5) Find the next interval.

We note that  $f(\frac{7}{2}) = \frac{1}{16} > 0$ . Plotting this on our graph we see the following.



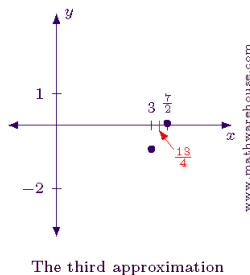
Finding a smaller interval

Examining this graph, we see that the root must lie between  $x = 3$  and  $x = \frac{7}{2}$ . This becomes our next interval.

Step 6) Find the third approximation and its associated error.

The midpoint of the interval  $[3, \frac{7}{2}]$  is at  $x = \frac{13}{4}$ , as shown on the graph below. This is the next approximation.

Max Error: The interval has a length of  $1/2$ , so the maximum possible error is  $\pm 1/4$  of a unit.



Answer The table below summarizes the approximations we found and their associated errors.

| Approximation   | $x$ -value         | Possible Error    |
|-----------------|--------------------|-------------------|
| 1 <sup>st</sup> | $x = 3$            | $\pm 1$           |
| 2 <sup>nd</sup> | $x = \frac{7}{2}$  | $\pm \frac{1}{2}$ |
| 3 <sup>rd</sup> | $x = \frac{13}{4}$ | $\pm \frac{1}{4}$ |

### Example 2

Use the bisection algorithm to approximate the solution to the equation below to within less than 0.1 of its real value. Assume  $x$  is in radians.

$$\sin x = 6 - x$$

### Solution

Step 1) Rewrite the equation so it is equal to 0.

$$x - 6 + \sin x = 0$$

The function we'll work with is  $f(x) = x - 6 + \sin x$ . Notice that the function is continuous everywhere.

Step 2) Find an initial interval to work with.

Setting up a table of values, we see the following.

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| $x$ | $f(x)$              |
|-----|---------------------|
| 0   | $f(0) = -6$         |
| 1   | $f(1) \approx -4.2$ |
| 2   | $f(2) \approx -3.1$ |
| 3   | $f(3) \approx -2.9$ |
| 4   | $f(4) \approx -2.8$ |
| 5   | $f(5) \approx -2$   |
| 6   | $f(6) \approx -0.3$ |
| 7   | $f(7) \approx 1.7$  |

The first time we see a positive function value is at  $x = 7$ . So, we can use  $[6, 7]$  as the initial interval.

Step 3) Find the first approximation and its associated error.

| Interval | Midpoint | Max Error |
|----------|----------|-----------|
| $[6, 7]$ | 6.5      | $\pm 0.5$ |

1st Approximation:  $x = 6.5$

Associated error:  $\pm 0.5$  units.

Step 4) Find the 2nd interval. Use this new interval to determine the 2nd approximation. The first line of the table is included for completeness. The new work is on the second line.

| Finding the New Interval |                       |                     |              | Next Approximation |            |
|--------------------------|-----------------------|---------------------|--------------|--------------------|------------|
| $f(\text{left})$         | $f(\text{mid})$       | $f(\text{right})$   | New Interval | Midpoint           | Max Error  |
| Starting Interval:       |                       |                     | $[6, 7]$     | 6.5                | $\pm 0.5$  |
| $f(6) \approx -0.28$     | $f(6.5) \approx 0.72$ | $f(7) \approx 1.66$ | $[6, 6.5]$   | 6.25               | $\pm 0.25$ |

2nd Interval:  $[6, 6.5]$

2nd Approximation:  $x = 6.25$

Associated Error:  $\pm 0.25$  units.

Step 5) Repeat Step 4 until the associated error is less than 0.1 units.

Finding the 3rd Approximation

| Finding the New Interval |                        |                       |              | Next Approximation |             |
|--------------------------|------------------------|-----------------------|--------------|--------------------|-------------|
| $f(\text{left})$         | $f(\text{mid})$        | $f(\text{right})$     | New Interval | Midpoint           | Max Error   |
| Starting Interval:       |                        |                       | $[6, 7]$     | 6.5                | $\pm 0.5$   |
| $f(6) \approx -0.28$     | $f(6.5) \approx 0.72$  | $f(7) \approx 1.66$   | $[6, 6.5]$   | 6.25               | $\pm 0.25$  |
| $f(6) \approx -0.28$     | $f(6.25) \approx 0.22$ | $f(6.5) \approx 0.72$ | $[6, 6.25]$  | 6.125              | $\pm 0.125$ |

3rd Interval:  $[6, 6.25]$

3rd Approximation:  $[6, 6.125]$

Associated Error:  $\pm 0.125$  units.

Finding the 4th Approximation

| Finding the New Interval |                          |                        |                 | Next Approximation |              |
|--------------------------|--------------------------|------------------------|-----------------|--------------------|--------------|
| $f(\text{left})$         | $f(\text{mid})$          | $f(\text{right})$      | New Interval    | Midpoint           | Max Error    |
| Starting Interval:       |                          |                        | $[6, 7]$        | 6.5                | $\pm 0.5$    |
| $f(6) \approx -0.28$     | $f(6.5) \approx 0.72$    | $f(7) \approx 1.66$    | $[6, 6.5]$      | 6.25               | $\pm 0.25$   |
| $f(6) \approx -0.28$     | $f(6.25) \approx 0.22$   | $f(6.5) \approx 0.72$  | $[6, 6.25]$     | 6.125              | $\pm 0.125$  |
| $f(6) \approx -0.28$     | $f(6.125) \approx -0.03$ | $f(6.25) \approx 0.22$ | $[6.125, 6.25]$ | 6.1875             | $\pm 0.0625$ |

4th Interval:  $[6, 6.25]$

4th Approximation:  $[6, 6.125]$

Associated Error:  $\pm 0.125$  units.

Answer The solution to the equation is approximately 6.1875. This approximation is accurate to within  $\pm 0.0625$  units.

Example 3

Find the third approximation from the bisection algorithm to approximate the value of  $\sqrt[3]{2}$ .

Solution

Step 1) Find (make) a non-linear function with a root at  $\sqrt[3]{2}$ .

We are interested in knowing the approximate value of  $x = \sqrt[3]{2}$ . We use this equation to build a non-linear function with a root at the appropriate value.

$$\begin{aligned}x &= \sqrt[3]{2} \\x^3 &= 2 \\x^3 - 2 &= 0\end{aligned}$$

We'll use the function  $f(x) = x^3 - 2$ . Notice that the function is continuous everywhere.

Step 2) Find a starting interval.

We notice that at  $x = 0$ , the function is negative, and at  $x = 2$  the function is positive. We'll use  $[0, 2]$  as our starting interval.

Step 3) Setup and work through the table as in the previous example. The approximations are in blue. The new intervals are in red.

| Finding the New Interval |                      |                   |              | Next Approximation |            |
|--------------------------|----------------------|-------------------|--------------|--------------------|------------|
| $f(\text{left})$         | $f(\text{mid})$      | $f(\text{right})$ | New Interval | Midpoint           | Max Error  |
| Starting Interval:       |                      |                   | $[0, 2]$     | 1                  | $\pm 1$    |
| $f(0) = -2$              | $f(1) = -1$          | $f(2) = 6$        | $[1, 2]$     | 1.5                | $\pm 0.5$  |
| $f(1) = -1$              | $f(1.5) \approx 1.4$ | $f(2) = 6$        | $[1, 1.5]$   | 1.25               | $\pm 0.25$ |

Answer  $\sqrt[3]{2} \approx 1.25$  with a possible error of  $\pm 0.25$ . (Even with only 3 approximations, we're pretty close! The calculator tells us  $\sqrt[3]{2} \approx 1.25992$ )

### Accuracy and Iterations

We know the first approximation is within  $0.5(b - a)$  of the actual value of the root. Similarly,

the second approximation is within  $0.5^2(b - a)$  of the actual value, and

the third approximation is within  $0.5^3(b - a)$  of the actual value.

the fourth approximation is within  $0.5^4(b - a)$  of the actual value.

In general, the  $n^{\text{th}}$  approximation will be within  $0.5^n(b - a)$  of the actual value. This allows us to determine ahead of time how many iterations are needed to achieve a desired degree of accuracy, as in the following example.

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Example 4

The function  $f(x) = x^4 - 5$  has a positive root that is less than 3. If we started the bisection algorithm with the interval  $[0, 3]$ , how many iterations would it take before our approximation is within  $10^{-4}$  of the actual value?

Solution

Step 1) Solve  $0.5^n(b - a) = 0.01$  for  $n$  when  $a = 0$  and  $b = 3$ .

$$0.5^n(3 - 0) = 10^{-4}$$

$$0.5^n(3) = 10^{-4}$$

$$0.5^n = \frac{10^{-4}}{3}$$

$$n \ln(0.5) = \ln\left(\frac{1}{30000}\right)$$

$$n \ln 2 = -\ln(30000)$$

$$n = \frac{-\ln 30000}{\ln 2}$$

$$\approx 14.87$$

We would need at least 15 iterations to ensure the accuracy desired.

### Questions

Question 1 Find the 4th approximation of the positive root of the function  $f(x) = x^4 - 7$  using the bisection algorithm.

Question 2 Find the third approximation of the root of the function  $f(x) = \frac{1}{2}x - \sqrt[3]{x} + 1$  using the bisection algorithm.

Question 3 Approximate the negative root of the function  $f(x) = x^2 - 7$  to within 0.1 of its actual value.

Question 4 Approximate the value of the root of  $f(x) = -3x^3 + 5x^2 + 14x - 16$  near  $x = 3$  to within 0.05 of its actual value.

Question 5 Find the 4th approximation to the solution of the equation below using the bisection algorithm.

$$x^2 - x - 2 = x$$

Question 6 Find the 5th approximation to the solution to the equation below, using the bisection algorithm.

$$x^4 - 2 = x + 1$$

Question 7 The equation below should have a solution that is larger than 5. Use the bisection algorithm to approximate this solution to within 0.1 of its actual value.

$$x^3 + 18x - 6 - 9x^2 - 2x + 7$$

Question 8 The only real solution to the equation below is negative. Approximate the value of this solution to within 0.05 units of its actual value.

$$x^3 + 7x^2 - 2 = 2x^2 - 7x - 7$$

Question 9 Use the bisection algorithm to approximate the value of  $\sqrt{71}$ . Find the 4th approximation.

Question 10 Use the bisection algorithm to approximate the value of  $\frac{1}{\sqrt[3]{3}}$ . Find the 3rd approximation.

Question 11 Use the bisection algorithm to approximate the value of  $\sqrt{125}$  to within 0.125 units of the actual value.

Question 12 Use the bisection algorithm to approximate the value of  $\frac{\sqrt[4]{12500}}{2}$  to within 0.1 units of the actual value.

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Question 13 Suppose we used the bisection method on  $f(x)$ , with an initial interval of  $[2, 5]$ . How many iterations would it take before the maximum error would be less than 0.01 units?

Question 14 Suppose we used the bisection method on  $f(x)$ , with an initial interval of  $[-1, 1]$ . How many iterations would it take before the maximum error would be less than 0.02 units?

## Answers

### Question 1 Solution

Step 1) Since the function is continuous everywhere, find an appropriate starting interval.

Then, notice that  $f(1) = -6 < 0$ , but  $f(2) = 9 > 0$ . Let's use  $[1, 2]$  as the starting interval.

Step 2) Set up and use the table of values as in the examples above. The approximations are in blue, the new intervals are in red.

| Finding the New Interval   |                                 |                              |                 | Next Approximation |              |
|----------------------------|---------------------------------|------------------------------|-----------------|--------------------|--------------|
| $f(\text{left})$           | $f(\text{mid})$                 | $f(\text{right})$            | New Interval    | Midpoint           | Max Error    |
| Starting Interval:         |                                 |                              | $[1, 2]$        | <b>1.5</b>         | $\pm 0.5$    |
| $f(1) = -6$                | $f(\text{1.5}) \approx -2$      | $f(2) = 9$                   | $[1.5, 2]$      | <b>1.75</b>        | $\pm 0.25$   |
| $f(\text{1.5}) \approx -2$ | $f(\text{1.75}) \approx 2.4$    | $f(2) = 9$                   | $[1.5, 1.75]$   | <b>1.625</b>       | $\pm 0.125$  |
| $f(1.5) \approx -2$        | $f(\text{1.625}) \approx -0.03$ | $f(\text{1.75}) \approx 2.4$ | $[1.625, 1.75]$ | <b>1.6875</b>      | $\pm 0.0625$ |

Answer The positive root of  $f(x) = x^4 - 7$  is at approximately  $x = 1.6875$ . This approximation is off by at most  $\pm 0.0625$  units.

### Question 2 Solution

Step 1) Since the function is continuous everywhere, determine an appropriate starting interval.

Set up a table of values to help us find an appropriate interval.

| $x$ | $f(x)$              |
|-----|---------------------|
| 0   | $f(0) = -1$         |
| 1   | $f(1) \approx -0.8$ |
| 2   | $f(2) \approx -0.4$ |
| 3   | $f(3) \approx -0.1$ |
| 4   | $f(4) \approx 0.3$  |

This table indicates the root is between  $x = 3$  and  $x = 4$ , so a good starting interval is  $[3, 4]$

Step 2) Set up and use a table to track the appropriate values.

| Finding the New Interval |                        |                      |              | Next Approximation |             |
|--------------------------|------------------------|----------------------|--------------|--------------------|-------------|
| $f(\text{left})$         | $f(\text{mid})$        | $f(\text{right})$    | New Interval | Midpoint           | Max Error   |
| Starting Interval:       |                        |                      | $[3, 4]$     | $3.5$              | $\pm 0.5$   |
| $f(3) \approx -0.1$      | $f(3.5) \approx 0.1$   | $f(4) \approx 0.3$   | $[3, 3.5]$   | $3.25$             | $\pm 0.25$  |
| $f(3) \approx -0.1$      | $f(3.25) \approx 0.01$ | $f(3.5) \approx 0.1$ | $[3, 3.25]$  | $3.125$            | $\pm 0.125$ |

Answer The function has a root at approximately  $x = 3.125$  with a maximum possible error of  $\pm 0.125$  units.

### Question 3 Solution

Step 1) Since the function is continuous everywhere, determine an appropriate starting interval.

At  $x = -2$  the function value is  $f(-2) = -3$ , and at  $x = -3$  the function value is  $f(-3) = 2$ . We'll use  $[-3, -2]$  as the starting interval.

Step 2) Determine the first approximation and the maximum possible error in using it.

1st Approximation: The midpoint of the first interval is  $x = -2.5$ .

Associated Error:  $\pm 0.5$  units.

Step 3) Determine the second interval, second approximation and the associated maximum error.

| Finding the 2nd Interval |        |                        |
|--------------------------|--------|------------------------|
|                          | $x$    | $f(x)$                 |
| Current left-endpoint    | $-3$   | $f(-3) = 2$            |
| Midpoint                 | $-2.5$ | $f(-2.5) \approx -0.8$ |
| Current right-endpoint   | $-2$   | $f(-2) = -3$           |

Second Interval:  $[-3, -2.5]$

Second Approximation: The midpoint of the second interval is  $x = -2.75$ .

Associated Error:  $\pm 0.25$  units

Step 4) Repeat Step 3 with the new interval. Continue to repeat until the maximum error is less than 0.1.

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Finding the 3rd Interval

|                        | $x$   | $f(x)$                 |
|------------------------|-------|------------------------|
| Current left-endpoint  | -3    | $f(-3) = 2$            |
| Midpoint               | -2.75 | $f(-2.75) \approx 0.6$ |
| Current right-endpoint | -2.5  | $f(-2.5) \approx -0.8$ |

Third interval:  $[-2.75, -2.5]$ .

Third approximation: The midpoint is  $x = -2.625$

Associated error:  $\pm 0.125$  units.

Finding the 4th Interval

|                        | $x$    | $f(x)$                   |
|------------------------|--------|--------------------------|
| Current left-endpoint  | -2.75  | $f(-2.75) \approx 0.6$   |
| Midpoint               | -2.625 | $f(-2.625) \approx -0.1$ |
| Current right-endpoint | -2.5   | $f(-2.5) \approx -0.8$   |

4th interval:  $[-2.75, -2.625]$ .

4th approximation: Midpoint is at  $x = -2.6875$

Associated error:  $\pm 0.0625$  units.

Answer The negative root of the function is at approximately  $x = -2.6875$  with a maximum error of only  $\pm 0.0625$  units.

Question 4 Solution

Step 1) Determine an appropriate starting interval, the first approximation and its associated maximum error value.

First, notice that the function is continuous everywhere. Then, since we're told that the root is near  $x = 3$  we can check that  $f(3) = -10$ .

Checking  $x = 4$  we find that  $f(4) = -72$ , but at  $x = 2$  the function value is  $f(2) = 8$ .

First Interval:  $[2, 3]$

First Approximation: The midpoint is at  $x = 2.5$

Associated error:  $\pm 0.5$  units

Step 2) Determine the second interval, the second approximation and the associated maximum error.

Finding the 2nd Interval

|                        | $x$ | $f(x)$             |
|------------------------|-----|--------------------|
| Current left-endpoint  | 2   | $f(2) = 8$         |
| Midpoint               | 2.5 | $f(2.5) \approx 3$ |
| Current right-endpoint | 3   | $f(3) = -10$       |

Second Interval:  $[2.5, 3]$

Second Approximation: The midpoint is at  $x = 2.75$

Associated Error:  $\pm 0.25$  units.

Step 3) Repeat Step 2 until the maximum possible error is less than 0.05 units.

Finding the 3rd Interval

|                        | $x$  | $f(x)$               |
|------------------------|------|----------------------|
| Current left-endpoint  | 2.5  | $f(2.5) \approx 3$   |
| Midpoint               | 2.75 | $f(2.75) \approx -2$ |
| Current right-endpoint | 3    | $f(3) = -10$         |

Third interval:  $[2.5, 2.75]$ .

Third approximation: The midpoint is at  $x = 2.625$

Associated error:  $\pm 0.125$  units.

Finding the 4th Interval

|                        | $x$   | $f(x)$                 |
|------------------------|-------|------------------------|
| Current left-endpoint  | 2.5   | $f(2.5) \approx 3$     |
| Midpoint               | 2.625 | $f(2.625) \approx 0.9$ |
| Current right-endpoint | 2.75  | $f(2.75) \approx -2$   |

4th interval:  $[2.625, 2.75]$

4th approximation: The midpoint is  $x = 2.6875$ .

Associated error of  $\pm 0.0625$  units.

Finding the 5th Interval

|                        | $x$    | $f(x)$                   |
|------------------------|--------|--------------------------|
| Current left-endpoint  | 2.625  | $f(2.625) \approx 0.9$   |
| Midpoint               | 2.6875 | $f(2.6875) \approx -0.5$ |
| Current right-endpoint | 2.75   | $f(2.75) \approx -2$     |

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5th interval:  $[2.625, 2.6875]$ .

5th approximation: The midpoint is  $x = 2.65625$

Associated error:  $\pm 0.03125$  units.

Answer The root of the function is approximately  $x = 2.65625$  and has an associated maximum error of only  $\pm 0.03125$  units.

#### Question 5 Solution

Step 1) Identify the function by getting the equation equal to zero.

$$x^2 - 2x - 2 = 0$$

The function we'll use is  $f(x) = x^2 - 2x - 2$ .

Step 2) Determine an appropriate starting interval, the first approximation and its associated maximum error.

At  $x = 0$  the function value is  $f(0) = -2$ , while at  $x = 3$  the function value is  $f(3) = 1$ .

1st Interval:  $[0, 3]$

1st Approximation: The midpoint is at  $x = 1.5$ .

Associated Error:  $\pm 1.5$  units.

Step 3) Determine the second interval, second approximation and its associated maximum error.

Finding the 2nd Interval

|                        | $x$ | $f(x)$           |
|------------------------|-----|------------------|
| Current left-endpoint  | 0   | $f(0) = -2$      |
| Midpoint               | 1.5 | $f(1.5) = -2.75$ |
| Current right-endpoint | 3   | $f(3) = 1$       |

2nd Interval:  $[1.5, 3]$

2nd Approximation:  $x = 2.25$

Associated Error:  $\pm 0.75$  units

Step 4) Repeat Step 3 twice to complete the iterations of the bisection method for this question.

Finding the 3rd Interval

|                        | $x$  | $f(x)$                 |
|------------------------|------|------------------------|
| Current left-endpoint  | 1.5  | $f(1.5) = -2.75$       |
| Midpoint               | 2.25 | $f(2.25) \approx -1.4$ |
| Current right-endpoint | 3    | $f(3) = 1$             |

3rd Interval:  $[2.25, 3]$

3rd Approximation:  $x = 2.625$

Associated Error:  $\pm 0.375$  units

Finding the 4th Interval

|                        | $x$   | $f(x)$                 |
|------------------------|-------|------------------------|
| Current left-endpoint  | 2.25  | $f(2.25) = -1.4375$    |
| Midpoint               | 2.625 | $f(2.625) = -0.359375$ |
| Current right-endpoint | 3     | $f(3) = 1$             |

4th Interval:  $[2.625, 3]$

4th Approximation:  $x = 2.8125$

Associated Error:  $\pm 0.1875$  units

Answer The solution to the equation is approximately  $x = 2.8125$  with a maximum error of 0.1875 units.

#### Question 6 Solution

Step 1) Identify the function we will use by rewriting the equation so it is set equal to zero.

$$x^4 - x - 3 = 0$$

The function we will use is  $f(x) = x^4 - x - 3$ .

Step 2) Identify the first interval, the first approximation and its associated maximum error.

Notice that at  $x = 0$  the function value is  $f(0) = -3$ .

Also, at  $x = 2$  the function value is  $f(2) = 11$ .

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1st Interval:  $[0, 2]$

1st Approximation:  $x = 1$

Associated Error:  $\pm 1$  unit

Step 3) Identify the 2nd interval, approximation and associated error.

Finding the 2nd Interval

|                        | $x$ | $f(x)$      |
|------------------------|-----|-------------|
| Current left-endpoint  | 0   | $f(0) = -3$ |
| Midpoint               | 1   | $f(1) = -3$ |
| Current right-endpoint | 2   | $f(2) = 11$ |

2nd Interval:  $[1, 2]$

2nd Approximation:  $x = 1.5$

Associated Error  $\pm 0.5$  units

Step 4) Repeat Step 3 until you've found the 5th approximation.

Finding the 3rd Interval

|                        | $x$ | $f(x)$               |
|------------------------|-----|----------------------|
| Current left-endpoint  | 1   | $f(1) = -3$          |
| Midpoint               | 1.5 | $f(1.5) \approx 0.6$ |
| Current right-endpoint | 2   | $f(2) = 11$          |

3rd Interval:  $[1, 1.5]$

3rd Approximation:  $x = 1.25$

Associated Error:  $\pm 0.25$  units

Finding the 4th Interval

|                        | $x$  | $f(x)$                 |
|------------------------|------|------------------------|
| Current left-endpoint  | 1    | $f(1) = -3$            |
| Midpoint               | 1.25 | $f(1.25) \approx -1.8$ |
| Current right-endpoint | 1.5  | $f(1.5) \approx 0.6$   |

4th Interval:  $[1.25, 1.5]$

4th Approximation:  $x = 1.375$

Associated Error:  $\pm 0.125$  units

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Finding the 5th Interval

|                        | $x$   | $f(x)$                  |
|------------------------|-------|-------------------------|
| Current left-endpoint  | 1.25  | $f(1.25) \approx -1.8$  |
| Midpoint               | 1.375 | $f(1.375) \approx -0.8$ |
| Current right-endpoint | 1.5   | $f(1.5) \approx 0.6$    |

- 5th Interval:  $[1.375, 1.5]$
- 5th Approximation:  $x = 1.4375$
- Associated Error:  $\pm 0.0625$  units

Answer The solution to the equation is approximately  $x = 1.4375$ . This approximation has an maximum error of at most 0.0625 units.

#### Question 7 Solution

Step 1) Identify the function we'll use by rewriting the equation so it is equal to zero.

$$x^3 - 9x^2 + 20x - 13 = 0$$

The function is  $f(x) = x^3 - 9x^2 + 20x - 13$ .

Step 2) Determine the first interval, 1st approximation, and its associated error.

We know the solution is larger than 5, but we don't know how much larger. We set up a small table of values to help us out.

| $x$ | $f(x)$ |
|-----|--------|
| 5   | -13    |
| 6   | -1     |
| 7   | 29     |

From this table we can select the first interval and determine the first approximation.

1st Interval:  $[6, 7]$

1st Approximation:  $x = 6.5$

Associated Error:  $\pm 0.5$  units

Step 3) Find the second interval, second approximation and the associated error.

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Finding the 2nd Interval

|                        | $x$ | $f(x)$                |
|------------------------|-----|-----------------------|
| Current left-endpoint  | 6   | $f(6) = -1$           |
| Midpoint               | 6.5 | $f(6.5) \approx 11.4$ |
| Current right-endpoint | 7   | $f(7) = 29$           |

2nd Interval:  $[6, 6.5]$

2nd Approximation  $x = 6.25$

Associated Error:  $\pm 0.25$  units

Step 4) Repeat Step 3 until the maximum error is less than the given tolerance of 0.1.

Finding the 3rd Interval

|                        | $x$  | $f(x)$                |
|------------------------|------|-----------------------|
| Current left-endpoint  | 6    | $f(6) = -1$           |
| Midpoint               | 6.25 | $f(6.25) \approx 4.6$ |
| Current right-endpoint | 6.5  | $f(6.5) \approx 11.4$ |

3rd Interval:  $[6, 6.25]$

3rd Approximation:  $x = 6.125$

Associated Error:  $\pm 0.125$  units

Finding the 4th Interval

|                        | $x$   | $f(x)$                 |
|------------------------|-------|------------------------|
| Current left-endpoint  | 6     | $f(6) = -1$            |
| Midpoint               | 6.125 | $f(6.125) \approx 1.6$ |
| Current right-endpoint | 6.25  | $f(6.25) \approx 4.6$  |

4th Interval:  $[6, 6.125]$

4th Approximation  $x = 6.0625$

Associated Error:  $\pm 0.0625$  units

Answer: The solution to the equation is approximately  $x = 6.0625$  with a maximum error of 0.0625 units.

Question 8 Solution

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Step 1) Rewrite the equation so we can identify the function we are working with.

$$x^3 + 5x^2 + 7x + 5 = 0$$

$$\text{So, } f(x) = x^3 + 5x^2 + 7x + 5$$

Step 2) Identify the first interval, the first approximation, and the associated error.

We know the solution is negative, but that is all. Let's set up a table of values to get an idea of where our first interval should be.

| $x$ | $f(x)$ |
|-----|--------|
| -1  | 2      |
| -2  | 3      |
| -3  | 2      |
| -4  | -7     |

1st Interval:  $[-3, -4]$

1st Approximation:  $x = -3.5$

Associated Error:  $\pm 0.5$  units

Step 3) Identify the 2nd interval, 2nd approximation and the associated maximum error.

Finding the 2nd Interval

|                        | $x$  | $f(x)$                 |
|------------------------|------|------------------------|
| Current left-endpoint  | -4   | $f(-4) = -7$           |
| Midpoint               | -3.5 | $f(-3.5) \approx -1.1$ |
| Current right-endpoint | -3   | $f(-3) = 2$            |

2nd Interval:  $[-3.5, -3]$

2nd Approximation:  $x = -3.25$

Associated Error:  $\pm 0.25$  units

Step 4) Repeat Step 3 until the maximum error is less 0.05 units.

Finding the 3rd Interval

|                        | $x$   | $f(x)$                 |
|------------------------|-------|------------------------|
| Current left-endpoint  | -3.5  | $f(-3.5) \approx -1.1$ |
| Midpoint               | -3.25 | $f(-3.25) \approx 0.7$ |
| Current right-endpoint | -3    | $f(-3) = 2$            |

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3rd Interval:  $[-3.5, -3.25]$

3rd Approximation:  $x = -3.375$

Associated Error:  $\pm 0.125$

Finding the 4th Interval

|                        | $x$    | $f(x)$                   |
|------------------------|--------|--------------------------|
| Current left-endpoint  | -3.5   | $f(-3.5) \approx -1.1$   |
| Midpoint               | -3.375 | $f(-3.375) \approx -0.1$ |
| Current right-endpoint | -3.25  | $f(-3.25) \approx 0.7$   |

4th Interval:  $[-3.375, -3.25]$

4th Approximation:  $x = -3.3125$

Associated Error:  $\pm 0.0625$  units

Finding the 5th Interval

|                        | $x$     | $f(x)$                   |
|------------------------|---------|--------------------------|
| Current left-endpoint  | -3.375  | $f(-3.375) \approx -0.1$ |
| Midpoint               | -3.3125 | $f(-3.3125) \approx 0.3$ |
| Current right-endpoint | -3.25   | $f(-3.25) \approx 0.7$   |

5th Interval:  $[-3.375, -3.3125]$

5th Approximation:  $x = -3.34275$

Associated Error:  $\pm 0.03125$  units

Answer The equation has a solution at approximately  $x = -3.34275$  with a maximum error in the approximation of at most 0.03125 units.

#### Question 9 Solution

Step 1) Find a non-linear function whose root is at  $\sqrt{7}$ .

$$\begin{aligned}x &= \sqrt{71} \\x^2 &= 71 \\x^2 - 71 &= 0\end{aligned}$$

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We'll use  $f(x) = x^2 - 71$ .

Step 2) Find the first interval, first approximation and its associated maximum error.

We know  $\sqrt{71}$  is larger than 8, but less than 9. We'll use  $[8, 9]$  as the first interval.

- 1st Interval:  $[8, 9]$
- 1st Approximation:  $x = 8.5$
- Associated Error:  $\pm 0.5$  units

Step 3) Find the second interval, second approximation and the associated maximum error.

Finding the 2nd Interval

|                        | $x$ | $f(x)$          |
|------------------------|-----|-----------------|
| Current left-endpoint  | 8   | $f(8) = -7$     |
| Midpoint               | 8.5 | $f(8.5) = 1.25$ |
| Current right-endpoint | 9   | $f(9) = 10$     |

2nd Interval:  $[8, 8.5]$

2nd Approximation:  $x = 8.25$

Associated Error:  $\pm 0.25$  units

Step 4) Repeat Step 3 until you've found the 4th approximation.

Finding the 3rd Interval

|                        | $x$  | $f(x)$                 |
|------------------------|------|------------------------|
| Current left-endpoint  | 8    | $f(8) = -7$            |
| Midpoint               | 8.25 | $f(8.25) \approx -2.9$ |
| Current right-endpoint | 8.5  | $f(8.5) = 1.25$        |

3rd Interval:  $[8.25, 8.5]$

3rd Approximation  $x = 8.375$

Associated Error:  $\pm 0.125$  units

Finding the 4th Interval

|                | $x$   | $f(x)$                  |
|----------------|-------|-------------------------|
| Left endpoint  | 8.25  | $f(8.25) \approx -2.9$  |
| Midpoint       | 8.375 | $f(8.375) \approx -0.9$ |
| Right endpoint | 8.5   | $f(8.5) = 1.25$         |

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4th Interval:  $[8.375, 8.5]$

4th Approximation:  $x = 8.4375$

Associated Error:  $\pm 0.0625$  units

Answer:  $\sqrt{71} \approx 8.4375$  with a maximum error in this approximation of  $0.0625$ .

#### Question 10 Solution

Step 1) Determine the nonlinear function we will use for the bisection algorithm.

$$\begin{aligned}x &= \frac{1}{\sqrt[5]{3}} \\x^5 &= \frac{1}{3} \\3x^5 &= 1 \\3x^5 - 1 &= 0\end{aligned}$$

We will use  $f(x) = 3x^5 - 1$ .

Step 2) Find the first interval, first approximation and the associated error.

Since  $0 < \frac{1}{\sqrt[5]{3}} < 1$ , we should be able to use  $[0, 1]$  as the first interval. A quick check of the function values confirms this.

| $x$ | $f(x)$ |
|-----|--------|
| 0   | -1     |
| 1   | 2      |

1st Interval:  $[0, 1]$

1st Approximation:  $x = 0.5$

Associated Error:  $\pm 0.5$  units

Step 3) Determine the second interval, second approximation and the associated error.

Finding the 2nd Interval

|                        | $x$ | $f(x)$                |
|------------------------|-----|-----------------------|
| Current left-endpoint  | 0   | $f(0) = -1$           |
| Midpoint               | 0.5 | $f(0.5) \approx -0.9$ |
| Current right-endpoint | 1   | $f(1) = 2$            |

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2nd Interval:  $[0.5, 1]$

2nd Approximation:  $x = 0.75$

Associated Error:  $\pm 0.25$  units

Step 4) Find the third interval, third approximation and its associated error.

Finding the 3rd Interval

|                        | $x$  | $f(x)$                 |
|------------------------|------|------------------------|
| Current left-endpoint  | 0.5  | $f(0.5) \approx -0.9$  |
| Midpoint               | 0.75 | $f(0.75) \approx -0.2$ |
| Current right-endpoint | 1    | $f(1) = 2$             |

Third Interval:  $[0.75, 1]$

Third Approximation:  $x = 0.875$  with an error of 0.125 units.

Answer:  $\sqrt[5]{3} \approx 0.875$  with a maximum error of 0.125 units.

Question 11 Solution

Step 1) Find a nonlinear function with a root at  $\sqrt{125}$ .

$$\begin{aligned}x &= \sqrt{125} \\x^2 &= 125 \\x^2 - 125 &= 0\end{aligned}$$

We'll use  $f(x) = x^2 - 125$ .

Step 2) Determine an appropriate starting interval. It's midpoint will be the first approximation.

Since  $11^2 = 121$  and  $12^2 = 144$  we know  $11 < \sqrt{125} < 12$ .

First Interval:  $[11, 12]$

First approximation:  $x = 11.5$

Associated Error:  $\pm 0.5$  units

Step 3) Determine the second interval, the second approximation, and the associated error value.

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Finding the 2nd Interval

|                        | $x$  | $f(x)$           |
|------------------------|------|------------------|
| Current left-endpoint  | 11   | $f(11) = -4$     |
| Midpoint               | 11.5 | $f(11.5) = 7.25$ |
| Current right-endpoint | 12   | $f(12) = 19$     |

2nd Interval:  $[11, 11.5]$

2nd Approximation:  $x = 11.25$  with a maximum error of 0.25 units.

Step 4) Determine the third interval, the third approximation, and the associated error value.

Finding the 3rd Interval

|                        | $x$   | $f(x)$                 |
|------------------------|-------|------------------------|
| Current left-endpoint  | 11    | $f(11) = -4$           |
| Midpoint               | 11.25 | $f(11.25) \approx 1.6$ |
| Current right-endpoint | 11.5  | $f(11.5) = 7.25$       |

Third Interval:  $[11, 11.25]$

Third Approximation:  $x = 11.125$  with a maximum error of 0.125.

Answer:  $\sqrt{125} \approx 11.125$  with a maximum error of 0.125 units.

Question 12

Step 1) Find a nonlinear function with a root at  $\frac{\sqrt[4]{12500}}{2}$ .

$$x = \frac{\sqrt[4]{12500}}{2}$$

$$x^4 = \frac{12500}{16}$$

$$x^4 = \frac{3125}{4}$$

$$4x^4 = 3125$$

$$4x^4 - 3125 = 0$$

We'll use the function  $f(x) = 4x^4 - 3125$ .

Step 2) Determine the appropriate starting interval, the first approximation and the associated error.

Since  $10^4 = 10,000$  is about the right size, we try  $f(10) = 36,875$

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By comparison,  $f(5) = -625$ , so the best starting interval is somewhere between  $x = 5$  and  $x = 10$ . Let's make a table of values to help us narrow things down.

| $x$ | $f(x)$ |
|-----|--------|
| 5   | -625   |
| 6   | 2059   |

Well, that was convenient.

1st Interval:  $[5, 6]$

1st Approximation:  $x = 5.5$

Associated Error:  $\pm 0.5$  units

Step 3) Find the second interval, approximation, and associated error.

Finding the 2nd Interval

|                        | $x$ | $f(x)$            |
|------------------------|-----|-------------------|
| Current left-endpoint  | 5   | $f(5) = -625$     |
| Midpoint               | 5.5 | $f(5.5) = 535.25$ |
| Current right-endpoint | 6   | $f(6) = 2059$     |

2nd Interval:  $[5, 5.5]$

2nd Approximation:  $x = 5.25$

Associated Error:  $\pm 0.25$  units

Step 4) Repeat Step 3 until the maximum error is smaller than the allowed tolerance.

Finding the 3rd Interval

|                        | $x$  | $f(x)$                  |
|------------------------|------|-------------------------|
| Current left-endpoint  | 5    | $f(5) = -625$           |
| Midpoint               | 5.25 | $f(5.25) \approx -86.2$ |
| Current right-endpoint | 5.5  | $f(5.5) = 535.25$       |

Third Interval:  $[5.25, 5.5]$

Third Approximation: The midpoint of the 3rd interval is  $x = 5.375$

Associated Error:  $\pm 0.125$  units

Finding the 4th Interval

|                        | $x$   | $f(x)$                   |
|------------------------|-------|--------------------------|
| Current left-endpoint  | 5.25  | $f(5.25) \approx -86.2$  |
| Midpoint               | 5.375 | $f(5.375) \approx 213.7$ |
| Current right-endpoint | 5.5   | $f(5.5) = 535.25$        |

Fourth Interval:  $[5.25, 5.375]$

Fourth Approximation: The midpoint of the 4th interval is  $x = 5.3125$

Associated Error:  $\pm 0.0625$  units

Answer:  $\frac{1}{2} \cdot \sqrt[4]{12500} \approx 5.3125$  with a maximum error of 0.0625.

Question 13 Solution

Step 1) Solve  $0.5^n(b - a)$  for  $n$  when  $a = 2$  and  $b = 5$ .

$$0.5^n(5 - 2) = 0.01$$

$$0.5^n \cdot 3 = \frac{1}{10}$$

$$\left(\frac{1}{2}\right)^n = \frac{1}{30}$$

$$n \cdot \ln\left(\frac{1}{2}\right) = \ln\left(\frac{1}{30}\right)$$

$$n \cdot (\ln 1 - \ln 2) \ln 1 - \ln 30$$

$$-n \ln 2 = -\ln 30$$

$$n = \frac{\ln 30}{\ln 2}$$

$$\approx 4.90732$$

Answer We will need to use at least 5 iterations in order to ensure the accuracy.

Question 14

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Step 1) Solve  $0.5^n(b-a) = 0.02$  for  $n$  when  $a = -1$  and  $b = 1$ .

$$0.5^n(1 - (-1)) = 0.02$$

$$\left(\frac{1}{2}\right)^n \cdot 2 = \frac{1}{50}$$

$$\left(\frac{1}{2}\right)^n = \frac{1}{100}$$

$$n \ln\left(\frac{1}{2}\right) = \ln\left(\frac{1}{100}\right)$$

$$n(\ln 1 - \ln 2) = \ln 1 - \ln 100$$

$$-n \ln 2 = -\ln 100$$

$$n = \frac{\ln 100}{\ln 2}$$

$$\approx 6.64473$$

Answer: We will need at least 7 iterations before the error tolerance is reached.

Next Lesson: What is a Derivative?

Previous Lesson: The Intermediate Value Theorem

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## ឯកសារយោង

### Reference

1. មូលដ្ឋានគ្រឹះនៃការវិភាគលេខ រៀបរៀងដោយ លោក ម៉ែន សុគន្ធី និងអេវ៉ាន អាស្ទី ឆ្នាំ ២០០១
2. Numerical Analysis (seventh edition) by RICHARD L. BURDEN AND J. DOUGLAS FAIRES

Fb Page: Mony Chantrea