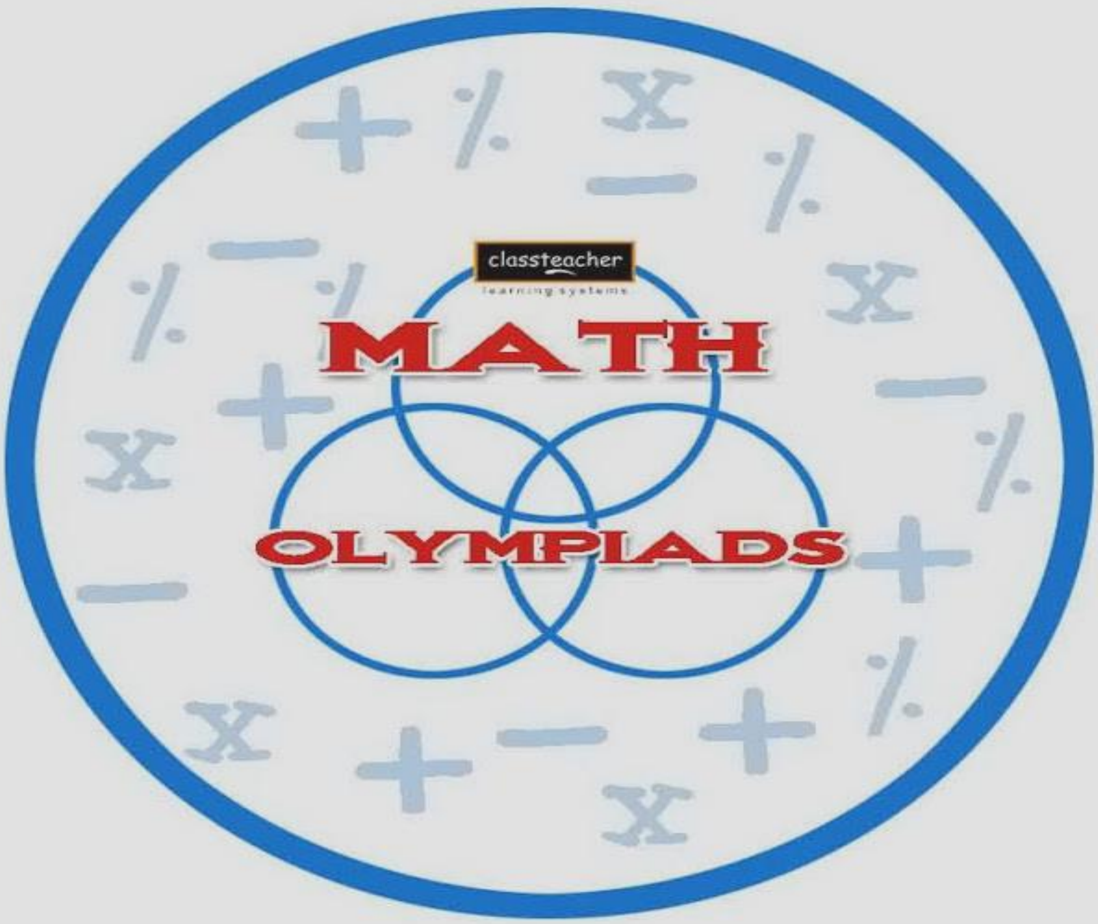
ក្រសួងអប់រំ យុវជន និង កីឡា

១០០ លំហាត់គណិតវិទ្យាអូឡាំពិច

100 problems in mathematical olympiad

ភាគទី១

APMO IMO



សម្រាប់ៈត្រៀមប្រឡងសិស្សពូកែនិងអាហារូបករណ៍

រៀបរៀងដោយៈ ផុន សុកាល់

និស្សិតវិស្វករនៃវិទ្យាស្ថានបច្ចេកវិទ្យាកម្ពុជា

១០០លំហាត់គណិតវិទ្យាអូឡាំពិច

រៀបរៀងដោយៈ

ផុន សុកាល់

និស្សិតនៃវិទ្យាស្ថានបច្ចេកវិទ្យាកម្ពុជា(សាលាតិចណូ)

រួមមានៈ

៙ លំហាត់គណិតវិទ្យាអូឡាំពិចតាមបណ្តាប្រទេសជុំវិញពិតពលោក

៙ លំហាត់គណិតវិទ្យាអូឡាំពិចអន្តរជាតិដូចជា IMO APMO PUTNAM ...

សម្រាប់ៈ

→ ត្រៀមប្រឡងសិស្សពូកែថ្នាក់ខេត្ត/រាជធានី និង ថ្នាក់ទូទាំងប្រទេស

→ ត្រៀមប្រឡងអាហារូបករណ៍ក្នុងស្រុក ក្រៅស្រុក

ភ្នំពេញ ថ្ងៃទី ២០ ខែ មករា ឆ្នាំ ២០១៤

អារម្ភកថា

ដោយមើលឃើញពីកង្វះខាតនៃឯកសារគណិតវិទ្យាអូឡាំពិចទើបជម្រុញចិត្តរូបខ្ញុំបាទឲ្យខិតខំរៀបរៀងជាសៀវភៅមួយក្បាលនេះឡើងដើម្បីទុកជាឯកសារសម្រាប់សិស្សានុសិស្ស សិក្សាស្រាវជ្រាវ ។

លំហាត់ក្នុងសៀវភៅនេះស្ទើរតែទាំងអស់សុទ្ធតែបកប្រែកែសម្រួលពីភាសាបរទេសទៅជាភាសាជាតិយើងដើម្បីសម្រួលដល់អ្នកសិក្សា ងាយស្រួលស្វែងយល់ ដែលក្នុងនោះលំហាត់មួយចំនួនត្រូវបានដកស្រង់ចេញពីសៀវភៅគណិតវិទ្យាអូឡាំពិចបរទេសល្បីៗហើយលំហាត់មួយចំនួនទៀតដកស្រង់ចេញពីអ៊ីនធឺណិតតាមរយៈវេបសាយគណិតវិទ្យាសំខាន់ៗ ។ លំហាត់ទាំងនោះភាគច្រើនជាលំហាត់ធ្លាប់ចេញប្រឡងគណិតវិទ្យាអូឡាំពិចនៅតាមបណ្តាប្រទេសជុំវិញពិភពលោកដូចជា សហរដ្ឋអាមេរិច ប្រទេសរូស្ស៊ី អង់គ្លេស កាណាដា អូស្ត្រាលី ឥណ្ឌា រ៉ូមេនី ប្រេស៊ីល ចិន កូរ៉េ ម៉ុងហ្គោលី សិង្ហបុរី ភីលីពីន ឥណ្ឌូនេស៊ី វៀតណាម ជាដើម ។ សល់ពីនោះជាលំហាត់មាននៅក្នុងការប្រឡងគណិតវិទ្យាអូឡាំពិចលក្ខណៈអន្តរជាតិរួមមាន: IMO APMO PUTNAM ... ។

ចំនុចពិសេសនៃសៀវភៅនេះគឺលំហាត់នីមួយៗសុទ្ធតែប្រដាប់ដោយដំណោះស្រាយយ៉ាងក្បោះក្បាយ ងាយយល់ ហើយប្រកបដោយគន្លឹះប្លែកៗដើម្បីទុកជាទុនក្នុងការត្រៀមខ្លួនប្រឡងសិស្សពូកែគណិតវិទ្យាថ្នាក់ជាតិ ក៏ដូចជាថ្នាក់អន្តរជាតិ ។

ជាទីអវសាន្ត ទោះបីជារូបខ្ញុំបាទខិតខំពិនិត្យពិច័យយ៉ាងណា ក៏ដោយក៏កំហុសឆ្គងច្បាស់ជាកើតមានជាអចេតនាជាក់ជាពុំខាន ។ ហេតុនេះ ខ្ញុំសង្ឃឹមថាមិត្តអ្នកសិក្សា នឹងយល់ហើយជួយកែតម្រូវ ។ ខ្ញុំបាទនឹងទន្ទឹងរង់ចាំពាក្យរិះគន់ក្នុងន័យស្ថាបនា ។

សូមជូនពរឲ្យមិត្តអ្នកសិក្សា ជួបតែសេចក្តីសុខ និង ទទួលជោគជ័យគ្រប់ភារកិច្ច ។

ភ្នំពេញ ថ្ងៃទី២០ ខែ មករា ឆ្នាំ ២០១៤

អ្នករៀបរៀង

ផុន សុកល់

និស្សិតនៃវិទ្យាស្ថានបច្ចេកវិទ្យាកម្ពុជា(សាលាតិចណូ)

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លំហាត់ទី១

គេឲ្យ x^3+ax^2+bx+c មានឫសបីជាចំនួនពិតខុសៗ ប៉ុន្តែ

$\left(x^2+x+2001\right)^3+a\left(x^2+x+2001\right)^2+b\left(x^2+x+2001\right)+c$

គ្មានឫសជាចំនួនពិតទេ ។ ចូរស្រាយបញ្ជាក់ថា:

$2001^3+a2001^2+b2001+c>\frac{1}{64}$ ។

(Russian Mathematical Olympiad 2001)

ដំណោះស្រាយ

តាង $f(x)=x^2+x+2001=(x+\frac{1}{2})^2+2001-\frac{1}{4}\geq 2001-\frac{1}{4}$

តាមបម្រាប់: $\left(f(x)\right)^3+a\left(f(x)\right)^2+bf(x)+c$ គ្មានឫសជាចំនួនពិត

នោះបើ x^3+ax^2+bx+c មានឫសជាចំនួនពិតលុះត្រាតែឫសទាំងនោះ

សុទ្ធតែមានតម្លៃ $<2001-\frac{1}{4}$ ។

តាង p,q,r ជាឫសទាំងបីនៃ x^3+ax^2+bx+c នោះមានន័យថា:

$p,q,r<2001-\frac{1}{4}\Rightarrow 2001-p>\frac{1}{4}, 2001-q>\frac{1}{4}$ និង

$2001-r>\frac{1}{4}$

$\Rightarrow (2001-p)(2001-q)(2001-r)>\frac{1}{64}$

$$\Leftrightarrow 2001^3-(p+q+r)2001^2+(pq+qr+rp)2001$$
$$-pqr>\frac{1}{64} \qquad (*)$$

តែដោយ p,q,r ជាឫសទាំងបីនៃ x^3+ax^2+bx+c នោះតាមទ្រឹស្តីបទវ្យែត

យើងបាន:

$$\begin{cases} p+q+r=-a \\ pq+qr+rp=b \\ pqr=-c \end{cases}$$

តាម (*) យើងបាន: $2001^3+a2001^2+b2001+c>\frac{1}{64}$

ដូចនេះ វិសមភាពត្រូវបានស្រាយបញ្ជាក់ ។

<u>លំហាត់ទី២</u>

តាង a,b,c,d ជាចំនួនគត់វិជ្ជមានដែលផ្ទៀងផ្ទាត់ ៖ $a>b>c>d$ និង $(a+b-c+d)|(ac+bd)$ ។

ចូរស្រាយបញ្ជាក់ថា: ចំពោះគ្រប់ចំនួនគត់វិជ្ជមាន m និង ចំនួនគត់សេស n នោះ $a^nb^m+c^md^n$ មិនមែនជាចំនួនបឋមទេ ។

(Mongolia ,Team Selection Test 2008)

<u>ដំណោះស្រាយ</u>

យើងមាន: $(a+b-c+d)|(ac+bd)$

នោះ $(a+b-c+d)|(ac+bd)+(a+b-c+d)a$

$\Leftrightarrow a+b-c+d|(a+b)(a+d) \qquad (*)$

នាំឲ្យ មាន $k\in\mathbb{N}$ ដែល $(a+b)(a+d)=k(a+b-c+d)$

សន្មតថា: $\gcd(a+d,a+b-c+d)=1$ នោះតាម $(*)$ យើងបាន:

$a+b-c+d|a+b$ នាំឲ្យ $a+b=l(a+b-c+d)$ ដែល $l\in\mathbb{N}$

បើ $l=1$ នោះ: $a+b=a+b-c+d \Leftrightarrow c=d$ មិនពិត ព្រោះ $c>d$

បើ $l\geq 2$ នោះ: $a+b\geq 2(a+b-c+d)$

$\Leftrightarrow 2c-2d\geq a+b>2c \Leftrightarrow -2d>0$ មិនពិត

នាំឲ្យការសន្មតថា: $\gcd(a+d, a+b-c+d)=1$ មិនពិត

នេះបញ្ជាក់ថា: $\gcd(a+d, a+b-c+d)=q>1$ ។

ម្យ៉ាងទៀត: $q|a+d$ និង $q|a+b-c+d \Rightarrow q|b-c$ នោះយើងបាន:

$a \equiv -d \pmod{q}$ និង $b \equiv c \pmod{q}$

$\Rightarrow a^n \equiv -d^n$ (ព្រោះ n ជា ចំនួនគត់សេស) និង $b^m \equiv c^m \pmod{q}$

$\Rightarrow a^n b^m \equiv -c^m d^n \pmod{q} \Leftrightarrow a^n b^m + c^m d^n \equiv 0 \pmod{q}$

មានន័យថា: $a^n b^m + c^m d^n$ ជាពហុគុណនៃ $q>1$ ។

ដូចនេះ $a^n b^m + c^m d^n$ មិនមែនជាចំនួនបឋមទេ ។

លំហាត់ទី៣

ចូរស្រាយបញ្ជាក់ថា បើ x និង y ជាចំនួនគត់វិជ្ជមានដែល: x^2+y^2-x ចែកដាច់នឹង $2xy$ នោះ x ត្រូវតែជាការេប្រាកដ ។

(British Mathematical Olympiad 1991)

ដំណោះស្រាយ

ដោយ x^2+y^2-x ចែកដាច់នឹង $2xy$ នោះមានចំនួនគត់វិជ្ជមាន k មួយដែល:

$x^2+y^2-x=2xyk \Leftrightarrow y^2-2xyk+x^2-x=0$

ដើម្បីឲ្យសមីការខាងលើនេះមានឫសជាចំនួនគត់វិជ្ជមានលុះត្រាតែ:

$\Delta=(-2xk)^2-4(x^2-x)=4x\left(x(k^2-1)+1\right)$ ជាការេប្រាកដ

$\Leftrightarrow x\left(x(k^2-1)+1\right)$ ជាការេប្រាកដ

តែដោយ $-(k^2-1)x+\left(x(k^2-1)+1\right)=1$ នោះតាមទ្រឹស្តីបទ *Bezout*

គេបាន: x និង $x(k^2-1)+1$ ជាចំនួនបឋមរវាងគ្នា

$\Rightarrow x$ និង $x(k^2-1)+1$ ត្រូវតែជាការេប្រាកដទាំងពីរ ។

ដូចនេះ x ជាការេប្រាកដ ។

លំហាត់ទី៤

គេឲ្យម៉ាទ្រីសនៃចំនួនពិតមិនអវិជ្ជមានមួយដែលមានលំដាប់ $(m\times n)$ ហើយ ជួរដេកនីមួយៗ និងជួររឈរនីមួយៗ មានយ៉ាងតិចធាតុវិជ្ជមានមួយ ។ ម្យ៉ាងទៀត បើជួរដេកមួយ និង ជួរឈរមួយប្រសព្វគ្នាត្រង់ធាតុជាចំនួនវិជ្ជមាន នោះផលបូកធាតុរបស់ពួកវាទាំងពីរ គឺ ស្មើគ្នា ។ ចូរស្រាយបញ្ជាក់ថា: $m=n$ ។

(Canadian Mathematical Olympiad 2006)

ដំណោះស្រាយ

តាង a_{ij} ជាធាតុស្ថិតនៅក្នុងជួរដេកទី i និង ជួរឈរទី j ហើយ តាង $S=\{(i,j):a_{ij}>0\}$ ។

ឧបមាថា r_i ជាផលបូកធាតុទាំងអស់ក្នុងជួរដេកទី i និង c_j ជាផលបូកធាតុក្នុងជួរឈរទី j នោះ $r_i=c_j$ ពេលណាដែល $(i,j)\in S$ ។

យើងបាន: $\sum\left\{\dfrac{a_{ij}}{r_i}:(i,j)\in S\right\}=\sum\left\{\dfrac{a_{ij}}{c_j}:(i,j)\in S\right\}$

ដោយ:

$$\sum\left\{\frac{a_{ij}}{r_i}:(i,j)\in S\right\}=\sum\left\{\frac{a_{ij}}{r_i}:1\le i\le m,1\le j\le n\right\}$$
$$=\sum_{i=1}^{m}\left(\frac{1}{r_i}\sum_{j=1}^{n}a_{ij}\right)=\sum_{i=1}^{m}\left(\frac{1}{r_i}\right)r_i=\sum_{i=1}^{m}1=m$$

ហើយ:

$$\sum\left\{\frac{a_{ij}}{c_j}:(i,j)\in S\right\}=\sum\left\{\frac{a_{ij}}{c_j}:1\le i\le m,1\le j\le n\right\}$$
$$=\sum_{j=1}^{n}\left(\frac{1}{c_j}\sum_{j=1}^{m}a_{ij}\right)=\sum_{j=1}^{n}\left(\frac{1}{c_j}\right)c_j=\sum_{j=1}^{n}1=n$$

ដូចនេះ $m=n$ ។

<u>លំហាត់ទី៨</u>

អនុគមន៍ $f:\mathbb{R}\to\mathbb{R}$ មួយផ្ទៀងផ្ទាត់លក្ខខណ្ឌ៖ $f\left(f\left(f(0)\right)\right)=0$ និង $\left|f(a)-f(b)\right|\le|a-b|$ ចំពោះ $a,b\in\mathbb{R}$ ។

ស្រាយបញ្ជាក់ថា $f(0)=0$ ។

(Mongolia Mathematical Olympiad 2000)

<u>ដំណោះស្រាយ</u>

ចំពោះ $a,b\in\mathbb{R}$, យើងមាន: $\left|f(a)-f(b)\right|\le|a-b|$ នោះយើងបាន:

$$\left|f(0)\right|=\left|f(0)-0\right|\ge\left|f\left(f(0)\right)-f(0)\right|$$
$$\ge\left|f\left(f\left(f(0)\right)\right)-f\left(f(0)\right)\right|=\left|f\left(f(0)\right)\right| \quad (1)$$

ហើយ: $\left|f\left(f(0)\right)\right|=\left|f\left(f(0)\right)-0\right|\ge\left|f\left(f\left(f(0)\right)\right)-f(0)\right|$
$$=\left|f(0)\right| \qquad (2)$$

តាម (1) និង (2) ទាញបាន: $\left|f(0)\right|=\left|f\left(f(0)\right)\right|$

នាំឲ្យ $f(0)=\pm f\left(f(0)\right)$

$*$បើ $f(0)=f\left(f(0)\right)\Rightarrow f(0)=f\left(f(0)\right)=f\left(f\left(f(0)\right)\right)=0$

$*$បើ $f(0)=-f\left(f(0)\right)$

$\Rightarrow \left|f(0)\right|=\left|f(0)-0\right|\ge\left|f\left(f(0)\right)-f(0)\right|=2\left|f(0)\right|$

$\Rightarrow\left|f(0)\right|\le 0\Rightarrow f(0)=0$ ។

សរុបទាំងពីរករណី , ដូចនេះ $f(0)=0$ ។

<u>លំហាត់ទី៦</u>

គេឲ្យ (a_n) ជាស្វ៊ីតនៃចំនួនពិតដែលផ្ទៀងផ្ទាត់: $|a_{k+m}-a_k-a_m|\leq 1$ ចំពោះគ្រប់ចំនួនគត់វិជ្ជមាន k និង m ។ ចូរស្រាយបញ្ជាក់ថា: ចំពោះគ្រប់ចំនួនគត់វិជ្ជមាន p និង q នោះគេបាន វិសមភាព:

$$\left|\frac{a_p}{p}-\frac{a_q}{q}\right|<\frac{1}{p}+\frac{1}{q}$$ ។

(IMO Longlists 1980)

<u>ដំណោះស្រាយ</u>

ចំពោះគ្រប់ចំនួនគត់វិជ្ជមាន k និង m យើងមាន: $|a_{k+m}-a_k-a_m|\leq 1$ នោះយើងបាន:

$$-1\leq a_{k+m}-a_k-a_m\leq 1\Leftrightarrow a_{k+m}-1\leq a_k+a_m\leq a_{k+m}+1$$

យើងនឹងស្រាយតាមវិចារដោយកំនើនធៀបនឹង q ថា:

$$a_{pq}-(q-1)\leq qa_p\leq a_{pq}+(q-1) \qquad (*)$$

- ចំពោះ $q=1\Rightarrow(*):a_p\leq a_p\leq a_p$ ជាវិសមភាពពិត ។
- ឧបមាថាវិសមភាពនេះពិតដល់ q គឺថា:

$$a_{pq}-(q-1)\leq qa_p\leq a_{pq}+(q-1)$$

- ពិនិត្យចំពោះ $q+1$:

យើងមាន: $(q+1)a_p=qa_p+a_p\leq a_p+a_{pq}+(q-1)\leq a_{p+pq}+1$ $+(q-1)\ =\ a_{(q+1)p}+q$

ហើយ $(q+1)a_p=qa_p+a_p\geq a_p+a_{pq}-(q-1)\geq\ a_{p+pq}-1$ $-(q-1)=a_{(q+1)p}-q\ \Rightarrow\ a_{(q+1)p}-q\leq(q+1)a_p\leq a_{(q+1)p}+q$

នេះមានន័យថា: វិសមភាព $(*)$ ពិតដល់ $q+1$ ។

ដូច្នេះ $a_{pq}-(q-1)\leq qa_p\leq a_{pq}+(q-1)$ ចំពោះ $\forall p,q\in\mathbb{N}$ ។

ស្រាយដូចគ្នាដែរតាមវិចារដោយកំនើនធៀបនឹង p គេបាន ៖

$a_{pq}-(p-1)\le pa_q\le a_{pq}+(p-1)$ ចំពោះ $\forall p,q\in\mathbb{N}$ ។

យើងបានៈ

$$\left|qa_p-pa_q\right|\le\max\left\{\left|a_{pq}+(p-1)-\left(a_{pq}-(q-1)\right)\right|,\right.$$

$$\left.\left|a_{pq}+(q-1)-\left(a_{pq}-(p-1)\right)\right|\right\}=p+q-2<p+q$$

$\Rightarrow\quad\left|\frac{a_p}{p}-\frac{a_q}{q}\right|<\frac{1}{p}+\frac{1}{q}$ ចំពោះ $\forall p,q\in\mathbb{N}$ ។

ដូចនេះ វិសមភាពត្រូវបានស្រាយបញ្ជាក់ ។

លំហាត់ទី៧

តាង a ជាចំនួនគត់សេសមួយ ។ ចូរស្រាយបញ្ជាក់ថាៈ $a^{2^n}+2^{2^n}$ និង $a^{2^m}+2^{2^m}$ ជាចំនួនបឋមរវាងគ្នាចំពោះគ្រប់ចំនួនគត់វិជ្ជមាន n និង m ដែល $n\neq m$ ។

(Baltic Way 2001)

ដំណោះស្រាយ

ដោយមិនធ្វើឲ្យបាត់បង់លក្ខណៈទូទៅ , យើងសន្មតថា $m>n$ ។

ចំពោះចំនួនបឋម p ណាមួយដែលចែកដាច់ $a^{2^n}+2^{2^n}$ នោះគេបានៈ

$a^{2^n}\equiv-2^{2^n}\pmod p$

លើកអង្គទាំងពីរនៃសមីការនេះជាការចំនួន $m-n$ ដង គេបានៈ

$a^{2^m}\equiv 2^{2^m}\pmod p$ ហើយដោយ a ជាចំនួនគត់សេសនោះ $p\neq 2$

$\Rightarrow 2^{2^m}+2^{2^m}=2^{2^m+1}\not\equiv 0\pmod p$

$\Rightarrow a^{2^m} \equiv 2^{2^m} \equiv -2^{2^m} \pmod p$

$\Rightarrow a^{2^m} + 2^{2^m} \not\equiv 0 \pmod p \Rightarrow p$ ចែកមិនដាច់ $a^{2^m} + 2^{2^m}$

នេះបញ្ជាក់ថាៈ គ្រប់កត្តាបឋមនៃ $a^{2^n} + 2^{2^n}$ គឺចែកមិនដាច់ $a^{2^m} + 2^{2^m}$ ។

ដូចនេះ $a^{2^n} + 2^{2^n}$ និង $a^{2^m} + 2^{2^m}$ ជាចំនួនបឋមរវាងគ្នាចំពោះគ្រប់ចំនួនគត់វិជ្ជមាន n និង m ដែល $n \neq m$ ។

លំហាត់ទី៨

តាង a,b,c ជាបីចំនួនគត់ខុសៗគ្នា ។ $P(x)$ ជាពហុធាដែលមានមេគុណជាចំនួនគត់ ។ ចូរបង្ហាញថាៈ មិនអាចមានករណីដែល $P(a)=b$, $P(b)=c$ និង $P(c)=a$ នោះទេ ។

(USAMO 1974)

ដំណោះស្រាយ

យើងមាន $P(x)$ ជាពហុធាដែលមានមេគុណជាចំនួនគត់ ហើយ a,b,c ជាបីចំនួនគត់ខុសៗគ្នា នោះតាមទ្រឹស្តីបទផលគុណកត្តាយើងបានៈ

$a-b \mid P(a)-P(b), b-c \mid P(b)-P(c)$ និង $c-a \mid P(c)-P(a)$

$\Rightarrow \dfrac{P(a)-P(b)}{a-b}, \dfrac{P(b)-P(c)}{b-c}, \dfrac{P(c)-P(a)}{c-a}$ ជាចំនួនគត់

ឧបមាថាៈ $P(a)=b$, $P(b)=c$ និង $P(c)=a$

$\Rightarrow \dfrac{b-c}{a-b}, \dfrac{c-a}{b-c}, \dfrac{a-b}{c-a}$ សុទ្ធតែជាចំនួនគត់ដែលមានផលគុណស្មើនឹង 1

នោះយើងទាញបាន តម្លៃដាច់ខាតរបស់វានីមួយៗត្រូវតែស្មើនឹង 1 មានន័យថាៈ

$$\left|\frac{b-c}{a-b}\right| = \left|\frac{c-a}{b-c}\right| = \left|\frac{a-b}{c-a}\right| = 1 \Rightarrow |a-b| = |b-c| = |c-a|$$

ជាករណីមិនអាច ព្រោះ a,b,c ជាចំនួនគត់ ខុសៗគ្នា , នេះបញ្ជាក់ថា: ការឧបមាខាងលើគឺខុស ។

ដូចនេះ មិនអាចមានករណីដែល: $P(a)=b$, $P(b)=c$ និង $P(c)=a$ នោះទេ ។

<u>លំហាត់ទី៩</u>

ស្វ៊ីត $p_1, p_2, \ldots$ ត្រូវបានកំណត់ដូចតទៅ: $p_1 = 2$ និង ចំពោះ $n \geq 2$, p_n ជាតួចែកបឋមដែលធំបំផុតនៃ $p_1 p_2 \ldots p_{n-1} + 1$ ។

ចូរស្រាយបញ្ជាក់ថា: 5 មិនមែនជាតួនៃស្វ៊ីតនេះទេ ។

(Australia Mathematical Olympiad 1982)

<u>ដំណោះស្រាយ</u>

យើងមាន: $p_1 = 2$ និង p_2(តួចែកបឋមធំបំផុតនៃ $p_1 + 1 = 3$)$= 3$

ហើយចំពោះ $k > 1$ យើងមាន:

$p_k =$ តួចែកបឋមធំបំផុតនៃ $p_1 p_2 \ldots p_{k-1} + 1$ នោះ $p_k > 2$

ព្រោះ $p_1 p_2 \ldots p_{k-1} + 1$ ជាចំនួនគត់សេស ។

ឧបមាថា: 5 ជាតួនៃស្វ៊ីត (p_n) មានន័យថា $p_n = 5$ ចំពោះ n ណាមួយ

គេបាន: $p_1 p_2 \ldots p_{n-1} + 1 = 3^m.5^l$ ព្រោះ $p_1 p_2 \ldots p_{n-1} + 1$ ជាចំនួនគត់សេស ហើយ 5 ជាតួចែកបឋមធំបំផុតរបស់វា ។

យើងសង្កេតឃើញថា: ចំពោះ $m \geq 1$ នោះ $3^m.5^l \equiv 0 \pmod 3$

តែ $p_1 p_2 \ldots p_{n-1} + 1 = 2 \cdot 3 \cdot p_3 \cdots p_{n-1} + 1 \equiv 1 \pmod 3$

$\Rightarrow p_1 p_2 \ldots p_{n-1} + 1 = 3^m.5^l$ ជាសមភាពផ្ទៀងផ្ទាត់ លុះត្រាតែ: $m = 0$

ក្នុងករណីនេះ គេបាន:

$p_1p_2...p_{n-1} = 2\cdot 3\cdot p_3\cdots p_{n-1} = 5^l - 1$

$$= (5-1)\left(5^{l-1}+5^{l-2}+\cdots+1\right)$$

គឺថា: $2\cdot 3\cdot p_3\cdots p_{n-1} = 4\left(5^{l-1}+5^{l-2}+\cdots+1\right)$

សមភាពនេះមិនពិតទេព្រោះ អង្គទី១ $\not\equiv 0 \pmod 4$

(ព្រោះ $p_3, p_4, ..., p_{n-1}$ សុទ្ធតែជាចំនួនបឋម>2)

តែអង្គទី២ $\equiv 0 \pmod 4$, នេះបញ្ជាក់ថា: ការឧបមាខាងលើគឺខុស

មានន័យថា: $p_n \neq 5$ ចំពោះគ្រប់ n ។

ដូចនេះ 5 មិនមែនជាតួនៃស្វ៊ីត(p_n) នោះទេ ។

<u>លំហាត់ទី១០</u>

តាង $f:\mathbb{N}\to\mathbb{N}$ ជាអនុគមន៍មួយដែលផ្ទៀងផ្ទាត់:

$f(n+1) > f(f(n))$ ចំពោះគ្រប់ចំនួនគត់វិជ្ជមាន n ។

ចូរស្រាយបញ្ជាក់ថា: $f(n) = n$ ចំពោះគ្រប់ចំនួនគត់វិជ្ជមាន n ។

(IMO 1977)

<u>ដំណោះស្រាយ</u>

តាង $P(n)$ ជាសំណើមួយដែលថា:“ បើ $m \geq n$ នោះ $f(m) \geq n$ ” ។

តាមវិចារដោយកំនើន , យើងនឹងស្រាយថា $P(n)$ ជាសំណើពិត:

ដោយចំពោះគ្រប់ $m \geq 1, f(m) \geq 1$ នោះបញ្ជាក់ថា: $P(1)$ ជាសំណើពិត

ឧបមាថាសំណើ $P(n)$ ពិតដល់ n គឺថា: បើ $m \geq n$ នោះ $f(m) \geq n$

យើងនឹងស្រាយថា: $P(n+1)$ ពិតគឺត្រូវស្រាយថា:

បើ $m \geq n+1$ នោះ $f(m) \geq n+1$ ។

ដោយ $m \geq n+1$ ឬ $m-1 \geq n$

$\Rightarrow f(m-1) \geq n \Rightarrow f(f(m-1)) \geq n$

យើងបាន: $f(m) > f(f(m-1)) \geq n$ ទាញបាន: $f(m) \geq n+1$

មានន័យថា: $P(n+1)$ ជាសំណើពិត ។

ដូច្នេះជាទូទៅ បើ $m \geq n$ នោះ $f(m) \geq n$ ។

ក្នុងករណីពិសេសយើងបាន: $f(n) \geq n$ ចំពោះគ្រប់ $n \geq 1$ ។

$\Rightarrow f(n+1) > f(f(n)) \geq f(n)$ ទាញបាន: f ជាអនុគមន៍កើនដាច់ខាត ។

យើងឧបមាថា: មាន k ណាមួយដែលធ្វើឲ្យ $f(k) \neq k$ ។

$\Rightarrow f(k) > k \Leftrightarrow f(k) \geq k+1$

$\Rightarrow f(k+1) > f(f(k)) \geq f(k+1)$ នេះជាសមភាពមិនពិត

ដែលបញ្ជាក់ថា: គ្មានចំនួនគត់ k ណាមួយធ្វើឲ្យ $f(k) \neq k$ នោះឡើយ ។

ដូចនេះ $f(n) = n$ ចំពោះគ្រប់ចំនួនគត់វិជ្ជមាន n ។

លំហាត់ទី១១

តាង p និង q ជាចំនួនកុំផ្លិចដែល: $q \neq 0$ ។ ចូរស្រាយបញ្ជាក់ថា: បើឫសទាំងអស់នៃសមីការ $x^2 + px + q^2 = 0$ មានតម្លៃដាច់ខាតដូចគ្នា នោះ $\frac{p}{q}$ ជាចំនួនពិត ។

(Romanian Mathematical Olympiad 1999)

ដំណោះស្រាយ

តាង x_1 និង x_2 ជាឫសនៃសមីការ ហើយ $r = |x_1| = |x_2|$ ។

តាមទ្រឹស្តីបទវ្យែត គេបាន: $x_1 + x_2 = -p$ និង $x_1 x_2 = q^2$

$$\Rightarrow \frac{p^2}{q^2} = \frac{(x_1+x_2)^2}{x_1x_2} = \frac{x_1}{x_2} + \frac{x_2}{x_1} + 2$$

$$= \frac{x_1\bar{x}_2}{|x_2|^2} + \frac{x_2\bar{x}_1}{|x_1|^2} + 2 = \frac{x_1\bar{x}_2}{r^2} + \frac{x_2\bar{x}_1}{r^2} + 2$$

$$= 2 + \frac{1}{r^2}(x_1\bar{x}_2 + x_2\bar{x}_1) = 2 + \frac{2}{r_2}\cdot \text{Re}(x_1\bar{x}_2)$$

ព្រោះតាមរូបមន្ត: $z + \bar{z} = 2\text{Re}(z)$

មានន័យថា: $\frac{p^2}{q^2}$ ជាចំនួនពិត (1)

ម្យ៉ាងទៀត យើងមាន: $\text{Re}\left(x_1\overline{x_2}\right) \geq -\left|x_1\overline{x_2}\right| = -|x_1|\left|\overline{x_2}\right| = -r^2$

$$\Rightarrow \frac{p^2}{q^2} \geq 0 \quad (2)$$

តាម (1) និង (2) ទាញបាន: $\frac{p}{q}$ ជាចំនួនពិត ។

ដូចនេះ: $\frac{p}{q}$ ជាចំនួនពិតមួយ , ត្រូវបានស្រាយបញ្ជាក់ ។

__លំហាត់ទី៦__

ស៊ីតហ្វីបូណាស៊ី $(F_n)_{n\geq 1}$ ត្រូវបានកំណត់ដោយ: $F_1 = F_2 = 1$ និង $F_{n+2} = F_{n+1} + F_n, n \geq 1$ ។ ចូរស្រាយបញ្ជាក់ថា:

$$F_{2n} = \frac{F_{2n+2}^3 + F_{2n-2}^3}{9} - 2F_{2n}^3 \quad \text{ចំពោះគ្រប់ } n \geq 2 \text{ ។}$$

(Korean Mathematics Competition 2000)

ដំណោះស្រាយ

យើងមាន:

$$F_{2n+2}=F_{2n+1}+F_{2n}=F_{2n}+F_{2n}+F_{2n-1}=2F_{2n}+F_{2n}-F_{2n-2}$$

$$\Rightarrow F_{2n+2}=3F_{2n}-F_{2n-2}$$

$$\Rightarrow F_{2n+2}^3=\left(3F_{2n}-F_{2n-2}\right)^3$$

$$=27F_{2n}^3-F_{2n-2}^3-9F_{2n}F_{2n-2}\left(3F_{2n}-F_{2n-2}\right)$$

$$\Rightarrow F_{2n+2}^3+F_{2n-2}^3=27F_{2n}^3-9F_{2n}F_{2n-2}F_{2n+2}\qquad(*)$$

ម្យ៉ាងទៀត យើងមាន: $F_{2n}+F_{2n-4}=3F_{2n-2}$

$$\Rightarrow F_{2n}\left(F_{2n}+F_{2n-4}\right)=3F_{2n}F_{2n-2}=\left(F_{2n+2}+F_{2n-2}\right)F_{2n-2}$$

$$\Rightarrow F_{2n}^2-F_{2n-2}F_{2n+2}=F_{2n-2}^2-F_{2n-4}F_{2n}=\cdots=F_4^2-F_2F_6$$

$$=3^2-1\cdot 8=1\quad\Rightarrow F_{2n-2}F_{2n+2}=F_{2n}^2-1$$

$\Rightarrow$ តាម $(*)$ គេបាន: $F_{2n+2}^3+F_{2n-2}^3=27F_{2n}^3-9F_{2n}\left(F_{2n}^2-1\right)$

$$\Leftrightarrow\ F_{2n}=\frac{F_{2n+2}^3+F_{2n-2}^3}{9}-2F_{2n}^3$$

ដូចនេះសមភាពត្រូវបានស្រាយបញ្ជាក់ ។

លំហាត់ទី១៣

ចូរស្រាយបញ្ជាក់ថាឫសទាំង5នៃសមីការ:

$a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0=0$ មិនអាចជាចំនួនពិតទាំងអស់បានទេ បើ $2a_2^4<5a_5a_3$ ។

(USAMO 1983)

ដំណោះស្រាយ

តាង r_i , $(i=1,2,3,4,5)$ ជាឫសទាំង 5 នៃសមីការ:

$a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0=0$ នោះតាមទ្រឹស្តីបទវ្យែត

គេបាន: $\sum_{i=1}^{5} r_i = -\frac{a_4}{a_5}$ ហើយ $\sum_{1\le i<j\le 5} r_ir_j = \frac{a_3}{a_5}$ ។

តាមលក្ខខ័ណ្ឌ: $2a_2^4 < 5a_5a_3 \Leftrightarrow 2\left(\frac{a_4}{a_5}\right)^2 < 5\cdot\frac{a_3}{a_5}$

$$\Leftrightarrow 2\left(\sum_{i=1}^{5} r_i\right)^2 < 5\cdot\sum_{1\le i<j\le 5} r_ir_j$$

$$\Leftrightarrow 2\cdot\sum_{i=1}^{5} r_i^2 + 4\cdot\sum_{1\le i<j\le 5} r_ir_j < 5\cdot\sum_{1\le i<j\le 5} r_ir_j$$

$$\Leftrightarrow 2\cdot\sum_{i=1}^{5} r_i^2 < \sum_{1\le i<j\le 5} r_ir_j \qquad (*)$$

យើងឧបមាថា: r_i , $(i=1,2,3,4,5)$ សុទ្ធតែជាចំនួនពិតនោះ

យើងបាន: $r_i^2+r_j^2 \ge 2r_ir_j \quad ,(1\le i<j\le 5)$

$$\Rightarrow \sum_{1\le i<j\le 5}\left(r_i^2+r_j^2\right) \ge 2\cdot\sum_{1\le i<j\le 5}\left(r_ir_j\right)$$

$$\Leftrightarrow 2\sum_{i=1}^{5} r_i^2 \ge \sum_{1\le i<j\le 5} r_ir_j$$ ផ្ទុយពីលក្ខខ័ណ្ឌ $(*)$

នេះមានន័យថា:ការឧបមាខាងលើគឺខុស ។

ដូចនេះ ឫសទាំង5នៃសមីការ:

$a_5x^5+a_4x^4+a_3x^3+a_2x^2+a_1x+a_0=0$ មិនអាចជាចំនួនពិតទាំងអស់បានទេ បើ $2a_2^4 < 5a_5a_3$ ។

<u>លំហាត់ទី១៨</u>

តាង x,y ជាចំនួនគត់ដែល $x,y\neq -1$ ហើយផ្ទៀងផ្ទាត់:

$\frac{`x^4-1}{y+1}+\frac{y^4-1}{x+1}\in\mathbb{Z}$ ។ ចូរស្រាយបញ្ជាក់ថា:

$x^4y^{44}-1$ ចែកដាច់នឹង $x+1$ ។

(Vietnam Mathematical Olympiad 2007)

<u>ដំណោះស្រាយ</u>

យើងតាង $\frac{x^4-1}{y+1}=\frac{a}{b}$ និង $\frac{y^4-1}{x+1}=\frac{c}{d}$ ដែល $b,d\neq 0$ និង $(a,b)=(c,d)=1$ ។

តាមសម្មតិកម្ម: $\frac{`x^4-1}{y+1}+\frac{y^4-1}{x+1}\in\mathbb{Z}$

$\Rightarrow \frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}\in\mathbb{Z}\Rightarrow bd\,|\,ad+bc\ \Rightarrow d\,|\,ad+bc\Rightarrow d\,|\,bc$

$\Rightarrow d\,|\,b$ (ព្រោះ $(c,d)=1$) $(*)$

ម្យ៉ាងទៀត: $\frac{ac}{bd}=(x+1)\left(x^2+1\right)(y+1)\left(y^2+1\right)\in\mathbb{Z}\ \Rightarrow bd\,|\,ac$

$\Rightarrow b\,|\,ac\Rightarrow b\,|\,c$ (ព្រោះ $(a,b)=1$) $(**)$

តាម $(*)$ និង $(**)$ គេបាន: $d\,|\,c\Rightarrow d=1\Rightarrow\frac{y^4-1}{x+1}=c\in\mathbb{Z}$

$\Rightarrow x+1\,|\,y^4-1$

យើងមាន:

$$x^4y^{44}-1=x^4y^{44}-y^{44}+y^{44}-1$$

$$=y^{44}\left(x^4-1\right)+\left(y^4-1\right)m\quad,(m\in\mathbb{Z})$$

$$= y^{44}(x-1)(x+1)\left(x^2+1\right)+\left(y^4-1\right)m$$

$$\Rightarrow\ x+1 \mid x^4 y^{44}-1$$

ដូចនេះ $x^4y^{44}-1$ ចែកដាច់នឹង $x+1$ ។

លំហាត់ទី១៥

តាង $\{a_n\}_{n\geq 0}$ ជាស្វ៊ីតនៃចំនួនពិតដែលផ្ទៀងផ្ទាត់: $a_{n+1}\geq a_n^2+\frac{1}{5}$ ចំពោះគ្រប់ $n\geq 0$ ។ ចូរស្រាយបញ្ជាក់ថា: $\sqrt{a_{n+5}}\geq a_{n-5}^2$ ចំពោះគ្រប់ $n\geq 5$ ។

(USA , Team Selection Test 2001)

ដំណោះស្រាយ

យើងមាន:

$$\begin{cases} a_{n+1}\geq a_n^2+\frac{1}{5} \\ a_{n+2}\geq a_{n+1}^2+\frac{1}{5} \\ a_{n+3}\geq a_{n+2}^2+\frac{1}{5} \\ a_{n+4}\geq a_{n+3}^2+\frac{1}{5} \\ a_{n+5}\geq a_{n+4}^2+\frac{1}{5} \end{cases}$$

បូកអង្គនិងអង្គនៃវិសមភាពទាំងអស់ខាងលើនេះ គេបាន:

$$a_{n+5} \geq a_n^2 + \left(a_{n+1}^2 - a_{n+1} + \frac{1}{4}\right) + \left(a_{n+2}^2 - a_{n+2} + \frac{1}{4}\right)$$

$$+\left(a_{n+3}^2 - a_{n+3} + \frac{1}{4}\right) + \left(a_{n+4}^2 - a_{n+4} + \frac{1}{4}\right)$$

$$= a_n^2 + \left(a_{n+1} - \frac{1}{2}\right)^2 + \left(a_{n+2} - \frac{1}{2}\right)^2 + \left(a_{n+3} - \frac{1}{2}\right)^2$$

$$+\left(a_{n+4} - \frac{1}{2}\right)^2 \geq a_n^2 \quad \Rightarrow a_{n+5} \geq a_n^2 \quad ,\forall\ n \geq 0$$

$\Rightarrow a_n \geq a_{n-5}^2 \ ,\forall n \geq 5 \quad \Rightarrow a_{n+5} \geq a_{n-5}^4 \ ,\forall n \geq 5$

$\Rightarrow \sqrt{a_{n+5}} \geq a_{n-5}^2 \ ,\forall n \geq 5$ ។

ដូចនេះ $\sqrt{a_{n+5}} \geq a_{n-5}^2$ ចំពោះគ្រប់ $n \geq 5$ ។

លំហាត់ទី១៦

គេតាង $\{x\}$ ជាផ្នែកទសភាគនៃចំនួនពិត x ។ ចូរស្រាយបញ្ជាក់ថា:

$\lim_{n\to\infty}\left\{\left(2+\sqrt{3}\right)^n\right\} = 1$ ។

(Australia mathematical Olympiad 1982)

ដំណោះស្រាយ

តាមរូបមន្តនៃការពន្លាតទ្វេធា *Newton*, យើងសង្កេតឃើញថា:

$\left(2-\sqrt{3}\right)^n = a - b\sqrt{3}$ និង $\left(2+\sqrt{3}\right)^n = a + b\sqrt{3}$

ដែល a,b ជាចំនួនគត់ ។

យើងបាន: $\left(2-\sqrt{3}\right)^n+\left(2+\sqrt{3}\right)^n=2a$ ជាចំនួនគត់

$$\Rightarrow \left\{\left(2-\sqrt{3}\right)^n\right\}+\left\{\left(2+\sqrt{3}\right)^n\right\}=1$$

$$\Rightarrow \left\{\left(2+\sqrt{3}\right)^n\right\}=1-\left\{\left(2-\sqrt{3}\right)^n\right\}$$

ដោយ $0<2-\sqrt{3}<1 \Rightarrow 0<\left(2-\sqrt{3}\right)^n<1$

$$\Rightarrow \left\{\left(2-\sqrt{3}\right)^n\right\}=\left(2-\sqrt{3}\right)^n$$

យើងបាន: $\left\{\left(2+\sqrt{3}\right)^n\right\}=1-\left(2-\sqrt{3}\right)^n$

$$\Rightarrow \lim_{n\to\infty}\left\{\left(2+\sqrt{3}\right)^n\right\}=1-\lim_{n\to\infty}\left(2-\sqrt{3}\right)^n=1$$

ព្រោះ $0<2-\sqrt{3}<1 \Rightarrow \lim_{n\to\infty}\left(2-\sqrt{3}\right)^n=0$ ។

ដូចនេះ $\lim_{n\to\infty}\left\{\left(2+\sqrt{3}\right)^n\right\}=1$ ត្រូវបានស្រាយបញ្ជាក់ ។

<u>លំហាត់ទី១៧</u>

តាង $f(x)=x^2+2007x+1$ ។ ចូរស្រាយបញ្ជាក់ថា:
ចំពោះគ្រប់ចំនួនគត់វិជ្ជមាន n , សមីការ $f\left(f\left(\ldots\left(f(x)\right)\ldots\right)\right)=0$
(មាន n ដងនៃ f) , មានយ៉ាងតិចឫសជាចំនួនពិតមួយ ។

(Brazil mathematical Olympiad 2007)

ដំណោះស្រាយ

តាង $G(x)=f\left(f\left(...\left(f(x)...\right)\right)\right)$, (មាន n ដងនៃ f) ។

យើងសង្កេតឃើញថា:

បើ $x\to\infty \Rightarrow f(x)=x^2+2007x+1$

$$=x^2\left(1+\frac{2007}{x}+\frac{1}{x^2}\right)\to+\infty$$

$\Rightarrow G(x)\to+\infty$

$\Rightarrow$ បើ $x\to\infty\Rightarrow G(x)\to+\infty$

នេះមានន័យថា: ប្រាកដជាមានចំនួនពិត α មានតម្លៃធំគ្រប់គ្រាន់មួយ ដែល $G(\alpha)>0 \quad (1)$

ម្យ៉ាងទៀត: យើងតាង β ជាឫសមួយនៃសមីការ:

$f(x)=x \Leftrightarrow x^2+2006x+1=0$ ដែលជាសមីការមានឫសសុទ្ធតែជាចំនួនពិតវិជ្ជមាននោះយើងទាញបាន: $\beta<0$ ។

$\Rightarrow G(\beta)=\beta<0 \quad (2)$

តាម (1) និង (2) គេបាន: $G(\alpha)\cdot G(\beta)<0$,

នោះតាមទ្រឹស្តីបទតម្លៃកណ្តាល: $\exists x$ មួយដែល $x\in(\alpha,\beta)$ ធ្វើឲ្យ $G(x)=0$ ។

ដូចនេះ ចំពោះគ្រប់ចំនួនគត់វិជ្ជមាន n ,

សមីការ $G(x)=f\left(f\left(...\left(f(x)\right)...\right)\right)=0$ (មាន n ដងនៃ f),

មានយ៉ាងតិចឫសជាចំនួនពិតមួយ ។

លំហាត់ទី១៨

គេឲ្យអនុគមន៍: $f(x)=\cos(a_1+x)+\frac{1}{2}\cos(a_2+x)+$ $\frac{1}{4}\cos(a_3+x)+\cdots+\frac{1}{2^{n-1}}\cos(a_n+x)$ ដែល a_i ជាចំនួនពិតថេរ និង អថេរពិត x, ចូរស្រាយបញ្ជាក់ថា: បើ $f(x_1)=f(x_2)=0$ នោះ គេបាន: x_1-x_2 ជាពហុគុណនៃ π ។

(IMO 1969)

ដំណោះស្រាយ

យើងមាន: $f(x)=\cos(a_1+x)+\frac{1}{2}\cos(a_2+x)+\frac{1}{4}\cos(a_3+x)$

$$+\cdots+\frac{1}{2^{n-1}}\cos(a_n+x) \qquad (*)$$

$$\Rightarrow f(-a_1)=\cos 0+\frac{1}{2}\cos(a_2-a_1)+\cdots+\frac{1}{2^{n-1}}\cos(a_n-a_1)$$

$$\Rightarrow f(-a_1)\geq 1-\frac{1}{2}-\frac{1}{4}-\cdots\frac{1}{2^{n-1}}=1-\left(1-\frac{1}{2^{n-1}}\right)>0$$

នេះបញ្ជាក់ថាអនុគមន៍ f(x) មិនមែនស្មើនឹង 0 គ្រប់ x នោះទេ ។

ម្យ៉ាងទៀត:

$$f(x)=\sum_{k=1}^{n}\frac{1}{2^{k-1}}\cdot\cos(a_k+x)$$

$$=\sum_{k=1}^{n}\left(\frac{1}{2^{k-1}}\cdot\cos a_k\cdot\cos x-\frac{1}{2^{k-1}}\cdot\sin a_k\cdot\sin x\right)$$

$$=\cos x\cdot\sum_{k=1}^{n}\frac{1}{2^{k-1}}\cdot\cos a_k-\sin x\cdot\sum_{k=1}^{n}\frac{1}{2^{k-1}}\cdot\sin a_k$$

តាង $A=\sum_{k=1}^{n}\frac{1}{2^{k-1}}\cdot\cos a_k$ និង $B=\sum_{k=1}^{n}\frac{1}{2^{k-1}}\cdot\sin a_k$

ដែល A និង B មិនអាចស្មើនឹង 0 ព្រមគ្នានោះទេ ព្រោះ បើ

$A=B=0$ នោះ $f(x)=0$ គ្រប់ x ដែលផ្ទុយពី $(*)$ ។

ក្នុងករណីនេះ , យើងបាន:

$$\begin{aligned} f(x) &= A\cdot\cos x - B\cdot\sin x \\ &= \sqrt{A^2+B^2}\left(\sin\alpha\cdot\cos x-\cos\alpha\cdot\sin x\right) \\ &= \sqrt{A^2+B^2}\cdot\sin(\alpha-x) \quad ,\left(\sqrt{A^2+B^2}\neq 0\right) \end{aligned}$$

ដែល $\sin\alpha=\frac{A}{\sqrt{A^2+B^2}}$ និង $\cos\alpha=\frac{B}{\sqrt{A^2+B^2}}$ ។

ដោយ $f(x_1)=f(x_2)=0 \Rightarrow \sin(\alpha-x_1)=\sin(\alpha-x_1)=0$

$$\Rightarrow \begin{cases}\alpha-x_1=k_1\pi \\ \alpha-x_2=k_2\pi\end{cases},(k_1,k_2\in\mathbb{Z}) \quad \Rightarrow x_1-x_2=(k_1-k_2)\pi$$

$\Rightarrow$ x_1-x_2 ជាពហុគុណ នៃ π ។

លំហាត់ទី១៩

តាង $n>3$ និង $a_1,a_2,...,a_n$ ជាចំនួនពិតដែលផ្ទៀងផ្ទាត់:

$a_1+a_2+\cdots+a_n\geq n$ និង ${a_1}^2+{a_2}^2+\cdots+{a_n}^2\geq n^2$ ។

ចូរស្រាយបញ្ជាក់ថា: $\max\{a_1,a_2,...,a_n\}\geq 2$ ។

(USAMO 1999)

ដំណោះស្រាយ

យើងឧបមាថា: $a_i < 2$ ចំពោះគ្រប់ $i = 1, 2, ..., n$ ។

តាង $x_i = 2 - a_i > 0$ គេបាន:

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} (2 - a_i) = \sum_{i=1}^{n} 2 - \sum_{i=1}^{n} a_i = 2n - \sum_{i=1}^{n} a_i \le 2n - n = n$$

ហើយ $n^2 \le \sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} (2 - x_i)^2 = 4n - 4\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i^2 \quad (1)$

តែចំពោះ $x_i > 0$ គេបាន: $\sum_{i=1}^{n} x_i^2 \le \left(\sum_{i=1}^{n} x_i\right)^2 \le n \cdot \sum_{i=1}^{n} x_i$

$\Rightarrow n^2 \le 4n - 4\sum_{i=1}^{n} x_i + n \cdot \sum_{i=1}^{n} x_i$

$\Leftrightarrow (n-4)\left(n - \sum_{i=1}^{n} x_i\right) \le 0$, វិសមភាពនេះមិនពិតព្រោះ $n \ge 4$ និង

$\sum_{i=1}^{n} x_i \le n$ ។

នាំឲ្យការឧបមាខាងលើគឺខុស មានន័យថា: ចំនួន $a_i, (i = 1, 2, ..., n)$ មិនអាចសុទ្ធតែតូចជាង 2 ទេគឺត្រូវមានយ៉ាងហោចណាស់ a_i មួយដែរដែល ≥ 2 ។

ដូចនេះ $\max\{a_1, a_2, ..., a_n\} \ge 2$ ។

លំហាត់ទី២០

គេកំណត់ស្វ៊ីតនៃអនុគមន៍ $\left(f_n(x)\right)_{n\in\mathbb{N}_0}$ ដូចតទៅ: $f_0(x)=1,$
$f_1(x)=x$ និង $\left(f_n(x)\right)^2-1=f_{n-1}(x)\cdot f_{n+1}(x)$,
ចំពោះគ្រប់ $n\geq 1$ ។ ចូរស្រាយបញ្ជាក់ថា: ចំពោះn នីមួយៗ ,
$f_n(x)$ ជាពហុធាដែលមានមេគុណជាចំនួនគត់ ។

(India Mathematical Olympiad 2012)

ដំណោះស្រាយ

ចំពោះគ្រប់ $n\geq 1,$ យើងមាន: $\left(f_n(x)\right)^2-1=f_{n-1}(x)\cdot f_{n+1}(x)$

$$\Rightarrow\left(f_{n+1}(x)\right)^2-1=f_n(x)\cdot f_{n+2}(x)$$

ដកអង្គនិងអង្គនៃសមភាពទាំងពីរនេះ, យើងបាន:

$$\left(f_n(x)\right)^2-\left(f_{n+1}(x)\right)^2=f_{n-1}(x)\cdot f_{n+1}(x)-f_n(x)\cdot f_{n+2}(x)$$

$$\Leftrightarrow f_n(x)\left(f_n(x)+f_{n+2}(x)\right)=f_{n+1}(x)\left(f_{n-1}(x)+f_{n+1}(x)\right)$$

$$\Rightarrow\frac{f_{n+2}(x)+f_n(x)}{f_{n+1}(x)}=\frac{f_{n+1}(x)+f_{n-1}(x)}{f_n(x)}=\cdots=\frac{f_2(x)+f_0(x)}{f_1(x)}$$

$$=x\quad\Rightarrow f_{n+1}(x)=xf_n(x)-f_{n-1}(x)\quad ,n\geq 1$$

តាមវិចារដោយកំនើន, យើងនឹងអាចស្រាយបានថា: $f_n(x)$ ជាពហុធាមានមេគុណជាចំនួនគត់ ។

ដូចនេះ ចំពោះn នីមួយៗ , $f_n(x)$ ជាពហុធាដែលមានមេគុណជាចំនួនគត់ ។

លំហាត់ទី២១

តាង k,m,n ជាចំនួនគត់វិជ្ជមានដែលផ្ទៀងផ្ទាត់: $k>n>1$ និង $(k,n)=1$ ។ ចូរស្រាយបញ្ជាក់ថា: បើ $k-n|k^m-n^{m-1}$ នោះ $k\le 2n-1$ ។

(Indonesia National Science Olympiad 2003)

ដំណោះស្រាយ

យើងមាន: $k^m-n^m=(k-n)\left(k^{m-1}+k^{m-2}\cdot n+\cdots+k\cdot n^{m-2}+n^{m-1}\right)$

$\Rightarrow k-n|k^m-n^m$ តែតាមសម្មតិកម្ម: $k-n|k^m-n^{m-1}$

$\Rightarrow k-n|\left(k^m-n^{m-1}\right)-\left(k^m-n^m\right)$

$\Leftrightarrow k-n|n^m-n^{m-1}=n^{m-1}(n-1) \qquad (1)$

ដោយ $(k,n)=1 \Rightarrow (k-n,n)=1 \Rightarrow \left(k-n,n^{m-1}\right)=1 \quad (2)$

តាម (1) និង (2) គេទាញបាន: $k-n|n-1$

ដោយសារតែ $k>n>1$, យើងបាន: $k-n\le n-1\Leftrightarrow k\le 2n-1$

ដូចនេះ $k\le 2n-1$ ត្រូវបានស្រាយបញ្ជាក់ ។

លំហាត់ទី២២

តាង a,b ជាចំនួនពិតដែលផ្ទៀងផ្ទាត់: $3^a+13^b=17^a$ និង $5^a+7^b=11^b$ ។ ចូរស្រាយបញ្ជាក់ថា: $a<b$ ។

(Romanian Mathematical Olympiad 2002)

ដំណោះស្រាយ

យើងឧបមាថា: $a \geq b$ នោះ $13^b \leq 13^a$ និង $5^b \leq 5^a$

តាមសម្មតិកម្ម: $3^a + 13^b = 17^a \Rightarrow 3^a + 13^a \geq 17^a$

$$\Leftrightarrow \left(\frac{3}{17}\right)^a + \left(\frac{13}{17}\right)^a \geq 1$$

ពិនិត្យអនុគមន៍: $f(x) = \left(\frac{3}{17}\right)^x + \left(\frac{13}{17}\right)^x$, $f(x)$ជាអនុគមន៍ចុះ

ជានិច្ច ហើយ $f(1) = \frac{3}{17} + \frac{13}{17} = \frac{16}{17} < 1 \Rightarrow f(a) \geq 1 > f(1)$

$\Rightarrow a < 1 \quad (*)$

ម្យ៉ាងទៀត: $5^a + 7^b = 11^b \Rightarrow 5^b + 7^b \leq 11^b$

$$\Leftrightarrow \left(\frac{5}{11}\right)^b + \left(\frac{7}{11}\right)^b \leq 1$$

ពិនិត្យអនុគមន៍: $g(x) = \left(\frac{5}{11}\right)^x + \left(\frac{7}{11}\right)^x$, $g(x)$ ជាអនុគមន៍ចុះ

ជានិច្ច ហើយ $g(1) = \frac{5}{11} + \frac{7}{11} = \frac{12}{11} > 1 \Rightarrow g(b) \leq 1 < g(1)$

$\Rightarrow b > 1 \quad (**)$

តាម $(*)$ និង $(**)$ គេទាញបាន: $a < 1 < b$,វិសមភាពនេះផ្ទុយពីការឧបមាខាងដើមដែលថា: $a \geq b$ ។

ដូចនេះ $a < b$ ត្រូវបានស្រាយបញ្ជាក់ ។

លំហាត់ទី២៣

តាង a,b,c ជាចំនួនគត់វិជ្ជមាន ។

ចូរស្រាយបញ្ជាក់ថា: បីចំនួន $a^2+b+c\,,b^2+c+a\,,c^2+a+b$ មិនអាចជាការេប្រាកដទាំងអស់បានទេ ។

(APMO 2011)

ដំណោះស្រាយ

ឧបមាថាបីចំនួន $a^2+b+c, b^2+c+a, c^2+a+b$ សុទ្ធតែជាការេប្រាកដ

ដោយ a^2+b+c ជាការេប្រាកដធំជាង a^2 $\Rightarrow a^2+b+c \geq (a+1)^2$

$\Leftrightarrow b+c \geq 2a+1$

សម្រាយដូចគ្នាដែរ គេបាន: $c+a \geq 2b+1$ និង $a+b \geq 2c+1$ ។

បូកអង្គនិងអង្គនៃវិសមភាពទាំងបីខាងលើ យើងបាន:

$2(a+b+c) \geq 2(a+b+c)+3$ ជាវិសមភាពមិនពិត ។

នេះបញ្ជាក់ថា: ការឧបមាខាងលើគឺខុស ។

ដូចនេះ បីចំនួន $a^2+b+c, b^2+c+a, c^2+a+b$ មិនអាចជាការេប្រាកដទាំងអស់បានទេ ។

លំហាត់ទី២៤

តាង $\{a_n\}$ ជាស្វ៊ីតមួយដែលផ្ទៀងផ្ទាត់: $a_1=5$ និង

$a_n=\sqrt[n]{a_{n-1}^{n-1}+2^{n-1}+2\cdot 3^{n-1}}$ ចំពោះគ្រប់ $n \geq 2$ ។

ចូរស្រាយបញ្ជាក់ថា: $\{a_n\}$ ជាស្វ៊ីតចុះ ។

(Vietnam Mathematical Olympiad 2010)

ដំណោះស្រាយ

ចំពោះគ្រប់ $n\geq 2$, យើងមាន: $a_n=\sqrt[n]{a_{n-1}^{n-1}+2^{n-1}+2\cdot 3^{n-1}}$

$\Leftrightarrow a_n^n=a_{n-1}^{n-1}+2^{n-1}+2\cdot 3^{n-1}$

យើងបាន:

$$a_2^2=a_1+2+2\cdot 3$$
$$a_3^3=a_2^2+2^2+2\cdot 3^2$$
$$a_4^4=a_3^3+2^3+2\cdot 3^3$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_n^n=a_{n-1}^{n-1}+2^{n-1}+2\cdot 3^{n-1}$$

បូកអង្គនិងអង្គ:

$$a_n^n=a_1+\left(2+2^2+\cdots+2^{n-1}\right)+2\left(3+3^2+\cdots+3^{n-1}\right)$$
$$=5+2\cdot\frac{2^{n-1}-1}{2-1}+2\cdot 3\cdot\frac{3^{n-1}-1}{3-1}=5+2^n-2+3^n-3=2^n+3^n$$

$\Rightarrow a_n=\sqrt[n]{2^n+3^n}=3\sqrt[n]{1+\left(\frac{2}{3}\right)^n}=3\sqrt[n]{1+f_n}$ ដែល $f_n=\left(\frac{2}{3}\right)^n$

ដោយ $0<\frac{2}{3}<1$ នោះ $0<f_n<f_{n-1}\Rightarrow 1<1+f_n<1+f_{n-1}$

យើងបាន: $1<(1+f_n)^{\frac{1}{n}}<(1+f_{n-1})^{\frac{1}{n}}\quad (1)$

ម្យ៉ាងទៀត: $\frac{1}{n}<\frac{1}{n-1}\quad ,\forall n\geq 2$

$\Rightarrow (1+f_{n-1})^{\frac{1}{n}}<(1+f_{n-1})^{\frac{1}{n-1}}\ ,\forall n\geq 2\ \ (2)$

តាម (1) និង (2) យើងបាន: $(1+f_n)^{\frac{1}{n}}<(1+f_{n-1})^{\frac{1}{n-1}}$

$\Rightarrow 3\sqrt[n]{1+f_n} < 3^{n-1}\sqrt{1+f_{n-1}} \Leftrightarrow a_n < a_{n-1}$, $\forall n \geq 2$

ដូចនេះ $\{a_n\}$ ជាស្វ៊ីតចុះ ។

លំហាត់ទី២៨

គេឲ្យ n, p ជាចំនួនគត់ដែល $n > 1$ និង p ជាចំនួនបឋម ។

បើ $n|p-1$ និង $p|n^3-1$, ចូរបង្ហាញថា: $4p-1$ ជាការេប្រាកដ ។

(Argentina ,Team Selection Test 2005)

ដំណោះស្រាយ

យើងមាន: $p|n^3-1 \Leftrightarrow p|(n-1)(n^2+n+1)$

ឧបមាថា: $p|n-1 \Rightarrow n-1 \geq p > p-1$, តែ $n|p-1 \Rightarrow p-1 \geq n$

$\Rightarrow n-1 > n$ ជាវិសមភាពមិនពិត ។

មានន័យថា: $p\nmid n-1$ នោះយើងទាញបាន: $p|n^2+n+1$, (p ជាចំនួនបឋម)

ដោយ $n|p-1$ នោះ $p-1 = nk \Leftrightarrow p = nk+1$ ចំពោះចំនួនគត់ $k \geq 1$

យើងបាន: $p|(n^2+n+1)-p \Leftrightarrow p|(n^2+n+1)-(nk+1)$

$\Leftrightarrow p|n(n-k+1)$

បើ $p|n$, នោះតាមសម្មតិកម្ម: $n|p-1$ គេទាញបាន: $p|p-1$ ដែលជាករណីមិនអាច , ដូច្នេះ យើងបាន: $p|n-k+1 \Rightarrow n-k+1 \geq p$ ឬ

$n-k+1=0$

បើ $n-k+1 \geq p = nk+1$ នោះ $n(k-1)+k \leq 0$, ជាវិសមភាពមិនពិត

នេះបញ្ជាក់ថា: $n-k+1=0$ នោះ $k=n+1$

$\Rightarrow p = nk+1 = n(n+1)+1 = n^2+n+1$

$\Rightarrow 4p-3=4n^2+4n+1=(2n+1)^2$

ដូចនេះ $4p-1$ ជាការេប្រាកដ ។

លំហាត់ទី២៦

តាង $x_1, x_2, \dots, x_{100}$ ជាចំនួនពិតមិនអវិជ្ជមានដែលផ្ទៀងផ្ទាត់:

$x_i + x_{i+1} + x_{i+2} \le 1$ ចំពោះគ្រប់ $i=1,2,\dots,100$ (យើងយក $x_{101}=x_1$, $x_{102}=x_2$) ។ ចូរកំណត់តម្លៃអតិបរមាដែលអាចនៃផលបូក:

$S=\sum_{i=1}^{100} x_i x_{i+2}$ ។

(IMO Shortlist 2010)

ដំណោះស្រាយ

ចំពោះគ្រប់ $i=1,2,\dots,100$ យើងមាន: $x_i + x_{i+1} + x_{i+2} \le 1$

$\Leftrightarrow x_{i+2} \le 1 - x_i - x_{i+1}$

យើងបាន:

$$S=\sum_{i=1}^{100} x_i x_{i+2} \le \sum_{i=1}^{100} x_i\left(1-x_i+x_{i+1}\right)=\sum_{i=1}^{100} x_i - \sum_{i=1}^{100} x_i^{\,2} - \sum_{i=1}^{100} x_i x_{i+1}$$

$$=\sum_{i=1}^{n} x_i - \frac{1}{2}\sum_{i=1}^{n}\left(x_i+x_{i+1}\right)^2$$

តាមវិសមភាព $Cauchy-schwarz:$

$$\sum_{i=1}^{100}\left(x_i+x_{i+1}\right)^2 \ge \frac{1}{100}\left(\sum_{i=1}^{100}\left(x_i+x_{i+1}\right)\right)^2$$

ដូច្នេះ $S \le \sum_{i=1}^{100} x_i - \frac{1}{200}\left(\sum_{i=1}^{100}(x_i + x_{i+1})\right)^2 = \sum_{i=1}^{100} x_i - \frac{2}{100}\left(\sum_{i=1}^{100} x_i\right)^2$

$$= \frac{2}{100}\left(\sum_{i=1}^{100} x_i\right)\left(\frac{100}{2} - \sum_{i=1}^{100} x_i\right)$$

ដោយប្រើវិសមភាព $AM - GM$ គេបាន:

$$S \le \frac{2}{100} \cdot \left(\frac{1}{2}\left(\sum_{i=1}^{100} x_i + \frac{100}{2} - \sum_{i=1}^{100} x_i\right)\right)^2 = \frac{2}{100}\left(\frac{100}{4}\right)^2 = \frac{25}{2}$$

ដូចនេះ $S_{\max} = \frac{25}{2}$ ។

<u>លំហាត់ទី២៧</u>

ចូររកគ្រប់អនុគមន៍ $f : \Re \to \Re$ ដែលផ្ទៀងផ្ទាត់: $\forall x \in \mathbb{R}$,

$f(x) = \max(2xy - f(y))$ ដែល $y \in \mathbb{R}$ ។

(Morocco Mathematical Olympiad 2011)

<u>ដំណោះស្រាយ</u>

តាមសម្មតិកម្ម: ចំពោះចំនួនពិត x ណាមួយ , យើងបាន:

$f(x) \ge 2xy - f(y)$ ចំពោះគ្រប់ $y \in \mathbb{R}$ ។

យក $x = y$ នោះ $f(x) \ge x^2$

កំណត់ $g(x) = f(x) - x^2 \ge 0$ នោះ $f(x) = g(x) + x^2$

យើងបាន: $f(x) = \max(2xy - f(y))$

$\Leftrightarrow g(x) + x^2 = \max(2xy - g(y) - y^2)$

$\Leftrightarrow g(x) = \max(2xy - y^2 - g(y)) - x^2$

$\Leftrightarrow g(x) = \max\left(-x^2 + 2xy - y^2 - g(y)\right)$

$\Leftrightarrow 0 \le g(x) = \max\left(-(x-y)^2 - g(y)\right) \le 0$

ទាញបាន: $g(x) = 0$ នោះ $f(x) = x^2, \forall x \in \mathbb{R}$ ។

ដូចនេះ $f(x) = x^2, \forall x \in \mathbb{R}$ ត្រូវបានកំណត់ ។

<u>លំហាត់ទី២៨</u>

គេឲ្យពហុធា $az^4 + bz^3 + cz^2 + dz^2 + e$ មានមេគុណសុទ្ធតែជាចំនួនគត់ (ដែល $a \neq 0$) និង មានឫស r_1, r_2, r_3, r_4 ដែល $r_1 + r_2$ ជាចំនួនសនិទាន ហើយ $r_3 + r_4 \neq r_1 + r_2$ ។ ចូរស្រាយបញ្ជាក់ថា: $r_1 r_2$ ជាចំនួនសនិទាន ។

(PUTNAM 2003)

<u>ដំណោះស្រាយ</u>

ដោយ r_1, r_2, r_3, r_4 ជាឫសនៃពហុធា $az^4 + bz^3 + cz^2 + dz^2 + e$ នោះតាមទ្រឹស្តីវ្យែត គេបាន:

$$\begin{cases} r_1 + r_2 + r_3 + r_4 = -\dfrac{b}{a} \\ r_1 r_2 + r_1 r_3 + r_1 r_4 + r_2 r_3 + r_2 r_4 + r_3 r_4 = \dfrac{c}{a} \\ r_1 r_2 r_3 + r_1 r_2 r_4 + r_2 r_3 r_4 + r_3 r_4 r_1 = -\dfrac{d}{a} \end{cases}$$

$$\Leftrightarrow \begin{cases} (r_1+r_2)+(r_3+r_4)=-\dfrac{b}{a} \\ (r_1+r_2)(r_3+r_4)+r_1r_2+r_3r_4=\dfrac{c}{a} \\ r_1r_2(r_3+r_4)+r_3r_4(r_1+r_2)=-\dfrac{d}{a} \end{cases}$$

តាង $m=r_1+r_2$ និង $n=r_3+r_4$, យើងបានៈ

$$\begin{cases} m+n=-\dfrac{b}{a} & (1) \\ mn+r_1r_2+r_3r_4=\dfrac{c}{a} & (2) \\ nr_1r_2+mr_3r_4=-\dfrac{d}{a} & (3) \end{cases}$$

ដោយ a,b,c,d ជាចំនួនគត់ , $(a\neq 0)$ នោះតាម (1) យើងបានៈ $m+n$ ជាចំនួនសនិទាន ទាញបានៈ n ជាចំនួនសនិទាន ព្រោះ $m=r_1+r_2$ ជាចំនួនសនិទាន នាំឲ្យ mn ជាចំនួនសនិទាន ហើយតាម (2) យើងបានៈ $mn+r_1r_2+r_3r_4$ ជាចំនួនសនិទាន នោះ $r_1r_2+r_3r_4$ ជាចំនួនសនិទាន នាំឲ្យ $m(r_1r_2+r_3r_4)$ ជាចំនួនសនិទាន តែតាម (3): $nr_1r_2+mr_3r_4$ ជាចំនួនសនិទាន នាំឲ្យ $m(r_1r_2+r_3r_4)-(nr_1r_2+mr_3r_4)=(m-n)r_1r_2$ ជាចំនួនសនិទាន តែដោយ $m=r_1+r_2\neq r_3+r_4=n$ នោះ $m-n\neq 0$ គេទាញបានៈ r_1r_2 ត្រូវតែជាចំនួនសនិទាន ។

ដូចនេះ r_1r_2 ជាចំនួនសនិទាន , ត្រូវបានស្រាយបញ្ជាក់ ។

លំហាត់ទី២៩

គេឲ្យ $a_0 = 1994$ និង $a_{n+1} = \dfrac{a_n^2}{a_n + 1}$ ចំពោះគ្រប់ចំនួនគត់មិនអវិជ្ជមាន n ,
ចូរស្រាយបញ្ជាក់ថា: $1994 - n$ ជាចំនួនគត់ធំបំផុត តូចជាងឬស្មើនឹង a_n
ចំពោះ $0 \le n \le 998$ ។

(IMO Shortlist 1994)

ដំណោះស្រាយ

យើងមាន: $a_n - a_{n+1} = a_n - \dfrac{a_n^2}{a_n + 1} = 1 - \dfrac{1}{a_n + 1} > 0$

$\Rightarrow a_n > a_{n+1}$ ចំពោះ $n = 0, 1, 2, \ldots$ គេបាន:

$a_0 > a_1 > a_2 > \cdots > a_n > \cdots$

ម្យ៉ាងទៀត: $a_n = a_0 + (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_n - a_{n-1})$

$$= 1994 - n + \frac{1}{a_0 + 1} + \frac{1}{a_1 + 1} + \cdots + \frac{1}{a_{n-1} + 1} > 1994 - n \qquad (1)$$

ហើយចំពោះ $0 \le n \le 998$, យើងមាន:

$$\frac{1}{a_0 + 1} + \frac{1}{a_1 + 1} + \cdots + \frac{1}{a_{n-1} + 1} < \frac{1}{a_{n-1} + 1} + \frac{1}{a_{n-1} + 1} + \cdots + \frac{1}{a_{n-1} + 1}$$

$$= \frac{n}{a_{n-1} + 1} < \frac{998}{a_{997} + 1} < \frac{998}{1994 - 997 + 1} = 1$$

$$\Rightarrow 1994 - n + \frac{1}{a_0 + 1} + \frac{1}{a_1 + 1} + \cdots + \frac{1}{a_{n-1} + 1} < (1994 - n) + 1$$

$\Leftrightarrow a_n < (1994 - n) + 1 \quad (2)$

តាម (1) និង (2) គេបាន: $1994 - n < a_n < (1994 - n) + 1$

$\Rightarrow \lfloor a_n \rfloor = 1994 - n$

ដូចនេះ $1994-n$ ជាចំនួនគត់ធំបំផុត តូចជាងឬស្មើនឹង a_n ចំពោះ $0\le n\le 998$ ។

<u>លំហាត់ទី៣០</u>

គេឲ្យចំនួនគត់ $n>1$ ហើយ a,b ជាចំនួនពិតវិជ្ជមានដែលផ្ទៀងផ្ទាត់:

$a^n=a+1$ និង $b^{2n}=b+3a$ ។

តើ a និង b មួយណាធំជាងមួយណា ?

(USAMO 1993)

<u>ដំណោះស្រាយ</u>

យើងមាន: $a^n=a+1>1$, (ព្រោះ $a>0$) $\Rightarrow a>1$

ហើយ $a^{2n}=(a+1)^2>4a$ (តាមវិសមភាព $AM-GM$, សមភាពមិនអាចកើតមានទេព្រោះ $a>1$) ។

ម្យ៉ាងទៀត: $b^{2n}=b+3a \Rightarrow \left(\dfrac{b}{a}\right)^{2n}=\dfrac{b+3a}{a^{2n}}<\dfrac{b+3a}{4a}$

យើងឧបមាថា: $b\ge a$ នោះ $\dfrac{b+3a}{4a}\le\dfrac{b+3b}{4a}=\dfrac{b}{a}$

ទាញបាន: $\left(\dfrac{b}{a}\right)^{2n}<\dfrac{b}{a}$ ជាវិសមភាពមិនពិតព្រោះ $n>1$ ។

នេះបញ្ជាក់ថា: ការឧបមាខាងលើគឺខុស មានន័យថា: $b>a$ ។

ដូចនេះ $b>a$ ។

លំហាត់ទី៣១

k ជាចំនួនគត់វិជ្ជមានមួយ ។ ស្វ៊ីត $a_1, a_2, a_3, \ldots$ ត្រូវបានកំណត់ដោយៈ $a_1 = k+1$ និង $a_{n+1} = a_n^2 - ka_n + k$ ។ ចូរបង្ហាញថាៈ a_m និង a_n ជាចំនួនបឋមរវាងគ្នា ចំពោះ $m \neq n$ ។

(Poland Mathematical Olympiad 2002)

ដំណោះស្រាយ

យើងមានៈ $a_{n+1} = a_n^2 - ka_n + k$

$$\Rightarrow a_{n+1} - k = a_n(a_n - k) = a_n a_{n-1}(a_{n-1} - k)$$

$$= \cdots = a_n a_{n-1} \cdots a_1 (a_1 - k)$$

$$= a_n a_{n-1} \cdots a_1 \quad , (\text{ព្រោះ } a_1 = k+1) \qquad (1)$$

ឧបមាថាៈ $n > m$, យើងតាង $d = \gcd(a_m, a_n)$ នោះតាម (1) គេបានៈ $d|k$ ដោយ $a_n - k = a_{n-1}^2 - ka_{n-1} \Rightarrow d|a_{n-1}^2$

តាង p ជាតួចែកបឋមនៃ d នោះៈ $p|a_{n-1}^2 \Rightarrow p|a_{n-1}$, នេះមានន័យថាៈ គ្រប់តួចែកបឋមទាំងអស់នៃ d ចែកដាច់ a_{n-1} នោះយើងទាញបានៈ $d|a_{n-1}$

ធ្វើដូចរបៀបនេះជាបន្តបន្ទាប់យើងនឹងបានៈ $d|a_1 = k+1$

ដោយ $d|k$ នោះទាញបានៈ $d|1 \Rightarrow d = 1$ ។

ដូចនេះ a_m និង a_n ជាចំនួនបឋមរវាងគ្នា ចំពោះ $m \neq n$ ។

លំហាត់ទី៣២

តាង $f:\mathbb{R}\to\mathbb{R}$ ជាអនុគមន៍មួយដែលផ្ទៀងផ្ទាត់:
$f\left(f\left(x\right)\right)=x^2-x+1$ ចំពោះគ្រប់ចំនួនពិត x ,
ចូរកំណត់តម្លៃ $f\left(0\right)$ ។

(Baltic Way 2011)

ដំណោះស្រាយ

ចំពោះគ្រប់ចំនួនពិត x យើងមាន: $f\left(f\left(x\right)\right)=x^2-x+1$ នោះ
$f\left(f\left(0\right)\right)=1$
ម្យ៉ាងទៀត: $f\left(x^2-x+1\right)=f\left(f\left(f\left(x\right)\right)\right)=\left(f\left(x\right)\right)^2-f\left(x\right)+1$
នោះ $\left(f\left(0\right)\right)^2-f\left(0\right)+1=\left(f\left(1\right)\right)^2-f\left(1\right)+1=f\left(1\right)$
នាំឲ្យ $\left(f\left(1\right)\right)^2-2f\left(1\right)+1=0$ ទាញបាន: $f\left(1\right)=1$
ហើយ $f\left(0\right)\left(f\left(0\right)-1\right)=0$
បើ $f\left(0\right)=0$ នោះ $1=f\left(f\left(0\right)\right)=f\left(0\right)=0$ មិនពិត
ទាញបាន: $f\left(0\right)-1=0$
ដូចនេះ $f\left(0\right)=1$ ។

លំហាត់ទី៣៣

ចូរកំណត់គ្រប់ស្វ៊ីតនៃចំនួនពិត $a_1, a_2, ..., a_{1995}$ ដែលផ្ទៀងផ្ទាត់:
$2\sqrt{a_n-(n-1)} \geq a_{n+1}-(n-1)$ ចំពោះ $n=1,2,3,...,1994$ និង
$2\sqrt{a_{1995}-1994} \geq a_1+1$ ។

(APMO 1995)

ដំណោះស្រាយ

យើងមាន:

$$\sum_{n=1}^{1995} 2\sqrt{a_n-(n-1)} = \sum_{n=1}^{1994} 2\sqrt{a_n-(n-1)} + 2\sqrt{a_{1995}-1994}$$

$$\geq \sum_{n=1}^{1994}\left[a_{n+1}-(n-1)\right]+a_1+1 \geq \sum_{n=1}^{1994} a_{n+1}+a_1-\sum_{n=1}^{1994}(n-1)+1$$

$$=\sum_{n=1}^{1995} a_n-\sum_{n=1}^{1995}\left[(n-1)-1\right]=\sum_{n=1}^{1995}\left[a_n-(n-1)+1\right]$$

$$\Rightarrow \sum_{n=1}^{1995}\left[a_n-(n-1)+1-2\sqrt{a_n-(n-1)}\right] \leq 0$$

$$\Leftrightarrow \sum_{n=1}^{1995}\left(\sqrt{a_n-(n-1)}-1\right)^2 \leq 0$$

គេទាញបាន: $\sqrt{a_n-(n-1)}-1=0$ ចំពោះគ្រប់ $n=1,2,3,...,1995$

$\Leftrightarrow a_n=n$ ចំពោះគ្រប់ $n=1,2,3,...,1995$ ។

ដូចនេះ ស្វ៊ីតដែលត្រូវរកគឺ: $a_n=n$, $n=1,2,3,...,1995$ ។

លំហាត់ទី៣៨

ចូរបង្ហាញថា ចំពោះ $1 < a \le b \le c$ គេបានវិសមភាព:

$\log_a b + \log_b c + \log_c a \le \log_b a + \log_c b + \log_a c$ ។

(Russia Mathematical Olympiad 2009)

ដំណោះស្រាយ

តាង $x = \log_a b$, $y = \log_b c$ និង $z = \log_c a$ គេបាន:

$$xyz = \log_a b.\log_b c.\log_c a = \frac{\log b}{\log a} \cdot \frac{\log c}{\log b} \cdot \frac{\log a}{\log c} = 1$$

ដោយ $1 < a \le b \le c$ នោះគេបាន:

$$\begin{cases} x = \log_a b \ge 1 \\ y = \log_b c \ge 1 \\ z = \log_c a \le 1 \end{cases} \Leftrightarrow \begin{cases} x - 1 \ge 0 \\ y - 1 \ge 0 \\ z - 1 \le 0 \end{cases}$$

$$\Rightarrow (x-1)(y-1)(z-1) \le 0$$

$$\Leftrightarrow xyz - 1 + x + y + z - xy - yz - zx \le 0$$

$$\Leftrightarrow x + y + z \le xy + yz + zx$$

$$\Leftrightarrow x + y + z \le \frac{xy + yz + zx}{xyz} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

$$\Leftrightarrow \log_a b + \log_b c + \log_c a \le \frac{1}{\log_a b} + \frac{1}{\log_b c} + \frac{1}{\log_c a}$$

ដូចនេះ $\log_a b + \log_b c + \log_c a \le \log_b a + \log_c b + \log_a c$,

ត្រូវបានបង្ហាញ ។

លំហាត់ទី៣៨

ពហុធា $x^4 - 2x^2 + ax + b$ មានឫសបួនជាចំនួនពិតខុសៗគ្នា ។
ចូរបង្ហាញថា តម្លៃដាច់ខាតនៃឫសនីមួយៗសុទ្ធតែតូចជាង $\sqrt{3}$ ។

(Hungary Mathematical Olympiad 1999)

ដំណោះស្រាយ

តាង p, q, r, s ជាឫសទាំងបួននៃ $x^4 - 2x^2 + ax + b$ នោះតាម
ទ្រឹស្តីបទវៀែត គេបាន: $p+q+r+s=0$ និង

$pq+qr+rs+pr+ps+qs=-2$

នោះ $p^2+q^2+r^2+s^2=0^2+4=4$

ដោយ $q \neq r \neq s$ នោះតាមវិសមភាព $Cauchy-schwarz$:

$(q+r+s)^2 < (1^2+1^2+1^2)(q^2+r^2+s^2)$

$\Leftrightarrow (-p)^2 < 3(4-p^2)$

$\Leftrightarrow p^2 < 12-3p^2$

$\Leftrightarrow p^2 < 3$

$\Rightarrow |p| < \sqrt{3}$

ស្រាយតាមរបៀបដូចគ្នាដែរ គេបាន: $|q|<\sqrt{3}$, $|r|<\sqrt{3}$, $|s|<\sqrt{3}$
ដូចនេះ តម្លៃដាច់ខាតនៃឫសនីមួយៗ សុទ្ធតែតូចជាង $\sqrt{3}$ ។

លំហាត់ទី៣៦

គេឲ្យ x, y, z ជាចំនួនពិតវិជ្ជមានដែលផ្ទៀងផ្ទាត់: $xyz \ge 1$,

ចូរបង្ហាញថា: $\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \ge 0$ ។

(IMO 2005)

ដំណោះស្រាយ

វិសមភាពដែលត្រូវស្រាយសមមូល:

$$\frac{x^2 - x^5}{x^5 + y^2 + z^2} + \frac{y^2 - y^5}{y^5 + z^2 + x^2} + \frac{z^2 - z^5}{z^5 + x^2 + y^2} \le 0$$

$$\Leftrightarrow \frac{x^2 - x^5}{x^5 + y^2 + z^2} + 1 + \frac{y^2 - y^5}{y^5 + z^2 + x^2} + 1 + \frac{z^2 - z^5}{z^5 + x^2 + y^2} + 1 \le 3$$

$$\Leftrightarrow \frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2} + \frac{x^2 + y^2 + z^2}{y^5 + z^2 + x^2} + \frac{x^2 + y^2 + z^2}{z^5 + x^2 + y^2} \le 3$$

តាមវិសមភាព **Cauchy-Schwarz** គេបាន:

$$\left(x^2 + y^2 + z^2\right)^2 = \left(\sqrt{x^5}.\sqrt{x^{-1}} + y.y + z.z\right)^2$$

$$\le \left(x^5 + y^2 + z^2\right)\left(\frac{1}{x} + y^2 + z^2\right) \le \left(x^5 + y^2 + z^2\right)\left(yz + y^2 + z^2\right) ,$$

$$\left(xyz \ge 1 \Rightarrow \frac{1}{x} \le yz\right)$$

$$\Rightarrow \quad \frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2} \le \frac{y^2 + z^2 + yz}{x^2 + y^2 + z^2}$$

$$\Rightarrow \sum_{cyclic} \frac{x^2 + y^2 + z^2}{x^5 + y^2 + z^2} \le \sum_{cyclic} \frac{y^2 + z^2 + yz}{x^2 + y^2 + z^2}$$

$$= \frac{2\left(x^2+y^2+z^2\right)+\left(xy+yz+zx\right)}{x^2+y^2+z^2} = 2+\frac{xy+yz+zx}{x^2+y^2+z^2} \le 2+1=3$$

$$\left(\left(x-y\right)^2+\left(y-z\right)^2+\left(z-x\right)^2 \ge 0 \Rightarrow x^2+y^2+z^2 \ge xy+yz+zx\right)$$

ដូចនេះ វិសមភាពត្រូវបានស្រាយបញ្ជាក់ ។

លំហាត់ទី៣៧

$f:\mathbb{R}\to\mathbb{R}$ ជាអនុគមន៍ដែលផ្ទៀងផ្ទាត់: $f\left(xf\left(y\right)\right)=yf\left(x\right)$
ចំពោះចំនួនពិត x,y មួយចំនួន ។ ចូរស្រាយបញ្ជាក់ថា:
$f\left(-x\right)=f\left(x\right)$ ចំពោះគ្រប់ចំនួនពិត x ។

(Kazakhstan Mathematical Olympiad 2012)

ដំណោះស្រាយ

យើងសង្កេតឃើញថា: បើ $f\left(x\right)=0$, $\forall x\in\mathbb{R}$ នោះ
$f\left(-x\right)=f\left(x\right)$ ចំពោះគ្រប់ $\forall x\in\mathbb{R}$ ។
យើងសិក្សាលើករណី $f\left(x\right)$ មិនស្មើនឹង 0 គ្រប់ x នោះត្រូវមាន x_0
មួយដែល $f\left(x_0\right)\neq 0$ ។
បើ $f\left(y_1\right)=f\left(y_2\right)$ នោះ

$$y_1 f\left(x_0\right)=f\left(x_0 f\left(y_1\right)\right)=f\left(x_0 f\left(y_2\right)\right)=y_2 f\left(x_0\right)$$

$\Rightarrow \left(y_1-y_2\right)f\left(x_0\right)=0 \Rightarrow y_1=y_2$
នេះបញ្ជាក់ថា: f ជាអនុវត្តប្រកាន់ ។
ចំពោះ $x=1$ គេបាន: $f\left(f\left(y\right)\right)=yf\left(1\right)$ នោះ $f\left(f\left(1\right)\right)=f\left(1\right)$
$\Rightarrow f\left(1\right)=1$ នោះយើងបាន: $f\left(f\left(y\right)\right)=y$ ឬ $f\left(f\left(x\right)\right)=x$
ដូច្នេះ $f\left(f\left(xy\right)\right)=xy \qquad (1)$

ដោយជំនួស x ដោយ $f(x)$ ក្នុងសមីការអនុគមន៍:

$$f(xf(y)) = yf(x)$$

គេបាន: $f(f(x)\cdot f(y)) = yf(f(x)) = yx \quad (2)$

តាម (1) និង (2) គេបាន: $f(f(x)\cdot f(y)) = f(f(xy))$

$\Rightarrow f(x)\cdot f(y) = f(xy)$

$\Rightarrow f^2(-1) = f(1) = 1$

$\Rightarrow f(-1) = -1 \quad$ (ព្រោះ $f(-1)\neq 1$)

គេបាន: $-f(x) = f(-1)\cdot f(x) = f(-x)$

ដូចនេះ $f(-x) = f(x)$ ចំពោះគ្រប់ចំនួនពិត x ។

លំហាត់ទី៣៨

តាង m,n ជាចំនួនគត់ធម្មជាតិដែល: $m+n+1$ ជាចំនួនបឋម និង ចែកដាច់ $2(m^2+n^2)-1$ ។ ចូរស្រាយបញ្ជាក់ថា: $m=n$ ។

(Swi ss Mathematical Olympiad 2010)

ដំណោះស្រាយ

យើងមាន:

$$m+n+1 \mid 2(m^2+n^2)-1 = (m-n)^2+(m+n)^2-1$$

$$= (m-n)^2+(m+n+1)(m+n-1)$$

$\Rightarrow m+n+1 \mid (m-n)^2$

$\Rightarrow m+n+1 \mid m-n \quad$ (ព្រោះ $m+n+1$ ជាចំនួនបឋម)

ដោយ $m+n+1>m-n \;\Rightarrow\; m-n=0 \Leftrightarrow\; m=n$

ដូចនេះ $m=n$ ត្រូវបានស្រាយបញ្ជាក់ ។

លំហាត់ទី៣៩

គេឲ្យ $P(x)=2x^3-3x^2+2$ និង តាង

$S=\{P(n):n\in\mathbb{N},n\le 1999\}$, $T=\{n^2+1:n\in\mathbb{N}\}$ និង

$U=\{n^2+2:n\in\mathbb{N}\}$ ។ ចូរស្រាយបញ្ជាក់ថា: $S\cap T$ និង $S\cap U$

មានចំនួនធាតុស្មើគ្នា ។

(Romania Mathematical Olympiad 1999)

ដំណោះស្រាយ

តាង $|X|$ ជាចំនួនធាតុនៃសំណុំ X

ដោយ $P(n)=2n^3-3n^2+2=(n-1)^2(2n+1)+1$

នោះ $|S\cap T|$ ជាចំនួននៃការេប្រាកដមានទម្រង់ $(n-1)^2(2n+1)$

ដែល $n\in\mathbb{N}$, $n\le 1999$ ហើយ ចំពោះ $n\le 1999$, $(n-1)^2(2n+1)$

ជាការេប្រាកដលុះត្រាតែ: $n=1$ ឬ

$$n\in\left\{\frac{1}{2}(k^2-1):k=1,3,5,...,63\right\}$$

គេបាន: $|S\cap T|=33$ ។

ដូចគ្នាដែរ: $|S\cap U|$ ជាចំនួននៃការេប្រាកដមានទម្រង់:

$2n^3-3n^2=n^2(2n-3)$ ដែល $n\in\mathbb{N}$, $n\le 1999$

ហើយចំពោះ $n\le 1999, n^2(2n-3)$ ការេប្រាកដលុះត្រាតែ: $n=0$ ឬ

$n \in \left\{ \frac{1}{2}\left(k^2+3\right) : k = 1,3,5,\ldots,63 \right\}$ គេបាន: $|S \cap U| = 33$ ។

ដូចនេះ $|S \cap T| = |S \cap U|$ ត្រូវបានស្រាយបញ្ជាក់ ។

លំហាត់ទី៤០

តាង r,s,t ជាឫសនៃសមីការ: $x(x-2)(3x-7)=2$ ។

ចូរបង្ហាញថា r,s,t ជាចំនួនពិតវិជ្ជមាន រួចហើយកំណត់តម្លៃនៃ:

$\arctan r + \arctan s + \arctan t$ ។

(Ibero American 1987)

ដំណោះស្រាយ

តាង $f(x) = x(x-2)(3x-7) - 2 = 3x^3 - 13x^2 + 14x - 2$

នោះ $f(0) = -2$, $f(1) = 2$, $f(2) = -2$, $f(3) = 4$

យើងបាន:

$f(0) \times f(1) = -4 < 0$

$f(1) \times f(2) = -4 < 0$

$f(2) \times f(3) = -8 < 0$

តាមទ្រឹស្តីបទតម្លៃកណ្តាលទាញបាន: សមីការ $f(x) = 0$ មានឫសបីគឺ ឫសមួយនៅចន្លោះ $(0,1)$, ឫសមួយទៀតនៅចន្លោះ $(1,2)$ ហើយ ឫសទីបីនៅ ចន្លោះ $(2,3)$, នេះបញ្ជាក់ថា: ឫសទាំងបី r,s,t នៃ សមីការ $f(x) = 0$ សុទ្ធតែជាចំនួនពិតវិជ្ជមាន ។

ម្យ៉ាងទៀត យើងតាង:

$a = \arctan r$ នោះ $r = \tan a$

$b=\arctan s$ នោះ $s=\tan b$

$c=\arctan t$ នោះ $t=\tan c$

តាមរូបមន្ត:

$$\tan(a+b+c)=\frac{\tan a+\tan b+\tan c-\tan a.\tan b.\tan c}{1-(\tan a.\tan b+\tan b.\tan c+\tan c.\tan a)}$$

យើងបាន: $\tan(\arctan r+\arctan s+\arctan t)=\dfrac{(r+s+t)-rst}{1-(rs+st+tr)}$

ដោយ r,s,t ជាឫសនៃសមីការ: $f(x)=0$ នោះតាមទ្រឹស្តីបទវ្យែត

គេបាន: $r+s+t=\dfrac{13}{3}$, $rs+st+tr=\dfrac{14}{3}$ និង $rst=\dfrac{2}{3}$

នាំឲ្យ $\tan(\arctan r+\arctan s+\arctan t)=\dfrac{\frac{13}{3}-\frac{2}{3}}{1-\frac{14}{3}}=-1$

នោះ $\arctan r+\arctan s+\arctan t=-\dfrac{\pi}{4}+k\pi$, $(k\in\mathbb{Z})$ **(1)**

ដោយ r,s,t ជាចំនួនពិតវិជ្ជមាន នោះមានន័យថា:

$$\begin{cases}\tan a>0\\ \tan b>0\\ \tan c>0\end{cases} \text{ នាំឲ្យ } \begin{cases}0<a<\dfrac{\pi}{2}\\ 0<b<\dfrac{\pi}{2}\\ 0<c<\dfrac{\pi}{2}\end{cases}$$

គេបាន: $0<a+b+c<\dfrac{3\pi}{2}$ ឬ

$0<\arctan r+\arctan s+\arctan t<\dfrac{3\pi}{2}$ **(2)**

តាម **(1)** និង **(2)** យើងបាន: $\arctan r+\arctan s+\arctan t=\dfrac{3\pi}{4}$

ដូចនេះ $\arctan r+\arctan s+\arctan t=\dfrac{3\pi}{4}$ ត្រូវបានកំណត់ ។

លំហាត់ទី៤១

តាង $n\geq 3$ ជាចំនួនគត់ និង តាង $a_2,a_3,\ldots,a_n$ ជាចំនួនពិតវិជ្ជមាន ដែល: $a_2a_3\ldots a_n=1$ ។ ចូរស្រាយបញ្ជាក់ថា:

$(1+a_2)^2(1+a_3)^3\ldots(1+a_n)^n>n^n$ ។

(IMO 2012)

ដំណោះស្រាយ

ដោយ $a_2a_3\ldots a_n=1$ នោះ យើងអាចតាង: $a_2=\dfrac{x_2}{x_3}$, $a_3=\dfrac{x_3}{x_4},\ldots,$

$a_n=\dfrac{x_n}{x_2}$ ដែល $x_2,x_3,\ldots,x_n>0$ ។

ដូច្នេះ វិសមភាពដែលត្រូវស្រាយសមមូល:

$(x_2+x_3)^2(x_3+x_4)^3\ldots(x_n+x_2)^n>n^nx_3^2x_4^3\ldots x_n^{n-1}x_2^n$

$\Leftrightarrow \prod_{k=2}^{n}(x_k+x_{k+1})^k>n^n\prod_{k=2}^{n}x_{k+1}^k$,(ក្នុងករណីនេះ , យក $x_{n+1}=x_2$)

តាមវិសមភាព $AM-GM$ យើងមាន:

$$(x_k+x_{k+1})^k=\left(x_k+(k-1)\frac{x_{k+1}}{k-1}\right)^k\geq k^kx_k\frac{x_{k+1}^{k-1}}{(k-1)^{k-1}}$$

$$=x_kx_{k+1}^{k-1}\frac{k^k}{(k-1)^{k-1}}$$

នាំឲ្យ $\prod_{k=2}^{n}\left(x_k+x_{k+1}\right)^k \ge \prod_{k=2}^{n} x_k x_{k+1}^{k-1}.\prod_{k=2}^{n}\frac{k^k}{\left(k-1\right)^{k-1}} = n^n\prod_{k=2}^{n} x_{k+1}^k$

សមភាពកើតមានកាលណា: $x_k = \frac{x_{k+1}}{k-1}$ ឬ $a_k = \frac{1}{k-1}$

ចំពោះគ្រប់ $k\in\{2,3,...,n\}$ ។

មានន័យថា: $a_2 = 1$ និង $a_k < 1$, $\forall k\in\{3,4,...,n\}$

យើងបាន: $a_2a_3...a_n < 1$ ផ្ទុយពីសម្មតិកម្មដែល $a_2a_3...a_n = 1$ ។

នេះបញ្ជាក់ថា សមភាពមិនអាចកើតមានបានទេ គឺថា:

$\prod_{k=2}^{n}\left(x_k+x_{k+1}\right)^k > n^n\prod_{k=2}^{n} x_{k+1}^k$ ពិត ។

ដូចនេះ $\left(1+a_2\right)^2\left(1+a_3\right)^3...\left(1+a_n\right)^n > n^n$ ត្រូវបានស្រាយបញ្ជាក់ ។

លំហាត់ទី៤២

តាង a,n ជាចំនួនគត់ និង p ជាចំនួនបឋមមួយដែល: $p>|a|+1$ ។

ចូរស្រាយបញ្ជាក់ថា: ពហុធា $f\left(x\right)=x^n+ax+p$ មិនអាចសរសេរជាផលគុណនៃពីរពហុធាមិនថេរដែលមានមេគុណជាចំនួនគត់ ។

(Romanian Mathematical Olympiad 1999)

ដំណោះស្រាយ

តាង z ជាឫសមួយនៃពហុធា $f\left(x\right)$, យើងនឹងស្រាយថា: $|z|>1$.

ឧបមាថា $|z|\le 1$, នោះ $z^n+az=-p$

ទាញបាន:

$p=\left|z^n+az\right|=|z|\left|z^{n-1}+a\right|\le\left|z^{n-1}\right|+|a|\le 1+|a|$, ផ្ទុយពីសម្មតិកម្ម

នេះបញ្ជាក់ថា $|z|>1$ ។

យើងឧបមាថាពហុធា $f(x)$ អាចសរសេរជាផលគុណនៃពីរពហុធាមិនថេរដែលមានមេគុណជាចំនួនគត់ គឺមានន័យថា $f(x)=g(x).h(x)$,
យើងបាន: $p=f(0)=g(0).h(0)$
ដោយ p ជាចំនួនបឋម នោះទាញបាន $|g(0)|=1$ ឬ $|h(0)|=1$ ។
ដោយមិនធ្វើឲ្យបាត់បង់លក្ខណៈទូទៅ , យើងសន្មតថា $|g(0)|=1$
បើ $z_1, z_2, ..., z_k$ ជាឫសនៃ $g(x)$ នោះពួកវាក៏ជាឫសនៃ $f(x)$ ដែរ
នាំឲ្យ $|z_1|, |z_2|, ..., |z_k|>1$
យើងបាន: $1=|g(0)|=|z_1z_2...z_k|=|z_1||z_2|...|z_k|>1$, មិនពិត ។
បញ្ជាក់ថា ការឧបមាខាងលើគឺ ខុស ។
ដូចនេះ ពហុធា $f(x)$ មិនអាចសរសេរជាផលគុណនៃពីរពហុធាមិនថេរដែលមានមេគុណជាចំនួនគត់ ។

លំហាត់ទី៤៣

ចូរស្រាយបញ្ជាក់ថា បើ n មិនមែនជាពហុគុណនៃ 3 នោះមុំ $\frac{\pi}{n}$ អាចពុះចែកជាបីមុំប៉ុនៗគ្នាដោយប្រើតែបន្ទាត់និងដៃកឈាន ។

(USAMO 1981)

ដំណោះស្រាយ

ដោយ n មិនមែនជាពហុគុណនៃ 3 នោះ n និង 3 ត្រូវតែជាចំនួនបឋមរវាងគ្នា, តាមទ្រឹស្តីបទ $Bezout$: មានចំនួនគត់ a និង b ដែល $an+3b=1$ សមមូល: $a\frac{\pi}{n}+b\frac{\pi}{3}=\frac{\pi}{3n}$ ដែលក្នុងនោះ a និង b ត្រូវមានសញ្ញាផ្ទុយគ្នា ។

ប្រើដែកឈាន គូសរង្វង់មួយដែលមានធ្នូស្កាត់មុំផ្ចិត $\frac{\pi}{n}$ និង $\frac{\pi}{3}$ រួចប្រើរង្វាស់ធ្នូទាំងពីរនេះ សង់ធ្នូមួយដែលមានរង្វាស់ស្មើនឹង $|a|$ ដងនៃធ្នូដែលស្កាត់មុំ $\frac{\pi}{3}$ ជុំវិញរង្វង់ដែលបានសង់រួចតាមទិសដៅមួយដោយយក X ជាចំនុចចាប់ផ្តើម (គល់) និង A ជាចំនុចចុង ។

តាមទិសដៅមួយផ្សេងទៀត, ចាប់ផ្តើមពីចំនុច A យើងសង់ធ្នូមួយទៀតដែលមានរង្វាស់ស្មើនឹង $|b|$ ដងនៃធ្នូដែលស្កាត់ដោយមុំ $\frac{\pi}{n}$ ហើយយក Y ជាចំនុចចុងនៃធ្នូនោះ ។ តាមសមភាព $a\frac{\pi}{n}+b\frac{\pi}{3}=\frac{\pi}{3n}$ គេបាន: រង្វាស់ធ្នូ $XY=\frac{\pi}{3n}$ តាមរយះរង្វាស់ធ្នូនេះ , គេអាចខ័ណ្ឌចែកធ្នូដែលស្កាត់ដោយមុំផ្ចិត $\frac{\pi}{n}$ ជាធ្នូបីតូចៗប៉ុនគ្នា ដែលនេះបញ្ជាក់ថា: មុំ $\frac{\pi}{n}$ អាចពុះចែកជាមុំបីប៉ុនៗគ្នាដោយប្រើតែ បន្ទាត់ និង ដែកឈាន ។

លំហាត់ទី៤៨

ស្វ៊ីតនៃចំនួនគត់ $u_0,u_1,u_2,\ldots$ ផ្ទៀងផ្ទាត់: $u_0=1$ និង $u_{n+1}u_{n-1}=ku_n$ ចំពោះ $n\geq 1$ ហើយដែល k ជាចំនួនគត់វិជ្ជមានថេរមួយ ។
បើ $u_{2000}=2000$, ចូរកំណត់តម្លៃដែលអាចនៃ k ។

(British Mathematical Olympiad 1994)

ដំណោះស្រាយ

តាង $u_1 = x$, ដោយ $u_{n+1}u_{n-1} = ku_n$, $n \geq 1$ និង $u_0 = 1$ នោះគេបាន:

$u_2 = kx$, $u_3 = k^2$, $u_4 = \frac{k^2}{x}$, $u_5 = \frac{k}{x}$, $u_6 = 1$, $u_7 = x$, ... ។

យើងសង្កេតឃើញថា: $u_0 = u_6$, $u_1 = u_7$, ...

នាំឲ្យ (u_n) ជាស្វ៊ីតខួបដែលមានខួបស្មើនឹង 6

គេបាន: $u_2 = u_8 = u_{14} = \cdots = u_{6\times 333+2} = u_{2000}$

គឺថា: $u_2 = u_{2000} \Leftrightarrow kx = 2000 \Leftrightarrow \frac{k}{x} \cdot x^2 = 2^4.5^3$

ដោយ k ជាចំនួនគត់វិជ្ជមាន ហើយ $\frac{k}{x} = u_5$ ជាចំនួនគត់

នោះយើងទាញបាន: $x^2 \in \{1, 2^2, 2^4, 2^2.5^2, 2^4.5^2, 5^2\}$

$\Rightarrow x \in \{1, 2, 4, 10, 20, 5\}$

$\Rightarrow k \in \{2000, 1000, 500, 200, 100, 400\}$

ដូចនេះ តម្លៃដែលអាចនៃ k គឺ:

$k \in \{2000, 1000, 500, 200, 100, 400\}$ ។

លំហាត់ទី៤៥

តាង p ជាចំនួនបឋមសេសមួយ ។ ចូរស្រាយបញ្ជាក់ថា:

$$\sum_{k=1}^{p-1} k^{2p-1} \equiv \frac{p(p+1)}{2} \pmod{p^2}$$ ។

(Canadian Mathematical Olympiad 2004)

ដំណោះស្រាយ

យើងមាន: $\sum_{k=1}^{p-1} k^{2p-1} = 1^{2p-1} + 2^{2p-1} + \cdots + (p-1)^{2p-1}$

$= \left(1^{2p-1} + (p-1)^{2p-1}\right) + \left(2^{2p-1} + (p-2)^{2p-1}\right) + \cdots$

$+ \left(\left(\frac{p-1}{2}\right)^{2p-1} + \left(\frac{p+1}{2}\right)^{2p-1}\right)$, ($p \pm 1$ ជាចំនួនគត់គូ)

$= \sum_{k=1}^{\frac{p-1}{2}} \left(k^{2p-1} + (p-k)^{2p-1}\right) \quad (*)$

តាមការពន្លាតទ្វេធា *Newton* គេបាន:

$(p-k)^{2p-1} = p^{2p-1} - \cdots - C_{2p-1}^{2} \cdot p^2 k^{2p-3} + C_{2p-1}^{1} \cdot pk^{2p-2} - k^{2p-1}$

ដោយ គ្រប់តួនៃអង្គខាងស្តាំនៃសមភាពនេះសុទ្ធតែចែកដាច់នឹង p^2 លើកលែងតែពីរតួចុងក្រោយ នោះគេបាន:

$(p-k)^{2p-1} \equiv C_{2p-1}^{1} \cdot pk^{2p-2} - k^{2p-1} \pmod{p^2}$

$\Leftrightarrow (p-k)^{2p-1} + k^{2p-1} \equiv (2p-1)pk^{2p-2} \pmod{p^2} \quad (1)$

ចំពោះ $1 \le k < p$, k ចែកមិនដាច់នឹង p នោះតាមទ្រឹស្តីបទ

Fermat : $k^{p-1} \equiv 1 \pmod{p}$

នាំឲ្យ $(2p-1)k^{2p-2} \equiv (2p-1)(1^2) \equiv -1 \pmod{p}$

គេបាន: $(2p-1)k^{2p-2} = mp - 1$ ចំពោះចំនួនគត់ m ណាមួយ ។

នោះ $(2p-1)pk^{2p-2} = mp^2 - p \equiv -p \pmod{p^2} \quad (2)$

តាម (1) និង (2) គេបាន:

$(p-k)^{2p-1} + k^{2p-1} \equiv -p \pmod{p^2} \quad (**)$

តាម $(*)$ និង $(**)$ នាំឲ្យ:

$$\sum_{k=1}^{p-1} k^{2p-1} \equiv \sum_{k=1}^{\frac{p-1}{2}} (-p) \equiv \left(\frac{p-1}{2}\right)(-p)$$
$$\equiv \frac{p-p^2}{2} + p^2 \equiv \frac{p(p+1)}{2} \pmod{p^2}$$

ដូចនេះ $\sum_{k=1}^{p-1} k^{2p-1} \equiv \frac{p(p+1)}{2} \pmod{p^2}$ ត្រូវបានស្រាយបញ្ជាក់ ។

លំហាត់ទី៤៦

(u_n) ជាស្វ៊ីតមួយដែលកំណត់ដោយ: $u_1 = 1, u_2 = 2$ និង $u_{n+1} = 3u_n - u_{n-1}$ ចំពោះ $n \geq 2$ ។ (v_n) ជាស្វ៊ីតមួយទៀតដែលផ្ទៀងផ្ទាត់: $v_n = \sum_{i=1}^{n} arc\cot u_i$, $n = 1, 2, 3, ...$ ។

ចូរគណនា $\lim_{n\to\infty} v_n$ ។

(Vietnam Mathematical Olympiad 1984)

ដំណោះស្រាយ

យើងសង្កេតឃើញថា:

$$u_1 = 1 = F_1$$
$$u_2 = 2 = F_3$$
$$u_3 = 5 = F_5$$
$$u_4 = 13 = F_7, ...$$

នេះបញ្ជាក់ថាតួនីមួយៗនៃស្វ៊ីត (u_n) ជាតួទីសេសនៃស្វ៊ីតហ្វីបូណាស៊ី (F_n) ដែលកំណត់ដោយ: $F_1 = F_2 = 1$ និង $F_{n+1} = F_n + F_{n-1}$, $n \geq 2$

ម្យ៉ាងទៀត យើងមាន:

$$arc\cot F_{2i} - arc\cot F_{2i+1} = arc\cot\frac{F_{2i}F_{2i+1}+1}{F_{2i+1}-F_{2i}}$$
$$= arc\cot\frac{F_{2i}F_{2i+1}+1}{F_{2i-1}} \qquad (*)$$

យើងនឹងស្រាយថា: $F_{n+1}F_{n+2} - F_nF_{n+3} = (-1)^n$ ចំពោះ $n \ge 1$:

យើងមាន: $F_{n+1}F_{n+2} - F_nF_{n+3} = F_{n+1}(F_{n+1}+F_n) - F_n(F_{n+1}+F_{n+2})$

$$= F_{n+1}^2 - F_nF_{n+2} = F_{n+1}^2 - F_n(F_n + F_{n+1}) = F_{n+1}^2 - F_n^2 - F_nF_{n+1}$$
$$= (F_{n+1}-F_n)(F_{n+1}+F_n) - F_nF_{n+1} = F_{n-1}F_{n+2} - F_nF_{n+1}$$
$$= -(F_nF_{n+1} - F_{n-1}F_{n+2}) = \cdots = (-1)^{n-1}(F_2F_3 - F_1F_4) = (-1)^n$$

ដូច្នេះ $F_{n+1}F_{n+2} - F_nF_{n+3} = (-1)^n$ ចំពោះ $n \ge 1$ ពិត ។

យក $n = 2i-1$ គេបាន: $F_{2i}F_{2i+1} - F_{2i-1}F_{2i+2} = -1$

នាំឲ្យ $\dfrac{F_{2i}F_{2i+1}+1}{F_{2i-1}} = F_{2i+2}$

តាម $(*)$: $\arctan F_{2i} - \arctan F_{2i+1} = \arctan F_{2i+2}$

$$\Leftrightarrow \arctan F_{2i+1} = \arctan F_{2i} - \arctan F_{2i+2}$$

$$\Leftrightarrow \sum_{i=1}^{n} \arctan F_{2i+1} = \sum_{i=1}^{n}(\arctan F_{2i} - \arctan F_{2i+2})$$

$$\Leftrightarrow \sum_{i=2}^{n} \arctan u_i = \arctan F_2 - \arctan F_{2n+2}$$

$$\Rightarrow \lim_{n\to\infty}\sum_{i=2}^{n} \arctan u_i = \lim_{n\to\infty}\arctan u_1 - \lim_{n\to\infty}\arctan F_{2n+2}$$

ដោយ $\lim_{n\to\infty} F_{2n+2} = +\infty$ នោះ $\lim_{n\to\infty} arc\cot F_{2n+1} = 0$

គេបាន: $\lim_{n\to\infty}\sum_{i=2}^{n} arc\cot u_i = arc\cot u_1 = arc\cot 1 = \dfrac{\pi}{4}$

ដូចនេះ $\lim_{n\to\infty}\sum_{i=1}^{n} arc\cot u_i = \dfrac{\pi}{4} + \dfrac{\pi}{4} = \dfrac{\pi}{2}$ ។

លំហាត់ទី៤៧

តាង u និង v ជាពីរចំនួនពិតដែលផ្ទៀងផ្ទាត់៖
$(u+u^2+u^3+\cdots+u^8)+10u^9=(v+v^2+v^3+\cdots+v^{10})+10v^{11}=8$ ។ ចូរកំណត់ថាតើ u ឬ v មួយណាធំជាង ។

(USAMO1989)

ដំណោះស្រាយ

គេពិនិត្យអនុគមន៍ពីរ៖ $f(x)=1+x+x^2+\cdots+x^8+10x^9$ និង $g(x)=1+x+x^2+\cdots+x^{10}+10x^{11}$ នោះតាមសម្មតិកម្ម យើងបាន៖ $f(u)=g(v)=9$ ។

ហើយ $g(x)-f(x)=10x^{11}+x^{10}-9x^9=x^9(10x-9)(x+1)$

យើងសង្កេតឃើញថា៖ $f(0)=1<9=f(u)$ ហើយ

$$f\left(\frac{9}{10}\right)=1+\frac{9}{10}+\cdots+\frac{9^8}{10^8}+10\cdot\frac{9^9}{10^9}=\frac{1-\frac{9^9}{10^9}}{1-\frac{9}{10}}+10\cdot\frac{9^9}{10^9}$$

$=10>9=f(u)$ នាំឲ្យ $f(0)<f(u)<f\left(\frac{9}{10}\right)$

ដោយ $f(x)$ ជាអនុគមន៍កើនជានិច្ច នោះទាញបាន៖ $0<u<\frac{9}{10}$

ដូច្នេះ $g(u)-f(u)=u^9(10u-9)(u+1)<0$

នោះ $g(u)<f(u)=g(v)$ នាំឲ្យ $u<v$ ព្រោះ $g(x)$ ជាអនុគមន៍កើនជានិច្ច

ដូចនេះ $u<v$ ។

លំហាត់ទី៤៨

ចូររកគ្រប់ចំនួនគត់វិជ្ជមាន k ដែលផ្ទៀងផ្ទាត់វិសមភាព:
$k(ab+bc+ca)>5(a^2+b^2+c^2)$ ក្នុងនោះ a,b,c ត្រូវតែជារង្វាស់ជ្រុងនៃត្រីកោណមួយ ។

(China Girl's Mathematical Olympiad 2002)

ដំណោះស្រាយ

យើងមាន: $k(ab+bc+ca)>5(a^2+b^2+c^2)\ \ (*)$

ដោយ $(a-b)^2+(b-c)^2+(c-a)^2\geq 0$

$\Leftrightarrow\ a^2+b^2+c^2\geq ab+bc+ca$

គេបាន: $k(ab+bc+ca)>5(ab+bc+ca)$

នាំឲ្យ $k>5$ ឬ $k\geq 6$ ព្រោះ $k\in\mathbb{N}\ \ (1)$

តែយើងដឹងហើយថា វិសមភាព $(*)$ ផ្ទៀងផ្ទាត់លុះត្រាតែ: a,b,c ត្រូវតែជារង្វាស់ជ្រុងនៃត្រីកោណមួយ ដូច្នេះបើយើងយក $\{a,b,c\}=\{1,1,2\}$ ដែលជាសំណុំត្រីធាតុមិនអាចផ្គុំជាត្រីកោណមួយបាននោះយើងនឹងបាន: $k(1\times 1+1\times 2+2\times 1)\leq 5(1^2+1^2+2^2)$

$\Leftrightarrow k\leq 6\ \ (2)$

តាម (1) និង (2) ទាញបាន: $k=6$ ។

យើងនឹងស្រាយបញ្ជាក់ថា: $k=6$ ពិតប្រាកដមែន ៖

ចំពោះ $k=6$ នោះគេបាន:

$6(ab+bc+ca)>5(a^2+b^2+c^2)\ \ (\Omega)$

$\Leftrightarrow 5a^2-6(b+c)a+5(b^2+c^2)-6bc<0$ ជាវិសមីការដឺក្រេទី២ មានអញ្ញាត a ហើយមាន:

$\Delta'=[3(b+c)]^2-5[5(b^2+c^2)-6bc]=-16b^2+48bc-16c^2$

$$= -16\left(b^2 - 3bc + c^2\right) = -16\left[(b-c)^2 - bc\right]$$

$$= -16(b-c)^2 + 16bc \le 16bc \le 4(b+c)^2$$

នាំឲ្យ $a < \dfrac{3(b+c)+\sqrt{\Delta'}}{5} \le \dfrac{3(b+c)+2(b+c)}{5}$

$\Leftrightarrow a < b+c$ ដូចគ្នាដែរ បើយើងប្រើវិសមីការ (Ω) ដោយយក b រួច c ជាអញ្ញាតវិញ នោះគេនឹងបាន: $b < c+a$ និង $c < a+b$

នេះបញ្ជាក់ថា: a,b,c ជារង្វាស់ជ្រុងនៃត្រីកោណមួយ ។

ដូចនេះ $k = 6$ មានតែមួយគត់ ។

លំហាត់ទី៤៩

គេឲ្យ $l\,,m$ និង n ជាចំនួនគត់វិជ្ជមានដែល $m-n$ ជាចំនួនបឋម និង $8\left(l^2 - mn\right) = 2\left(m^2+n^2\right) + 5(m+n)l$ ។ ចូរស្រាយបញ្ជាក់ថា: $11l+3$ ជាការេប្រាកដ ។

(Bulgarian Mathematical Olympiad , Regional round 2006)

ដំណោះស្រាយ

តាង $m-n=p$ និង $m+n=q$ $\Rightarrow m = \frac{1}{2}(p+q)$ និង $n = \frac{1}{2}(q-p)$

យើងបាន: $mn = \frac{1}{4}\left(q^2-p^2\right)$ និង $m^2+n^2 = \frac{1}{2}\left(p^2+q^2\right)$

សមភាពដែលគេឲ្យសមមូល: $8\left(l^2 - \frac{1}{4}\left(q^2-p^2\right)\right) = \left(p^2+q^2\right) + 5ql$

$\Leftrightarrow 8l^2 - 2q^2 + 2p^2 = p^2 + q^2 + 5ql \Leftrightarrow 3q^2 + 5ql - 8l^2 = p^2$

$\Leftrightarrow (q-l)(3q+8l) = p^2$

ដោយ $p = m-n$ ជាចំនួនបឋម ហើយ $3q+8l > q-l$ នោះគេទាញបាន:

$$\begin{cases} q-l=1 \\ 3q+8l=p^2 \end{cases} \Rightarrow 3(1+l)+8l=p^2 \Leftrightarrow 11l+3=p^2$$

ដូចនេះ $11l+3$ ជាការេប្រាកដ ។

លំហាត់ទី៥០

ចូរស្រាយបញ្ជាក់ថា ស្វ៊ីតដែលកំណត់ដោយ: $y_0=1$ និង

$y_{n+1}=\frac{1}{2}\left(3y_n+\sqrt{5y_n^2-4}\right)$ ចំពោះ $n\geq 0$ មានតួសុទ្ធតែជាចំនួនគត់ ។

(British Mathematical Olympiad 2002)

ដំណោះស្រាយ

ចំពោះគ្រប់ $n\geq 0$ យើងមាន: $y_{n+1}=\frac{1}{2}\left(3y_n+\sqrt{5y_n^2-4}\right)$

$\Leftrightarrow\ 2y_{n+1}-3y_n=\sqrt{5y_n^2-4}\quad (*)$

$\Leftrightarrow\ 4y_{n+1}^2-12y_{n+1}y_n+9y_n^2=5y_n^2-4$

$\Leftrightarrow\ y_{n+1}^2-3y_{n+1}y_n+y_n^2+1=0\quad (1)$

$\Rightarrow\ y_n^2-3y_ny_{n-1}+y_{n-1}^2+1=0\quad (2)$

ដកអង្គនិងអង្គនៃ (1) និង (2) គេបាន: $y_{n+1}^2-y_{n-1}^2-3y_{n+1}y_n+3y_ny_{n-1}=0$

$\Leftrightarrow\ (y_{n+1}-y_{n-1})(y_{n+1}+y_{n-1})-3y_n(y_{n+1}-y_{n-1})=0$

$\Leftrightarrow\ (y_{n+1}-y_{n-1})(y_{n+1}+y_{n-1}-3y_n)=0$

តែតាម $(*)$: $2y_{n+1}-3y_n\geq 0\Rightarrow 2y_{n+1}\geq 3y_n\Rightarrow y_{n+1}>y_n$

$\Rightarrow\ y_{n+1}>y_n>y_{n-1}\ \Rightarrow\ y_{n+1}-y_n>0$

$\Rightarrow\ y_{n+1}-3y_n+y_{n-1}=0$

យើងមាន: $y_0=1$ នោះ $y_1=\frac{1}{2}\left(3y_0+\sqrt{5y_0^2-4}\right)=2$

សង្កេតឃើញថា: $y_0\ ,\ y_1\in\mathbb{Z}$

ឧបមាថា: $y_{n-1}\,,\,y_n\in\mathbb{Z} \quad\Rightarrow\; y_{n+1}=3y_n-y_{n-1}\in\mathbb{Z}$

នោះតាមវិចារអនុមានរួមគណិតវិទ្យា គេបាន: $y_n\in\mathbb{Z}$ គ្រប់ $n\geq 0$ ។

ដូចនេះ គ្រប់តួនៃស្វ៊ីត $(y_n)_{n\geq 0}$ ជាចំនួនគត់ ។

<u>លំហាត់ទី៥១</u>

ចំនួនពិតវិជ្ជមាន a,b,c ផ្ទៀងផ្ទាត់សមភាព: $(a+b)(b+c)(c+a)=8$ ។

ចូរស្រាយបញ្ជាក់ថា: $\dfrac{a+b+c}{3}\geq\sqrt[27]{\dfrac{a^3+b^3+c^3}{3}}$ ។

(Macedonia Mathematical Olympiad 2008)

<u>ដំណោះស្រាយ</u>

គេមានឯកលក្ខណះភាព:

$$\begin{aligned}(a+b+c)^3 &= a^3+b^3+c^3+3(a+b)(b+c)(c+a)\\ &= a^3+b^3+c^3+24 \quad (\text{ព្រោះ } (a+b)(b+c)(c+a)=8\ )\\ &= a^3+b^3+c^3+3+3+3+3+3+3+3+3\\ &\geq 9\sqrt[9]{(a^3+b^3+c^3)3^8} \quad (\text{ ប្រើវិសមភាព } Cauchy\ 9 \text{ តួ })\end{aligned}$$

$$\Leftrightarrow\; a+b+c\geq 3\sqrt[27]{\frac{a^3+b^3+c^3}{3}}$$

$$\Leftrightarrow\; \frac{a+b+c}{3}\geq\sqrt[27]{\frac{a^3+b^3+c^3}{3}}$$

ដូចនេះវិសមភាពត្រូវបានស្រាយបញ្ជាក់ ។

លំហាត់ទី៨២

គេឲ្យ f ជាអនុគមន៍មានតម្លៃជាចំនួនពិតហើយផ្ទៀងផ្ទាត់:

$f(x,y)=f(x,z)-2f(y,z)-2z$ ចំពោះគ្រប់ចំនួនពិត x,y និង z ។

ចូរកំណត់តម្លៃនៃ $f(2005,1000)$ ។

(Singapore Mathematical Olympiad 2005)

ដំណោះស្រាយ

ចំពោះគ្រប់ចំនួនពិត x,y និង z យើងមាន:

$f(x,y)=f(x,z)-2f(y,z)-2z$

យក $x=y=z$ នោះ $f(x,x)=-x$ ចំពោះគ្រប់ចំនួនពិត x ។

យក $x=y$ នោះ $f(x,x)=f(x,z)-2f(x,z)-2z$

$\Rightarrow f(x,z)=-f(x,x)-2z=x-2z$

យក $x=2005$ និង $z=1000$ នោះ $f(2005,1000)=2005-2\times1000=5$

ដូចនេះ $f(2005,1000)=5$ ។

លំហាត់ទី៨៣

ចូរស្រាយបញ្ជាក់ថា: គ្មានចំនួនដែលមានទម្រង់ $44\cdots4443$ ណាមួយចែកដាច់នឹង 13 នោះទេ ។

(Dutch Mathematical Olympiad 2007)

ដំណោះស្រាយ

ឧបមាថា: មានចំនួនមួយដែលមានទម្រង់ $\underbrace{44\cdots4443}_{k}$ ចែកដាច់នឹង 13

យើងបាន: $\underbrace{44\cdots4443}_{k}-13=\underbrace{44\cdots44430}_{k-1}=\underbrace{44\cdots4443}_{k-1}\times10$ ក៏ចែកដាច់នឹង

13 ដែរ នាំឲ្យ $\underbrace{44\cdots4443}_{k-1}$ ចែកដាច់នឹង 13 ព្រោះ 10 និង 13 ជាចំនួនបឋមរវាងគ្នា ។ ដោយចេះតែបន្តធ្វើតាមរបៀបនេះ យើងនឹងបានៈ 43 ចែកដាច់នឹង 13 តែនេះជាករណីមិនពិត ។ បញ្ជាក់ថា ការឧបមាខាងលើគឺខុស ។ ដូចនេះគ្មានចំនួនដែលមានទម្រង់ $44\cdots4443$ ណាមួយចែកដាច់នឹង 13 នោះទេ ។

លំហាត់ទី៤៨

គេឲ្យ A ជាចំនួនពិត និង n ជាចំនួនគត់ដែល $2\le n\le 19$ ។ រកគ្រប់ពហុធា $P(x)$ ដែលមានមេគុណជាចំនួនពិតហើយផ្ទៀងផ្ទាត់ៈ
$P\left(P\left(P(x)\right)\right)=Ax^n+19x+99$ ។

(Austria Mathematical Olympiad 1999)

ដំណោះស្រាយ

តាង $m=\deg P(x)$ នោះ $\deg P\left(P\left(P(x)\right)\right)=m^3$

គេមានៈ $P\left(P\left(P(x)\right)\right)=Ax^n+19x+99$

បើ $A\neq 0$ នោះ $n=m^3$

ដោយ $2\le n\le 19 \Leftrightarrow 2\le m^3\le 19 \Rightarrow m=2$ (ព្រោះ m ជាចំនួនគត់)

$\Rightarrow n=8$ គេបានៈ $P(x)=ax^2+bx+c$ ដែល $a,b,c\in\mathbb{R}$ និង $a\neq 0$

$$\Rightarrow P\left(P(x)\right)=a\left(ax^2+bx+c\right)^2+b\left(ax^2+bx+c\right)+c$$

$$=a^3x^4+2a^2bx^3+\cdots$$

$$\Rightarrow P\left(P\left(P(x)\right)\right)=a^3\left(ax^2+bx+c\right)^4+2a^2b\left(ax^2+bx+c\right)^3+\cdots$$

$$=a^7x^8+4a^6bx^7+\cdots$$

តែ $P\left(P\left(P(x)\right)\right)=Ax^8+19x+99$ ទាញបានៈ $4a^6b=0\Rightarrow b=0$

ព្រោះ $a \neq 0$ នោះយើងបាន: $P(x) = ax^2 + c \Rightarrow P(-x) = P(x)$

$\Rightarrow P(P(P(-x))) = P(P(P(x)))$ នាំឲ្យ $P(P(P(x)))$ ជាអនុគមន៍គូ ផ្ទុយពី

$P(P(P(x))) = Ax^8 + 19x + 99$ ដែលមិនមែនជាអនុគមន៍គូ , នេះបញ្ជាក់ថា:

A ត្រូវតែស្មើនឹង 0 ។

ក្នុងករណីនេះ គេបាន: $P(P(P(x))) = 19x + 99$ នាំឲ្យ $\deg P(x) = 1$

នោះគេតាង $P(x) = \alpha x + \beta \quad , \alpha \neq 0$ គេបាន:

$$P(P(x)) = \alpha P(x) + \beta = \alpha(\alpha x + \beta) + \beta = \alpha x^2 + \alpha\beta + \beta$$

$$\Rightarrow P(P(P(x))) = \alpha^2 P(x) + \alpha\beta + \beta = \alpha^2(\alpha x + \beta) + \alpha\beta + \beta$$

$$= \alpha^3 x + (\alpha^2 + \alpha + 1)\beta$$

ដោយ $P(P(P(x))) = 19x + 99$

$$\Rightarrow \begin{cases} \alpha^3 = 19 \\ (\alpha^2 + \alpha + 1)\beta = 99 \end{cases} \Rightarrow \begin{cases} \alpha = \sqrt[3]{19} \\ \beta = \dfrac{99}{\alpha^2 + \alpha + 1} = \dfrac{99(\alpha - 1)}{\alpha^3 - 1} = \dfrac{99(\sqrt[3]{19} - 1)}{18} \end{cases}$$

ដូចនេះ: $P(x) = \sqrt[3]{19}x + \dfrac{99}{18}(\sqrt[3]{19} - 1)$ ។

លំហាត់ទី៤៨

បើ a ជាឫសពិតនៃសមីការ: $x^5 - x^3 + x - 2 = 0$ ។ បង្ហាញថា $\lfloor a^6 \rfloor = 3$ ។

(India Mathematical Olympiad 2004)

ដំណោះស្រាយ

ដោយ a ជាឫសពិតមួយនៃសមីការ: $x^5 - x^3 + x - 2 = 0$ នោះមានន័យថា:

$$a^5 - a^3 + a - 2 = 0 \quad , \quad (a \neq 0)$$

$$\Leftrightarrow a^5 = a^3 - a + 2$$

$$\Rightarrow a^7 = a^5 - a^3 + 2a^2$$
$$= a^3 - a + 2 - a^3 + 2a^2$$
$$= 2a^2 - a + 2$$
$$\Leftrightarrow a^7 + a = 2a^2 + 2$$
$$\Rightarrow a^6 + 1 = 2a + \frac{2}{a} \ge 2\sqrt{(2a)\left(\frac{2}{a}\right)} = 4 \text{ (តាមវិសមភាព } Cauchy)$$
$$\Rightarrow a^6 \ge 3 \quad (1)$$

ហើយ $a^5 + a = a^3 + 2$

$$\Rightarrow \frac{a^3+2}{a^3} = \frac{a^5+a}{a^3} = a^2 + \frac{1}{a^2} > 2\sqrt{a^2 \cdot \frac{1}{a^2}} = 2$$

(សមភាពមិនអាចកើតមានបានទេព្រោះ $a \neq \pm 1$ ផ្ទុយមកវិញ បើ $a = \pm 1$ នោះ a មិនមែនជាឫសនៃសមីការគេឲ្យនោះទេ)

$$\Leftrightarrow 1 + \frac{2}{a^3} > 2 \Rightarrow a^3 < 2 \Rightarrow a^6 < 4 \quad (2)$$

តាម (1) និង (2) គេបាន: $3 \le a^6 < 4$ នោះ $\lfloor a^6 \rfloor = 3$

ដូចនេះ $\lfloor a^6 \rfloor = 3$ ។

<u>លំហាត់ទី៥៦</u>

$a_1, a_2, a_3, \cdots$ ជាស្វ៊ីតមួយដែលផ្ទៀងផ្ទាត់: $a_1 = 2$, $a_2 = 5$ និង $a_{n+2} = (2 - n^2)a_{n+1} + (2 + n^2)a_n$ ចំពោះគ្រប់ $n \ge 1$ ។ តើមានសន្ទស្សន៍ p, q, r ដែលផ្ទៀងផ្ទាត់: $a_p a_q = a_r$ ដែរឬទេ ?

(1995 Czech-Slovak Match)

ដំណោះស្រាយ

យើងសង្កេតឃើញថា: $a_1 = 2 \equiv 2(\mathrm{mod}3)$ និង $a_2 = 5 \equiv 2(\mathrm{mod}3)$

ឧបមាថា: $a_n \equiv 2(\mathrm{mod}3)$ និង $a_{n+1} \equiv 2(\mathrm{mod}3)$ គេបាន:

$a_{n+2} \equiv (2-n^2)2 + (2+n^2)2 = 8 \equiv 2(\mathrm{mod}3)$

នោះតាមវិចារអនុមានរួមគណិតវិទ្យា គេបាន: $a_n \equiv 2(\mathrm{mod}3)$ ចំពោះគ្រប់ $n \ge 1$

ក្នុងនោះ $a_p \equiv 2(\mathrm{mod}3)$, $a_q \equiv 2(\mathrm{mod}3)$ $\Rightarrow$ $a_p a_q \equiv 4 \equiv 1(\mathrm{mod}3)$

ហើយ $a_r \equiv 2(\mathrm{mod}3)$ $\Rightarrow$ $a_p a_q \not\equiv a_r(\mathrm{mod}3)$ $\Rightarrow$ $a_p a_q \ne a_r$ ចំពោះគ្រប់ p,q,r ។

ដូចនេះ មិនមានសន្ទស្សន៍ p,q,r ដែលផ្ទៀងផ្ទាត់: $a_p a_q = a_r$ នោះទេ ។

លំហាត់ទី៨៧

គេឲ្យ a និង b ជាពីរចំនួនគត់វិជ្ជមានដែល $a > b$ ។

គេដឹងថា: $\gcd(a-b, ab+1) = 1$ និង $\gcd(a+b, ab-1) = 1$,

ចូរស្រាយបញ្ជាក់ថា: $(a-b)^2 + (ab+1)^2$ មិនមែនជាការេប្រាកដ ។

(Iran Mathematical Olympiad 2010)

ដំណោះស្រាយ

គេមាន: $(a-b)^2 + (ab+1)^2 = a^2 - 2ab + b^2 + a^2b^2 + 2ab + 1$

$$= (a^2+1)(b^2+1)$$

យើងនឹងស្រាយឲ្យឃើញថា: $\gcd(a^2+1, b^2+1) = 1$:

តាង $\delta = \gcd(a^2+1, b^2+1)$ នោះ $\delta | a^2+1 \wedge \delta | b^2+1$

$\Rightarrow \delta | (a^2+1) - (b^2+1)$ $\Rightarrow \delta | (a-b)(a+b)$

ឧបមាថា: $\delta \ne 1$ នោះយើងតាង d ជាតួចែកបឋមមួយនៃ δ , $(d > 1)$

$\Rightarrow d|\delta \Rightarrow d|(a-b)(a+b) \Rightarrow d|a-b \vee d|a+b$, (d ជាចំនួនបឋម)

• បើ $d|a-b$:

យើងមាន: $\delta|a^2+1 \wedge \delta|b^2+1$ ហើយ $d|\delta \Rightarrow d|a^2+1 \wedge d|b^2+1$

$\Rightarrow d|(a^2+1)(b^2+1)=(a-b)^2+(ab+1)^2$

$\Rightarrow d|(ab+1)^2$ ព្រោះ $d|a-b$

$\Rightarrow d|ab+1$ ព្រោះ d ជាចំនួនបឋម

$\Rightarrow d|\gcd(a-b,ab+1)=1$ មិនពិតព្រោះ $d>1$

• បើ $d|a+b$:

យើងមាន: $d|(a^2+1)(b^2+1)=(a+b)^2+(ab-1)^2$

$\Rightarrow d|(ab-1)^2$

$\Rightarrow d|ab-1$

$\Rightarrow d|\gcd(a+bb,ab-1)=1$ មិនពិតព្រោះ $d>1$

ទាំងពីរករណីនេះបញ្ជាក់ថា: ការឧបមាដែលថា $\delta\neq 1$ គឺខុស ។

ដូច្នេះ δ ត្រូវតែស្មើនឹង 1 មានន័យថា: a^2+1 និង b^2+1 ជាចំនួនបឋមរវាងគ្នា នាំឲ្យ បើ $(a-b)^2+(ab+1)^2=(a^2+1)(b^2+1)$ ជាការេប្រាកដលុះត្រាតែ a^2+1 និង b^2+1 ត្រូវតែជាការេប្រាកដទាំងពីរ តែនេះជាករណីមិនអាច ។

ដូចនេះ $(a-b)^2+(ab+1)^2$ មិនមែនជាការេប្រាកដនោះទេ ។

លំហាត់ទី៤៨

រង្វាស់ជ្រុង a , b , c និង u , v , w នៃពីរត្រីកោណ ABC និង UVW ផ្ទៀងផ្ទាត់នូវប្រព័ន្ធសមីការៈ $u(v+w-u)=a^2$, $v(w+u-v)=b^2$ និង $w(u+v-w)=c^2$ ។ ចូរស្រាយបញ្ជាក់ថា ABC ជាត្រីកោណមានមុំក្នុងទាំងបីជាមុំស្រួចហើយសរសេរមុំ U , V , W ជាអនុគមន៍នៃ A , B , C ។

(British Mathematical Olympiad 1996)

ដំណោះស្រាយ

យើងមានៈ

$$\begin{cases} u(v+w-u)=a^2 \\ v(w+u-v)=b^2 \\ w(u+v-w)=c^2 \end{cases} \Leftrightarrow \begin{cases} a^2=uv+uw-u^2 \\ b^2=vw+vu-v^2 \\ c^2=wu+wv-w^2 \end{cases}$$

$$\Rightarrow a^2+b^2-c^2=2uv-u^2-v^2+w^2$$

$$=w^2-(u-v)^2=(w-u+v)(w+u-v)>0$$

(ព្រោះ $w+v>u$, $w+u>v$ វិសមភាពក្នុងត្រីកោណ UVW)

$$\Rightarrow \cos C=\frac{a^2+b^2-c^2}{2ab}>0$$

ស្រាយដូចគ្នាដែរៈ $\cos B>0$, $\cos A>0$

នាំឲ្យ A , B , C សុទ្ធតែជាមុំស្រួច ។

ម្យ៉ាងទៀត $\cos C=\sqrt{\frac{(a^2+b^2-c^2)^2}{4a^2b^2}}$

$$=\sqrt{\frac{(w-u+v)^2(w+u-v)^2}{4uv(v+w-u)(w+u-v)}}$$

$$=\sqrt{\frac{(w-u+v)(w+u-v)}{4uv}}$$

$$= \sqrt{\frac{2uv - u^2 - v^2 + w^2}{4uv}}$$

$$= \frac{1}{\sqrt{2}}\sqrt{1 - \frac{u^2 + v^2 - w^2}{2uv}}$$

$$= \frac{1}{\sqrt{2}}\sqrt{1 - \cos W}$$

$\Rightarrow \sqrt{2}\cos C = \sqrt{1 - \cos W}$

$\Leftrightarrow \cos W = 1 - 2\cos^2 C = -\cos 2C = \cos(\pi - 2C)$

$\Rightarrow W = \pi - 2C$

ដូចគ្នា: $V = \pi - 2B$, $U = \pi - 2A$

ដូចនេះ A , B , C ជាមុំស្រួច ហើយ $U = \pi - 2A$, $V = \pi - 2B$ និង $W = \pi - 2C$ ។

<u>លំហាត់ទី៥៩</u>

គេឲ្យ a , b , c ជាចំនួនពិតវិជ្ជមានដែលផ្ទៀងផ្ទាត់: $a^3 + b^3 = c^3$ ។

ចូរស្រាយបញ្ជាក់ថា: $a^2 + b^2 - c^2 > 6(c-a)(c-b)$ ។

(India National Mathematical Olympiad 2009)

<u>ដំណោះស្រាយ</u>

តាមវិសមភាព $AM - GM$ គេមាន:

$bc^2 + cb^2 + b^3 + ac^2 + ca^2 + a^3 \geq 6\sqrt[6]{(bc^2)(cb^2)(b^3)(ac^2)(ca^2)(a^3)} = 6abc$

ដែលសមភាពកើតមានកាលណា: $bc^2 = cb^2 = b^3 = ac^2 = ca^2 = a^3$

នាំឲ្យ $a = b = c$ តែ $a^3 + b^3 = c^3$ ទាញបាន: $a = b = c = 0$ មិនពិតព្រោះ $a,b,c \in \mathbb{R}^+$ នេះបញ្ជាក់ថាវិសមភាពខាងលើមិនអាចក្លាយជាសមភាពទេ

គឺថា: $bc^2 + cb^2 + b^3 + ac^2 + ca^2 + a^3 > 6abc$

ចែកអង្គទាំងពីរនៃវិសមភាពនេះនឹង $ab>0$ គេបាន:

$$\frac{c^2+cb+b^2}{a}+\frac{c^2+ca+a^2}{b}>6c$$

គុណអង្គទាំងពីរនៃវិសមភាពនេះនឹង $(c-a)(c-b)>0$ គេបាន:

$$\frac{(c-a)(c^3-b^3)}{a}+\frac{(c-b)(c^3-a^3)}{b}>6c(c-a)(c-b)$$

ដោយ $a^3+b^3=c^3$ នាំឲ្យ $c^3-a^3=b^3$ និង $c^3-b^3=a^3$ គេបាន:

$$\frac{a^3(c-a)}{a}+\frac{b^3(c-b)}{b}>6c(c-a)(c-b)$$

$$\Leftrightarrow\ a^2(c-a)+b^2(c-b)>6c(c-a)(c-b)$$

$$\Leftrightarrow\ a^2c-a^3+b^2c-b^3>6c(c-a)(c-b)$$

$$\Leftrightarrow\ a^2+b^2-\frac{a^3+b^3}{c}>6(c-a)(c-b)$$

តែ $a^3+b^3=c^3 \Rightarrow \dfrac{a^3+b^3}{c}=c^2$

ដូចនេះ: $a^2+b^2-c^2>6(c-a)(c-b)$ ត្រូវបានស្រាយបញ្ជាក់ ។

លំហាត់ទី៦០

គេឲ្យពហុធា $P(x)=x^3+ax^2+bx+c$ មានឫសសុទ្ធតែជាចំនួនពិត ហើយ គេតាង $Q(x)=5x^2-16x+2004$ ។ ឧបមាថា $P(Q(x))=0$ គ្មានឫសជាចំនួនពិត , ចូរស្រាយបញ្ជាក់ថា: $P(2004)>2004$ ។

(India International Mathematical Olympiad Training Camp 2005)

ដំណោះស្រាយ

តាង $P(x)=(x-x_1)(x-x_2)(x-x_3)$ ដែល x_1 , x_2 , x_3 ជាឫសពិតទាំងបីនៃ $P(x)$

ដោយ $P(Q(x))=0$ ជាសមីការគ្មានឫសជាចំនួនពិត នោះទាញបាន:

$Q(x)=x_i \quad ; \quad i=1,2,3$ ត្រូវតែជាសមីការគ្មានឫសជាចំនួនពិត នោះ $\Delta<0$ ។

យើងមាន: $Q(x)=x_i \Leftrightarrow 5x^2-16x+2004=x_i$

$\Leftrightarrow 5x^2-16x+2004-x_i=0 \quad \Rightarrow \Delta=16^2-20(2004-x_i)<0$

$\Leftrightarrow 2004-x_i>\dfrac{64}{4}$ ចំពោះ $i=1,2,3$

$\Rightarrow P(2004)=(2004-x_1)(2004-x_2)(2004-x_3)>\left(\dfrac{64}{5}\right)^3>2004$

ដូចនេះ $P(2004)>2004$ ត្រូវបានស្រាយបញ្ជាក់ ។

លំហាត់ទី៦១

ស្វ៊ីតពីរ (x_n) និង (y_n) ត្រូវបានកំណត់ដោយ: $x_0=365$,

$x_{n+1}=x_n\left(x_n^{1986}+1\right)+1622$ ចំពោះគ្រប់ $n\geq 0$ និង $y_0=16$,

$y_{n+1}=y_n\left(y_n^3+1\right)-1952$ ចំពោះគ្រប់ $n\geq 0$ ។ ចូរស្រាយបញ្ជាក់ថា:

$\left|x_p-x_q\right|>0$ ចំពោះគ្រប់ $p,q\geq 1$ ។

(Vietnam Mathematical Olympiad 1987)

ដំណោះស្រាយ

កត់សម្គាល់ឃើញថា: (x_n) និង (y_n) ជាស្វ៊ីតនៃចំនួនគត់វិជ្ជមាន ។

គេមាន $x_{n+1}=x_n\left(x_n^{1986}+1\right)+1622=x_n^{1987}+x_n+1622$

$\Rightarrow x_{n+1}-x_n=x_n^{1987}+1622$

$\Rightarrow x_1-x_0=x_0^{1987}+1622=\left(365^{1987}-365\right)+1987$

ដោយ 1987 ជាចំនួនបឋម នោះតាមទ្រឹស្តីបទ *Fermat* គេបាន:

$365^{1987}\equiv 365(\text{mod}1987) \Leftrightarrow 365^{1987}-365\equiv 0(\text{mod}1987)$

$\Rightarrow\ x_1 - x_0 \equiv 0 \pmod{1987}$

$\Leftrightarrow\ x_1 \equiv x_0 \pmod{1987}$

ដូចគ្នាដែរ:

$x_2 - x_1 = x_1^{1987} + 1622 \equiv x_0^{1987} + 1622 = \left(365^{1987} - 365\right) + 1987 \equiv 0$
$\pmod{1987}\ \ \Rightarrow\ x_2 \equiv x_1 \equiv x_0 \pmod{1987}$

តាមរបៀបនេះយើងបាន: $x_n \equiv x_0 \pmod{1987}$ គ្រប់ $n \geq 1$

ដោយ $x_0 = 365$ នោះ $x_n \equiv 365 \pmod{198}$ គ្រប $n \geq 1$ ។

ម្យ៉ាងទៀត $y_{n+1} = y_n\left(y_n^3 + 1\right) - 1952 = y_n^4 + y_n - 1952$

$\Leftrightarrow\ y_{n+1} - y_n = y_n^4 - 1952$

$\Rightarrow\ y_1 - y_0 = y_0^4 - 1952 = 16^4 - 1952 = 63584 = 32 \cdot 1987$

$\Rightarrow\ y_1 - y_0 \equiv 0 \pmod{1987}\ \Leftrightarrow\ y_1 \equiv y_0 \pmod{1987}$

ដូចគ្នាដែរ: $y_2 - y_1 = y_1^4 - 1952 \equiv y_0^4 - 1952 \equiv 0 \pmod{1987}$

$\Rightarrow\ y_2 \equiv y_1 \equiv y_0 \pmod{1987}$

តាមលំនាំនេះយើងបាន: $y_n \equiv y_0 \pmod{1987}$ គ្រប់ $n \geq 1$

ដោយ $y_0 = 16$ នោះ $y_n \equiv 16 \pmod{1987}$ គ្រប់ $n \geq 1$ ។

ចំពោះគ្រប់ $p, q \geq 1$ យើងមាន: $x_p \equiv 365 \pmod{1987}$ និង

$y_q \equiv 16 \pmod{1987}$

$\Rightarrow\ x_p - x_q \equiv 365 - 16 \equiv 349 \pmod{1987}$

នេះបញ្ជាក់ថា $x_p \neq y_q$ គ្រប់ $p, q \geq 1$ ។

ដូចនេះ $\left|x_p - x_q\right| > 0$ ចំពោះគ្រប់ $p, q \geq 1$ ។

លំហាត់ទី៦២

គេឲ្យ n ជាចំនួនគត់មួយ ។ បង្ហាញថា: បើ $2+2\sqrt{1+12n^2}$ ជាចំនួនគត់ នោះ វាត្រូវតែជាការេប្រាកដ ។

(British Mathematical Olympiad 2006)

ដំណោះស្រាយ

តាង $2+2\sqrt{1+12n^2}=m$ ជាចំនួនគត់

$\Leftrightarrow\ 4\left(1+12n^2\right)=(m-2)^2$ ក្នុងករណីនេះ , m ត្រូវតែជាចំនួនគត់គូ គឺ

$m=2i$

គេបាន: $1+12n^2=(i-1)^2$ ទាញបាន: i ត្រូវតែជាចំនួនគត់គូ គឺ $i=2j$

គេបាន: $1+12n^2=(2j-1)^2\ \Leftrightarrow\ 1+12n^2=4j^2-4j+1$

$\Leftrightarrow\ 3n^2=j(j-1)$

ដោយ $\gcd(j,j-1)=1$ នោះ $j=3k^2$, $j-1=l^2$ ឬ $j=k^2$, $j-1=3l^2$

* បើ $j=3k^2$, $j-1=l^2$ នោះ $l^2+1=3k^2\ \Rightarrow\ l^2+1\equiv 0(\mathrm{mod}\,3)$

$\Leftrightarrow\ l^2\equiv -1(\mathrm{mod}\,3)$ មិនពិតទេព្រោះ ចំពោះគ្រប់ចំនួនពិត l , $l^2\not\equiv -1(\mathrm{mod}\,3)$

* បើ $j=k^2$, $j-1=3l^2$ នោះ $m=2i=4j=4k^2=(2k)^2$ ជាការេប្រាកដ ។

ដូចនេះ ចំពោះគ្រប់ចំនួនគត់ n , បើ $2+2\sqrt{1+12n^2}$ ជាចំនួនគត់នោះ វាត្រូវតែជាការេប្រាកដ ។

លំហាតទី៦៣

ចូររកគ្រប់ចំនួនគត់ x និង ចំនួនបឋម p ដែលផ្ទៀងផ្ទាត់: $x^8+2^{2^x+2}=p$ ។

(Greece Mathematical Olympiad 2008)

ដំណោះស្រាយ

យើងមាន: $x^8 + 2^{2^x+2} = p$

យើងសង្កេតឃើញថា បើ $x<0$ នោះ $2^x \in \mathbb{Q} \Rightarrow 2^{2^x+2} \in \overline{\mathbb{Q}}$

$\Rightarrow p = x^8 + 2^{2^x+2} \notin \mathbb{N}$ មិនពិត ដូច្នេះ $x \geq 0$

បើ $x=0$ នោះ $p=2^3$ មិនមែនជាចំនួនបឋម ។

បើ $x=1$ នោះ $p=17$ ជាចំនួនបឋម គេបាន $x=1$ និង $p=17$ ជាចម្លើយមួយ

បើ $x \geq 2$:

$$p = \left(x^4\right)^2 + 4\cdot 2^{2^x}$$

$$= \left(x^4 + 2\cdot 2^{2^{x-1}}\right)^2 - 4x^4\cdot 2^{2^{x-1}}$$

$$= \left(x^4 + 2\cdot 2^{2^{x-1}} - 2x^2\cdot 2^{2^{x-2}}\right)\left(x^4 + 2\cdot 2^{2^{x-1}} + 2x^2\cdot 2^{2^{x-2}}\right)$$

ដើម្បីឲ្យ p អាចជាចំនួនបឋមលុះត្រាតែ: $x^4 + 2\cdot 2^{2^{x-1}} - 2x^2\cdot 2^{2^{x-2}} = 1$

$\Leftrightarrow \left(x^4 - 2x^2\cdot 2^{2^{x-2}} + 2^{2^{x-1}}\right) + 2^{2^{x-1}} = 1$

$\Leftrightarrow \left(x^2 - 2^{2^{x-2}}\right)^2 + 2^{2^{x-1}} = 1$ ជាករណីមិនអាច ។

ដូចនេះ $x=1$ និង $p=17$ ជាចម្លើយតែមួយគត់ ។

លំហាត់ទី៦៨

គេឲ្យ a_1, a_2, a_3, a_4, a_5 ជាចំនួនពិតដែលផ្ទៀងផ្ទាត់សមីការខាងក្រោម:

$\dfrac{a_1}{k^2+1} + \dfrac{a_2}{k^2+2} + \dfrac{a_3}{k^2+3} + \dfrac{a_4}{k^2+4} + \dfrac{a_5}{k^2+5} = \dfrac{1}{k^2}$ ចំពោះ $k = 1,2,3,4,5$ ។

ចូរកំណត់តម្លៃនៃ $\dfrac{a_1}{37} + \dfrac{a_2}{38} + \dfrac{a_3}{39} + \dfrac{a_4}{40} + \dfrac{a_5}{41}$ ។

(APMO 2009)

ដំណោះស្រាយ

តាង $P(x)=x^2(x^2+1)(x^2+2)(x^2+3)(x^2+4)(x^2+5)\left(\frac{a_1}{x^2+1}+\frac{a_2}{x^2+2}+\frac{a_3}{x^2+3}+\frac{a_4}{x^2+4}+\frac{a_5}{x^2+5}-\frac{1}{x^2}\right)$ ជាពហុធាដឺក្រេទី 10

ដោយ $\frac{a_1}{k^2+1}+\frac{a_2}{k^2+2}+\frac{a_3}{k^2+3}+\frac{a_4}{k^2+4}+\frac{a_5}{k^2+5}=\frac{1}{k^2}$ ចំពោះ $k=1,2,3,4,5$

នោះ $P(\pm1)=P(\pm2)=P(\pm3)=P(\pm4)=P(\pm5)=0$

នាំឲ្យ $x=\pm1,\pm2,\pm3,\pm4,\pm5$ គេបាន:

$$P(x)=\lambda(x^2-1^2)(x^2-2^2)(x^2-3^2)(x^2-4^2)(x^2-5^2)$$

នោះ $P(0)=-\lambda(5!)^2$

តែ: $P(x)=x^2(x^2+1)(x^2+2)(x^2+3)(x^2+4)(x^2+5)\left(\frac{a_1}{x^2+1}+\frac{a_2}{x^2+2}+\frac{a_3}{x^2+3}+\frac{a_4}{x^2+4}+\frac{a_5}{x^2+5}\right)-(x^2+1)(x^2+2)(x^2+3)(x^2+4)(x^2+5)$

នោះ $P(0)=-(5!)$ នាំឲ្យ $-\lambda(5!)^2=-(5!)\Rightarrow \lambda=\frac{1}{5!}$

គេបាន: $P(x)=\frac{1}{5!}(x^2-1^2)(x^2-2^2)(x^2-3^2)(x^2-4^2)(x^2-5^2)$

ដោយ: $P(x)=x^2(x^2+1)(x^2+2)(x^2+3)(x^2+4)(x^2+5)\left(\frac{a_1}{x^2+1}+\frac{a_2}{x^2+2}+\frac{a_3}{x^2+3}+\frac{a_4}{x^2+4}+\frac{a_5}{x^2+5}-\frac{1}{x^2}\right)$

នាំឲ្យ $\frac{a_1}{x^2+1}+\frac{a_2}{x^2+2}+\frac{a_3}{x^2+3}+\frac{a_4}{x^2+4}+\frac{a_5}{x^2+5}$

$$=\frac{\frac{1}{5!}(x^2-1^2)(x^2-2^2)(x^2-3^2)(x^2-4^2)(x^2-5^2)}{x^2(x^2+1)(x^2+2)(x^2+3)(x^2+4)(x^2+5)}+\frac{1}{x^2}$$

យក $x=6$ គេបាន:

$$\frac{a_1}{37}+\frac{a_2}{38}+\frac{a_3}{39}+\frac{a_4}{40}+\frac{a_5}{41}$$

$$=\frac{\frac{1}{5!}\left(6^2-1^2\right)\left(6^2-2^2\right)\left(6^2-3^2\right)\left(6^2-4^2\right)\left(6^2-5^2\right)}{6^2\left(6^2+1\right)\left(6^2+2\right)\left(6^2+3\right)\left(6^2+4\right)\left(6^2+5\right)}+\frac{1}{6^2}$$

$$=\frac{187465}{6744582}$$

ដូចនេះ $\frac{a_1}{37}+\frac{a_2}{38}+\frac{a_3}{39}+\frac{a_4}{40}+\frac{a_5}{41}=\frac{187465}{6744582}$ ត្រូវបានកំណត់ ។

លំហាត់ទី៦៥

រកគ្រប់អនុគមន៍ $f:\mathbb{N}\to\mathbb{N}$ ដែលផ្ទៀងផ្ទាត់: $f(1)=1$,

$f(mn)=\frac{f(m)f(n)}{f(\gcd(m,n))}$ និង $\underbrace{(f\circ f\circ\cdots\circ f)}_{2000}(m)=m$

ចំពោះគ្រប់ $m\ ,\ n\in\mathbb{N}$ ។

(Iran Mathematical Olympiad 2000)

ដំណោះស្រាយ

ចំពោះគ្រប់ $m\ ,\ n\in\mathbb{N}$ យើងមាន: $f(mn)=\frac{f(m)f(n)}{f(\gcd(m,n))}$

នោះយើងបាន: $f(4)=f(2\times 2)=\frac{f(2)f(2)}{f(\gcd(2,2))}=f(2)$

ម្យ៉ាងទៀត $\underbrace{(f\circ f\circ\cdots\circ f)}_{2000}(m)=m$ ចំពោះគ្រប់ $m\in\mathbb{N}$

នោះ $2=\underbrace{(f\circ f\circ\cdots\circ f)}_{2000}(2)=\underbrace{(f\circ f\circ\cdots\circ f)}_{1999}(f(2))$

$=\underbrace{(f\circ f\circ\cdots\circ f)}_{1999}(f(4))=\underbrace{(f\circ f\circ\cdots\circ f)}_{2000}(4)=4$ ជាករណីមិនអាចមាន ។

ដូចនេះ គ្មានអនុគមន៍ f ណាផ្ទៀងផ្ទាត់លក្ខខណ្ឌប្រធានលំហាត់នោះទេ ។

លំហាត់ទី៦៦

គេឲ្យ a , b , c , d ជាចំនួនពិតវិជ្ជមានដែល:

$\frac{1}{1+a^4}+\frac{1}{1+b^4}+\frac{1}{1+c^4}+\frac{1}{1+d^4}=1$ ។ ចូរស្រាយបញ្ជាក់ថា: $abcd\geq 3$ ។

(Latvia 2002)

ដំណោះស្រាយ

តាង $a^2=\tan A$, $b^2=\tan B$, $c^2=\tan C$, $d^2=\tan D$

ដូច្នេះសមភាពដែលគេឲ្យសមមូល:

$$\frac{1}{1+\tan^2 A}+\frac{1}{1+\tan^2 B}+\frac{1}{1+\tan^2 C}+\frac{1}{1+\tan^2 D}=1$$

$$\Leftrightarrow\ \cos^2 A+\cos^2 B+\cos^2 C+\cos^2 D=1$$

$$\Rightarrow\ 1-\cos^2 A=\cos^2 B+\cos^2 C+\cos^2 D\geq 3\sqrt[3]{\cos^2 B\cdot\cos^2 C\cdot\cos^2 D}$$

$$\Leftrightarrow\ \sin^2 A\geq 3\sqrt[3]{\cos^2 B\cdot\cos^2 C\cdot\cos^2 D}$$

ដូចគ្នាដែរ: $\sin^2 B\geq 3\sqrt[3]{\cos^2 A\cdot\cos^2 C\cdot\cos^2 D}$

$$\sin^2 C\geq 3\sqrt[3]{\cos^2 A\cdot\cos^2 B\cdot\cos^2 D}$$

$$\sin^2 D\geq 3\sqrt[3]{\cos^2 A\cdot\cos^2 B\cdot\cos^2 C}$$

គុណអង្គនិងអង្គនៃវិសមភាពទាំងបួនខាងលើ គេបាន:

$$\sin^2 A\cdot\sin^2 B\cdot\sin^2 C\cdot\sin^2 D\geq 3^4\cos^2 A\cdot\cos^2 B\cdot\cos^2 C\cdot\cos^2 D$$

$$\Leftrightarrow\ \tan^2 A\cdot\tan^2 B\cdot\tan^2 C\cdot\tan^2 D\geq 3^4$$

$$\Leftrightarrow\ a^4b^4c^4d^4\geq 3^4$$

$\Rightarrow \quad abcd \geq 3$

ដូចនេះ $abcd \geq 3$ ត្រូវបានស្រាយបញ្ជាក់ ។

លំហាត់ទី៦៧

ចូរស្រាយបញ្ជាក់ថា បើ m, n និង r ជាចំនួនគត់វិជ្ជមាន ហើយ
$1+m+n\sqrt{3}=\left(2+\sqrt{3}\right)^{2r-1}$ នោះ m ជាការេប្រាកដ ។

(British Mathematical Olympiad 1986)

ដំណោះស្រាយ

តាមរូបមន្តពន្លាតទ្វេធា *Newton* គេមាន:

$$\left(2+\sqrt{3}\right)^{2r-1}=\sum_{k=0}^{2r-1} C_{2r-1}^{k}\cdot 2^{(2r-1)-k}\cdot\left(\sqrt{3}\right)^{k}$$
$$=\left(2^{2r-1}+3C_{2r-1}^{2}\cdot 2^{2r-3}+\cdots+3^{r-1}C_{2r-1}^{2r-2}\cdot 2\right)$$
$$+\left(C_{2r-1}^{1}\cdot 2^{2r-2}+3C_{2r-1}^{3}\cdot 2^{2r-4}+\cdots+3^{r-1}C_{2r-1}^{2r-1}\right)\sqrt{3}$$

ហើយដូចគ្នាដែរចំពោះ

$$\left(2-\sqrt{3}\right)^{2r-1}=\sum_{k=0}^{2r-1} C_{2r-1}^{k}\cdot 2^{(2r-1)-k}\cdot\left(-\sqrt{3}\right)^{k}$$
$$=\left(2^{2r-1}+3C_{2r-1}^{2}\cdot 2^{2r-3}+\cdots+3^{r-1}C_{2r-1}^{2r-2}\cdot 2\right)$$
$$-\left(C_{2r-1}^{1}\cdot 2^{2r-2}+3C_{2r-1}^{3}\cdot 2^{2r-4}+\cdots+3^{r-1}C_{2r-1}^{2r-1}\right)\sqrt{3}$$

តាង $a=2^{2r-1}+3C_{2r-1}^{2}\cdot 2^{2r-3}+\cdots+3^{r-1}C_{2r-1}^{2r-2}\cdot 2$

ហើយ $b=C_{2r-1}^{1}\cdot 2^{2r-2}+3C_{2r-1}^{3}\cdot 2^{2r-4}+\cdots+3^{r-1}C_{2r-1}^{2r-1}$ ដែល a និង b សុទ្ធតែ ជាចំនួនគត់វិជ្ជមាន ក្នុងនោះ $a+1=\left(2^{2r-1}+1\right)+3C_{2r-1}^{2}\cdot 2^{2r-3}+\cdots+3^{r-1}C_{2r-1}^{2r-2}\cdot 2$ $=3x$ ដែល $x=\left(2^{2r-2}-2^{2r-3}+\cdots-2+1\right)+\left(C_{2r-1}^{2}\cdot 2^{2r-3}+\cdots+3^{r-2}C_{2r-1}^{2r-1}\cdot 2\right)$ ជាចំនួនគត់សេស យើងបាន: $\left(2+\sqrt{3}\right)^{2r-1}=a+b\sqrt{3}$ និង

$$\left(2-\sqrt{3}\right)^{2r-1}=a-b\sqrt{3}$$

នាំឲ្យ $\left(2+\sqrt{3}\right)^{2r-1}\left(2-\sqrt{3}\right)^{2r-1}=\left(a+b\sqrt{3}\right)\left(a-b\sqrt{3}\right)$

$\Leftrightarrow\ a^2-3b^2=1\ \Leftrightarrow\ (a-1)\left(\dfrac{a+1}{3}\right)=b^2\quad(*)$

យើងនឹងស្រាយថា៖ $\gcd\left(a-1,\dfrac{a+1}{3}\right)=1$

តាង $d=\gcd\left(a-1,\dfrac{a+1}{3}\right)$ នោះ $a-1=pd$ និង $\dfrac{a+1}{3}=qd$ ដែល $\gcd(p,q)=1$

នាំឲ្យ $(a-1)-3\left(\dfrac{a+1}{3}\right)=pd-3qd$ ឬ $(3q-p)d=2$ នាំឲ្យ $d|2$

ដោយ $a-1=3x-2$ និង $\dfrac{a+1}{3}=x$ សុទ្ធតែជាចំនួនគត់សេស នោះ

$d=\gcd\left(a-1,\dfrac{a+1}{3}\right)$ ក៏ត្រូវតែជាចំនួនគត់សេសដែរ យើងទាញបាន៖ $d=1$

មានន័យថា $a-1$ និង $\dfrac{a+1}{3}$ ជាចំនួនបឋមរវាងគ្នា តាម $(*)$ នាំឲ្យ $a-1$ និង $\dfrac{a+1}{3}$ ត្រូវតែជាការេប្រាកដទាំងពីរ ។

ដោយ $\left(2+\sqrt{3}\right)^{2r-1}=a+b\sqrt{3}$ តែតាមសម្មតិកម្ម៖

$$1+m+n\sqrt{3}=\left(2+\sqrt{3}\right)^{2r-1}$$

ទាញបាន៖ $1+m=a$ នោះ $m=a-1$ ជាការេប្រាកដ ។

ដូចនេះ m ជាការេប្រាកដ ត្រូវបានស្រាយបញ្ជាក់ ។

លំហាត់ទី៦៨

ចូរស្រាយបញ្ជាក់ថា ប្រព័ន្ធសមីការ $x^3+y^3+z^3=x+y+z$ និង $x^2+y^2+z^2=xyz$ គ្មានចម្លើយ (x,y,z) ជាចំនួនពិតវិជ្ជមានទេ ។

(Canadian Mathematical Olympiad 2003)

ដំណោះស្រាយ

តាង $f(x,y,z)=(x^3-x)+(y^3-y)+(z^3-z)$

ដោយ $x^3+y^3+z^3=x+y+z$ នោះ $f(x,y,z)=0$

បើ $x,y,z\geq 1$ នោះ $f(x,y,z)\geq 0$ ដែលសមភាពកើតមានកាលណា: $x=y=z=1$ តែបើ $x=y=z=1$ នោះសមីការ $x^2+y^2+z^2=xyz$ មិនផ្ទៀងផ្ទាត់ នេះបញ្ជាក់ថា បើប្រព័ន្ធសមីការមានចម្លើយ (x,y,z) ជាចំនួនពិតវិជ្ជមាន នោះយ៉ាងតិចបំផុតមានមួយនៃអញ្ញាតទាំងបីត្រូវតែតូចជាង 1 យើងសន្មតថា: $x<1$ គេបាន: $x^2+y^2+z^2>y^2+z^2\geq 2yz>yz>xyz$ ផ្ទុយពីសម្មតិកម្ម: $x^2+y^2+z^2=xyz$ ។

ដូចនេះ យើងអាចសន្និដ្ឋានបានថា: ប្រព័ន្ធសមីការគ្មានចម្លើយ (x,y,z) ជាចំនួនពិតវិជ្ជមាននោះទេ ។

លំហាត់ទី៦៩

គេឲ្យ $f(x)$ ជាពហុធា *monic* ដឺក្រេទី 1991 ដែលមានមេគុណជាចំនួនគត់ ហើយគេតាង $g(x)=\left(f(x)\right)^2-9$ ។ ចូរស្រាយបញ្ជាក់ថា: ចំនួននៃឫសដែលជាចំនួនគត់ខុសៗគ្នារបស់សមីការ $g(x)=0$ មិនអាចលើសពី 1995 នោះទេ ។

(IMO Shortlist 1991)

ដំណោះស្រាយ

យើងមាន: $g(x)=\left(f(x)\right)^2-9=\left(f(x)-3\right)\left(f(x)+3\right)$

តាង $x_1, x_2, x_3, \ldots, x_k$ ជាឫសគត់ខុសៗគ្នានៃសមីការ $f(x)-3=0$ និង $y_1, y_2, y_3, \ldots, y_h$ ជាជាឫសគត់ខុសៗគ្នានៃសមីការ $f(x)+3=0$ ហើយយើងសន្មតថា: $x_1<x_2<x_3<\cdots<x_k$, $y_1<y_2<y_3<\cdots<y_h$ និង $x_i\neq y_j$ ។

ឧបមាថា: ចំនួននៃឫសគត់ខុសៗគ្នារបស់សមីការ $g(x)=0$ លើសពី 1995 នោះមានន័យថា: $k+h>1995$ ឬ $k+h\geq 1996$

ដោយ $f(x)$ មានដឺក្រេ 1991 នោះឫសនៃសមីការ $f(x)-3=0$ និង $f(x)+3=0$ មានចំនួនមិនលើសពី 1991 ទេ គឺថា: $k\leq 1991$ និង $h\leq 1991$

នាំឲ្យ $k, h\geq 5$

ដោយឫសនៃសមីការ $g(x)=0$ សុទ្ធតែជាចំនួនគត់ខុសៗគ្នា នោះច្បាស់ជាមាន i, j ដែល $\left|x_i-y_j\right|\geq 7$ $(*)$

តាង $f(x)=x^{1991}+a_{1990}x^{1990}+\cdots+a_0$,($f(x)$ ជាពហុធា *monic* ដឺក្រេទី 1991)

នោះ $f(x_i)-3=0$ គ្រប់ $i=1,2,3,\ldots,k$ និង $f\left(y_j\right)+3=0$ គ្រប់ $j=1,2,3,\ldots,h$ ។

គេបាន:

$$\begin{cases} x_i^{1991} + a_{1990}x_i^{1990} + \cdots + a_0 = 3 \\ y_j^{1991} + a_{1990}y_j^{1990} + \cdots + a_0 = -3 \end{cases}$$

ដកអង្គនិងអង្គគេបាន: $\left(x_i^{1991} - y_j^{1991}\right) + \left(x_i^{1990} - y_j^{1990}\right)a_{1990} + \cdots + \left(x_i - y_j\right)a_1 = 6$

ដោយ $x_i - y_j \mid x_i^m - y_j^m$ ចំពោះគ្រប់ $m = 1,2,3,\ldots,1991$

នាំឲ្យ $x_i - y_j \mid 6$ នោះ $\left|x_i - y_j\right| \le 6$ ចំពោះគ្រប់ $i = 1,2,3,\ldots,k$ និង $j = 1,2,3,\ldots,h$ ផ្ទុយពី $(*)$ នេះបញ្ជាក់ថា: ការឧបមាខាងលើគឺខុស ។

ដូចនេះ ចំនួនឫសនៃសមីការ $g(x) = 0$ មិនអាចលើសពី 1995 នោះទេ ។

<u>លំហាត់ទី៧០</u>

គេឲ្យ a, b, c ជាចំនួនពិតវិជ្ជមានដែល $a + b + c = 1$ ។

ចូរបង្ហាញថា: $\sqrt{a^{1-a}b^{1-b}c^{1-c}} \le \dfrac{1}{3}$ ។

(Austria Mathematical Olympiad 2008)

<u>ដំណោះស្រាយ</u>

របៀបទី១

តាមលក្ខណៈស៊ីមេទ្រី យើងសន្មតថា: $a \ge b \ge c > 0$

នាំឲ្យ $\log a \ge \log b \ge \log c$ គេបាន: $(a-b)(\log a - \log b) \ge 0$,

$(b-c)(\log b - \log c) \ge 0$ និង $(a-c)(\log a - \log c) \ge 0$

នោះ: $(a-b)(\log a - \log b) + (b-c)(\log b - \log c) + (a-c)(\log a - \log c) \ge 0$

$$\Leftrightarrow 2(a\log a + b\log b + c\log c) \ge (\log b + \log c)a + (\log c + \log a)b + (\log a + \log b)c$$

$$\Leftrightarrow 2\left(\log a^a + \log b^b + \log c^c\right) \ge a\log bc + b\log ca + c\log ab$$

$\Leftrightarrow\ 2\log a^a b^b c^c \geq \log b^a c^a + \log c^b a^b + \log a^c b^c$

$\Leftrightarrow\ 2\log a^a b^b c^c + \log a^a b^b c^c \quad \geq \log b^{a+c} c^{b+a} a^{b+c} + \log a^a b^b c^c$

$\Leftrightarrow\ 3\log a^a b^b c^c \geq \log a^{a+b+c} b^{a+b+c} c^{a+b+c}$

$\Leftrightarrow\ \log\left(a^a b^b c^c\right)^3 \geq \log abc$ ព្រោះ $a+b+c=1$

$\Leftrightarrow \left(a^a b^b c^c\right)^3 \geq abc$

$\Leftrightarrow\ a^a b^b c^c \geq (abc)^{\frac{1}{3}} = (abc)(abc)^{-\frac{2}{3}}$

$\Leftrightarrow\ \dfrac{abc}{a^a b^b c^c} \leq (abc)^{\frac{2}{3}}$

$\Leftrightarrow\ \sqrt{a^{1-a} b^{1-b} c^{1-c}} \leq \sqrt[3]{abc} \leq \dfrac{a+b+c}{3} = \dfrac{1}{3}$

ដូចនេះវិសមភាព $\sqrt{a^{1-a} b^{1-b} c^{1-c}} \leq \dfrac{1}{3}$ ត្រូវបានស្រាយបញ្ជាក់ ។

របៀបទី២

តាមលក្ខណះស៊ីមេទ្រី យើងសន្មតថាៈ $a \geq b \geq c > 0$

នាំឲ្យ $\log a \geq \log b \geq \log c$ ដោយប្រើវិសមភាព *Chebyshev* គេបានៈ

$\dfrac{a+b+c}{3} \cdot \dfrac{\log a + \log b + \log c}{3} \leq \dfrac{a\log a + b\log b + c\log c}{3}$

$\Leftrightarrow\ \log abc \leq \log\left(a^a b^b c^c\right)^3$ ព្រោះ $a+b+c=1$

$\Leftrightarrow\ abc \leq \left(a^a b^b c^c\right)^3 \Rightarrow \sqrt{a^{1-a} b^{1-b} c^{1-c}} \leq \sqrt[3]{abc} \leq \dfrac{a+b+c}{3} = \dfrac{1}{3}$

ដូចនេះវិសមភាព $\sqrt{a^{1-a} b^{1-b} c^{1-c}} \leq \dfrac{1}{3}$ ត្រូវបានស្រាយបញ្ជាក់ ។

លំហាត់ទី៧១

តាង f ជាអនុគមន៍មួយកំណត់ពីសំណុំនៃចំនួនគត់ទៅសំណុំនៃចំនួនគត់វិជ្ជមាន , គេសន្មតថា ចំពោះពីរចំនួនគត់ m និង n នោះផលសង $f(m)-f(n)$ ចែកដាច់នឹង $f(m-n)$ ។ ចូរស្រាយបញ្ជាក់ថា ចំពោះគ្រប់ចំនួនគត់ m និង n ដែល $f(m)\le f(n)$ នោះ ចំនួន $f(n)$ ចែកដាច់នឹង $f(m)$ ។

(IMO 2011)

ដំណោះស្រាយ

ចំពោះគ្រប់ចំនួនគត់ m និង n គេមាន: $f(m-n)|f(m)-f(n)$

យក $n=0$ នោះ $f(m)|f(m)-f(0)$ នាំឲ្យ $f(m)|f(0)$ ចំពោះគ្រប់ចំនួនគត់ m ។

យក $m=0$ នោះ $f(-n)|f(0)-f(n)$ នាំឲ្យ $f(-n)|f(n)$ ហើយគេក៏បាន: $f(n)|f(-n)$ ទាញបាន: $f(-n)=f(n)$ ចំពោះគ្រប់ចំនួនគត់ n

ហើយ $f(m+n)=f\left(m-(-n)\right)|f(m)-f(-n)=f(m)-f(n)$ ។

ចំពោះចំនួនគត់ m និង n ដែល $f(m)\le f(n)$ នោះគេបាន:

$f(m+n)\le f(n)-f(m)$

ម្យ៉ាងទៀត $f(n)=f\left((m+n)-m\right)|f(m+n)-f(m)$

នាំឲ្យ $f(n)\le|f(m+n)-f(m)|$ ឬ $f(m+n)-f(m)=0$

*ករណី $f(n)\le|f(m+n)-f(m)|$ នោះ $f(n)\le f(m+n)-f(m)$ ឬ $-f(n)\ge f(m+n)-f(m)$

បើ $f(n)\le f(m+n)-f(m)$

នាំឲ្យ $f(n)+f(m)\le f(m+n)\le f(n)-f(m)$

គេបាន: $2f(m)\le 0$ មិនពិតព្រោះ f មានតម្លៃជាចំនួនគត់វិជ្ជមាន ។

បើ $-f(n)\geq f(m+n)-f(m)$

នាំឲ្យ $f(m)\leq f(n)\leq f(m)-f(m+n)$

គេបាន: $f(m+n)\leq 0$ មិនពិតព្រោះ f មានតម្លៃជាចំនួនគត់វិជ្ជមាន ។

*ករណីចុងក្រោយ $f(m+n)-f(m)=0$ ឬ $f(m+n)=f(m)$

ដោយ $f(m+n)|f(m)-f(n) \Leftrightarrow f(m)|f(m)-f(n)$

នាំឲ្យ $f(m)|f(n)$

ដូចនេះ ចំពោះគ្រប់ចំនួនគត់ m និង n ដែល $f(m)\leq f(n)$ នោះចំនួន $f(n)$ ចែកដាច់នឹង $f(m)$ ។

លំហាត់ទី៧២

គេឲ្យ a និង b ជាចំនួនគត់វិជ្ជមាន ។ ចូរស្រាយបញ្ជាក់ថា បើ $4ab-1$ ជាតួចែកនៃ $(4a^2-1)^2$ នោះ $a=b$ ។

(IMO 2007)

ដំណោះស្រាយ

បើ $a=b$ នោះ $4ab-1|(4a^2-1)^2$ ពិត ។

ឧបមាថា: មានគូ $(a,b)\in\mathbb{N}^2$ ដែល $a\neq b$ ហើយផ្ទៀងផ្ទាត់ $4ab-1|(4a^2-1)^2$

ជ្រើសរើសគូ (a,b) មួយណាដែលធ្វើឲ្យ $2a+b$ មានតម្លៃតូចជាងគេ

គេពិនិត្យ: $4b^2-1=4b^2-(4ab)^2+(4ab)^2-1$

$$=4b^2-(4ab)^2+(4ab-1)(4ab+1)$$

$$\equiv 4b^2(1-4a^2)\ (\operatorname{mod} 4ab-1)$$

$$\Rightarrow\ (4b^2-1)^2\equiv 16b^4(4a^2-1)^2\equiv 0(\operatorname{mod} 4ab-1)$$

$\Rightarrow\ 4ab-1|(4b^2-1)^2$ បញ្ជាក់ថា: (b,a) ក៏ជាគូដែលផ្ទៀងផ្ទាត់លក្ខខណ្ឌ

លំហាត់ដែរ គេបាន: $2b+a>2a+b$ នោះ $b>a$ ។

ម្យ៉ាងទៀត: $4ab-1 \mid (4a^2-1)^2$

$\Rightarrow (4a^2-1)^2=(4ab-1)k$ ដែល $k\in\mathbb{N}$

ដោយ $4ab-1\equiv -1 \pmod{4a} \Rightarrow k(4ab-1)\equiv -k \pmod{4a}$

$\Rightarrow (4a^2-1)^2\equiv -k \pmod{4a}$ តែ $(4a^2-1)^2\equiv 1 \pmod{4a}$

ទាញបាន: $k=4al-1$ ដែល $l\in\mathbb{N}$ ហើយ $(4a^2-1)^2=(4ab-1)k$

នោះ $k=4al-1 \mid (4a^2-1)^2$ បញ្ជាក់ថា: (a,l) ក៏ជាគូដែលផ្ទៀងផ្ទាត់លក្ខខណ្ឌ

លំហាត់ដែរ គេបាន: $2a+l>2a+b \Rightarrow l>b>a \quad (*)$

ដោយ $(4a^2-1)^2=(4ab-1)(4al-1)$

$\Leftrightarrow 16a^4-8a^2+1=16a^2bl-4ab-4al+1$

$\Rightarrow 4a^3-2a=4abl-b-l=(4al-1)b-l>(4al-1)a-l$

$\Leftrightarrow 4a^3-2a>4a^2l-a-l$

$\Leftrightarrow 4a^3-a>4a^2l-l$

$\Leftrightarrow (4a^2-1)a>(4a^2-1)l$

$\Rightarrow a>l$ ផ្ទុយពី $(*)$ នេះបញ្ជាក់ថា: ការឧបមាដែលថាមានគូ $(a,b)\in\mathbb{N}^2$

ដែល $a\neq b$ ហើយផ្ទៀងផ្ទាត់ $4ab-1 \mid (4a^2-1)^2$ មិនពិតទេ ។

ដូចនេះ បើ $4ab-1$ ជាតួចែកនៃ $(4a^2-1)^2$ លុះត្រាតែ $a=b$ ។

លំហាត់ទី៧៣

គេឲ្យ $a_1, a_2, \dots, a_n, \dots$ ជាស្វ៊ីតនៃចំនួនពិតដែលផ្ទៀងផ្ទាត់: $0\le a_n\le 1$ និង $a_n-2a_{n+1}+a_{n+2}\ge 0$ ចំពោះ $n=1,2,3,\dots$ ។ ចូរស្រាយបញ្ជាក់ថា: $0\le (n+1)(a_n-a_{n+1})\le 2$ ចំពោះគ្រប់ $n=1,2,3,\dots$ ។

(IMO Shortlist 1975)

ដំណោះស្រាយ

ចំពោះ $n=1,2,3,\ldots$ យើងមាន: $a_n-2a_{n+1}+a_{n+2}\ge 0$

$\Leftrightarrow\ a_n-a_{n+1}\ge a_{n+1}-a_{n+2}$

តាង $\Delta a_n=a_n-a_{n+1}$ នាំឲ្យ $\Delta a_n\ge \Delta a_{n+1}$ នោះចំពោះ $k\ge n$, $\Delta a_k\le \Delta a_n$

គេបាន: $a_n-a_{n+m}=\left(a_n-a_{n+1}\right)+\left(a_{n+1}-a_{n+2}\right)+\cdots+\left(a_{n+m-1}-a_{n+m}\right)$

$$=\Delta a_n+\Delta a_{n+1}+\cdots+\Delta a_{n+m-1}\le\ m\Delta a_n$$

$\Leftrightarrow\ a_n-a_{n+m-1}\le\ m\Delta a_n$

ឧបមាថា: $\Delta a_n<0$ ចំពោះ n ខ្លះ នោះចំពោះ m មានតម្លៃធំគ្រប់គ្រាន់ គេបាន:

$a_n-a_{n+m}<-1$ មិនពិតព្រោះ $0\le a_n\le 1$, $n=1,2,3,\ldots$ ។

ដូច្នេះការឧបមាខាងលើគឺខុស មានន័យថា: $\Delta a_n\ge 0$ ចំពោះគ្រប់ $n=1,2,3,\ldots$

គេបាន $(n+1)\Delta a_n\ge 0$ ចំពោះគ្រប់ $n=1,2,3,\ldots$ (1)

ម្យ៉ាងទៀត: $0\le a_n\le 1$, $n=1,2,3,\ldots$

នោះ $n\ge a_1+a_2+a_3+\cdots+a_n$

$$=\left(a_1-a_2\right)+2\left(a_2-a_3\right)+\cdots+n\left(a_n-a_{n+1}\right)+na_{n+1}$$

$$=\Delta a_1+2\Delta a_2+\cdots+n\Delta a_n+na_{n+1}$$

$$\ge\Delta a_n+2\Delta a_n+\cdots+n\Delta a_n=\left(1+2+\cdots+n\right)\Delta a_n=\frac{n\left(n+1\right)}{2}\Delta a_n$$

នាំឲ្យ $(n+1)\Delta a_n\le 2$ (2)

តាម (1) និង (2) គេបាន: $0\le(n+1)\Delta a_n\le 2$

ដូចនេះ $0\le(n+1)\left(a_n-a_{n+1}\right)\le 2$ ចំពោះគ្រប់ $n=1,2,3,\ldots$ ។

លំហាត់ទី៧៤

គេឲ្យ $a_1,a_2,\dots,a_n$ ជាចំនួនពិត ។ ចំពោះគ្រប់ i $(1\le i\le n)$ គេកំណត់:

$d_i=\max\{a_i:1\le j\le i\}-\min\{a_j:i\le j\le n\}$ ហើយគេតាង

$d=\max\{d_i:1\le i\le n\}$ ។

ក/ ចូរស្រាយបញ្ជាក់ថា ចំពោះចំនួនពិត $x_1\le x_2\le\cdots\le x_n$ នោះ

$\max\{|x_i-a_i|:1\le i\le n\}\ge\dfrac{d}{2}$ $(*)$ ។

ខ/ ចូរបង្ហាញថា មានចំនួនពិត $x_1\le x_2\le\cdots\le x_n$ ដែលធ្វើឲ្យ $(*)$ ក្លាយជាសមភាព ។

(IMO 2007)

ដំណោះស្រាយ

ក/. ស្រាយបញ្ជាក់ថា ចំពោះចំនួនពិត $x_1\le x_2\le\cdots\le x_n$ នោះ

$\max\{|x_i-a_i|:1\le i\le n\}\ge\dfrac{d}{2}$ $(*)$:

យើងសន្មតថា: $d=\max\{d_i:1\le i\le n\}=d_m$ ចំពោះសន្ទស្សន៍ m , $(1\le m\le n)$ ណាមួយ ហើយតាង k និង l , $(1\le k\le m\le l\le n)$ ជាសន្ទស្សន៍ដែល $d_m=a_k-a_l$ នោះយើងបាន: $d_m=a_k-a_l\le(a_k-x_k)+(x_l-a_l)$

(ព្រោះដោយ $k\le l$ នោះ $x_k\le x_l$ ឬ $x_l-x_k\ge 0$)

យើងទាញបាន: $a_k-x_k\ge\dfrac{d_m}{2}=\dfrac{d}{2}$ ឬ $x_l-a_l\ge\dfrac{d_m}{2}=\dfrac{d}{2}$

ដូចនេះ $\max\{|x_i-a_i|:1\le i\le n\}\ge\dfrac{d}{2}$ ចំពោះចំនួនពិត $x_1\le x_2\le\cdots\le x_n$ ។

ខ/.មានចំនួនពិត $x_1\le x_2\le\cdots\le x_n$ ដែលធ្វើឲ្យ $(*)$ ក្លាយជាសមភាព:

តាង $M_i=\max\{a_j:1\le j\le i\}$ និង $m_i=\min\{a_j:i\le j\le n\}$

យើងយក $x_i=\dfrac{m_i+M_i}{2}$ នោះ $\{x_i\}$ ជាស្វ៊ីតមិនចុះ ព្រោះ $\{m_i\}$ និង $\{M_i\}$ សុទ្ធ

តែជាស្វ៊ីតមិនចុះ ។

ដោយ $m_i \le a_i \le M_i$ និង $d_i = M_i - m_i$ នោះយើងបាន:

$$-\frac{d_i}{2} = \frac{m_i - M_i}{2} = \frac{m_i + M_i - 2M_i}{2} = \frac{m_i + M_i}{2} - M_i = x_i - M_i \le x_i - a_i$$

គឺថា: $x_i - a_i \ge -\frac{d_i}{2} \quad (1)$

ហើយ $\frac{d_i}{2} = \frac{M_i - m_i}{2} = \frac{M_i + m_i - 2m_i}{2} = \frac{M_i + m_i}{2} - m_i = x_i - m_i \ge x_i - a_i$

គឺថា: $x_i - a_i \le \frac{d_i}{2} \quad (2)$

តាម (1) និង (2) យើងបាន: $-\frac{d_i}{2} \le x_i - a_i \le \frac{d_i}{2}$ នោះ $|x_i - a_i| \le \frac{d_i}{2}$

នាំឲ្យ $\max\{|x_i - a_i| : 1 \le i \le n\} \le \max\left\{\frac{d_i}{2} : 1 \le i \le n\right\} = \frac{d}{2}$

តែតាម $(*)$ គេទាញបាន: $\max\{|x_i - a_i| : 1 \le i \le n\} = \frac{d}{2}$ ចំពោះស្វ៊ីត $\{x_i\}$ ។

ដូចនេះ មានចំនួនពិត $x_1 \le x_2 \le \cdots \le x_n$ ដែលធ្វើឲ្យ $(*)$ ក្លាយជាសមភាព ។

__លំហាត់ទី៧៨__

រកគ្រប់ចំនួនបឋម p ដែលផ្ទៀងផ្ទាត់: $2^p + p^2$ ក៏ជាចំនួនបឋមដែរ ។

(Albanian IMO TST 2011)

__ដំណោះស្រាយ__

បើ $p = 2$ នោះ $2^p + p^2 = 8$ មិនមែនជាចំនួនបឋម ។

បើ $p = 3$ នោះ $2^p + p^2 = 17$ ជាចំនួនបឋមនាំឲ្យ $p = 3$ ជាចម្លើយមួយ ។

បើ $p > 3$ (p ត្រូវតែជាចំនួនសេស) នោះ $2^p \equiv (-1)^p \equiv -1 \pmod 3$

ហើយតាមទ្រឹស្តីបទ $Fermat$: $p^2 \equiv 1 \pmod 3$ គេបាន: $2^p + p^2 \equiv 0 \pmod 3$

នោះ $2^p + p^2$ មិនមែនជាចំនួនបឋមទេ ។

ដូចនេះ $p = 3$ ជាចំនួនបឋមតែមួយគត់ដែលធ្វើឲ្យ $2^p + p^2$ ក៏ជាចំនួនបឋមដែរ

លំហាត់ទី៧៦

តាង a, b, c ជាចំនួនពិត ។ ចូរស្រាយបញ្ជាក់ថា:

$$ab + bc + ca + \max\{|a-b|, |b-c|, |c-a|\} \leq 1 + \frac{1}{3}(a+b+c)^2 \quad ។$$

(Baltic Way 2012)

ដំណោះស្រាយ

វិសមភាពដែលត្រូវស្រាយអាចសរសេរជា:

$$\Leftrightarrow \ 4\left((a+b+c)^2 - 3(ab+bc+ca)\right) + 12 \geq 12\max\{|a-b|, |b-c|, |c-a|\}$$

$$\Leftrightarrow \ T = 2\left((a-b)^2 + (b-c)^2 + (c-a)^2\right) + 12 \geq 12\max\{|a-b|, |b-c|, |c-a|\}$$

តាមវិសមភាព $Cauchy - Schwarz$:

$$(a-b)^2 = \left((a-c)+(c-b)\right)^2 \leq 2\left((a-c)^2 + (c-b)^2\right)$$

$$\Leftrightarrow 2\left((a-b)^2 + (b-c)^2 + (c-a)^2\right) \geq 3(a-b)^2$$

ស្រាយដូចគ្នា: $2\left((a-b)^2 + (b-c)^2 + (c-a)^2\right) \geq 3(b-c)^2$

និង $2\left((a-b)^2 + (b-c)^2 + (c-a)^2\right) \geq 3(c-a)^2$

គេបាន: $T \geq 3(a-b)^2 + 12 = 3\left((a-b)^2 + 4\right) \geq 3 \cdot 2\sqrt{(a-b)^2 \cdot 4} = 12|a-b|$

ដូចគ្នាដែរ: $T \geq 12|b-c|$ និង $T \geq 12|c-a|$

យើងទាញបាន: $T \geq 12\max\{|a-b|, |b-c|, |c-a|\}$ ពិត ។

ដូចនេះវិសមភាពត្រូវបានស្រាយបញ្ជាក់ ។

លំហាត់ទី៧៧

ចូរកំណត់គ្រប់ពហុធា $P(x)$ ដែលមានមេគុណជាចំនួនពិតហើយផ្ទៀងផ្ទាត់: $(x+1)P(x-1)-(x-1)P(x)$ ជាពហុធាថេរ ។

(Canadian Mathematical Olympiad 2013)

ដំណោះស្រាយ

តាង $R(x)=(x+1)P(x-1)-(x-1)P(x)$

$\Rightarrow R(-1)=2P(-1)$ និង $R(1)=2P(0)$

ដោយ $R(x)$ ជាពហុធាថេរ នោះ $R(-1)=R(1)$

$\Rightarrow P(-1)=P(0)=k$ ចំពោះចំនួនថេរ k ណាមួយ

ដូច្នេះយើងអាចតាង $P(x)=x(x+1)Q(x)+k$ ដែល $Q(x)$ ជាពហុធាមួយ

នោះ $P(x-1)=x(x-1)Q(x-1)+k$

យើងបាន: $R(x)=(x+1)x(x-1)Q(x-1)+(x+1)k-(x-1)x(x+1)Q(x)$

$-(x-1)k=x(x-1)(x+1)\left(Q(x-1)-Q(x)\right)+2k$

$R(x)$ ជាពហុធាថេរលុះត្រាតែ: $Q(x-1)=Q(x)$ ចំពោះគ្រប់ x

$\Rightarrow Q(x-1)=Q(x)=c$ ចំពោះចំនួនថេរ c ណាមួយ

ដូចនេះ $P(x)=cx(x+1)+k$ ដែល $c\,,k$ ជាចំនួនថេរ ។

លំហាត់ទី៧៨

គេឲ្យ $34!=295232799cd96041408476186096435ab000000$ ។

ចូរកំណត់លេខខ្ទង់ $a\,,b\,,c\,,d$ ។

(British Mathematical Olympiad 2002)

ដំណោះស្រាយ

តាមសម្មតិកម្ម: $34!=295232799cd96041408476186096435ab\times10^{6}$

តែ $34!=1\cdot2\cdot3\cdots33\cdot34=2^{32}\cdot3^{15}\cdot5^{7}\cdot7^{4}\cdot11^{3}\cdot13^{2}\cdot17^{2}\cdot19\cdot23\cdot29\cdot31$

$=19\cdot23\cdot29\cdot31\cdot2^{25}\cdot3^{15}\cdot7^{4}\cdot11^{3}\cdot13^{2}\cdot17^{2}\cdot10^{7}$

នាំឲ្យ $295232799cd96041408476186096435ab$

$=19\cdot23\cdot29\cdot31\cdot2^{25}\cdot3^{15}\cdot7^{4}\cdot11^{3}\cdot13^{2}\cdot17^{2}\cdot10 \qquad (*)$

ទាញបាន: $b=0$ ហើយ $a\neq0$ ។

យើងសង្កេតឃើញថា: អង្គទី២នៃ $(*)$ ចែកដាច់នឹង 4 នោះអង្គទី១នៃ $(*)$ ក៏ត្រូវចែកដាច់នឹង 4 ដែរ នាំឲ្យ $\overline{5a}$ ចែកដាច់នឹង 4 ទាញបាន: $a=2$ ។

ម្យ៉ាងទៀត អង្គទី២នៃ $(*)$ ចែកដាច់នឹង 9 ផង 11ផង នោះអង្គទី១នៃ $(*)$ ក៏ត្រូវមានលក្ខណះ ដូច្នេះដែរ គេបាន:

$$\begin{cases}141+c+d\equiv0(\text{mod}9)\\(80+d)-(61+c)\equiv0(\text{mod}11)\end{cases}$$

$$\Leftrightarrow\begin{cases}c+d\equiv3(\text{mod}9)\\d-c\equiv3(\text{mod}11)\end{cases}$$

ដោយ $0\le c,d\le9$ នោះ $-9\le d-c\le9$ និង $0\le c+d\le18$

$$\Rightarrow\begin{cases}c+d=\{3,12\}\\d-c=\{-8,3\}\end{cases}$$

$*$ករណី $\begin{cases}c+d=3\\d-c=-8\end{cases}\Rightarrow 2d=-5$ មិនពិត

$*$ករណី $\begin{cases}c+d=12\\d-c=3\end{cases}\Rightarrow 2d=15$ មិនពិត

$*$ករណី $\begin{cases}c+d=12\\d-c=-8\end{cases}\Rightarrow 2d=4\Rightarrow d=2\Rightarrow c=10$ មិនពិត ព្រោះ $0\le c\le9$

$*$ករណី $\begin{cases}c+d=3\\d-c=3\end{cases}\Rightarrow 2d=6\Rightarrow d=3\Rightarrow c=0$ យក

ដូចនេះ $a=2,b=0,c=0,d=3$ ។

លំហាត់ទី៧៩

ចូរស្រាយបញ្ជាក់ថាបើ a , b , c ជាចំនួនគត់ធម្មជាតិដែលផ្ទៀងផ្ទាត់:
$a^{2007}+b^{2007}=c^{2007}$ នោះ $\min\{a,b\}\geq 2007$ ។

(Vietnam Olympic 2007)

ដំណោះស្រាយ

ឧបមាថា: $a\leq b$ នោះ $\min\{a,b\}=a$ (1)

តាមសម្មតិកម្ម: $a^{2007}+b^{2007}=c^{2007}$

ទាញបាន: $c>b$ ឬ $c\geq b+1$ (ព្រោះ $b,c\in\mathbb{N}^*$)

$$\Rightarrow c^{2007}\geq (b+1)^{2007}=b^{2007}+2007b^{2006}+\cdots+1$$
$$\geq b^{2007}+2007b^{2006}$$

$$\Rightarrow c^{2007}-b^{2007}\geq 2007b^{2006}\geq 2007a^{2006}$$

$$\Rightarrow a^{2007}\geq 2007a^{2006}$$

$$\Rightarrow a\geq 2007 \quad (2)$$

តាម (1) និង (2) យើងបាន: $\min\{a,b\}\geq 2007$

ដូចនេះ $\min\{a,b\}\geq 2007$ ត្រូវបានស្រាយបញ្ជាក់ ។

លំហាត់ទី៨០

គេឲ្យ a, b, n ជាចំនួនគត់ដែលធំជាង 1 និង $0 \le x_i \le b$ ចំពោះ $i = 0,1,2,\ldots,n$ ហើយ $x_n \ne 0, x_{n-1} \ne 0$ ។ $\overline{x_n x_{n-1} \ldots x_1 x_0}$ តាងឲ្យចំនួន A_n នៅក្នុងប្រព័ន្ធរបាប់គោល a ហើយតាងឲ្យ B_n ក្នុងប្រព័ន្ធរបាប់គោល b , ដូចគ្នាដែរ $\overline{x_{n-1} x_{n-2} \ldots x_1 x_0}$ តាងឲ្យចំនួន A_{n-1} នៅក្នុងប្រព័ន្ធរបាប់គោល a ហើយតាងឲ្យ B_{n-1} ក្នុងប្រព័ន្ធរបាប់ គោល b ។ ចូរស្រាយបញ្ជាក់ថា: $a < b$ លុះត្រាតែ $\frac{A_{n-1}}{A_n} < \frac{B_{n-1}}{B_n}$ ។

(IMO 1970)

ដំណោះស្រាយ

តាមសម្មតិកម្ម: $A_n = \overline{x_n x_{n-1} \ldots x_1 x_0}_{(a)} = x_n a^n + x_{n-1} a^{n-1} + \cdots + x_0$

$$A_{n-1} = \overline{x_{n-1} x_{n-2} \ldots x_1 x_0}_{(a)} = x_{n-1} a^{n-1} + x_{n-2} a^{n-2} + \cdots + x_0$$

$$B_n = \overline{x_n x_{n-1} \ldots x_1 x_0}_{(b)} = x_n b^n + x_{n-1} b^{n-1} + \cdots + x_0$$

$$B_{n-1} = \overline{x_{n-1} x_{n-2} \ldots x_1 x_0}_{(b)} = x_{n-1} b^{n-1} + x_{n-2} a^{n-2} + \cdots + x_0$$

តាង $F(t) = \dfrac{x_n t^n + x_{n-1} t^{n-1} + \cdots + x_1 t + x_0}{x_{n-1} t^{n-1} + x_{n-2} t^{n-2} + \cdots + x_1 t + x_0}$

$$= 1 + \frac{x_n t^n}{x_{n-1} t^{n-1} + x_{n-2} t^{n-2} + \cdots + x_1 t + x_0}$$

$= 1 + \dfrac{x_n}{G(t)}$ ដែល $G(t) = \dfrac{x_{n-1}}{t} + \dfrac{x_{n-2}}{t^2} + \cdots + \dfrac{x_1}{t^{n-1}} + \dfrac{x_0}{t^n}$

ដោយចំពោះ $i = 0,1,2,\ldots,n : x_i \ge 0$ នោះ $G(t)$ ជាអនុគមន៍ចុះ នាំឲ្យ $F(t)$ ជាអនុគមន៍កើន យើងបាន:

- បើ $a > b$ នោះ $F(a) > F(b)$ ឬ $\dfrac{A_n}{A_{n-1}} > \dfrac{B_n}{B_{n-1}}$ ឬ $\dfrac{A_{n-1}}{A_n} < \dfrac{B_{n-1}}{B_n}$

• បើ $\frac{A_{n-1}}{A_n} < \frac{B_{n-1}}{B_n}$ ឬ $\frac{A_n}{A_{n-1}} > \frac{B_n}{B_{n-1}}$ ឬ $F(a) > F(b)$ នោះ $a > b$

ដូចនេះ $a < b$ លុះត្រាតែ $\frac{A_{n-1}}{A_n} < \frac{B_{n-1}}{B_n}$ ។

លំហាត់ទី៨១

គេឲ្យស្វ៊ីត $a_0 , a_1 , a_2 , \dots$ និង $b_0 , b_1 , b_2 ,\dots$ ដែល $a_0 , b_0 > 0$ ហើយ

$a_{n+1} = a_n + \frac{1}{2b_n}$, $b_{n+1} = b_n + \frac{1}{2a_n}$ ។ ចូរស្រាយបញ្ជាក់ថា:

$Max\{a_{2006}, b_{2006}\} > \sqrt{2007}$ ។

(Vietnam Olympic 2007)

ដំណោះស្រាយ

គេមាន: $a_{n+1}b_{n+1} = \left(a_n + \frac{1}{2b_n}\right)\left(b_n + \frac{1}{2a_n}\right)$

$$= a_n b_n + \frac{1}{4a_n b_n} + 1$$

$$= a_{n-1}b_{n-1} + \frac{1}{4a_{n-1}b_{n-1}} + \frac{1}{4a_n b_n} + 2$$

$$= \cdots = a_0 b_0 + \sum_{i=0}^{n} \frac{1}{4a_i b_i} + n + 1$$

នាំឲ្យ $a_{2006} \cdot b_{2006} = a_0 b_0 + \sum_{i=0}^{2005} \frac{1}{4a_i b_i} + 2006$

$$> a_0 b_0 + \frac{1}{4a_0 b_0} + 2006 \geq 2\sqrt{a_0 b_0 \cdot \frac{1}{4a_0 b_0}} + 2006 = 2007 \quad (1)$$

ម្យ៉ាងទៀត: $Max\{a_{2006}, b_{2006}\} \geq a_{2006} > 0$ ហើយ

$Max\{a_{2006}, b_{2006}\} \geq b_{2006} > 0$

នាំឲ្យ $\left(Max\{a_{2006}, b_{2006}\}\right)^2 \geq a_{2006} \cdot b_{2006} \quad (2)$

តាម (1) និង (2) ទាញបាន: $\left(Max\{a_{2006},b_{2006}\}\right)^2 > 2007$

ដូចនេះ $Max\{a_{2006},b_{2006}\} > \sqrt{2007}$ ត្រូវបានស្រាយបញ្ជាក់ ។

លំហាត់ទី៨២

គេឲ្យ $f(x) = x^n + 5x^{n-1} + 3$ ដែល $n > 1$ ជាចំនួនគត់ ។

ចូរស្រាយបញ្ជាក់ថា: $f(x)$ មិនអាចសរសេរជាផលគុណនៃពីរពហុធាដែលមាន មេគុណជាចំនួនគត់ហើយមានដឺក្រេយ៉ាងតិចបំផុត 1 នោះឡើយ ។

(IMO1993)

ដំណោះស្រាយ

ឧបមាថា $f(x) = g(x)\cdot h(x)$ ដែល $g(x)$ និង $h(x)$ ជាពហុធាមិនថេរមាន មេគុណជាចំនួនគត់ ។

គេមាន: $f(x) = x^n + 5x^{n-1} + 3 \Rightarrow f(0) = g(0)\cdot h(0) = 3$

នោះច្បាស់ជាមានមួយក្នុងចំណោម $g(0)$ និង $h(0)$ ស្មើនឹង ± 1 , សន្មតថា: $g(0) = \pm 1$ ដូច្នេះ យើងអាចតាង $g(x) = x^r + \cdots \pm 1$

ដោយ $f(\pm 1) \neq 0$ នោះ $r > 1$ ឬ $r \geq 2$ ។

តាង $z_1, z_2, \ldots, z_r$ ជាឫសនៃពហុធា $g(x)$ នោះ

$g(x) = (x - z_1)(x - z_2)\cdots(x - z_r)$ គេបាន: $g(0) = (-z_1)(-z_2)\cdots(z_r)$

$\Rightarrow |z_1 z_2 \cdots z_r| = |g(0)| = 1$ ឬ $\left|\prod_{i=1}^{r} z_i^{n-1}\right| = 1$

ហើយ: $|g(-5)| = |(-5 - z_1)(-5 - z_2)\cdots(-5 - z_r)|$

$= |(5 + z_1)(5 + z_2)\cdots(5 + z_r)| = \left|\prod_{i=0}^{r}(5 + z_i)\right| = \left|\prod_{i=0}^{r}(5 + z_i)\right|\left|\prod_{i=0}^{r} z_i^{n-1}\right|$

$= \left|\prod_{i=0}^{r} z_i^{n-1}(5 + z_i)\right|$

ដោយ z_i ជាឫសនៃ $g(x)$ នោះពួកវាក៏ជាឫសនៃ $f(x)$ គេបាន: $f(z_i)=0$

$\Rightarrow z_i^n+5z_i^{n-1}+3=0 \Leftrightarrow z_i^{n-1}(z_i+5)=-3$

$$\Rightarrow |g(-5)|=\left|\prod_{i=0}^{r}(-3)\right|=3^r\geq 3^2=9 \quad (*)$$

តែ $g(-5)\cdot h(-5)=f(-5)=3 \Rightarrow |g(-5)|\leq 3$ ផ្ទុយពី $(*)$ នេះបញ្ជាក់ថា: ការឧបមាខាងលើគឺខុស ។

ដូចនេះ: $f(x)$ មិនអាចសរសេរជាផលគុណនៃពីរពហុធាដែលមានមេគុណជាចំនួនគត់ហើយមាដឺក្រេយ៉ាងតិចបំផុត 1 នោះឡើយ ។

លំហាត់ទី៨៣

ចំពោះពីរចំនួនគត់វិជ្ជមាន a និង b ដែលជាចំនួនបឋមរវាងគ្នា , ចូររកគ្រប់តួចែករួមធំបំផុតដែលអាចនៃ $a+b$ និង $\frac{a^{2005}+b^{2005}}{a+b}$ ។

(Korean Mathematical Olympiad 2005)

ដំណោះស្រាយ

តាង $d=\gcd\left(a+b,\frac{a^{2005}+b^{2005}}{a+b}\right)$

$\Rightarrow a+b\equiv 0(\bmod d) \Leftrightarrow a\equiv -b(\bmod d)$

ហើយ $\frac{a^{2005}+b^{2005}}{a+b}\equiv 0(\bmod d)$

$\Leftrightarrow a^{2004}-a^{2003}b+a^{2002}b^2-\cdots+b^{2004}\equiv 0(\bmod d)$

$\Rightarrow b^{2004}+b^{2004}+\cdots+b^{2004}\equiv 0(\bmod d)$

$\Rightarrow 2005b^{2004}\equiv 0(\bmod d) \quad \Rightarrow 2005a^{2004}\equiv 0(\bmod d)$

$\Rightarrow d\,|\,2005a^{2004} \wedge d\,|\,2005b^{2004}$

$$\Rightarrow\ d \mid \gcd\left(2005a^{2004}, 2005b^{2004}\right) = 2005\gcd\left(a^{2004}, b^{2004}\right)$$

$$= 2005\left(\gcd(a,b)\right)^{2004} = 2005 \quad (\text{ព្រោះ } \gcd(a,b)=1)$$

$$\Rightarrow d \in \{1, 5, 401, 2005\}$$

ដូចនេះតម្លៃដែលអាចនៃ d គឺ $d \in \{1, 5, 401, 2005\}$ ។

លំហាត់ទី៨៤

ចូរស្រាយបញ្ជាក់ថា: ចំពោះគ្រប់ $n \in \mathbb{N}$ នោះគេបាន:

$$\sum_{k=0}^{n} \frac{(2n)!}{(k!)^2\left((n-k)!\right)^2} = \left(C_{2n}^{n}\right)^2 \quad ។$$

(IMO Longlist 1982)

ដំណោះស្រាយ

យើងពិនិត្យពហុធា: $(1+x)^{2n} = (1+x)^n \cdot (1+x)^n$

តាមការពន្លាតទ្វេធា *Newton* យើងបាន:

$$C_{2n}^{0} + C_{2n}^{1} \cdot x + C_{2n}^{2} \cdot x^2 + \cdots + C_{2n}^{2n} \cdot x^{2n}$$

$$= \left(C_n^0 + C_n^1 \cdot x + \cdots + C_n^n \cdot x^n\right)\left(C_n^0 + C_n^1 \cdot x + \cdots + C_n^n \cdot x^n\right)$$

តាមរបៀបប្រដូចលេខមេគុណនៃ x^n នោះយើងបាន:

$$C_{2n}^{n} = C_n^0 \cdot C_n^n + C_n^1 \cdot C_n^{n-1} + \cdots + C_n^{n-1} \cdot C_n^1 + C_n^n \cdot C_n^0$$

$$\Leftrightarrow C_{2n}^{n} = \left(C_n^0\right)^2 + \left(C_n^1\right)^2 + \cdots + \left(C_n^n\right)^2$$

$$(\text{ព្រោះ } C_n^k = C_n^{n-k} \quad ; \quad k = 0, 1, 2, \ldots, n)$$

ដូច្នេះ

$$\sum_{k=0}^{n} \frac{(2n)!}{(k!)^2\left((n-k)!\right)^2} = \sum_{k=0}^{n} \frac{(n!)^2 (2n)!}{(n!)^2 (k!)^2\left((n-k)!\right)^2}$$

$$= \frac{(2n)!}{n!n!} \sum_{k=0}^{n} \frac{(n!)^2}{(k!)^2\left((n-k)!\right)^2}$$

$$= C_{2n}^{n} \cdot \sum_{k=0}^{n} \left(C_n^k\right)^2$$

$$= C_{2n}^{n} \left(\left(C_n^0\right)^2 + \left(C_n^1\right)^2 + \cdots + \left(C_n^n\right)^2 \right)$$

$$= C_{2n}^{n} \cdot C_{2n}^{n} = \left(C_{2n}^{n}\right)^2$$

ដូចនេះ $\sum_{k=0}^{n} \frac{(2n)!}{(k!)^2 ((n-k)!)^2} = \left(C_{2n}^{n}\right)^2$ ត្រូវបានស្រាយបញ្ជាក់ ។

លំហាត់ទី៨៥

តាង $a\,,b\,,c\,,d\,,e\,,\,f$ ជាចំនួនពិតដែលផ្ទៀងផ្ទាត់:
$P(x) = x^8 - 4x^7 + 7x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ អាចដាក់ជាផលគុណ 8 កត្តាលីនេអ៊ែ $x - x_i$ ដែល $x_i > 0$ ចំពោះ $i = 0,1,2,\ldots,8$ ។
ចូរកំណត់គ្រប់តម្លៃដែលអាចនៃ f ។

(APMO 2003)

ដំណោះស្រាយ

ដោយពហុធា $P(x) = x^8 - 4x^7 + 7x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ អាចដាក់ជាផលគុណ 8 កត្តាលីនេអ៊ែ $x - x_i\,,\ i = 0,1,2,\ldots,8$

មានន័យថា: $x_i\,,\ i = 0,1,2,\ldots,8$ ជាឫសទាំង 8 នៃពហុធា $P(x)$

នោះតាមទ្រឹស្តីបទវ្យែតគេបាន: $\sum_{i=1}^{8} x_i = 4$ និង $\sum_{1 \le i < j \le 8} x_i x_j = 7$

នោះ: $$\sum_{i=1}^{8} x_i^2 = \left(\sum_{i=1}^{8} x_i \right)^2 - 2 \sum_{1 \le i < j \le 8} x_i x_j = 2$$

តែតាមវិសមភាព $Cauchy - Schwarz$: $$\left(\sum_{i=1}^{8} x_i \right)^2 \le 8 \left(\sum_{i=1}^{8} x_i^2 \right) = 16$$

$\Rightarrow \sum_{i=1}^{8} x_i \leq 4$ តែ $\sum_{i=1}^{8} x_i = 4$

នោះសមភាពកើតមានកាលណា: $x_1 = x_2 = \cdots = x_8 = \frac{1}{2}$

ដូចនេះ $f = \prod_{i=1}^{8} x_i = \left(\frac{1}{2}\right)^8 = \frac{1}{256}$ ។

លំហាត់ទី៨៦

ចូរបង្ហាញថា:ចំពោះគ្រប់ចំនួនគត់វិជ្ជមាន n , $\sum_{k=0}^{n} C_{2n+1}^{2k+1} \cdot 8^k$ ចែកមិនដាច់នឹង 5 នោះឡើយ ។ (IMO 1974)

ដំណោះស្រាយ

យើងពិនិត្យ: $(1+x)^{2n+1} = \sum_{k=0}^{2n+1} C_{2n+1}^{k} \cdot x^k = \sum_{k=0}^{n} C_{2n+1}^{2k} \cdot x^{2k} + \sum_{k=0}^{n} C_{2n+1}^{2k+1} \cdot x^{2k+1}$

$$= \sum_{k=0}^{n} C_{2n+1}^{2k} \cdot x^{2k} + x \sum_{k=0}^{n} C_{2n+1}^{2k+1} \cdot x^{2k}$$

លំនាំដូចគ្នាដែរ: $(1-x)^{2n+1} = \sum_{k=0}^{n} C_{2n+1}^{2k} \cdot x^{2k} - x \sum_{k=0}^{n} C_{2n+1}^{2k+1} \cdot x^{2k}$

យក $x = \sqrt{8}$ និង តាង $m = \sum_{k=0}^{n} C_{2n+1}^{2k} \cdot 8^k$, $n = \sum_{k=0}^{n} C_{2n+1}^{2k+1} \cdot 8^k$ នោះយើងបាន:

$$\left(1+\sqrt{8}\right)^{2n+1} = m + n\sqrt{8} \quad \text{និង} \quad \left(1-\sqrt{8}\right)^{2n+1} = m - n\sqrt{8}$$

$\Rightarrow \left(1+\sqrt{8}\right)^{2n+1}\left(1-\sqrt{8}\right)^{2n+1} = \left(m+n\sqrt{8}\right)\left(m-n\sqrt{8}\right)$

$\Leftrightarrow (-7)^{2n+1} = m^2 - 8n^2$

$\Leftrightarrow m^2 = 8n^2 - 7^{2n+1}$

បើ n ចែកដាច់នឹង 5 នោះ $m^2 \equiv -7^{2n+1} (\text{mod} 5)$

តែ $-7^{2n+1} = -7 \cdot 49^n \equiv 2(-1)^{n+1} \equiv \pm 2 \ (\text{mod} 5)$ មិនពិតព្រោះ

$m^2 \equiv \{0,1,4\} \pmod 5$ ចំពោះគ្រប់ $m \in \mathbb{N}$ ។

ដូចនេះ $n = \sum_{k=0}^{n} C_{2n+1}^{2k+1} \cdot 8^k$ ចែកមិនដាច់នឹង 5 ។

លំហាត់ទី៨៧

គេឲ្យ a និង b ជាពីរចំនួនគត់វិជ្ជមាន ។ ចូរស្រាយបញ្ជាក់ថា:

បើ $lcm(a,a+5) = lcm(b,b+5)$ នោះ $a = b$ ។

ដំណោះស្រាយ

គេមាន: $\gcd(a,a+5) = \gcd(a,5) \in \{1,5\}$ ហើយ

$\gcd(b,b+5) = \gcd(b,5) \in \{1,5\}$

តាមសម្មតិកម្ម: $lcm(a,a+5) = lcm(b,b+5)$

$$\Leftrightarrow \quad \frac{a(a+5)}{\gcd(a,a+5)} = \frac{b(b+5)}{\gcd(b,b+5)} \qquad (*)$$

• បើ $\gcd(a,a+5) = 1$ និង $\gcd(b,b+5) = 5$

នោះ $(*)$: $b(b+5) = 5a(a+5)$

ដោយ $\gcd(b,b+5) = \gcd(b,5) = 5$ នោះ $5 | b$

នាំឲ្យ $25 | b(b+5) \Rightarrow 25 | 5a(a+5) \Rightarrow 5 | a(a+5) = a^2 + 5a$

$\Rightarrow 5 | a^2 \Rightarrow 5 | a$ (ព្រោះ 5 ជាចំនួនបឋម)

$\Rightarrow \gcd(a,a+5) = \gcd(a,5) = 5$ មិនពិតទេព្រោះយើងកំពុងសិក្សាករណី $\gcd(a,a+5) = 1$ ។

ស្រាយដូចគ្នាដែរចំពោះករណី $\gcd(a,a+5) = 5$ និង $\gcd(b,b+5) = 1$ ក៏មិនពិតដែរ ។

• បើ $\gcd(a,a+5) = \gcd(b,b+5) = 1$ ឬ $\gcd(a,a+5) = \gcd(b,b+5) = 5$

នោះ $(*)$: $b(b+5)=a(a+5)\Leftrightarrow a^2-b^2+5(a-b)=0$

$\Leftrightarrow (a-b)(a+b+5)=0$

$\Rightarrow\ a=b$

ដូចនេះ បើ $lcm(a,a+5)=lcm(b,b+5)$ នោះ $a=b$ ។

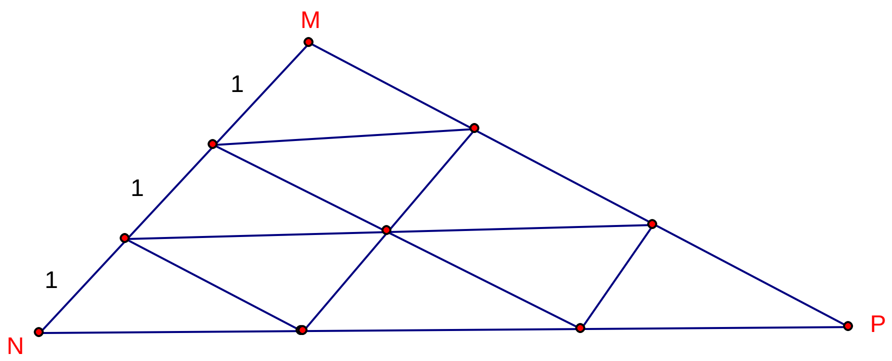
លំហាត់ទី៨៨

ចូរស្រាយបញ្ជាក់ថា: គឺមានត្រីកោណដែលអាចឲ្យគេកាត់ជា 2005 ត្រីកោណតូចប៉ុនៗគ្នា ។ (APMO 2005)

ដំណោះស្រាយ

យើងត្រូវដឹងថា: បើត្រីកោណមួយមានជ្រុងមួយប្រវែង n នោះគេច្បស់ជាអាចកាត់វាជា n^2 ត្រីកោណតូចប៉ុនៗគ្នា ដែលដូចទៅនឹងត្រីកោណដើមនោះ ហើយជ្រុងនីមួយៗដែលត្រូវនឹងជ្រុងប្រវែង n គឺ មានប្រវែងស្មើ 1 ។

ជាឧទាហរណ៍: គេអាចកាត់ត្រីកោណដែលមានជ្រុងមួយប្រវែង 3 ជាត្រីកោណប៉ុនៗគ្នាចំនួន $3^2=9$ ហើយជ្រុងនីមួយៗដែលត្រូវនឹងជ្រុងប្រវែង 3 គឺមានប្រវែងស្មើ 1 ។



ពេលនេះយើងត្រូវវិភាគលើលំហាត់របស់យើងវិញ , កត់សម្គាល់ឃើញថាៈ

$$2005 = 5\times 401 = \left(2^2+1\right)\left(20^2+1\right) = 40^2+1+20^2+2^2$$
$$= (40-1)^2 + 2\cdot 20 + 20^2 + 2^2$$
$$= 39^2 + 22^2$$

តាង ABC ជាត្រីកោណកែងមួយដែលមានជ្រុងជាប់មុំកែង AB និង BC មានប្រវែង 39 និង 22 រៀងគ្នា ។

យើងគូសកម្ពស់ BK ដែលចែកត្រីកោណ ABC ជាពីរត្រីកោណដូចគ្នាគឺ ΔBCK និង ΔABK ។

ដោយ ΔABK មានជ្រុង $AB=39$ នោះគេអាចកាត់វាជាត្រីកោណប៉ុនៗគ្នាចំនួន 39^2 ហើយ ចំពោះ ΔBCK ដែលមាន $BC=22$ គេអាចកាត់វាជាត្រីកោណប៉ុនៗ គ្នាចំនួន 22^2 ។

ម្យ៉ាងទៀតៈ ដោយ ΔABK និង ΔBCK ជាត្រីកោណដូចគ្នានោះយើងទាញបានៈ ត្រីកោណទាំង $39^2+22^2=2005$ សុទ្ធតែប៉ុនៗគ្នា ។

ដូចនេះ គឺមានត្រីកោណដែលអាចឲ្យគេកាត់ជា 2005 ត្រីកោណតូចប៉ុនៗគ្នា ។

លំហាត់ទី៨៩

តាង r និង s ជាចំនួនពិតវិជ្ជមានដែលផ្ទៀងផ្ទាត់ៈ

$(r+s-rs)(r+s+rs)=rs$, ចូររកតម្លៃអប្បបរមានៃ $r+s-rs$ និង $r+s+rs$

(The Philippines Mathematical Olympiad 2013)

ដំណោះស្រាយ

សម្មតិកម្មៈ $(r+s-rs)(r+s+rs)=rs$

$\Leftrightarrow\ rs = (r+s)^2 - (rs)^2 \geq 4rs - (rs)^2$

$\Leftrightarrow (rs)^2 - 3rs \geq 0$

$\Rightarrow\ rs \geq 3$

ម្យ៉ាងទៀត: $rs = (r+s)^2 - (rs)^2 \Rightarrow (r+s)^2 = (rs)^2 + rs$

$\Rightarrow\ r+s+rs = \sqrt{(rs)^2+rs} + rs \geq \sqrt{3^2+3} + 3 = 2\sqrt{3}+3$ និង

$$r+s-rs = \sqrt{(rs)^2+rs} - rs = \frac{1}{\sqrt{1+\frac{1}{rs}}+1} \geq \frac{1}{\sqrt{1+\frac{1}{3}}+1} = 2\sqrt{3}-3$$

ដូចនេះ $\min(r+s+rs) = 2\sqrt{3}+3$ និង $\min(r+s-rs) = 2\sqrt{3}-3$ ។

លំហាត់ទី៩០

ចំពោះចំនួនគត់វិជ្ជមាន k មួយ , គេតាង $f_1(k)$ ជាការេនៃផលបូកលេខខ្ទង់នៃ k ហើយគេកំណត់ $f_{n+1} = f_1 \circ f_n$ ។ ចូរគណនា $f_{2007}\left(2^{2006}\right)$ ។

(Hong Kong Mathematical Olympiad 2006)

ដំណោះស្រាយ

តាង $n(x)$ ជាចំនួនលេខខ្ទង់នៃ x ហើយ $s(x)$ ជាផលបូកលេខខ្ទង់នៃនៃ x

នោះគេបាន: $x \geq 10^{n(x)-1} \ \Rightarrow\ n(x) \leq 1+\log x$

ហើយ $s(x) \leq 9\cdot n(x) \leq 9(1+\log x)$

យើងបាន: $f_1(x) \leq 81(1+\log x)^2$

$$\Rightarrow f_1\left(2^{2006}\right) \leq 81(1+2006\log 2)^2 < 81\left(1+\frac{2006}{3}\right)^2 < 81\cdot 700^2 < 10^8$$

$$\Rightarrow f_2\left(2^{2006}\right) = f_1\left(f_1\left(2^{2006}\right)\right) \leq 81\left(1+\log f_1\left(2^{2006}\right)\right)^2 \leq 81(1+8)^2 < 10^4$$

$$\Rightarrow f_3\left(2^{2006}\right) = f_1\left(f_2\left(2^{2006}\right)\right) \leq 81\left(1+\log f_2\left(2^{2006}\right)\right)^2 < 81(1+4)^2 = 2025$$

$$\Rightarrow f_4\left(2^{2006}\right) \leq (1+9+9+9)^2 = 784$$

$\Rightarrow\ f_5\left(2^{2006}\right)\le(6+9+9)^2=576$

$\Rightarrow\ f_6\left(2^{2006}\right)\le(4+9+9)^2=484$

$\Rightarrow\ f_7\left(2^{2006}\right)\le(3+9+9)^2=441$

$\Rightarrow\ f_8\left(2^{2006}\right)\le(3+9+9)^2=441,\ldots$

យើងទាញបាន: ចំពោះ $n\ge7$ នោះ $f_n\left(2^{2006}\right)\le441$

$\Rightarrow\ s\left(f_n\left(2^{2006}\right)\right)\le3+9+9=21$

ដោយ $2^{2006}\equiv4\ (\text{mod}9)\ \Rightarrow\ f_n\left(2^{2006}\right)\ \equiv\{4,7\}\ (\text{mod}9)$

$\Rightarrow s\left(f_n\left(2^{2006}\right)\right)\equiv\{4,7\}\ (\text{mod}9)$

យើងបាន: $s\left(f_n\left(2^{2006}\right)\right)\in\{4,7,13,16\}$ ចំពោះ $n\ge7$ ។

* បើ $s\left(f_n\left(2^{2006}\right)\right)=4$

នោះ $f_{n+1}\left(2^{2006}\right)=4^2=16$

$f_{n+2}\left(2^{2006}\right)=(1+6)^2=49$

$f_{n+3}\left(2^{2006}\right)=(4+9)^2=169$

$f_{n+4}\left(2^{2006}\right)=(1+6+9)^2=256$

$f_{n+5}\left(2^{2006}\right)=(2+5+6)^2=169,\ldots$

$\Rightarrow\ f_n\left(2^{2006}\right)\in\{169,256\}$ ចំពោះ $n\ge10$ ។

* បើ $s\left(f_n\left(2^{2006}\right)\right)=7$, ស្រាយដូចគ្នាដែរ នោះ $f_n\left(2^{2006}\right)\in\{169,256\}$

ចំពោះ $n\ge9$ ។

* បើ $s\left(f_n\left(2^{2006}\right)\right)=13$ ឬ $s\left(f_n\left(2^{2006}\right)\right)=16$ នោះ $f_n\left(2^{2006}\right)\in\{169,256\}$

ចំពោះ $n\ge7$ ។

សរុបមក: បើ $s\left(f_n\left(2^{2006}\right)\right)\in\{4,7,13,16\}$ ចំពោះ $n\ge7$ នោះ

$f_n\left(2^{2006}\right)\in\{169,256\}$ ចំពោះ $n\ge10$ ។

យក $n=2007$ នោះ $f_{2007}\left(2^{2006}\right)\in\{169,256\}$ $(*)$

ម្យ៉ាងទៀតៈ ដោយ $2^{2006}\equiv 4\ (\text{mod}9)$

នោះ

$$f_1\left(2^{2006}\right)\equiv 7\ (\text{mod}9)$$

$$f_2\left(2^{2006}\right)\equiv 4\ (\text{mod}9)$$

$$f_3\left(2^{2006}\right)\equiv 7\ (\text{mod}9)$$

$$f_4\left(2^{2006}\right)\equiv 4\ (\text{mod}9)\ ,\ldots$$

យើងអាចសន្និដ្ឋានបានថាៈ ចំពោះ $p\in\mathbb{N}^*$ នោះ $f_{2p}\left(2^{2006}\right)\equiv 4\,(\text{mod}9)$

ហើយ $f_{2p+1}\left(2^{2006}\right)\equiv 7\,(\text{mod}9)$

នេះបញ្ជាក់ថា $f_{2007}\left(2^{2006}\right)\equiv 7\,(\text{mod}9)$ $(**)$

តាម $(*)$ និង $(**)$ គេបានៈ $f_{2007}\left(2^{2006}\right)=169$

ដូចនេះ $f_{2007}\left(2^{2006}\right)=169$ ។

លំហាត់ទី៩១

គេឲ្យ ABC ជាត្រីកោណមួយដែលមានកាំរង្វង់ចារឹកក្រៅ R , បរិមាត្រ P និង ផ្ទៃក្រឡា K ។ ចូរកំណត់តម្លៃអតិបរមានៃ $\dfrac{KP}{R^3}$ ។

(Canadian Mathematical Olympiad 2005)

ដំណោះស្រាយ

តាមទ្រឹស្តីបទស៊ីនុសអនុវត្តក្នុងត្រីកោណ ABC : $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=2R$

នោះ $a=2R\sin A$, $b=2R\sin B$ និង $c=2R\sin C$

យើងបានៈ $K=\dfrac{1}{2}ab\sin C=\dfrac{1}{2}\cdot 2R\sin A\cdot 2R\sin B\cdot sibC$

$$=2R^2\cdot\sin A\cdot\sin B\cdot\sin C$$

ហើយ $P=a+b+c=2R(\sin A+\sin B+\sin C)$

នាំឲ្យ $\dfrac{KP}{R^3}=\dfrac{4R^3\sin A\cdot\sin B\cdot\sin C(\sin A+\sin B+\sin C)}{R^3}$

$$=4\sin A\cdot\sin B\cdot\sin C(\sin A+\sin B+\sin C)$$

តាមវិសមភាព *Cauchy* គេបាន:

$$\sin A\cdot\sin B\cdot\sin C\le\left(\frac{\sin A+\sin B+\sin C}{3}\right)^3$$

យើងបាន: $\dfrac{KP}{R^3}\le\dfrac{4}{27}(\sin A+\sin B+\sin C)^4$

ដោយ $\sin X$ ជាអនុគមន៍ផតចំពោះ $X\in(0,\pi)$ នោះតាមវិសមភាព *Jensen*

គេបាន: $\dfrac{\sin A+\sin B+\sin C}{3}\le\sin\left(\dfrac{A+B+C}{3}\right)=\sin\dfrac{\pi}{3}=\dfrac{\sqrt{3}}{2}$

$$\Rightarrow\ \sin A+\sin B+\sin C\ \le\ \frac{3\sqrt{3}}{2}$$

ទាញបាន: $\dfrac{KP}{R^3}\le\dfrac{4}{27}\left(\dfrac{3\sqrt{3}}{2}\right)^4=\dfrac{27}{4}$

សមភាពកើតមានក្នុងករណី: $\sin A=\sin B=\sin C\ \Rightarrow\ A=B=C$

ដូចនេះ តម្លៃអតិបរមានៃ $\dfrac{KP}{R^3}$ គឺ $\dfrac{27}{4}$ ។

លំហាត់ទី៩២

គេឲ្យ a,b,c ជាចំនួនគត់ ហើយគេសន្មតសមីការ: $x^{2003}+ax^2+bx+c=0$ មានឫសជាចំនួនគត់ x_1,x_2,x_3 ។ ចូរស្រាយបញ្ជាក់ថា:

$$2003\,|\,(a^{2003}+b^{2003}+c^{2003}-1)(x_1-x_2)(x_2-x_3)(x_3-x_1)$$ ។

(Vietnam Olympic 2003)

ដំណោះស្រាយ

x_1, x_2, x_3 ជាឫសនៃសមីការ: $x^{2003}+ax^2+bx+c=0$ នោះគេបាន:

$x_i^{2003}+ax_i^2+bx_i+c=0$ ចំពោះ $i=1,2,3$

$\Leftrightarrow x_i^{2003}-x_i+\left(ax_i^2+(b-1)x_i+c\right)=0$

តាង $f(x)=ax^2+(b-1)x+c$ នោះ $x_i^{2003}-x_i+f(x_i)=0$

តែតាមទ្រឹស្តីបទ $Fermat$: $x_i^{2003}\equiv x_i\pmod{2003} \Leftrightarrow x_i^{2003}-x_i\equiv 0$

$\pmod{2003} \Rightarrow f(x_i)\equiv 0\pmod{2003}$ ចំពោះ $i=1,2,3$

បើ $(x_1-x_2)(x_2-x_3)(x_3-x_1)\equiv 0\pmod{2003}$ នោះសំណើពិត ។

បើ $(x_1-x_2)(x_2-x_3)(x_3-x_1)\not\equiv 0\pmod{2003}$ នោះ

$x_1-x_2\not\equiv 0\pmod{2003}$, $x_2-x_3\not\equiv 0\pmod{2003}$ និង

$x_3-x_1\not\equiv 0\pmod{2003}$ (ព្រោះ 2003 ជាចំនួនបឋម)

យើងមាន: $f(x_1)\equiv 0\pmod{2003}$ និង $f(x_2)\equiv 0\pmod{2003}$

$\Rightarrow f(x_1)-f(x_2)\equiv 0\pmod{2003}$

$\Leftrightarrow (x_1-x_2)\left(a(x_1+x_2)+b-1\right)\equiv 0\pmod{2003}$

$\Rightarrow a(x_1+x_2)+b-1\equiv 0\pmod{2003}$

ដូចគ្នា: $a(x_2+x_3)+b-1\equiv 0\pmod{2003}$

ដកអង្គនិងអង្គ: $a(x_3-x_1)\equiv 0\pmod{2003}$

$\Rightarrow a\equiv 0\pmod{2003} \Rightarrow b-1\equiv 0\pmod{2003}$

ដោយ $f(x_i)=ax_i^2+(b-1)x_i+c\equiv 0\pmod{2003}$

$\Rightarrow c\equiv 0\pmod{2003} \Rightarrow a+b+c-1\equiv 0\pmod{2003}$

តាមទ្រឹស្តីបទ $Fermat$: $a^{2003}\equiv a\pmod{2003}$, $b^{2003}\equiv b\pmod{2003}$ និង

$c^{2003}\equiv c\pmod{2003}$

$\Rightarrow a^{2003}+b^{2003}+c^{2003}-1\equiv a+b+c-1\equiv 0\pmod{2003}$

$$\Rightarrow (x_1-x_2)(x_2-x_3)(x_3-x_1)\left(a^{2003}+b^{2003}+c^{2003}-1\right)\equiv 0 \pmod{2003}$$

សរុបមកៈ $2003\left|\left(a^{2003}+b^{2003}+c^{2003}-1\right)(x_1-x_2)(x_2-x_3)(x_3-x_1)\right.$ ។

លំហាត់ទី៩៣

គេឲ្យចំនួនពិតវិជ្ជមាន a , b , c និង x , y , z ផ្ទៀងផ្ទាត់ៈ

$a+x=b+y=c+z=1$ ។ ចូរស្រាយបញ្ជាក់ថាៈ

$$(abc+xyz)\left(\frac{1}{ay}+\frac{1}{bz}+\frac{1}{cx}\right)\geq 3$$

(Russian Mathematical Olympiad 2002)

ដំណោះស្រាយ

យើងមានៈ $a+x=b+y=c+z=1$ នោះ $x=1-a$, $y=1-b$ និង $z=1-c$

យើងបានៈ $abc+xyz=abc+(1-a)(1-b)(1-c)$

$$=abc-a(1-b)(1-c)+(1-b)(1-c)$$

$$=abc-a(1-c-b+bc)+(1-b)(1-c)$$

$$=ac-(1-b)a+(1-b)(1-c)$$

ចែកអង្គទាំងពីរនៃសមភាពនេះនឹង $ay=a(1-b)$ នោះ

$$\frac{abc+xyz}{ay}=\frac{c}{1-b}+\frac{1-c}{a}-1$$

ស្រាយលំនាំដូចគ្នាដែរៈ $$\frac{abc+xyz}{ay}=\frac{a}{1-c}+\frac{1-a}{b}-1$$

$$\frac{abc+xyz}{ay}=\frac{b}{1-a}+\frac{1-b}{c}-1$$

បូកអង្គនិងអង្គគេបានៈ

$$(abc+xyz)\left(\frac{1}{ay}+\frac{1}{bz}+\frac{1}{cx}\right)$$

$$=\left(\frac{c}{1-b}+\frac{1-b}{c}\right)+\left(\frac{1-c}{a}+\frac{a}{1-c}\right)+\left(\frac{1-a}{b}+\frac{b}{1-a}\right)-3$$

$$\geq 2+2+2-3=3$$ (តាមវិសមភាព $AM-GM$)

ដូចនេះ $(abc+xyz)\left(\frac{1}{ay}+\frac{1}{bz}+\frac{1}{cx}\right)\geq 3$ ត្រូវបានស្រាយបញ្ជាក់ ។

លំហាត់ទី៩៨

ចូររកគ្រប់គូមានលំដាប់ (a,b) នៃចំនួនគត់វិជ្ជមានដែលធ្វើឲ្យ: $\frac{a^3b-1}{a+1}$ និង $\frac{b^3a+1}{b-1}$ សុទ្ធតែជាចំនួនគត់វិជ្ជមាន ។

(Junior Balkan MO 2013)

ដំណោះស្រាយ

តាមសម្មតិកម្ម: $\frac{a^3b-1}{a+1}$ ជាចំនួនគត់វិជ្ជមាននោះ

$$a+1 \mid a^3b-1=b\left(a^3+1\right)-(b+1) \Rightarrow a+1 \mid b+1$$

ស្រាយដូចគ្នាដែរចំពោះ: $\frac{b^3a+1}{b-1}$ ជាចំនួនគត់វិជ្ជមាន នោះគេបាន: $b-1 \mid b+1$

$$\Rightarrow b-1 \mid (b+1)-(b-1)=2 \Rightarrow b-1\in\{1,2\} \Rightarrow b\in\{2,3\}$$

បើ $b=2 \Rightarrow a+1 \mid 3 \Rightarrow a=2$

បើ $b=3 \Rightarrow a+1 \mid 4 \Rightarrow a\in\{1,3\}$

សរុបចម្លើយមាន: $(a=2,b=2)$; $(a=1,b=3)\wedge(a=3,b=3)$

យើងយកគូចម្លើយទាំងនេះទៅជំនួសក្នុងសម្មតិកម្ម , ឃើញថាពិតជាផ្ទៀងផ្ទាត់

ដូចនេះ $(a,b)\in\{(2,2),(1,3),(3,3)\}$ ។

លំហាត់ទី៩៥

គេឲ្យ (u_n) ជាស្វ៊ីតដែលកំណត់ដោយ $u_{n+2}=u_{n+1}-u_n$ ហើយគេតាង

$S_n=\sum_{i=0}^{n}u_i$ បើគេដឹងថា: $u_2-u_1=S_{2003}^2-S_{2003}+1$,

ចូរកំណត់តម្លៃនៃ S_{2003} ។

(Vietnam Olympic 2003)

ដំណោះស្រាយ

គេមាន: $u_{n+2}=u_{n+1}-u_n$ បើគេតាង $u_1=x$ និង $u_2=y$ គេបាន: $u_3=y-x$,

$u_4=-x$, $u_5=-y$, $u_6=x-y$, $u_7=x$, $u_8=y$, ...

យើងសង្កេតឃើញថា: $u_1=u_7$, $u_2=u_8$, ... នោះបញ្ជាក់ថា ស្វ៊ីត (u_n) ជាស្វ៊ីតខួបដែលមានខួបស្មើនឹង 6 នោះ $u_n=u_{n+6m}$, $m\in\mathbb{N}$ ។

ម្យ៉ាងទៀតយើងសង្កេត: $S_6=0$ នាំឲ្យ $S_{6k}=0$ ដែល $k\in\mathbb{N}$ គេបាន:

$$\begin{aligned}S_{2003}&=S_{1998}+u_{1999}+u_{2000}+u_{2001}+u_{2002}+u_{2003}\\&=S_{6\times333}+u_{6\times333+1}+u_{6\times333+2}+u_{6\times333+3}+u_{6\times333+4}+u_{6\times333+5}\\&=0+u_1+u_2+u_3+u_4+u_5\\&=x+y+y-x-x-y\\&=y-x=u_2-u_1\end{aligned}$$

តាមសម្មតិកម្ម: $u_2-u_1=S_{2003}^2-S_{2003}+1$

$$\Leftrightarrow\ S_{2003}=S_{2003}^2-S_{2003}+1$$

$$\Leftrightarrow\ (S_{2003}-1)^2=0\ \Rightarrow\ S_{2003}=1$$

ដូចនេះ $S_{2003}=1$ ។

លំហាត់ទី៩៦

គេឲ្យ ABC ជាត្រីកោណមួយដែលផ្ទៀងផ្ទាត់:

$(8AB-7BC-3CA)^2=6\left(AB^2-BC^2-CA^2\right)$ ។

ចូរស្រាយបញ្ជាក់ថា: $A=60^0$ ។

(Korean Mathematical Competition 2002)

ដំណោះស្រាយ

តាង $a=BC$, $b=CA$ និង $c=AB$ នោះសមភាពដែលគេឲ្យសមមូល:

$$(8c-7a-3b)^2=6\left(c^2-a^2-b^2\right)$$

$$\Leftrightarrow 15b^2+2b(21a-24c)+55a^2-112ac+58c^2=0$$

នេះជាសមីការដឺក្រេទី២, អញ្ញាត b ត្រូវតែមានលក្ខខណ្ឌ: $\Delta\geq 0$

$$\Leftrightarrow -6\left(64a^2-112ac+49c^2\right)\geq 0 \Leftrightarrow (8a-7c)^2\leq 0 \Rightarrow 8a=7c$$

ជំនួសក្នុងសមភាពដើមគេបាន: $3a=7b$ យើងទាញបាន: $\frac{a}{7}=\frac{b}{3}=\frac{c}{8}$

នេះបញ្ជាក់ថា: ΔABC ដូចនឹង $\Delta A'B'C'$ ដែលមានជ្រុង $7,3,8$ ដែលក្នុង

$\Delta A'B'C'$ នេះមាន: $\cos A'=\dfrac{3^2+8^2-7^2}{2\cdot 3\cdot 8}=\dfrac{1}{2} \Rightarrow A'=60^0$

ដូចនេះ $A=A'=60^0$ ។

លំហាត់ទី៩៧

គេឲ្យ a, b, c, d ជាចំនួនគត់ដែល $a>b>c>d>0$ ។ ដោយដឹងថា: $ac+bd=(b+d+a-c)(b+d-a+c)$, ចូរស្រាយបញ្ជាក់ថា: $ab+cd$ មិនមែនជាចំនួនបឋមទេ ។

(IMO 2001)

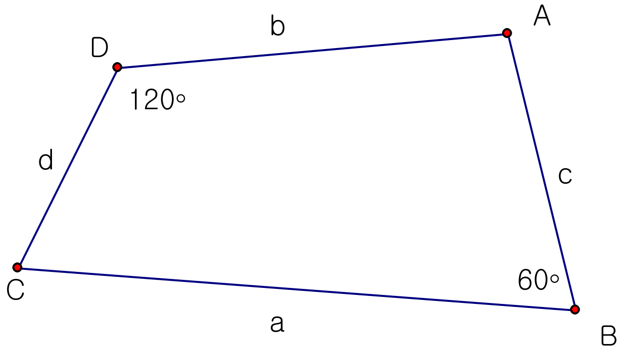
ដំណោះស្រាយ

តាមសម្មតិកម្ម: $ac+bd=(b+d+a-c)(b+d-a+c)$

$$\Leftrightarrow \quad a^2-ac+c^2=b^2+bd+d^2$$

$$\Leftrightarrow \quad a^2+c^2-2acCos60^0=b^2+d^2-2bdCos120^0$$

សមភាពនេះបញ្ជាក់ថា: a, b, c, d ជារង្វាស់ជ្រុងនៃចតុកោណប៉ោង $ABCD$ មួយដែល $BC=a$, $DA=b$, $AB=c$, $CD=d$, $\angle ABC=60^0$ និង $\angle ADC=120^0$



តាមទ្រឹស្តីបទកូស៊ីនុសក្នុងត្រីកោណ ABD និង BCD គេបាន:

$$b^2+c^2-2bcCosA=BD^2=a^2+d^2-2adCosC$$

តែ $A+C=180^0$ ទាញបាន: $CosA=\dfrac{b^2+c^2-a^2-d^2}{2bc+2ad}$

នាំឲ្យ $BD^2=a^2+d^2+2ad\cdot\dfrac{b^2+c^2-a^2-d^2}{2bc+2ad}=\dfrac{(ac+bd)(ab+cd)}{bc+ad}$

តែតាមទ្រឹស្តីបទ *Ptolemy* អនុវត្តក្នុងចតុកោណប៉ោង $ABCD$ ដែលជាចតុកោណចារឹកក្នុងរង្វង់មួយគេបានៈ

$$BD \cdot AC = BC \cdot AD + AB \cdot CD = ab + cd$$

$$\Leftrightarrow \quad BD^2 \cdot AC^2 = (ab+cd)^2$$

$$\Leftrightarrow \quad \frac{(ac+bd)(ab+cd)}{bc+ad} \cdot \left(b^2 + bd + d^2\right) = (ab+cd)^2$$

$$\Leftrightarrow (ac+bd)\left(b^2 + bd + d^2\right) = (ab+cd)(bc+ad)$$

$$\Rightarrow ac+bd \,|\, (ab+cd)(bc+ad) \qquad (1)$$

ដោយ $a > b > c > d > 0$ នោះ $ab+cd > ac+bd > bc+ad \qquad (2)$

បើ $ab+cd$ ជាចំនួនបឋម នោះតាមទំនាក់ទំនង (1) និង (2) គេទាញបានៈ

$ac+bd \,|\, bc+ad$ មិនពិតព្រោះ $ac+bd > bc+ad$ ។

ដូចនេះ $ab+cd$ មិនមែនជាចំនួនបឋមទេ ។

លំហាត់ទី៩៨

គេឲ្យ $f(x,y) = \sqrt{\frac{2003}{2}} \cdot \cos 2(x+y) + a\cos(x+y+\alpha)$ ចំពោះគ្រប់ a, $\alpha \in \mathbb{R}$ ។ ចូរស្រាយបញ្ជាក់ថាៈ $\left(\max f(x,y)\right)^2 + \left(\min f(x,y)\right)^2 \geq 2003$

(Vietnam Olympic 2003)

ដំណោះស្រាយ

ចំពោះគ្រប់ a , $\alpha \in \mathbb{R}$ យើងមានៈ

$$f(x,y) = \sqrt{\frac{2003}{2}} \cdot \cos 2(x+y) + a\cos(x+y+\alpha)$$

យើងបានៈ $f(0,0) = \sqrt{\frac{2003}{2}} + a\cos\alpha$

ហើយ $f\left(\frac{\pi}{2},\frac{\pi}{2}\right)=\sqrt{\frac{2003}{2}}-a\cos\alpha$

នោះ $f(0,0)+f\left(\frac{\pi}{2},\frac{\pi}{2}\right)=2\sqrt{\frac{2003}{2}}$

នាំឲ្យ $\max f(x,y) \geq \max\left\{f(0,0), f\left(\frac{\pi}{2},\frac{\pi}{2}\right)\right\} \geq \sqrt{\frac{2003}{2}}$

យើងបាន: $\left(\max f(x,y)\right)^2 \geq \frac{2003}{2}$ (1)

ម្យ៉ាងទៀតយើងមាន: $f\left(\frac{\pi}{4},\frac{\pi}{4}\right)=-\sqrt{\frac{2003}{2}}-a\sin\alpha$ ហើយ

$$f\left(-\frac{\pi}{4},-\frac{\pi}{4}\right)=-\sqrt{\frac{2003}{2}}+a\sin\alpha$$

នោះ $f\left(\frac{\pi}{4},\frac{\pi}{4}\right)+f\left(-\frac{\pi}{4},-\frac{\pi}{4}\right)=-\sqrt{\frac{2003}{2}}$

នាំឲ្យ $\min f(x,y) \leq \min\left\{f\left(\frac{\pi}{4},\frac{\pi}{4}\right), f\left(-\frac{\pi}{4},-\frac{\pi}{4}\right)\right\} \leq -\sqrt{\frac{2003}{2}}$

យើងបាន: $\left(\min f(x,y)\right)^2 \geq \frac{2003}{2}$ (2)

បូក (1) និង (2) គេបាន: $\left(\max f(x,y)\right)^2+\left(\min f(x,y)\right)^2 \geq 2003$ ។

<u>លំហាត់ទី៩៩</u>

តាង P ជាចំណុចមួយនៅក្នុងប្លង់នៃត្រីកោណ ABC ដែលអង្កត់ PA, PB និង PC អាចផ្គុំជាត្រីកោណដែលមានមុំមួយជាមុំទាល ។ គេសន្មតថា: នៅក្នុងត្រីកោណនេះ មុំទាលគឺឈមនឹងជ្រុង PA ។ ចូរស្រាយបញ្ជាក់ថា $\angle BAC$ ជា មុំស្រួច ។

(USAMO 2001)

ដំណោះស្រាយ

តាមទ្រឹស្តីបទកូស៊ីនុសក្នុងត្រីកោណដែលផ្គុំដោយអង្កត់ PA, PB និង PC គេបាន: $AP^2 = PB^2 + PC^2 - 2PB\cdot PC\cdot\cos\omega$ ដែល ω ជាមុំដែលឈមនឹងជ្រុង PA

តាមសម្មតិកម្ម: $\omega\in\left(90^o,180^o\right)$ នោះ $\cos\omega<0$ គេបាន:

$$AP^2 > PB^2 + PC^2$$

$$\Leftrightarrow \quad \overrightarrow{AP}^2 > \overrightarrow{PB}^2 + \overrightarrow{PC}^2$$

$$\Leftrightarrow \quad \overrightarrow{AP}^2 > \left(\overrightarrow{AB}-\overrightarrow{AP}\right)^2 + \left(\overrightarrow{AC}-\overrightarrow{AP}\right)^2$$

$$\Leftrightarrow \quad \overrightarrow{AP}^2 > \left(\overrightarrow{AB}^2 - 2\overrightarrow{AB}\cdot\overrightarrow{AP} + \overrightarrow{AP}^2\right) + \left(\overrightarrow{AC}^2 - 2\overrightarrow{AC}\cdot\overrightarrow{AP} + \overrightarrow{AP}^2\right)$$

$$\Leftrightarrow \; 2\overrightarrow{AB}\cdot\overrightarrow{BC} > \overrightarrow{AB}^2 + \overrightarrow{AC}^2 + \overrightarrow{AP}^2 - 2\overrightarrow{AB}\cdot\overrightarrow{AP} - 2\overrightarrow{AC}\cdot\overrightarrow{AP} + 2\overrightarrow{AB}\cdot\overrightarrow{BC}$$

$$= \left(\overrightarrow{AP} - \overrightarrow{AB} - \overrightarrow{AC}\right)^2 \geq 0$$

$$\Rightarrow \; \overrightarrow{AB}\cdot\overrightarrow{BC} \; > 0 \quad \Rightarrow \; \cos\angle BAC = \frac{\overrightarrow{AB}\cdot\overrightarrow{BC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{BC}\right|} > 0$$

ដូចនេះ $\angle BAC$ ជាមុំស្រួច ។

លំហាត់ទី១០០

តាង T ជាសំណុំនៃគ្រប់តួចែកគត់វិជ្ជមាននៃ 2004^{100} ។ ចូរកំណត់ចំនួនធាតុច្រើនបំផុតដែលអាច ដែលសំណុំរង S នៃ T អាចមាន បើគ្មានធាតុនៃ S ណាជា ពហុគុណនៃធាតុមួយផ្សេងទៀតនៃ S ទេ ។

(Canadian Mathematical Olympiad 2004)

ដំណោះស្រាយ

ដោយ $2004^{100}=2^{200}\cdot 3^{100}\cdot 167^{100}$ នោះគេបាន:

$T=\left\{2^a3^b167^c:0\le a\le 200,0\le b,c\le 100\right\}$, $(a,b,c\in\mathbb{N})$

តាង $S=\left\{2^{200-b-c}\cdot 3^b\cdot 167^c:0\le b,c\le 100\right\}$

ចំពោះ $0\le b,c\le 100$ នោះ $0\le 200-b-c\le 200$

នេះបញ្ជាក់ថា: S ជាសំណុំរងមួយនៃ T ។

ដោយ b មាន101តម្លៃដែលអាចហើយ c មាន101តម្លៃដែលអាចនោះ S មាន 101^2 ធាតុ ។

យើងនឹងស្រាយថា: គ្មានធាតុនៃ S ណាជាពហុគុណនៃធាតុមួយផ្សេងទៀតទេ ហើយក៏គ្មានសំណុំរង នៃ T ផ្សេងទៀតណាដែលមានចំនួនធាតុច្រើនជាង S នោះឡើយ:

ឧបមាថា: $2^{200-b-c}\cdot 3^b\cdot 167^c$ ជាពហុគុណនៃ $2^{200-j-k}\cdot 3^j\cdot 167^k$

នោះ $200-b-c\ge 200-j-k$, $b\ge j$, $c\ge k$

នាំឲ្យ $b+c\le j+k$, $b\ge j$, $c\ge k$ ទាញបាន: $b=j$ និង $c=k$

នេះបញ្ជាក់ថាគ្មានធាតុនៃ S ណាជាពហុគុណនៃធាតុមួយផ្សេងទៀតនៃ S ទេ

តាង U ជាសំណុំរងមួយនៃ T ដែលមានធាតុលើសពី 101^2

ដោយមានត្រឹមតែ 101^2 គូខុសគ្នានៃ (b,c) ដែល $0\le b,c\le 100$

នោះតាមគោលការណ៍រន្ធព្រាប $(\textit{The pigeon hole principle})$, U ត្រូវមានធាតុពីរ $u_1=2^{a_1}3^{b_1}167^{c_1}$ និង $u_2=2^{a_2}3^{b_2}167^{c_2}$ ដែល $b_1=b_2$, $c_1=c_2$ និង $a_1\ne a_2$ ។

បើ $a_1>a_2$ នោះ u_1 ជាពហុគុណនៃ u_2 និង បើ $a_1<a_2$ នោះ u_2 ជាពហុគុណនៃ u_1 នេះបញ្ជាក់ថា: U មិនបំពេញលក្ខខណ្ឌដែលត្រូវការនោះទេ ។

ដូចនេះចំនួនធាតុច្រើនបំផុតដែលអាច ដែលសំណុំរងមួយនៃ T អាចមានគឺ $101^2=10201$ ។

XVII Asian Pacific Mathematics Olympiad

Time allowed: 4 hours
Each problem is worth 7 points

∗ *The contest problems are to be kept confidential until they are posted on the official APMO website. Please do not disclose nor discuss the problems over the internet until that date. No calculators are to be used during the contest.*

Problem 1. Prove that for every irrational real number a, there are irrational real numbers b and b' so that $a+b$ and ab' are both rational while ab and $a+b'$ are both irrational.

Problem 2. Let a, b and c be positive real numbers such that $abc = 8$. Prove that

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \geq \frac{4}{3}.$$

Problem 3. Prove that there exists a triangle which can be cut into 2005 congruent triangles.

Problem 4. In a small town, there are $n \times n$ houses indexed by (i, j) for $1 \leq i, j \leq n$ with $(1, 1)$ being the house at the top left corner, where i and j are the row and column indices, respectively. At time 0, a fire breaks out at the house indexed by $(1, c)$, where $c \leq \frac{n}{2}$. During each subsequent time interval $[t, t+1]$, the fire fighters defend a house which is not yet on fire while the fire spreads to all undefended *neighbors* of each house which was on fire at time t. Once a house is defended, it remains so all the time. The process ends when the fire can no longer spread. At most how many houses can be saved by the fire fighters? A house indexed by (i, j) is a *neighbor* of a house indexed by (k, ℓ) if $|i-k| + |j-\ell| = 1$.

Problem 5. In a triangle ABC, points M and N are on sides AB and AC, respectively, such that $MB = BC = CN$. Let R and r denote the circumradius and the inradius of the triangle ABC, respectively. Express the ratio MN/BC in terms of R and r.

XVII APMO - March, 2005

Problems and Solutions

Problem 1. Prove that for every irrational real number a, there are irrational real numbers b and b' so that $a+b$ and ab' are both rational while ab and $a+b'$ are both irrational.

(Solution) Let a be an irrational number. If a^2 is irrational, we let $b = -a$. Then, $a+b=0$ is rational and $ab=-a^2$ is irrational. If a^2 is rational, we let $b = a^2 - a$. Then, $a+b=a^2$ is rational and $ab = a^2(a-1)$. Since

$$a = \frac{ab}{a^2} + 1$$

is irrational, so is ab.

Now, we let $b' = \frac{1}{a}$ or $b' = \frac{2}{a}$. Then $ab' = 1$ or 2, which is rational. Note that

$$a + b' = \frac{a^2+1}{a} \quad \text{or} \quad a + b' = \frac{a^2+2}{a}.$$

Since,

$$\frac{a^2+2}{a} - \frac{a^2+1}{a} = \frac{1}{a},$$

at least one of them is irrational.

Problem 2. Let a, b and c be positive real numbers such that $abc = 8$. Prove that

$$\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \ \geq \ \frac{4}{3}\,.$$

(Solution) Observe that

$$\frac{1}{\sqrt{1+x^3}} \geq \frac{2}{2+x^2}\,. \tag{1}$$

In fact, this is equivalent to $(2+x^2)^2 \geq 4(1+x^3)$, or $x^2(x-2)^2 \geq 0$. Notice that equality holds in (1) if and only if $x = 2$.

We substitute x by a, b, c in (1), respectively, to find

$$\begin{aligned} &\frac{a^2}{\sqrt{(1+a^3)(1+b^3)}} + \frac{b^2}{\sqrt{(1+b^3)(1+c^3)}} + \frac{c^2}{\sqrt{(1+c^3)(1+a^3)}} \\ &\geq \frac{4a^2}{(2+a^2)(2+b^2)} + \frac{4b^2}{(2+b^2)(2+c^2)} + \frac{4c^2}{(2+c^2)(2+a^2)}\,. \end{aligned} \tag{2}$$

We combine the terms on the right hand side of (2) to obtain

$$\text{Left hand side of (2)} \geq \frac{2S(a,b,c)}{36+S(a,b,c)} = \frac{2}{1+36/S(a,b,c)}\,, \tag{3}$$

where $S(a,b,c) := 2(a^2+b^2+c^2) + (ab)^2 + (bc)^2 + (ca)^2$. By AM-GM inequality, we have

$$\begin{aligned} a^2+b^2+c^2 &\geq 3\sqrt[3]{(abc)^2} = 12\,, \\ (ab)^2+(bc)^2+(ca)^2 &\geq 3\sqrt[3]{(abc)^4} = 48\,. \end{aligned}$$

Note that the equalities holds if and only if $a = b = c = 2$. The above inequalities yield

$$S(a,b,c) = 2(a^2+b^2+c^2) + (ab)^2 + (bc)^2 + (ca)^2 \geq 72\,. \tag{4}$$

Therefore

$$\frac{2}{1+36/S(a,b,c)} \geq \frac{2}{1+36/72} = \frac{4}{3}\,, \tag{5}$$

which is the required inequality.

Problem 3. Prove that there exists a triangle which can be cut into 2005 congruent triangles.

(Solution) Suppose that one side of a triangle has length n. Then it can be cut into n^2 congruent triangles which are similar to the original one and whose corresponding sides to the side of length n have lengths 1.

Since $2005 = 5 \times 401$ where 5 and 401 are primes and both primes are of the type $4k+1$, it is representable as a sum of two integer squares. Indeed, it is easy to see that

$$\begin{aligned} 2005 &= 5 \times 401 = (2^2+1)(20^2+1) \\ &= 40^2 + 20^2 + 2^2 + 1 \\ &= (40-1)^2 + 2 \times 40 + 20^2 + 2^2 \\ &= 39^2 + 22^2. \end{aligned}$$

Let ABC be a right-angled triangle with the legs AB and BC having lengths 39 and 22, respectively. We draw the altitude BK, which divides ABC into two similar triangles. Now we divide ABK into 39^2 congruent triangles as described above and BCK into 22^2 congruent triangles. Since ABK is similar to BKC, all 2005 triangles will be congruent.

Problem 4. In a small town, there are $n \times n$ houses indexed by (i, j) for $1 \le i, j \le n$ with $(1, 1)$ being the house at the top left corner, where i and j are the row and column indices, respectively. At time 0, a fire breaks out at the house indexed by $(1, c)$, where $c \le \frac{n}{2}$. During each subsequent time interval $[t, t+1]$, the fire fighters defend a house which is not yet on fire while the fire spreads to all undefended *neighbors* of each house which was on fire at time t. Once a house is defended, it remains so all the time. The process ends when the fire can no longer spread. At most how many houses can be saved by the fire fighters? A house indexed by (i, j) is a *neighbor* of a house indexed by (k, ℓ) if $|i-k| + |j-\ell| = 1$.

(Solution) At most $n^2 + c^2 - nc - c$ houses can be saved. This can be achieved under the following order of defending:

$$\begin{gathered}(2, c), (2, c+1); (3, c-1), (3, c+2); (4, c-2), (4, c+3); \ldots \\ (c+1, 1), (c+1, 2c); (c+1, 2c+1), \ldots, (c+1, n).\end{gathered} \tag{6}$$

Under this strategy, there are

2 columns (column numbers $c, c+1$) at which $n-1$ houses are saved
2 columns (column numbers $c-1, c+2$) at which $n-2$ houses are saved
...
2 columns (column numbers $1, 2c$) at which $n-c$ houses are saved
$n-2c$ columns (column numbers $n-2c+1, \ldots, n$) at which $n-c$ houses are saved

Adding all these we obtain:

$$2[(n-1) + (n-2) + \cdots + (n-c)] + (n-2c)(n-c) = n^2 + c^2 - cn - c. \tag{7}$$

We say that a house indexed by (i, j) is at level t if $|i-1| + |j-c| = t$. Let $d(t)$ be the number of houses at level t defended by time t, and $p(t)$ be the number of houses at levels greater than t defended by time t. It is clear that

$$p(t) + \sum_{i=1}^{t} d(i) \le t \text{ and } p(t+1) + d(t+1) \le p(t) + 1.$$

Let $s(t)$ be the number of houses at level t which are not burning at time t. We prove that

$$s(t) \le t - p(t) \le t$$

for $1 \le t \le n-1$ by induction. It is obvious when $t = 1$. Assume that it is true for $t = k$. The union of the neighbors of any $k - p(k) + 1$ houses at level $k+1$ contains at least $k - p(k) + 1$ vertices at level k. Since $s(k) \le k - p(k)$, one of these houses at level k is burning. Therefore, at most $k - p(k)$ houses at level $k+1$ have no neighbor burning. Hence we have

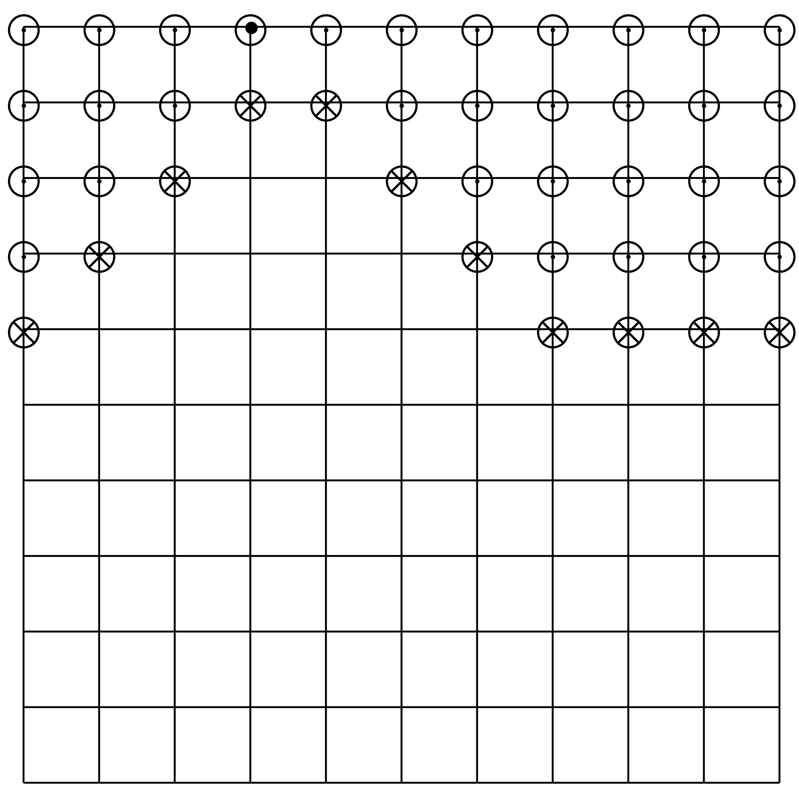
$$\begin{aligned} s(k+1) &\le k - p(k) + d(k+1) \\ &= (k+1) - (p(k) + 1 - d(k+1)) \\ &\le (k+1) - p(k+1). \end{aligned}$$

We now prove that the strategy given above is optimal. Since

$$\sum_{t=1}^{n-1} s(t) \leq \binom{n}{2},$$

the maximum number of houses at levels less than or equal to $n-1$, that can be saved under any strategy is at most $\binom{n}{2}$, which is realized by the strategy above. Moreover, at levels bigger than $n-1$, every house is saved under the strategy above.

The following is an example when $n = 11$ and $c = 4$. The houses with $\bigcirc$ mark are burned. The houses with $\otimes$ mark are blocked ones and hence those and the houses below them are saved.



Problem 5. In a triangle ABC, points M and N are on sides AB and AC, respectively, such that $MB = BC = CN$. Let R and r denote the circumradius and the inradius of the triangle ABC, respectively. Express the ratio MN/BC in terms of R and r.

(Solution) Let ω, O and I be the circumcircle, the circumcenter and the incenter of ABC, respectively. Let D be the point of intersection of the line BI and the circle ω such that $D \neq B$. Then D is the midpoint of the arc AC. Hence $OD \perp CN$ and $OD = R$.

We first show that triangles MNC and IOD are similar. Because $BC = BM$, the line BI (the bisector of $\angle MBC$) is perpendicular to the line CM. Because $OD \perp CN$ and $ID \perp MC$, it follows that

$$\angle ODI = \angle NCM \tag{8}$$

Let $\angle ABC = 2\beta$. In the triangle BCM, we have

$$\frac{CM}{NC} = \frac{CM}{BC} = 2\sin\beta \tag{9}$$

Since $\angle DIC = \angle DCI$, we have $ID = CD = AD$. Let E be the point of intersection of the line DO and the circle ω such that $E \neq D$. Then DE is a diameter of ω and $\angle DEC = \angle DBC = \beta$. Thus we have

$$\frac{DI}{OD} = \frac{CD}{OD} = \frac{2R\sin\beta}{R} = 2\sin\beta. \tag{10}$$

Combining equations (8), (9), and (10) shows that triangles MNC and IOD are similar. It follows that

$$\frac{MN}{BC} = \frac{MN}{NC} = \frac{IO}{OD} = \frac{IO}{R}\,. \tag{11}$$

The well-known Euler's formula states that

$$OI^2 = R^2 - 2Rr. \tag{12}$$

Therefore,

$$\frac{MN}{BC} = \sqrt{1 - \frac{2r}{R}}\,. \tag{13}$$

(Alternative Solution) Let a (resp., b, c) be the length of BC (resp., AC, AB). Let α (resp., β, γ) denote the angle $\angle BAC$ (resp., $\angle ABC$, $\angle ACB$). By introducing coordinates $B = (0,0)$, $C = (a,0)$, it is immediate that the coordinates of M and N are

$$M = (a\cos\beta, a\sin\beta), \quad N = (a - a\cos\gamma, a\sin\gamma), \tag{14}$$

respectively. Therefore,

$$\begin{aligned}(MN/BC)^2 &= [(a - a\cos\gamma - a\cos\beta)^2 + (a\sin\gamma - a\sin\beta)^2]/a^2 \\ &= (1 - \cos\gamma - \cos\beta)^2 + (\sin\gamma - \sin\beta)^2 \\ &= 3 - 2\cos\gamma - 2\cos\beta + 2(\cos\gamma\cos\beta - \sin\gamma\sin\beta) \\ &= 3 - 2\cos\gamma - 2\cos\beta + 2\cos(\gamma+\beta) \\ &= 3 - 2\cos\gamma - 2\cos\beta - 2\cos\alpha \\ &= 3 - 2(\cos\gamma + \cos\beta + \cos\alpha).\end{aligned} \tag{15}$$

Now we claim

$$\cos\gamma + \cos\beta + \cos\alpha = \frac{r}{R} + 1. \tag{16}$$

From

$$\begin{aligned} a &= b\cos\gamma + c\cos\beta \\ b &= c\cos\alpha + a\cos\gamma \\ c &= a\cos\beta + b\cos\alpha \end{aligned} \tag{17}$$

we get

$$a(1+\cos\alpha) + b(1+\cos\beta) + c(1+\cos\gamma) = (a+b+c)(\cos\alpha + \cos\beta + \cos\gamma). \tag{18}$$

Thus

$$\begin{aligned} &\cos\alpha + \cos\beta + \cos\gamma \\ &= \frac{1}{a+b+c}\left(a(1+\cos\alpha) + b(1+\cos\beta) + c(1+\cos\gamma)\right) \\ &= \frac{1}{a+b+c}\left(a\left(1 + \frac{b^2+c^2-a^2}{2bc}\right) + b\left(1 + \frac{a^2+c^2-b^2}{2ac}\right) + c\left(1 + \frac{a^2+b^2-c^2}{2ab}\right)\right) \\ &= \frac{1}{a+b+c}\left(a+b+c+\frac{a^2(b^2+c^2-a^2) + b^2(a^2+c^2-b^2) + c^2(a^2+b^2-c^2)}{2abc}\right) \\ &= 1 + \frac{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}{2abc(a+b+c)}. \end{aligned} \tag{19}$$

On the other hand, from $R = \dfrac{a}{2\sin\alpha}$ it follows that

$$\begin{aligned} R^2 &= \frac{a^2}{4(1-\cos^2\alpha)} = \frac{a^2}{4\left(1 - \left(\dfrac{b^2+c^2-a^2}{2bc}\right)^2\right)} \\ &= \frac{a^2b^2c^2}{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}. \end{aligned} \tag{20}$$

Also from $\frac{1}{2}(a+b+c)r = \frac{1}{2}bc\sin\alpha$, it follows that

$$\begin{aligned} r^2 &= \frac{b^2c^2(1-\cos^2\alpha)}{(a+b+c)^2} = \frac{b^2c^2\left(1-\left(\frac{b^2+c^2-a^2}{2bc}\right)^2\right)}{(a+b+c)^2} \\ &= \frac{2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4}{4(a+b+c)^2}. \end{aligned} \tag{21}$$

Combining (19), (20) and (21), we get (16) as desired.

Finally, by (15) and (16) we have

$$\frac{MN}{BC} = \sqrt{1-\frac{2r}{R}}. \tag{22}$$

Another proof of (16) from R.A. Johnson's "Advanced Euclidean Geometry"[1]:

Construct the perpendicular bisectors OD, OE, OF, where D, E, F are the midpoints of BC, CA, AB, respectively. By Ptolemy's Theorem applied to the cyclic quadrilateral $OEAF$, we get

$$\frac{a}{2}\cdot R = \frac{b}{2}\cdot OF + \frac{c}{2}\cdot OE.$$

Similarly

$$\frac{b}{2}\cdot R = \frac{c}{2}\cdot OD + \frac{a}{2}\cdot OF, \quad \frac{c}{2}\cdot R = \frac{a}{2}\cdot OE + \frac{b}{2}\cdot OD.$$

Adding, we get

$$sR = OD\cdot\frac{b+c}{2} + OE\cdot\frac{c+a}{2} + OF\cdot\frac{a+b}{2}, \tag{23}$$

where s is the semiperimeter. But also, the area of triangle OBC is $OD\cdot\frac{a}{2}$, and adding similar formulas for the areas of triangles OCA and OAB gives

$$rs = \triangle ABC = OD\cdot\frac{a}{2} + OE\cdot\frac{b}{2} + OF\cdot\frac{c}{2} \tag{24}$$

Adding (23) and (24) gives $s(R+r) = s(OD+OE+OF)$, or

$$OD+OE+OF = R+r.$$

Since $OD = R\cos A$ *etc.*, (16) follows.

[1] This proof was introduced to the coordinating country by Professor Bill Sands of Canada.

XVII Asian Pacific Mathematics Olympiad
(The Result)

Medal	Student Name	
Gold Medal	Ramiro Lafuente(Argentina)	Chris Leong (Australia)
	Peng Shi(Canada)	Ho Chung Siu(HongKong)
	Masaki Watanabe(Japan)	Andrey Kim(Kazakhstan)
	Gyu Min Jeon(Korea)	Chen Soncco(Peru)
	Jingyi, Kenneth Tay(Singapore)	Cedric Lin(Taiwan R.O.C.)
	Saran Ahuja(Thailand)	Yi Sun(USA)
Silver Medal	Lucas Andisco(Argentina)	Graham White(Australia)
	Vinoth Nandakumar(Australia)	Richard Peng(Canada)
	Elyot Grant(Canada)	Esteban Gonzalez(Colombia)
	Pascual Restrepo(Colombia)	Yun Pui Tsoi(HongKong)
	On Yip Chung(HongKong)	Andre Yohannes Wibisono(Indonesia)
	Yuki Yoshida(Japan)	Akinori Mitani(Japan)
	Nurbol Duisenbayev(Kazakhstan)	Maksat Abishev(Kazakhstan)
	Yulhee Nam(Korea)	Chang hyun Kwack(Korea)
	Pablo Soberon(Mexico)	Ameya Velingker(USA)
	John Kim(USA)	Heather Macbeth(New Zealand)
	Nai-Chung Kuo(Taiwan R.O.C.)	Pi-Horng Liu(Taiwan R.O.C.)
	Napat Rujeerapaiboon(Thailand)	Wuttisak Trongsiriwat(Thailand)
Bronze Medal	German Gieczewski(Argentina)	Pablo Blanc(Argentina)
	Brian Leung(Australia)	Matthew Ng(Australia)
	Stephen Muirhead(Australia)	Sam Chow(Australia)
	Kent Huynh(Canada)	Oleg Ivrii(Canada)
	Boris Braverman(Canada)	David Rhee(Canada)
	Sebastian Libedinsky(Chile)	Siu Sing Chow(HongKong)
	Jack Hui(HongKong)	Ming Fong Poon(HongKong)
	Man Ho Tsang(HongKong)	Sho Sugiura(Japan)
	Tsukasa Kuribayashi(Japan)	Tomohiro Mitani(Japan)
	Yusuke Kinoshita(Japan)	Kairat Zhubayev(Kazakhstan)
	Yerzhan Baimbetov(Kazakhstan)	Rashid Unbayev(Kazakhstan)
	Kuandyk Yerzhanov(Kazakhstan)	Wonho Jeong(Korea)
	YoungSeok Kim(Korea)	Sang Yeon Kim(Korea)
	Jeha Yang(Korea)	Hector Garcia(Mexico)

	Tom Wang(New Zealand)	Oscar Garay(Peru)
	Yong Sheng Soh(Singapore)	Jun Feng, Jack Ho(Singapore),
	Wei Quan Lim(Singapore)	Chung-Heng Yeh(Taiwan R.O.C.)
	Tsung-Che Wu(Taiwan R.O.C.)	Shinn-Yih Huang(Taiwan R.O.C.)
	Cheng-Chiang Tsai(Taiwan R.O.C.)	Patid Hanchaipibulkul(Thailand)
	Wutichai Chongchitmate(Thailand)	Nithi Rungthanapirom(Thailand)
	Amornsit Atchariyabodee(Thailand)	Brian Lawrence(USA)
	Tae Hyeon Ko(USA)	Eric Price(USA)
	Thomas Mildorf(USA)	
Hon Men	Federico Felguer(Argentina)	Javier Corti(Argentina)
	Julian Eisenschlos(Argentina)	Nicolas Komanski(Argentina)
	Leopoldo Taravilse(Argentina)	Maximiliano Camporino(Argentina)
	Matt Libling(Australia)	Kim Ramchen(Australia)
	Charles Li(Australia)	Yufei Zhao(Canada)
	William Fu(Canada)	Francis Chung(Canada)
	Yat Wui Cheung(HongKong)	Chiu Wai Wong(HongKong)
	Wing Yin Yim(HongKong)	Jose Gabriel Acevedo(Colombia)
	Fernando Gomez(Ecuador)	David Hartanto(Indonesia)
	Dimas Yusuf Danurwenda(Indonesia)	Kenta Noguchi(Japan)
	Atsushi Yano(Japan)	Kouta Yoshisato(Japan)
	Daniyar Kelbetov(Kazakhstan)	Alen Bayev(Kazakhstan)
	Alexei Pak(Kazakhstan)	Whan Ghang(Korea)
	Woo-Jin Chang(Korea)	Min Sung Chung(Korea)
	Paul Gallegos(Mexico)	Joshua Hernandez(Mexico)
	Andres Ruiz(Mexico)	Galo Higuera(Mexico)
	Manuel Guevara(Mexico)	Francisco Ibarra(Mexico)
	Mario Huicochea(Mexico)	Issac Buenrostro(Mexico)
	Eric Kang(New Zealand)	James Liley(New Zealand)
	Renzo Monteza(Peru)	Franco Vargas(Peru)
	Jia Min, Charmaine Sia(Singapore)	WeiCheng Chin(Singapore)
	Jiawei Wu(Singapore)	Yan Zhao(Singapore)
	Jun Wei Ho(Singapore)	Kuen-Yew, Bryan Hooi(Singapore)
	Yu-Shiang Cheng(Taiwan R.O.C.)	Pi-Hsun Shih(Taiwan R.O.C.)
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	Wich Huengwattanakul(Thailand)	Teerawut Wonnaparhun(Thailand)
	Tedrick Leung(USA)	Adam Hesterberg(USA)
	Joshua Horowitz(USA)	

XVIII Asian Pacific Mathematics Olympiad

Time allowed: 4 hours
Each problem is worth 7 points

∗ *The contest problems are to be kept confidential until they are posted on the official APMO website. Please do not disclose nor discuss the problems over the internet until that date. No calculators are to be used during the contest.*

Problem 1. Let n be a positive integer. Find the largest nonnegative real number $f(n)$ (depending on n) with the following property: whenever $a_1, a_2, \ldots, a_n$ are real numbers such that $a_1 + a_2 + \cdots + a_n$ is an integer, there exists some i such that $|\, a_i - \frac{1}{2} \,| \geq f(n)$.

Problem 2. Prove that every positive integer can be written as a finite sum of distinct integral powers of the golden mean $\tau = \frac{1+\sqrt{5}}{2}$. Here, an integral power of τ is of the form τ^i, where i is an integer (not necessarily positive).

Problem 3. Let $p \geq 5$ be a prime and let r be the number of ways of placing p checkers on a $p \times p$ checkerboard so that not all checkers are in the same row (but they may all be in the same column). Show that r is divisible by p^5. Here, we assume that all the checkers are identical.

Problem 4. Let A, B be two distinct points on a given circle O and let P be the midpoint of the line segment AB. Let O_1 be the circle tangent to the line AB at P and tangent to the circle O. Let ℓ be the tangent line, different from the line AB, to O_1 passing through A. Let C be the intersection point, different from A, of ℓ and O. Let Q be the midpoint of the line segment BC and O_2 be the circle tangent to the line BC at Q and tangent to the line segment AC. Prove that the circle O_2 is tangent to the circle O.

Problem 5. In a circus, there are n clowns who dress and paint themselves up using a selection of 12 distinct colours. Each clown is required to use at least five different colours. One day, the ringmaster of the circus orders that no two clowns have exactly the same set of colours and no more than 20 clowns may use any one particular colour. Find the largest number n of clowns so as to make the ringmaster's order possible.

Problem 1. Let n be a positive integer. Find the largest nonnegative real number $f(n)$ (depending on n) with the following property: whenever $a_1, a_2, \ldots, a_n$ are real numbers such that $a_1 + a_2 + \cdots + a_n$ is an integer, there exists some i such that $\mid a_i - \frac{1}{2} \mid \geq f(n)$.

(Solution) The answer is

$$f(n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{1}{2n} & \text{if } n \text{ is odd.} \end{cases}$$

First, assume that n is even. If $a_i = \frac{1}{2}$ for all i, then the sum $a_1 + a_2 + \cdots + a_n$ is an integer. Since $\mid a_i - \frac{1}{2} \mid = 0$ for all i, we may conclude $f(n) = 0$ for any even n.

Now assume that n is odd. Suppose that $\mid a_i - \frac{1}{2} \mid < \frac{1}{2n}$ for all $1 \leq i \leq n$. Then, since $\sum_{i=1}^{n} a_i$ is an integer,

$$\frac{1}{2} \leq \left| \sum_{i=1}^{n} a_i - \frac{n}{2} \right| \leq \sum_{i=1}^{n} \left| a_i - \frac{1}{2} \right| < \frac{1}{2n} \cdot n = \frac{1}{2},$$

a contradiction. Thus $|a_i - \frac{1}{2}| \geq \frac{1}{2n}$ for some i, as required. On the other hand, putting $n = 2m+1$ and $a_i = \frac{m}{2m+1}$ for all i gives $\sum a_i = m$, while

$$\left| a_i - \frac{1}{2} \right| = \frac{1}{2} - \frac{m}{2m+1} = \frac{1}{2(2m+1)} = \frac{1}{2n}$$

for all i. Therefore, $f(n) = \frac{1}{2n}$ is the best possible for any odd n.

Problem 2. Prove that every positive integer can be written as a finite sum of distinct integral powers of the golden mean $\tau = \frac{1+\sqrt{5}}{2}$. Here, an integral power of τ is of the form τ^i, where i is an integer (not necessarily positive).

(Solution) We will prove this statement by induction using the equality

$$\tau^2 = \tau + 1.$$

If $n = 1$, then $1 = \tau^0$. Suppose that $n-1$ can be written as a finite sum of integral powers of τ, say

$$n - 1 = \sum_{i=-k}^{k} a_i \tau^i, \tag{1}$$

where $a_i \in \{0, 1\}$ and $n \geq 2$. We will write (1) as

$$n - 1 = a_k \cdots a_1 a_0 . a_{-1} a_{-2} \cdots a_{-k}. \tag{2}$$

For example,

$$1 = 1.0 = 0.11 = 0.1011 = 0.101011.$$

Firstly, we will prove that we may assume that in (2) we have $a_i a_{i+1} = 0$ for all i with $-k \le i \le k-1$. Indeed, if we have several occurrences of 11, then we take the leftmost such occurrence. Since we may assume that it is preceded by a 0, we can replace 011 with 100 using the identity $\tau^{i+1} + \tau^i = \tau^{i+2}$. By doing so repeatedly, if necessary, we will eliminate all occurrences of two 1's standing together. Now we have the representation

$$n - 1 = \sum_{i=-K}^{K} b_i \tau^i, \tag{3}$$

where $b_i \in \{0, 1\}$ and $b_i b_{i+1} = 0$.

If $b_0 = 0$ in (3), then we just add $1 = \tau^0$ to both sides of (3) and we are done.

Suppose now that there is 1 in the unit position of (3), that is $b_0 = 1$. If there are two 0's to the right of it, i.e.

$$n - 1 = \cdots 1.00 \cdots ,$$

then we can replace 1.00 with 0.11 because $1 = \tau^{-1} + \tau^{-2}$, and we are done because we obtain 0 in the unit position. Thus we may assume that

$$n - 1 = \cdots 1.010 \cdots .$$

Again, if we have $n - 1 = \cdots 1.0100 \cdots$, we may rewrite it as

$$n - 1 = \cdots 1.0100 \cdots = \cdots 1.0011 \cdots = \cdots 0.1111 \cdots$$

and obtain 0 in the unit position. Therefore, we may assume that

$$n - 1 = \cdots 1.01010 \cdots .$$

Since the number of 1's is finite, eventually we will obtain an occurrence of 100 at the end, i.e.

$$n - 1 = \cdots 1.01010 \cdots 100.$$

Then we can shift all 1's to the right to obtain 0 in the unit position, i.e.

$$n - 1 = \cdots 0.11 \cdots 11,$$

and we are done.

Problem 3. Let $p \ge 5$ be a prime and let r be the number of ways of placing p checkers on a $p \times p$ checkerboard so that not all checkers are in the same row (but they may all be in the same column). Show that r is divisible by p^5. Here, we assume that all the checkers are identical.

(Solution) Note that $r = \binom{p^2}{p} - p$. Hence, it suffices to show that

$$(p^2 - 1)(p^2 - 2) \cdots (p^2 - (p-1)) - (p-1)! \equiv 0 \pmod{p^4}. \tag{1}$$

Now, let

$$f(x) := (x-1)(x-2)\cdots(x-(p-1)) = x^{p-1} + s_{p-2}x^{p-2} + \cdots + s_1 x + s_0. \qquad (2)$$

Then the congruence equation (1) is same as $f(p^2) - s_0 \equiv 0 \pmod{p^4}$. Therefore, it suffices to show that $s_1 p^2 \equiv 0 \pmod{p^4}$ or $s_1 \equiv 0 \pmod{p^2}$.

Since $a^{p-1} \equiv 1 \pmod{p}$ for all $1 \le a \le p-1$, we can factor

$$x^{p-1} - 1 \equiv (x-1)(x-2)\cdots(x-(p-1)) \pmod{p}. \qquad (3)$$

Comparing the coefficients of the left hand side of (3) with those of the right hand side of (2), we obtain $p \,|\, s_i$ for all $1 \le i \le p-2$ and $s_0 \equiv -1 \pmod{p}$. On the other hand, plugging p for x in (2), we get

$$f(p) = (p-1)! = s_0 = p^{p-1} + s_{p-2}p^{p-2} + \cdots + s_1 p + s_0,$$

which implies

$$p^{p-1} + s_{p-2}p^{p-2} + \cdots + s_2 p^2 = -s_1 p.$$

Since $p \ge 5$, $p \,|\, s_2$ and hence $s_1 \equiv 0 \pmod{p^2}$ as desired.

Problem 4. Let A, B be two distinct points on a given circle O and let P be the midpoint of the line segment AB. Let O_1 be the circle tangent to the line AB at P and tangent to the circle O. Let ℓ be the tangent line, different from the line AB, to O_1 passing through A. Let C be the intersection point, different from A, of ℓ and O. Let Q be the midpoint of the line segment BC and O_2 be the circle tangent to the line BC at Q and tangent to the line segment AC. Prove that the circle O_2 is tangent to the circle O.

(Solution) Let S be the tangent point of the circles O and O_1 and let T be the intersection point, different from S, of the circle O and the line SP. Let X be the tangent point of ℓ to O_1 and let M be the midpoint of the line segment XP. Since $\angle TBP = \angle ASP$, the triangle TBP is similar to the triangle ASP. Therefore,

$$\frac{PT}{PB} = \frac{PA}{PS}.$$

Since the line ℓ is tangent to the circle O_1 at X, we have

$$\angle SPX = 90^\circ - \angle XSP = 90^\circ - \angle APM = \angle PAM$$

which implies that the triangle PAM is similar to the triangle SPX. Consequently,

$$\frac{XS}{XP} = \frac{MP}{MA} = \frac{XP}{2MA} \quad \text{and} \quad \frac{XP}{PS} = \frac{MA}{AP}.$$

From this and the above observation follows

$$\frac{XS}{XP} \cdot \frac{PT}{PB} = \frac{XP}{2MA} \cdot \frac{PA}{PS} = \frac{XP}{2MA} \cdot \frac{MA}{XP} = \frac{1}{2}\,. \tag{1}$$

Let A' be the intersection point of the circle O and the perpendicular bisector of the chord BC such that A, A' are on the same side of the line BC, and N be the intersection point of the lines $A'Q$ and CT. Since

$$\angle NCQ = \angle TCB = \angle TCA = \angle TBA = \angle TBP$$

and

$$\angle CA'Q = \frac{\angle CAB}{2} = \frac{\angle XAP}{2} = \angle PAM = \angle SPX,$$

the triangle NCQ is similar to the triangle TBP and the triangle $CA'Q$ is similar to the triangle SPX. Therefore

$$\frac{QN}{QC} = \frac{PT}{PB} \quad \text{and} \quad \frac{QC}{QA'} = \frac{XS}{XP}\,.$$

and hence $QA' = 2QN$ by (1). This implies that N is the midpoint of the line segment QA'. Let the circle O_2 touch the line segment AC at Y. Since

$$\angle ACN = \angle ACT = \angle BCT = \angle QCN$$

and $|CY| = |CQ|$, the triangles YCN and QCN are congruent and hence $NY \perp AC$ and $NY = NQ = NA'$. Therefore, N is the center of the circle O_2, which completes the proof.

Remark: Analytic solutions are possible: For example, one can prove for a triangle ABC inscribed in a circle O that $AB = k(2+2t), AC = k(1+2t), BC = k(1+4t)$ for some positive numbers k, t if and only if there exists a circle O_1 such that O_1 is tangent to the side AB at its midpoint, the side AC and the circle O.

One obtains $AB = k'(1+4t'), AC = k'(1+2t'), BC = k'(2+2t')$ by substituting $t = 1/4t'$ and $k = 2k't'$. So, there exists a circle O_2 such that O_2 is tangent to the side BC at its midpoint, the side AC and the circle O.

In the above, $t = \tan^2\alpha$ and $k = \frac{4R\tan\alpha}{(1+\tan^2\alpha)(1+4\tan^2\alpha)}$, where R is the radius of O and $\angle A = 2\alpha$. Furthermore, $t' = \tan^2\gamma$ and $k' = \frac{4R\tan\gamma}{(1+\tan^2\gamma)(1+4\tan^2\gamma)}$, where $\angle C = 2\gamma$. Observe that $\sqrt{tt'} = \tan\alpha \cdot \tan\gamma = \frac{XS}{XP} \cdot \frac{PT}{PB} = \frac{1}{2}$, which implies $tt' = \frac{1}{4}$. It is now routine easy to check that $k = 2k't'$.

Problem 5. In a circus, there are n clowns who dress and paint themselves up using a selection of 12 distinct colours. Each clown is required to use at least five different colours. One day, the ringmaster of the circus orders that no two clowns have exactly the same set of colours and no more than 20 clowns may use any one particular colour. Find the largest number n of clowns so as to make the ringmaster's order possible.

(Solution) Let C be the set of n clowns. Label the colours $1, 2, 3, \ldots, 12$. For each $i = 1, 2, \ldots, 12$, let E_i denote the set of clowns who use colour i. For each subset S of $\{1, 2, \ldots, 12\}$, let E_S be the set of clowns who use exactly those colours in S. Since $S \neq S'$ implies $E_S \cap E_{S'} = \emptyset$, we have

$$\sum_S |E_S| = |C| = n,$$

where S runs over all subsets of $\{1, 2, \ldots, 12\}$. Now for each i,

$$E_S \subseteq E_i \quad \text{if and only if} \quad i \in S,$$

and hence

$$|E_i| = \sum_{i \in S} |E_S|.$$

By assumption, we know that $|E_i| \leq 20$ and that if $E_S \neq \emptyset$, then $|S| \geq 5$. From this we obtain

$$20 \times 12 \geq \sum_{i=1}^{12} |E_i| = \sum_{i=1}^{12} \left(\sum_{i \in S} |E_S| \right) \geq 5 \sum_S |E_S| = 5n.$$

Therefore $n \leq 48$.

Now, define a sequence $\{c_i\}_{i=1}^{52}$ of colours in the following way:

1 2 3 4 | 5 6 7 8 | 9 10 11 12 |
4 1 2 3 | 8 5 6 7 | 12 9 10 11 |
3 4 1 2 | 7 8 5 6 | 11 12 9 10 |
2 3 4 1 | 6 7 8 5 | 10 11 12 9 | 1 2 3 4

The first row lists $c_1, \ldots, c_{12}$ in order, the second row lists $c_{13}, \ldots, c_{24}$ in order, the third row lists $c_{25}, \ldots, c_{36}$ in order, and finally the last row lists $c_{37}, \ldots, c_{52}$ in order. For each j, $1 \leq j \leq 48$, assign colours $c_j, c_{j+1}, c_{j+2}, c_{j+3}, c_{j+4}$ to the j-th clown. It is easy to check that this assignment satisfies all conditions given above. So, 48 is the largest for n.

Remark: The fact that $n \leq 48$ can be obtained in a much simpler observation that

$$5n \leq 12 \times 20 = 240.$$

There are many other ways of constructing 48 distinct sets consisting of 5 colours. For example, consider the sets

$$\{1,2,3,4,5,6\}, \quad \{3,4,5,6,7,8\}, \quad \{5,6,7,8,9,10\}, \quad \{7,8,9,10,11,12\},$$
$$\{9,10,11,12,1,2\}, \quad \{11,12,1,2,3,4\}, \quad \{1,2,5,6,9,10\}, \quad \{3,4,7,8,11,12\}.$$

Each of the above 8 sets has 6 distinct subsets consisting of exactly 5 colours. It is easy to check that the 48 subsets obtained in this manner are all distinct.

Medal	Student Name	
Gold Award	Brian Lawrence (USA) Cheng-Chiang Tsai (Taiwan) Yufei Zhao (Canada) Panupong Pasupat (Thailand) Guevara Manuel Guevara (Mexico) Daniel Soncco (Peru)	Seok Hyeong Lee (Korea) Toshiki Kataoka (Japan) Pok Wai Fong (Hong Kong) Andrey Kim (Kazakhstan) Yan Zhao (Singapore)
Silver Award	Arnav Tripathy (USA) Ju Gang Nam (Korea) Shinn-Yih Huang (Taiwan) Masaki Watanabe (Japan) Peng Shi (Canada) Yun Pui Tsoi (Hong Kong) Teeradej Kittipassorn (Thailand) Graham White (Australia) Daulet Zhanguzin (Kazakhstan) Issac Buenrostro (Mexico) German Gieczewski (Argentina) Franco Vargas (Peru) Martin Spencer (New Zealand)	Alexander Zhal (USA) Ga Hye Jung (Korea) Pi-Hsun Shih (Taiwan) Tukasa Kuribayashi (Japan) Jennifer Park (Canada) Chiu Wai Wong (Hong Kong) Tanawit Saesue (Thailand) Mark Norrish (Australia) Nurbol Duysenbaev (Kazakhstan) Joshua Ivan Hernández (Mexico) Wei Quan Lim (Singapore) José Alejandro Samper (Colombia) Andrew Jonathan (Indonesia)
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Honourable Mention	Adam Hesterberg (USA) Tony Liu (USA) Whan Ghang (Korea) Yi-Hsien Liu (Taiwan) Cedric Lin (Taiwan) Tomohiro Mitani (Japan) Richard Peng (Canada) Alex Remorov (Canada) Cheuk Ting Li (Hong Kong) Warut Suksompong (Thailand) Bodin Ponvilawan (Thailand) Charles Li (Australia) Erbol Palzhanov (Kazakhstan) Aizhan Mukataeva (Kazakhstan) Aaron Escalera (Mexico) Rodrigo Mendoza (Mexico) Nicolas Komanski (Argentina) Emilio Martinez (Argentina) Hao Chuien Hang (Singapore) Kuen-Yew, Bryan Hooi (Singapore) Miguel Acosta (Colombia) Santiago Cuellar (Colombia) Ronald Chan (New Zealand) Rupert Nelson (New Zealand) Wei-Jian Chuah (Malaysia) Maerzad Mohd Nazmi (Malaysia) David Narvaez (Panama) Antonio Fan (Panama) Fernando Gomez (Ecuador)	Jason Larue (USA) Young Seok Kim (Korea) Ji Gu Kim (Korea) Yi-Wei Chan (Taiwan) Kazuhiro Hosaka (Japan) Teruhisa Koshikawa (Japan) Allen Zhang (Canada) Kwun Tat Chan (Hong Kong) Ka Wai NG (Hong Kong) Sarita Bunsupha (Thailand) Vinh Pham (Australia) Giles Gardam (Australia) Viktor Sosin (Kazakhstan) Valente Ramirez (Mexico) Fernando Campos (Mexico) Jan Contreras (Mexico) Nicolas Ponieman (Argentina) Fernando Vidal (Argentina) Jun Ren Lim (Singapore) Feng Zhu (Singapore) Fabián Prada (Colombia) Federico Castillo (Colombia) James Liley (New Zealand) Toto Suryo Efar (Indonesia) Tee-Jin Cheong (Malaysia) Yu Wei Lim (Malaysia) Luis Pena (Panama) Sebastian Libedinsky (Chile)

XIX Asian Pacific Mathematics Olympiad

Time allowed: 4 hours
Each problem is worth 7 points

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Problem 1. Let S be a set of 9 distinct integers all of whose prime factors are at most 3. Prove that S contains 3 distinct integers such that their product is a perfect cube.

Problem 2. Let ABC be an acute angled triangle with $\angle BAC = 60^\circ$ and $AB > AC$. Let I be the incenter, and H the orthocenter of the triangle ABC. Prove that

$$2\angle AHI = 3\angle ABC.$$

Problem 3. Consider n disks $C_1, C_2, \ldots, C_n$ in a plane such that for each $1 \le i < n$, the center of C_i is on the circumference of C_{i+1}, and the center of C_n is on the circumference of C_1. Define the *score* of such an arrangement of n disks to be the number of pairs (i, j) for which C_i properly contains C_j. Determine the maximum possible score.

Problem 4. Let x, y and z be positive real numbers such that $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$. Prove that

$$\frac{x^2 + yz}{\sqrt{2x^2(y+z)}} + \frac{y^2 + zx}{\sqrt{2y^2(z+x)}} + \frac{z^2 + xy}{\sqrt{2z^2(x+y)}} \ge 1.$$

Problem 5. A regular (5×5)-array of lights is defective, so that toggling the switch for one light causes each adjacent light in the same row and in the same column as well as the light itself to change state, from on to off, or from off to on. Initially all the lights are switched off. After a certain number of toggles, exactly one light is switched on. Find all the possible positions of this light.

XIX Asian Pacific Mathematics Olympiad

Problem 1. Let S be a set of 9 distinct integers all of whose prime factors are at most 3. Prove that S contains 3 distinct integers such that their product is a perfect cube.

Solution. Without loss of generality, we may assume that S contains only positive integers. Let

$$S = \{2^{a_i}3^{b_i} \mid a_i, b_i \in \mathbb{Z},\ a_i, b_i \geq 0,\ 1 \leq i \leq 9\}.$$

It suffices to show that there are $1 \leq i_1, i_2, i_3 \leq 9$ such that

$$a_{i_1} + a_{i_2} + a_{i_3} \equiv b_{i_1} + b_{i_2} + b_{i_3} \equiv 0 \pmod 3. \qquad (\dagger)$$

For $n = 2^a 3^b \in S$, let's call $(a \pmod 3, b \pmod 3)$ the *type* of n. Then there are 9 possible types :

$$(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2).$$

Let $N(i,j)$ be the number of integers in S of type (i,j). We obtain 3 distinct integers whose product is a perfect cube when

(1) $N(i,j) \geq 3$ for some i, j, or

(2) $N(i,0)N(i,1)N(i,2) \neq 0$ for some $i = 0,1,2$, or

(3) $N(0,j)N(1,j)N(2,j) \neq 0$ for some $j = 0,1,2$, or

(4) $N(i_1,j_1)N(i_2,j_2)N(i_3,j_3) \neq 0$, where $\{i_1,i_2,i_3\} = \{j_1,j_2,j_3\} = \{0,1,2\}$.

Assume that none of the conditions (1)∼(3) holds. Since $N(i,j) \leq 2$ for all (i,j), there are at least five $N(i,j)$'s that are nonzero. Furthermore, among those nonzero $N(i,j)$'s, no three have the same i nor the same j. Using these facts, one may easily conclude that the condition (4) should hold. (For example, if one places each nonzero $N(i,j)$ in the (i,j)-th box of a regular 3×3 array of boxes whose rows and columns are indexed by 0,1 and 2, then one can always find three boxes, occupied by at least one nonzero $N(i,j)$, whose rows and columns are all distinct. This implies (4).)

Second solution. Up to (†), we do the same as above and get 9 possible types:

$$(a \pmod 3), b \pmod 3)) = (0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)$$

for $n = 2^a 3^b \in S$.

Note that (i) among any 5 integers, there exist 3 whose sum is 0 (mod 3), and that (ii) if $i, j, k \in \{0,1,2\}$, then $i+j+k \equiv 0 \pmod 3$ if and only if $i = j = k$ or $\{i,j,k\} = \{0,1,2\}$.

Let's define

T: the set of types of the integers in S;
$N(i)$: the number of integers in S of the type $(i, \cdot)$;
$M(i)$: the number of integers $j \in \{0,1,2\}$ such that $(i,j) \in T$.

If $N(i) \geq 5$ for some i, the result follows from (i). Otherwise, for some permutation (i,j,k) of $(0,1,2)$,

$$N(i) \geq 3, \quad N(j) \geq 3, \quad N(k) \geq 1.$$

If $M(i)$ or $M(j)$ is 1 or 3, the result follows from (ii). Otherwise $M(i) = M(j) = 2$. Then either

$$(i,x),(i,y),(j,x),(j,y) \in T \quad \text{or} \quad (i,x),(i,y),(j,x),(j,z) \in T$$

for some permutation (x,y,z) of $(0,1,2)$. Since $N(k) \geq 1$, at least one of (k,x), (k,y) and (k,z) contained in T. Therefore, in any case, the result follows from (ii). (For example, if $(k,y) \in T$, then take $(i,y),(j,y),(k,y)$ or $(i,x),(j,z),(k,y)$ from T.)

Problem 2. Let ABC be an acute angled triangle with $\angle BAC = 60^\circ$ and $AB > AC$. Let I be the incenter, and H the orthocenter of the triangle ABC. Prove that

$$2\angle AHI = 3\angle ABC.$$

Solution. Let D be the intersection point of the lines AH and BC. Let K be the intersection point of the circumcircle O of the triangle ABC and the line AH. Let the line through I perpendicular to BC meet BC and the minor arc BC of the circumcircle O at E and N, respectively. We have

$$\angle BIC = 180^\circ - (\angle IBC + \angle ICB) = 180^\circ - \frac{1}{2}(\angle ABC + \angle ACB) = 90^\circ + \frac{1}{2}\angle BAC = 120^\circ$$

and also $\angle BNC = 180^\circ - \angle BAC = 120^\circ = \angle BIC$. Since $IN \perp BC$, the quadrilateral $BICN$ is a kite and thus $IE = EN$.

Now, since H is the orthocenter of the triangle ABC, $HD = DK$. Also because $ED \perp IN$ and $ED \perp HK$, we conclude that $IHKN$ is an isosceles trapezoid with $IH = NK$.

Hence

$$\angle AHI = 180^\circ - \angle IHK = 180^\circ - \angle AKN = \angle ABN.$$

Since $IE = EN$ and $BE \perp IN$, the triangles IBE and NBE are congruent. Therefore

$$\angle NBE = \angle IBE = \angle IBC = \angle IBA = \frac{1}{2}\angle ABC$$

and thus

$$\angle AHI = \angle ABN = \frac{3}{2}\angle ABC.$$

Second solution. Let P, Q and R be the intersection points of BH, CH and AH with AC, AB and BC, respectively. Then we have $\angle IBH = \angle ICH$. Indeed,

$$\angle IBH = \angle ABP - \angle ABI = 30^\circ - \frac{1}{2}\angle ABC$$

and

$$\angle ICH = \angle ACI - \angle ACH = \frac{1}{2}\angle ACB - 30^\circ = 30^\circ - \frac{1}{2}\angle ABC,$$

because $\angle ABH = \angle ACH = 30^\circ$ and $\angle ACB + \angle ABC = 120^\circ$. (Note that $\angle ABP > \angle ABI$ and $\angle ACI > \angle ACH$ because AB is the longest side of the triangle ABC under the given conditions.) Therefore $BIHC$ is a cyclic quadrilateral and thus

$$\angle BHI = \angle BCI = \frac{1}{2}\angle ACB.$$

On the other hand,

$$\angle BHR = 90^\circ - \angle HBR = 90^\circ - (\angle ABC - \angle ABH) = 120^\circ - \angle ABC.$$

Therefore,

$$\begin{aligned}\angle AHI &= 180^\circ - \angle BHI - \angle BHR = 60^\circ - \frac{1}{2}\angle ACB + \angle ABC \\ &= 60^\circ - \frac{1}{2}(120^\circ - \angle ABC) + \angle ABC = \frac{3}{2}\angle ABC.\end{aligned}$$

Problem 3. Consider n disks $C_1, C_2, \ldots, C_n$ in a plane such that for each $1 \leq i < n$, the center of C_i is on the circumference of C_{i+1}, and the center of C_n is on the circumference of C_1. Define the *score* of such an arrangement of n disks to be the number of pairs (i, j) for which C_i properly contains C_j. Determine the maximum possible score.

Solution. The answer is $(n-1)(n-2)/2$.

Let's call a set of n disks satisfying the given conditions an *n-configuration.* For an n-configuration $\mathcal{C} = \{C_1, \ldots, C_n\}$, let $S_{\mathcal{C}} = \{(i, j) \mid C_i \text{ properly contains } C_j\}$. So, the score of an n-configuration $\mathcal{C}$ is $|S_{\mathcal{C}}|$.

We'll show that (i) there is an n-configuration $\mathcal{C}$ for which $|S_{\mathcal{C}}| = (n-1)(n-2)/2$, and that (ii) $|S_{\mathcal{C}}| \leq (n-1)(n-2)/2$ for any n-configuration $\mathcal{C}$.

Let C_1 be any disk. Then for $i = 2, \ldots, n-1$, take C_i inside C_{i-1} so that the circumference of C_i contains the center of C_{i-1}. Finally, let C_n be a disk whose center is on the circumference of C_1 and whose circumference contains the center of C_{n-1}. This gives $S_{\mathcal{C}} = \{(i, j) \mid 1 \leq i < j \leq n-1\}$ of size $(n-1)(n-2)/2$, which proves (i).

For any n-configuration $\mathcal{C}$, $S_{\mathcal{C}}$ must satisfy the following properties:

(1) $(i, i) \notin S_{\mathcal{C}}$,

(2) $(i+1, i) \notin S_{\mathcal{C}}$, $(1, n) \notin S_{\mathcal{C}}$,

(3) if $(i, j), (j, k) \in S_{\mathcal{C}}$, then $(i, k) \in S_{\mathcal{C}}$,

(4) if $(i, j) \in S_{\mathcal{C}}$, then $(j, i) \notin S_{\mathcal{C}}$.

Now we show that a set G of ordered pairs of integers between 1 and n, satisfying the conditions (1)$\sim$(4), can have no more than $(n-1)(n-2)/2$ elements. Suppose that there exists a set G that satisfies the conditions (1)$\sim$(4), and has more than $(n-1)(n-2)/2$ elements. Let n be the least positive integer with which there exists such a set G. Note that G must have $(i, i+1)$ for some $1 \leq i \leq n$ or $(n, 1)$, since otherwise G can have at most

$$\binom{n}{2} - n = \frac{n(n-3)}{2} < \frac{(n-1)(n-2)}{2}$$

elements. Without loss of generality we may assume that $(n, 1) \in G$. Then $(1, n-1) \notin G$, since otherwise the condition (3) yields $(n, n-1) \in G$ contradicting the condition (2). Now let $G' = \{(i, j) \in G \mid 1 \leq i, j \leq n-1\}$, then G' satisfies the conditions (1)$\sim$(4), with $n-1$.

We now claim that $|G - G'| \leq n-2$:

Suppose that $|G - G'| > n-2$, then $|G - G'| = n-1$ and hence for each $1 \leq i \leq n-1$, either (i, n) or (n, i) must be in G. We already know that $(n, 1) \in G$ and $(n-1, n) \in G$ (because $(n, n-1) \notin G$) and this implies that $(n, n-2) \notin G$ and $(n-2, n) \in G$. If we keep doing this process, we obtain $(1, n) \in G$, which is a contradiction.

Since $|G - G'| \leq n - 2$, we obtain

$$|G'| \geq \frac{(n-1)(n-2)}{2} - (n-2) = \frac{(n-2)(n-3)}{2}.$$

This, however, contradicts the minimality of n, and hence proves (ii).

Problem 4. Let x, y and z be positive real numbers such that $\sqrt{x}+\sqrt{y}+\sqrt{z}=1$. Prove that

$$\frac{x^2+yz}{\sqrt{2x^2(y+z)}}+\frac{y^2+zx}{\sqrt{2y^2(z+x)}}+\frac{z^2+xy}{\sqrt{2z^2(x+y)}}\geq 1.$$

Solution. We first note that

$$\begin{aligned}\frac{x^2+yz}{\sqrt{2x^2(y+z)}} &= \frac{x^2-x(y+z)+yz}{\sqrt{2x^2(y+z)}}+\frac{x(y+z)}{\sqrt{2x^2(y+z)}}\\ &= \frac{(x-y)(x-z)}{\sqrt{2x^2(y+z)}}+\sqrt{\frac{y+z}{2}}\\ &\geq \frac{(x-y)(x-z)}{\sqrt{2x^2(y+z)}}+\frac{\sqrt{y}+\sqrt{z}}{2}. \qquad (1)\end{aligned}$$

Similarly, we have

$$\frac{y^2+zx}{\sqrt{2y^2(z+x)}}\geq\frac{(y-z)(y-x)}{\sqrt{2y^2(z+x)}}+\frac{\sqrt{z}+\sqrt{x}}{2}, \qquad (2)$$

$$\frac{z^2+xy}{\sqrt{2z^2(x+y)}}\geq\frac{(z-x)(z-y)}{\sqrt{2z^2(x+y)}}+\frac{\sqrt{x}+\sqrt{y}}{2}. \qquad (3)$$

We now add (1)$\sim$(3) to get

$$\begin{aligned}&\frac{x^2+yz}{\sqrt{2x^2(y+z)}}+\frac{y^2+zx}{\sqrt{2y^2(z+x)}}+\frac{z^2+xy}{\sqrt{2z^2(x+y)}}\\ &\geq \frac{(x-y)(x-z)}{\sqrt{2x^2(y+z)}}+\frac{(y-z)(y-x)}{\sqrt{2y^2(z+x)}}+\frac{(z-x)(z-y)}{\sqrt{2z^2(x+y)}}+\sqrt{x}+\sqrt{y}+\sqrt{z}\\ &= \frac{(x-y)(x-z)}{\sqrt{2x^2(y+z)}}+\frac{(y-z)(y-x)}{\sqrt{2y^2(z+x)}}+\frac{(z-x)(z-y)}{\sqrt{2z^2(x+y)}}+1.\end{aligned}$$

Thus, it suffices to show that

$$\frac{(x-y)(x-z)}{\sqrt{2x^2(y+z)}}+\frac{(y-z)(y-x)}{\sqrt{2y^2(z+x)}}+\frac{(z-x)(z-y)}{\sqrt{2z^2(x+y)}}\geq 0. \qquad (4)$$

Now, assume without loss of generality, that $x\geq y\geq z$. Then we have

$$\frac{(x-y)(x-z)}{\sqrt{2x^2(y+z)}}\geq 0$$

and

$$\frac{(z-x)(z-y)}{\sqrt{2z^2(x+y)}}+\frac{(y-z)(y-x)}{\sqrt{2y^2(z+x)}}=\frac{(y-z)(x-z)}{\sqrt{2z^2(x+y)}}-\frac{(y-z)(x-y)}{\sqrt{2y^2(z+x)}}$$
$$\geq \frac{(y-z)(x-y)}{\sqrt{2z^2(x+y)}}-\frac{(y-z)(x-y)}{\sqrt{2y^2(z+x)}}=(y-z)(x-y)\left(\frac{1}{\sqrt{2z^2(x+y)}}-\frac{1}{\sqrt{2y^2(z+x)}}\right).$$

The last quantity is non-negative due to the fact that

$$y^2(z+x)=y^2z+y^2x\geq yz^2+z^2x=z^2(x+y).$$

This completes the proof.

Second solution. By Cauchy-Schwarz inequality,

$$\left(\frac{x^2}{\sqrt{2x^2(y+z)}}+\frac{y^2}{\sqrt{2y^2(z+x)}}+\frac{z^2}{\sqrt{2z^2(x+y)}}\right)$$
$$\times\left(\sqrt{2(y+z)}+\sqrt{2(z+x)}+\sqrt{2(x+y)}\right)\geq(\sqrt{x}+\sqrt{y}+\sqrt{z})^2=1, \tag{5}$$

and

$$\left(\frac{yz}{\sqrt{2x^2(y+z)}}+\frac{zx}{\sqrt{2y^2(z+x)}}+\frac{xy}{\sqrt{2z^2(x+y)}}\right)$$
$$\times\left(\sqrt{2(y+z)}+\sqrt{2(z+x)}+\sqrt{2(x+y)}\right)\geq\left(\sqrt{\frac{yz}{x}}+\sqrt{\frac{zx}{y}}+\sqrt{\frac{xy}{z}}\right)^2. \tag{6}$$

We now combine (5) and (6) to find

$$\left(\frac{x^2+yz}{\sqrt{2x^2(y+z)}}+\frac{y^2+zx}{\sqrt{2y^2(z+x)}}+\frac{z^2+xy}{\sqrt{2z^2(x+y)}}\right)$$
$$\times\left(\sqrt{2(x+y)}+\sqrt{2(y+z)}+\sqrt{2(z+x)}\right)$$
$$\geq 1+\left(\sqrt{\frac{yz}{x}}+\sqrt{\frac{zx}{y}}+\sqrt{\frac{xy}{z}}\right)^2\geq 2\left(\sqrt{\frac{yz}{x}}+\sqrt{\frac{zx}{y}}+\sqrt{\frac{xy}{z}}\right).$$

Thus, it suffices to show that

$$2\left(\sqrt{\frac{yz}{x}}+\sqrt{\frac{zx}{y}}+\sqrt{\frac{xy}{z}}\right)\geq\sqrt{2(y+z)}+\sqrt{2(z+x)}+\sqrt{2(x+y)}. \tag{7}$$

Consider the following inequality using AM-GM inequality

$$\left[\sqrt{\frac{yz}{x}}+\left(\frac{1}{2}\sqrt{\frac{zx}{y}}+\frac{1}{2}\sqrt{\frac{xy}{z}}\right)\right]^2\geq 4\sqrt{\frac{yz}{x}}\left(\frac{1}{2}\sqrt{\frac{zx}{y}}+\frac{1}{2}\sqrt{\frac{xy}{z}}\right)=2(y+z),$$

or equivalently

$$\sqrt{\frac{yz}{x}}+\left(\frac{1}{2}\sqrt{\frac{zx}{y}}+\frac{1}{2}\sqrt{\frac{xy}{z}}\right)\geq\sqrt{2(y+z)}\,.$$

Similarly, we have

$$\sqrt{\frac{zx}{y}}+\left(\frac{1}{2}\sqrt{\frac{xy}{z}}+\frac{1}{2}\sqrt{\frac{yz}{x}}\right)\geq\sqrt{2(z+x)}\,,$$
$$\sqrt{\frac{xy}{z}}+\left(\frac{1}{2}\sqrt{\frac{yz}{x}}+\frac{1}{2}\sqrt{\frac{zx}{y}}\right)\geq\sqrt{2(x+y)}\,.$$

Adding the last three inequalities, we get

$$2\left(\sqrt{\frac{yz}{x}}+\sqrt{\frac{zx}{y}}+\sqrt{\frac{xy}{z}}\right)\geq\sqrt{2(y+z)}+\sqrt{2(z+x)}+\sqrt{2(x+y)}\,.$$

This completes the proof.

Problem 5. A regular (5×5)-array of lights is defective, so that toggling the switch for one light causes each adjacent light in the same row and in the same column as well as the light itself to change state, from on to off, or from off to on. Initially all the lights are switched off. After a certain number of toggles, exactly one light is switched on. Find all the possible positions of this light.

Solution. We assign the following *first labels* to the 25 positions of the lights:

1	1	0	1	1
0	0	0	0	0
1	1	0	1	1
0	0	0	0	0
1	1	0	1	1

For each on-off combination of lights in the array, define its *first value* to be the sum of the first labels of those positions at which the lights are switched on. It is easy to check that toggling any switch always leads to an on-off combination of lights whose first value has the same parity(the remainder when divided by 2) as that of the previous on-off combination.

The $90°$ rotation of the first labels gives us another labels (let us call it the *second labels*) which also makes the parity of the *second value*(the sum of the second labels of those positions at which the lights are switched on) invariant under toggling.

1	0	1	0	1
1	0	1	0	1
0	0	0	0	0
1	0	1	0	1
1	0	1	0	1

Since the parity of the first and the second values of the initial status is 0, after certain number of toggles the parity must remain unchanged with respect to the first labels and the second labels as well. Therefore, if exactly one light is on after some number of toggles, the label of that position must be 0 with respect to both labels. Hence according to the above pictures, the possible positions are the ones marked with $*_i$'s in the following picture:

	$*_2$		$*_1$	
		$*_0$		
	$*_3$		$*_4$	

Now we demonstrate that all five positions are possible:

Toggling the positions checked by t (the order of toggling is irrelevant) in the first picture makes the center($*_0$) the only position with light on and the second picture makes the position $*_1$ the only position with light on. The other $*_i$'s can be obtained by rotating the second picture appropriately.

			t	t
		t		
	t	t		t
t				t
t		t	t	

	t		t	
t	t		t	t
	t			
		t	t	t
			t	

AWARD	STUDENT NAME	COUNTRY
Gold Award	Ignacio Rossi	Argentina
	Dimitri Dziabenko	Canada
	Daniel Campos	Costa Rica
	Chiu Wai Wong	Hongkong
	Masaki Watanabe	Japan
	Je ha Yang	Korea
	Jia Han Chiam	Singapore
	Pao-Yu Chien	Taiwan
	Pasin Manurangsi	Thailand
	Arnav Tripathy	USA
Silver Award	Mavro Schilman	Argentina
	Julian Eisenschlos	Argentina
	Andrew Elvey Price	Australia
	Danny Shi	Canada
	Jarno Sun	Canada
	Diego Cifuentes	Colombia
	Miguel Acosta	Colombia
	Fernando Gomez	Ecuador
	Jack Hui	Hongkong
	Pok Wai Fong	Hongkong
	Han Beng Koe	Indonesia
	Raymond Christopher Sitorus	Indonesia
	Toshiki Kataoka	Japan
	Makoto Soejima	Japan
	Igor Stasiy	Kazakhstan
	Alen bayev	Kazakhstan
	Jin a Lim	Korea
	Jung ho Koh	Korea
	Isaac Buenrostro	Mexico
	Rupert Nelson	New Zealand
	Daniel Soncco	Peru
	Gar Goei Loke	Singapore
	Dominic Lee	Singapore
	Yi-Hsiu Chen	Taiwan
	Kuan-Yu Lin	Taiwan
	Paphon Kiatsakuldecha	Thailand
	Tanapong Jarusirupipat	Thailand
	Tony Lia	USA
	Adam Hesterberg	USA
	Fernando Vidal	Argentina
	Magall Giaroll	Argentina
	Nicolas Rodriguez Castro	Argentina
	Roberto Morales	Argentina
	Vathryn Zealand	Australia
	Max Menzies	Australia

Bronze Award	Anthony Morris	Australia
	Sen Lin	Australia
	Ildar Gasin	Australia
	Sampson Wong	Australia
	Steven Karp	Canada
	Kent Huynh	Canada
	Johnson Mo	Canada
	Julian Sun	Canada
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	Chak Hei Yuen	Hongkong
	Ping Neai Chung	Hongkong
	Cho Ho Lam	Hongkong
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	Nugroho Seto Saputra	Indonesia
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	Yosuke Morita	Japan
	Yuta Ohashi	Japan
	Kazuhiro Hosaka	Japan
	Yeskendir Kassenov	Kazakhstan
	Nazerke Bakhytzhan	Kazakhstan
	Aydyn Kunekeyev	Kazakhstan
	Aset Mussagaliev	Kazakhstan
	Kyeong sik Nam	Korea
	Seok hyeong Lee	Korea
	Yo han Lee	Korea
	Ho min Lee	Korea
	Andrés Góneg	Mexico
	Fernando Canpos	Mexico
	Erick Gallegos	Mexico
	Marco Avila	Mexico
	Cristian Oliva	Mexico
	Manuel Novelo	Mexico
	Tomas Angles	Peru
	Dennis Alaro	Peru
	Mingyan Simon Lin	Singapore
	Rachabattuni Chaitanya	Singapore
	Yi-Wei Chan	Taiwan
	Pi-Hsun Shih	Taiwan
	Po-Hsiang Hao	Taiwan
	Lun-Kai Hsu	Taiwan
	PanuPong Pasupat	Thailand
	Suthee Ruangwises	Thailand
	Wijit Youngjit	Thailand
	Pongpak Bhumiwat	Thailand
	Chandresh Ramlagan	Trinidad and Tobago
	Shaunak Kishorz	USA
	Alex Zhai	USA

	Michael Gottlieb	USA
	Jeremy Haha	USA
Honourable Mention	Pablo Zimmermann	Argentina
	Federico Cogornd	Argentina
	Joe Kileel	Canada
	Yan Li	Canada
	Max Zhou	Canada
	Pablo Serrano	Ecuador
	Sze Wai Wong	Hongkong
	On Yip Chung	Hongkong
	Pak Hw Lee	Hongkong
	Andika Sutanto	Indonesia
	Hilman Fathurrahman	Indonesia
	Nurfitri Anbarsanti	Indonesia
	Kosuke Suzuki	Japan
	Yuki Kawaguchi	Japan
	Yuki Yoshida	Japan
	Sanzhar Orazbayev	Kazakhstan
	Yerker Tusupbekov	Kazakhstan
	Azat Utepbayev	Kazakhstan
	Ga hye Jeong	Korea
	Whan Ghang	Korea
	Soo hong Lee	Korea
	Eduardo Velasco	Mexico
	Malors Espinosa	Mexico
	Ilya Chevyrev	New Zealand
	Juan Chong	Panama
	Antonio Fan	Panama
	Lili Luo	Panama
	Muhammad Fahd Waseem	Pakistan
	Mohsin Ali Khan	Pakistan
	Wilbert Pumacay	Peru
	Cesar Cuenca	Peru
	Eduardo salas	Peru
	Lei Lei	Singapore
	Chong Min Tan	Singapore
	Pei-Lun Kuo	Taiwan
	Kuan-Chieh Liao	Taiwan
	Yu-Wei Fan	Taiwan
	Potcharapol Suteparuk	Thailand
	Warut Suksompong	Thailand
	Wich Huengwattanakul	Thailand
	Vinod Sookram	Trinidad and Tobago
	Tedrick Leung	USA
	Zarathastra Brady	USA
	Seungsoo Kim	USA

XX Asian Pacific Mathematics Olympiad

March, 2008

Time allowed: 4 hours
Each problem is worth 7 points

∗ *The contest problems are to be kept confidential until they are posted on the official APMO website. Please do not disclose nor discuss the problems over the internet until that date. No calculators are to be used during the contest.*

Problem 1. Let ABC be a triangle with $\angle A < 60^\circ$. Let X and Y be the points on the sides AB and AC, respectively, such that $CA + AX = CB + BX$ and $BA + AY = BC + CY$. Let P be the point in the plane such that the lines PX and PY are perpendicular to AB and AC, respectively. Prove that $\angle BPC < 120^\circ$.

Problem 2. Students in a class form groups each of which contains exactly three members such that any two distinct groups have at most one member in common. Prove that, when the class size is 46, there is a set of 10 students in which no group is properly contained.

Problem 3. Let Γ be the circumcircle of a triangle ABC. A circle passing through points A and C meets the sides BC and BA at D and E, respectively. The lines AD and CE meet Γ again at G and H, respectively. The tangent lines of Γ at A and C meet the line DE at L and M, respectively. Prove that the lines LH and MG meet at Γ.

Problem 4. Consider the function $f : \mathbb{N}_0 \to \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of all non-negative integers, defined by the following conditions:

$$\text{(i) } f(0) = 0, \text{ (ii) } f(2n) = 2f(n) \text{ and (iii) } f(2n+1) = n + 2f(n) \text{ for all } n \geq 0.$$

(a) Determine the three sets $L := \{\, n \mid f(n) < f(n+1) \,\}$, $E := \{\, n \mid f(n) = f(n+1) \,\}$, and $G := \{\, n \mid f(n) > f(n+1) \,\}$.

(b) For each $k \geq 0$, find a formula for $a_k := \max\{f(n) : 0 \leq n \leq 2^k\}$ in terms of k.

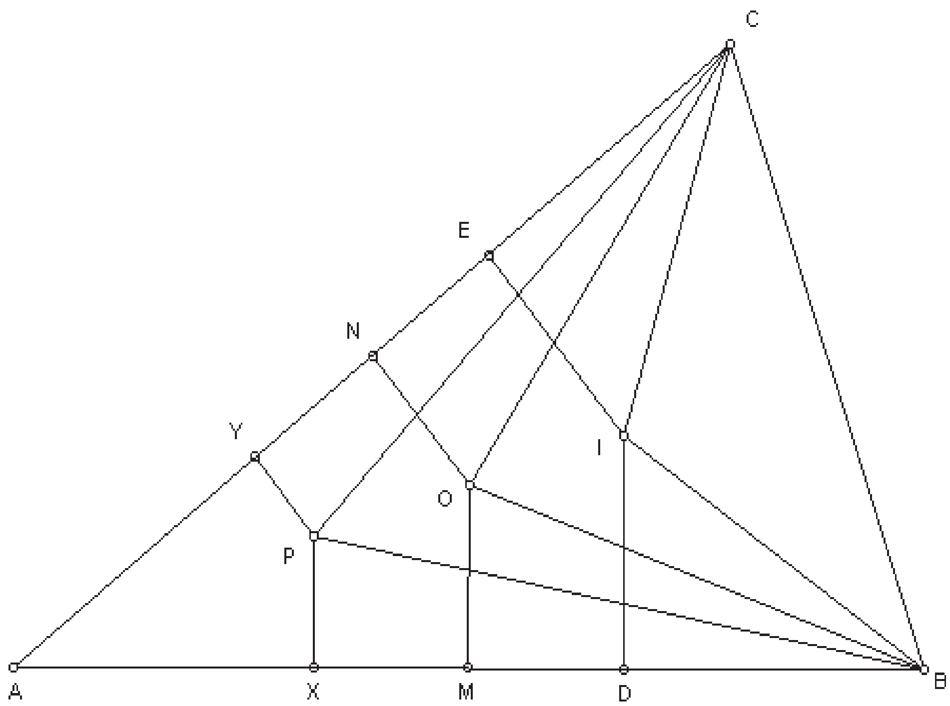
Problem 5. Let a, b, c be integers satisfying $0 < a < c - 1$ and $1 < b < c$. For each k, $0 \leq k \leq a$, let r_k, $0 \leq r_k < c$, be the remainder of kb when divided by c. Prove that the two sets $\{r_0, r_1, r_2, \ldots, r_a\}$ and $\{0, 1, 2, \ldots, a\}$ are different.

XX Asian Pacific Mathematics Olympiad

March, 2008

Problem 1. Let ABC be a triangle with $\angle A < 60^\circ$. Let X and Y be the points on the sides AB and AC, respectively, such that $CA + AX = CB + BX$ and $BA + AY = BC + CY$. Let P be the point in the plane such that the lines PX and PY are perpendicular to AB and AC, respectively. Prove that $\angle BPC < 120^\circ$.

(Solution) Let I be the incenter of $\triangle ABC$, and let the feet of the perpendiculars from I to AB and to AC be D and E, respectively. (Without loss of generality, we may assume that AC is the longest side. Then X lies on the line segment AD. Although P may or may not lie inside $\triangle ABC$, the proof below works for both cases. Note that P is on the line perpendicular to AB passing through X.) Let O be the midpoint of IP, and let the feet of the perpendiculars from O to AB and to AC be M and N, respectively. Then M and N are the midpoints of DX and EY, respectively.



The conditions on the points X and Y yield the equations

$$AX = \frac{AB + BC - CA}{2} \qquad \text{and} \qquad AY = \frac{BC + CA - AB}{2}.$$

From $AD = AE = \dfrac{CA + AB - BC}{2}$, we obtain

$$BD = AB - AD = AB - \frac{CA + AB - BC}{2} = \frac{AB + BC - CA}{2} = AX.$$

Since M is the midpoint of DX, it follows that M is the midpoint of AB. Similarly, N is the midpoint of AC. Therefore, the perpendicular bisectors of AB and AC meet at O, that is, O is the circumcenter of $\triangle ABC$. Since $\angle BAC < 60°$, O lies on the same side of BC as the point A and

$$\angle BOC = 2\angle BAC.$$

We can compute $\angle BIC$ as follows:

$$\begin{aligned} \angle BIC &= 180° - \angle IBC - \angle ICB = 180° - \frac{1}{2}\angle ABC - \frac{1}{2}\angle ACB \\ &= 180° - \frac{1}{2}(\angle ABC + \angle ACB) = 180° - \frac{1}{2}(180° - \angle BAC) = 90° + \frac{1}{2}\angle BAC \end{aligned}$$

It follows from $\angle BAC < 60°$ that

$$2\angle BAC < 90° + \frac{1}{2}\angle BAC, \quad \text{i.e.,} \quad \angle BOC < \angle BIC.$$

From this it follows that I lies inside the circumcircle of the isosceles triangle BOC because O and I lie on the same side of BC. However, as O is the midpoint of IP, P must lie outside the circumcircle of triangle BOC and on the same side of BC as O. Therefore

$$\angle BPC < \angle BOC = 2\angle BAC < 120°.$$

Remark. If one assumes that $\angle A$ is smaller than the other two, then it is clear that the line PX (or the line perpendicular to AB at X if $P = X$) runs through the excenter I_C of the excircle tangent to the side AB. Since $2\angle ACI_C = \angle ACB$ and $BC < AC$, we have $2\angle PCB > \angle C$. Similarly, $2\angle PBC > \angle B$. Therefore,

$$\angle BPC = 180° - (\angle PBC + \angle PCB) < 180° - \left(\frac{\angle B + \angle C}{2}\right) = 90 + \frac{\angle A}{2} < 120°.$$

In this way, a special case of the problem can be easily proved.

Problem 2. Students in a class form groups each of which contains exactly three members such that any two distinct groups have at most one member in common. Prove that, when the class size is 46, there is a set of 10 students in which no group is properly contained.

(Solution) We let C be the set of all 46 students in the class and let

$$s := \max\{\,|S| \,:\, S \subseteq C \text{ such that } S \text{ contains no group properly}\,\}.$$

Then it suffices to prove that $s \geq 10$. (If $|S| = s > 10$, we may choose a subset of S consisting of 10 students.)

Suppose that $s \leq 9$ and let S be a set of size s in which no group is properly contained. Take any student, say v, from outside S. Because of the maximality of s, there should be a group containing the student v and two other students in S. The number of ways to choose two students from S is

$$\binom{s}{2} \leq \binom{9}{2} = 36.$$

On the other hand, there are at least $37 = 46 - 9$ students outside of S. Thus, among those 37 students outside, there is at least one student, say u, who does not belong to any group containing two students in S and one outside. This is because no two distinct groups have two members in common. But then, S can be enlarged by including u, which is a contradiction.

Remark. One may choose a subset S of C that contains no group properly. Then, assuming $|S| < 10$, prove that there is a student outside S, say u, who does not belong to any group containing two students in S. After enlarging S by including u, prove that the enlarged S still contains no group properly.

Problem 3. Let Γ be the circumcircle of a triangle ABC. A circle passing through points A and C meets the sides BC and BA at D and E, respectively. The lines AD and CE meet Γ again at G and H, respectively. The tangent lines of Γ at A and C meet the line DE at L and M, respectively. Prove that the lines LH and MG meet at Γ.

(Solution) Let MG meet Γ at P. Since $\angle MCD = \angle CAE$ and $\angle MDC = \angle CAE$, we have $MC = MD$. Thus

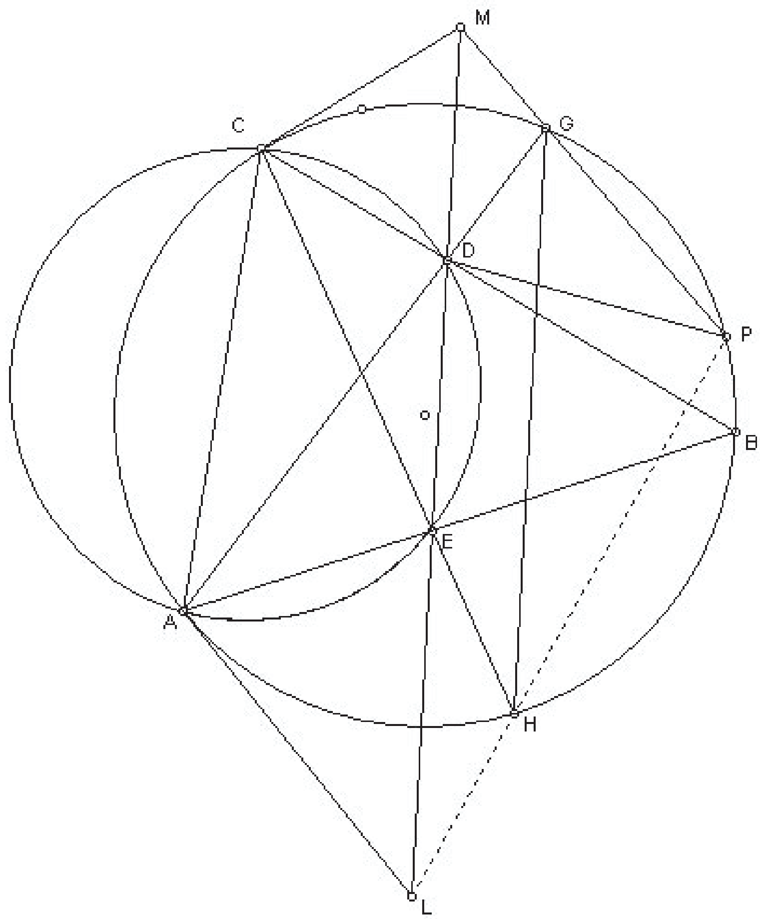
$$MD^2 = MC^2 = MG \cdot MP$$

and hence MD is tangent to the circumcircle of $\triangle DGP$. Therefore $\angle DGP = \angle EDP$.

Let Γ' be the circumcircle of $\triangle BDE$. If $B = P$, then, since $\angle BGD = \angle BDE$, the tangent lines of Γ' and Γ at B should coincide, that is Γ' is tangent to Γ from inside. Let $B \neq P$. If P lies in the same side of the line BC as G, then we have

$$\angle EDP + \angle ABP = 180^\circ$$

because $\angle DGP + \angle ABP = 180^\circ$. That is, the quadrilateral $BPDE$ is cyclic, and hence P is on the intersection of Γ' with Γ.



Otherwise,

$$\angle EDP = \angle DGP = \angle AGP = \angle ABP = \angle EBP.$$

Therefore the quadrilateral $PBDE$ is cyclic, and hence P again is on the intersection of Γ' with Γ.

Similarly, if LH meets Γ at Q, we either have $Q = B$, in which case Γ' is tangent to Γ from inside, or $Q \neq B$. In the latter case, Q is on the intersection of Γ' with Γ. In either case, we have $P = Q$.

Problem 4. Consider the function $f : \mathbb{N}_0 \to \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of all non-negative integers, defined by the following conditions:

$$\text{(i) } f(0) = 0, \text{ (ii) } f(2n) = 2f(n) \text{ and (iii) } f(2n+1) = n + 2f(n) \text{ for all } n \geq 0.$$

(a) Determine the three sets $L := \{ n \mid f(n) < f(n+1) \}$, $E := \{ n \mid f(n) = f(n+1) \}$, and $G := \{ n \mid f(n) > f(n+1) \}$.

(b) For each $k \geq 0$, find a formula for $a_k := \max\{f(n) : 0 \leq n \leq 2^k\}$ in terms of k.

(Solution) (a) Let

$$L_1 := \{2k : k > 0\}, \quad E_1 := \{0\} \cup \{4k+1 : k \geq 0\}, \quad \text{and} \quad G_1 := \{4k+3 : k \geq 0\}.$$

We will show that $L_1 = L, E_1 = E$, and $G_1 = G$. It suffices to verify that $L_1 \subseteq E$, $E_1 \subseteq E$, and $G_1 \subseteq G$ because L_1, E_1, and G_1 are mutually disjoint and $L_1 \cup E_1 \cup G_1 = \mathbb{N}_0$.

Firstly, if $k > 0$, then $f(2k) - f(2k+1) = -k < 0$ and therefore $L_1 \subseteq L$.

Secondly, $f(0) = 0$ and

$$\begin{aligned} f(4k+1) &= 2k + 2f(2k) = 2k + 4f(k) \\ f(4k+2) &= 2f(2k+1) = 2(k + 2f(k)) = 2k + 4f(k) \end{aligned}$$

for all $k \geq 0$. Thus, $E_1 \subseteq E$.

Lastly, in order to prove $G_1 \subset G$, we claim that $f(n+1) - f(n) \leq n$ for all n. (In fact, one can prove a stronger inequality: $f(n+1) - f(n) \leq n/2$.) This is clearly true for even n from the definition since for $n = 2t$,

$$f(2t+1) - f(2t) = t \leq n.$$

If $n = 2t+1$ is odd, then (assuming inductively that the result holds for all nonnegative $m < n$), we have

$$\begin{aligned} f(n+1) - f(n) &= f(2t+2) - f(2t+1) = 2f(t+1) - t - 2f(t) \\ &= 2(f(t+1) - f(t)) - t \leq 2t - t = t < n. \end{aligned}$$

For all $k \geq 0$,

$$\begin{aligned} &f(4k+4) - f(4k+3) = f(2(2k+2)) - f(2(2k+1)+1) \\ &= 4f(k+1) - (2k+1+2f(2k+1)) = 4f(k+1) - (2k+1+2k+4f(k)) \\ &= 4(f(k+1) - f(k)) - (4k+1) \leq 4k - (4k+1) < 0. \end{aligned}$$

This proves $G_1 \subseteq G$.

(b) Note that $a_0 = a_1 = f(1) = 0$. Let $k \geq 2$ and let $N_k = \{0, 1, 2, \ldots, 2^k\}$. First we claim that the maximum a_k occurs at the largest number in $G \cap N_k$, that is, $a_k = f(2^k - 1)$. We use mathematical induction on k to prove the claim. Note that $a_2 = f(3) = f(2^2 - 1)$.

Now let $k \geq 3$. For every even number $2t$ with $2^{k-1} + 1 < 2t \leq 2^k$,

$$f(2t) = 2f(t) \leq 2a_{k-1} = 2f(2^{k-1} - 1) \tag{†}$$

by induction hypothesis. For every odd number $2t+1$ with $2^{k-1} + 1 \leq 2t + 1 < 2^k$,

$$\begin{aligned} f(2t+1) &= t + 2f(t) \leq 2^{k-1} - 1 + 2f(t) \\ &\leq 2^{k-1} - 1 + 2a_{k-1} = 2^{k-1} - 1 + 2f(2^{k-1} - 1) \end{aligned} \tag{‡}$$

again by induction hypothesis. Combining (†), (‡) and

$$f(2^k - 1) = f(2(2^{k-1} - 1) + 1) = 2^{k-1} - 1 + 2f(2^{k-1} - 1),$$

we may conclude that $a_k = f(2^k - 1)$ as desired.

Furthermore, we obtain

$$a_k = 2a_{k-1} + 2^{k-1} - 1$$

for all $k \geq 3$. Note that this recursive formula for a_k also holds for $k \geq 0, 1$ and 2. Unwinding this recursive formula, we finally get

$$\begin{aligned}
a_k &= 2a_{k-1} + 2^{k-1} - 1 = 2(2a_{k-2} + 2^{k-2} - 1) + 2^{k-1} - 1 \\
&= 2^2 a_{k-2} + 2 \cdot 2^{k-1} - 2 - 1 = 2^2(2a_{k-3} + 2^{k-3} - 1) + 2 \cdot 2^{k-1} - 2 - 1 \\
&= 2^3 a_{k-3} + 3 \cdot 2^{k-1} - 2^2 - 2 - 1 \\
&\vdots \\
&= 2^k a_0 + k2^{k-1} - 2^{k-1} - 2^{k-2} - \ldots - 2 - 1 \\
&= k2^{k-1} - 2^k + 1 \quad \text{for all } k \geq 0.
\end{aligned}$$

Problem 5. Let a, b, c be integers satisfying $0 < a < c-1$ and $1 < b < c$. For each k, $0 \le k \le a$, let r_k, $0 \le r_k < c$, be the remainder of kb when divided by c. Prove that the two sets $\{r_0, r_1, r_2, \ldots, r_a\}$ and $\{0, 1, 2, \ldots, a\}$ are different.

(Solution) Suppose that two sets are equal. Then $\gcd(b, c) = 1$ and the polynomial

$$f(x) := (1 + x^b + x^{2b} + \cdots + x^{ab}) - (1 + x + x^2 + \cdots + x^{a-1} + x^a)$$

is divisible by $x^c - 1$. (This is because: $m = n + cq \implies x^m - x^n = x^{n+cq} - x^n = x^n(x^{cq} - 1)$ and $(x^{cq} - 1) = (x^c - 1)((x^c)^{q-1} + (x^c)^{q-2} + \cdots + 1)$.) From

$$f(x) = \frac{x^{(a+1)b} - 1}{x^b - 1} - \frac{x^{a+1} - 1}{x - 1} = \frac{F(x)}{(x-1)(x^b - 1)} \ ,$$

where $F(x) = x^{ab+b+1} + x^b + x^{a+1} - x^{ab+b} - x^{a+b+1} - x$, we have

$$F(x) \equiv 0 \pmod{x^c - 1} .$$

Since $x^c \equiv 1 \pmod{x^c - 1}$, we may conclude that

$$\{ab + b + 1, b, a + 1\} \equiv \{ab + b, a + b + 1, 1\} \pmod{c}. \qquad (\dagger)$$

Thus,

$$b \equiv \ ab + b, \ a + b + 1 \ \text{ or } \ 1 \pmod{c}.$$

But neither $b \equiv 1 \pmod{c}$ nor $b \equiv a + b + 1 \pmod{c}$ are possible by the given conditions. Therefore, $b \equiv ab + b \pmod{c}$. But this is also impossible because $\gcd(b, c) = 1$.

Award	Names of Contestant	Country
GOLD AWARD	Stefanich Germán	Argentina
	Cheung Paul	Australia
	Sun Chen	Canada
	Acosta Miguel	Colombia
	Wong Chiu Wai	Hong Kong
	Soejima Makoto	Japan
	Pacchiano Aldo	Mexico
	Lee Soo Hong	Republic of Korea
	Korb Dmitry	Russian Federation
	Chiam Jia-Han	Singapore
	Shih Pi-Hsun	Taiwan
	Ruangwises Suthee	Thailand
	Gauthier Gregory	USA
SILVER AWARD	Morales Roberto	Argentina
	Zimmermann Pablo	Argentina
	Elvey Price Andrew	Australia
	Gardam Giles	Australia
	Lvov Nikita	Canada
	Schneider Jonathan	Canada
	Paez Gustavo Nicolas	Colombia
	Han Qi	Ecuador
	Fok Pak Hei	Hong Kong
	Li Cheuk Ting	Hong Kong
	Obaja Aldrian	Indonesia
	Kataoka Toshiki	Japan
	Yoshida Yuki	Japan
	Daliyev Asset	Kazakhstan
	Khajimuratov Nursultan	Kazakhstan
	Uteshev Dilshat	Kazakhstan
	Loke Zhi Kin	Malaysia
	Manrique Fernando	Peru
	Ramos Ricardo	Peru
	Lee Jae Hun	Republic of Korea
	Shim Jae Yong	Republic of Korea
	Kudyk Nikita	Russian Federation
	Kuznetsov Timofey	Russian Federation
	Lei Lei	Singapore
	Loh Yao Chen Ivan	Singapore
	Chien Pao-Yu	Taiwan
	Hsu Lun-Kai	Taiwan
	Suksompong Warut	Thailand
	Suteparuk Potcharapol	Thailand
	Meredov Azat	Turkmenistan
	Kishore Shaunak	USA
	Sankar Krishanu	USA

BRONZE AWARD	Blanc Pablo	Argentina
	Cogorno Federico	Argentina
	Eisenschlos Julián	Argentina
	Ponieman Nicolás	Argentina
	Chong Aaron	Australia
	Lo Irene	Australia
	Menzies Max	Australia
	Zhang Bonnie	Australia
	Abdullah M.M Sayeef	Bangladesh
	Se Dara	Cambodia
	Cheng Robin	Canada
	Kileel Joe	Canada
	Li Yan	Canada
	Remorov Alexander	Canada
	Cuellar Santiago	Colombia
	Olarte Jorge Alberto	Colombia
	Chung Ping Ngai	Hong Kong
	Lee Pak Hin	Hong Kong
	Tai Yee Ka	Hong Kong
	Wong Man Lok	Hong Kong
	Choiri Henra Hadhil	Indonesia
	Pramayoga Satria Stanza	Indonesia
	Arano Yuki	Japan
	Kawashima Yumehito	Japan
	Matsuda Fuyuki	Japan
	Seki Norifumi	Japan
	Bakhytzhan Nazerke	Kazakhstan
	Kanapin Nurlan	Kazakhstan
	Sailauov Tolebi	Kazakhstan
	Yeksembaev Abylai	Kazakhstan
	Lim Yu Wei	Malaysia
	Saw Yi Hui	Malaysia
	Anguiano Marcelino	Mexico
	Espinoza Malors	Mexico
	López Guillerno	Mexico
	Velasco Eduardo	Mexico
	Kornfeld Ben	New Zealand
	Yon Tom	New Zealand
	Cuenca Cesar	Peru
	Munoz Ivan	Peru
	Ochoa Oswaldo	Peru
	Velez Amilcar	Peru
	Velez Luis	Peru
	Narciso Tobit James	Philippines
	Lee Yo Han	Republic of Korea
	Lim SunKyu	Republic of Korea
	Sohn Ji hoon	Republic of Korea

	Song Sang Hun	Republic of Korea
	Beloshapko Georgy	Russian Federation
	Gusev Anton	Russian Federation
	Tarantsev Gennady	Russian Federation
	Voropaev Alexey	Russian Federation
	Kang Zi Yang	Singapore
	Liew Xiao Tian	Singapore
	Tan Juanhe	Singapore
	Tng Barry	Singapore
	Huang Chao-Chien	Taiwan
	Tong Yu-Fang	Taiwan
	Wang Chih-Wei	Taiwan
	Wang Szu-Po	Taiwan
	Chaturapruek Sorathan	Thailand
	Kamtue Supanat	Thailand
	Manurangsi Pasin	Thailand
	Siltham Sira	Thailand
	Akyniyazov Dowlet	Turkmenistan
	Mao Haitao	USA
	Meng Delong	USA
	Rosett Max	USA
	Superdock Matthew	USA
HONOURABLE MENTION	Dodyk Juan	Argentina
	Sadofschi Costa Iván	Argentina
	Umfurer Alfredo	Argentina
	Gardiner Sean	Australia
	Wong Sampson	Australia
	Sun Jarno	Canada
	Chan Kwun Tat	Hong Kong
	Ching Tak Wing	Hong Kong
	Wong Sze Wai	Hong Kong
	Imamura Shirou	Japan
	Inoue Shutaro	Japan
	Inoue Takuya	Japan
	Klochkov Yegor	Kazakhstan
	Han Man Woong	Republic of Korea
	Jang Junyeong	Republic of Korea
	Park Sunggi	Republic of Korea
	Blinov Alexander	Russian Federation
	Kosmodemyansky Daniil	Russian Federation
	Nikiforov Djulus	Russian Federation
	Arifin Tjeng Vincent Brian	Singapore
	Lim Jun Ren	Singapore
	Wong Gabriel	Singapore
	Liao Kuan-Chieh	Taiwan
	Liu Yu-Ting	Taiwan
	Lu Han	Taiwan

	Pornnopparat Donlapark	Thailand
	Bernstein Sergei	USA
	Firoiu Vald	USA
	Hance Travis	USA

XXI Asian Pacific Mathematics Olympiad

March, 2009

Time allowed: 4 hours *Each problem is worth 7 points*

∗ The contest problems are to be kept confidential until they are posted on the official APMO website (http://www.kms.or.kr/Competitions/APMO). Please do not disclose nor discuss the problems over the internet until that date. Calculators are not allowed to use.

Problem 1. Consider the following operation on positive real numbers written on a blackboard: *Choose a number r written on the blackboard, erase that number, and then write a pair of positive real numbers a and b satisfying the condition $2r^2 = ab$ on the board.*

Assume that you start out with just one positive real number r on the blackboard, and apply this operation $k^2 - 1$ times to end up with k^2 positive real numbers, not necessarily distinct. Show that there exists a number on the board which does not exceed kr.

Problem 2. Let a_1, a_2, a_3, a_4, a_5 be real numbers satisfying the following equations:

$$\frac{a_1}{k^2+1} + \frac{a_2}{k^2+2} + \frac{a_3}{k^2+3} + \frac{a_4}{k^2+4} + \frac{a_5}{k^2+5} = \frac{1}{k^2} \quad \text{for } k = 1, 2, 3, 4, 5.$$

Find the value of $\dfrac{a_1}{37} + \dfrac{a_2}{38} + \dfrac{a_3}{39} + \dfrac{a_4}{40} + \dfrac{a_5}{41}$. (Express the value in a single fraction.)

Problem 3. Let three circles $\Gamma_1, \Gamma_2, \Gamma_3$, which are non-overlapping and mutually external, be given in the plane. For each point P in the plane, outside the three circles, construct six points $A_1, B_1, A_2, B_2, A_3, B_3$ as follows: For each $i = 1, 2, 3$, A_i, B_i are distinct points on the circle Γ_i such that the lines PA_i and PB_i are both tangents to Γ_i. Call the point P *exceptional* if, from the construction, three lines A_1B_1, A_2B_2, A_3B_3 are concurrent. Show that every exceptional point of the plane, if exists, lies on the same circle.

Problem 4. Prove that for any positive integer k, there exists an arithmetic sequence

$$\frac{a_1}{b_1}, \quad \frac{a_2}{b_2}, \quad \ldots, \quad \frac{a_k}{b_k}$$

of rational numbers, where a_i, b_i are relatively prime positive integers for each $i = 1, 2, \ldots, k$, such that the positive integers $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ are all distinct.

Problem 5. Larry and Rob are two robots travelling in one car from Argovia to Zillis. Both robots have control over the steering and steer according to the following algorithm: Larry makes a 90° left turn after every ℓ kilometer driving from start; Rob makes a 90° right turn after every r kilometer driving from start, where ℓ and r are relatively prime positive integers. In the event of both turns occurring simultaneously, the car will keep going without changing direction. Assume that the ground is flat and the car can move in any direction.

Let the car start from Argovia facing towards Zillis. For which choices of the pair (ℓ, r) is the car guaranteed to reach Zillis, regardless of how far it is from Argovia?

XXI Asian Pacific Mathematics Olympiad

March, 2009

Problem 1. Consider the following operation on positive real numbers written on a blackboard:

> Choose a number r written on the blackboard, erase that number, and then write a pair of positive real numbers a and b satisfying the condition $2r^2 = ab$ on the board.

Assume that you start out with just one positive real number r on the blackboard, and apply this operation $k^2 - 1$ times to end up with k^2 positive real numbers, not necessarily distinct. Show that there exists a number on the board which does not exceed kr.

(Solution) Using AM-GM inequality, we obtain

$$\frac{1}{r^2} = \frac{2}{ab} = \frac{2ab}{a^2b^2} \le \frac{a^2+b^2}{a^2b^2} = \frac{1}{a^2} + \frac{1}{b^2}\,. \qquad (*)$$

Consequently, if we let S_ℓ be the sum of the squares of the reciprocals of the numbers written on the board after ℓ operations, then S_ℓ increases as ℓ increases, that is,

$$S_0 \le S_1 \le \cdots \le S_{k^2-1}. \qquad (**)$$

Therefore if we let s be the smallest real number written on the board after $k^2 - 1$ operations, then $\frac{1}{s^2} \ge \frac{1}{t^2}$ for any number t among k^2 numbers on the board and hence

$$k^2 \times \frac{1}{s^2} \ge S_{k^2-1} \ge S_0 = \frac{1}{r^2}\,,$$

which implies that $s \le kr$ as desired. □

Remark. The nature of the problem does not change at all if the numbers on the board are restricted to be positive integers. But that may mislead some contestants to think the problem is a number theoretic problem rather than a combinatorial problem.

Problem 2. Let a_1, a_2, a_3, a_4, a_5 be real numbers satisfying the following equations:

$$\frac{a_1}{k^2+1} + \frac{a_2}{k^2+2} + \frac{a_3}{k^2+3} + \frac{a_4}{k^2+4} + \frac{a_5}{k^2+5} = \frac{1}{k^2} \quad \text{for } k = 1, 2, 3, 4, 5.$$

Find the value of $\dfrac{a_1}{37} + \dfrac{a_2}{38} + \dfrac{a_3}{39} + \dfrac{a_4}{40} + \dfrac{a_5}{41}$. (Express the value in a single fraction.)

(Solution) Let $R(x) := \dfrac{a_1}{x^2+1} + \dfrac{a_2}{x^2+2} + \dfrac{a_3}{x^2+3} + \dfrac{a_4}{x^2+4} + \dfrac{a_5}{x^2+5}$. Then $R(\pm 1) = 1$, $R(\pm 2) = \dfrac{1}{4}$, $R(\pm 3) = \dfrac{1}{9}$, $R(\pm 4) = \dfrac{1}{16}$, $R(\pm 5) = \dfrac{1}{25}$ and $R(6)$ is the value to be found. Let's put $P(x) := (x^2+1)(x^2+2)(x^2+3)(x^2+4)(x^2+5)$ and $Q(x) := R(x)P(x)$. Then for $k = \pm 1, \pm 2, \pm 3, \pm 4, \pm 5$, we get $Q(k) = R(k)P(k) = \dfrac{P(k)}{k^2}$, that is, $P(k) \quad k^2 Q(k) = 0$. Since $P(x) \quad x^2 Q(x)$ is a polynomial of degree 10 with roots $\pm 1, \pm 2, \pm 3, \pm 4, \pm 5$, we get

$$P(x) \quad x^2 Q(x) = A(x^2 \quad 1)(x^2 \quad 4)(x^2 \quad 9)(x^2 \quad 16)(x^2 \quad 25). \qquad (*)$$

Putting $x = 0$, we get $A = \dfrac{P(0)}{(\quad 1)(\quad 4)(\quad 9)(\quad 16)(\quad 25)} = \quad \dfrac{1}{120}$. Finally, dividing both sides of $(*)$ by $P(x)$ yields

$$1 \quad x^2 R(x) = 1 \quad x^2 \frac{Q(x)}{P(x)} = \quad \frac{1}{120} \quad \frac{(x^2 \quad 1)(x^2 \quad 4)(x^2 \quad 9)(x^2 \quad 16)(x^2 \quad 25)}{(x^2+1)(x^2+2)(x^2+3)(x^2+4)(x^2+5)}$$

and hence that

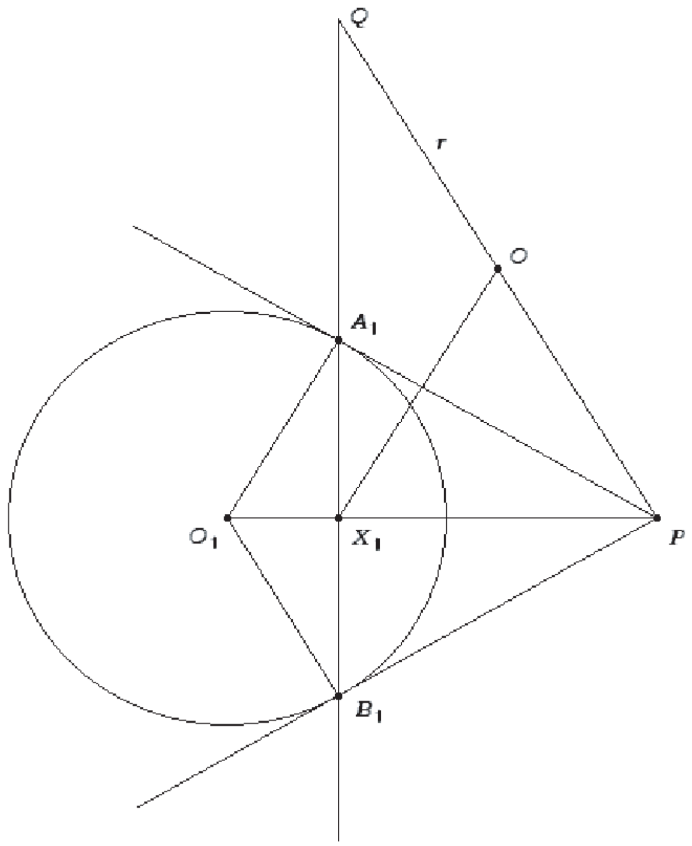
$$1 \quad 36R(6) = \quad \frac{35 \times 32 \times 27 \times 20 \times 11}{120 \times 37 \times 38 \times 39 \times 40 \times 41} = \quad \frac{3 \times 7 \times 11}{13 \times 19 \times 37 \times 41} = \quad \frac{231}{374699},$$

which implies $R(6) = \dfrac{187465}{6744582}$.

Remark. We can get $a_1 = \dfrac{1105}{72}, a_2 = \quad \dfrac{2673}{40}, a_3 = \dfrac{1862}{15}, a_4 = \quad \dfrac{1885}{18}, a_5 = \dfrac{1323}{40}$ by solving the given system of linear equations, which is extremely messy and takes a lot of time.

Problem 3. Let three circles $_1$, $_2$, $_3$, which are non-overlapping and mutually external, be given in the plane. For each point P in the plane, outside the three circles, construct six points $A_1, B_1, A_2, B_2, A_3, B_3$ as follows: For each $i = 1, 2, 3$, A_i, B_i are distinct points on the circle $_i$ such that the lines PA_i and PB_i are both tangents to $_i$. Call the point P *exceptional* if, from the construction, three lines A_1B_1, A_2B_2, A_3B_3 are concurrent. Show that every exceptional point of the plane, if exists, lies on the same circle.

(Solution) Let O_i be the center and r_i the radius of circle $_i$ for each $i = 1, 2, 3$. Let P be an exceptional point, and let the three corresponding lines A_1B_1, A_2B_2,A_3B_3 concur at Q. Construct the circle with diameter PQ. Call the circle , its center O and its radius r. We now claim that all exceptional points lie on .



Let PO_1 intersect A_1B_1 in X_1. As $PO_1 \perp A_1B_1$, we see that X_1 lies on . As PA_1 is a tangent to ${}_1$, triangle PA_1O_1 is right-angled and similar to triangle $A_1X_1O_1$. It follows that

$$\frac{O_1X_1}{O_1A_1} = \frac{O_1A_1}{O_1P}, \quad \text{i.e.,} \quad O_1X_1 \;\; O_1P = O_1A_1{}^2 = r_1{}^2.$$

On the other hand, $O_1X_1 \;\; O_1P$ is also the power of O_1 with respect to , so that

$$r_1^2 = O_1X_1 \;\; O_1P = (O_1O \quad r)(O_1O + r) = O_1O^2 \quad r^2, \tag{$*$}$$

and hence

$$r^2 = OO_1^2 \quad r_1^2 = (OO_1 \quad r_1)(OO_1 + r_1).$$

Thus, r^2 is the power of O with respect to ${}_1$. By the same token, r^2 is also the power of O with respect to ${}_2$ and ${}_3$. Hence O must be the radical center of the three given circles. Since r, as the square root of the power of O with respect to the three given circles, does not depend on P, it follows that all exceptional points lie on . □

Remark. In the event of the radical point being at in nity (and hence the three radical axes being parallel), there are no exceptional points in the plane, which is consistent with the statement of the problem.

Problem 4. Prove that for any positive integer k, there exists an arithmetic sequence

$$\frac{a_1}{b_1}, \quad \frac{a_2}{b_2}, \quad \ldots, \quad \frac{a_k}{b_k}$$

of rational numbers, where a_i, b_i are relatively prime positive integers for each $i = 1, 2, \ldots, k$, such that the positive integers $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ are all distinct.

(Solution) For $k = 1$, there is nothing to prove. Henceforth assume $k \geq 2$.
Let $p_1, p_2, \ldots, p_k$ be k distinct primes such that

$$k < p_k < \cdots < p_2 < p_1$$

and let $N = p_1 p_2 \cdots p_k$. By Chinese Remainder Theorem, there exists a positive integer x satisfying

$$x \equiv -i \pmod{p_i}$$

for all $i = 1, 2, \ldots, k$ and $x > N^2$. Consider the following sequence:

$$\frac{x+1}{N}, \quad \frac{x+2}{N}, \quad , \ldots, \quad \frac{x+k}{N}.$$

This sequence is obviously an arithmetic sequence of positive rational numbers of length k. For each $i = 1, 2, \ldots, k$, the numerator $x + i$ is divisible by p_i but not by p_j for $j \neq i$, for otherwise p_j divides $|i - j|$, which is not possible because $p_j > k > |i - j|$. Let

$$a_i := \frac{x+i}{p_i}, \quad b_i := \frac{N}{p_i} \qquad \text{for all } \ i = 1, 2, \ldots, k.$$

Then

$$\frac{x+i}{N} = \frac{a_i}{b_i}, \quad \gcd(a_i, b_i) = 1 \qquad \text{for all } \ i = 1, 2, \ldots, k,$$

and all b_i's are distinct from each other. Moreover, $x > N^2$ implies

$$a_i = \frac{x+i}{p_i} > \frac{N^2}{p_i} > N > \frac{N}{p_j} = b_j \qquad \text{for all } \ i, j = 1, 2, \ldots, k$$

and hence all a_i's are distinct from b_i's. It only remains to show that all a_i's are distinct from each other. This follows from

$$a_j = \frac{x+j}{p_j} > \frac{x+i}{p_j} > \frac{x+i}{p_i} = a_i \qquad \text{for all } \ i < j$$

by our choice of $p_1, p_2, \ldots, p_k$. Thus, the arithmetic sequence

$$\frac{a_1}{b_1}, \quad \frac{a_2}{b_2}, \quad \ldots, \quad \frac{a_k}{b_k}$$

of positive rational numbers satisfies the conditions of the problem. □

Remark. Here is a much easier solution:

For any positive integer $k \geq 2$, consider the sequence

$$\frac{(k!)^2+1}{k!}, \frac{(k!)^2+2}{k!}, \ldots, \frac{(k!)^2+k}{k!}.$$

Note that $\gcd(k!, (k!)^2+i) = i$ for all $i = 1, 2, \ldots, k$. So, taking

$$a_i := \frac{(k!)^2+i}{i}, \quad b_i := \frac{k!}{i} \qquad \text{for all } \ i = 1, 2, \ldots, k,$$

we have $\gcd(a_i, b_i) = 1$ and

$$a_i = \frac{(k!)^2+i}{i} > a_j = \frac{(k!)^2+j}{j} > b_i = \frac{k!}{i} > b_j = \frac{k!}{j}$$

for any $1 \leq i < j \leq k$. Therefore this sequence satisfies every condition given in the problem.

Problem 5. Larry and Rob are two robots travelling in one car from Argovia to Zillis. Both robots have control over the steering and steer according to the following algorithm: Larry makes a 90° left turn after every ℓ kilometer driving from start; Rob makes a 90° right turn after every r kilometer driving from start, where ℓ and r are relatively prime positive integers. In the event of both turns occurring simultaneously, the car will keep going without changing direction. Assume that the ground is flat and the car can move in any direction.

Let the car start from Argovia facing towards Zillis. For which choices of the pair (ℓ, r) is the car guaranteed to reach Zillis, regardless of how far it is from Argovia?

(Solution) Let Zillis be d kilometers away from Argovia, where d is a positive real number. For simplicity, we will position Argovia at $(0, 0)$ and Zillis at $(d, 0)$, so that the car starts out facing east. We will investigate how the car moves around in the period of travelling the first ℓr kilometers, the second ℓr kilometers, ..., and so on. We call each period of travelling ℓr kilometers a *section*. It is clear that the car will have identical behavior in every section except the direction of the car at the beginning.

`Case 1:` $\ell - r \equiv 2 \pmod 4$. After the first section, the car has made $\ell - 1$ right turns and $r - 1$ left turns, which is a net of $2(\equiv \ell - r \pmod 4)$ right turns. Let the displacement vector for the first section be (x, y). Since the car has rotated 180°, the displacement vector for the second section will be $(-x, -y)$, which will take the car back to $(0, 0)$ facing east again. We now have our original situation, and the car has certainly never travelled further than ℓr kilometers from Argovia. So, the car cannot reach Zillis if it is further apart from Argovia.

`Case 2:` $\ell - r \equiv 1 \pmod 4$. After the first section, the car has made a net of 1 right turn. Let the displacement vector for the first section again be (x, y). This time the car has rotated 90° clockwise. We can see that the displacements for the second, third and fourth section will be $(y, -x)$, $(-x, -y)$ and $(-y, x)$, respectively, so after four sections the car is back at $(0, 0)$ facing east. Since the car has certainly never travelled further than $2\ell r$ kilometers from Argovia, the car cannot reach Zillis if it is further apart from Argovia.

`Case 3:` $\ell - r \equiv 3 \pmod 4$. An argument similar to that in `Case 2` (switching the roles of left and right) shows that the car cannot reach Zillis if it is further apart from Argovia.

`Case 4:` $\ell \equiv r \pmod 4$. The car makes a net turn of 0° after each section, so it must be facing east. We are going to show that, after traversing the first section, the car will be at $(1, 0)$. It will be useful to interpret the Cartesian plane as the complex plane, i.e. writing $x + iy$ for (x, y), where $i = \sqrt{-1}$. We will denote the k-th kilometer of movement by m_{k-1}, which takes values from the set $\{1, i, \;\; 1, \;\; i\}$, depending on the direction. We then just have to show that

$$\sum_{k=0}^{\ell r \;\; 1} m_k = 1,$$

which implies that the car will get to Zillis no matter how far it is apart from Argovia.

`Case 4a:` $\ell \;\; r \;\; 1 \pmod 4$. First note that for $k = 0, 1, \ldots, \ell r \;\; 1$,

$$m_k = i^{\lfloor k/\ell \rfloor}(\;\; i)^{\lfloor k/r \rfloor}$$

since $\lfloor k/\ell \rfloor$ and $\lfloor k/r \rfloor$ are the exact numbers of left and right turns before the $(k+1)$st kilometer, respectively. Let $a_k(\;\; k \pmod \ell)$ and $b_k(\;\; k \pmod r)$ be the remainders of k when divided by ℓ and r, respectively. Then, since

$$a_k = k \;\; \left\lfloor \frac{k}{\ell} \right\rfloor \ell \;\; k \;\; \left\lfloor \frac{k}{\ell} \right\rfloor \pmod 4 \quad \text{and} \quad b_k = k \;\; \left\lfloor \frac{k}{r} \right\rfloor r \;\; k \;\; \left\lfloor \frac{k}{r} \right\rfloor \pmod 4,$$

we have $\lfloor k/\ell \rfloor \;\; k \;\; a_k \pmod 4$ and $\lfloor k/r \rfloor \;\; k \;\; b_k \pmod 4$. We therefore have

$$m_k = i^{k \;\; a_k}(\;\; i)^{k \;\; b_k} = (\;\; i^2)^k i^{\;\; a_k}(\;\; i)^{\;\; b_k} = (\;\; i)^{a_k} i^{b_k}.$$

As ℓ and r are relatively prime, by Chinese Remainder Theorem, there is a bijection between pairs $(a_k, b_k) = (k(\mathrm{mod}\ \ell), k(\mathrm{mod}\ r))$ and the numbers $k = 0, 1, 2, \ldots, \ell r \;\; 1$. Hence

$$\sum_{k=0}^{\ell r \;\; 1} m_k = \sum_{k=0}^{\ell r \;\; 1} (\;\; i)^{a_k} i^{b_k} = \left(\sum_{k=0}^{\ell \;\; 1} (\;\; i)^{a_k} \right) \left(\sum_{k=0}^{r \;\; 1} i^{b_k} \right) = 1 \times 1 = 1$$

as required because $\ell \;\; r \;\; 1 \pmod 4$.

`Case 4b:` $\ell \;\; r \;\; 3 \pmod 4$. In this case, we get

$$m_k = i^{a_k}(\;\; i)^{b_k},$$

where $a_k(\;\; k \pmod \ell)$ and $b_k(\;\; k \pmod r)$ for $k = 0, 1, \ldots, \ell r \;\; 1$. Then we can proceed analogously to `Case 4a` to obtain

$$\sum_{k=0}^{\ell r \;\; 1} m_k = \sum_{k=0}^{\ell r \;\; 1} (\;\; i)^{a_k} i^{b_k} = \left(\sum_{k=0}^{\ell \;\; 1} (\;\; i)^{a_k} \right) \left(\sum_{k=0}^{r \;\; 1} i^{b_k} \right) = i \times (\;\; i) = 1$$

as required because $\ell \;\; r \;\; 3 \pmod 4$.

Now clearly the car traverses through all points between $(0, 0)$ and $(1, 0)$ during the rst section and, in fact, covers all points between $(n \;\; 1, 0)$ and $(n, 0)$ during the n-th section. Hence it will eventually reach $(d, 0)$ for any positive d.

To summarize: (ℓ, r) satis es the required conditions if and only if

$$\ell \quad r \quad 1 \quad \text{or} \quad \ell \quad r \quad 3 \pmod 4. \qquad \square$$

Remark. In case $\gcd(\ell, r) = d \neq 1$, the answer is:

$$\frac{\ell}{d} \quad \frac{r}{d} \quad 1 \quad \text{or} \quad \frac{\ell}{d} \quad \frac{r}{d} \quad 3 \pmod 4.$$

Award	Names of Contestant	Country
GOLD AWARD	Stefanich Germán	Argentina
	Elvey Price Andrew	Australia
	Sun Chen	Canada
	Chung Ping Ngai	Hong Kong
	Soejima Makoto	Japan
	Yim Joon Hyuk	Korea
	Loke Zhi Kin	Malaysia
	Mejia Cdrdero Julian	Peru
	Fernandes Kenrick Orrin	Qatar
	Tong Yu-Fan	Taiwan
	Manurangsi Pasin	Thailand
	Cao Wenyu	U.S.A.
SILVER AWARD	Sadofschi Costa Iván	Argentina
	Varanese Axel	Argentina
	Chong Aaron	Australia
	Schneider Jonathan	Canada
	Spink Hunter	Canada
	Olarte Jorge Alberto	Colombia
	Rodriguez Arguedas Rafael Angel	Costa Rica
	Cheng Kwun Yee	Hong Kong
	Hung Ka Kin Kenneth	Hong Kong
	Hosaka Kazuhiro	Japan
	Takigiku Motoki	Japan
	Hajimuratov Nursultan	Kazakhstan
	Klochkov Egor	Kazakhstan
	Kim Bum Soo	Korea
	Lee Sang Hun	Korea
	Kornfeld Ben	New Zealand
	Angles Larico Tomas	Peru
	Saha Arpan	Qatar
	Ang Yan Sheng	Singapore
	Chiam Jia-Han	Singapore
	Loh Yao Chen Ivan	Singapore
	Shih Pi-Hsun	Taiwan
	Wu Yi-Chan	Taiwan
	Mirzaev Inomzhon	Tajikistan
	Kamtue Supanat	Thailand
	Siranart Nopphon	Thailand
	Gu Albert	U.S.A.
	Pan Qinxuan	U.S.A.
	Ponieman Nicolás	Argentina
	Prillo Sebastián	Argentina
	Rodriguez Castro Nicolás	Argentina
	Umfurer Alfredo	Argentina
	Lei Yitao	Australia

BRONZE AWARD	Lu Colin	Australia
	Ma Dana	Australia
	Ryba Christopher	Australia
	Wang Zejun	Australia
	Touch Sopheak	Cambodia
	Chen Yuhan	Canada
	Kim Junghoo	Canada
	Shi Danny	Canada
	Sun Jarno	Canada
	Castañeda Angela Maria	Colombia
	Cortis Jorge	Colombia
	Ayala Menjivar Julio César	El Salvador
	Lam Cho Ho	Hong Kong
	Ng Fung Ming	Hong Kong
	Yip Hok Pan	Hong Kong
	Yu Tak Hei	Hong Kong
	Andreas Joseph	Indonesia
	Oktovin Raja	Indonesia
	Hara Masaki	Japan
	Higaki Motohide	Japan
	Ishikawa Suguru	Japan
	Wakatsuki Shun	Japan
	Aimoldin Anuar	Kazakhstan
	Bolat Arman	Kazakhstan
	Kanybekuly Akzhol	Kazakhstan
	Kuzhagulov Azamat	Kazakhstan
	Nikiforov Alexandr	Kazakhstan
	Ahn Tae Joo	Korea
	Kang Tae Gu	Korea
	Lyoo Young Wook	Korea
	Song Sang Hun	Korea
	Lim Kai Tsen,Joshua	Malaysia
	Lim Yu Wei	Malaysia
	Mohamad Fadhlillah Mohamad Farzan	Malaysia
	Saw Yihui	Malaysia
	Granville Malcolm	New Zealand
	Tariq Isfar	Pakistan
	Waqar Ali Shah Syed	Pakistan
	Alcala Ramos Hector	Peru
	Ccopa Yugra Jesus	Peru
	Guerra Rios Percy	Peru
	Ramos Castillo Jesus	Peru
	Warton Cordero Alejandro	Peru
	Adajar Carlo Francisco	Philippines
	Encarnacion Immanuel	Philippines
	Gacuan Jake	Philippines
	Arasteh Ehsan	Qatar

	Pinto Lerrel Joseph	Qatar
	Valappil Amith Anil	Qatar
	Ang Jie Jun	Singapore
	Lin Wayne	Singapore
	Tan Douglas	Singapore
	Tng Jia Hao Barry	Singapore
	Hsu Lun-Kai	Taiwan
	Ko Shao-Heng	Taiwan
	Lai Yen-Lin	Taiwan
	Wang Szu-Po	Taiwan
	Pirahmad Olimjon	Tajikistan
	Chaiyakan Songkomkrit	Thailand
	Jiradilok Pakawut	Thailand
	Pitimanaaree Nipun	Thailand
	Ruangwises Suthee	Thailand
	Ramakrishnan Prithvi	Trinidad & Tobago
	Atayev Serdar	Turkmenistan
	Emirov Nazar	Turkmenistan
	Samedov Agageldi	Turkmenistan
	Baek Gye Hyun	U.S.A.
	Berman John	U.S.A.
	Hance Travis	U.S.A.
	Meng Delong	U.S.A.
HONOURABLE MENTION	Dodyk Juan	Argentina
	González Carolina	Argentina
	Saucedo Matias	Argentina
	Park Kiho	Australia
	Wormell John	Australia
	Gurram Neil	Canada
	Zhou Jonathan	Canada
	Zung Jonathan	Canada
	Lo Chun Tung	Hong Kong
	Ng Ngai Fung	Hong Kong
	Wo Bar Wai Barry	Hong Kong
	Kanna Raio	Japan
	Noyama Tomoyuki	Japan
	Seki Norifumi	Japan
	Gabdilalimov Alibek	Kazakhstan
	Sailauov Tolebi	Kazakhstan
	Sefullin Marat	Kazakhstan
	Hong Seung Wan	Korea
	Min Won June	Korea
	Park Seung Hyun	Korea
	Arifin Tieng Vincent Brian	Singapore
	Lam Jun Wei	Singapore
	Lee You Jun	Singapore
	Huang Chang-Ming	Taiwan

	Wang Chien-Yi	Taiwan
	Yang Hung-I	Taiwan
	Bhudisaksang Theerawat	Thailand
	Bhumiwat Pongpak	Thailand
	Yangjit Wijit	Thailand
	Bernstein Sergei	U.S.A.
	Phan Toan Duc	U.S.A.
	Superdock Matthew	U.S.A.

XXII Asian Pacific Mathematics Olympiad

March, 2010

Time allowed: 4 hours *Each problem is worth 7 points*

**The contest problems are to be kept confidential until they are posted on the official APMO website (http://www.mmjp.or.jp/competitions/APMO). Please do not disclose nor discuss the problems over the internet until that date. Calculators are not allowed to use.*

Problem 1. Let ABC be a triangle with $\angle BAC \neq 90^\circ$. Let O be the circumcenter of the triangle ABC and let Γ be the circumcircle of the triangle BOC. Suppose that Γ intersects the line segment AB at P different from B, and the line segment AC at Q different from C. Let ON be a diameter of the circle Γ. Prove that the quadrilateral $APNQ$ is a parallelogram.

Problem 2. For a positive integer k, call an integer a *pure k-th power* if it can be represented as m^k for some integer m. Show that for every positive integer n there exist n distinct positive integers such that their sum is a pure 2009-th power, and their product is a pure 2010-th power.

Problem 3. Let n be a positive integer. n people take part in a certain party. For any pair of the participants, either the two are acquainted with each other or they are not. What is the maximum possible number of the pairs for which the two are not acquainted but have a common acquaintance among the participants?

Problem 4. Let ABC be an acute triangle satisfying the condition $AB > BC$ and $AC > BC$. Denote by O and H the circumcenter and the orthocenter, respectively, of the triangle ABC. Suppose that the circumcircle of the triangle AHC intersects the line AB at M different from A, and that the circumcircle of the triangle AHB intersects the line AC at N different from A. Prove that the circumcenter of the triangle MNH lies on the line OH.

Problem 5. Find all functions f from the set $\mathbf{R}$ of real numbers into $\mathbf{R}$ which satisfy for all $x, y, z \in \mathbf{R}$ the identity

$$f(f(x) + f(y) + f(z)) = f(f(x) - f(y)) + f(2xy + f(z)) + 2f(xz - yz).$$

SOLUTIONS FOR 2010 APMO PROBLEMS

Problem 1. Let ABC be a triangle with $\angle BAC \neq 90°$. Let O be the circumcenter of the triangle ABC and let Γ be the circumcircle of the triangle BOC. Suppose that Γ intersects the line segment AB at P different from B, and the line segment AC at Q different from C. Let ON be a diameter of the circle Γ. Prove that the quadrilateral $APNQ$ is a parallelogram.

Solution: From the assumption that the circle Γ intersects both of the line segments AB and AC, it follows that the 4 points N, C, Q, O are located on Γ in the order of N, C, Q, O or in the order of N, C, O, Q. The following argument for the proof of the assertion of the problem is valid in either case. Since $\angle NQC$ and $\angle NOC$ are subtended by the same arc $\overset{\frown}{NC}$ of Γ at the points Q and O, respectively, on Γ, we have $\angle NQC = \angle NOC$. We also have $\angle BOC = 2\angle BAC$, since $\angle BOC$ and $\angle BAC$ are subtended by the same arc $\overset{\frown}{BC}$ of the circum-circle of the triangle ABC at the center O of the circle and at the point A on the circle, respectively. From $OB = OC$ and the fact that ON is a diameter of Γ, it follows that the triangles OBN and OCN are congruent, and therefore we obtain $2\angle NOC = \angle BOC$. Consequently, we have $\angle NQC = \frac{1}{2}\angle BOC = \angle BAC$, which shows that the 2 lines AP, QN are parallel.

In the same manner, we can show that the 2 lines AQ, PN are also parallel. Thus, the quadrilateral $APNQ$ is a parallelogram.

Problem 2. For a positive integer k, call an integer a *pure k-th power* if it can be represented as m^k for some integer m. Show that for every positive integer n there exist n distinct positive integers such that their sum is a pure 2009-th power, and their product is a pure 2010-th power.

Solution: For the sake of simplicity, let us set $k = 2009$.

First of all, choose n distinct positive integers $b_1, \cdots, b_n$ suitably so that their product is a pure $k+1$-th power (for example, let $b_i = i^{k+1}$ for $i = 1, \cdots, n$). Then we have $b_1 \cdots b_n = t^{k+1}$ for some positive integer t. Set $b_1 + \cdots + b_n = s$.

Now we set $a_i = b_i s^{k^2-1}$ for $i = 1, \cdots, n$, and show that $a_1, \cdots, a_n$ satisfy the required conditions. Since $b_1, \cdots, b_n$ are distinct positive integers, it is clear that so are $a_1, \cdots, a_n$. From

$$a_1 + \cdots + a_n = s^{k^2-1}(b_1 + \cdots + b_n) = s^{k^2} = (s^k)^{2009},$$

$$a_1 \cdots a_n = (s^{k^2-1})^n b_1 \cdots b_n = (s^{k^2-1})^n t^{k+1} = (s^{(k-1)n} t)^{2010}$$

we can see that $a_1, \cdots, a_n$ satisfy the conditions on the sum and the product as well. This ends the proof of the assertion.

Remark: We can find the appropriate exponent $k^2 - 1$ needed for the construction of the a_i's by solving the simultaneous congruence relations: $x \equiv 0 \pmod{k+1}$, $x \equiv -1 \pmod{k}$.

Problem 3. Let n be a positive integer. n people take part in a certain party. For any pair of the participants, either the two are acquainted with each other or they are not. What is the maximum possible number of the pairs for which the two are not acquainted but have a common acquaintance among the participants?

Solution: When 1 participant, say the person A, is mutually acquainted with each of the remaining $n-1$ participants, and if there are no other acquaintance relationships among the participants, then for any pair of participants not involving A, the two are not mutual acquaintances, but they have a common acquaintance, namely A, so any such pair satisfies the requirement. Thus, the number desired in this case is $\frac{(n-1)(n-2)}{2} = \frac{n^2-3n+2}{2}$.

Let us show that $\frac{n^2-3n+2}{2}$ is the maximum possible number of the pairs satisfying the requirement of the problem. First, let us observe that in the process of trying to find the maximum possible number of such pairs, if we split the participants into two non-empty subsets T and S which are disjoint, we may assume that there is a pair consisting of one person chosen from T and the other chosen from S who are mutual acquaintances. This is so, since if there are no such pair for some splitting T and S, then among the pairs consisting of one person chosen from T and the other chosen from S, there is no pair for which the two have a common acquaintance among participants, and therefore, if we arbitrarily choose a person $A \in T$ and $B \in S$ and declare that A and B are mutual acquaintances, the number of the pairs satisfying the requirement of the problem does not decrease.

Let us now call a set of participants a *group* if it satisfies the following 2 conditions:

• One can connect any person in the set with any other person in the set by tracing a chain of mutually acquainted pairs. More precisely, for any pair of people A, B in the set there exists a sequence of people $A_0, A_1, \cdots, A_n$ for which $A_0 = A$, $A_n = B$ and, for each $i : 0 \le i \le n-1$, A_i and A_{i+1} are mutual acquaintances.

• No person in this set can be connected with a person not belonging to this set by tracing a chain of mutually acquainted pairs.

In view of the discussions made above, we may assume that the set of all the participants to the party forms a group of n people. Let us next consider the following lemma.

Lemma. In a group of n people, there are at least $n-1$ pairs of mutual acquaintances.

Proof: If you choose a mutually acquainted pair in a group and declare the two in the pair are not mutually acquainted, then either the group stays the same or splits into 2 groups. This means that by changing the status of a mutually acquainted pair in a group to that of a non-acquainted pair, one can increase the number of groups at most by 1. Now if in a group of n people you change the status of all of the mutually acquainted pairs to that of non-acquainted pairs, then obviously, the number of groups increases from 1 to n. Therefore, there must be at least $n-1$ pairs of mutually acquainted pairs in a group consisting of n people. □

The lemma implies that there are at most $\frac{n(n-1)}{2} - (n-1) = \frac{n^2-3n+2}{2}$ pairs satisfying the condition of the problem. Thus the desired maximum number of pairs satisfying the requirement of the problem is $\frac{n^2-3n+2}{2}$.

Remark: One can give a somewhat different proof by separating into 2 cases depending on whether there are at least $n-1$ mutually acquainted pairs, or at most $n-2$ such pairs. In the former case, one can argue in the same way as the proof above, while in the latter case, the Lemma above implies that there would be 2 or more groups to start with, but then, in view of the comment made before the definition of a group above, these groups can be combined to form one group, thereby one can reduce the argument to the former case.

Alternate Solution 1: The construction of an example for the case for which the number $\frac{n^2-3n+2}{2}$ appears, and the argument for the case where there is only 1 group would be the same as in the preceding proof.

Suppose, then, n participants are separated into $k (k \geq 2)$ groups, and the number of people in each group is given by $a_i, i = 1, \cdots, k$. In such a case, the number of pairs for which paired people are not mutually acquainted but have a common acquaintance is at most $\sum_{i=1}^{k} {}_{a_i}C_2$, where we set ${}_1C_2 = 0$ for convenience. Since ${}_aC_2 + {}_bC_2 \leq {}_{a+b}C_2$ holds for any pair of positive integers a, b, we have $\sum_{i=1}^{k} {}_{a_i}C_2 \leq {}_{a_1}C_2 + {}_{n-a_1}C_2$. From

$${}_{a_1}C_2 + {}_{n-a_1}C_2 = a_1^2 - na_1 + \frac{n^2-n}{2} = \left(a_1 - \frac{n}{2}\right)^2 + \frac{n^2-2n}{4}$$

it follows that ${}_{a_1}C_2 + {}_{n-a_1}C_2$ takes its maximum value when $a_1 = 1, n-1$. Therefore, we have $\sum_{i=1}^{k} {}_{a_i}C_2 \leq {}_{n-1}C_2$, which shows that in the case where the number of groups are 2 or more, the number of the pairs for which paired people are not mutually acquainted but have a common acquaintance is at most ${}_{n-1}C_2 = \frac{n^2-3n+2}{2}$, and hence the desired maximum number of the pairs satisfying the requirement is $\frac{n^2-3n+2}{2}$.

Alternate Solution 2: Construction of an example would be the same as the preceding proof.

For a participant, say A, call another participant, say B, a *familiar face* if A and B are not mutually acquainted but they have a common acquaintance among the participants, and in this case call the pair A, B a familiar pair.

Suppose there is a participant P who is mutually acquainted with d participants. Denote by S the set of these d participants, and by T the set of participants different from P and not belonging to the set S. Suppose there are e pairs formed by a person in S and a person in T who are mutually acquainted.

Then the number of participants who are familiar faces to P is at most e. The number of pairs formed by two people belonging to the set S and are mutually acquainted is at most ${}_dC_2$. The number of familiar pairs formed by two people belonging to the set T is at most ${}_{n-d-1}C_2$. Since there are e pairs formed by a person in the set S and a person in the set T who are mutually acquainted (and so the pairs are not familiar pairs), we have at most $d(n-1-d)-e$ familiar pairs formed by a person chosen from S and a person chosen from T. Putting these together we conclude that there are at most $e + {}_dC_2 + {}_{n-1-d}C_2 + d(n-1-d) - e$ familiar pairs. Since

$$e + {}_dC_2 + {}_{n-1-d}C_2 + d(n-1-d) - e = \frac{n^2-3n+2}{2},$$

the number we seek is at most $\frac{n^2-3n+2}{2}$, and hence this is the desired solution to the problem.

Problem 4. Let ABC be an acute triangle satisfying the condition $AB > BC$ and $AC > BC$. Denote by O and H the circumcenter and the orthocenter, respectively, of the triangle ABC. Suppose that the circumcircle of the triangle AHC intersects the line AB at M different from A, and that the circumcircle of the triangle AHB intersects the line AC at N different from A. Prove that the circumcenter of the triangle MNH lies on the line OH.

Solution: In the sequel, we denote $\angle BAC = \alpha, \angle CBA = \beta, \angle ACB = \gamma$. Let O' be the circumcenter of the triangle MNH. The lengths of line segments starting from the point H will be treated as signed quantities.

Let us denote by M', N' the point of intersection of CH, BH, respectively, with the circumcircle of the triangle ABC (distinct from C, B, respectively.) From the fact that 4 points A, M, H, C lie on the same circle, we see that $\angle MHM' = \alpha$ holds. Furthermore, $\angle BM'C, \angle BN'C$ and α are all subtended by the same arc $\overset{\frown}{BC}$ of the circumcircle of the triangle ABC at points on the circle, and therefore, we have $\angle BM'C = \alpha$, and $\angle BN'C = \alpha$ as well. We also have $\angle ABH = \angle ACN'$ as they are subtended by the same arc $\overset{\frown}{AN'}$ of the circumcircle of the triangle ABC at points on the circle. Since $HM' \perp BM, HN' \perp AC$, we conclude that

$$\angle M'HB = 90^\circ - \angle ABH = 90^\circ - \angle ACN' = \alpha$$

is valid as well. Putting these facts together, we obtain the fact that the quadrilateral $HBM'M$ is a rhombus. In a similar manner, we can conclude that the quadrilateral $HCN'N$ is also a rhombus. Since both of these rhombuses are made up of 4 right triangles with an angle of magnitude α, we also see that these rhombuses are similar.

Let us denote by P, Q the feet of the perpendicular lines on HM and HN, respectively, drawn from the point O'. Since O' is the circumcenter of the triangle MNH, P, Q are respectively, the midpoints of the line segments HM, HN. Furthermore, if we denote by R, S the feet of the perpendicular lines on HM and HN, respectively, drawn from the point O, then since O is the circumcenter of both the triangle $M'BC$ and the triangle $N'BC$, we see that R is the intersection point of HM and the perpendicular bisector of BM', and S is the intersection point of HN and the perpendicular bisector of CN'.

We note that the similarity map ϕ between the rhombuses $HBM'M$ and $HCN'N$ carries the perpendicular bisector of BM' onto the perpendicular bisector of CN', and straight line HM onto the straight line HN, and hence ϕ maps R onto S, and P onto Q. Therefore, we get $HP : HR = HQ : HS$. If we now denote by X, Y the intersection points of the line HO' with the line through R and perpendicular to HP, and with the line through S and perpendicular to HQ, respectively, then we get

$$HO' : HX = HP : HR = HQ : HS = HO' : HY$$

so that we must have $HX = HY$, and therefore, $X = Y$. But it is obvious that the point of intersection of the line through R and perpendicular to HP with the line through S and perpendicular to HQ must be O, and therefore, we conclude that $X = Y = O$ and that the points H, O', O are collinear.

Alternate Solution: Deduction of the fact that both of the quadrilaterals $HBM'M$ and $HCN'N$ are rhombuses is carried out in the same way as in the preceding proof.

We then see that the point M is located in a symmetric position with the point B with respect to the line CH, we conclude that we have $\angle CMB = \beta$. Similarly, we have $\angle CNB = \gamma$. If we now put $x = \angle AHO'$, then we get

$$\angle O' = \beta - \alpha - x, \ \ \angle MNH = 90^\circ - \beta - \alpha + x,$$

from which it follows that

$$\angle ANM = 180^\circ - \angle MNH - (90^\circ - \alpha) = \beta - x.$$

Similarly, we get

$$\angle NMA = \gamma + x.$$

Using the laws of sines, we then get

$$\frac{\sin(\gamma+x)}{\sin(\beta-x)} = \frac{AN}{AM} = \frac{AC}{AM}\cdot\frac{AB}{AC}\cdot\frac{AN}{AB}$$
$$= \frac{\sin\beta}{\sin(\beta-\alpha)}\cdot\frac{\sin\gamma}{\sin\beta}\cdot\frac{\sin(\gamma-\alpha)}{\sin\gamma} = \frac{\sin(\gamma-\alpha)}{\sin(\beta-\alpha)}.$$

On the other hand, if we let $y = \angle AHO$, we then get

$$\angle OHB = 180^\circ - \gamma - y,\ \ \angle CHO = 180^\circ - \beta + y,$$

and since

$$\angle HBO = \gamma - \alpha,\ \ \angle OCH = \beta - \alpha,$$

using the laws of sines and observing that $OB = OC$, we get

$$\frac{\sin(\gamma-\alpha)}{\sin(\beta-\alpha)} = \frac{\sin\angle HBO}{\sin\angle OCH} = \frac{\sin(180^\circ-\gamma-y)\cdot\frac{OH}{OB}}{\sin(180^\circ-\beta+y)\cdot\frac{OH}{OC}}$$
$$= \frac{\sin(180^\circ-\gamma-y)}{\sin(180^\circ-\beta+y)} = \frac{\sin(\gamma+y)}{\sin(\beta-y)}.$$

We then get $\sin(\gamma+x)\sin(\beta-y) = \sin(\beta-x)\sin(\gamma+y)$. Expanding both sides of the last identity by using the addition formula for the sine function and after factoring and using again the addition formula we obtain that $\sin(x-y)\sin(\beta+\gamma) = 0$. This implies that $x-y$ must be an integral multiple of 180°, and hence we conclude that H, O, O' are collinear.

Problem 5. Find all functions f from the set $\mathbf{R}$ of real numbers into $\mathbf{R}$ which satisfy for all $x, y, z \in \mathbf{R}$ the identity

$$f(f(x)+f(y)+f(z)) = f(f(x)-f(y)) + f(2xy+f(z)) + 2f(xz-yz).$$

Solution: It is clear that if f is a constant function which satisfies the given equation, then the constant must be 0. Conversely, $f(x) = 0$ clearly satisfies the given equation, so, the identically 0 function is a solution. In the sequel, we consider the case where f is not a constant function.

Let $t \in \mathbf{R}$ and substitute $(x,y,z) = (t,0,0)$ and $(x,y,z) = (0,t,0)$ into the given functional equation. Then, we obtain, respectively,

$$f(f(t)+2f(0)) = f(f(t)-f(0)) + f(f(0)) + 2f(0),$$
$$f(f(t)+2f(0)) = f(f(0)-f(t)) + f(f(0)) + 2f(0),$$

from which we conclude that $f(f(t)-f(0)) = f(f(0)-f(t))$ holds for all $t \in \mathbf{R}$. Now, suppose for some pair u_1, u_2, $f(u_1) = f(u_2)$ is satisfied. Then by substituting $(x,y,z) = (s,0,u_1)$ and $(x,y,z) = (s,0,u_2)$ into the functional equation and comparing the resulting identities, we can easily conclude that

$$f(su_1) = f(su_2) \qquad (*)$$

holds for all $s \in \mathbf{R}$. Since f is not a constant function there exists an s_0 such that $f(s_0)-f(0) \neq 0$. If we put $u_1 = f(s_0) - f(0), u_2 = -u_1$, then $f(u_1) = f(u_2)$, so we have by $(*)$

$$f(su_1) = f(su_2) = f(-su_1)$$

for all $s \in \mathbf{R}$. Since $u_1 \neq 0$, we conclude that

$$f(x) = f(-x)$$

holds for all $x \in \mathbf{R}$.

Next, if $f(u) = f(0)$ for some $u \neq 0$, then by $(*)$, we have $f(su) = f(s0) = f(0)$ for all s, which implies that f is a constant function, contradicting our assumption. Therefore, we must have $f(s) \neq f(0)$ whenever $s \neq 0$.

We will now show that if $f(x) = f(y)$ holds, then either $x = y$ or $x = -y$ must hold. Suppose on the contrary that $f(x_0) = f(y_0)$ holds for some pair of non-zero numbers x_0, y_0 for which $x_0 \neq y_0, x_0 \neq -y_0$. Since $f(-y_0) = f(y_0)$, we may assume, by replacing y_0 by $-y_0$ if necessary, that x_0 and y_0 have the same sign. In view of $(*)$, we see that $f(sx_0) = f(sy_0)$ holds for all s, and therefore, there exists some $r > 0, r \neq 1$ such that

$$f(x) = f(rx)$$

holds for all x. Replacing x by rx and y by ry in the given functional equation, we obtain

$$f(f(rx) + f(ry) + f(z)) = f(f(rx) - f(ry)) + f(2r^2xy + f(z)) + 2f(r(x-y)z) \qquad \text{(i)},$$

and replacing x by r^2x in the functional equation, we get

$$f(f(r^2x) + f(y) + f(z)) = f(f(r^2x) - f(y)) + f(2r^2xy + f(z)) + 2f((r^2x - y)z) \quad \text{(ii)}.$$

Since $f(rx) = f(x)$ holds for all $x \in \mathbf{R}$, we see that except for the last term on the right-hand side, all the corresponding terms appearing in the identities (i) and (ii) above are equal, and hence we conclude that

$$f(r(x-y)z) = f((r^2x - y)z)) \qquad \text{(iii)}$$

must hold for arbitrary choice of $x, y, z \in \mathbf{R}$. For arbitrarily fixed pair $u, v \in \mathbf{R}$, substitute $(x, y, z) = (\frac{v-u}{r^2-1}, \frac{v-r^2u}{r^2-1}, 1)$ into the identity (iii). Then we obtain $f(v) = f(ru) = f(u)$, since $x - y = u, r^2x - y = v, z = 1$. But this implies that the function f is a constant, contradicting our assumption. Thus we conclude that if $f(x) = f(y)$ then either $x = y$ or $x = -y$ must hold.

By substituting $z = 0$ in the functional equation, we get

$$f(f(x) + f(y) + f(0)) = f(f(x) - f(y) + f(0)) = f((f(x) - f(y)) + f(2xy + f(0)) + 2f(0).$$

Changing y to $-y$ in the identity above and using the fact that $f(y) = f(-y)$, we see that all the terms except the second term on the right-hand side in the identity above remain the same. Thus we conclude that $f(2xy + f(0)) = f(-2xy + f(0))$, from which we get either $2xy + f(0) = -2xy + f(0)$ or $2xy + f(0) = 2xy - f(0)$ for all $x, y \in \mathbf{R}$. The first of these alternatives says that $4xy = 0$, which is impossible if $xy \neq 0$. Therefore the second alternative must be valid and we get that $f(0) = 0$.

Finally, let us show that if f satisfies the given functional equation and is not a constant function, then $f(x) = x^2$. Let $x = y$ in the functional equation, then since $f(0) = 0$, we get

$$f(2f(x) + f(z)) = f(2x^2 + f(z)),$$

from which we conclude that either $2f(x) + f(z) = 2x^2 + f(z)$ or $2f(x) + f(z) = -2x^2 - f(z)$ must hold. Suppose there exists x_0 for which $f(x_0) \neq x_0^2$, then from the second alternative, we see that $f(z) = -f(x_0) - x_0^2$ must hold for all z, which means that f must be a constant function, contrary to our assumption. Therefore, the first alternative above must hold, and we have $f(x) = x^2$ for all x, establishing our claim.

It is easy to check that $f(x) = x^2$ does satisfy the given functional equation, so we conclude that $f(x) = 0$ and $f(x) = x^2$ are the only functions that satisfy the requirement.

- **Summary of Results**

Rank	2010 Participants	# of Contestant	Total Score	Gold Awards	Silver Awards	Bronze Awards	Hon. Men.
1	Republic of KOREA	10	265	1	2	4	3
2	RUSSIA	10	264	1	2	4	3
3	United States of America	10	236	1	2	4	3
4	THAILAND	10	223	1	2	4	3
5	JAPAN	10	221	1	2	4	3
6	CANADA	10	215	1	2	4	3
7	HONG KONG	10	190	1	2	4	3
8	SINGAPORE	10	188	1	2	4	3
9	AUSTRALIA	10	185	1	2	4	3
10	TAIWAN	10	183	1	2	4	3
11	KAZAKHSTAN	10	179	1	2	4	3
12	PERU	10	166	1	1	5	3
13	ARGENTINA	10	162	1	2	4	3
14	BRAZIL	10	143	0	3	4	0
15	MEXICO	10	143	0	2	5	1
16	AZERBAIJAN	10	124	0	2	2	0
17	MALAYSIA	10	122	0	2	2	1
18	COLOMBIA	10	103	0	0	3	0
19	TURKMENISTAN	10	97	0	0	3	0
20	NEW ZEALAND	10	86	0	0	1	0
21	KYRGYZ	10	84	0	0	1	0
22	CAMBODIA	10	83	0	0	1	0
23	INDONESIA	10	79	0	0	0	1
24	BANGLADESH	10	76	0	0	0	0
25	PHILIPPINES	9	75	0	0	1	0
26	TAJIKISTAN	9	70	0	0	0	0
27	PANAMA	6	55	0	0	1	0
28	EL SALVADOR	7	41	0	0	0	0
29	COSTA RICA	7	35	0	0	0	0
30	ECUADOR	5	35	0	0	1	1
31	URUGUAY	5	35	0	0	1	0
32	TRINIDAD AND TOBAGO	10	21	0	0	0	0
33	QATAR	10	♦	0	0	0	0
	Total	**298**	**4184**	**13**	**34**	**79**	**43**

m : 14.040 / σ : 7.393	m + σ = 21.433	m + σ /3 = 16.504	m - σ /3 = 11.576
Awards Cut-offs :	**Gold ≥ 22**	**Silver ≥ 17**	**Bronze ≥ 14**

# of Problems	**1**	**2**	**3**	**4**	**5**	**Total**
Mean(m) :	6.35	2.05	3.67	1.06	0.9	14.04

♦ Coordination of the marking not complete hence the number of the participants from Qatar are excluded from the total above.

• **Award**

Award	Names of Contestant	Country
GOLD AWARD	Prilo Sebastian	Argentina
	Liang Alfred	Australia
	Deng Calvin	Canada
	Ching Tak Wing	Hong Kong
	Hosaka Kazuhiro	Japan
	Satylkhanov Kanat	Kazakhstan
	Song Sanghun	Korea
	Raul Arturo Chavez Sarmiento	Peru
	Matdinov Marsel	Russia
	Loh Yao ChenIvan	Singapore
	Wu Yi-Chan	Taiwan
	Manurangsi Pasin	Thailand
	Gu Albert	U.S.A.
SILVER AWARD	Stefanich German Ezequiel	Argentina
	Zylber Ariel Ricardo	Argentina
	Large Timothy	Australia
	Chong Aaron	Australia
	Shukurlu Altun	Azerbaijan
	Mustafayev Elshad	Azerbaijan
	De Sa Oliveira Sales Marcelo Tadeu	Brazil
	Secco Torres da Silva Matheus	Brazil
	Camelo Sa Joao Lucas	Brazil
	Sun Chen	Canada
	Sardarli Mariya	Canada
	Hung Ka Kin	Hong Kong
	Chan Kwun Tat	Hong Kong
	Kishikawa Akio	Japan
	Minegishi Ryu	Japan
	Kalsin Sergey	Kazakhstan
	Ovchinnikov Denis	Kazakhstan
	Kim Bumsoo	Korea
	Lyoo Youngwook	Korea
	How Si Yu	Malaysia
	Tham Ying Hong	Malaysia
	Perales Daniel	Mexico
	Hernandez Flavio	Mexico
	Jose Gustavo Garcia Sulca	Peru
	Mokin Vasiliy	Russia
	Erokhin Stanislav	Russia
	Aw JinAlan	Singapore
	Ang Jie Jun	Singapore
	Hsia Chih-Yang	Taiwan
	Wang Szu-Po	Taiwan
	Kamtue Supanat	Thailand
	Charoenngam Nipith	Thailand

	Phan Toan Duc	U.S.A.
	Keller Sam	U.S.A.
BRONZE AWARD	Cogorno Nicolas Alejandro	Argentina
	Cufre Juan Manuel	Argentina
	Umfurer Alfredo Raul	Argentina
	Sadofschi Costa Ivan	Argentina
	Wong Sampson	Australia
	Gorey Declan	Australia
	Law Stacey	Australia
	De Silva Raveen	Australia
	Rustamli Subhan	Azerbaijan
	Bagirov Naib	Azerbaijan
	Lisboa Empinotti Gustavo	Brazil
	Barbosa Alves Deborah	Brazil
	Mendes Silva Maria Clara	Brazil
	De Andrade Silva Carios Henrique	Brazil
	Vuthea Vong	Cambodia
	Zung Jonathan	Canada
	Rickards James	Canada
	Zhu Yuqi	Canada
	Cheng Robin	Canada
	Azuero Juan Camilo	Colombia
	Castaneda Angela	Colombia
	Arcila David	Colombia
	Xavier Andrew Soriano Diaz	Ecuador
	Wong Mon Lok	Hong Kong
	Lo Jing Hoi	Hong Kong
	Lo Chun Tung	Hong Kong
	Chow Tseung Man	Hong Kong
	Takigiku Motoki	Japan
	Noyama Tomoyuki	Japan
	Koshiyama Hiroki	Japan
	Matsuo Yoshinori	Japan
	Zhakupov Kairulla	Kazakhstan
	Yermakhanov Nurbergen	Kazakhstan
	Zhanbulatufy Medet	Kazakhstan
	Duisenbayev Azamat	Kazakhstan
	Yim Joonhyuk	Korea
	Jee Eunsoo	Korea
	Hwang Sunghyun	Korea
	Park Hyunyoung	Korea
	Mamarasul uulu Mamatkasim	Kyrgyzstan
	How Si Wei	Malaysia
	Lim Joshua Kai Tsen	Malaysia
	Anorve Fernando	Mexico
	Miranda Jose-Luis	Mexico
	Calderon Irving	Mexico

	Guardiola Jose	Mexico
	Diaz Julio	Mexico
	Zhang Robert	New Zealand
	Fan Antonio	Panama
	Juan Paucar Zanabria	Peru
	Jesus Angel Advincula Altamirano	Peru
	Josue Benjamin Garcia Piscoya	Peru
	Jose Ruben Matta Ramos	Peru
	Alejandro Omar Loyola Bartra	Peru
	Lao Carmela Antoinette	Philippines
	Pakharev Alexei	Russia
	Ivlev Fedor	Russia
	Menshchikov Andrei	Russia
	Omelyanenko Victor	Russia
	Chan Jun NengRyan	Singapore
	Tng Jia HaoBarry	Singapore
	Wong Ying LinGabriel	Singapore
	Lim Jeck	Singapore
	Chen Brain	Taiwan
	Chen Wei-Jung	Taiwan
	Chen Yung-Hsin	Taiwan
	Lin Wu-Cheng	Taiwan
	Ruangwises Suthee	Thailand
	Tantipongpipat Uthaipon	Thailand
	Vachiraprasith Kamron	Thailand
	Jiradilok Pakawut	Thailand
	Atayev Serdar	Turkmenistan
	Hajyyev Rovshen	Turkmenistan
	Samedov Agageldi	Turkmenistan
	Rodriguez Ismael	Uruguay
	Chu Tim	U.S.A.
	He Xiaoyu	U.S.A.
	Yang David	U.S.A.
	Wu Tiangi	U.S.A.
	Pedraza Lucia Ines	Argentina
	Juncal Mariano	Argentina
	Lang Carolina	Argentina
	Li Michael	Australia
	Yu Angel	Australia
	Ryba Christopher	Australia
	Song Alex	Canada
	Hemmati Soroosh	Canada
	Jiang Heinrich	Canada
	Bruno Giuseppe Poggi Cevallos	Ecuador
	Tam Ka Yu	Hong Kong
	Wong Ching	Hong Kong
	Lee Man Ching	Hong Kong

H.MEN AWARD	Lingga Ivan Wangsa Cipta @	Indonesia
	Nakasuka Kengo	Japan
	Higaki Motohide	Japan
	Yoshida Kensuke	Japan
	Khadjimuratov Nursultan	Kazakhstan
	Nurgabulov Yerniyaz	Kazakhstan
	Rakhymzhan Altai	Kazakhstan
	Kim Sungho	Korea
	Park Sunggi	Korea
	Lee Sanghoon	Korea
	Teo Wee Seong	Malaysia
	Arreola Fernando	Mexico
	Hector Moises Alcala Ramos	Peru
	Edison Jordy Meza Ramos	Peru
	Javier Perales Valerio	Peru
	Tyshchuk Konstantin	Russia
	Egorov Dmitry	Russia
	Bernshtein Anton	Russia
	Kuan Joseph	Singapore
	Lee You Jun	Singapore
	Lin Wayne	Singapore
	Wu Sheng-Hsien	Taiwan
	Chen Ting-Wei	Taiwan
	Lin Yu-Hsuan	Taiwan
	Sothanaphan Nat	Thailand
	Siltham Sira	Thailand
	Ponvilawan Ben	Thailand
	Firoiu Vlad	U.S.A.
	Yang Dai	U.S.A.
	Lee Mitchell	U.S.A.

The Mathematical Olympiad Foundation of Japan

2011 APMO PROBLEMS

Time allowed: 4 hours *Each problem is worth 7 points*

**The contest problems are to be kept confidential until they are posted on the official APMO website (http://www.mmjp.or.jp/competitions/APMO). Please do not disclose nor discuss the problems over the internet until that date. Calculators are not allowed to use.*

Problem 1. Let a, b, c be positive integers. Prove that it is impossible to have all of the three numbers $a^2 + b + c,\ b^2 + c + a,\ c^2 + a + b$ to be perfect squares.

Problem 2. Five points A_1, A_2, A_3, A_4, A_5 lie on a plane in such a way that no three among them lie on a same straight line. Determine the maximum possible value that the minimum value for the angles $\angle A_i A_j A_k$ can take where i, j, k are distinct integers between 1 and 5.

Problem 3. Let ABC be an acute triangle with $\angle BAC = 30°$. The internal and external angle bisectors of $\angle ABC$ meet the line AC at B_1 and B_2, respectively, and the internal and external angle bisectors of $\angle ACB$ meet the line AB at C_1 and C_2, respectively. Suppose that the circles with diameters B_1B_2 and C_1C_2 meet inside the triangle ABC at point P. Prove that $\angle BPC = 90°$.

Problem 4. Let n be a fixed positive odd integer. Take $m + 2$ **distinct** points $P_0, P_1, \cdots, P_{m+1}$ (where m is a non-negative integer) on the coordinate plane in such a way that the following 3 conditions are satisfied:
(1) $P_0 = (0, 1),\ P_{m+1} = (n+1, n)$, and for each integer i, $1 \le i \le m$, both x- and y- coordinates of P_i are integers lying in between 1 and n (1 and n inclusive).
(2) For each integer i, $0 \le i \le m$, P_iP_{i+1} is parallel to the x-axis if i is even, and is parallel to the y-axis if i is odd.
(3) For each pair i, j with $0 \le i < j \le m$, line segments P_iP_{i+1} and P_jP_{j+1} share at most 1 point.

Determine the maximum possible value that m can take.

Problem 5. Determine all functions $f : \mathbf{R} \to \mathbf{R}$, where $\mathbf{R}$ is the set of all real numbers, satisfying the following 2 conditions:
(1) There exists a real number M such that for every real number x, $f(x) < M$ is satisfied.
(2) For every pair of real numbers x and y,

$$f(xf(y)) + yf(x) = xf(y) + f(xy)$$

is satisfied.

SOLUTIONS FOR 2011 APMO PROBLEMS

Problem 1.

Solution: Suppose all of the 3 numbers a^2+b+c, b^2+c+a and c^2+a+b are perfect squares. Then from the fact that a^2+b+c is a perfect square bigger than a^2 it follows that $a^2+b+c \geq (a+1)^2$, and therefore, $b+c \geq 2a+1$. Similarly we obtain $c+a \geq 2b+1$ and $a+b \geq 2c+1$.

Adding the corresponding sides of the preceding 3 inequalities, we obtain $2(a+b+c) \geq 2(a+b+c)+3$, a contradiction. This proves that it is impossible to have all the 3 given numbers to be perfect squares.

Alternate Solution: Since the given conditions of the problem are symmetric in a, b, c, we may assume that $a \geq b \geq c$ holds. From the assumption that a^2+b+c is a perfect square, we can deduce as in the solution above the inequality $b+c \geq 2a+1$. But then we have

$$2a \geq b+c \geq 2a+1,$$

a contradiction, which proves the assertion of the problem.

Problem 2.

Solution: We will show that 36° is the desired answer for the problem.

First, we observe that if the given 5 points form a regular pentagon, then the minimum of the angles formed by any triple among the five vertices is 36°, and therefore, the answer we seek must be bigger than or equal to 36°.

Next, we show that for any configuration of 5 points satisfying the condition of the problem, there must exist an angle smaller than or equal to 36° formed by a triple chosen from the given 5 points. For this purpose, let us start with any 5 points, say A_1, A_2, A_3, A_4, A_5, on the plane satisfying the condition of the problem, and consider the smallest convex subset, call it Γ, in the plane containing all of the 5 points. Since this convex subset Γ must be either a triangle or a quadrilateral or a pentagon, it must have an interior angle with 108° or less. We may assume without loss of generality that this angle is $\angle A_1A_2A_3$. By the definition of Γ it is clear that the remaining 2 points A_4 and A_5 lie in the interior of the angular region determined by $\angle A_1A_2A_3$, and therefore, there must be an angle smaller than or equal to $\frac{1}{3} \cdot 108^\circ = 36^\circ$, which is formed by a triple chosen from the given 5 points, and this proves that 36° is the desired maximum.

Problem 3.

Solution: Since $\angle B_1BB_2 = 90^\circ$, the circle having B_1B_2 as its diameter goes through the points B, B_1, B_2. From $B_1A : B_1C = B_2A : B_2C = BA : BC$, it follows that this circle is the Apolonius circle with the ratio of the distances from the points A and C being $BA : BC$. Since the point P lies on this circle, we have

$$PA : PC = BA : BC = \sin C : \sin A,$$

from which it follows that $PA \sin A = PC \sin C$. Similarly, we have $PA \sin A = PB \sin B$, and therefore, $PA \sin A = PB \sin B = PC \sin C$.

Let us denote by D, E, F the foot of the perpendicular line drawn from P to the line segment BC, CA and AB, respectively. Since the points E, F lie on a circle having PA as its diameter, we have by the law of sines $EF = PA \sin A$. Similarly, we have $FD = PB \sin B$ and $DE = PC \sin C$. Consequently, we conclude that DEF is an equilateral triangle. Furthermore, we have $\angle CPE = \angle CDE$, since the quadrilateral $CDPE$ is cyclic. Similarly, we have $\angle FPB = \angle FDB$. Putting these together, we get

$$\begin{aligned}\angle BPC &= 360^\circ - (\angle CPE + \angle FPB + \angle EPF)\\ &= 360^\circ - \{(\angle CDE + \angle FDB) + (180^\circ - \angle FAE)\}\\ &= 360^\circ - (120^\circ + 150^\circ) = 90^\circ,\end{aligned}$$

which proves the assertion of the problem.

Alternate Solution: Let O be the midpoint of the line segment B_1B_2. Then the points B and P lie on the circle with center at O and going through the point B_1. From

$$\angle OBC = \angle OBB_1 - \angle CBB_1 = \angle OB_1B - \angle B_1BA = \angle BAC$$

it follows that the triangles OCD and OBA are similar, and therefore we have that $OC \cdot OA = OB^2 = OP^2$. Thus we conclude that the triangles OCP and OPA are similar, and therefore, we have $\angle OPC = \angle PAC$. Using this fact, we obtain

$$\begin{aligned}\angle PBC - \angle PBA &= (\angle B_1BC + \angle PBB_1) - (\angle ABB_1 - \angle PBB_1)\\ &= 2\angle PBB_1 = \angle POB_1 = \angle PCA - \angle OPC\\ &= \angle PCA - \angle PAC,\end{aligned}$$

from which we conclude that $\angle PAC + \angle PBC = \angle PBA + \angle PCA$. Similarly, we get $\angle PAB + \angle PCB = \angle PBA + \angle PCA$. Putting these facts together and taking into account the fact that

$$(\angle PAC + \angle PBC) + (\angle PAB + \angle PCB) + (\angle PBA + \angle PCA) = 180^\circ,$$

we conclude that $\angle PBA + \angle PCA = 60^\circ$, and finally that

$$\angle BPC = (\angle PBA + \angle PAB) + (\angle PCA + \angle PAC) = \angle BAC + (\angle PBA + \angle PCA) = 90^\circ,$$

proving the assertion of the problem.

Problem 4.

Solution: We will show that the desired maximum value for m is $n(n-1)$.

First, let us show that $m \le n(n-1)$ always holds for any sequence $P_0, P_1, \cdots, P_{m+1}$ satisfying the conditions of the problem.

Call a point a **turning point** if it coincides with P_i for some i with $1 \le i \le m$. Let us say also that 2 points $\{P, Q\}$ are **adjacent** if $\{P, Q\} = \{P_{i-1}, P_i\}$ for some i with $1 \le i \le m$, and **vertically adjacent** if, in addition, PQ is parallel to the y-axis.

Any turning point is vertically adjacent to exactly one other turning point. Therefore, the set of all turning points is partitioned into a set of pairs of points using the relation of "vertical adjacency". Thus we can conclude that if we fix $k \in \{1, 2, \cdots, n\}$, the number of turning points having the x-coordinate k must be even, and hence it is less than or equal to $n - 1$. Therefore, altogether there are less than or equal to $n(n-1)$ turning points, and this shows that $m \le n(n-1)$ must be satisfied.

It remains now to show that for any positive odd number n one can choose a sequence for which $m = n(n-1)$. We will show this by using the mathematical induction on n. For $n = 1$, this is clear. For $n = 3$, choose

$$P_0 = (0,1), \qquad P_1 = (1,1), \qquad P_2 = (1,2), \qquad P_3 = (2,2),$$

$$P_4 = (2,1), \qquad P_5 = (3,1), \qquad P_6 = (3,3), \qquad P_7 = (4,3).$$

It is easy to see that these points satisfy the requirements (See fig. 1 below).

figure 1

Let n be an odd integer ≥ 5, and suppose there exists a sequence satisfying the desired conditions for $n-4$. Then, it is possible to construct a sequence which gives a configuration indicated in the following diagram (fig. 2), where the configuration inside of the dotted square is given by the induction hypothesis:

figure 2

By the induction hypothesis, there are exactly $(n-4)(n-5)$ turning points for the configuration inside of the dotted square in the figure 2 above, and all of the lattice points in the figure 2 lying outside of the dotted square except for the 4 points $(n, 2)$, $(n-1, n-2)$, $(2, 3)$, $(1, n-1)$ are turning points. Therefore, the total number of turning points in this configuration is

$$(n-4)(n-5) + (n^2 - (n-4)^2 - 4) = n(n-1),$$

showing that for this n there exists a sequence satisfying the desired properties, and thus completing the induction process.

Problem 5.

Solution: By substituting $x = 1$ and $y = 1$ into the given identity we obtain $f(f(1)) = f(1)$. Next, by substituting $x = 1$ and $y = f(1)$ into the given identity and using $f(f(1)) = f(1)$, we get $f(1)^2 = f(1)$, from which we conclude that either $f(1) = 0$ or $f(1) = 1$. But if $f(1) = 1$, then substituting $y = 1$ into the given identity, we get $f(x) = x$ for all x, which contradicts the condition (1). Therefore, we must have $f(1) = 0$.

By substituting $x = 1$ into the given identity and using the fact $f(1) = 0$, we then obtain $f(f(y)) = 2f(y)$ for all y. This means that if a number t belongs to the range of the function f, then so does $2t$, and by induction we can conclude that for any non-negative integer n, $2^n t$ belongs to the range of f if t does. Now suppose that there exists a real number a for which $f(a) > 0$, then for any non-negative integer n $2^n f(a)$ must belong to the range of f, which leads to a contradiction to the condition (1). Thus we conclude that $f(x) \leq 0$ for any real number x.

By substituting $\frac{x}{2}$ for x and $f(y)$ for y in the given identity and using the fact that $f(f(y)) = 2f(y)$, we obtain

$$f(xf(y)) + f(y)f\left(\frac{x}{2}\right) = xf(y) + f\left(\frac{x}{2}f(y)\right),$$

from which it follows that $xf(y) - f(xf(y)) = f(y)f\left(\frac{x}{2}\right) - f\left(\frac{x}{2}f(y)\right) \geq 0$, since the values of f are non-positive. Combining this with the given identity, we conclude that $yf(x) \geq f(xy)$. When $x > 0$, by letting y to be $\dfrac{1}{x}$ and using the fact that $f(1) = 0$, we get $f(x) \geq 0$. Since $f(x) \leq 0$ for any real number x, we conclude that $f(x) = 0$ for any positive real number x. We also have $f(0) = f(f(1)) = 2f(1) = 0$.

If f is identically 0, i.e., $f(x) = 0$ for all x, then clearly, this f satisfies the given identity. If f satisfies the given identity but not identically 0, then there exists a $b < 0$ for which $f(b) < 0$. If we set $c = f(b)$, then we have $f(c) = f(f(b)) = 2f(b) = 2c$. For any negative real number x, we have $cx > 0$ so that $f(cx) = f(2cx) = 0$, and by substituting $y = c$ into the given identity, we get

$$f(2cx) + cf(x) = 2cx + f(cx),$$

from which it follows that $f(x) = 2x$ for any negative real x.

We therefore conclude that if f satisfies the given identity and is not identically 0, then f is of the form $f(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ 2x & \text{if } x < 0. \end{cases}$ Finally, let us show that the function f of the form shown above does satisfy the conditions of the problem. Clearly, it satisfies the condition (1). We can check that f satisfies the condition (2) as well by separating into the following 4 cases depending on whether x, y are non-negative or negative.

- when both x and y are non-negative, both sides of the given identity are 0.
- when x is non-negative and y is negative, we have $xy \leq 0$ and both sides of the given identity are $4xy$.

- when x is negative and y is non-negative, we have $xy \leq 0$ and both sides of the given identity are $2xy$.
- when both x and y are negative, we have $xy > 0$ and both sides of the given identity are $2xy$.

Summarizing the arguments above, we conclude that the functions f satisfying the conditions of the problem are

$$f(x) = 0 \qquad \text{and} \qquad f(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ 2x & \text{if } x < 0. \end{cases}$$

The following up-dated reports of the results of APMO 2011 are changed slightly from the version reported at the time of the General Meeting of APMO 2011 held at the time of 52nd IMO. The change was necessitated by the inclusion of the results of the Costa Rica team, consisting of 5 students (including 1 gold medalist and 2 copper medalists). We verified that the test was administered on the specified day (March 7) and the marking of the papers was done properly, but for some reason we did not receive the report of the results until late May, which was too late for the preparation of our initial report.

• Summary of Results

Rank	2011 Participants	# of Contestant	Total Score	Gold Awards	Silver Awards	Bronze Awards	Hon. Men.
1	KOREA	10	289	1	2	4	3
2	USA	10	269	1	2	4	3
3	THAILAND	10	228	1	2	4	3
4	PERU	10	223	1	2	4	3
5	TAIWAN	10	220	1	2	4	3
6	JAPAN	10	208	1	2	4	3
7	RUSSIA	10	205	1	2	4	3
8	SINGAPORE	10	205	1	2	4	3
9	BRAZIL	10	190	1	2	4	3
10	HONG KONG	10	173	1	2	4	3
11	CANADA	10	164	1	2	4	3
12	AUSTRALIA	10	156	0	3	4	3
13	KAZAKHSTAN	10	154	0	3	4	3
14	MEXICO	10	141	0	3	4	3
15	INDONESIA	10	135	0	2	5	3
16	NEW ZEALAND	10	128	1	1	5	1
17	ARGENTINA	10	121	0	2	5	2
18	MALAYSIA	10	108	1	0	4	1
19	AZERBAIJAN	10	101	0	2	4	0
20	TAJIKISTAN	10	82	0	0	3	0
21	TURKMENISTAN	10	78	0	1	2	0
22	SAUDI ARABIA	10	60	0	1	1	1
23	PHILIPPINES	10	59	0	0	2	0
24	COSTA RICA	5	49	1	0	2	0
25	COLOMBIA	10	41	0	0	0	1
26	CAMBODIA	10	41	0	0	0	1
27	BANGLADESH	5	26	0	0	0	1
28	ECUADOR	10	24	0	0	0	1
29	URUGUAY	2	20	0	0	1	0
30	TRINIDAD AND TOBAGO	10	18	0	0	0	0
31	EL SALVADOR	9	16	0	0	1	0
32	COTE D'IVOIRE	3	11	0	0	0	0
33	PAKISTAN	10	8	0	0	0	0
34	SYRIA	10	5	0	0	0	0
35	QATAR	7	1	0	0	0	0
♦	Total	317	3961	14	41	92	53

m : 12.495 / σ : 8.892	m + σ = 21.387	m + σ /3 = 15.459	m - σ /3 = 9.531
Awards Cut-offs :	**Gold ≥ 22**	**Silver ≥ 16**	**Bronze ≥ 10**

# of Problems	**1**	**2**	**3**	**4**	**5**	**Total**
Mean(m) :	4.79	4.08	1.61	0.6	1.41	12.5

♦ Panama has participated in the competition, but decided to withdraw from the consideration for ranking and awards. For this reason, data for Panama are not included from the tabulation above.

• Award

GOLD AWARD	Kim Young Ho	KOREA
	He Xiaoyu	USA
	Lim Jeck	SINGAPORE
	Chavez Sarmiento Raul Arturo	PERU
	Hua Shiang-Chih	TAIWAN
	Kishikawa Akio	JAPAN
	Pakharev Alexey	RUSSIA
	Kurutach Thanard	THAILAND
	Camelo Sa Joao Lucas	BRAZIL
	Song Alex	CANADA
	Granville Malcolm	NEW ZEALAND
	Li Yau Wing	HONG KONG
	Tham Ying Hong	MALAYSIA
	Jimenez Humberto	COSTA RICA
SILVER AWARD	Cao Wenyu	USA
	Park Jun Oh	KOREA
	Hwang Sung Hyun	KOREA
	Chen Wei-Jung	TAIWAN
	Garcia Sulca Jose Gustavo	PERU
	Bhutisaksang Theerawat	THAILAND
	O'dorney Evan	USA
	Kitamura Takuma	JAPAN
	Liao Wei-En	TAIWAN
	Braga Costa Andre Macieira	BRAZIL
	Lisboa Empinotti Gustavo	BRAZIL
	Koshiyama Hiroki	JAPAN
	Loyola Bartra Alejandro	PERU
	Skutin Alexander	RUSSIA
	Kuan Joseph	SINGAPORE
	Kor Ryan	SINGAPORE
	Charoenngam Nipith	THAILAND
	Grigorev Mikhail	RUSSIA
	Mustafayev Elshad	AZERBAIJAN
	Wo Bar Wai Barry	HONG KONG
	Lu Colin	AUSTRALIA
	Perales Daniel	MEXICO

	Zylber Ariel	ARGENTINA
	Brennan Matthew	CANADA
	Lingga Ivan Wangsa Cipta	INDONESIA
	Nurzhigit Nurbolat	KAZAKHSTAN
	Large Tim	AUSTRALIA
	Hernandez Flavio	MEXICO
	Spink Hunter	CANADA
	Ching Tak Wing	HONG KONG
	Wael Hussain Al Saeed	SAUDI ARABIA
	Prillo Sebastian	ARGENTINA
	Huseynli Shahin	AZERBAIJAN
	Khou Victor	AUSTRALIA
	Azvero Juan Camilo	COLOMBIA
	Gunardi Johan	INDONESIA
	Nurgabilov Ernyaz	KAZAKHSTAN
	Drobax Andrei	KAZAKHSTAN
	Rogue Diego	MEXICO
	Shanmuganathan Arun Chockalingam	NEW ZEALAND
	Hajyyev Rovshen	TURKMENISTAN
	Chang Jae Won	KOREA
	Moon Han Wool	KOREA
	Na Sang Hoon	KOREA
	Whang Sung Seob	KOREA
	Gu Albert	USA
	Deng Calvin	USA
	Yang David	USA
	Gunby Benjamin	USA
	Fiuza do Nascimento Henriqve Gasparini	BRAZIL
	Kasai Yumi	JAPAN
	Luyo Carbonero Paul Michael	PERU
	Gutierrez Taipe Gianmarco Jaime	PERU
	Yeh Chih-Kuan	TAIWAN
	Siltham Sira	THAILAND
	Figueroa Curo Jesus Alberto	PERU
	Yangjit Wijit	THAILAND
	Pitimanaaree Nipun	THAILAND
	Karntikoon Kritkorn	THAILAND
	Adachi Satoru	JAPAN
	Meza Ramos Edison Jordy	PERU
	Shabanov Lev	RUSSIA
	Krachun Dmitry	RUSSIA
	Ang Yan Sheng	SINGAPORE
	Ding Yue	SINGAPORE
	Chen Yu-An	TAIWAN
	Tsigler Alexander	RUSSIA
	Ang Jie Jun	SINGAPORE
	Chiang Hung	TAIWAN

BRONZE AWARD	Gorelov Ivan	RUSSIA
	Chang Lai-Ho	TAIWAN
	Kasaura Kazumi	JAPAN
	Yoshida Kensuke	JAPAN
	Chan Jun Neng, Ryan	SINGAPORE
	Lima Rossi Hanon Guy	BRAZIL
	Wong Sze Nga	HONG KONG
	Lum Kai Chun	HONG KONG
	Gorey Declan	AUSTRALIA
	Xu Yunning	AUSTRALIA
	Mendes Silva Maria Clara	BRAZIL
	Porto Lira Caique	BRAZIL
	Jiang Heinrich	CANADA
	Sun Susan	CANADA
	Au Lawrence	HONG KONG
	Tam Ka Yu	HONG KONG
	Musaev Temirlan	KAZAKHSTAN
	Gonzalez Carolina	ARGENTINA
	Gafarzade Miraga	AZERBAIJAN
	Shukurlu Altun	AZERBAIJAN
	Wong Rachel	AUSTRALIA
	Yu Angel	AUSTRALIA
	Spivak Daniel	CANADA
	Zhou Kaiven	CANADA
	Moektijono Tobi	INDONESIA
	Ibraimov Akhzhol	KAZAKHSTAN
	Daurihan Muhyt	KAZAKHSTAN
	Ezhebay Ashim	KAZAKHSTAN
	Belanger Georges	MEXICO
	Salgado Gaston	ARGENTINA
	Alekberzade Elvin	AZERBAIJAN
	Cifventes Jesus David	COLOMBIA
	Putra Tryan Aditya	INDONESIA
	Ananto Pramudya	INDONESIA
	Teh Anzo Zhao Yang	MALAYSIA
	Khong Yi Kye	MALAYSIA
	Medrano Adun	MEXICO
	Acevedo Joshva	MEXICO
	Anorve Feinando	MEXICO
	Zhang Robert	NEW ZEALAND
	Morrissey Benedict	NEW ZEALAND
	Allen James	NEW ZEALAND
	Morco Henry Jefferson	PHILIPPINES
	Sharipov Dyhakhongir	TAJIKISTAN
	Shodavlat Saidmuhammad	TAJIKISTAN
	Muqaymkhonov Musokhon	TAJIKISTAN
	Hojamammedov Kerven	TURKMENISTAN

	Agamyradov Palvan	TURKMENISTAN
	Peraza Javier	URUGUAY
	Heredia Ezequiel	COSTA RICA
	Bagirov Naib	AZERBAIJAN
	Harmanto Bivan Alzacky	INDONESIA
	Nassirudin Muhammad	INDONESIA
	Yan Tom	NEW ZEALAND
	Shin Ha Young	NEW ZEALAND
	Urbina Sanchez Gerardo Augusto	EL SALVADOR
	How Si Yu	MALAYSIA
	How Si Wei	MALAYSIA
	Assenza Franco	ARGENTINA
	Capretto Margarita	ARGENTINA
	Sever Jeremias	ARGENTINA
	Sy Amiel	PHILIPPINES
	Al-Yazeed Basyoni	SAUDI ARABIA
	Calderon Tomas	COSTA RICA
H.MEN AWARD	Lee Dong Hwan	KOREA
	Ki Do Hyeong	KOREA
	Jee Eun Soo	KOREA
	Ruangwises Suthee	USA
	Sothanaphan Nat	THAILAND
	Babbitt Matthew	USA
	Lee Mitchell	USA
	Cardenas Barriga Pablo Conzalo	PERU
	Tantipongpipat Uthaipon	THAILAND
	Jiradilok Pakawut	THAILAND
	Gol'tsova Nadezhda	RUSSIA
	Chiang Cheng-Min	TAIWAN
	Chen Brian	TAIWAN
	Kurozumi Atumasa	JAPAN
	Minegishi Ryu	JAPAN
	Simizu Genki	JAPAN
	Warton Cordero Alejandro Miguel	PERU
	Burova Olga	RUSSIA
	Wang Hsin-Po	TAIWAN
	Rocha de Melo Rafael Rodrigues	BRAZIL
	Nishida Kawai Daniel Eiti	BRAZIL
	de Oliveira Bitaraes Victor	BRAZIL
	Cheung Lap Chung Ivan	HONG KONG
	Cheung Hil Fung Harry	HONG KONG
	Ccopa Yugra Jesus Marcos	PERU
	Bolbachan Vasily	RUSSIA
	Ke Yuxuan	SINGAPORE
	Lin Wayne	SINGAPORE
	Aw Alan	SINGAPORE
	Chua Alexander	AUSTRALIA

	Chow Chi Hong	HONG KONG
	Fu Nancy	AUSTRALIA
	Chen Yuzhou	CANADA
	Sardarli Mariya	CANADA
	Zhou Kevin	CANADA
	Seylov Termirlan	KAZAKHSTAN
	Amankeldyin Akezhan	KAZAKHSTAN
	Zhanakhimetov Sultan	KAZAKHSTAN
	Al Akbar Gulbuddin Hikmatyar	INDONESIA
	Han George	NEW ZEALAND
	Papatoniou John	AUSTRALIA
	Gunawan Jeremiah Riker	INDONESIA
	Zaky Ahmad	INDONESIA
	Rivera Jose	MEXICO
	Domingvez Angel	MEXICO
	Orfiz Juan	MEXICO
	Umfurer Alfredo	ARGENTINA
	Rojas Damian	ARGENTINA
	Rattanak Rith	CAMBODIA
	Cortez Lemos Carlos Andres	ECUADOR
	Soh Chin Lip	MALAYSIA
	Hassan Alsibyani	SAUDI ARABIA
	Hasan Abid	BGD

The Mathematical Olympiad Foundation of Japan

2012 APMO PROBLEMS

Time allowed: 4 hours *Each problem is worth 7 points*

**The contest problems are to be kept confidential until they are posted on the official APMO website (http://www.mmjp.or.jp/competitions/APMO). Please do not disclose nor discuss the problems over the internet until that date. Calculators are not allowed to use.*

Problem 1. Let P be a point in the interior of a triangle ABC, and let D, E, F be the point of intersection of the line AP and the side BC of the triangle, of the line BP and the side CA, and of the line CP and the side AB, respectively. Prove that the area of the triangle ABC must be 6 if the area of each of the triangles PFA, PDB and PEC is 1.

Problem 2. Into each box of a 2012×2012 square grid, a real number greater than or equal to 0 and less than or equal to 1 is inserted. Consider splitting the grid into 2 non-empty rectangles consisting of boxes of the grid by drawing a line parallel either to the horizontal or the vertical side of the grid. Suppose that for at least one of the resulting rectangles the sum of the numbers in the boxes within the rectangle is less than or equal to 1, no matter how the grid is split into 2 such rectangles. Determine the maximum possible value for the sum of all the 2012×2012 numbers inserted into the boxes.

Problem 3. Determine all the pairs (p, n) of a prime number p and a positive integer n for which $\frac{n^p+1}{p^n+1}$ is an integer.

Problem 4. Let ABC be an acute triangle. Denote by D the foot of the perpendicular line drawn from the point A to the side BC, by M the midpoint of BC, and by H the orthocenter of ABC. Let E be the point of intersection of the circumcircle Γ of the triangle ABC and the half line MH, and F be the point of intersection (other than E) of the line ED and the circle Γ. Prove that $\frac{BF}{CF} = \frac{AB}{AC}$ must hold.

Here we denote by XY the length of the line segment XY.

Problem 5. Let n be an integer greater than or equal to 2. Prove that if the real numbers $a_1, a_2, \cdots, a_n$ satisfy $a_1^2 + a_2^2 + \cdots + a_n^2 = n$, then

$$\sum_{1 \le i < j \le n} \frac{1}{n - a_i a_j} \le \frac{n}{2}$$

must hold.

SOLUTIONS FOR 2012 APMO PROBLEMS

Problem 1.

Solution: Let us denote by $\triangle XYZ$ the area of the triangle XYZ. Let $x = \triangle PAB$, $y = \triangle PBC$ and $z = \triangle PCA$.

From

$$y : z = \triangle BCP : \triangle ACP = BF : AF = \triangle BPF : \triangle APF = (x-1) : 1$$

follows that $z(x-1) = y$, which yields $(z+1)x = x+y+z$. Similarly, we get $(x+1)y = x+y+z$ and $(y+1)z = x+y+z$. Thus, we obtain $(x+1)y = (y+1)z = (z+1)x$.

We may assume without loss of generality that $x \le y, z$. If we assume that $y > z$ holds, then we get $(y+1)z > (z+1)x$, which is a contradiction. Similarly, we see that $y < z$ leads to a contradiction $(x+1)y < (y+1)z$. Therefore, we must have $y = z$. Then, we also get from $(y+1)z = (z+1)x$ that $x = z$ must hold. We now obtain from $(x-1) : 1 = y : z = 1 : 1$ that $x = y = z = 2$ holds. Therefore, we conclude that the area of the triangle ABC equals $x+y+z = 6$.

Problem 2.

Solution: If we insert numbers as in the figure below ($0'$s are to be inserted in the remaining blank boxes), then we see that the condition of the problem is satisfied and the total number of all the numbers inserted is 5.

0	1	0	
1	1	1	
0	1	0	

We will show that the sum of all the numbers to be inserted in the boxes of the given grid cannot be more than 5 if the distribution of the numbers has to satisfy the requirement of the problem. Let $n = 2012$. Let us say that the row number (the column number) of a box in the given grid is i (j, respectively) if the box lies on the i-th row and the j-th column. For a pair of positive integers x and y, denote by $R(x, y)$ the sum of the numbers inserted in all of the boxes whose row number is greater than or equal to x and less than or equal to y (assign the value 0 if $x > y$).

First let a be the largest integer satisfying $1 \le a \le n$ and $R(1, a-1) \le 1$, and then choose the smallest integer c satisfying $a \le c \le n$ and $R(c+1, n) \le 1$. It is possible to choose such a pair a, c since $R(1, 0) = 0$ and $R(n+1, n) = 0$. If $a < c$, then we have $a < n$ and so, by the maximality of a, we must have $R(1, a) > 1$, while from the minimality of c, we must have $R(a+1, n) > 1$. Then by splitting the grid into 2 rectangles by means of the horizontal line bordering the a-th row and the $a+1$-th row, we get the splitting contradicting the requirement of the problem. Thus, we must have $a = c$.

Similarly, if for any pair of integers x, y we define $C(x, y)$ to be the sum of the numbers inserted in all of the boxes whose column number is greater than or equal to x and less than or equal to y ($C(x, y) = 0$ if $x > y$), then we get a number b for which

$$C(1, b-1) \le 1, \quad C(b+1, n) \le 1, \quad 1 \le b \le n.$$

If we let r be the number inserted in the box whose row number is a and the column number is b, then since $r \leq 1$, we conclude that the sum of the numbers inserted into all of the boxes is

$$\leq R(1, a-1) + R(a+1, n) + C(1, b-1) + C(b+1, n) + r \leq 5.$$

Problem 3.

Solution

For integers a, b and a positive integer m, let us write $a \equiv b \pmod{m}$ if $a - b$ is divisible by m. Since $\frac{n^p+1}{p^n+1}$ must be a positive integer, we see that $p^n \leq n^p$ must hold. This means that if $p = 2$, then $2^n \leq n^2$ must hold. As it is easy to show by induction that $2^n > n^2$ holds if $n \geq 5$, we conclude that if $p = 2$, then $n \leq 4$ must be satisfied. And we can check that $(p, n) = (2, 2), (2, 4)$ satisfy the condition of the problem, while $(2, 3)$ does not.

Next, we consider the case where $p \geq 3$.

Suppose s is an integer satisfying $s \geq p$. If $s^p \leq p^s$ for such an s, then we have

$$\begin{aligned}
(s+1)^p = s^p\left(1+\frac{1}{s}\right)^p &\leq p^s\left(1+\frac{1}{p}\right)^p \\
&= p^s \sum_{r=0}^{p} {}_pC_r \frac{1}{p^r} < p^s \sum_{r=0}^{p} \frac{1}{r!} \\
&\leq p^s\left(1 + \sum_{r=1}^{p} \frac{1}{2^{r-1}}\right) \\
&< p^s(1+2) \leq p^{s+1}
\end{aligned}$$

Thus we have $(s+1)^p < p^{s+1}$, and by induction on n, we can conclude that if $n > p$, then $n^p < p^n$. This implies that we must have $n \leq p$ in order to satisfy our requirement $p^n \leq n^p$.

We note that since $p^n + 1$ is even, so is $n^p + 1$, which, in turn implies that n must be odd and therefore, $p^n + 1$ is divisible by $p + 1$, and $n^p + 1$ is also divisible by $p+1$. Thus we have $n^p \equiv -1 \pmod{(p+1)}$, and therefore, $n^{2p} \equiv 1 \pmod{(p+1)}$.

Now, let e be the smallest positive integer for which $n^e \equiv 1 \pmod{(p+1)}$. Then, we can write $2p = ex + y$, where x, y are non-negative integers and $0 \leq y < e$, and we have

$$1 \equiv n^{2p} = (n^e)^x \cdot n^y \equiv n^y \pmod{(p+1)},$$

which implies, because of the minimality of e, that $y = 0$ must hold. This means that $2p$ is an integral multiple of e, and therefore, e must equal one of the numbers $1, 2, p, 2p$.

Now, if $e = 1, p$, then we get $n^p \equiv 1 \pmod{(p+1)}$, which contradicts the fact that p is an odd prime. Since n and $p + 1$ are relatively prime, we have by Euler's Theorem that $n^{\varphi(p+1)} \equiv 1 \pmod{(p+1)}$, where $\varphi(m)$ denotes the number of integers $j (1 \leq j \leq m)$ which are relatively prime with m. From $\varphi(p+1) < p+1 < 2p$ and the minimality of e, we can then conclude that $e = 2$ must hold.

From $n^2 \equiv 1 \pmod{(p+1)}$, we get

$$-1 \equiv n^p = n^{(2 \cdot \frac{p-1}{2} + 1)} \equiv n \pmod{(p+1)},$$

which implies that $p+1$ divides $n+1$. Therefore, we must have $p \leq n$, which, together with the fact $n \leq p$, show that $p = n$ must hold.

It is clear that the pair (p, p) for any prime $p \geq 3$ satisfies the condition of the problem, and thus, we conclude that the pairs (p, n) which satisfy the condition of the problem must be of the form $(2, 4)$ and (p, p) with any prime p.

Alternate Solution. Let us consider the case where $p \geq 3$. As we saw in the preceding solution, n must be odd if the pair (p, n) satisfy the condition of the problem. Now, let q be a prime factor of $p+1$. Then, since $p+1$ divides p^n+1, q must be a prime factor of p^n+1 and of n^p+1 as well. Suppose $q \geq 3$. Then, from $n^p \equiv -1 \pmod{q}$, it follows that $n^{2p} \equiv 1 \pmod{q}$ holds. If we let e be the smallest positive integer satisfying $n^e \equiv 1 \pmod{q}$, then by using the same argument as we used in the preceding solution, we can conclude that e must equal one of the numbers $1, 2, p, 2p$. If $e = 1, p$, then we get $n^p \equiv 1 \pmod{q}$, which contradicts the assumption $q \geq 3$. Since n is not a multiple of q, by Fermat's Little Theorem we get $n^{q-1} \equiv 1 \pmod{q}$, and therefore, we get by the minimality of e that $e = 2$ must hold. From $n^2 \equiv 1 \pmod{q}$, we also get

$$n^p = n^{(2 \cdot \frac{p-1}{2} + 1)} \equiv n \pmod{q},$$

and since $n^p \equiv -1 \pmod{q}$, we have $n \equiv -1 \pmod{q}$ as well.

Now, if $q = 2$ then since n is odd, we have $n \equiv -1 \pmod{q}$ as well. Thus, we conclude that for an arbitrary prime factor q of $p+1$, $n \equiv -1 \pmod{q}$ must hold.

Suppose, for a prime q, q^k for some positive integer k is a factor of $p+1$. Then q^k must be a factor of n^p+1 as well. But since

$$n^p + 1 = (n+1)(n^{p-1} - n^{p-2} + \cdots - n + 1) \quad \text{and}$$

$$n^{p-1} - n^{p-2} + \cdots - n + 1 \equiv (-1)^{p-1} - (-1)^{p-2} + \cdots - (-1) + 1 \not\equiv 0 \pmod{q},$$

we see that q^k must divide $n+1$. By applying the argument above for each prime factor q of $p+1$, we can then conclude that $n+1$ must be divisible by $p+1$, and as we did in the preceding proof, we can conclude that $n = p$ must hold.

Problem 4.

Solution: If $AB = AC$, then we get $BF = CF$ and the conclusion of the problem is clearly satisfied. So, we assume that $AB \neq AC$ in the sequel.

Due to symmetry, we may suppose without loss of generality that $AB > AC$. Let K be the point on the circle Γ such that AK is a diameter of this circle. Then, we get

$$\angle BCK = \angle ACK - \angle ACB = 90^\circ - \angle ACB = \angle CBH$$

and

$$\angle CBK = \angle ABK - \angle ABC = 90^\circ - \angle ABC = \angle BCH,$$

from which we conclude that the triangles BCK and CBH are congruent. Therefore, the quadrilateral $BKCH$ is a parallelogram, and its diagonal HK passes through the center M of the other diagonal BC. Therefore, the 3 points H, M, K lie on the same straight line, and we have $\angle AEM = \angle AEK = 90^\circ$.

From $\angle AED = 90^\circ = \angle ADM$, we see that the 4 points A, E, D, M lie on the circumference of the same circle, from which we obtain $\angle AMB = \angle AED = \angle AEF = \angle ACF$. Putting this fact together with the fact that $\angle ABM = \angle AFC$, we conclude that the triangles ABM and AFC are similar, and we get $\frac{AM}{BM} = \frac{AC}{FC}$. By a similar argument, we get that the triangles ACM and AFB are similar, and therefore, that $\frac{AM}{CM} = \frac{AB}{FB}$ holds. Noting that $BM = CM$, we also get $\frac{AC}{FC} = \frac{AB}{FB}$, from which we can conclude that $\frac{BF}{CF} = \frac{AB}{AC}$, proving the assertion of the problem.

Problem 5.

Solution: Let us note first that if $i \neq j$, then since $a_i a_j \leq \frac{a_i^2 + a_j^2}{2}$, we have

$$n - a_i a_j \geq n - \frac{a_i^2 + a_j^2}{2} \geq n - \frac{n}{2} = \frac{n}{2} > 0.$$

If we set $b_i = |a_i|$ $(i = 1, 2, \ldots, n)$, then we get $b_1^2 + b_2^2 + \cdots + b_n^2 = n$ and $\frac{1}{n - a_i a_j} \leq \frac{1}{n - b_i b_j}$, which shows that it is enough to prove the assertion of the problem in the case where all of $a_1, a_2, \cdots, a_n$ are non-negative. Hence, we assume from now on that $a_1, a_2, \cdots, a_n$ are all non-negative.

By multiplying by n the both sides of the desired inequality we get the inequality:

$$\sum_{1 \leq i < j \leq n} \frac{n}{n - a_i a_j} \leq \frac{n^2}{2}$$

and since $\frac{n}{n - a_i a_j} = 1 + \frac{a_i a_j}{n - a_i a_j}$, we obtain from the inequality above by subtracting $\frac{n(n-1)}{2}$ from both sides the following inequality:

$$\text{(i)} \qquad \sum_{1 \leq i < j \leq n} \frac{a_i a_j}{n - a_i a_j} \leq \frac{n}{2}$$

We will show that this inequality (i) holds.

If for some i the equality $a_i^2 = n$ is valid, then $a_j = 0$ must hold for all $j \neq i$ and the inequality (i) is trivially satisfied. So, we assume from now on that $a_i^2 < n$ is valid for each i.

Let us assume that $i \neq j$ from now on. Since $0 \leq a_i a_j \leq \left(\frac{a_i + a_j}{2}\right)^2 \leq \frac{a_i^2 + a_j^2}{2}$ holds, we have

$$\text{(ii)} \qquad \frac{a_i a_j}{n - a_i a_j} \leq \frac{a_i a_j}{n - \frac{a_i^2 + a_j^2}{2}} \leq \frac{\left(\frac{a_i + a_j}{2}\right)^2}{n - \frac{a_i^2 + a_j^2}{2}} = \frac{1}{2} \cdot \frac{(a_i + a_j)^2}{(n - a_i^2) + (n - a_j^2)}.$$

Since $n - a_i^2 > 0$, $n - a_j^2 > 0$, we also get from the Cauchy-Schwarz inequality that

$$\left(\frac{a_j^2}{n - a_i^2} + \frac{a_i^2}{n - a_j^2} \right) ((n - a_i^2) + (n - a_j^2)) \geq (a_i + a_j)^2,$$

from which it follows that

$$\text{(iii)} \qquad \frac{(a_i + a_j)^2}{(n - a_i^2) + (n - a_j^2)} \leq \left(\frac{a_j^2}{n - a_i^2} + \frac{a_i^2}{n - a_j^2} \right)$$

holds. Combining the inequalities (ii) and (iii), we get

$$\begin{aligned}\sum_{1\le i<j\le n}\frac{a_ia_j}{n-a_ia_j} &\le \frac{1}{2}\sum_{1\le i<j\le n}\left(\frac{a_j^2}{n-a_i^2}+\frac{a_i^2}{n-a_j^2}\right)\\ &= \frac{1}{2}\sum_{i\ne j}\frac{a_j^2}{n-a_i^2}\\ &= \frac{1}{2}\sum_{i=1}^{n}\frac{n-a_i^2}{n-a_i^2}\\ &= \frac{n}{2},\end{aligned}$$

which establishes the desired inequality (i).

• **Summary of Results**

Rank	2012 Participants	# of Contestant	Total Score	Gold Awards	Silver Awards	Bronze Awards	Hon. Men.
1	KOREA	10	333	1	2	4	3
2	USA	10	283	1	2	4	3
3	THAILAND	10	279	1	2	4	3
4	RUSSIA	10	249	1	2	4	3
5	JAPAN	10	241	1	2	4	3
6	TAIWAN	10	232	1	2	4	3
7	PERU	10	227	1	2	4	3
8	SINGAPORE	10	217	1	2	4	3
9	HONG KONG	10	214	1	2	4	3
10	CANADA	10	209	1	2	4	3
11	BRAZIL	10	202	1	2	4	3
12	MEXICO	10	177	1	2	4	3
13	INDONESIA	10	176	1	2	4	3
14	KAZAKHSTAN	10	167	0	3	4	3
15	AUSTRALIA	10	154	1	1	5	3
16	MALAYSIA	10	150	1	2	2	0
17	ARGENTINA	10	126	0	1	4	5
18	TURKMENISTAN	10	115	0	1	4	2
19	BANGLADESH	10	105	0	1	2	2
20	PHILIPPINES	10	97	0	1	1	4
21	NEW ZEALAND	10	81	0	0	2	4
22	SAUDI ARABIA	10	78	1	0	0	3
23	COLOMBIA	10	76	0	1	0	3
24	AZERBAIJAN	9	68	0	0	1	4
25	URUGUAY	5	66	0	0	3	2
26	TAJIKISTAN	9	63	0	0	2	1
27	PAKISTAN	10	60	0	0	1	2
28	MOROCOO	10	49	0	0	0	2
29	EL SALVADOR	4	46	0	1	1	0
30	CAMBODIA	10	45	0	0	0	1
31	Costa Rica	10	45	0	0	0	3
32	SYRIA	6	35	0	0	0	2
33	ECUADOR	8	28	0	1	0	0
34	Kyrgyz Republic	4	28	0	0	0	0
35	PANAMA	5	21	0	0	0	0
36	TRINIDAD AND TOBAGO	10	9	0	0	0	0
37	QATAR	3	5	0	0	0	0
	Total	333	4756	16	39	84	85

M : 14.369 / σ : 9.498	$m + \sigma = 23.866$	$m + \sigma/3 = 17.534$	$m - \sigma/3 = 11.203$
Awards Cut-offs :	**Gold ≥ 24**	**Silver ≥ 18**	**Bronze ≥ 12**

# of Problems	**1**	**2**	**3**	**4**	**5**	**Total**
Mean(m) :	5.08	3.76	2.92	2.59	0.47	14.37

The Mathematical Olympiad Foundation of Japan

•**Award**

Award	Names of Contestant	Country
GOLD AWARD	Park Tae Hwan	KOREA
	Yarg David	USA
	Laksanabunsong Pongsatorn	THAILAND
	How Si Yu	MALAYSIA
	Chow Chi Hong	HONG KONG
	Yeh Chih-Kuan	TAIWAN
	Angels Rodrigo Sanches	BRAZIL
	Brennan Matthew	CANADA
	Yumi Kasai	JAPAN
	Advincula Altamirano Jesus Angel	PERU
	Klyuev Daniil	RUSSIA
	Basyoni Al-Yazeed	SAUDI ARABIA
	Ang Yan Sheng	SINGAPORE
	Xu Yanning	AUSTRALIA
	Lie Stefanus	INDONESIA
	Garza Vargas Jorge	MEXICO
SILVER AWARD	Park Sung Jin	KOREA
	Kang Seungyeon	KOREA
	Rickards James	CANADA
	Ryu Minegishi	JAPAN
	Genki Shimizu	JAPAN
	Anzo The Zhao Yang	MALAYSIA
	Luyo Carbonero Paul	PERU
	Altamirano Modesto Christian Omar	PERU
	Ryabtseva Maria	RUSSIA
	Krachun Dmitry	RUSSIA
	Lim Jeck	SINGAPORE
	Kor Ryan	SINGAPORE
	Chang Keng-Chien	TAIWAN
	Lin Kuan-Yu	TAIWAN
	Jiradilok Pakawut	THAILAND
	Komolsiripakdi Tanat	THAILAND
	Tas James	USA
	Gao Zijing	USA
	Song Zhou Qun(Alex)	CANADA
	Camelo Sa Joas Lvcas	BRAZIL
	Tham Ying Hong	MALAYSIA
	Li Yau Wing	HONG KONG
	Loo Andy	HONG KONG
	Mendes Silva Maria Clara	BRAZIL
	Salazar Veronica	COLOMBIA
	Escobar Benitez Byron Thonatiu	EL SALVADOR
	Roque Montoya Diego	MEXICO
	Morco Henry Jefferson	PHILIPPINES

	Fu Nancy	AUSTRALIA
	Biswas Dhananjoy	BANGLADESH
	Cortez Lemos Carlos Andres	ECUADOR
	Zaky Ahmad	INDONESIA
	Dushayev Saken	KAZAKHSTAN
	Ibraimov Akzhol	KAZAKHSTAN
	Chiu Han Enrique	MEXICO
	Moektijono Tobi	INDONESIA
	Zylber Ariel Ricardo	ARGENTINA
	Imanmalik Yerzhan	KAZAKHSTAN
	Soltanov Merdan	TURKMENISTAN
	Kim Dong Hyo	KOREA
	Moon Hanwool	KOREA
	Kim Jae Joong	KOREA
	Jung Jong Wook	KOREA
	Kensuke Yoshida	JAPAN
	Pitimanaaree Nipun	THAILAND
	Brakensiek Joshua	USA
	Jagadeesan Ravi	USA
	Li Ray	USA
	Derg Calvin	USA
	Chavez Sarmiento Raul Arturo	PERU
	Krokhmal Nikolay	RUSSIA
	Anunrojwong Jirawat	THAILAND
	Charoenngam Nipith	THAILAND
	Karntikoon Kritkorn	THAILAND
	Grigorev Miknail	RUSSIA
	Baev Budimir	RUSSIA
	Wong Sze Nga	HONG KONG
	How Si Wei	MALAYSIA
	Perales Valerio Javier	PERU
	Shogo Murai	JAPAN
	Takuma Kitamura	JAPAN
	Hiroki Komatsu	JAPAN
	Napa Bernuy Angel Gerardo	PERU
	Ishkuvatov Ruslan	RUSSIA
	Liao Wei-En	TAIWAN
	Tsai Che-Ping	TAIWAN
	Wu Fei	CANADA
	Lee You Jun	SINGAPORE
	Ling Yan Hao	SINGAPORE
	Braga Costa Andre Macieira	BRAZIL
	Whalhey Alexander	CANADA
	Kung Man kit	HONG KONG
	Warton Cordero Alejandro Miguel	PERU
	Chen Ho-Chien	TAIWAN
	Yu Jia-Ruei	TAIWAN

BRONZE AWARD	Lai Leo	CANADA
	Yap Jit Wu	SINGAPORE
	Zhang Aidi	SINGAPORE
	Fiuza do Nascimento Henrigve Gasparini	BRAZIL
	Miyazaki Rafael Kazuhiro	BRAZIL
	de Alencar Severo Franco Mathevs	BRAZIL
	Au Lawrence	HONG KONG
	Tang Dik Man Damian	HONG KONG
	Lesmana Nixie Sapphira	INDONESIA
	Bonifacio Mariano Carlos	ARGENTINA
	Chua Alexander	AUSTRALIA
	Sanjaya Stephen	INDONESIA
	Koswara Ivan Adrian	INDONESIA
	Susan Fransisca	INDONESIA
	Nurgabylov Yerniyaz	KAZAKHSTAN
	Gonzalez Cazares Jorge	MEXICO
	Aztiazaran Tobin Alberto	MEXICO
	Ortiz Rhoton Juan	MEXICO
	Garcia Alejo	URUGUAY
	Kwong Jason	AUSTRALIA
	Khou Victor	AUSTRALIA
	Huseyuli Shahin	AZERBAIJAN
	Shorif Mugdho Mirza Md.Tanjim	BANGLADESH
	Liu Yi	CANADA
	Urbina Sanchez Gerardo Augusto	EL SALVADOR
	Kalmyrzayev Sergazy	KAZAKHSTAN
	Abdybayev Azamat	KAZAKHSTAN
	Medrano Martindel Campo Adan	MEXICO
	Shan Syed Waqar Ali	PAKISTAN
	Co Kenneth	PHILIPPINES
	Shodaolat Saidmuhammad	TAJIKISTAN
	Agamyradov Palvan	TURKMENISTAN
	Assensa Franco	ARGENTINA
	Chapman Kaimyn	AUSTRALIA
	Sailanbayev Alibek	KAZAKHSTAN
	Correa Ignacio	URUGUAY
	Staffa Bruno	ARGENTINA
	Gunning Alexander	AUSTRALIA
	Das Sourav	BANGLADESH
	Khong Yi Kye	MALAYSIA
	Han George	NEW ZEALAND
	Shanmugantahan Arun Chockalingam	NEW ZEALAND
	Shamsiddin Shahyodshohi	TAJIKISTAN
	Rahmanov Nazar	TURKMENISTAN
	Hojamammedov Kerven	TURKMENISTAN
	Goldsztajn Diego	URUGUAY
	Valiyev Ilshat	TURKMENISTAN

	Cogorno Nicolas Alejandro	ARGENTINA
HONOURABLE MENTION	Ki Do Hyeong	KOREA
	Schin Seung Won	KOREA
	Jung In Sung	KOREA
	Shen Bobby	USA
	Neeranartvong Weerachai	THAILAND
	Suaysom Natchanon	THAILAND
	Thongsai Poon	THAILAND
	Yarg Patrick	USA
	Lee Mitchell	USA
	Rukhovich Alexey	RUSSIA
	Kazumi Kasaura	JAPAN
	Yo Mitani	JAPAN
	Ostanin Alexander	RUSSIA
	Matushkin Alexander	RUSSIA
	Liao Yu-Hong	TAIWAN
	Cheng Li-Wei	TAIWAN
	Yam Yuen Sum	HONG KONG
	Lau Chun Ting	HONG KONG
	Chan Long Tin	HONG KONG
	Kenta Shinya	JAPAN
	Wu Yu-Sheng	TAIWAN
	Nishida Kawai Daniel Eiti	BRAZIL
	Tanadi Raymond	INDONESIA
	Ccopa Yugra Jesus Marcos	PERU
	Li Lawrence	SINGAPORE
	Tan Sheldon Kieren	SINGAPORE
	de Oliveira Pacanowski Alessandro A.P.	BRAZIL
	Zelazny Michel Rozenberg	BRAZIL
	Chen Letian	CANADA
	Zhou Kevin Kai Qi	CANADA
	Toribio Dionicio Ricardo Gabriel	PERU
	Yang Allen	CANADA
	Kulboldin Daniyar	KAZAKHSTAN
	Rojas Salvador Luiggui Angel	PERU
	Tan Way	SINGAPORE
	Young Peter Tirtowijoyo	INDONESIA
	Orynkul Batyrkhan	KAZAKHSTAN
	Amankeldi Akezhan	KAZAKHSTAN
	Diaz Calderon Julio	MEXICO
	Sanchez Gomez Jose	MEXICO
	Teran Rios Diego	MEXICO
	Lang Carolina	ARGENTINA
	Shu Mel	AUSTRALIA
	Hasanzade Suleyman	AZERBAIJAN
	Hasan Adib	BANGLADESH
	Rveda Maria Ximena	COLOMBIA

	Jimcnet Humberto	Costa Rica
	Calderon Tomas	Costa Rica
	Harmanto Bivan Alzacky	INDONESIA
	Allabayev Shanur	TURKMENISTAN
	Candioti Alejandro Marcelo	ARGENTINA
	Salgado Gaston	ARGENTINA
	Ragavan Seyoon	AUSTRALIA
	Calisa Vaishnavi	AUSTRALIA
	Aliyeu Izmir	AZERBAIJAN
	Hasanzade Hasan	AZERBAIJAN
	Ouaki Mohamed Mehdi	MOROCOO
	Jeng Hao	NEW ZEALAND
	Cho Byung-Cheol	NEW ZEALAND
	Awad Al-Kaream Siddig	SAUDI ARABIA
	Elmurod Khusraoi	TAJIKISTAN
	Chiazzo Ignacio	URUGUAY
	Siock Maximiliano	URUGUAY
	Sclar Melanie	ARGENTINA
	Bombau Ignacio Javier	ARGENTINA
	Acosia Francisco	COLOMBIA
	Ben Ajiba Mhamed Amine	MOROCOO
	Allen James	NEW ZEALAND
	Jamy Misha Nasir	PAKISTAN
	Sy Adrian Reginald	PHILIPPINES
	Uy Mikaela Angelina	PHILIPPINES
	Dee Camille Tyrene	PHILIPPINES
	Chuatak John Thomas	PHILIPPINES
	Abdal Jalil Eid Husun	SAUDI ARABIA
	Abdullayev Mustafa	AZERBAIJAN
	Kabir Ehsanul	BANGLADESH
	Thy So Thea	CAMBODIA
	Florez Leonardo	COLOMBIA
	Quesada Oscar	Costa Rica
	Qi Vincent	NEW ZEALAND
	Malik Usama Zaid	PAKISTAN
	Hamdi Yassine	SAUDI ARABIA
	Hamshow Mohammad Yazen	SYRIA
	Khadour Mohamed	SYRIA
	Bayramov Kerven	TURKMENISTAN

The Mathematical Olympiad Foundation of Japan

XXV Asian Pacific Mathematics Olympiad

Time allowed: 4 hours *Each problem if worth 7 points*

Problem 1. Let ABC be an acute triangle with altitudes AD, BE and CF, and let O be the center of its circumcircle. Show that the segments OA, OF, OB, OD, OC, OE dissect the triangle ABC into three pairs of triangles that have equal areas.

Problem 2. Determine all positive integers n for which $\dfrac{n^2+1}{[\sqrt{n}]^2+2}$ is an integer. Here $[r]$ denotes the greatest integer less than or equal to r.

Problem 3. For $2k$ real numbers $a_1, a_2, \ldots, a_k$, $b_1, b_2, \ldots, b_k$ define the sequence of numbers X_n by

$$X_n = \sum_{i=1}^{k} [a_i n + b_i] \quad (n = 1, 2, \ldots).$$

If the sequence X_n forms an arithmetic progression, show that $\sum_{i=1}^{k} a_i$ must be an integer. Here $[r]$ denotes the greatest integer less than or equal to r.

Problem 4. Let a and b be positive integers, and let A and B be finite sets of integers satisfying:

(i) A and B are disjoint;

(ii) if an integer i belongs either to A or to B, then $i + a$ belongs to A or $i - b$ belongs to B.

Prove that $a|A| = b|B|$. (Here $|X|$ denotes the number of elements in the set X.)

Problem 5. Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of AC such that PB and PD are tangent to ω. The tangent at C intersects PD at Q and the line AD at R. Let E be the second point of intersection between AQ and ω. Prove that B, E, R are collinear.

Solutions of APMO 2013

Problem 1. Let ABC be an acute triangle with altitudes AD, BE and CF, and let O be the center of its circumcircle. Show that the segments OA, OF, OB, OD, OC, OE dissect the triangle ABC into three pairs of triangles that have equal areas.

Solution. Let M and N be midpoints of sides BC and AC, respectively. Notice that $\angle MOC = \frac{1}{2}\angle BOC = \angle EAB$, $\angle OMC = 90^\circ = \angle AEB$, so triangles OMC and AEB are similar and we get $\frac{OM}{AE} = \frac{OC}{AB}$. For triangles ONA and BDA we also have $\frac{ON}{BD} = \frac{OA}{BA}$. Then $\frac{OM}{AE} = \frac{ON}{BD}$ or $BD \cdot OM = AE \cdot ON$.

Denote by $S(\Phi)$ the area of the figure Φ. So, we see that $S(OBD) = \frac{1}{2}BD \cdot OM = \frac{1}{2}AE \cdot ON = S(OAE)$. Analogously, $S(OCD) = S(OAF)$ and $S(OCE) = S(OBF)$.

Alternative solution. Let R be the circumradius of triangle ABC, and as usual write A, B, C for angles $\angle CAB$, $\angle ABC$, $\angle BCA$ respectively, and a, b, c for sides BC, CA, AB respectively. Then the area of triangle OCD is

$$S(OCD) = \tfrac{1}{2} \cdot OC \cdot CD \cdot \sin(\angle OCD) = \tfrac{1}{2}R \cdot CD \cdot \sin(\angle OCD).$$

Now $CD = b\cos C$, and

$$\angle OCD = \frac{180^\circ - 2A}{2} = 90^\circ - A$$

(since triangle OBC is isosceles, and $\angle BOC = 2A$). So

$$S(OCD) = \tfrac{1}{2}Rb\cos C\sin(90^\circ - A) = \tfrac{1}{2}Rb\cos C\cos A.$$

A similar calculation gives

$$\begin{aligned} S(OAF) &= \tfrac{1}{2}OA \cdot AF \cdot \sin(\angle OAF) \\ &= \tfrac{1}{2}R \cdot (b\cos A)\sin(90^\circ - C) \\ &= \tfrac{1}{2}Rb\cos A\cos C, \end{aligned}$$

so OCD and OAF have the same area. In the same way we find that OBD and OAE have the same area, as do OCE and OBF.

Problem 2. Determine all positive integers n for which $\frac{n^2+1}{[\sqrt{n}]^2+2}$ is an integer. Here $[r]$ denotes the greatest integer less than or equal to r.

Solution. We will show that there are no positive integers n satisfying the condition of the problem.

Let $m = [\sqrt{n}]$ and $a = n - m^2$. We have $m \geq 1$ since $n \geq 1$. From $n^2+1 = (m^2+a)^2+1 \equiv (a-2)^2+1 \pmod{(m^2+2)}$, it follows that the condition of the problem is equivalent to the fact that $(a-2)^2+1$ is divisible by m^2+2. Since we have

$$0 < (a-2)^2+1 \leq \max\{2^2, (2m-2)^2\} + 1 \leq 4m^2 + 1 < 4(m^2+2),$$

we see that $(a-2)^2+1=k(m^2+2)$ must hold with $k=1,2$ or 3. We will show that none of these can occur.

Case 1. When $k=1$. We get $(a-2)^2-m^2=1$, and this implies that $a-2=\pm1$, $m=0$ must hold, but this contradicts with fact $m\geq 1$.

Case 2. When $k=2$. We have $(a-2)^2+1=2(m^2+2)$ in this case, but any perfect square is congruent to $0,1,4$ mod 8, and therefore, we have $(a-2)^2+1\equiv 1,2,5 \pmod 8$, while $2(m^2+2)\equiv 4,6 \pmod 8$. Thus, this case cannot occur either.

Case 3. When $k=3$. We have $(a-2)^2+1=3(m^2+2)$ in this case. Since any perfect square is congruent to 0 or 1 mod 3, we have $(a-2)^2+1\equiv 1,2 \pmod 3$, while $3(m^2+2)\equiv 0 \pmod 3$, which shows that this case cannot occur either.

Problem 3. For $2k$ real numbers $a_1,a_2,\ldots,a_k,\ b_1,b_2,\ldots,b_k$ define the sequence of numbers X_n by

$$X_n=\sum_{i=1}^{k}[a_in+b_i] \quad (n=1,2,\ldots).$$

If the sequence X_n forms an arithmetic progression, show that $\sum_{i=1}^{k}a_i$ must be an integer. Here $[r]$ denotes the greatest integer less than or equal to r.

Solution. Let us write $A=\sum_{i=1}^{k}a_i$ and $B=\sum_{i=1}^{k}b_i$. Summing the corresponding terms of the following inequalities over i,

$$a_in+b_i-1<[a_in+b_i]\leq a_in+b_i,$$

we obtain $An+B-k<X_n<An+B$. Now suppose that $\{X_n\}$ is an arithmetic progression with the common difference d, then we have $nd=X_{n+1}-X_1$ and $A+B-k<X_1\leq A+B$ Combining with the inequalities obtained above, we get

$$A(n+1)+B-k<nd+X_1<A(n+1)+B,$$

or

$$An-k\leq An+(A+B-X_1)-k<nd<An+(A+B-X_1)<An+k,$$

from which we conclude that $|A-d|<\frac{k}{n}$ must hold. Since this inequality holds for any positive integer n, we must have $A=d$. Since $\{X_n\}$ is a sequence of integers, d must be an integer also, and thus we conclude that A is also an integer.

Problem 4. Let a and b be positive integers, and let A and B be finite sets of integers satisfying:

(i) A and B are disjoint;

(ii) if an integer i belongs either to A or to B, then $i+a$ belongs to A or $i-b$ belongs to B.

Prove that $a|A|=b|B|$. (Here $|X|$ denotes the number of elements in the set X.)

Solution. Let $A^*=\{n-a : n\in A\}$ and $B^*=\{n+b : n\in B\}$. Then, by (ii), $A\cup B\subseteq A^*\cup B^*$ and by (i),

$$|A\cup B|\leq|A^*\cup B^*|\leq|A^*|+|B^*|=|A|+|B|=|A\cup B|. \qquad (1)$$

Thus, $A \cup B = A^* \cup B^*$ and A^* and B^* have no element in common. For each finite set X of integers, let $\sum(X) = \sum_{x \in X} x$. Then

$$\begin{aligned}\sum(A) + \sum(B) &= \sum(A \cup B) \\ &= \sum(A^* \cup B^*) = \sum(A^*) + \sum(B^*) \\ &= \sum(A) - a|A| + \sum(B) + b|B|, \end{aligned} \tag{2}$$

which implies $a|A| = b|B|$.

Alternative solution. Let us construct a directed graph whose vertices are labelled by the members of $A \cup B$ and such that there is an edge from i to j iff $j \in A$ and $j = i + a$ or $j \in B$ and $j = i - b$. From (ii), each vertex has out-degree ≥ 1 and, from (i), each vertex has in-degree ≤ 1. Since the sum of the out-degrees equals the sum of the in-degrees, each vertex has in-degree and out-degree equal to 1. This is only possible if the graph is the union of disjoint cycles, say $G_1, G_2, \ldots, G_n$. Let $|A_k|$ be the number of elements of A in G_k and $|B_k|$ be the number of elements of B in G_k. The cycle G_k will involve increasing vertex labels by a a total of $|A_k|$ times and decreasing them by b a total of $|B_k|$ times. Since it is a cycle, we have $a|A_k| = b|B_k|$. Summing over all cycles gives the result.

Problem 5. Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of AC such that PB and PD are tangent to ω. The tangent at C intersects PD at Q and the line AD at R. Let E be the second point of intersection between AQ and ω. Prove that B, E, R are collinear.

Solution. To show B, E, R are collinear, it is equivalent to show the lines AD, BE, CQ are concurrent. Let CQ intersect AD at R and BE intersect AD at R'. We shall show $RD/RA = R'D/R'A$ so that $R = R'$.

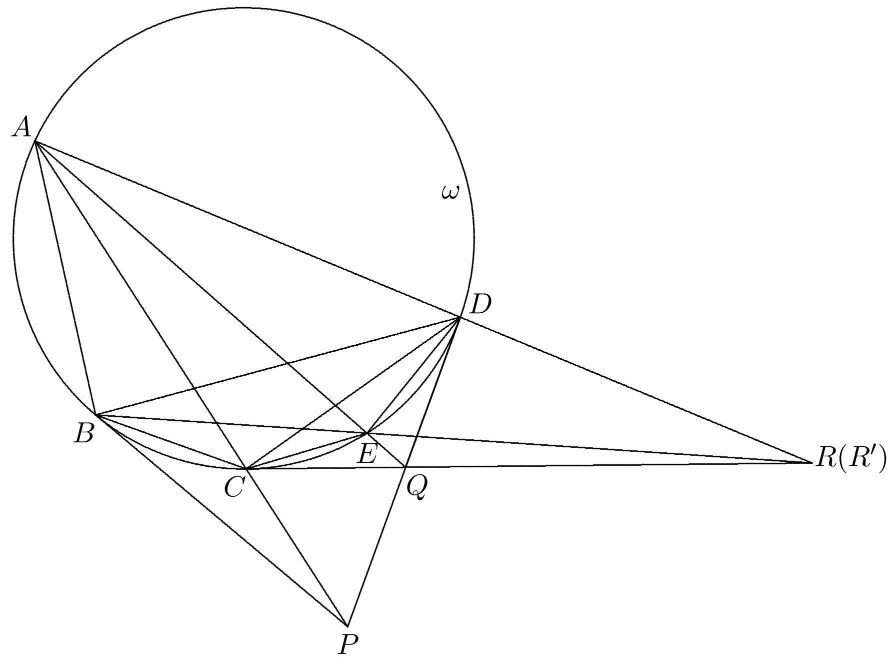
Since $\triangle PAD$ is similar to $\triangle PDC$ and $\triangle PAB$ is similar to $\triangle PBC$, we have $AD/DC = PA/PD = PA/PB = AB/BC$. Hence, $AB \cdot DC = BC \cdot AD$. By Ptolemy's theorem, $AB \cdot DC = BC \cdot AD = \frac{1}{2} CA \cdot DB$. Similarly $CA \cdot ED = CE \cdot AD = \frac{1}{2} AE \cdot DC$.

Thus

$$\frac{DB}{AB} = \frac{2DC}{CA}, \tag{3}$$

and

$$\frac{DC}{CA} = \frac{2ED}{AE}. \tag{4}$$



Since the triangles RDC and RCA are similar, we have $\frac{RD}{RC} = \frac{DC}{CA} = \frac{RC}{RA}$. Thus using (4)

$$\frac{RD}{RA} = \frac{RD \cdot RA}{RA^2} = \left(\frac{RC}{RA}\right)^2 = \left(\frac{DC}{CA}\right)^2 = \left(\frac{2ED}{AE}\right)^2. \tag{5}$$

Using the similar triangles ABR' and EDR', we have $R'D/R'B = ED/AB$. Using the similar triangles DBR' and EAR' we have $R'A/R'B = EA/DB$. Thus using (3) and (4),

$$\frac{R'D}{R'A} = \frac{ED \cdot DB}{EA \cdot AB} = \left(\frac{2ED}{AE}\right)^2. \tag{6}$$

It follows from (5) and (6) that $R = R'$.

XXV APMO Summary of Results

Rank	2013 Participants	# of Contestant	Total Score	Gold Awards	Silver Awards	Bronze Awards	Hon. Men.
1	KOREA	10	350	1	2	4	3
1	RUSSIA	10	350	1	2	4	3
1	USA	10	350	1	2	4	3
4	THAILAND	10	313	1	2	4	3
5	TAIWAN	10	288	1	2	4	3
6	JAPAN	10	267	1	2	4	3
7	CANADA	10	257	1	2	4	3
7	SINGAPORE	10	257	1	2	4	3
9	BRAZIL	10	248	1	2	4	3
10	HONG KONG	10	237	1	2	4	3
11	KAZAKHSTAN	10	222	1	2	4	3
12	INDONESIA	10	214	0	3	4	3
13	MALAYSIA	10	213	1	2	4	3
14	MEXICO	10	194	1	2	4	3
15	AUSTRALIA	10	191	1	2	4	3
16	BANGLADESH	10	148	0	1	6	3
17	ARGENTINA	10	118	0	2	2	3
18	SAUDI ARABIA	10	115	0	1	3	6
19	TURKMENISTAN	10	114	0	1	2	7
20	COLOMBIA	9	105	0	3	0	3
21	TAJIKISTAN	10	94	0	1	1	8
22	Costa Rica	9	91	1	0	2	2
23	AZERBAIJAN	10	90	0	0	3	5
24	NEW ZEALAND	10	85	0	0	3	5
25	PHILIPPINES	10	68	0	0	1	6
26	ECUADOR	10	50	0	1	1	1
27	PAKISTAN	10	46	0	0	0	5
28	Kyrgyz Republic	5	39	0	0	1	3
29	CAMBODIA	10	36	0	0	0	3
30	SYRIA	6	25	0	0	0	2
31	PANAMA	3	16	0	0	1	0
32	URUGUAY	1	14	0	0	1	0
33	EL SALVADOR	4	13	0	0	0	1
34	TRINIDAD AND TOBAGO	10	7	0	0	0	1
	Total	307	5225	15	41	87	109

m : 17.020 / σ : 11.572	m + σ = 28.592	m + σ/3 = 20.87	m - σ/3 = 13.163
Awards Cut-offs :	Gold ≥ 29	Silver ≥ 21	Bronze ≥ 14

# of Problems	1	2	3	4	5	Total
Mean(m) :	5.53	4.31	2.26	2.53	2.39	17.02

Country code	Names of Contestants		Medal
	Last	First	
AUS 1	Gunning	Alex	Gold
BRA 1	Angelo	Rodrigo Sanches	Gold
CAN 1	Deng	Calvin	Gold
CRI 1	Jimenez Alvarado	Humberto	Gold
HKG 1	Lau	Chun Ting	Gold
JPN 1	Mayuko	Yamashita	Gold
KAZ 1	Tortai	Alisher	Gold
KOR 1	Lee	You Seong	Gold
MAS 1	Lim	Kai Ze Justin	Gold
MEX 1	Roque Montoya	Diego Alonso	Gold
RUS 1	Chernega	Nikita	Gold
SGP 1	Lim	Jeck	Gold
TWN 1	Chiang	Hung	Gold
THA 1	Pitimanaaree	Nipun	Gold
USA 1	Chen	Evan	Gold
ARG 1	Darago	Ignacio Martin	Silver
ARG 2	Staffa	Bruno	Silver
AUS 2	Zheng	Jonathan	Silver
AUS 3	Chua	Alexander	Silver
BGD 1	Das	Sourav	Silver
BRA 2	Zanarella	Murilo Corato	Silver
BRA 3	Miyazaki	Rafael Kazuhiro	Silver
CAN 2	Song	Alex	Silver
CAN 3	Sun	Kevin	Silver
COL 1	Salazar	Veronica	Silver
COL 2	Florez	Leonardo	Silver
COL 3	Espinoza	Camilo	Silver
ECU 1	Cortez Lemos	Carlos Andres	Silver
HKG 2	Wong	Nihg Shing	Silver
HKG 3	Man	Siu Hang	Silver
IDN 1	Harmanto	Bivan	Silver
IDN 2	Wibisono	Kevin	Silver
IDN 3	Ulum	Mohammad Yasya	Silver
JPN 2	Tasuki	Kinjo	Silver
JPN 3	Takuma	Kitamura	Silver
KAZ 2	Kalmyrzayev	Sergazy	Silver
KAZ 3	Abuov	Nurbol	Silver
KOR 2	Kim	Juhan	Silver
KOR 3	Lee	Jongwon	Silver
MAS 2	Tham	Ying Hong	Silver
MAS 3	How	Si Yu	Silver
MEX 2	Beuchot Castellano	Kevin	Silver
MEX 3	Chiu Han	Enrique	Silver
RUS 2	Kochkin	Ivan	Silver
RUS 3	Klyuev	Daniil	Silver
SAU 1	Basyoni	Al-Yazeed	Silver
SGP 2	Ling	Yan Hao	Silver
SGP 3	Yap	Jit Wu	Silver
TWN 2	Chen	Brian	Silver
TWN 3	Chao	William Ting-Wei	Silver
TJK 1	Oqilnazarzoda	Sadi	Silver
THA 2	Karntikoon	Kritkorn	Silver

THA 3	Wattanawongwibul	Phusit	Silver
TKM 1	Ovlyagulyyev	Dovlet	Silver
USA 2	Brakensiek	Joshua	Silver
USA 3	Sellke	Mark	Silver
ARG 3	Salvatore	Alejo Manuel	Bronze
ARG 4	Candioti	Alejandro	Bronze
AUS 4	Ragavan	Seyoon	Bronze
AUS 5	Zelich	Ivan	Bronze
AUS 6	Shu	Mel	Bronze
AUS 7	Kwong	Jason	Bronze
AZE 1	Hasanzade	Hasan	Bronze
AZE 2	Hasanzade	Suleyman	Bronze
AZE 3	Maharranov	Mehdi	Bronze
BGD 2	Abrar	Mohammad Nadimul	Bronze
BGD 3	Hasan	MD. Zahidul	Bronze
BGD 4	Shafiullah	Nur Muhammad	Bronze
BGD 5	Hasan	Tahmid	Bronze
BGD 6	Ferdous	Fahim	Bronze
BGD 7	Hasan	Adib	Bronze
BRA 4	de Alencar Severo	Franco Matheus	Bronze
BRA 5	Oliveira Reis	Victor	Bronze
BRA 6	Da Silva Reis	Lucas	Bronze
BRA 7	De Oliveira Bitaraes	Victor	Bronze
CAN 4	Whatley	Alexander	Bronze
CAN 5	Ma	Zhiyao	Bronze
CAN 6	Spivak	Daniel	Bronze
CAN 7	Wu	Fei	Bronze
CRI 2	Villalobos Carballo	Kimberly	Bronze
CRI 3	Villegas Segura	Juleana	Bronze
ECU 2	Paladines Valverde	Omar Nicolas	Bronze
HKG 4	Yam	Yuen Sum	Bronze
HKG 5	Tse	Shun Chung	Bronze
HKG 6	Chan	Long Tin	Bronze
HKG 7	Yu	Hoi Wai	Bronze
IDN 4	Young	Peter	Bronze
IDN 5	Sanjaya	Stephen	Bronze
IDN 6	Rifai	Wiwit	Bronze
IDN 7	Susan	Fransisca	Bronze
JPN 4	Shogo	Murai	Bronze
JPN 5	Hitoshi	Ito	Bronze
JPN 6	Yo	Mitani	Bronze
JPN 7	Daisuke	Miyamoto	Bronze
KAZ 4	Abdrakhmanov	Alen	Bronze
KAZ 5	Kulboldin	Daniyar	Bronze
KAZ 6	Kalysh	Ileskhan	Bronze
KAZ 7	Saylanbayev	Alibek	Bronze
KOR 4	Ju	Jung Hun	Bronze
KOR 5	Baek	Seung Yoon	Bronze
KOR 6	Kim	Min Hyuk	Bronze
KOR 7	Han	Kihyun	Bronze
KGZ 1	Eshbolot uulu	Kayrat	Bronze
MAS 4	Teh	Zhao Yang Anzo	Bronze
MAS 5	How	Si Wei	Bronze
MAS 6	Khong	Yi Kye	Bronze

MAS 7	Lim	Yun Kai	Bronze
MEX 4	Ortiz Rhoton	Juan Carlos	Bronze
MEX 5	Ramos Tormo	Luis Xavier	Bronze
MEX 6	Medrano Martin del C.	Adan	Bronze
MEX 7	Medrano Martin del C.	Olga	Bronze
NZL 1	Han	George	Bronze
NZL 2	Cho	Byung-Cheol	Bronze
NZL 3	Chen	Natalia	Bronze
PAN 1	Moreno	Felipe	Bronze
PHI 1	Sy	Adrian Reginald	Bronze
RUS 4	Krachun	Dmitry	Bronze
RUS 5	Ershov	Stanislav	Bronze
RUS 6	Volostnov	Alexey	Bronze
RUS 7	Belov	Dmitry	Bronze
SAU 2	Alharbi	Abdulrahman	Bronze
SAU 3	Eid	Husain	Bronze
SAU 4	Khan	Ibrahim	Bronze
SGP 4	Lin	Kewei, David	Bronze
SGP 5	Howe	Choong Yin	Bronze
SGP 6	Teh	Jiun Harn	Bronze
SGP 7	Tan	Sheldon Kieren	Bronze
TWN 4	Lu	Yan-Der	Bronze
TWN 5	Yu	Hung-Hsun	Bronze
TWN 6	Hseih	Yu-Guan	Bronze
TWN 7	Mao	Yao-Wen	Bronze
TJK 2	Makhmudov	Alisher	Bronze
THA 4	Vivatsethachai	Suchan	Bronze
THA 5	Sothanaphan	Nat	Bronze
THA 6	Komolsiripakdi	Tanat	Bronze
THA 7	Neeranartvong	Weerachai	Bronze
TKM 2	Soyunov	Nuryagdy	Bronze
TKM 3	Rahmanov	Nazar	Bronze
URY 1	Correa	Ignacio	Bronze
USA 4	Shen	Bobby	Bronze
USA 5	Stoner	David	Bronze
USA 6	Wang	Victor	Bronze
USA 7	Tao	James	Bronze
ARG 5	Salvador	Alejo Antonio	H.Men
ARG 6	Mazzocato	Agustin	H.Men
ARG 7	Salgado	Gaston	H.Men
AUS 8	So	Patrick	H.Men
AUS 9	He	Patrick	H.Men
AUS 10	Wijerathna	Praveen	H.Men
AZE 4	Bayramov	Isa	H.Men
AZE 5	Agayev	Ismat	H.Men
AZE 6	Aliyev	Izmir	H.Men
AZE 8	Shirinov	Mahammad	H.Men
AZE 9	Verdiyev	Farid	H.Men
BGD 8	Kabir	Ehsanul	H.Men
BGD 9	Turzo	Sazid Akhter	H.Men
BGD 10	Das	Sowmitra	H.Men
BRA 8	De Faveri	Alexandre Perozim	H.Men
BRA 9	Zelazny	Michel Rozenberg	H.Men
BRA 10	Pacanowski	Alessandro	H.Men

KHM 1	Khon	Vanny	H.Men
KHM 2	La	Sokchamroeun	H.Men
KHM 3	Um	Chanpiset	H.Men
CAN 8	Lai	Leo	H.Men
CAN 9	Du	Richard	H.Men
CAN 10	Lim	Ursula Anne	H.Men
COL 4	Diaz	Juan Sebastian	H.Men
COL 5	Acosia	Francisco	H.Men
COL 6	Lopez	Hector	H.Men
CRI 4	Leon Jimenez	Daniel	H.Men
CRI 5	Ferreira Lor	Rosaria	H.Men
ECU 3	Perez Lopez	Javier	H.Men
SLV 1	Mundo Duenas	Manuel Alejandro	H.Men
HKG 8	Ho	Wui Hang	H.Men
HKG 9	Hon	Pun Yat	H.Men
HKG 10	Cheung	Ka Wai	H.Men
IDN 8	Soesanto	Christa	H.Men
IDN 9	Said	Septian	H.Men
IDN 10	Suyitno	Edwin	H.Men
JPN 8	Kento	Nomura	H.Men
JPN 9	Satoshi	Hayakawa	H.Men
JPN 10	Atsumasa	Kurozumi	H.Men
KAZ 8	Adilkhanuly	Taken	H.Men
KAZ 9	Zhunis	Abylay	H.Men
KAZ 10	Toktasynov	Yeldar	H.Men
KOR 8	Song	Young Keun	H.Men
KOR 9	Jung	Chong Wook	H.Men
KOR 10	Hwang	Injae	H.Men
KGZ 2	Mullahashim	Simai	H.Men
KGZ 3	Moskalenko	Alim	H.Men
KGZ 4	Imankulov	Ermek	H.Men
MAS 8	Tan	Kin Aun	H.Men
MAS 9	Noorman	Muhammad Afiq	H.Men
MAS 10	Tam	Sheng Cong Bryan	H.Men
MEX 8	Gomez Casarez	Axel	H.Men
MEX 9	Espinoza Ruiz	Demian	H.Men
MEX 10	Rojas Cuadra	Maria Cecilia	H.Men
NZL 4	Guo	Siyuan	H.Men
NZL 5	Kim	Jaehwan	H.Men
NZL 6	Seong	Ian	H.Men
NZL 7	Kim	Su Jeong	H.Men
NZL 8	Shen	Kevin	H.Men
PAK 1	Awais Muhammad	Chishti	H.Men
PAK 2	Mohammad Faaiz	Taufiq	H.Men
PAK 3	Ammar	Abbas	H.Men
PAK 4	Shahzaib	Ali	H.Men
PAK 5	Muhammad	Adnan	H.Men
PHI 2	Uy	Mikaela Angelina	H.Men
PHI 3	Ang	Clyde Wesley	H.Men
PHI 4	Sy	Andrew Lawrence	H.Men
PHI 5	Lim Tiong Soon	Kelsey	H.Men
PHI 6	Lao	Ma. Czarina Angela	H.Men
PHI 7	Chuatak	John Thomas	H.Men
RUS 8	Baev	Budimir	H.Men

RUS 9	Matushkin	Alexander	H.Men
RUS 10	Shabanov	Lev	H.Men
SAU 5	Zawawi	Sameh	H.Men
SAU 6	Alsaeed	Wael	H.Men
SAU 7	Hbibullah	Alzubair	H.Men
SAU 8	Eid	Hasan	H.Men
SAU 9	Hamdi	Yassine	H.Men
SAU10	Alsheikh Saleh	Mahdi	H.Men
SGP 8	Tan	Pin Lin	H.Men
SGP 9	Zhang	Wen	H.Men
SGP 10	Tan	Siah Yong	H.Men
SYR 1	Atieh	Fadi	H.Men
SYR 2	Rahma	Ahmad Rami	H.Men
TWN 8	Lin	Hau-Yi	H.Men
TWN 9	Lai	Cheng-Chien	H.Men
TWN 10	Yu	Jia-Ruei	H.Men
TJK 3	Emomov	Golibjon	H.Men
TJK 4	Mirzoev	Manuchekhr	H.Men
TJK 5	Karimov	Farrukh	H.Men
TJK 6	Shahzodshokhi	Shamsiddin	H.Men
TJK 7	Muqayumkhonov	Musokhon	H.Men
TJK 8	Halimov	Shuhrat	H.Men
TJK 9	Murodov	Firuz	H.Men
TJK 10	Bobizoda	Sobiz	H.Men
THA 8	Pitimanaaree	Paween	H.Men
THA 9	Ngamsangrat	Thee	H.Men
THA 10	Sanguanmoo	Sivakorn	H.Men
TTO 1	Ramakrishnan	Prasanna	H.Men
TKM 4	Valiyev	Ilshat	H.Men
TKM 5	Soltanov	Merdan	H.Men
TKM 6	Iljanov	Perman	H.Men
TKM 7	Soyunjov	Alshir	H.Men
TKM 8	Gutlygeldiyev	Babanazar	H.Men
TKM 9	Halmedov	Bazarbay	H.Men
TKM 10	Shamyradov	Seyitmuhammet	H.Men
USA 8	Jagadeesan	Ravi	H.Men
USA 9	Swayze	Thomas	H.Men
USA 10	Schneider	Eric	H.Men

ឯកសារយោង:

១. 360 Problems for Mathematical Contests

២. Bulgarian Mathematical Competitions 2003-2006

៣. Elementary Number Theory

៤. Australian Mathematical Olympiads 1979-1995

៥. Five hundred mathematical challenges

៦. Junior Balkan Mathematical Olympiads

៧. Romanian Mathematical Olympiads

៨. 103 Trigonometry Problems

៩. 104 Number Theory Problems

១០. Mathematical Olympiad in China - Problems and Solutions

១១. Selected-Problems-Of-The-Vietnamese-Mathematical-Olympiad-1962-2009

១៣. The IMO Compendium (fist edition)

១៤. សៀវភៅល្បីៗជាច្រើនទៀត

១៥. និង៍តាម Internet ។