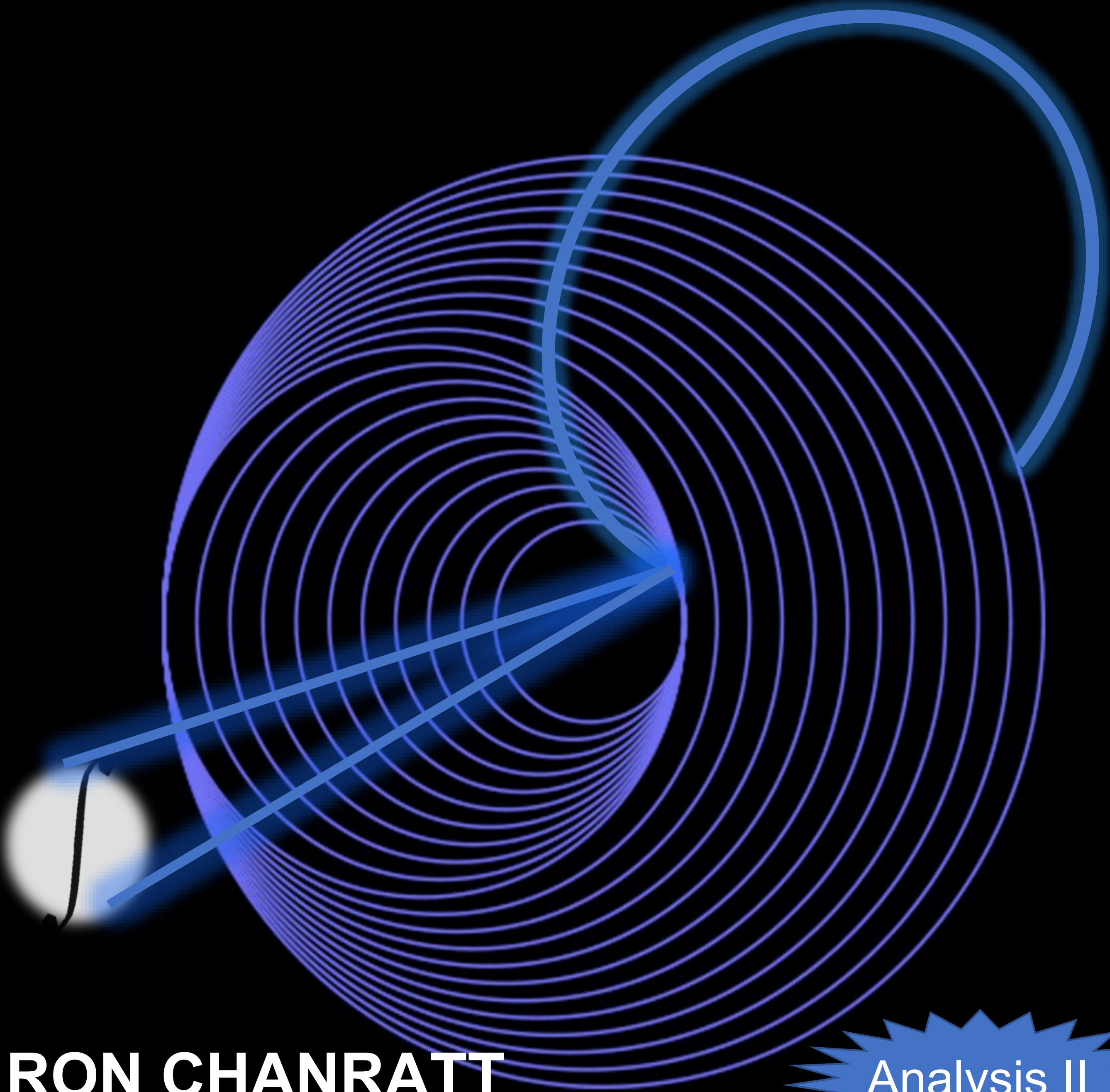
INTEGRAL

Mathematic



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Analysis II

<u>អារម្ភកថា</u>

សូមគោរពជម្រាបសួរដល់លោកគ្រូអ្នកគ្រូ ប្រិយមិត្តអ្នកសិក្សាទាំងអស់ជាទីស្រលាញ់និងជាទីរាប់អាន !

សៀវភៅកំណែលំហាត់អាំងតេក្រាលឆ្នាំទី២នេះខ្ញុំបាទបានរៀបជាផ្នែកលំហាត់និងដំណោះស្រាយជាពិសេសលើកយកគ្រប់ករណីពិសេសៗជាច្រើនសម្រាប់ជាគន្លឹះក្នុងការដោះស្រាយលំហាត់បានយ៉ាងទូលំទូលាយ។

រាល់កំហុសខុសឆ្គងដោយអចេតនាក៏សូមមានការអភ័យទោសពីសំណាក់អ្នកសិក្សាគ្រប់មជ្ឈដ្ឋាន។

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និស្សិតឆ្នាំទី២នៃសាកលវិទ្យាល័យភូមិន្ទភ្នំពេញ

ឆ្នាំសិក្សា៖ ២០២៣

ផ្នែកលំហាត់

I. គណនាអាំងតេក្រាល៖

- ករណី: អាំងតេក្រាលប្តូរអថេរ៖

$$1.I=\int\frac{cosx}{\sqrt{1+sinx}}dx$$

$$2.I=\int\frac{1+arcsin5x}{\sqrt{1-25x^2}}dx$$

- ករណី: អាំងតេក្រាលដោយផ្នែក៖

$$1.I=\int\frac{(2x+3)dx}{x^2+3x-4}$$

$$2.I=\int\frac{e^x}{(e^x+1)^2}dx$$

- ករណី: អាំងតេក្រាលមានរាងជាកន្សោមនៃ ax^2+bx+c

$$1.I=\int\frac{dx}{2x^2+x+4}$$

$$2.I=\int\frac{dx}{x^2+2x+3}$$

- ករណី: អាំងតេក្រាលដែលមានទម្រង់៖ $\int\frac{(Ax+B)dx}{ax^2+bx+c}$

$$1.I=\int\frac{(3x+2)}{\sqrt{2+\frac{x}{2}+3x^2}}dx$$

$$2.I=\int\frac{(x+5)}{\sqrt{4x^2-\frac{x}{4}+2}}dx$$

- ករណី: អាំងតេក្រាលដែលមានទម្រង់៖ $\int\frac{dx}{(Ax+B)\sqrt{ax^2+bx+c}}$

$$1.I=\int\frac{dx}{\left(\frac{1}{2}x+3\right)\sqrt{2x^2+4x+\frac{1}{2}}}$$

$$2.I=\int\frac{dx}{(3x+2)\sqrt{1-x-x^2}}$$

- ករណី: អាំងតេក្រាលតាមវិធីសាស្រ្ត *Euler* ៖
  - ចំពោះ: *Euler* (1)

$$1.I=\int\frac{dx}{x\sqrt{x^2-x+2}}$$

$$2.I=\int\frac{dx}{x\sqrt{2x^2+x+2}}$$

  - ចំពោះ: *Euler* (2)

$$1.I=\int\frac{dx}{x\sqrt{3x^2-2x+1}}$$

$$2.I=\int\frac{dx}{\sqrt{2x^2-4x+4}}$$

  - ចំពោះ: *Euler* (3)

$$1.I=\int\frac{dx}{x\sqrt{2x^2-3x-2}}$$

$$2.I=\int\frac{dx}{\sqrt{x^2-2x-8}}$$

- ករណី: អាំងតេក្រាលតាមវិធីសាស្រ្តជំនួសដោយត្រីកោណមាត្រ
  - បើអាំងតេក្រាលមានរាងជាប៉ាឌីកាល់ $\sqrt{a^2-x^2}$

$$1.I=\int\frac{dx}{x^2\sqrt{9-x^2}}$$

$$2.I=\int\frac{dx}{x^2\sqrt{16-x^2}}$$

- បើអាំងតេក្រាលមានរាងជារ៉ាឌីកាល់ $\sqrt{x^2-a^2}$

$$1.I=\int\frac{\sqrt{x^2-4}}{x}dx$$

$$2.I=\int\frac{\sqrt{x^2-36}}{x}dx$$

- បើអាំងតេក្រាលមានរាងជារ៉ាឌីកាល់ $\sqrt{x^2+a^2}$

$$1.I=\int\frac{dx}{\sqrt{x^2+4}}$$

$$2.I=\int\frac{dx}{\sqrt{x^2+16}}$$

- ករណី: អាំងតេក្រាលប្រភាគសនិទាន

$$1.I=\int\frac{x^2+2x-2}{x-1}dx$$

$$2.I=\int\frac{(2x+1)\,dx}{x(x-1)\,(x-1)^2}$$

- ករណី: អាំងតេក្រាលភាគបែងមានឬសលំដាប់ខ្ពស់ច្រើន ដោយប្រើវិធីសាស្ត្រ $(Ostrogradsky)$

$$1.I=\int\frac{4x^3-3x^2-2}{(x^3-1)^2}dx$$

- ករណី: អាំងតេក្រាលនៃទ្វេធាឌីផេរ៉ង់ស្យែល

$$1.I=\int x^{-2}(1+x^2)^{-\frac{1}{2}}dx$$

$$2.I=\int\frac{dx}{\sqrt{x}\left(1+\sqrt{x}\right)^2}$$

- ករណី: អាំងតេក្រាលនៃកន្សោមត្រីកោណមាត្រខ្លះៗ

$$1.I=\int\frac{dx}{4-5\sin x}$$

$$2.I=\int\frac{dx}{5-3\cos x}$$

ផ្នែកដំណោះស្រាយ

I. គណនាអាំងតេក្រាល៖

- ករណី: អាំងតេក្រាលប្តូរអថេរ៖

$$1.I=\int\frac{\cos x}{\sqrt{1+\sin x}}dx$$

តាង $t=1+\sin x \Longrightarrow dt=\cos x dx$

$$\Longrightarrow\int\frac{\cos x}{\sqrt{1+\sin x}}dx=\int\frac{dt}{\sqrt{t}}=\int t^{-\frac{1}{2}}dt$$

$$=2\sqrt{t}+C=2\sqrt{1+\sin x}+C$$

ដូចនេះ: $I=2\sqrt{1+\sin x}+C$

$$2.I=\int\frac{1+\arcsin 5x}{\sqrt{1-25x^2}}dx$$

តាង $t=1+\arcsin 5x\Longrightarrow dt=-\dfrac{5}{\sqrt{1-(5x)^2}}dx$

$$\Longrightarrow\int\frac{1+\arcsin 5x}{\sqrt{1-25x^2}}dx=-\frac{1}{5}\int dt=-\frac{1}{5}\times t=-\frac{1}{5}(1+\arcsin 5x)+C$$

ដូចនេះ: $I=-\dfrac{1}{5}(\arcsin 5x)+C$

- ករណី: អាំងតេក្រាលដោយផ្នែក៖

$$1.I=\int\frac{(2x+3)dx}{x^2+3x-4}$$

តាង $u=x^2+3x-4 \Longrightarrow du=(2x+3)dx$

$$\Longrightarrow\int\frac{(2x+3)dx}{x^2+3x-4}=\int\frac{1}{u}du=\ln|u|=\ln|x^2+3x-4|+C$$

ដូចនេះ: $I=\ln\left|x^2+3x-4\right|+C$

$$2.I=\int\frac{e^x}{(e^x+1)^2}dx$$

តាង $u=e^x+1 \Longrightarrow du=e^x dx$

$$\Longrightarrow\int\frac{e^x}{(e^x+1)^2}=\int\frac{du}{u^2}=\int u^{-2}du$$

$$=-\frac{1}{u}=-\frac{1}{e^x+1}+C$$

ដូចនេះ: $I=-\frac{1}{e^x+1}+C$

- ករណី: អាំងតេក្រាលមានរាងជាកន្សោមនៃ ax^2+bx+c

$$1.I=\int\frac{dx}{2x^2+x+4}$$

យើងបាន៖

$$\int\frac{dx}{2x^2+x+4}=\frac{1}{2}\int\frac{dx}{x^2+\frac{x}{2}+2}=\frac{1}{2}\int\frac{dx}{\left(x+\frac{1}{4}\right)^2+\left(\frac{31}{16}\right)}$$

$$=\frac{1}{2}\cdot\int\frac{dx}{\left(x+\frac{1}{4}\right)^2+\left(\sqrt{\frac{31}{16}}\right)^2}=\frac{1}{2}\times\arctan\left(\frac{x+\frac{1}{4}}{\frac{\sqrt{31}}{4}}\right)=\frac{1}{2}\left[\arctan\left(\frac{4x+1}{\sqrt{31}}\right)\right]+C$$

ដូចនេះ: $I=\frac{1}{2}\left[\arctan\left(\frac{4x+1}{\sqrt{31}}\right)\right]+C$

$$2.I=\int\frac{dx}{x^2+2x+3}$$

យើងបាន៖

$$\int\frac{dx}{x^2+2x+3}=\int\frac{dx}{(x+1)^2+2}=\int\frac{dx}{(x+1)^2+\left(\sqrt{2}\right)^2}$$

$$=\int\frac{d(x+1)}{(x+1)^2+\left(\sqrt{2}\right)^2}=\frac{1}{\sqrt{2}}\arctan\left(\frac{x+1}{\sqrt{2}}\right)+C$$

ដូចនេះ: $I=\frac{1}{\sqrt{2}}\arctan\left(\frac{x+1}{\sqrt{2}}\right)+C$

- ករណី: អាំងតេក្រាលដែលមានទម្រង់៖ $\int\frac{(Ax+B)dx}{ax^2+bx+c}$

$1.I=\int\frac{(3x+2)}{\sqrt{2+\frac{x}{2}+3x^2}}dx$

យើបាន៖

$$\int\frac{(3x+2)}{\sqrt{2+\frac{x}{2}+3x^2}}dx=\int\frac{(3x+2)dx}{\sqrt{3x^2+\frac{x}{2}+2}}=\frac{1}{2}\int\frac{6x+4}{3x^2+\frac{x}{2}+2}dx$$

$$=\frac{1}{2}\int\frac{\left(6x+\frac{1}{2}-\frac{1}{2}+4\right)}{\sqrt{3x^2+\frac{x}{2}+2}}dx=\frac{1}{2}\int\frac{6x+\frac{1}{2}}{\sqrt{3x^2+\frac{x}{2}+2}}dx-\frac{7}{2}\int\frac{dx}{\sqrt{3x^2+\frac{x}{2}+2}}$$

$$=\frac{1}{2}\frac{\left(3x^2+\frac{x}{2}+2\right)^{\frac{1}{2}}}{\frac{1}{2}}-\frac{7}{2\sqrt{3}}\int\frac{dx}{\sqrt{\left(x+\frac{1}{12}\right)^2+\frac{95}{144}}}$$

$$=\sqrt{3x^2+\frac{x}{2}+2}-\frac{7}{2\sqrt{3}}\ln\left[\left(x+\frac{1}{12}\right)+\sqrt{\left(x+\frac{1}{12}\right)^2+\frac{95}{144}}\right]+C$$

ដូចនេះ: $I=\sqrt{3x^2+\frac{x}{2}+2}-\frac{7}{2\sqrt{3}}\ln\left[\left(x+\frac{1}{12}\right)+\sqrt{\left(x+\frac{1}{12}\right)^2+\frac{95}{144}}\right]+C$

$2.I=\int\frac{(x+5)}{\sqrt{4x^2-\frac{x}{4}+2}}dx$

យើងបាន៖

$$\int \frac{(x+5)}{\sqrt{4x^2-\frac{x}{4}+2}}dx=\frac{1}{8}\int \frac{8x+45}{\sqrt{4x^2-\frac{x}{4}+2}}dx$$

$$=\frac{1}{8}\int \frac{\left(8x-\frac{1}{4}+\frac{1}{4}+45\right)dx}{\sqrt{4x^2-\frac{x}{4}+2}}=\frac{1}{8}\int \frac{\left(8x-\frac{x}{4}+2\right)dx}{\sqrt{4x^2-\frac{x}{4}+2}}+\frac{1}{8}\int \frac{\frac{181}{4}}{\sqrt{4x^2-\frac{x}{4}+2}}dx$$

$$=\frac{1}{8}\sqrt{4x^2-\frac{x}{4}+2}+\frac{181}{64}\int \frac{dx}{\sqrt{x^2-\frac{x}{16}+\left(\frac{x}{32}\right)^2-\frac{1}{1024}+\frac{1}{2}}}$$

$$=\frac{1}{4}\int \sqrt{4x^2-\frac{x}{4}+2}+\frac{181}{64}\int \frac{dx}{\sqrt{\left(x-\frac{1}{32}\right)^2+\frac{501}{1024}}}$$

$$=\frac{1}{4}\sqrt{4x^2-\frac{x}{4}+2}+\frac{181}{64}\ln\left[\left(x-\frac{1}{32}\right)+\sqrt{\left(x-\frac{1}{32}\right)^2+\frac{501}{1024}}\right]+C$$

ដូចនេះ $I=\frac{1}{4}\sqrt{4x^2-\frac{x}{4}+2}+\frac{181}{64}\ln\left[\left(x-\frac{1}{32}\right)+\sqrt{\left(x-\frac{1}{32}\right)^2+\frac{501}{1024}}\right]+C$

- ករណី: អាំងតេក្រាលដែលមានទម្រង់៖ $\int \frac{dx}{(Ax+B)\sqrt{ax^2+bx+c}}$

$$1.I=\int \frac{dx}{\left(\frac{1}{2}x+3\right)\sqrt{2x^2+4x+\frac{1}{2}}}$$

តាង $\frac{x}{2}+3=\frac{1}{t}\Longleftrightarrow \frac{x}{2}=\frac{1}{t}-3\Longrightarrow \frac{2}{t}-6$

$$d\left(\frac{x}{2}+3\right)=d\left(\frac{1}{t}\right)$$

$$\frac{dx}{2}=-\frac{1}{t^2}dt\Longrightarrow dx=-\frac{2}{t^2}dt$$

$$\Longrightarrow I=\int \frac{-\frac{2}{t^2}\mathrm{dt}}{\frac{1}{t}\sqrt{2\cdot\left(\frac{2}{t}-6\right)^2+4\left(\frac{2}{t}-6\right)+\frac{1}{2}}}$$

$$=-2\int \frac{dt}{t\times\sqrt{2\times\left(\frac{4}{t^2}-\frac{24}{t}+36\right)+\frac{8}{t}-24+\frac{1}{2}}}$$

$$=-2\int \frac{dt}{t\times\sqrt{\frac{8}{t^2}-\frac{48t}{t^2}+\frac{72t^2}{t^2}+\frac{8t}{t^2}-\frac{47t^2}{t^2}}}$$

$$=-2\int \frac{dt}{\sqrt{8-48t+72t^2+8t-47t^2}}$$

$$=-2\int \frac{dt}{\sqrt{25t^2-40t+8}}=-\frac{2}{5}\int \frac{dt}{\sqrt{t^2-\frac{8t}{5}+\frac{8}{25}}}$$

$$=-\frac{2}{5}\int \frac{dt}{\sqrt{\left(t-\frac{8}{10}\right)^2-\frac{32}{100}}}=-\frac{2}{5}\ln\left|t-\frac{8}{10}+\sqrt{\left(t-\frac{8}{10}\right)^2-\frac{32}{100}}\right|+C$$

$$=-\frac{2}{5}\ln\left|\frac{2}{x+6}-\frac{8}{10}+\sqrt{\left(\frac{2}{x+6}-\frac{8}{10}\right)^2-\frac{32}{100}}\right|+C$$

ដូចនេះ: $I=-\frac{2}{5}\ln\left|\frac{2}{x+6}-\frac{8}{10}+\sqrt{\left(\frac{2}{x+6}-\frac{8}{10}\right)^2-\frac{32}{100}}\right|+C$

$$2.I=\int \frac{dx}{(3x+2)\sqrt{1-x-x^2}}$$

តាង $3x+2=\frac{1}{t}\Longleftrightarrow 3x=\frac{1}{t}-2 \quad\Longrightarrow x=\frac{1-2t}{3t}\Longrightarrow dx=-\frac{1}{3t^2}dt$

$$=\int \frac{-\frac{1}{3t}}{\sqrt{1-\frac{1-2t}{3t}-\frac{1-4t+4t^2}{9t^2}}}$$

$$=\int \frac{-\frac{1}{3t}dt}{\sqrt{\frac{9t^2-3t+6t^2-1+4t-4t^2}{9t^2}}}$$

$$=\int \frac{dt}{\sqrt{11t^2+t-1}}=-\frac{1}{\sqrt{11}}\int \frac{dt}{\sqrt{t^2+\frac{t}{11}-\frac{1}{11}}}$$

$$= -\frac{1}{\sqrt{11}}\int \frac{dt}{\sqrt{t^2 + 2\cdot\frac{t}{22} + \left(\frac{1}{22}\right)^2 - \left(\frac{1}{12}\right)^2 - \frac{1}{11}}}$$

$$= -\frac{\sqrt{11}}{11}\int \frac{dt}{\sqrt{\left(t+\frac{1}{22}\right)^2 - \frac{1}{484} - \frac{1}{11}}} = -\frac{\sqrt{11}}{11}\int \frac{dt}{\sqrt{\left(t+\frac{1}{22}\right)^2 - \left(\frac{\sqrt{45}}{22}\right)^2}}$$

$$= -\frac{\sqrt{11}}{11}\ln\left[t + \frac{1}{22} + \sqrt{\left(t+\frac{1}{22}\right)^2 - \left(\frac{\sqrt{45}}{22}\right)^2}\right]$$

$$= -\frac{\sqrt{11}}{11}\ln\left[\frac{1}{3x+2} + \sqrt{\left(\frac{1}{3x+2}+\frac{1}{22}\right)^2 - \left(\frac{\sqrt{45}}{22}\right)^2}\right] + C$$

ដូចនេះ $I = -\frac{\sqrt{11}}{11}\ln\left[\frac{1}{3x+2} + \frac{1}{22} + \sqrt{\left(\frac{1}{3x+2}+\frac{1}{22}\right)^2 - \left(\frac{\sqrt{45}}{22}\right)^2}\right] + C$

- ករណី: អាំងតេក្រាលតាមវិធីសាស្រ្ត *Euler*
  - ចំពោះ *Euler* (1)

    $1.I = \int \frac{dx}{x\sqrt{x^2 - x + 2}}$

    ដោយ $a = 1 > 0, \Delta < 0$

    តាង $\sqrt{x^2 - x + 2} = x + t$

$$\Longrightarrow x^2-x+2=x^2+2xt+t^2$$

$$\Longrightarrow -x-2xt=t^2-2$$

$$\Longrightarrow x(-1-2t)=t^2-2 \Longrightarrow x=\frac{t^2-2}{-1-2t}$$

$$\Longrightarrow dx=\frac{-2t^2-2t-4}{(-1-2t)^2}dt$$

$$\Longrightarrow I=\int\frac{\frac{-2t^2-2t-4}{(-1-2t)^2}dt}{\frac{t^2-2}{-1-2t}\left[\frac{t^2-2}{-t-2t}+t\right]}=\int\frac{(-2t^2-2t-4)\,dt}{(t^2-2)\,(t^2-2)+t(-1-2t)}$$

$$=\int\frac{(-2t^2-2t-4)\,dt}{(t^2-2)\,(-t^2-t-2)}=\int\frac{2}{t^2-2}dt=\frac{2}{2\sqrt{2}}\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right|+C$$

$$=\frac{2}{2\sqrt{2}}\ln\left|\frac{\frac{\sqrt{x^2-x+2}}{x}-2}{\frac{\sqrt{x^2-x+2}}{x}+2}\right|+C$$

ដូចនេះ $I=\frac{2}{2\sqrt{2}}\ln\left|\frac{\frac{\sqrt{x^2-x+2}}{x}-2}{\frac{\sqrt{x^2-x+2}}{x}+2}\right|+C$

$$2.I=\int\frac{dx}{x\sqrt{2x^2+x+2}}$$

ដោយ $a>0$, $\Delta<0$

តាង $\sqrt{2x^2+x+2}=\sqrt{2}\,x+t$

$$2x^2+x+2=2x^2+2\sqrt{2}\,xt+t^2$$

$$x-2\sqrt{2}\,xt=t^2-2$$

$$x=\frac{t^2-2}{1-2\sqrt{2}\,t}$$

$$\Longrightarrow dx=\frac{2t-2\sqrt{2}\,t^2-4\sqrt{2}}{\left(1-2\sqrt{2}\,t\right)^2}dt$$

$$\Longrightarrow I=\int\frac{\dfrac{2t-2\sqrt{2}\,t^2-4\sqrt{2}}{\left(1-2\sqrt{2}\,t\right)^2}}{\dfrac{(t^2-2)}{\left(1-2\sqrt{2}\,t\right)}\left[\sqrt{2}\left(\dfrac{t^2-2}{1-2\sqrt{2}\,t}\right)+t\right]}$$

$$=\int\frac{\left(2t-2\sqrt{2}\,t^2-4\sqrt{2}\right)dt}{(t^2-2)\left(\sqrt{2}\,t^2-2\sqrt{2}+t-2\sqrt{2}\,t^2\right)}$$

$$=\int\frac{2\left(t-\sqrt{2}\,t^2-2\sqrt{2}\right)dt}{(t^2-2)\left(t-\sqrt{2}\,t^2-2\sqrt{2}\right)}$$

$$=2\int\frac{dt}{t^2-\left(\sqrt{2}\right)^2}=\frac{2}{2\sqrt{2}}\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right|+C$$

$$=\frac{\sqrt{2}}{2}\ln\left|\frac{\dfrac{\sqrt{2x^2+x+2}}{\sqrt{2}\,x}-\sqrt{2}}{\dfrac{\sqrt{2x^2+x+2}}{\sqrt{2}\,x}+\sqrt{2}}\right|+C$$

ដូចនេះ $I=\dfrac{\sqrt{2}}{2}\ln\left|\dfrac{\dfrac{\sqrt{2x^2+x+2}}{\sqrt{2}\,t}-\sqrt{2}}{\dfrac{\sqrt{2x^2+x+2}}{\sqrt{2}\,x}+\sqrt{2}}\right|+C$

- ចំពោះ *Euler* (2)

$$1.I=\int\frac{dx}{x\sqrt{3x^2-2x+1}}$$

ដោយ $c>0$, $\Delta<0$

តាង $\sqrt{3x^2-2x+1}=xt+1$

$\sqrt{3x^2-2x+1}=xt+1$

$\Longleftrightarrow 3x^2-2x+1=(xt+1)^2$

$3x^2-2x+1=x^2t^2+2xt+1$

$x(3x-2-xt^2-2t)=0$

$x=0$ $(don't\ take)$

$3x-2-xt^2-2t=0$

$\Longrightarrow x=\dfrac{2+2t}{-t^2+3}$

$\Longrightarrow dx=\dfrac{2t^2+4t+6}{(3-t^2)^2}dt$

$$\Longrightarrow I=\int\frac{\left[\frac{2t^2+4t+6}{(3-t^2)^2}\right]dt}{\left(\frac{2+2t}{-t^2+3}\right)\left[\left(\frac{2+2t}{-t^2+3}\right)\times t+1\right]}$$

$$=\int\frac{(2t^2+4t+6)\,dt}{(2+2t)\left[(2+2t)t+(3-t^2)\right]}$$

$$=2\int\frac{(t^2+2t+3)\,dt}{(2+2t)\,(t^2+2t+3)}$$

$$=\int\frac{dt}{1+t}=\ln|1+t|+C=\ln\left|1+\frac{\sqrt{3x^2-2x+1}-1}{x}\right|+C$$

ដូចនេះ: $I=\ln\left|1+\dfrac{\sqrt{3x^2-2x+1}-1}{x}\right|+$

$2.I=\displaystyle\int\frac{dx}{\sqrt{2x^2-4x+4}}$

ដោយ $c>0$, $\Delta>0$

តាង $\sqrt{2x^2-4x+4}=xt+2$

$$\Longrightarrow 2x^2-4x+4=(xt+2)^2=x^2t^2+4xt+4$$

$$x(2x-xt^2-4-4t)=0$$

$$x=0\ (don't\ take)$$

$$2x-xt^2-4-4t=0$$

$$\Longrightarrow x=\frac{4-4t}{2-t^2}$$

$$\Longrightarrow dx=\frac{(4t^2+8t+8)dt}{(2-t^2)^2}$$

$$\Longrightarrow I=\int\frac{\left[\frac{4t^2+8t+8}{(2-t^2)^2}\right]dt}{\left(\frac{4+4t}{2-t^2}\right)\times t+2}$$

$$=4\int\frac{(t^2+2t+2)dt}{(2-t^2)(2t^2+4t+4)}$$

$$2\int\frac{dt}{2-t^2}=\frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{2}+t}{\sqrt{2}-t}\right|+C$$

$$=\frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{2}+\frac{\sqrt{2x^2-4x+4}}{x}-2}{\sqrt{2}-\frac{\sqrt{2x^2-4x+4}}{x}-2}\right|+C$$

ដូចនេះ $I=\frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{2}+\frac{\sqrt{2x^2-4x+4}}{x}-2}{\sqrt{2}-\frac{\sqrt{2x^2-4x+4}}{x}-2}\right|+C$

- ចំពោះ $Euler\ (3)$

$$1.I=\int\frac{dx}{x\sqrt{2x^2-3x-2}}$$

ដោយ $\Delta>0$, $2x^2-3x-2$ មានឫស $x_1=2$, $x_2=-\frac{1}{2}$

តាង $\sqrt{2x^2-3x-2}=(x-2)t$

$$\Longleftrightarrow 2x^2-3x-2=(x-2)^2t^2$$

$$\Longleftrightarrow 2x+1=xt^2-2t^2$$

$$\Longleftrightarrow x=\frac{2t^2+1}{t^2-2}$$

$$\Longrightarrow dx=\frac{-10tdt}{(t^2-2)^2}$$

$$\Longrightarrow I=\int\frac{\frac{-10tdt}{(t^2-2)^2}}{\left(\frac{2t^2+1}{t^2-2}\right)\left(\frac{2t^2+1}{t^2-2}-2\right)t}$$

$$=-10\int\frac{tdt}{(2t^2+1)5t}$$

$$=-2\int\frac{dt}{2t^2+1}=-\int\frac{dt}{t^2+\left(\sqrt{\frac{1}{2}}\right)^2}$$

$$=-\sqrt{2}\arctan\sqrt{2}\,t+C=-\sqrt{2}\arctan\sqrt{2}\left(\frac{\sqrt{2x^2-3x-2}}{x-2}\right)+C$$

ដូចនេះ $I=-\sqrt{2}\arctan\sqrt{2}\left(\frac{\sqrt{2x^2-3x-2}}{x-2}\right)+C$

$$2.I=\int\frac{dx}{\sqrt{x^2-2x-8}}$$

ដោយ $\Delta>0$, x^2-2x-8 មានឫស $x_1=4$, $x_2=-2$

តាង $\sqrt{2x^2-2x-8}=(x+2)t$

$$\sqrt{x^2-2x-8}=(x+2)t$$

$$\Longrightarrow x^2-2x-8=(x-2)^2t^2$$

$$\Longrightarrow x=\frac{2t^2+4}{1-t^2}$$

$$\Longrightarrow dx=\frac{12tdt}{(1-t^2)^2}$$

$$\Longrightarrow I=\frac{\frac{12t}{(1-t^2)}dt}{\left(\frac{2t^2+4}{1-t^2}+2\right)t}=\int\frac{12tdt}{(1-t^2)\times 6t}$$

$$=2\int\frac{dt}{1-t^2}=\ln\left|\frac{1+\frac{\sqrt{x^2-2x-8}}{x+2}}{1-\frac{\sqrt{x^2-2x-8}}{x+2}}\right|+C$$

ដូចនេះ $I=\ln\left|\frac{1+\frac{\sqrt{x^2-2x-8}}{x+2}}{1-\frac{\sqrt{x^2-2x-8}}{x+2}}\right|+C$

- ករណី: អាំងតេក្រាលតាមវិធីសាស្រ្តជំនួសដោយត្រីកោណមាត្រ
  - បើអាំងតេក្រាលមានរាងជារ៉ាឌីកាល់ $\sqrt{a^2-x^2}$

$1.I=\int\frac{dx}{x^2\sqrt{9-x^2}}$

តាង $x=3\sin t\Longrightarrow dx=3\cos tdt$

ហើយ $\sqrt{3^2-x^2}=3\cos t$

$$\Longrightarrow I=\int\frac{3\cos tdt}{9\sin^2 t\times 3\cos t}=\frac{1}{9}\int\frac{\cos tdt}{\sin^2 t\cos t}=\frac{1}{9}\int\frac{dt}{\sin^2 t}=-\frac{1}{9}\cot t+C$$

$$=-\frac{1}{9}\left(\frac{\cos t}{\sin t}\right)+C=-\frac{1}{9}\frac{\frac{\sqrt{9-x^2}}{3}}{\frac{x}{3}}+C=-\frac{1}{9}\frac{\sqrt{9-x^2}}{x}+C$$

ដូចនេះ $I=-\frac{1}{9}\frac{\sqrt{9-x^2}}{x}+C$

$2.I=\int\frac{dx}{x^2\sqrt{16-x^2}}$

តាង $x=4\sin t\Longrightarrow dx=4\cos tdt$

ហើយ $\sqrt{4^2-x^2}=4\cos t$

$$I=\int \frac{4\cos tdt}{16\sin^2 t\times 4\cos tdt}$$

$$=\frac{1}{16}\int \frac{dt}{\sin^2 t}=-\frac{1}{16}\left(\frac{\cos t}{\sin t}\right)+C$$

$$=-\frac{1}{16}\frac{\frac{\sqrt{16-x^2}}{4}}{\frac{x}{4}}+C=-\frac{1}{16}\times\frac{\sqrt{16-x^2}}{x}+C$$

ដូចនេះ $I=-\frac{1}{16}\times\frac{\sqrt{16-x^2}}{x}+C$

- អាំងតេក្រាលមានរាងជាអ៊ីកាល់ $\sqrt{x^2-a^2}$

$$1.I=\int \frac{\sqrt{x^2-4}}{x}dx$$

តាង $x=\frac{2}{\cos t}\Longrightarrow dx=\frac{2\sin tdt}{\cos^2 t}$

ហើយ $\sqrt{x^2-4}=2\tan t$

$$\Longrightarrow I=\int \frac{\left(2\tan t\times\frac{2\sin t}{\cos^2 t}\right)dt}{\frac{2}{\cos t}}=2\int \tan t\times\frac{\sin t}{\cos t}$$

$$=2\int \tan^2 \mathrm{tdt}=2\int (1+\tan^2 \mathrm{t}-1)\mathrm{dt}$$

$$=2\left[\int (1+\tan^2 \mathrm{t})-\int \mathrm{dt}\right]=2(\mathrm{tant}-\mathrm{t})+\mathrm{C}$$

$$=\sqrt{x^2-4}-2\arccos\frac{2}{x}+C$$

ដូចនេះ $I=\sqrt{x^2-4}-2\arccos\frac{2}{x}+C$

$$2.I=\int \frac{\sqrt{x^2-36}}{x}dx$$

តាង $x=\frac{6}{\cos t}\Longrightarrow dx=\frac{6\sin tdt}{\cos^2 t}$

ហើយ $\sqrt{x^2-36}=6\tan t$

$$\Longrightarrow I=\int \frac{6\tan t\times \frac{6\sin t}{\cos^2 t}}{\frac{6}{\cos t}}dt=6\int \tan^2 t dt$$

$$=6\int (1+\tan^2 t-1)\,dt=6(\tan t-t)+C$$

$$=\sqrt{x^2-6^2}-6\arctan\frac{x}{6}+C$$

ដូចនេះ៖ $I=\sqrt{x^2-6^2}-6\arctan\frac{x}{6}+C$

- បើអាំងតេក្រាលមានរាងជារ៉ាឌីកាល់ $\sqrt{x^2+a^2}$

$$1.I=\int \frac{dx}{\sqrt{x^2+4}}$$

តាង $x=2\tan t\Longrightarrow dx=\frac{2}{\cos^2 t}dt$

ហើយ $\sqrt{x^2-2^2}=\frac{2}{\cos t}$

$$\Longrightarrow I=\int \frac{2}{\cos^2 t}\times\frac{1}{\frac{2}{\cos t}}dt$$

$$I=\int \frac{dt}{\cos t}=\ln\left|\tan t+\frac{1}{\cos t}\right|+C$$

$$=\ln\left|\frac{x}{2}+\frac{\sqrt{x^2+2^2}}{2}\right|+C=\ln\left|x+\sqrt{x^2-2^2}\right|+C$$

ដូចនេះ៖ $I=\ln\left|x+\sqrt{x^2-2^2}\right|+C$

$$2.I=\int \frac{dx}{\sqrt{x^2+16}}$$

តាង $x=4\tan t\Longrightarrow dx=\frac{4}{\cos^2 t}dt$

ហើយ $\sqrt{x^2-4^2}=\frac{4}{\cos t}$

$$\Longrightarrow I=\int \frac{4}{\cos^2 t}\times\frac{1}{\frac{4}{\cos t}}dt$$

$$I=\int \frac{dt}{\cos t}=\ln\left|\tan t+\frac{1}{\cos t}\right|+C$$

$$=\ln\left|\frac{x}{4}+\frac{\sqrt{x^2+4^2}}{}\right|+C=\ln\left|x+\sqrt{x^2-4^2}\right|+C$$

ដូចនេះ៖ $I=\ln\left|x+\sqrt{x^2-4^2}\right|+C$

- ករណី: អាំងតេក្រាលប្រភាគសនិទាន

$$1.I=\int \frac{x^2+2x-2}{x-1}dx$$

$$\Longrightarrow I=\int (x+3)+\frac{1}{x-1}dx$$

$$=\frac{x^2}{2}+3x+\ln|x-1|+C$$

ដូចនេះ៖ $I=\frac{x^2}{2}+3x+\ln|x-1|+C$

$$2.I=\int \frac{2x+1}{x(x-1)(x-1)^2}dx$$

ដោយ $\frac{2x+1}{x(x-1)^3}=\frac{A}{x}+\frac{B}{(x-1)^3}+\frac{C}{(x-1)^2}+\frac{D}{(x-1)}$

ចំពោះ៖ $x=1\Longrightarrow B=3$

$x=0 \Longleftrightarrow A=-1$

$x=-12 \Longleftrightarrow A+2B+2C+2D=5$

$C+D=0 \ (1)$

$x=3 \Longleftrightarrow 8A+3B+2C+12D=7$

$C+2D=1 \ (2)$

$$\begin{array}{l}\begin{cases} C+D=0 \\ C+2D=1 \end{cases} \\ \hline \quad -D=-1 \end{array}$$

$\Longrightarrow D=1 \Longrightarrow C=-1$

$$\Longrightarrow I=\int\left(-\frac{1}{x}+\frac{3}{(x-1)^3}-\frac{1}{(x-1)^2}+\frac{1}{(x-1)}\right)dx$$

$$=-\ln|x|-\frac{3}{2(x-1)^2}-\frac{1}{(x-1)}+\ln|x-1|+C$$

ដូចនេះ $I=-\ln|x|-\dfrac{3}{2(x-1)^2}-\dfrac{1}{(x-1)}+\ln|x-1|+C$

- ករណី: អាំងតេក្រាលភាគបែងមានឫសលំដាប់ខ្ពស់ច្រើន ដោយប្រើវិធីសាស្ត្រ (*Ostrogradsky*)

$1.I=\int \dfrac{4x^3-3x^2-2}{(x^3-1)^2}dx$

តាង $\begin{cases} p(x)=4x^3-3x^2-2 \\ Q(x)=(x^3-1)^2 \Longrightarrow Q(x)=6x^2(x^3-1) \end{cases}$

គេបាន៖ $\begin{cases} Q_1(x)=x^3+1 \\ Q_2(x)=\dfrac{Q(x)}{Q_1(x)}=x^3+1 \\ P_1(x)=Ax^2+Bx+C \\ P_1(x)=Dx^2+Ex+F \end{cases}$

តាម $\int \dfrac{P(x)}{Q(x)}dx=\dfrac{P_1(x)}{Q_1(x)}+\dfrac{P_2(x)}{Q_2(x)}dx$

នោះ $\int \frac{4x^3-3x^2-2}{(x^3-1)^2}dx = \frac{Ax^2+Bx+C}{x^3-1} + \int \frac{Dx^2+Ex+F}{x^3-1}dx$

$$\Longleftrightarrow \frac{x^4-x^2+2}{(x^3-1)^2} = \frac{(x^3-1)(2Ax+B)-3x^2(Ax^2+Bx+C)}{(x^3-1)^2} + \frac{Dx^2+Ex+C}{x^3-1}$$

$$\Longleftrightarrow x^4-x^2+2 = Dx^5+(E-A)x^4+(F-2B)x^3+(-D-3C)x^2$$
$$+(-2A-E)x+(-F-B)$$

$$\begin{cases} D=0 \\ E-A=1 & (1) \\ F-2B=0 & (2) \\ -3C-D=-1 & (3) \\ -2A-E=0 & (4) \\ -B-F=2 & (5) \end{cases}$$

$(3):-3C-D=1 \Longrightarrow C=\frac{1}{3}$

$(2)+(5):B=-\frac{2}{3}$

$(5):\Longrightarrow F=-\frac{4}{3}$

$(1)+(4):\Longrightarrow A=-\frac{1}{3}$

$(4):\Longrightarrow E=\frac{2}{3}$

គេបាន៖ $\int \frac{x^4-x^2+2}{(x^3-1)^2}dx = \frac{-\frac{1}{3}x^2-\frac{2}{3}x+\frac{1}{3}}{x^3-1}\int \frac{\frac{2}{3}x-\frac{4}{3}}{x^3-1}dx$

$$-\frac{1}{3}\left(\frac{x^2+2x-1}{x^3-1}\right)+\frac{1}{3}I_1$$

គណនា $I_1=\int \frac{2x-4}{(x^3-1)}dx = \int \frac{2x-4}{(x-1)(x^2+x+1)}dx$

ដោយ $\frac{2x-4}{(x-1)(x^2+x+1)} = \frac{a}{(x-1)} + \frac{bx+}{x^2+x+1}$

$$\begin{cases} a=-b & (6) \\ 2a+c=2 & (7) \\ a-c=-4 & (8) \end{cases} \Longleftrightarrow \begin{cases} a=-\frac{2}{3} \\ b=\frac{2}{3} \\ c=\frac{10}{3} \end{cases}$$

គេបាន $I_1=\int \frac{2x-4}{x^2+x+1}dx = \int\left[\frac{-\frac{2}{3}}{x-1}+\frac{\frac{2}{3}+\frac{10}{3}}{x^2+x+1}\right]dx$

$$\Longrightarrow I_1=-\frac{2}{3}\ln|x-1|+\frac{1}{3}\ln|x^2+x+1|+2\sqrt{3}\arctan\left(\frac{2\sqrt{3}x+\sqrt{3}}{3}\right)+C$$

ដូចនេះ

$$\Longrightarrow I_1=-\frac{2}{3}\ln|x-1|+\frac{1}{3}\ln|x^2+x+1|+2\sqrt{3}\arctan\left(\frac{2\sqrt{3}x+\sqrt{3}}{3}\right)+C$$

- ករណី: អាំងតេក្រាលនៃទ្វេធាឌីផេរ៉ង់ស្យែល

$1.I=\int x^{-2}(1+x^2)^{-\frac{1}{2}}dx$

ដោយ $-\frac{1}{2}\notin\mathbb{Z}$, $\frac{m+1}{n}=\frac{-2+1}{-\frac{1}{2}}=2\in\mathbb{Z}$

$K=2$

$$x=\frac{1+x^2}{x^2}=t^2$$

តាង $\Longrightarrow x=\frac{1}{\sqrt{t^2-1}}\Longrightarrow dx=-\frac{tdt}{(t^2-1)\left(\sqrt{t^2-1}\right)}$

$$\Longrightarrow I=\int (t^2-1)\left(1+\frac{1}{t^2-1}\right)^{-\frac{1}{2}}-\frac{tdt}{(t^2-1)\left(\sqrt{t^2-1}\right)}dt$$

$$=-\int dt=-t=-\sqrt{\frac{1+x^2}{x^2}}+C$$

ដូចនេះ $I=-\sqrt{\dfrac{1+x^2}{x^2}}+C$

$2.I=\displaystyle\int\frac{dx}{\sqrt{x}\left(1+\sqrt{x}\right)^2}$

ដោយ $P=-2\in\mathbb{Z}$

នោះ $K=2$ ជាភាគបែងនៃ $\dfrac{m+1}{n}$

តាង $x=t^2\Longrightarrow dx=2tdt$

$x^{-\frac{1}{2}}=(t^2)^{-\frac{1}{2}}=\dfrac{1}{t}$

$x^{\frac{1}{2}}=(t^2)^{\frac{1}{2}}=t$

$\Longrightarrow I=\displaystyle\int\frac{1}{t}(1+t)^{-2}2tdt$

$\quad=\displaystyle\int\frac{2dt}{(1+t^2)}=-\frac{2}{1+\sqrt{x}}+C$

ដូចនេះ $I=\displaystyle\int\frac{2dt}{(1+t^2)}=-\frac{2}{1+\sqrt{x}}+C$

- ករណី: អាំងតេក្រាលនៃកន្សោមត្រីកោណមាត្រខ្លះៗ

$1.I=\displaystyle\int\frac{dx}{4-5\sin x}$

តាង $t=\tan\dfrac{x}{2}\Longrightarrow dx=\dfrac{2dt}{1+t^2}$, $\sin x=\dfrac{2t}{1+t^2}$

គេបាន $I=\displaystyle\int\frac{\frac{2dt}{1+t^2}}{4-5\left(\frac{2t}{1+t^2}\right)}=2\int\frac{dt}{4+4t^2-10t}$

$=\dfrac{1}{2}\displaystyle\int\frac{dt}{t^2-\frac{10}{4}t+1}=\frac{2}{5}\int\frac{dt}{\left(t-\frac{5}{4}\right)^2-\left(\frac{3}{4}\right)^2}=-\frac{4}{15}\ln\left|\frac{t-\frac{5}{4}-\frac{3}{4}}{t-\frac{5}{4}+\frac{3}{4}}\right|$

$=-\dfrac{4}{5}\ln\left|\dfrac{4\tan\left(\frac{x}{2}\right)-8}{4\tan\left(\frac{x}{2}\right)-2}\right|+C$

ដូចនេះ $I = -\frac{4}{5}\ln\left|\frac{4\tan\left(\frac{x}{2}\right)-8}{4\tan\left(\frac{x}{2}\right)-2}\right| + C$

$$2.I = \int \frac{dx}{5-3\cos x}$$

តាង $t = \tan\frac{x}{2} \Longrightarrow dx = \frac{2dt}{1+t^2}$ $, \cos x = \frac{1-t^2}{1+t^2}$

$$\Longrightarrow I = \int \frac{\frac{2dt}{1+t^2}}{5-3\left(\frac{1-t^2}{1+t^2}\right)} = 2\int \frac{dt}{5+5t^2-3+3t^2}$$

$$= 2\int \frac{dt}{8t^2+2} = \frac{1}{4}\int \frac{dt}{t^2+\left(\frac{1}{2}\right)^2} = \frac{1}{2}\arctan 2t + C$$

$$= \frac{1}{2}\arctan\left(2\tan\frac{x}{2}\right) + C$$

ដូចនេះ $I = \frac{1}{2}\arctan\left(2\tan\frac{x}{2}\right) + C$

=========✂=========

សូមអរគុណ !